Optimal asset allocation for a DC plan with partial information under inflation and mortality risks

Calisto Guambe[1,3], Rodwell Kufakunesu[1*], Augustinus van Zyl[1], Conrad Beyers[2]

[1] *Department of Mathematics and Applied Mathematics, University of Pretoria, 0002, South Africa*

[2] *Department of Actuarial Science, University of Pretoria, 0002, South Africa*

[3] *Department of Mathematics and Informatics, Eduardo Mondlane University, 257, Mozambique*

Abstract. We study an asset allocation stochastic problem for a defined-contribution pension plan during the accumulation phase. We consider a financial market composed of a risk-free asset, an inflation-linked bond and the risky asset. The fund manager aims to maximize the expected power utility derived from the terminal wealth. Our solution allows one to incorporate a clause which allows for the distribution of a member's premiums to his surviving dependents, should the member die before retirement. Besides the mortality risk, our optimization problem takes into account the salary and the inflation risks. We then obtain closed form solutions for the asset allocation problem using a sufficient maximum principle approach for the problem with partial information. Finally, we give a numerical example.

1. Introduction

The asset allocation problem for pension funds asset allocation problem has become a very important area of research in recent years. This is motivated by different reasons; for instance, the average age of the employees when they join a pension plan and the life expectancy has increased in the last decade. In the area of pension funds, we distinguish two types of pension plans: a Defined Benefit (DB) plan, where the benefits are known in advance and the contributions are adjusted in time to ensure that the fund remains in balance and a Defined Contribution (DC) plan, where the contributions are defined in advance and the benefits depend on the return of the fund, with the risk taken by the plan members. We refer to Antolin et al. (2008) or Devolder et al. (2013) for a thorough

E-mail address: calistoguambe@yahoo.com.br, rodwell.kufakunesu@up.ac.za, gusti.vanzyl@up.ac.za, conrad.beyers@up.ac.za.

Key words and phrases. DC pension plan, Maximum principle, Stochastic income, inflation risks, mortality risks.

* Corresponding author

discussion on the theory of pension funds. Since most of the developing countries have moved or are moving from DB to DC plans, where the employee is directly exposed to the financial risks, the study of optimization problems in the context of pension funds it is of great importance. This is because the solution of such problems will help both the pension plan members and the pension fund managers in their allocation of funds in different assets in order to achieve the best retirement savings, even during the periods of market fluctuations or lack of information.

There is a vast of literature dealing with optimization of pension funds problem, for instance, under the expected utility maximization framework, Sun et al. (2018), considered a robust portfolio choice for DC pension plan with stochastic income and interest rate. Sun et al. (2016) studied the jump diffusion case of a DC investment plan. Osu et al. (2017) studied the effect of stochastic extra contribution on DC pension funds, and references therein. This problem has also been considered in the mean variance framework, see, e.g., He & Liang (2013) and references therein. All the above references solved the DC pension fund problem using a dynamic programming approach under the setting of complete information. Otherwise, Battocchio & Menoncin (2004) considered a martingale method for a DC investment problem. Chen & Delong (2015), studied a DC pension fund problem with regime switching using the techniques of backward stochastic differential equations with quadratic growth.

To the best of our knowledge, in almost all the literature on DC investment problems, the partial information case in the control has not been considered. However, like other investment problems, in the pension fund investment problem, the information about the state control is not always available on time of the decision, which leads to delayed information about the investment strategy. Thus, one needs to consider the case of DC investments with partial information. We assume that the investment strategy is adapted to a given sub-filtration of the filtration generated by the underlying diffusion processes. Therefore, the dynamic programming approach is not applicable. We use a sufficient maximum principle for such a DC investment problem. In the literature, this method has been widely studied. See, for instance, An & Øksendal (2008), Baghery & Øksendal (2007), Framstad et al. (2004) and references therein.

In this paper, we study an asset allocation stochastic problem for a defined-contribution pension plan during the accumulation phase. We consider a financial market composed by a risk-free asset, the inflation-linked bond and the risky asset, where a fund manager aims to maximize the expected power utility derived from the terminal wealth. In order to protect the rights of a member who dies before retirement, we introduce a clause which allows the member to withdraw his premiums and the difference is distributed among the survival members. Besides the mortality risk, the fund manager takes into account the salary and the inflation risks. Furthermore, due to the ultimate aim of the pension fund and to prevent the members from losing all their savings, we introduce a restriction in their investment choices. This restriction forces the plan members to put a certain minimum proportion of their savings in a risk-free investment.

This paper unifies the inflation risks, the stochastic salary and the mortality risks on an optimal investment problem for a DC pension plan under partial information and study the optimization problem under partial information case.

The rest of the paper is organized as follows: in Section 2, we introduce the setting assumptions of the financial market, namely, the inflation linked-bond and the risky asset. We also consider the existence of stochastic income and we state the main optimization problem under study. In Section 3, we solve the asset allocation problem of the pension fund manager with partial information using the maximum principle approach presented in the Appendix. Finally, we give a numerical example in Section 4.

2. The model formulation

Consider two independent Brownian motions $\{W_I(t); W_S(t),\ 0 \leq t \leq T\}$ associated to the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Let a fixed investment horizon of a defined contribution pension fund be $T < \infty$, with a retirement date denoted by $t_0 + T < \infty$. We assume the existence of a financial market composed by three assets: a risk-free asset, an inflation linked index and a risky asset. The risk-free asset has price $S_0(t)$ defined by

$$dS_0(t) = r(t)S_0(t)dt\,, \tag{2.1}$$

where $r(t) \in \mathbb{R}_0^+$ is the risk-free interest rate.

In order to capture the inflation risks, we consider an inflation index $I(t)$ given by

$$dI(t) = I(t)[\mu_I(t)dt + \sigma_I(t)dW_I(t)]\,,$$

with deterministic expected rate of inflation $\mu_I(t) \in \mathbb{R}$ and volatility $\sigma_I(t) \in \mathbb{R}$ satisfying the following integrability condition.

$$\int_0^T \left[|\mu_I(t)| + \sigma_I^2(t)\right] dt < \infty, \quad \text{a.s.}$$

The inflation-linked bond price is defined by

$$B(t) = I(t)S_0(t)\,.$$

Then,

$$dB(t) = B(t)[(r(t) + \mu_I(t))dt + \sigma_I(t)dW_I(t)]\,. \tag{2.2}$$

Finally, assume that the fund manager may also allocate funds to a risky asset defined by the following geometric diffusion process

$$dS(t) = S(t)\left[\mu_S(t)dt + \sigma(t)dW_I(t) + \sigma_S(t)dW_S(t)\right]\,,$$

where the mean rate of return $\mu_S(t) := r(t) + \mu(t)$, with $\mu(t) > 0$ denoting a risk premium. The volatilities $\sigma(t)$, $\sigma_S(t)$ are deterministic functions, satisfying the following integrability condition

$$\int_0^T \left[|\mu_S(t)| + \sigma^2(t) + \sigma_S^2(t)\right] dt < \infty, \quad \text{a.s.} \tag{2.3}$$

We suppose that a pension member has a stochastic income salary driven by:

$$d\ell(t) = \ell(t)\left[(\mu_\ell(t) + r(t))dt + \sigma_1(t)dW_I(t) + \sigma_2(t)dW_S(t)\right], \tag{2.4}$$

where $\mu_\ell(t) + r(t)$ is the expected growth rate of income, with μ_ℓ representing the income growth interest rate or the inflation compensation. Usually, the income increases more rapidly during the boom of the economy than the recession period. σ_1 and σ_2 are the instantaneous volatilities arising from the inflation index and from the price process of the risky asset respectively. We also assume that μ_ℓ, σ_1 and σ_2 are deterministic functions satisfying the integrability condition as in (2.3).

Moreover, suppose that the pension member contributes an amount of $\delta\ell(t)$, at time t, where $\delta \in (0,1)$ is the proportion of the salary contributed to the pension plan. We assume that the accumulation period of the fund starts from age $t_0 > 0$ of the member, until the retirement age $t_0 + T$. In order to protect the rights of the plan members who die before retirement, we adopt the withdrawal of the premiums for the member who dies, as in He & Liang (2013).

Let M_0 be the number of members who are still alive in the pension at time t, with age $t_0 + t$. Then, the expected number of members who will die during the time interval $(t, t + \Delta t)$ is $M_0 P^{\Delta t}_{t_0+t}$, where $P^{\Delta t}_{t_0+t}$ is the probability that a person alive at the age $t_0 + t$ will die in the following time period of length Δt. Now, $\int_0^t \delta\ell(s)ds$ is the accumulated premium at time t. Hence, the premium returned to a died member from time t to $t + \Delta t$ is $\int_0^t \delta\ell(s)ds P^{\Delta t}_{t_0+t}$. After returning the premium, the difference between the accumulation and the return is equally distributed to the surviving members. The expected number of members who are alive at time $t + \Delta t$ is $M_0(1 - P^{\Delta t}_{t_0+t})$, which is a deterministic function of time.

Based on He & Liang (2013), we adopt the De Moivre mortality model, i.e., the deterministic force of mortality $\beta_{t_0}(t) = \frac{1}{\tau-(t_0+t)}$, where $\tau > 0$ is the maximal age of the life table. Then,

$$P^{\Delta t}_{t_0+t} = 1 - \exp\Big\{-\int_0^{\Delta t} \beta_{t_0}(u)du\Big\} = \frac{\Delta t}{\tau - t}, \quad 0 \le \Delta t \le \tau - t.$$

We consider a sub-filtration

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \quad \forall\, t \in [0,T],$$

where $\mathcal{E}_t$ represents the amount of the information available to the pension manager at time t.

Since we are modeling an investment plan for pension funds, we assume that there is a minimum proportion of the pension members wealth stipulated by the regulator to be invested in a risk-free asset. This means that in the wealth of the pension member, a certain proportion is assumed a priori to be in a risk-free asset. We denote that fraction by κ. We assume this fraction to avoid the possibility of investing the total wealth in risky assets. Let $\pi_1(t)$, $\pi_2(t)$ be the $\{\mathcal{E}_t\}_{t\in[0,T]}$-adapted processes denoting the proportions of the wealth invested in the inflation-linked bond and the risky asset respectively. Then $\pi_0(t) := 1 - \kappa - \pi_1(t) - \pi_2(t) \in \mathcal{E}_t$ is the additional proportion of the wealth invested in a risk-free asset.

Let $X(t)$ denote the wealth process of the pension plan member. Similar to He & Liang (2013) and Sun et al. (2016), we adopt a return premium clause, when a pension member dies during the accumulation phase. Thus, after deducting the expected return of the premiums for the members who died during the time interval $(t, t+\Delta t)$, the total wealth of the pension members is given by

$$\begin{aligned}\hat{X}(t+\Delta t) &= M_0 X(t)\Big[(1-\kappa-\pi_1(t)-\pi_2(t))\frac{S_0(t+\Delta t)}{S_0(t)} + \pi_1(t)\frac{B(t+\Delta t)}{B(t)} \\ &\quad +\pi_2(t)\frac{S(t+\Delta t)}{S(t)}\Big] + M_0 P^{\Delta t}_{t_0+t} - \varepsilon t\delta\ell(t)P^{\Delta t}_{t_0+t}\,.\end{aligned}$$

Here ε is a parameter with values 0 or 1. If $\varepsilon = 0$, the pension member obtains nothing during the accumulation phase, while if $\varepsilon = 1$, the premiums are returned to the member when he dies. Then, the total wealth is equally distributed to the surviving members and each of them has the pension wealth of

$$\begin{aligned}X(t+\Delta t) &= \frac{\hat{X}(t+\Delta t)}{M_0(1-P^{\Delta t}_{t_0+t})} \\ &= X(t)\Big[(1-\kappa-\pi_1(t)-\pi_2(t))\frac{S_0(t+\Delta t)}{S_0(t)} + \pi_1(t)\frac{B(t+\Delta t)}{B(t)} \\ &\quad +\pi_2(t)\frac{S(t+\Delta t)}{S(t)}\Big] + \delta\ell(t)\Delta t - \beta_{t_0}(t)\Delta t[\varepsilon t\delta\ell(t) - X(t)] + o(\Delta t)\,.\end{aligned}$$

Dividing by Δt and taking the limit, when $\Delta t \to 0$, we have the following wealth process in continuous time:

$$\begin{aligned}dX(t) &= \Big[X(t)\Big((1-\kappa)r(t) + \mu_I(t)\pi_1(t) + \mu(t)\pi_2(t) + \beta_{t_0}(t)\Big) \\ &\quad +(1-\varepsilon t\beta_{t_0}(t))\delta\ell(t)\Big]dt + (\pi_1(t)\sigma_I(t) + \pi_2(t)\sigma(t))X(t)dW_I(t) \\ &\quad +\pi_2(t)\sigma_S(t)X(t)dW_S(t)\,.\end{aligned}$$

Suppose that the income salary $\ell(t)$ is given as a numeraire. we define the relative wealth process by $Y(t) = \frac{X(t)}{\ell(t)}$. Then by Itô's formula, we have

$$\begin{aligned}dY(t) &= \Big\{Y(t)\Big[\mu_I(t)\pi_1(t) + \mu(t)\pi_2(t) - \kappa r(t) + \beta_{t_0}(t) - \mu_\ell(t) \\ &\quad +(\sigma_1^2(t) + \sigma_2^2(t)) - \pi_2(t)\sigma_S(t)\sigma_2(t) - \sigma_1(t)(\sigma_I(t)\pi_1(t) + \sigma(t)\pi_2(t))\Big] \\ &\quad +(1-\varepsilon t\beta_{t_0}(t))\delta\Big\}dt + (\sigma_I(t)\pi_1(t) + \sigma(t)\pi_2(t) - \sigma_1(t))Y(t)dW_I(t) \\ &\quad +(\sigma_S(t)\pi_2(t) - \sigma_2(t))Y(t)dW_S(t)\,.\end{aligned} \tag{2.5}$$

Define $\mathcal{A} := \{(\pi_1, \pi_2) := (\pi_1(t), \pi_2(t))_{t\in[0,T]}\}$ as a set of admissible strategies if $(\pi_1(t), \pi_2(t)) \in \{\mathcal{E}_t\}_{t\in[0,T]}$ and the SDE (2.5) has a unique strong solution such that $Y(t) \geq 0$, $\mathbb{P}$-a.s.

Let $U:(0,\infty)\mapsto\mathbb{R}$ be the utility function measuring the investor's preference. The main objective of the pension fund manager is to maximize the following functional:

$$\mathcal{J}(t,y,\pi_1,\pi_2)=\mathbb{E}_{t,y}[U(Y(T))]\,.$$

Then, the value function of the pension manager is given by

$$V(t,y)=\sup_{(\pi_1,\pi_2)\in\mathcal{A}}\mathcal{J}(t,y,\pi_1,\pi_2)\,. \tag{2.6}$$

3. Solution of the pension fund manager optimization problem

Since we consider an asset allocation problem with partial information, the classical dynamic programming approach applied, for instance, in Battocchio & Menoncin (2004), Federico (2008), Di Giacinto et al. (2011), Sun et al. (2018) is not applicable.

Applying a sufficient maximum principle approach for diffusion process model with partial information (see the results in the Appendix), we define the Hamiltonian $\mathcal{H}$: $[0,T]\times\mathbb{R}\times\mathcal{A}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by:

$$\begin{aligned}
&\mathcal{H}(t,Y(t),\pi_1(t),\pi_2(t),A(t),B_1(t),B_2(t)) \qquad (3.1)\\
=\;&\Big\{Y(t)\Big[\mu_I(t)\pi_1(t)+\mu(t)\pi_2(t)-\kappa r(t)+\beta_{t_0}(t)-\mu_\ell(t)+(\sigma_1^2(t)+\sigma_2^2(t))\\
&-\pi_2(t)\sigma_S(t)\sigma_2(t)-\sigma_1(t)(\sigma_I(t)\pi_1(t)+\sigma(t)\pi_2(t))\Big]+(1-\varepsilon t\beta_{t_0})\delta\Big\}A(t)\\
&+(\pi_1(t)\sigma_I(t)+\pi_2(t)\sigma(t)-\sigma_1(t))Y(t)B_1(t)+(\pi_2(t)\sigma_S(t)-\sigma_2(t))Y(t)B_2(t)\,.
\end{aligned}$$

The adjoint equation corresponding to the admissible strategy (π_1,π_2) is given by the following backward stochastic differential equation

$$\begin{aligned}
dA(t) &= -\Big\{\Big[\mu_I(t)\pi_1(t)+\mu_S(t)\pi_2(t)-\kappa r(t)+\beta_{t_0}(t)-\mu_\ell(t)\\
&\quad+\sigma_1^2(t)+\sigma_2^2(t)-\pi_2(t)\sigma_S(t)\sigma_2(t)-\sigma_1(t)(\pi_1(t)\sigma_I(t)+\pi_2(t)\sigma(t))\Big]A(t)\\
&\quad+(\pi_1(t)\sigma_I(t)+\pi_2(t)\sigma(t)-\sigma_1(t))B_1(t)+(\pi_2(t)\sigma_S(t)-\sigma_2(t))B_2(t)\Big\}dt\\
&\quad+B_1(t)dW_I(t)+B_2(t)dW_S(t)\,, \qquad (3.2)\\
A(T) &= U'(Y(T))\,.
\end{aligned}$$

Applying the first order conditions of optimality to the Hamiltonian with respect to (π_1,π_2), given the information available $\{\mathcal{E}_t\}_{t\in[0,T]}$, we have the following equations

$$\begin{cases}
(\mu_I(t)-\sigma_1(t)\sigma_I(t))\mathbb{E}[A^*(t)\mid\mathcal{E}_t]+\sigma_I(t)\mathbb{E}[B_1^*(t)\mid\mathcal{E}_t]\;=\;0,\\
(\mu_S(t)-(\sigma(t)\sigma_1(t)+\sigma_2(t)\sigma_S(t)))\mathbb{E}[A^*(t)\mid\mathcal{E}_t]\\
+\sigma(t)\mathbb{E}[B_1^*(t)\mid\mathcal{E}_t]+\sigma_S(t)\mathbb{E}[B_2^*(t)\mid\mathcal{E}_t]\;=\;0,
\end{cases} \tag{3.3}$$

where A^*,B_1^* and B_2^* are the adjoint processes corresponding to the optimal controls (π_1^*,π_2^*). For this optimal controls, the adjoint equation becomes

$$
\begin{aligned}
dA^*(t) &= -\Big\{[\beta_{t_0}(t) - \kappa r(t) - \mu_\ell(t) + \sigma_1^2(t) + \sigma_2^2(t)]A^*(t) - \sigma_1(t)B_2^*(t) - \sigma_2(t)B_2^*(t)\Big\}dt \\
&\quad + B_1^*(t)dW_I(t) + B_2^*(t)dW_S(t)\,, \qquad (3.4) \\
A^*(T) &= U'(Y(T))\,.
\end{aligned}
$$

In order to solve our optimization problem, we consider a power utility function of the form $U(y) = \frac{y^\alpha}{\alpha}$, where $\alpha \in (-\infty, 1)\setminus\{0\}$. Then the terminal condition for the first adjoint equation becomes $A^*(T) = Y(T)^{\alpha-1}$. From this form, we try the solution of the BSDE (3.4) to be of the form

$$
A^*(t) = (Y(t))^{\alpha-1}\phi(t)\,, \quad \phi(T) = 1\,. \qquad (3.5)
$$

Applying Itô's formula, we have

$$
\begin{aligned}
dA^*(t) &= Y^{\alpha-1}(t)\Big\{\phi'(t) + (\alpha-1)\phi(t)\Big[\mu_I(t)\pi_1^*(t) + \mu(t)\pi_2^*(t) \\
&\quad -\kappa r(t) + \beta_{t_0}(t) - \mu_\ell(t) + \sigma_1^2(t) + \sigma_2^2(t) - \sigma_S(t)\sigma_2(t)\pi_2^*(t) \\
&\quad -\sigma_1(t)(\pi_1^*(t)\sigma_I(t) + \pi_2^*(t)\sigma(t)) + (1 - \varepsilon t\beta_{t_0}(t))\delta(y(t))^{-1} \\
&\quad +\frac{1}{2}(\alpha-2)\Big[(\pi_2^*(t)\sigma_S(t) - \sigma_2(t))^2 + (\sigma_I(t)\pi_1^*(t) + \sigma(t)\pi_2^*(t) - \sigma_1(t))^2\Big]\Big]\Big\}dt \\
&\quad +(\alpha-1)\phi(t)Y^{\alpha-1}(t)(\sigma_I(t)\pi_1^*(t) + \sigma_2(t)\pi_2^*(t) - \sigma_1(t))dW_I(t) \\
&\quad +(\alpha-1)\phi(t)Y^{\alpha-1}(t)(\pi_2^*(t)\sigma_S(t) - \sigma_2(t))dW_S(t)\,.
\end{aligned}
$$

Comparing with the adjoint equation (3.4), we obtain the following relations

$$
\begin{aligned}
B_1^*(t) &= (\alpha-1)(\sigma_I(t)\pi_1^*(t) + \sigma_2(t)\pi_2^*(t) - \sigma_1(t))A^*(t)\,; \qquad (3.6) \\
B_2^*(t) &= (\alpha-1)(\pi_2^*(t)\sigma_S(t) - \sigma_2(t))A^*(t)\,. \qquad (3.7)
\end{aligned}
$$

Moreover, the function $\phi(t)$ solves the following backward ordinary differential equation

$$
(\phi'(t) + K(t)\phi(t))Y^{\alpha-1}(t) = Q(t)A^*(t) - \sigma_1(t)B_2^*(t) - \sigma_2(t)B_2^*(t)\Big\}\,, \qquad (3.8)
$$

where

$$
\begin{aligned}
K(t) &= (\alpha-1)\Big[\mu_I(t)\pi_1^*(t) + \mu(t)\pi_2^*(t) \\
&\quad -\kappa r(t) + \beta_{t_0}(t) - \mu_\ell(t) + \sigma_1^2(t) + \sigma_2^2(t) - \sigma_S(t)\sigma_2(t)\pi_2^*(t) \\
&\quad -\sigma_1(t)(\pi_1^*(t)\sigma_I(t) + \pi_2^*(t)\sigma(t)) + (1 - \varepsilon t\beta_{t_0}(t))\delta(y(t))^{-1} \\
&\quad +\frac{1}{2}(\alpha-2)\Big[(\pi_2^*(t)\sigma_S(t) - \sigma_2(t))^2 + (\sigma_I(t)\pi_1^*(t) + \sigma(t)\pi_2^*(t) - \sigma_1(t))^2\Big]\Big]
\end{aligned}
$$

and

$$
Q(t) = \kappa r(t) + \mu_\ell(t) - \beta_{t_0}(t) - \sigma_1^2(t) - \sigma_2^2(t)\,.
$$

Substituting (3.5), (3.6)–(3.7) into (3.3), we obtain the following optimal solutions:

$$
\pi_1^*(t) = \frac{\mu_I(t) - \sigma_1(t)\sigma_I(t)}{(1-\alpha)\sigma_I^2(t)} - \frac{1}{\sigma_I(t)}\Big(\sigma(t)\pi_2^*(t) - \sigma_1(t)\Big)\,; \qquad (3.9)
$$

$$\pi_2^*(t) \quad = \quad \frac{\mu(t) - \alpha\sigma_S(t)\sigma_2(t) - \frac{\sigma(t)}{\sigma_I(t)}\mu_I(t)}{(1-\alpha)\sigma_S^2(t)} \,. \tag{3.10}$$

Furthermore, from (3.5), (3.6) and (3.7), we can transform (3.8) to the following linear ODE

$$\phi'(t) + \mathcal{K}(t)\phi(t) = 0 \,,$$

where

$$\mathcal{K}(t) \quad = \quad K(t) - [Q(t) + (\alpha - 1)(\sigma_I(t)\pi_1^*(t) + \sigma_2(t)\pi_2^*(t) + \pi_2^*(t)\sigma_S(t) - \sigma_1(t) - \sigma_2(t))] \,,$$

which gives the following solution

$$\phi(t) = \exp\Big\{ -\int_t^T \mathcal{K}(s)ds \Big\} \,, \qquad t \in [0,T] \,.$$

This completes the solution (3.5), (3.6) and (3.7) respectively.

We then conclude this section summarizing our results in the following theorem.

Theorem 3.1. *Under the power utility function, the optimal strategies for a defined contribution problem (2.6), based on the information flow $\{\mathcal{E}_t\}_{t\in[0,T]}$, are given by*

$$\pi_1^*(t) \quad = \quad \frac{\mu_I(t) - \sigma_1(t)\sigma_I(t)}{(1-\alpha)\sigma_I^2(t)} - \frac{1}{\sigma_I(t)}\Big(\sigma(t)\pi_2^*(t) - \sigma_1(t)\Big) \,;$$

$$\pi_2^*(t) \quad = \quad \frac{\mu(t) - \alpha\sigma_S(t)\sigma_2(t) - \frac{\sigma(t)}{\sigma_I(t)}\mu_I(t)}{(1-\alpha)\sigma_S^2(t)} \,.$$

4. Numerical example

In this section, we consider a numerical application of our results, in order to show the behavior of the optimal portfolio strategy derived in the previous section. We assume the following parameters consistent with the numerical analysis in Battocchio & Menoncin (2004). Figure 1 shows how the funds are allocated, when we assume constant parameters. We can see that the investment in the risk-free and risky assets continue rising throughout the investment period. Figures 2-3 illustrate the effect of appreciation rate and inflation rate, respectively. We can see in both figures that the funds that the risk-free and inflation-linked assets have an opposite correlation. In Figure 3, for instance, when the inflation rate rises, the fund manager pays more attention on the inflation-linked asset to help to reduce its effect.

$$\begin{pmatrix} \mu & r & \sigma & \sigma_S & T & \sigma_I & \mu_I \\ 0.06 + \frac{9}{8000}t & 0.03 & 0.19 & 0.06 & 35 & 0.015 & -0.01 \\ & & & \mu_\ell & \sigma_1 & \sigma_2 & \ell(0) \\ & & & 0.01 & 0.014 & 0.171 & 100 \end{pmatrix}$$

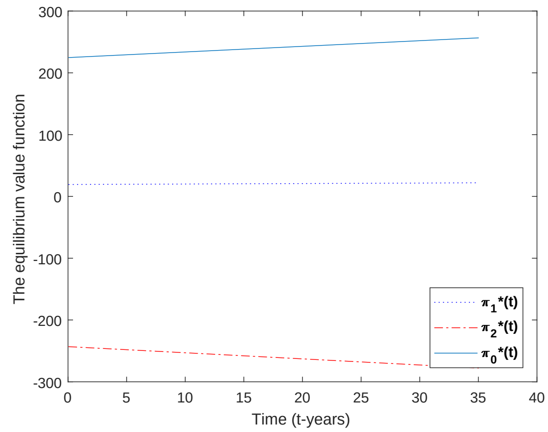

FIGURE 1. The graph shows how the fund manager should pursue with the portfolio allocation for $\alpha = -3$.

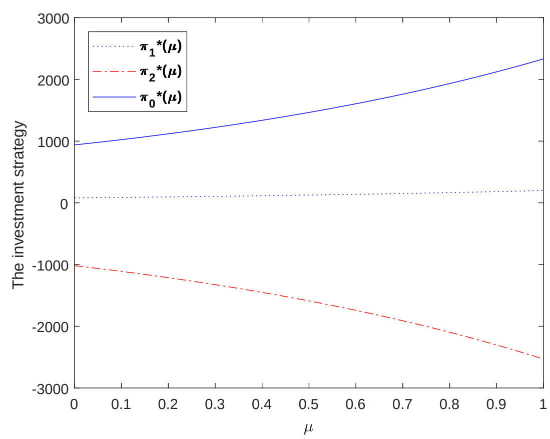

FIGURE 2. The effect of μ on the investment strategy.

Acknowledgment. The first author would like to express a deep gratitude to the University of Pretoria Absa Chair in Actuarial Science for financial support.

APPENDIX

We introduce a version of a maximum principle approach for stochastic volatility model under diffusion with partial information, which is mainly based on the results in Kufakunesu & Guambe (2018). On a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, suppose that the dynamics of the state process is given by the following stochastic differential equation (SDE)

$$\begin{aligned} dX(t) &= b(t, X(t), Y(t), \pi(t))dt + \sigma(t, X(t), Y(t), \pi(t))dW_1(t) \\ &\quad + \beta(t, X(t), Y(t), \pi(t))dW_2(t)\,; \end{aligned} \tag{4.1}$$

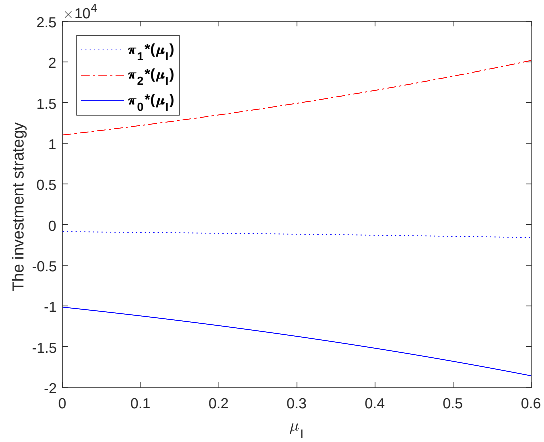

FIGURE 3. The effect of μ_I on the investment strategy

$$X(0) \quad = \quad x \in \mathbb{R}\,,$$

where the external economic factor Y is given by

$$dY(t) = \varphi(Y(t))dt + \phi(Y(t))dW_2(t)\,. \tag{4.2}$$

We assume that the functions $b, \sigma, \beta : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$; $\varphi, \phi : \mathbb{R} \to \mathbb{R}$ are given predictable processes, such that (4.1) and (4.2) are well defined and (4.1) has a unique solution for each $\pi \in \mathcal{A}$. Here, $\mathcal{A}$ is a given closed set in $\mathbb{R}$. We assume that the control process π is adapted to a given filtration $\{\mathcal{E}_t\}_{t\in[0,T]}$, where

$$\mathcal{E}_t \subseteq \mathcal{F}_t\,, \quad \forall\, t \in [0,T]\,.$$

The sub-filtration $\{\mathcal{E}_t\}_{t\in[0,T]}$ denotes the amount of the information available to the controller at time t about the state of the system.

Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$ be a continuous function and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a concave function. We define the performance criterion by

$$\mathcal{J}(t) = \mathbb{E}\Big[\int_0^T f(t, X(t), Y(t), \pi(t))dt + g(X(T), Y(T))\Big]\,. \tag{4.3}$$

We say that $\pi \in \mathcal{A}$ is an admissible strategy if (4.1) has a unique strong solution and

$$\mathbb{E}\Big[\int_0^T |f(t, X(t), Y(t), \pi(t))|dt + |g(X(T), Y(T))|\Big] < \infty\,.$$

The partial information control problem is to find $\pi^* \in \mathcal{A}$ such that

$$\mathcal{J}(\pi^*) = \sup_{\pi\in\mathcal{A}} \mathcal{J}(\pi)\,.$$

The control π^* is called an optimal control if it exists.

In order to solve this stochastic optimal control problem with stochastic volatility, we use the so called maximum principle approach. The beauty of this method is that it solves a stochastic control problem in a more general situation, that is, for both Markovian and non-Markovian cases. We point out that, due to the nature of the partial information $\{\mathcal{E}\}_{t\in[0,T]}$, the dynamic programming approach for a stochastic volatility model by Pham (2002) is not applicable. Our approach may be considered as an extension of the maximum principle approach for a stochastic control problem with partial information in Baghery & Øksendal (2007) to the stochastic volatility case.

We define the Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by:

$$\begin{aligned} & \mathcal{H}(t, X(t), Y(t), \pi(t), A_1(t), A_2(t), B_1(t), B_2(t)) \\ = \; & f(t, X(t), Y(t), \pi(t)) + b(t, X(t), Y(t), \pi(t))A_1(t) + \varphi(Y(t))A_2(t) \\ & +\sigma(t, X(t), Y(t), \pi(t))B_1(t) + \beta(t, X(t), Y(t), \pi(t))B_2(t) + \phi(Y(t))B_3(t) , \end{aligned} \tag{4.4}$$

From now on, we assume that the Hamiltonian $\mathcal{H}$ is continuously differentiable w.r.t. x and y. Then, the adjoint equations corresponding to the admissible strategy $\pi \in \mathcal{A}$ are given by the following $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted backward stochastic differential equations (BSDEs)

$$\begin{aligned} dA_1(t) &= -\frac{\partial \mathcal{H}}{\partial x}(t, X(t), Y(t), \pi(t), A_1(t), A_2(t), B_1(t), B_2(t))dt \\ &\quad +B_1(t)dW_1(t) + B_2(t)dW_2(t) , \end{aligned} \tag{4.5}$$

$$A_1(T) = \frac{\partial g}{\partial x}(X(T), Y(T)) \tag{4.6}$$

and

$$\begin{aligned} dA_2(t) &= -\frac{\partial \mathcal{H}}{\partial y}(t, X(t), Y(t), \pi(t), A_1(t), A_2(t), B_1(t), B_2(t))dt \\ &\quad +B_3(t)dW_1(t) + B_4(t)dW_2(t) , \end{aligned} \tag{4.7}$$

$$A_2(T) = \frac{\partial g}{\partial y}(X(T), Y(T)) . \tag{4.8}$$

The verification theorem associated to our problem is stated as follows:

Theorem 4.1. (Sufficient maximum principle) *Let $\pi^* \in \mathcal{A}$ with the corresponding wealth process X^*. Suppose that the pairs $(A_1^*(t), B_1^*(t), B_2^*(t))$ and $(A_2^*(t), B_3^*(t), B_4^*(t))$ are the solutions of the adjoint equations (4.5) and (4.7), respectively. Moreover, suppose that the following inequalities hold:*

(i) *The function $(x, y) \to g(x, y)$ is concave;*
(ii) *The function $\mathcal{H}(t) = \sup_{\pi\in\mathcal{A}} \mathcal{H}(t, X(t), Y(t), \pi, A_1^*(t), A_2^*(t), B_1^*(t), B_2^*(t))$ is concave and*

$$\mathbb{E}\Big[\mathcal{H}(t, X, Y, \pi^*, A_1^*, A_2^*, B_1^*, B_2^*) \mid \mathcal{E}_t\Big] = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\mathcal{H}(t, X, Y, \pi, A_1^*, A_2^*, B_1^*, B_2^*) \mid \mathcal{E}_t\Big] .$$

Furthermore, we assume the following:

$$\mathbb{E}\Big[\int_0^T (X^*(t))^2\Big((B_1^*(t))^2 + (B_2^*(t))^2\Big)dt\Big] < \infty ;$$

$$\mathbb{E}\Big[\int_0^T (Y(t))^2\Big((B_3^*(t))^2 + (B_4^*(t))^2\Big)dt\Big] < \infty\,;$$

$$\mathbb{E}\Big[\int_0^T \Big\{(A_1^*(t))^2\Big((\sigma(t,X(t),Y(t),\pi(t)))^2 + (\beta(t,X(t),Y(t),\pi(t)))^2\Big) + (A_2^*(t))^2(\phi(Y(t)))^2\Big]dt\Big] \;<\; \infty\,,$$

for all $\pi \in \mathcal{A}$.

Then, $\pi^* \in \mathcal{A}$ *is an optimal strategy with the corresponding optimal state process* X^*.

Proof. Let $\pi \in \mathcal{A}$ be an admissible strategy and $X(t)$ the corresponding wealth process. Then, following Framstad et al. (2004), Theorem 2.1., we have:

$$\begin{aligned}
\mathcal{J}(\pi^*) - \mathcal{J}(\pi) &= \mathbb{E}\Big[\int_0^T (f(t,X^*(t),Y^*(t),\pi^*(t)) - f(t,X(t),Y(t),\pi(t)))dt \\
&\quad + (g(X^*(T),Y^*(T)) - g(X(T),Y(T)))\Big] \\
&= \mathcal{J}_1 + \mathcal{J}_2\,.
\end{aligned}$$

By condition (i) and the integration by parts rule (Øksendal & Sulem (2007), Lemma 3.6.), we have

$$\begin{aligned}
\mathcal{J}_2 &= \mathbb{E}\Big[g(X^*(T),Y^*(T)) - g(X(T),Y(T))\Big] \\
&\geq \mathbb{E}\Big[(X^*(T) - X(T))A_1^*(T) + (Y^*(T) - Y(T))A_2^*(T)\Big] \\
&= \mathbb{E}\Big[\int_0^T (X^*(t) - X(t))dA_1^*(t) + \int_0^T A_1^*(t)(dX^*(t) - dX(t)) \\
&\quad + \int_0^T (Y^*(t) - Y(t))dA_2^*(t) + \int_0^T A_2^*(t)(dY^*(t) - dY(t)) \\
&\quad + \int_0^T [(\sigma(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_1^*(t) \\
&\quad + (\beta(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_2^*(t)]dt \\
&\quad + \int_0^T (\phi(Y^*(t)) - \phi(Y(t)))B_3^*(t)dt\Big] \\
&= \mathbb{E}\Big[-\int_0^T (X^*(t) - X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt - \int_0^T (Y^*(t) - Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt \\
&\quad + \int_0^T (A_1^*(t)b(t,X^*(t),Y^*(t),\pi^*(t)) - b(t,X(t),Y(t),\pi(t)))dt \\
&\quad + \int_0^T (\varphi(Y^*(t)) - \varphi(Y(t)))A_2^*(t)dt + \int_0^T (\phi(Y^*(t)) - \phi(Y(t)))B_3^*(t)dt
\end{aligned}$$

$$+\int_0^T [(\sigma(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_1^*(t)$$
$$+(\beta(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_2^*(t)]dt\Big],$$

where we have used the notation

$$\mathcal{H}^*(t) = \mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t)).$$

On the other hand, by definition of $\mathcal{H}$ in (4.4), we see that

$$\begin{aligned}
\mathcal{J}_1 &= \mathbb{E}\Big[\int_0^T (f(t,X^*(t),Y^*(t),\pi^*(t)) - f(t,X(t),Y(t),\pi(t)))dt\Big] \\
&= \mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t)) \\
&\quad -\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t))]dt \\
&\quad -\int_0^T A_1^*(t)(A_1^*(t)b(t,X^*(t),Y^*(t),\pi^*(t)) - b(t,X(t),Y(t),\pi(t)))dt \\
&\quad -\int_0^T (\varphi(Y^*(t)) - \varphi(Y(t)))A_2^*(t)dt + \int_0^T (\phi(Y^*(t)) - \phi(Y(t)))B_3^*(t)dt \\
&\quad -\int_0^T [(\sigma(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_1^*(t) \\
&\quad -(\beta(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_2^*(t)]dt\Big].
\end{aligned}$$

Then, summing the above two expressions, we obtain

$$\begin{aligned}
&\mathcal{J}_1 + \mathcal{J}_2 \\
= &\ \mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t)) \\
&-\mathcal{H}(t,X(t),Y(t),\pi(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t))]dt \\
&-\int_0^T (X^*(t) - X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt - \int_0^T (Y^*(t) - Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt.
\end{aligned}$$

By the concavity of $\mathcal{H}$, i.e., conditions (i) and (ii), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t)) \\
&-\mathcal{H}(t,X(t),Y(t),\pi(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t))]dt \mid \mathcal{E}_t\Big] \\
\geq &\ \mathbb{E}\Big[\int_0^T (X^*(t) - X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt + \int_0^T (Y^*(t) - Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt
\end{aligned}$$

$$+\int_0^T (\pi^*(t)-\pi(t))\frac{\partial \mathcal{H}^*}{\partial \pi}(t)dt \mid \mathcal{E}_t\Big].$$

Then, by the maximality of the strategy $\pi^* \in \{\mathcal{E}_t\}$-measurable and the concavity of the Hamiltonian $\mathcal{H}$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t)) \\
&-\mathcal{H}(t,X(t),Y(t),\pi(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t))]dt\Big] \\
\geq\ &\mathbb{E}\Big[\int_0^T (X^*(t)-X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt + \int_0^T (Y^*(t)-Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt\Big].
\end{aligned}$$

Hence $\mathcal{J}(\pi^*)-\mathcal{J}(\pi)=\mathcal{J}_1+\mathcal{J}_2\geq 0$. Therefore, $\mathcal{J}(\pi^*)\geq\mathcal{J}(\pi)$, for any strategy $\pi\in\mathcal{A}$. Then $\pi^*\in\mathcal{A}$ is optimal. □

References

An, T. T. & Øksendal, B. (2008), 'Maximum principle for stochastic differential games with partial information', *Journal of Optimization Theory and Applications* **139**(3), 463–483.

Antolin, P., Pugh, C. & Stewart, F. (2008), 'Forms of benefit payment at retirement', *OECD Working Paper on Insurance and Private Pensions No.26* p. DOI:10.1787/238013082545.

Baghery, F. & Øksendal, B. (2007), 'A maximum principle for stochastic control with partial information', *Stochastic Analysis and Applications* **25**(3), 705–717.

Battocchio, P. & Menoncin, F. (2004), 'Optimal pension management in a stochastic framework', *Insurance: Mathematics and Economics* **34**(1), 79–95.

Chen, A. & Delong, Ł. (2015), 'Optimal investment for a defined-contribution pension scheme under a regime switching model', *ASTIN Bulletin: The Journal of the IAA* **45**(2), 397–419.

Devolder, P., Janssen, J. & Manca, R. (2013), *Stochastic methods for pension funds*, John Wiley & Sons.

Di Giacinto, M., Federico, S. & Gozzi, F. (2011), 'Pension funds with a minimum guarantee: a stochastic control approach', *Finance and Stochastics* **15**(2), 297–342.

Federico, S. (2008), 'A pension fund in the accumulation phase: a stochastic control approach', *Banach Cent. Publ. Adv. Math. Finance* **83**, 61–83.

Framstad, N., Øksendal, B. & Sulem, A. (2004), 'Sufficient stochastic maximum principle for the optimal control of jump diffusions and applications to finance', *Journal of Optimization Theory and Applications* **121**(1), 77–98.

He, L. & Liang, Z. (2013), 'Optimal investment strategy for the dc plan with the return of premiums clauses in a mean–variance framework', *Insurance: Mathematics and Economics* **53**(3), 643–649.

Kufakunesu, R. & Guambe, C. (2018), 'On the optimal investment-consumption and life insurance selection problem with an external stochastic factor', *arXiv preprint arXiv:1808.04608* .

Øksendal, B. K. & Sulem, A. (2007), *Applied stochastic control of jump diffusions*, Vol. 498, Springer.

Osu, B., Akpanibah, E. & Njoku, K. (2017), 'On the effect of stochastic extra contribution on optimal investment strategies for stochastic salary under the affine interest rate model in a dc pension fund', *General Letters in Mathematic* **2**(3), 138–149.

Pham, H. (2002), 'Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints', *Applied Mathematics and Optimization* **46**(1), 55–78.

Sun, J., Li, Y. & Zhang, L. (2018), 'Robust portfolio choice for a defined contribution pension plan with stochastic income and interest rate', *Communications in Statistics-Theory and Methods* **47**(17), 4106–4130.

Sun, J., Li, Z. & Zeng, Y. (2016), 'Precommitment and equilibrium investment strategies for defined contribution pension plans under a jump–diffusion model', *Insurance: Mathematics and Economics* **67**, 158–172.