A NOTE ON THE STOCHASTIC ERICKSEN-LESLIE EQUATIONS FOR NEMATIC LIQUID CRYSTALS

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ABSTRACT. In this note we prove the existence and uniqueness of local maximal smooth solution of the stochastic simplified Ericksen-Leslie systems modelling the dynamics of nematic liquid crystals under stochastic perturbations.

1. **Introduction.** Liquid crystal, which is a state of matter that has properties between amorphous liquid and crystalline solid can be classified into two groups according to the form of their molecules. Liquid crystals with rod-shaped molecules are called calamitics while those with disc-like molecules are referred to discotics. In its turn, the calamitics can be divided into two phases: nematic and smectic. The nematic phase, referred to as nematic liquid crystal, is the simplest of liquid

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crystal phases. Nematic liquid crystals tend to align along a particular direction denoted by a unit vector $\mathbf{d}$, called the optical director axis. Most of the interesting phenomenology of nematic liquid crystals are linked to the geometry and dynamics of this director. We refer to [10] and [15] for a comprehensive treatment of the physics of liquid crystals.

To model the hydrodynamics of nematic liquid crystals, most scientists use the continuum theory developed by Ericksen [17] and Leslie [33]. From this theory, F. Lin and C. Liu [34] derived the most basic and simplest form of the dynamical system describing the motion of nematic liquid crystals flowing in $\mathbb{R}^d (d = 2, 3)$. This system is given by

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + \nabla p = -\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \tag{1.1a}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.1b}$$

$$\mathbf{d}_t + (\mathbf{v} \cdot \nabla)\mathbf{d} = \gamma(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}), \tag{1.1c}$$

$$|\mathbf{d}|^2 = 1. \tag{1.1d}$$

Here $p : \mathbb{R}^d \to \mathbb{R}$, $\mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{d} : \mathbb{R}^d \to \mathbb{R}^3$ represent the pressure, velocity of the fluid and the optical director, respectively. The symbol $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ stands for a square $d \times d$-matrix with entries given by

$$[\nabla \mathbf{d} \odot \nabla \mathbf{d}]_{i,j} = \sum_{k=1}^{3} \frac{\partial \mathbf{d}^k}{\partial x_i} \frac{\partial \mathbf{d}^k}{\partial x_j}, \quad \text{for any } i, j = 1, \dots, d.$$

Since the work of Lin and Liu [34], the Ginzburg-Landau system (1.1) itself, the approximation of the Ginzburg-Landau system, in which the term $|\mathbf{d}|^2 \mathbf{d}$ is replaced by $-\frac{1}{\varepsilon}^2 (|\mathbf{d}|^2 - 1)\mathbf{d}$ where $\varepsilon > 0$, and its several generalizations, have been the subjects of intensive mathematical studies. We refer, among others, to [2, 9, 11, 19, 24, 34, 35, 37, 39, 40, 50] for results obtained prior to 2013, and to [14, 18, 22, 26, 27, 28, 38, 49, 51, 52] for results obtained after 2014. For a detailed review of the literature about the mathematical theory of nematic liquid crystals and other related models, we recommend the review articles [36, 12] and the recent paper [23].

In this paper we consider the following system of stochastic partial differential equations (SPDEs)

$$d\mathbf{v} + \Big[(\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + \nabla p\Big] dt = \Big[-\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})\Big] dt + Q(\mathbf{v})\, dW \text{ in } \mathscr{O} \times (0, T] \tag{1.2a}$$

$$\nabla \cdot \mathbf{v} = 0 \text{ in } \mathscr{O} \times [0, T], \tag{1.2b}$$

$$\int_{\mathscr{O}} \mathbf{v}(t, x) dx = 0 \text{ for all } t \in [0, T] \tag{1.2c}$$

$$\partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} + (\mathbf{d} \times \mathbf{h}) \circ d\eta \text{ in } \mathscr{O} \times (0, T] \tag{1.2d}$$

$$|\mathbf{d}|^2 = 1 \text{ in } \mathscr{O} \times [0, T]. \tag{1.2e}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 \text{ and } \mathbf{d}(0, \cdot) = \mathbf{d}_0 \text{ in } \mathscr{O}, \tag{1.2f}$$

where we denote by $\mathcal{O}$ the d-dimensional torus $[-\pi, \pi]^d$, $d = 2, 3$, the mapping $\mathbf{h} : \mathbb{R}^d \to \mathbb{R}^3$ is a given function, W a cylindrical Wiener process evolving on a separable Hilbert space K_1, η is a one-dimensional standard Brownian motion, and Q is a nonlinear map satisfying several conditions specified later on.

Throughout this paper we assume that $\mathbf{v}, \mathbf{d}, p$, as well as $\mathbf{h}$ are 2π-periodic in the following sense:

$$u(x + 2\pi e_i) = u(x), \quad u \in \{\mathbf{v}, \mathbf{d}, p, \mathbf{h}\}, \quad x \in \mathbb{R}^d, i \in \{1, \ldots, d\}, \tag{1.3}$$

where $\{e_i, i = 1, \ldots, d\}$ is the canonical basis of $\mathbb{R}^d$. In what follows, when we refer to problem (1.2), we refer to the system of equations (1.2) with the boundary condition given in (1.3).

The system of SPDEs (1.2) describes the dynamics of nematic liquid crystal with a stochastic perturbation. Our investigation is motivated by the need for a mathematical analysis of the effect of the stochastic external perturbation on the dynamics of nematic liquid crystals. While the role of noise on the dynamics of $\mathbf{d}$ has been the subject of numerous theoretical and experimental studies in physics, see, for instance, [29, 45, 46], in which it is found that the time needed by the system to leave an unstable state diminishes in the presence of fluctuating magnetic fields, there are almost no rigorous mathematical results in his direction of research. The works [29, 45, 46] and the mathematical paper we cited earlier neglected either the effect of the velocity $\mathbf{v}$ or the stochastic external perturbation, although, de Gennes and Prost [15] noted that $\mathbf{v}$ plays an essential role in the dynamics of $\mathbf{d}$. It is this gap in knowledge that is the motivation for our mathematical study. The current authors established in [7] some existence, uniqueness and a maximum principle results for the stochastic version of a Ginzburg-Landau approximation of the system (1.2) without the sphere condition (1.2e).

In this paper we study the local resolvability of problem (1.2). Our result can be summarized as follows. Given a number $\alpha > \frac{d}{2}$ and a square integrable $\mathbb{H}^\alpha_{\mathrm{sol}} \times \mathbb{H}^{\alpha+1}$-valued random variable $(\mathbf{v}_0, \mathbf{d}_0)$ we can find a stopping time $\tilde{\tau}_\infty$ which can be approximated by an increasing sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$ and a unique local stochastic process $(\mathbf{v}, \mathbf{d}) = (\mathbf{v}(t), \mathbf{d}(t)), 0 \leqslant t < \tilde{\tau}_\infty$ satisfying the following conditions

1. $\tilde{\tau}_\infty > 0$ with positive probability,
2. $(\mathbf{v}(\cdot \wedge \tau_m), \mathbf{d}(\cdot \wedge \tau_m)) \in C([0, T]; \mathbb{H}^\alpha_{\mathrm{sol}} \times \mathbb{H}^{\alpha+1}) \cap \mathbb{L}^2(0, T; \mathbb{H}^{\alpha+1} \times \mathbb{H}^{\alpha+2})$ for any $m \in \mathbb{N}$, with probability 1;
3. and for all $t \in (0, T]$ and $m \in \mathbb{N}$ we have $\mathbb{P}$-a.s. $|\mathbf{d}(t \wedge \tau_m, x)| = 1$ for all $x \in \mathcal{O}$,
4. the process $(\mathbf{v}, \mathbf{d}) = (\mathbf{v}(t), \mathbf{d}(t)), 0 \leqslant t < \tilde{\tau}_\infty$ is a unique local solution to problem (1.2), see Definitions 3.3 and 3.4.

Moreover, we established probabilistic lower bounds on the lifespan $\tilde{\tau}_\infty$ of the local maximal solution.

These results extend to the stochastic case the local existence and uniqueness results for (1.1) obtained for the deterministic model by Wang et al. in [50]. Our proof consists of two steps. In the first one, we apply earlier results obtained in [7] to prove the existence and uniqueness of a maximal local solution satisfying the mild form of equations (1.2a)-(1.2d). In the second one, we prove that when properly localised the local solution preserves the sphere condition (1.2e).

The structure of the paper is as follows. In section 2 we present the main notation and standing assumptions we will be using in the whole paper. In section 3, we introduce the concept of a solution and state our main results. The proof of the main theorems are given in section 4 and section 5.

2. **Functional spaces and hypotheses.** We begin by introducing the necessary definitions of functional spaces frequently used in this work. We denote by $\mathscr{O}$ the d-dimensional torus $d = 2, 3$. Functions defined on $\mathscr{O}$ will be frequently identified with functions defined on the set $[-\pi, \pi]^d$ satisfying appropriate to their regularity periodic boundary conditions, for example, (1.3).

Throughout this paper we denote by $L^p(\mathscr{O})$ and $W^{m,p}(\mathscr{O})$, $p \in [1, \infty]$, $m \in \mathbb{N}$, the Lebesgue and Sobolev spaces of real valued functions defined on $\mathscr{O}$, see e.g. the monograph [48] by Temam (compare [3]). The corresponding spaces of $\mathbb{R}^d$(or some cases $\mathbb{R}^3$)-valued functions, will be denoted by the black-board fonts, e.g. the space $L^p(\mathscr{O}, \mathbb{R}^d)$ will be denoted by $\mathbb{L}^p(\mathscr{O})$.

For non-integer $r > 0$ the Sobolev spaces $H^{r,p}(\mathscr{O})$ and $\mathbb{H}^{r,p}(\mathscr{O})$ are defined by using the complex interpolation method. We will also use the notation $H^r(\mathscr{O}) := W^{r,2}(\mathscr{O})$. We simply skip the symbol of the torus $\mathscr{O}$, when there is no risk of ambiguity. For instance we will write L^p, resp. $\mathbb{L}^p$ of $\mathbb{W}^{m,p}$ instead of $L^p(\mathscr{O})$, resp. $\mathbb{L}^p(\mathscr{O})$ or $\mathbb{W}^{m,p}(\mathscr{O})$.

Given two Banach spaces K and H, we denote by $\mathscr{L}(K, H)$ the space of bounded linear operators. For two Hilbert space K and H we denote by $\mathscr{L}_2(K, H)$ the Hilbert space of all Hilbert-Schmidt operators from K to H. For $K = H$ we just write $\mathscr{L}(K)$ instead of $\mathscr{L}(K, K)$.

Following [48] we also introduce the following spaces

$$\begin{aligned} \mathbb{L}^2_0 =& \left\{ \mathbf{u} \in L^2(\mathscr{O}, \mathbb{R}^d) : \int_{\mathscr{O}} \mathbf{u}(x)dx = 0 \right\}, \\ \mathrm{H} =& \left\{ \mathbf{u} \in \mathbb{L}^2_0 : \operatorname{div} \mathbf{u} = 0 \right\}, \\ \mathbb{H}^r_{\mathrm{sol}} =& \mathrm{H} \cap \mathbb{H}^r, \quad r \in (0, \infty), \quad \mathrm{V} = \mathbb{H}^1_{\mathrm{sol}}. \end{aligned}$$

In the above formula, the divergence is understood in the weak sense. Note that $\mathbb{H}^0_{\mathrm{sol}} = \mathrm{H}$.

In (1.2), it is convenient to eliminate the pressure p by applying the Helmholtz-Leray projector operator $\Pi : \mathbb{L}^2 \to \mathrm{H}$ which projects into divergence free vectors and annihilates gradients. One of the remarkable properties of Π is that $\Pi \in \mathscr{L}(\mathbb{H}^r, \mathbb{H}^r_{\mathrm{sol}})$, $r > 0$, see [4]. We will frequently use this property without further notice.

Next, we define the Stokes operator, denoted by A, which is an unbounded linear operator on H, as follows.

$$\begin{cases} D(\mathrm{A}) & := \mathbb{H}^2_{\mathrm{sol}} \\ \mathrm{A}u & := -\Pi\Delta u, \; u \in D(\mathrm{A}). \end{cases} \tag{2.1}$$

It is well known that A is a strictly positive self-adjoint operator in H and that $D(\mathrm{A}^{1/2}) = \mathrm{V}$. It is also true that A is a strictly positive self-adjoint operator in every space $\mathbb{H}^r_{\mathrm{sol}}$, $r > 0$.

We will also need a version of the Laplace operator acting on $\mathbb{R}^3$-valued functions defined on $\mathscr{O}$, i.e.

$$\begin{cases} D(\mathrm{A}_2) & := \mathbb{H}^2(\mathscr{O}, \mathbb{R}^3), \\ \mathrm{A}_2 u & := -\Delta u, \ u \in D(\mathrm{A}_2). \end{cases} \tag{2.2}$$

It is well known that A_2 is a non-negative self-adjoint operator in $L^2(\mathscr{O}, \mathbb{R}^3)$. It is also true that A_2 is a non-negative self-adjoint operator in every space $H^r(\mathscr{O}, \mathbb{R}^3)$, $r > 0$.

It is well-known that $-\mathrm{A}$ (resp. $-\mathrm{A}_2$) is the infinitesimal generator analytic C_0-semigroup of contractions on H, resp. $L^2(\mathscr{O}, \mathbb{R}^3)$. These semigroups will be denoted by $\{\mathbf{S}(t) : t \geqslant 0\}$ and $\{\mathbf{T}(t) : t \geqslant 0\}$. Moreover, for $s' > s$ there exists a constant M (depending on the difference $s' - s$ and p) such that we have (compare Lemma 1.2 in the Kato-Ponce's paper [31])

$$\|\mathbf{S}(t)\|_{\mathscr{L}(\mathbb{H}^s_{\mathrm{sol}}; \mathbb{H}^{s'}_{\mathrm{sol}})} \leqslant M(1 + t^{-(s'-s)/2}), \quad t > 0, \tag{2.3}$$

and

$$\|\mathbf{T}(t)\|_{\mathscr{L}(\mathbb{H}^s; \mathbb{H}^{s'})} \leqslant M(1 + t^{-(s'-s)/2}), \quad t > 0. \tag{2.4}$$

Let us note the following inequality involving fractional Sobolev norms.

$$\|fg\|_{\mathbb{H}^s} \leqslant c_0(\|f\|_{\mathbb{L}^\infty}\|g\|_{\mathbb{H}^s} + \|f\|_{\mathbb{H}^s}\|g\|_{\mathbb{L}^\infty}), \ \ f, g \in \mathbb{L}^\infty \cap \mathbb{H}^s. \tag{2.5}$$

Now let $\alpha > d/2$. For any $\mathbf{u}, \mathbf{v} \in \mathbb{H}^\alpha_{\mathrm{sol}}$ and $\mathbf{d}, \mathbf{m} \in \mathbb{H}^{\alpha+1}$, where now $\mathbb{H}^{\alpha+1} = H^{\alpha+1}(\mathscr{O}, \mathbb{R}^3)$, we set

$$B(\mathbf{u}, \mathbf{v}) = \Pi(\mathbf{u} \cdot \nabla \mathbf{v}), \tag{2.6}$$

$$M(\mathbf{d}, \mathbf{m}) = -\Pi(\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{m})), \tag{2.7}$$

$$\tilde{B}(\mathbf{v}, \mathbf{d}) = \mathbf{v} \cdot \nabla \mathbf{d}. \tag{2.8}$$

Later on, we will state and prove few crucial properties of these nonlinear maps.

Let us fix $\mathbf{d} \in \mathbb{H}^{\alpha+1}$ and set

$$G(\mathbf{h}) = \mathbf{h} \times \mathbf{d}, \quad \mathbf{h} \in \mathbb{H}^{\alpha+1}. \tag{2.9}$$

It is easy to see that $G \in \mathscr{L}(\mathbb{H}^{\alpha+1})$. Let us note that the map G^2, also an element of $\mathscr{L}(\mathbb{H}^{\alpha+1})$, is of the following form

$$G^2(\mathbf{d}) = G \circ G(\mathbf{d}) = (\mathbf{d} \times \mathbf{h}) \times \mathbf{h}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \geqslant 0\}$ satisfying the usual condition. Let $\tilde{W} = (\tilde{W}(t))_{t \geqslant 0}$ be a cylindrical Wiener process evolving on a separable Hilbert space K_1 such that it is formally written as a series

$$\tilde{W}(t) = \sum_{k=1}^{\infty} \tilde{w}_k(t) \varphi_k, \ \ \forall t \geqslant 0,$$

where $(\tilde{w}_k(t))_{k\in\mathbb{N},t\geqslant 0}$ is a family of i.i.d. standard Brownian motions and $\{\varphi_k; k \in \mathbb{N}\}$ is an orthonormal basis of K_1. The above series does not converge in K_1, but it does converges in a separable Hilbert space $\tilde{\mathrm{K}}_1$ such that the embedding $\mathrm{K}_1 \subset \tilde{\mathrm{K}}_1$ is Hilbert-Schmidt. It is well-known also that $\tilde{W}$ has a modification, still denoted by W, whose trajectories are continuous $\tilde{\mathrm{K}}_1$-valued functions. Let η be a standard one dimensional Brownian motion and h be a smooth vector fields.

We now introduce the assumption on the coefficient Q of the noise.

Assumptions 1. We fix $\alpha > d/2$ and we assume that

$$Q : \mathbb{H}^\alpha_{\mathrm{sol}} \to \mathscr{L}_2(\mathrm{K}_1, \mathbb{H}^\alpha_{\mathrm{sol}})$$

is a globally Lipschitz map. In particular, there exists $\ell_0 \geqslant 0$ such that

$$\|Q(\mathbf{u})\|^2_{\mathscr{L}_2(\mathrm{K}_1,\mathbb{H}^\alpha_{\mathrm{sol}})} \leqslant \ell_0(1 + \|\mathbf{u}\|^2_{\mathbb{H}^\alpha_{\mathrm{sol}}}), \quad \text{for any } \mathbf{u} \in \mathbb{H}^\alpha_{\mathrm{sol}}.$$

Hereafter we set

$$\begin{aligned} \mathbf{H}_\alpha &= \mathbb{H}^{\alpha-1}_{\mathrm{sol}} \times \mathbb{H}^\alpha, \\ \mathbf{V}_\alpha &= \mathbb{H}^\alpha_{\mathrm{sol}} \times \mathbb{H}^{\alpha+1}, \\ \mathbf{E}_\alpha &= \mathbb{H}^{\alpha+1}_{\mathrm{sol}} \times \mathbb{H}^{\alpha+2}. \end{aligned} \tag{2.10}$$

The stochastic equations for nematic liquid crystal (1.2) can be rewritten as a stochastic evolution equation in the space $\mathbf{H}_\alpha$:

$$d\mathbf{y}(t) + \mathbf{A}\mathbf{y}(t)dt + \mathbf{F}(\mathbf{y}(t))dt + \mathbf{L}(\mathbf{y}(t))dt = \mathbf{G}(\mathbf{y}(t))dW(t), \quad t \geqslant 0, \tag{2.11}$$

where, for $\mathbf{y} = (\mathbf{v}, \mathbf{d}) \in \mathbf{E}_\alpha$ and $k = (k_1, k_2) \in \mathrm{K} := \mathrm{K}_1 \times \mathbb{R}$, we have

$$\mathbf{A}\mathbf{y} = \begin{pmatrix} \mathrm{A}\mathbf{v} \\ \mathrm{A}_2\mathbf{d} \end{pmatrix}, \qquad \mathbf{F}(\mathbf{y}) = \begin{pmatrix} B(\mathbf{v}, \mathbf{v}) + M(\mathbf{d}) \\ \tilde{B}(\mathbf{v}, \mathbf{d}) + |\nabla \mathbf{d}|^2\mathbf{d} \end{pmatrix}, \tag{2.12}$$

$$\mathbf{L}(\mathbf{y}) = \begin{pmatrix} 0 \\ -\frac{1}{2}G^2(\mathbf{d}) \end{pmatrix}, \qquad \mathbf{G}(\mathbf{y})k = \begin{pmatrix} Q(\mathbf{u})k_1 \\ G(\mathbf{d})k_2 \end{pmatrix}. \tag{2.13}$$

The process W is a cylindrical Wiener process on K such that for any $t \geqslant 0$

$$W(t) = \begin{pmatrix} \tilde{W}(t) \\ \eta(t) \end{pmatrix}, \quad t \geqslant 0.$$

3. **Existence and uniqueness of local maximal solution.** We first recall several definitions and concepts which are given in the following notations/definitions and are borrowed from [5] or [32].

Definition 3.1. (compare [32, p. 45]) For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a given right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, a stopping time τ is called accessible iff there exists an increasing sequence of stopping times τ_n such that a.s. $\tau_n < \tau$ and $\lim_{n\to\infty} \tau_n = \tau$.

Notation. For a stopping time τ we set

$$\Omega_t(\tau) := \{\omega \in \Omega : t < \tau(\omega)\},$$

$$[0, \tau) \times \Omega := \{(t, \omega) \in [0, \infty) \times \Omega : 0 \leqslant t < \tau(\omega)\}.$$

Definition 3.2. If X is a topological space, then an X-valued process $\eta : [0,\tau) \times \Omega \to X$, is *admissible* iff

(i) it is adapted, i.e. $\eta|_{\Omega_t(\tau)} : \Omega_t(\tau) \to X$ is $\mathcal{F}_t$ measurable, for any $t \geqslant 0$;
(ii) for almost all $\omega \in \Omega$, the function $[0, \tau(\omega)) \ni t \mapsto \eta(t,\omega) \in X$ is continuous.

We will also use for an admissible process $\eta : [0,\tau) \times \Omega \to X$ the notation $\{\eta(t), t < \tau\}$ or (η, τ).

A process $\eta : [0,\tau) \times \Omega \to X$ is *progressively measurable* iff for any $t > 0$ the map

$$[0, t \wedge \tau) \times \Omega \ni (s,\omega) \mapsto \eta(s,\omega) \in X$$

is $\mathcal{B}_{t\wedge\tau} \times \mathcal{F}_{t\wedge\tau}$ measurable.
Two processes $\eta_i : [0,\tau_i) \times \Omega \to X$, $i = 1,2$ are called *equivalent*, iff $\tau_1 = \tau_2$ a.s. and for any $t > 0$ the following holds

$$\eta_1(\cdot,\omega) = \eta_2(\cdot,\omega) \text{ on } [0,t] \text{ for a.a. } \omega \in \Omega_t(\tau_1) \cap \Omega_t(\tau_2).$$

We will use for two equivalent processes η_1 and η_2 the notation $(\eta_1, \tau_1) \sim (\eta_2, \tau_2)$)

Note that if processes $\eta_i : [0,\tau_i) \times \Omega \to X$, $i = 1,2$ are admissible and for any $t > 0$ $\eta_1(t)|_{\Omega_t(\tau_1)} = \eta_2(t)|_{\Omega_t(\tau_2)}$ a.s. then they are equivalent.

We now define some concepts of solution to (2.11), see [8, Def. 4.2] or [41, Def. 2.1].

Definition 3.3. Let $\mathbf{y}_0$ be a $\mathbf{V}_\alpha$-valued $\mathcal{F}_0$–measurable random variable such that $\mathbb{E}\|\mathbf{y}_0\|^2_{\mathbf{V}_\alpha} < \infty$. A local mild solution to problem (2.11) with initial condition $\mathbf{y}(0) = \mathbf{y}_0$ is a pair $(\mathbf{y}, \tau)$ such that

1. τ is an accessible $\mathbb{F}$-stopping time,
2. $\mathbf{y} : [0,\tau) \times \Omega \to \mathbf{V}_\alpha$ is an admissible process,
3. there exists an approximating sequence $(\tau_m)_{m\in\mathbb{N}}$ of finite $\mathbb{F}$-stopping times such that $\tau_m \nearrow \tau$ a.s. and, for every $m \in \mathbb{N}$ and $t \geqslant 0$, we have

$$\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_m]} \|\mathbf{y}(s)\|^2_{\mathbf{V}_\alpha} + \int_0^{t\wedge\tau_m} \|\mathbf{y}(s)\|^2_{\mathbf{E}_\alpha}\, ds \Big) < \infty, \tag{3.1}$$

$$\begin{aligned} \mathbf{y}(t\wedge\tau_m) &= \mathbb{S}(t\wedge\tau_m)\mathbf{y}_0 - \int_0^{t\wedge\tau_m} \mathbb{S}(t\wedge\tau_m - s)[\mathbf{F}(\mathbf{y}(s)) + \mathbf{L}(\mathbf{y}(s)]\, ds \qquad (3.2)\\ &+ \int_0^{t\wedge\tau_m} \mathbb{S}(t\wedge\tau_m - s)\mathbf{G}(\mathbf{y}(s))\, dW(s) \text{ in } \mathbf{H}_\alpha \ \mathbb{P}\text{-a.s.} \end{aligned}$$

4. The stopped processes $\mathbf{d}(\cdot\wedge\tau_m)$, $m \in \mathbb{N}$, satisfies: for all $t \in [0,T]$, $m \in \mathbb{N}$, $\mathbb{P}$-a.s.

$$|\mathbf{d}(t\wedge\tau_m, x, \omega)|^2 = 1, \tag{3.3}$$

for all $x \in \mathscr{O}$.

We also introduce the notion of maximal local solution and its lifespan.

Definition 3.4. Let us denote the family of all local mild solution (u,τ) to the problem (2.11) by $\mathcal{LS}$. For two elements $(u,\tau), (v,\sigma) \in \mathcal{LS}$ we write $(u,\tau) \preceq (v,\sigma)$, iff $\tau \leqslant \sigma$ a.s. and $v_{|[0,\tau)\times\Omega} \sim u$. We write $(u,\tau) \prec (v,\sigma)$, iff $(u,\tau) \preceq (v,\sigma)$ and $\tau < \sigma$

with positive probability. If there exists a maximal element (u, τ) in the set $(\mathcal{LS}, \preceq)$ is called a maximal local mild solution to the problem (2.11). If (u, τ) is a maximal local mild solution to equation (2.11), the stopping time τ is called its lifetime.

Remark 3.5. ◦ Note that if $(u, \tau) \preceq (v, \sigma)$ and $(v, \sigma) \preceq (u, \tau)$, then $(u, \tau) \sim (v, \sigma)$.
◦ The pair $(\mathcal{LS}, \preceq)$ is a partially ordered set in which, according to the Elworthy's Amalgamation Lemma, see [16, Lemmata III 6A and 6B], every non-empty chain has a least upper bound.

Having defined our solution concept, we can now state and prove the existence of a maximal local solution for our model. We also give a lower estimate and a characterisation of the local solution's lifespan.

Theorem 3.6. *Let $d \in \{2, 3\}$, $\alpha > d/2$, $\mathbf{h} \in \mathbb{H}^{\alpha+1}$. If Assumption 1 is satisfied, then for all $\mathcal{F}_0$-measurable and square integrable $\mathbb{H}^{\alpha}_{\text{sol}} \times \mathbb{H}^{\alpha+1}$-valued random variables $\mathbf{y}_0 = (\mathbf{v}_0, \mathbf{d}_0)$ the problem (2.11) for the stochastic liquid crystals has a unique local maximal strong solution $((\mathbf{v}; \mathbf{d}), \tilde{\tau}_\infty)$ satisfying the following properties:*

1. *given $R > 0$ and $\varepsilon > 0$ there exists $\tau(\varepsilon, R) > 0$, such that for every $\mathcal{F}_0$-measurable $\mathbb{H}^{\alpha}_{\text{sol}} \times \mathbb{H}^{\alpha+1}$-valued random variable $(\mathbf{v}_0, \mathbf{d}_0)$ satisfying $\mathbb{E}\|(\mathbf{v}_0, \mathbf{d}_0)\|^2_{\mathbb{H}^\alpha \times \mathbb{H}^{\alpha+1}} \leqslant R^2$, one has*
$$\mathbb{P}\big(\tilde{\tau}_\infty \geqslant \tau(\varepsilon, R)\big) \geqslant 1 - \varepsilon.$$
2. *We also have*
$$\mathbb{P}\left(\{\tilde{\tau}_\infty < \infty\} \cap \{\|(\mathbf{v}, \mathbf{d})\|_{C([0,T];\mathbb{H}^{\alpha}_{\text{sol}} \times \mathbb{H}^{\alpha+1})} < \infty\}\right) = 0, \tag{3.4}$$
$$\limsup_{t \nearrow \tilde{\tau}_\infty} \|\mathbf{v}(t)\|_{\mathbb{H}^{\alpha}_{\text{sol}}} + \|\mathbf{d}(t)\|_{\mathbb{H}^{\alpha+1}} = \infty \text{ a.s. on } \{\tilde{\tau}_\infty < \infty\}. \tag{3.5}$$

We will show in the next theorem that the local solution from Theorem 3.6 satisfies (1.2e).

Theorem 3.7. *Assume that all the assumption of Theorem 3.6 are satisfied. Let $\mathbf{y}_0 = (\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V}_\alpha$ such that $|\mathbf{d}_0(\omega, x)|^2 = 1$ for all $\omega \in \Omega$ and all $x \in \mathcal{O}$. Let $(\mathbf{y}; \tau) = ((\mathbf{v}, \mathbf{d}); \tau)$ be a local solution to (2.11) and $(\tau_m)_{m\in\mathbb{N}}$ an increasing sequence of stopping times approximating τ. Then, for all $t \in (0, T]$ $\mathbb{P}$-a.s. $|\mathbf{d}(t \wedge \tau_m, x, \omega)|^2 = 1$ for all $x \in \mathcal{O}$.*

Remark 3.8. We suspect that, if $d \in \{2, 3\}$, $\alpha > d/2$, then under reasonable assumptions about the noise, there exists a local maximal solution for every initial data $\mathbf{y}_0 = (\mathbf{v}_0, \mathbf{d}_0) \in \mathbb{H}^{\alpha-1}_{\text{sol}} \times \mathbb{H}^{\alpha}$. We also suspect that the existence of a local solution is mainly due to the mathematical analysis. We limited ourselves to the analysis of local solution as we were not able to derive proper estimates yielding global existence. We, however, have the conjecture that under smallness condition on the initial data one should be able to prove global existence of solution; this is the case for the deterministic model, see [50]. These questions will be investigated in subsequent papers.

The proofs of these two theorems are given in sections 4 and 5, respectively.

4. **Proof of Theorem 3.6.** In order to prove the results in Theorem 3.6 we will use the general results proved in [7, Theorem 5.15 and 5.16]. For this purpose we establish several crucial estimates for the nonlinear terms in (1.2) in the following lemmata.

Lemma 4.1. *Assume that $\alpha > d/2$. Then, there exist $\delta \in [0,1)$ and $C > 0$ such that for any $\mathbf{u} \in \mathbb{H}^\alpha_{\text{sol}}, \mathbf{v} \in \mathbb{H}^{\alpha+1}_{\text{sol}}$ and $\mathbf{d}, \mathbf{m} \in \mathbb{H}^{\alpha+1}$*

$$\|B(\mathbf{u},\mathbf{v})\|_{\mathbb{H}^{\alpha-1}} \leqslant C\left(\|\mathbf{u}\|_{\mathbb{L}^\infty}\|\mathbf{v}\|_{\mathbb{H}^\alpha} + \|\mathbf{u}\|_{\mathbb{H}^{\alpha-1}}\|\mathbf{v}\|^\delta_{\mathbb{H}^{\alpha+1}}\|\mathbf{v}\|^{1-\delta}_{\mathbb{H}^\alpha}\right), \tag{4.1}$$

$$\|\tilde{B}(\mathbf{v},\mathbf{d})\|_{\mathbb{H}^\alpha} \leqslant C\|\mathbf{v}\|_{\mathbb{H}^\alpha}\|\mathbf{d}\|_{\mathbb{H}^{\alpha+1}}, \tag{4.2}$$

$$\|M(\mathbf{d},\mathbf{m})\|_{\mathbb{H}^{\alpha-1}} \leqslant C\|\mathbf{d}\|_{\mathbb{H}^{\alpha+1}}\|\mathbf{m}\|_{\mathbb{H}^{\alpha+1}}. \tag{4.3}$$

Proof of Lemma 4.1. Let $\mathbf{u} \in \mathbb{H}^\alpha_{\text{sol}}, \mathbf{v} \in \mathbb{H}^{\alpha+1}_{\text{sol}}$ and $\mathbf{d}, \mathbf{m} \in \mathbb{H}^{\alpha+1}$. In what follows we will denote by C various generic constants not depending neither on $\mathbf{u}, \mathbf{v}, \mathbf{d}$ nor $\mathbf{m}$. By the inequality (2.5), we get

$$\|\mathbf{u}\cdot\nabla\mathbf{v}\|_{\mathbb{H}^{\alpha-1}} \leqslant C\left(\|\mathbf{u}\|_{\mathbb{L}^\infty}\|\nabla\mathbf{v}\|_{\mathbb{H}^{\alpha-1}} + \|\mathbf{u}\|_{\mathbb{H}^{\alpha-1}}\|\nabla\mathbf{v}\|_{\mathbb{L}^\infty}\right),$$

Since $\alpha > d/2$, one can find a positive constant $\delta \in (0,1)$ such that $\alpha - \delta > d/2$. Thus, by the Sobolev embedding $\mathbb{H}^{\alpha-\delta} \subset \mathbb{L}^\infty$ and the Gagliardo-Nirenberg inequality we have

$$\|\nabla\mathbf{g}\|_{\mathbb{L}^\infty} \leqslant C\|\nabla\mathbf{g}\|_{\mathbb{H}^{\alpha-\delta}} \leqslant C\|\nabla\mathbf{g}\|^{1-\delta}_{\mathbb{H}^{\alpha-1}}\|\nabla\mathbf{g}\|^\delta_{\mathbb{H}^\alpha}, \tag{4.4}$$

from which we infer that

$$\|\mathbf{u}\cdot\nabla\mathbf{v}\|_{\mathbb{H}^{\alpha-1}} \leqslant C\left(\|\mathbf{u}\|_{\mathbb{L}^\infty}\|\mathbf{v}\|_{\mathbb{H}^\alpha} + \|\mathbf{u}\|_{\mathbb{H}^{\alpha-1}}\|\mathbf{v}\|^\delta_{\mathbb{H}^{\alpha+1}}\|\mathbf{v}\|^{1-\delta}_{\mathbb{H}^\alpha_{\text{sol}}}\right).$$

The first estimate in our lemma easily follows from this last line and the fact that (as we are on the torus) the Leray-Helmhotz projection operator Π belongs to $\mathscr{L}(\mathbb{H}^{\alpha-1}, \mathbb{H}^{\alpha-1}_{\text{sol}})$.

We now prove the second estimate. As $\alpha > d/2$, $\mathbb{H}^\alpha_{\text{sol}}$ is an algebra and we can easily infer that

$$\|\mathbf{v}\cdot\nabla\mathbf{d}\|_{\mathbb{H}^\alpha_{\text{sol}}} \leqslant \|\mathbf{v}\|_{\mathbb{H}^\alpha}\|\mathbf{d}\|_{\mathbb{H}^{\alpha+1}},$$

from which the second estimate in our lemma easily follows.

Now we deal with third estimate where the nonlinear map M is involved. Since $\Pi \in \mathscr{L}(\mathbb{H}^{\alpha-1}_{\text{sol}})$ we get

$$\|M(\mathbf{d},\mathbf{m})\|_{\mathbb{H}^{\alpha-1}} \leqslant C\|\nabla\mathbf{d}\odot\nabla\mathbf{m}\|_{\mathbb{H}^\alpha_{\text{sol}}}.$$

Using an argument similar to the proof of (4.2) yields (4.3). □

We will also need the following lemma.

Lemma 4.2. *Let $\alpha > d/2$. Then, there exists a constant $C > 0$ such that for any $\mathbf{d}, \mathbf{m} \in \mathbb{H}^{\alpha+1}$*

$$\begin{aligned}\||\nabla\mathbf{d}|^2\mathbf{d} - |\nabla\mathbf{m}|^2\mathbf{m}\|_{\mathbb{H}^\alpha} \leqslant C\|\mathbf{d}-\mathbf{m}\|_{\mathbb{H}^{\alpha+1}}[\|\mathbf{d}\|_{\mathbb{H}^{\alpha+1}} + \|\mathbf{m}\|_{\mathbb{H}^{\alpha+1}}]\|\mathbf{d}\|_{\mathbb{H}^\alpha}\\ +C\|\mathbf{m}\|^2_{\mathbb{H}^{\alpha+1}}\|\mathbf{d}-\mathbf{m}\|_{\mathbb{H}^\alpha_{\text{sol}}}\end{aligned} \tag{4.5}$$

Proof of Lemma 4.2. Let us fix $\mathbf{d},\mathbf{m}\in\mathbb{H}^{\alpha+1}$. Again, since $\mathbb{H}^{\alpha}$ is an algebra, we easily deduce from the inequality

$$\||\nabla\mathbf{d}|^2\mathbf{d}-|\nabla\mathbf{m}|^2\mathbf{m}\|_{\mathbb{H}^{\alpha}}\leqslant\|[(|\nabla\mathbf{d}|-|\nabla\mathbf{m}|)(|\nabla\mathbf{d}|+|\nabla\mathbf{m}|)]\mathbf{d}\|_{\mathbb{H}^{\alpha}_{\text{sol}}}+\||\nabla\mathbf{m}|^2(\mathbf{d}-\mathbf{m})\|_{\mathbb{H}^{\alpha}_{\text{sol}}},$$

the inequality (4.5). □

We now are ready to embark on the promised proof of Theorem 3.6.

Proof of Theorem 3.6. Since the maps M, B and $\tilde{B}$ are bilinear, we infer from the Lemmata 4.1 and 4.2 that the nonlinear term $\mathbf{F}$ defined in (2.12) satisfies the following property: There exist two constants $\delta\in(0,1)$ and $C>0$ such that for any $\mathbf{y}_1,\mathbf{y}_2\in\mathbf{E}_\alpha$ we have

$$\begin{aligned}\|\mathbf{F}(\mathbf{y}_1)-\mathbf{F}(\mathbf{y}_2)\|_{\mathbf{H}_\alpha}\leqslant& C\|\mathbf{y}_1-\mathbf{y}_2\|_{\mathbf{V}_\alpha}\left(\|\mathbf{y}_1\|^{1-\delta}_{\mathbf{V}_\alpha}\|\mathbf{y}_2\|^{\delta}_{\mathbf{E}_\alpha}+\sum_{k=1}^{2}\left[\|\mathbf{y}_1\|^k_{\mathbf{V}_\alpha}+\|\mathbf{y}_2\|^k_{\mathbf{V}_\alpha}\right]\right)\\&+C\|\mathbf{y}_1-\mathbf{y}_2\|^{1-\delta}_{\mathbf{V}_\alpha}\|\mathbf{y}_1-\mathbf{y}_2\|^{\delta}_{\mathbf{E}_\alpha}\|\mathbf{y}_2\|_{\mathbf{V}_\alpha}.\end{aligned}\tag{4.6}$$

From the definition 2.9 of the map G and the assumption on $\mathbf{h}$ it follows that $\mathbf{L}\in\mathscr{L}(\mathbb{H}^{\alpha}_{\text{sol}}\times\mathbb{H}^{\alpha+1})$ from which, along with (4.6), we infer that $\mathbf{F}+\mathbf{L}$ satisfies Assumption 2 of Theorem A.1 (see also [7, Assumption 5.1]).

Because of Assumption 1 and the fact that $G\in\mathscr{L}(\mathbb{H}^{\alpha+1})$ it is clear that $\mathbf{G}$ satisfies Assumption 3 of Theorem A.1.

Now, let X_T be the Banach space

$$X_T:=C([0,T];\mathbf{V}_\alpha)\cap L^2(0,T;\mathbf{E}_\alpha)\tag{4.7}$$

with the norm defined by

$$|\mathbf{u}|^2_{X_T}=\sup_{s\in[0,T]}\|\mathbf{u}(s)\|^2_{\mathbf{V}_\alpha}+\int_0^T|\mathbf{u}(s)|^2_{\mathbf{E}_\alpha}\,ds.\tag{4.8}$$

It is know from [43, Lemma 1.2] or [31, Lemma 1.5] that the linear map $\mathbb{S}*:\mathbb{L}^2(0,T;\mathrm{H}_\alpha)\to X_T$ defined by

$$(\mathbb{S}*f)(\cdot)=\int_0^{\cdot}\mathbb{S}(\cdot-s)f(s)ds,\quad f\in\mathbb{L}^2(0,T;\mathbb{H}_\alpha),$$

is bounded.

It is also know, see [43, Lemma 1.4] or [44, Chapitre2, Lemma 2.1], that the linear map $\mathbb{S}\diamond:M^2(0,T;\mathscr{L}_2(\mathrm{K},\mathbf{V}_\alpha))\to M^2(X_T)$ defined by

$$(\mathbb{S}\diamond g)(\cdot)=\int_0^{\cdot}\mathbb{S}(\cdot-s)g(s)dW(s),\;\;g\in M^2(0,T;\mathscr{L}_2(\mathrm{K},\mathbf{V}_\alpha)),$$

is also bounded.

From the observations above, Assumption 1 and the assumption on the initial data $\mathbf{y}_0$ we infer that the problem (2.11) satisfies all the assumptions of Theorems A.1 and A.2 (see also [7, Theorem 5.15 and 5.16]) from which we easily complete the proof of the Theorem 3.6. □

5. Proof of Theorem 3.7.

In this section we give the proof of the sphere constraint.

Proof of Theorem 3.7. The proof will be divided into two steps.

Let $\varphi : \mathbb{R} \to [-1, 0]$ be a $\mathcal{C}^\infty$ class increasing function such that

$$\varphi(s) = \begin{cases} -1 & \text{iff } s \in (-\infty, -2], \\ 0 & \text{iff } s \in [-1, +\infty). \end{cases} \tag{5.1}$$

Let $\{\tilde{\varphi}_\ell : \ell \in \mathbb{N}\}$ and $\{\tilde{\phi}_\ell : \ell \in \mathbb{N}\}$ be two sequences of function $\mathbb{R}$ defined by

$$\tilde{\varphi}_\ell(a) = \varphi(\ell a), \quad a \in \mathbb{R}, \tag{5.2}$$

$$\tilde{\phi}_\ell(a) = a^2 \varphi(\ell a), \quad a \in \mathbb{R}. \tag{5.3}$$

We also set

$$\varphi_\ell(\mathbf{d}) = \tilde{\varphi}_\ell(|\mathbf{d}|^2 - 1), \ \mathbf{d} \in \mathbb{R}^3, \tag{5.4}$$

$$\phi_\ell(\mathbf{d}) = \tilde{\phi}_\ell(|\mathbf{d}|^2 - 1), \ \mathbf{d} \in \mathbb{R}^3. \tag{5.5}$$

Now, let $\alpha > \frac{d}{2}$ be a fixed number. For each $\ell \in \mathbb{N}$ we define a function

$$\begin{aligned} &\Psi_\ell : \mathbb{H}^\alpha \to \mathbb{R} \\ &\Psi_\ell(\mathbf{d}) = \|\phi_\ell \circ \mathbf{d}\|_{\mathbb{L}^1} = \int_{\mathcal{O}} (|\mathbf{d}(x)|^2 - 1)^2 [\varphi_\ell(\mathbf{d}(x))] dx, \ \mathbf{d} \in \mathbb{H}^\alpha. \end{aligned} \tag{5.6}$$

One can show that since $\mathbb{H}^\alpha \subset \mathbb{L}^\infty$ (as $\alpha > \frac{d}{2}$), the map Ψ_ℓ is twice (Fréchet) differentiable[1] and its first and second derivatives satisfy, for $\mathbf{d} \in \mathbb{H}^\alpha$ and $\mathbf{k}, \mathbf{f} \in \mathbb{H}^\alpha$,

$$\begin{aligned} \Psi'_\ell(\mathbf{d})(\mathbf{k}) = & 4 \int_{\mathcal{O}} \Big((|\mathbf{d}(x)|^2 - 1) \varphi_\ell(\mathbf{d}(x)) [\mathbf{d}(x) \cdot \mathbf{k}(x)] \Big) dx \\ & + 2\ell \int_{\mathcal{O}} (|\mathbf{d}(x)|^2 - 1)^2 \varphi'(\ell(|\mathbf{d}(x)|^2 - 1))(\mathbf{d}(x) \cdot \mathbf{k}(x)) dx, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \Psi''_\ell(\mathbf{d})(\mathbf{k}, \mathbf{f}) = 4\ell^2 & \int_{\mathcal{O}} \Big[(|\mathbf{d}(x)|^2 - 1)^2 \varphi''_\ell(\ell(|\mathbf{d}(x)|^2 - 1))(\mathbf{d}(x) \cdot \mathbf{k}(x))(\mathbf{d}(x) \cdot \mathbf{f}(x)) \Big] dx \\ & + 16\ell \int_{\mathcal{O}} \Big[(|\mathbf{d}(x)|^2 - 1) \varphi'(\ell(|\mathbf{d}(x)|^2 - 1))(\mathbf{d}(x) \cdot \mathbf{k}(x))(\mathbf{d}(x) \cdot \mathbf{f}(x)) \Big] dx \\ & + 8 \int_{\mathcal{O}} \Big[\varphi_\ell(\mathbf{d}(x))(\mathbf{d}(x) \cdot \mathbf{k}(x))(\mathbf{d}(x) \cdot \mathbf{f}(x)) \Big] dx \\ & + 2\ell \int_{\mathcal{O}} \Big[|\mathbf{d}(x)|^2 - 1)^2 \varphi'(\ell(|\mathbf{d}(x)|^2 - 1))(\mathbf{k}(x) \cdot \mathbf{f}(x) \Big] dx \\ & + 4 \int_{\mathcal{O}} \Big[\varphi_\ell(\mathbf{d}(x))(|\mathbf{d}(x)|^2 - 1)(\mathbf{k}(x) \cdot \mathbf{f}(x)) \Big] dx. \end{aligned} \tag{5.8}$$

In particular, if $\mathbf{d} \in \mathbb{H}^\alpha$ and $\mathbf{k}, \mathbf{f} \in \mathbb{H}^\alpha$ are such that

$$\mathbf{k}(x) \perp \mathbf{d}(x) \text{ and } \mathbf{f}(x) \perp \mathbf{d}(x) \text{ for all } x \in \mathcal{O},$$

---

[1] One might think that since Φ_ℓ is well defined on the space $\mathbb{H}^1$, it would also be twice differentiable in $\mathbb{H}^1$. However, for this to hold, we need to restrict it to the space $\mathbb{H}^\alpha$ for $\alpha > \frac{d}{2}$ as in this case $\mathbb{H}^\alpha \subset \mathbb{L}^\infty$. This is fact related to the properties of Nemytski maps, see the papers by the first named authour [5] and [8].

then

$$\Psi'_\ell(\mathbf{d})(\mathbf{k}) = 0, \tag{5.9}$$

and

$$\begin{aligned}\Psi''_\ell(\mathbf{d})(\mathbf{k},\mathbf{f}) =& 4\int_{\mathscr{O}} \Big[(|\mathbf{d}(x)|^2-1)\varphi_\ell(\mathbf{d}(x))(\mathbf{k}(x)\cdot\mathbf{f}(x))\Big]\,dx \\ &+ 2\ell\int_{\mathscr{O}} \Big[(|\mathbf{d}(x)|^2-1)^2\varphi'(\ell(|\mathbf{d}(x)|^2-1))(\mathbf{k}(x)\cdot\mathbf{f}(x))\Big]\,dx.\end{aligned} \tag{5.10}$$

Since the local solution $\mathbf{d}$ given by Theorem 3.6 satisfies the following integral equation in $\mathbb{H}^\alpha$, for all $t\in[0,T]$, all $m\in\mathbb{N}$, $\mathbb{P}$-a.s.

$$\mathbf{d}(t\wedge\tau_m) = \mathbf{d}_0 + \int_0^{t\wedge\tau_m} (\Delta\mathbf{d}(s)+|\nabla\mathbf{d}(s)|^2\mathbf{d}(s)-\mathbf{v}(s)\cdot\nabla\mathbf{d}(s))ds + \int_0^{t\wedge\tau_m}(\mathbf{d}(s)\times\mathbf{h})\circ d\eta,$$

it follows from the Itô formula, see [43, Theorem I.3.3.2] and [20, Theorem 1], that for any $m\in\mathbb{N}$, for all $t\in[0,T]$, $\mathbb{P}-a.s.$,

$$\begin{aligned}\Psi_\ell(\mathbf{d})(t\wedge\tau_m)-\Psi_\ell(\mathbf{d})(0) =& \int_0^{t\wedge\tau_m}\Psi'_\ell(\mathbf{d})(s)\left(\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}-\mathbf{v}\cdot\nabla\mathbf{d}+\frac{1}{2}G^2(\mathbf{d})\right)(s)ds \\ &+\int_0^{t\wedge\tau_m}\frac{1}{2}\Psi''_\ell(\mathbf{d})(s)(G(\mathbf{d})(s,G(\mathbf{d})(s)ds.\end{aligned}$$

Note that the stochastic integral vanishes because $G(\mathbf{d}(s,x))\perp\mathbf{d}(s,x)$ for all $s\in[0,\tau)$ and $x\in\mathscr{O}$.
Since $G^2(\mathbf{d})=(\mathbf{d}\times\mathbf{h})\times\mathbf{h}$ and $G(\mathbf{d})=\mathbf{d}\times\mathbf{h}$, we infer from (5.7) and the identity

$$-|a\times b|^2_{\mathbb{R}^3} = a\cdot((a\times b)\times b)\,, a,b\in\mathbb{R}^3,$$

that

$$\begin{aligned}\Psi'(\mathbf{d})(G^2(\mathbf{d})) = &-2\ell\int_{\mathscr{O}}(|\mathbf{d}(x)|^2-1)\varphi'(\ell(|\mathbf{d}(x)|^2-1))|G(\mathbf{d}(x))|^2dx \\ &-4\int_{\mathscr{O}}(|\mathbf{d}(x)|^2-1)\varphi_\ell(\mathbf{d}(x))|G(\mathbf{d}(x))|^2dx,\end{aligned}$$

which along with the fact that $G(\mathbf{d}(x))\perp\mathbf{d}(x)$ for any $x\in\mathscr{O}$ and (5.10) we infer that

$$\frac{1}{2}\Psi''_\ell(G(\mathbf{d}),G(\mathbf{d}))+\frac{1}{2}\Psi'_\ell(G^2(\mathbf{d}))=0.$$

Hence, for every $m\in\mathbb{N}$, for all $t\in[0,T]$, $\mathbb{P}$-a.s.,

$$\Psi_\ell(\mathbf{d}(t\wedge\tau_m))-\Psi_\ell(\mathbf{d}(0)) = \int_0^{t\wedge\tau_m}\Psi'_\ell(\mathbf{d}(s))\left(\Delta\mathbf{d}(s)+|\nabla\mathbf{d}(s)|^2\mathbf{d}(s)-\mathbf{v}(s)\cdot\nabla\mathbf{d}(s)\right)ds. \tag{5.11}$$

Now, observe that from the assumptions on the function φ and the definition of the sequence $\tilde{\phi}_\ell$, $\ell\in\mathbb{N}$ we infer that, with $a_-:=\max(-a,0)$, for any $a\in\mathbb{R}$,

$$\tilde{\phi}_\ell(a)\to(a_-)^2 \text{ and } \ell\varphi'(\ell a)\to 0 \text{ as } \ell\to\infty. \tag{5.12}$$

Observe also that there exists a constant $C>0$ such that for all $\ell\in\mathbb{N}$ and $a\in\mathbb{R}$

$$|\tilde{\phi}_\ell(a)|\leqslant Ca^2 \text{ and } |\ell\varphi'(\ell a)|\leqslant C|a|. \tag{5.13}$$

Therefore we infer from (5.12), (5.13) and the Lebesgue Dominated Convergence Theorem that for $\mathbf{d} \in \mathbb{H}^\alpha, \mathbf{k} \in \mathbb{H}^\alpha$

$$\lim_{\ell\to\infty} \Psi_\ell(\mathbf{d}) = \| \left(|\mathbf{d}|^2 - 1\right)_- \|^2,$$

$$\lim_{\ell\to\infty} \Psi'_\ell(\mathbf{d})(\mathbf{k}) = 4\int_{\mathcal{O}} [\left(|\mathbf{d}(x)|^2 - 1\right)_- (\mathbf{d}(x)\cdot \mathbf{k}(x))]\, dx.$$

Hence, setting $y(t) = \| (|\mathbf{d}(t)|^2 - 1)_- \|^2_{\mathbb{L}^2}$ and letting $\ell \to \infty$ in (5.11) we obtain that for every $m \in \mathbb{N}$, for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$y(t\wedge\tau_m) - y(0) + 4\int_0^{t\wedge\tau_m} \left(\int_{\mathcal{O}} \Big[-\Delta\mathbf{d}(s,x) - |\nabla\mathbf{d}(s,x)|^2\mathbf{d}(s,x) + \mathbf{v}(s,x)\cdot\nabla\mathbf{d}(s,x)\Big] \times \Big[\mathbf{d}(s,x)\left(|\mathbf{d}(s,x)|^2 - 1\right)_-\Big] dx\right) ds = 0.$$

Using the identities

$$\nabla|\mathbf{d}|^2 = 2\nabla\mathbf{d}\mathbf{d}, \tag{5.14}$$

$$\Delta|\mathbf{d}|^2 = 2\Delta\mathbf{d}\cdot\mathbf{d} + 2|\nabla\mathbf{d}|^2, \tag{5.15}$$

we deduce that for every $m \in \mathbb{N}$, for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$y(t\wedge\tau_m) - y(0) + 2\int_0^{t\wedge\tau_m} \left(\int_{\mathcal{O}} \Big[-\Delta|\mathbf{d}(s,x)|^2 - 2|\nabla\mathbf{d}(s,x)|^2(|\mathbf{d}(s,x)|^2 - 1) + \mathbf{v}(s,x)\cdot\nabla|\mathbf{d}(s,x)|^2\Big]\Big[\left(|\mathbf{d}(s,x)|^2 - 1\right)_-\Big] dx\right) ds = 0.$$

Now observe that from the definition of $\zeta := \left(|\mathbf{d}|^2 - 1\right)_-$ we have, for $\mathbf{d} \in \mathbb{H}^{\alpha+2}$,

$$\begin{aligned}
&\int_{\mathcal{O}} \Big[-\Delta|\mathbf{d}(x)|^2 - 2|\nabla\mathbf{d}(x)|^2(|\mathbf{d}(x)|^2 - 1)\Big]\zeta(x)dx \\
&\quad = -\int_{\mathcal{O}} \left(\Delta\zeta(x)\cdot\zeta(x) - 2\mathbb{1}_{|\mathbf{d}(x)|^2\leqslant 1}|\nabla\mathbf{d}(x|^2\zeta^2(x)\right) dx \\
&\quad \geqslant \int_{\mathcal{O}} |\nabla\zeta(x)|^2 dx.
\end{aligned}$$

Observe also that since $\nabla\cdot\mathbf{v} = 0$ we have[2]

$$\int_{\mathcal{O}} \mathbf{v}(x)\cdot\nabla|\mathbf{d}(x)|^2\zeta(x)dx = \int_{\mathcal{O}} \mathbf{v}(x)\cdot\nabla\zeta(x)\zeta(x)dx = 0.$$

Bearing in mind the two remarks above, we infer that for every $m \in \mathbb{N}$, $y(t\wedge\tau_m)$ satisfies for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$y(t\wedge\tau_m) + 2\int_0^{t\wedge\tau_m} |\nabla\zeta(s)|^2_{L^2} ds \leqslant y(0).$$

Since the second term in the left hand side of the above inequality is positive and $y(0) = \|(|\mathbf{d}_0|^2 - 1)_-\|^2$ and by assumption $|\mathbf{d}_0(x,\omega)|^2 = 1$ for all $x \in \mathcal{O}$ and $\omega \in \Omega$ we deduce that, for every $m \in \mathbb{N}$, for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$y(t\wedge\tau_m) = 0.$$

[2]Note that by [1, Exercise 7.1.5, p 283], $\zeta \in H^1$ if $\mathbf{d} \in \mathbb{H}^1$.

Since $\mathbb{H}^{\alpha+1} \subset C^1(\mathscr{O})$ as $\alpha > \frac{d}{2}$, we infer that for all $m \in \mathbb{N}$, $t \in [0,T]$, $\mathbb{P}$-a.s.

$$(|\mathbf{d}(t \wedge \tau_m, x, \omega)|^2 - 1)_- = 0 \text{ for all } x \in \mathscr{O}. \tag{5.16}$$

Thus, to complete the proof it is sufficient to show that for all $m \in \mathbb{N}$, for all $t \in [0,T]$ we have , $\mathbb{P}$-a.s.

$$(|\mathbf{d}(t \wedge \tau_m, x)|^2 - 1)_+ = 0 \text{ for all } x \in \mathscr{O}. \tag{5.17}$$

For this purpose we set

$$\begin{aligned} \xi(t,x) &:= \left(|\mathbf{d}(t,x)|^2 - 1\right)_+, \quad (t,x) \in [0,\tau) \times \mathscr{O}, \\ z(t) &= \|\xi(t)\|^2_{\mathbb{L}^2}, \quad t \in [0,T], \end{aligned}$$

and construct a sequence of functions Ψ_ℓ very similar to the one defined in (5.6). First let us define an increasing function $\varphi : \mathbb{R} \to [0,1]$ belonging to $\mathcal{C}^\infty$ satisfying

$$\varphi(s) = \begin{cases} 1 \text{ iff } s \in [2,\infty), \\ 0 \text{ iff } s \in (-\infty, 1]. \end{cases} \tag{5.18}$$

Now, we replace in the definition of Ψ_ℓ given by (5.6) the old function φ_l by the function defined above. With this new definition we can show by arguing as before that for $\mathbf{d} \in \mathrm{H}^2$ and $\mathbf{k} \in \mathbb{L}^2$ we have

$$\begin{aligned} \lim_{m\to\infty} \Psi_\ell(\mathbf{d}) &= \| \left(|\mathbf{d}|^2 - 1\right)_+ \|^2_{L^2}, \\ \lim_{m\to\infty} \Psi'_\ell(\mathbf{d})(\mathbf{k}) &= 4 \int_{\mathscr{O}} [\left(|\mathbf{d}(x)|^2 - 1\right)_+ \mathbf{d}(x) \cdot \mathbf{k}(x)]\, dx, \end{aligned}$$

and, for every $m \in \mathbb{N}$, for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$z(t \wedge \tau_m) - z(0) + 2\int_0^{t\wedge\tau_m} \int_{\mathscr{O}} |\nabla \xi(s,x)|^2 dxds - 4\int_0^{t\wedge\tau_m} \int_{\mathscr{O}} |\nabla \mathbf{d}(s,x)|^2 |\xi(s,x)|^2 dxds = 0. \tag{5.19}$$

Since the third term in the left hand side of the above identity is negative we cannot neglect this term. Before proceed with the proof we observe that from the assumption $\alpha > d/2$ and (3.1) we infer that for any $\varepsilon > 0$ there exists a constant $N > 0$ such that

$$\begin{aligned} \mathbb{P}(\Omega_{m,N}) &\geqslant 1 - \varepsilon, \quad \text{where} \\ \Omega_{m,N} &= \{\omega \in \Omega : \sup_{s \in [0, t\wedge\tau_m]} \|\nabla \mathbf{d}(s)\|_{\mathbb{L}^\infty} \leqslant N\}. \end{aligned}$$

Let us observe that for all $m \in \mathbb{N}$, in view of (5.19), we have on $\Omega_{m,N}$

$$\begin{aligned} z(t \wedge \tau_m) - z(0) + 2\int_0^{t\wedge\tau_m} \int_{\mathscr{O}} |\nabla \xi(s,x)|^2 dxds &\leqslant 4N^2 \int_0^{t\wedge\tau_m} \int_{\mathscr{O}} |\xi(s,x)|^2 dxds \\ &\leqslant 4N^2 \int_0^{t\wedge\tau_m} z(s)ds. \end{aligned} \tag{5.20}$$

Taking the expectation (over the set $\Omega_{m,N}$), because for a nonnegative function z, $\int_0^{t\wedge\tau} z(s)\, ds \leqslant \int_0^t z(s \wedge \tau)\, ds$, from the above inequality we get

$$\mathbb{E}\big[z(t \wedge \tau_m) 1_{\Omega_{m,N}}\big] \leqslant \mathbb{E}\big[z(0) 1_{\Omega_{m,N}}\big] + 4N^2 \int_0^t \mathbb{E}\big[z(s \wedge \tau_m) 1_{\Omega_{m,N}}\big]\, ds. \tag{5.21}$$

Applying the Gronwall Lemma we infer that

$$\mathbb{E}\big[z(t\wedge\tau_m)1_{\Omega_{m,N}}\big] \leqslant \mathbb{E}\big[z(t\wedge\tau_m)1_{\Omega_{m,N}}\big]e^{4N^2T} = 0, \quad t\in[0,T].$$

Hence we infer that $1_{\Omega_{m,N}}(|\mathbf{d}(t\wedge\tau_m)|^2-1)_+ = 0$ for every $t\in[0,T]$, $\mathbb{P}$-a.s. and therefore we deduce that for every $t\in[0,T]$ and for every $\varepsilon>0$

$$\mathbb{P}\left((|\mathbf{d}t\wedge\tau_m|^2-1)_+ = 0\right) \geqslant 1-\varepsilon.$$

From this last estimate and the first part of the proof infer that for all $t\in[0,T]$, $m\in\mathbb{N}$, $\mathbb{P}$-a.s.

$$(|\mathbf{d}(t\wedge\tau_m,x,\omega)|-1)_+ = (|\mathbf{d}(t\wedge\tau_m,x,\omega)|-1)_- = 0 \text{ for all } x\in\mathscr{O},$$

which implies (3.3). ☐

Appendix A. **Local strong solution for an abstract stochastic evolution equation.** The goal of this section is to recall general results about the existence of a local and maximal solution to an abstract stochastic partial differential equation with locally Lipschitz continuous coefficients. These results were proved in [7] utilising some truncation and fixed point methods. The proofs are highly technical, and hence we refer the reader to [7] for the details.

To start with let us fix some notations and assumptions. Let V, E and H be separable Banach spaces such that $E\subset V$ continuously. We denote the norm in V by $\|\cdot\|$ and we put

$$X_T := C([0,T];V)\cap L^2(0,T;E) \tag{A.1}$$

with the norm $|\cdot|_{X_T}$ satisfying

$$|u|^2_{X_T} = \sup_{s\in[0,T]}\|u(s)\|^2 + \int_0^T |u(s)|^2_E\,ds. \tag{A.2}$$

Let F and G be two nonlinear mappings satisfying the following sets of conditions.

Assumptions 2. Suppose that $F: E\to H$ is such that $F(0)=0$ and there exist $p,q\geqslant 1$, $\alpha,\gamma\in[0,1)$ and $C>0$ such that

$$\begin{aligned}|F(y)-F(x)|_H \leqslant C\Big[&\|y-x\|\|y\|^{p-\alpha}|y|^\alpha_E + |y-x|^\alpha_E\|y-x\|^{1-\alpha}\|x\|^p\Big]\\ &+C\Big[\|y-x\|\|y\|^{q-\gamma}|y|^\gamma_E + |y-x|^\gamma_E\|y-x\|^{1-\gamma}\|x\|^q\Big],\end{aligned} \tag{A.3}$$

for any $x,y\in E$.

Let K be a separable Hilbert space and $\mathscr{L}_2(\mathrm{K},V)$ the space of Hilbert-Schmidt operators from K onto V. For the sake of simplicity we denote by $\|\cdot\|_{\mathscr{L}_2}$ the norm in $\mathscr{L}_2(\mathrm{K},V)$.

Assumptions 3. Assume that $G: E\to\mathscr{L}_2(\mathrm{K},V)$ such that $G(0)=0$ and there exists $k\geqslant 1$, $\beta\in[0,1)$ and $C_G>0$ such that

$$\|G(y)-G(x)\|_{\mathscr{L}_2} \leqslant C_G\Big[\|y-x\|\|y\|^{k-\beta}|y|^\beta_E + |y-x|^\beta_E\|y-x\|^{1-\beta}\|x\|^k\Big], \tag{A.4}$$

for any $x,y\in E$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \geqslant 0\}$ satisfying the usual condition. By $M^2(X_T)$ we denote the space of all progressively measurable E-values processes whose trajectories belong to X_T almost surely endowed with a norm $|\cdot|_{M^2(X_T)}$ satisfying

$$|u|^2_{M^2(X_T)} = \mathbb{E}\Big[\sup_{s\in[0,T]} \|u(s)\|^2 + \int_0^T |u(s)|^2_E \, ds\Big]. \tag{A.5}$$

Let us also formulate the following assumptions.

Assumptions 4. Suppose that the embeddings $E \subset V \subset H$ are continuous. Consider (for simplicity) a one-dimensional Wiener process $W = \{W(t) : t \geqslant 0\}$.
Assume that $\{S(t) : t \in [0, \infty)\}$, is a family of bounded linear operators on the space H such that there exists two positive constants C_1 and C_2 with the following properties:
(i) For every $T > 0$ and every $f \in L^2(0, T; H)$ a function $u = S * f$ defined by

$$u(t) = \int_0^T S(t - r)f(r)\, dt, \quad t \in [0, T],$$

belongs to X_T and

$$|u|_{X_T} \leqslant C_1 |f|_{L^2(0,T;H)}. \tag{A.6}$$

(ii) For every $T > 0$ and every process $\xi \in M^2(0, T; \mathscr{L}_2(\mathrm{K}, V))$ a process $u = S \diamond \xi$ defined by

$$u(t) = \int_0^T S(t - r)\xi(r)\, dW(r), \quad t \in [0, T]$$

belongs to $M^2(X_T)$ and

$$|u|_{M^2(X_T)} \leqslant C_2 |\xi|_{M^2(0,T;\mathscr{L}_2(\mathrm{K},V))}.$$

(iii) For every $T > 0$ and every $u_0 \in V$, a function $u = Su_0$ defined by

$$u(t) = S(t)u_0, \quad t \in [0, T]$$

belongs to X_T. Moreover, for every $T_0 > 0$ there exist $C_0 > 0$ such that for all $T \in (0, T_0]$,

$$|u|_{X_T} \leqslant C_0 \|u_0\|. \tag{A.7}$$

Now let us consider a semigroup $\{S(t) : t \in [0, \infty)\}$ as above and the abstract SPDEs

$$u(t) = S(t)u_0 + \int_0^t S(t - s)F(u(s))ds + \int_0^t S(t - s)G(u(s))dW(s), \quad \text{for any } t > 0, \tag{A.8}$$

which is a mild version of the problem

$$\begin{cases} du(t) &= Au(t)\, dt + F\big(u(t)\big)\, dt + G\big(u(t)\big)dW(t), \quad t > 0, \\ u(0) &= u_0. \end{cases} \tag{A.9}$$

Here A is the infinitesimal generator of the semigroup $\{S(t) : t \geqslant 0\}$.

We will not recall the definitions of local and maximal solutions since they are the same as the ones introduced definition 3.3 and definition 3.4. We directly give the main theorems that are of interest to us. The first one is about the existence and uniqueness of a local solution and a probabilistic lower bound of the solution's lifespan.

Theorem A.1. *Suppose that Assumption 2, Assumption 3, and Assumption 4 hold. Then for every $\mathcal{F}_0$-measurable V-valued square integrable random variable u_0 there exists a local process $u = \big(u(t), t \in [0, T_1)\big)$ which is the unique local mild solution to our problem. Moreover, given $R > 0$ and $\varepsilon > 0$ there exists a stopping time $\tau(\varepsilon, R) > 0$, such that for every $\mathcal{F}_0$-measurable V-valued random variable u_0 satisfying $\mathbb{E}\|u_0\|^2 \leqslant R^2$, one has*

$$\mathbb{P}\big(T_1 \geqslant \tau(\varepsilon, R)\big) \geqslant 1 - \varepsilon.$$

The next result is about the existence and uniqueness of a maximal solution and the characterization of its lifespan.

Theorem A.2. *For every $u_0 \in L^2(\Omega, \mathcal{F}_0, V)$, the process $u = (u(t)\,,\, t < \tau_\infty)$ defined above is the unique local maximal solution to our equation. Moreover, $\mathbb{P}\big(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_V < \infty\}\big) = 0$ and on $\{\tau_\infty < \infty\}$, $\limsup_{t\to\tau_\infty} |u(t)|_V = +\infty$ a.s.*

The proofs of both theorems are highly nontrivial and technical, we refer to [7, Section 5] for the details.

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