SIBLING CURVES OF CUBIC POLYNOMIALS

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Abstract. Sibling curves are a new way to represent the zeroes of any complex-valued function. Interesting results are already known for the sibling curves of quadratic polynomials. In this article a complete investigation of sibling curves of cubic polynomials are given. This article concludes by providing a new general result for the sibling curves of complex polynomials.

Key words: Complex numbers, polynomials, cubic polynomials, sibling curves.

1. Introduction

One of the topics that arises when studying complex numbers is to find a way to represent the zeroes of complex-valued functions. Several methods of representing the zeroes are known [1].

The use of sibling curves [2] is a very robust method of representing the zeroes. The sibling curves of $f : \mathbb{C} \to \mathbb{C}$ is the set of all complex numbers c such that $f(c) \in \mathbb{R}$. This restriction gives a way to represent the zeroes of any complex-valued function in three dimensions.

It has been proven whenever f is a polynomial of degree n, then f has exactly n sibling curves [3]. Therefore if f is a quadratic polynomial, then it has precisely two sibling curves. It is known that two possible outcomes exist [4]. If the two sibling curves intersect, then the sibling curves are two parabolas. If the two sibling curves do not intersect, then neither of these curves have the shape of a parabola. In either case we get two sibling curves that are isometric.

In this paper we fully explore the sibling curves of cubic polynomials. Section 2 focuses on the possibilities of sibling curves for real cubic polynomials. This work assists section 3 to completely investigate the possibilities for complex cubic polynomials. Finally, in section 4 the author provides a brand-new result for the sibling curves of any complex polynomial.

2. Real case

This section only concentrates on the sibling curves of real cubic polynomials. This sections start by providing a standard form for cubic polynomials in Lemma 2.1. This standard form is then fully treated in Lemma 2.2. Combining these two lemmas the author now considers general real cubic polynomials. This is done in Theorem 2.3 to provide an easy criteria to classify the the possibilities for real cubic polynomials.

Lemma 2.1. *The sibling curves of* $f(x) = ax^3 + bx^2 + cx + d$ *is the sibling curves of* $g(z) = z^3 + ez + f$ *up to a linear transformation.*

Proof. Using scaling, substitute $x = \frac{1}{\sqrt[3]{a}}y$. Therefore $f(x) = f(\frac{1}{\sqrt[3]{a}}y) = y^3 + \frac{b}{\sqrt[3]{a^2}}y^2 + \frac{c}{\sqrt[3]{a}}y + d$. Now translate, by letting $y = z - \frac{b}{3\sqrt[3]{a^2}}$, that is $x = \frac{z}{\sqrt[3]{a}} - \frac{b}{3a}$. Therefore

$$\begin{aligned} f(x) &= f(\frac{z}{\sqrt[3]{a}} - \frac{b}{3a}) \\ &= a(\frac{z}{\sqrt[3]{a}} - \frac{b}{3a})^3 + b(\frac{z}{\sqrt[3]{a}} - \frac{b}{3a})^2 + c(\frac{z}{\sqrt[3]{a}} - \frac{b}{3a}) + d \\ &= z^3 + (\frac{6ac - 2b^2}{3a\sqrt[3]{a}})z + (\frac{2b^3 - 9abc + 27a^2d}{27a^2}) \end{aligned}$$

Thus $e = \frac{6ac-2b^2}{3a\sqrt[3]{a}}$ and $f = \frac{2b^3-9abc+27a^2d}{27a^2}$. □

Lemma 2.2. *Suppose* e, f *are real numbers. The sibling curves of* $g(z) = z^3 + ez + f$ *has three possibilities:*
(1) If $e = 0$, *then all the three sibling curves intersect at the same point.*
(2) If $e > 0$, *then the sibling curves never intersect.*
(3) If $e < 0$, *then there are two pairs of sibling curves where each pair has an unique point of intersection.*

Proof. Suppose $z = x + iy$ for some real numbers x, y. Then $g(z) = g(x + iy) = (x^3 - 3xy^2 + f) + (3x^2y - y^3 + ey)i$. If $g(z)$ is a real number, then $3x^2y - y^3 + ey = 0$. So $y(3x^2 - y^2 + e) = 0$. This shows that $y = 0$ is always a solution producing the sibling curve $(t, t^3 + et + f)$ where t is a real number. Now we consider three cases depending on the value of e.

<u>Case 1:</u> If $e = 0$, then $3x^2 - y^2 = 0$ or $y = \pm\sqrt{3}x$. This gives two sibling curves $(t + \sqrt{3}ti, f - 8t^2)$ and $(t - \sqrt{3}ti, f - 8t^2)$. Note each sibling curve contains the point $(0, f)$.

<u>Case 2:</u> If $e > 0$, then we have $3x^2 - y^2 = -e$. This is a hyperbola that is the projection of two sibling curves on the horizontal plane that do not intersect. The third sibling curve satisfies $y = 0$. However if $y = 0$, then $3x^2 = -e$, which do not have a real solution when $e > 0$. This proves that in this case

none of the sibling curves intersect.

<u>Case 3:</u> If $e < 0$, then we have $3x^2 - y^2 = -e$. This is again a hyperbola which is the projection of two sibling curves on the horizontal plane that do not intersect. However, note that the sibling curve in the plane $y = 0$ does intersect each sibling curve on the hyperbola at $x = \pm\sqrt{\frac{-e}{3}}$. The points are $(\sqrt{\frac{-e}{3}}, \frac{-e}{3}\sqrt{\frac{-e}{3}} + f)$ on one pair of sibling curves and $(-\sqrt{\frac{-e}{3}}, \frac{e}{3}\sqrt{\frac{-e}{3}} + f)$ on the other pair of sibling curves. These are the only points of intersections. $\square$

Figures 1, 2 and 3 demonstrates cubic polynomials of cases 1, 2 and 3 respectively.

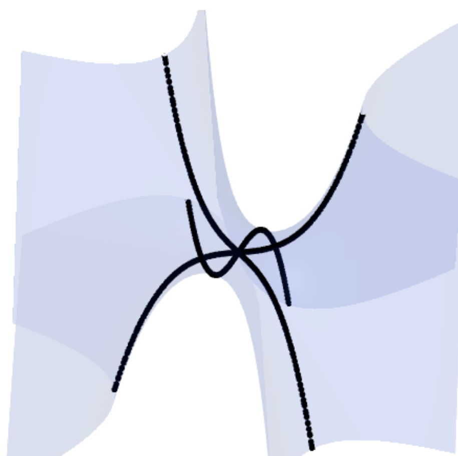

FIGURE 1. The sibling curves of $f(x) = x^3 + 3$.

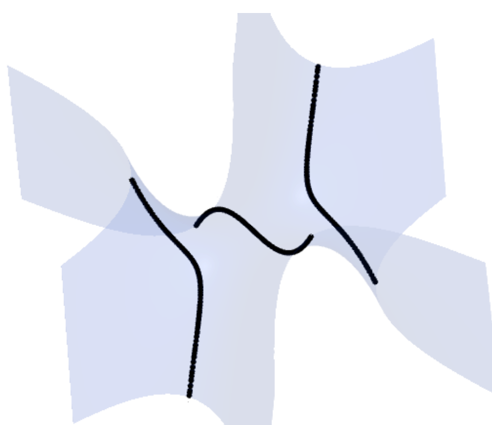

FIGURE 2. The sibling curves of $f(x) = x^3 + x + 1$.

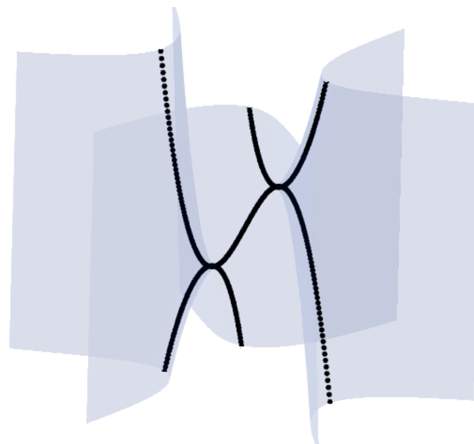

FIGURE 3. The sibling curves of $f(x) = x^3 - x$.

Theorem 2.3. *Suppose $f(x) = ax^3 + bx^2 + cx + d$ is a real cubic polynomial. The sibling curves has three possibilities depending on the value of $\triangle = 3ac - b^2$.*
(1) If $\triangle = 0$, then all the three sibling curves intersect at the same point.
(2) If $\triangle > 0$, then the sibling curves never intersect.
(3) If $\triangle < 0$, then there are two pairs of sibling curves where each pair has an unique point of intersection.

Proof. This is immediate from Lemma 2.1 and Lemma 2.2 where $e = 6ac - 2b^2$. □

Therefore Figures 1-3 exhaust all the possibilities for the real cubic case. It should be noted when $\triangle = 0$ we get three sibling curves that are isometric. If $\triangle \neq 0$ then there is always a pair of isometric sibling curves.

3. Complex case

This section focuses on the complex cubic polynomial case. Theorem 3.1 provides a way of determining the number of intersections that occurs given the polynomial.

Theorem 3.1. *Suppose $f(z) = az^3 + bz^2 + cz + d$ where $a \neq 0$. Let z_1, z_2 be the two complex roots of $3az^2+2bz+c = 0$. The number of intersections of the three sibling curves of f is the cardinality of the set $\{z_i : \frac{6ac-2b^2}{9a} z_i + \frac{9ad-bc}{9a} \in \mathbb{R}\}$.*

Proof. It was shown in [3] that polynomial sibling curves intersect iff there is a complex number z such that $f'(z) = 0$ and $f(z) \in \mathbb{R}$. Applying polynomial division it follows that $f(z) = f'(z).(\frac{z}{3} + \frac{b}{9a}) + \frac{6ac-2b^2}{9a}z + \frac{9ad-bc}{9a}$, where $f'(z) = 3az^2+2bz+c$. Noting for any z_i that $f(z_i) = 0$, the result follows immediately. □

It should be remarked that this result easily treats the case when a, b, c, d are real numbers. If $b^2 = 3ac$ then the quadratic $f'(z)$ has two equal roots. In this case we get only one point that is on each of the sibling curves. If $b^2 - 3ac > 0$, then the roots are complex which shows that the sibling curves never intersect. Lastly, if $b^2 - 3ac < 0$, then there are two distinct solutions whose evaluation at f is real.

So the question begs, is there a new possibility for the sibling curves of complex cubic polynomials that can occur that was not observed when studying real cubic polynomials in section 2. This is confirmed in the next example.

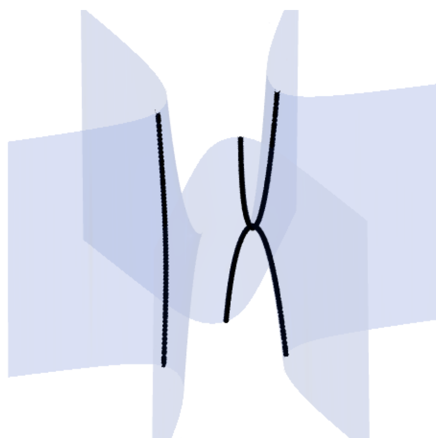

FIGURE 4. The sibling curves of $f(x) = x^3 + 3x - 2i$.

Consider the polynomial $f(x) = x^3 + 3x - 2i$. Solving $f'(x) = 0$, we get $x = \pm i$. Note $f(i) = 0$, but $f(-i) \neq 0$. Therefore by Theorem 3.1, the number of intersections of the sibling curves of f is a single point. This observation is confirmed in Figure 4. This example is a new possibility that was impossible in the real case.

4. A NEW GENERAL PROPERTY

Theorem 4.1. *If $f(z)$ is a complex polynomial of degree n, then the projection onto the horizontal plane has n straight line asymptotes.*

Proof. Suppose $f(z) = c_n z^n + c_{n-1}z^{n-1} + \ldots + c_0$ where $c_j \in \mathbb{C}$. Let $c_j = a_j \operatorname{cis}(b_j) = a_j e^{ib_j}$ for some real numbers a_j, b_j. Let $z = r \operatorname{cis}(\theta)$ for some real numbers r, θ. Solving $\operatorname{Im} f(z) = 0$ produces $0 = a_n.r^n \sin(n\theta + b_j) + \ldots + a_2.r^2 \sin(2\theta + b_2) + a_1.r \sin(\theta + b_1) + a_0$. Therefore

$$\begin{aligned}
\lim_{r\to\infty} a_n.r^n \sin(n\theta + b_j) + \ldots + a_2.r^2 \sin(2\theta + b_2) + a_1.r \sin(\theta + b_1) + a_0 &= 0 \\
\lim_{r\to\infty} \sin(n\theta + b_j) + \ldots + \frac{a_2}{a_n.r^{n-2}} \sin(2\theta + b_2) + \frac{a_1}{a_n.r^{n-1}} \sin(\theta + b_1) + a_0 &= 0 \\
\therefore \sin(n\theta + b_j) &= 0
\end{aligned}$$

This gives the n asymptotes defined by $y = \tan(\theta)x$ where $\sin(n\theta + b_j) = 0$. □

Theorem 4.2. *Suppose $f(z)$ is a complex polynomial of degree n. Then the projections of the sibling curves of f onto the horizontal plane contains no loops.*

Proof. Suppose $\operatorname{Im} f(z) = 0$ has a closed continuous curve C with no points in the interior of the set $\operatorname{Im} f(z) = 0$. So either all the interior points have $\operatorname{Im} f(z) > 0$ or f can be replaced with $-f$.

For each point z in the interior of C, we consider $\operatorname{Im} f(z)$. Suppose w is an interior point of C such that $\operatorname{Im} f(z)$ reaches a maximum. This must exist as the curve C is closed and bounded. Now let $g(z) = f(z + w)$. Suppose $\operatorname{Im} g(0) = M$, where M stands for the maximum value.

Let r be the smallest real number $r > 0$ such that there exists an angle θ such that $\operatorname{Im} g(r.e^{i\theta}) = 0$. It can be shown that $g(r.e^{i\theta}) + g(r.e^{i(\theta + \frac{2\pi}{n+1})}) + \ldots + g(r.e^{i(\theta + \frac{2n\pi}{n+1})}) = (n+1)M$. Hence one of the angles α satisfies $\operatorname{Im} g(r.e^{i\alpha}) \geq \frac{n+1}{n} M > M$ since $g(r.e^{i\theta}) = 0$ and $n \geq 1$. This gives a contradiction, since the maximum of $\operatorname{Im} g(z)$ occurs at $z = 0$. This contradiction proves the claim. □

From this result, it follows whenever $f(z)$ is a complex polynomial of degree n, then $g(x, y) = \operatorname{Im} f(x + iy)$ is a bivariate polynomial. This bivariate polynomial will never have a zero-loop: that is a non-trivial closed curve that satisfies $g(x, y) = 0$. For example, if $f(z) = z^3 + iz$, then $f(x + iy) = (x^3 - 3xy^2 - y) + i(3x^2y - y^3 + x)$. For this polynomial, the projection is the bivariate polynomial $3x^2y - y^3 + x = 0$. It can be shown that $3x^2y - y^3 + x = 0$ has no loop in the $xy-$plane.

As a further consequence from this result, it follows that it is impossible for two sibling curves to intersect at more than one point.

5. Conclusion

We know that a cubic polynomial has three sibling curves. From section 4, it follows that it is impossible for any pair of sibling curves to intersect at more than one point. Therefore up to homotopy, we know that there are four possibilities as shown in Figure 5.

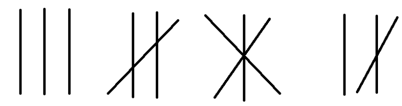

Figure 5. Homotopy possibilities of the sibling curves of cubic polynomials.

In section 2 and 3, the author demonstrated that each homotopy possibility for cubic polynomials occurs. Actually, infinitely many examples exist for each homotopy type. As remarked earlier, let z_1, z_2 be two roots of f'. If the roots are repeated and $f(z_1) \neq \mathbb{R}$ then the first homotopy possibility occurs, otherwise $f(z_1) \in \mathbb{R}$ and then the third homotopy possibility occurs. If the roots are distinct, let n be the cardinality of set $\{f(z_1), f(z_2)\} \cap \mathbb{R}$. If $n = 0$ then homotopy possibility 1 occurs. If $n = 1$ then homotopy possibility 4 occurs. If $n = 2$ then homotopy possibility 2 occurs.

The reader may be interested in the homotopy possibilities for quartic polynomials. In Figure 6 and 7, two possibilites are demonstrated.

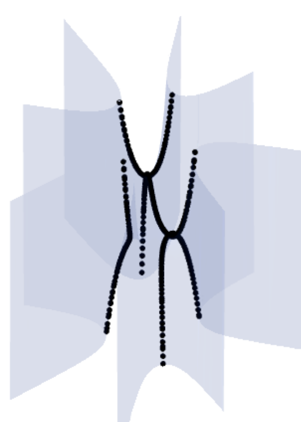

Figure 6. The sibling curves of $f(z) = (-1-2i)z^4 + 4iz^3 + (2-2i)z^2$.

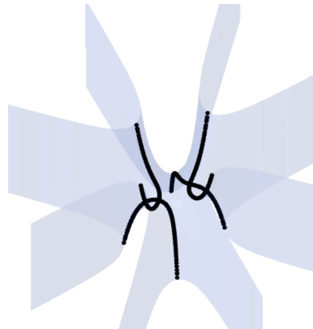

FIGURE 7. The sibling curves of $f(z) = (-i)z^4 + 2iz^3 + (-i)z^2$.

To summarize, the author proved in section 4 that two distinct sibling curves of the same polynomial can never intersect at more than one point. Combining this fact with the fact that a polynomial of degree n has n sibling curves, allows you to list all the homotopy possibilities. For the cubic case, there are four homotopy possibilities. Theorem 2.3 and Theorem 3.1 provides easy-to-check criteria on how to determine which homotopy possibility occurs. The results in section 2 and 3 is therefore a full investigation of the sibling curves of cubic polynomials.

So an open question now remains, given a polynomial of degree n, are all configurations (that can be drawn up to homotopy - just like the cubic case) possible?

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