The multiplicative reasoning proficiency of learners with learning difficulties

by

Elizma Louw

Submitted in partial fulfilment of the requirements for the degree of Philosophiae Doctor in the Faculty of Education, University of Pretoria

Supervisor: Prof Gerrit Stols
Co-supervisors: Dr Grietjie Haupt
Dr David Sekao

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DECLARATION OF ORIGINALITY

I declare that the thesis, which I hereby submit for the degree Philosophiae Doctor in the Faculty of Education at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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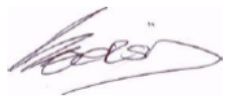

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ABSTRACT

In mathematics there is a conceptual shift from additive to multiplicative reasoning that learners in the Intermediate Phase (Grades 4 to 6) need to make to understand more complex mathematical concepts in secondary school. The aim of this study was to explore and describe the multiplicative proficiency of Grade 6 learners with learning difficulties by exploring the current status of their conceptual knowledge of multiplication, as well as their level of procedural fluency and the nature of their strategic competence. A single-case study design was used, with fifteen Grade 6 learners selected from three special needs schools in Pretoria, South Africa, forming together a group as the unit of analysis. During individualised task-based interviews, participants were required to solve ten classes of multiplication problems. They were further required to solve the problems by using different representations, namely abstract, semi-concrete and concrete representations for a multi-dimensional approach that allowed for in-depth understanding of their reasoning.

The findings of this study indicated that only a few participants made the conceptual shift from additive to multiplicative reasoning, mainly when answering the less cognitively complex questions, since they showed conceptual understanding of and procedural fluency in the way they dealt with those questions. However, only two of the participants answered the less cognitively complex questions in a way that demonstrated proficiency in multiplicative reasoning and showed conceptual understanding, procedural fluency and strategic competence. The majority of the participants were not proficient in multiplicative reasoning and did not make the shift from additive to multiplicative reasoning, especially for the more cognitively complex questions. They still thought in additive terms and struggled to solve cognitively complex multiplication problems. However, some of the participants could solve these problems with either semi-concrete or concrete representations, but not with abstract representations. More participants showed procedural fluency in solving classes of problems they had already learned to solve, even if they were cognitively more complex. The three main reasons identified for this lack of proficiency were misconceptions, misrepresentations and calculation errors, which could inform mathematics teachers’ instructional practice to help learners make the shift from additive to multiplicative reasoning.

Key words: conceptual understanding; mathematics; multiplicative reasoning; procedural fluency; representations; special needs education; teaching and learning

LANGUAGE EDITOR DISCLAIMER

Editor's statement

F.J. Opper – Translator and Language Editor

PO Box 162
ST FRANCIS BAY
6312

Tel 042 298 0330 / 082 5326 015

To whom it may concern

Herewith I, FJ Opper, confirm that I undertook the language editing of Ms Elizma Louw's doctoral thesis titled:

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LIST OF ABBREVIATIONS

2D	Two Dimensional
3D	Three Dimensional
ADD	Attention Deficit Disorder
ADHD	Attention Deficit Hyperactivity Disorder
ANA	Annual National Assessments
CAPS	Curriculum and Assessment Policy Statement
CSA	Concrete, Semi-concrete, Abstract
DBE	Department of Basic Education
DoE	Department of Education
FET	Further Education and Training
HRW	Human Rights Watch
LSEN	Learners with special educational needs
NRC	National Research Council
USA	United States of America
WEF	World Economic Forum

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CHAPTER 1

INTRODUCTION AND CONTEXTUALISATION

1.1 Background to the study

A major conceptual shift that needs to take place during the Intermediate Phase (Grades 4 to 6) is the shift from additive to multiplicative reasoning (Hurst, 2015; Hurst & Hurrell, 2014; Tzur, Xin, Si, Kenney, & Guebert, 2010). If learners are unable to make this conceptual shift they will struggle with mathematics in the Senior Phase (Grades 7 to 9) and Further Education and Training (FET) Phase (Grades 10 to 12), as multiplicative reasoning is seen by many authors as the foundation and a pre-requisite for most mathematics learned in primary and secondary school (Brown, Kuchemann, & Hodgen, 2010; Hurst & Hurrell, 2014; 2015; 2016; Vergnaud, 1983). This may be one of the reasons why the results of the Annual National Assessments (ANA) are so poor. The most recent ANA report revealed a bleak picture of mathematics proficiency in South Africa (DBE, 2014). The pass rate for Grade 3, Grade 6 and Grade 9 mathematics in mainstream schools were 56%, 43% and 11% respectively. In special needs schools the pass rate for mathematics were 53% in Grade 3 and 37% in Grade 6, with no statistics available for Grade 9. This clearly shows a decline in the pass rate from Grade 3 to Grade 9 in both mainstream and special needs schools.

Mathematical content presented in the Intermediate Phase plays a vital role in learners' ability to successfully learn mathematical concepts in subsequent phases (Brown et al., 2010; Hurst & Hurrell, 2014; 2015; 2016). During this phase, learners need to make a conceptual shift from additive to multiplicative reasoning, which is an important step in their cognitive development and enables them to understand more abstract and complex mathematical content, such as proportional and algebraic reasoning. However, learners with learning difficulties do not necessarily make this shift. Society expects the government to provide every learner with the best opportunity to develop to his or her full potential. In 2001, the United States of America (USA) introduced the No Child Left Behind Act, which was later changed to become the Every Student Succeeds Act of 2015 and essentially entails that all learners, including those with disabilities, must receive high-quality education (US Department of Education, n.d.). Unfortunately, no such legislation exists in South Africa. In 2017, there were 461 special needs schools and 848 full-service or inclusive schools in South Africa (DBE, 2018). The Human Rights Watch (HRW) estimates that over half a million, or 70 per cent of learners with special needs of school-going age, are not yet attending any type of school (Fortune, 2017; HRW, 2016). The goal of the Department of Basic Education is to integrate learners with special needs into mainstream schools (DoE, 2001). In 2007, South Africa was one of the first countries that adopted the United Nations Disability Rights Treaty, which requires governments to promote inclusive education in their countries. However, the large number of learners with special needs who are still not attending schools shows that the government is not yet doing enough in this regard. Fortune (2017) speculates that one reason for the problem is that there is a gap between written policy and its implementation.

To ensure that learners with special needs are not left behind, policies should be effectively implemented and teachers should receive appropriate training for teaching learners with special needs. In 2015, HRW found that teachers lack the training required to understand learners' disabilities and how they should be taught. Furthermore, learners with special needs start school later than other learners and many leave schools before completing Grade 12 (HRW, 2015). Therefore, they do not reach their full potential and are often left behind. Effective teacher training is particularly important in the case of mathematics education in South Africa. For this reason, my research focused on a specific category of special needs, namely learners with learning difficulties. Learning difficulties is a broad concept that implies that although learners are capable of learning, they experience various barriers to learning that can range from mild to severe (Mercer, Mercer, & Pullen, 2014). This study focused on barriers to learning that include difficulty with memorisation, limited attention span, processing of information and language, abstract thinking and metacognitive thinking (Allsopp, Kyger, & Lovin, 2007; Dednam, 2011; Miller & Mercer, 1997).

1.2 Multiplicative reasoning

In order to assess the proficiency of learners, the teacher should focus on the mental activities of learners (the cognitive processes) and not only on the answer (the product) (Vergnaud, 2013a). Vergnaud (1982), who coined the term conceptual field, defined and classified these cognitive processes into classes of multiplication problems, the mastering of which requires different concepts, calculation techniques and symbolic representations which are intricately connected with one another. Based on the work of Vergnaud, the authors Siemon, Breed and Virgona (2010, p. 2) identify the following three characteristics of multiplicative reasoning: The "capacity to work flexibly and efficiently with an extended range of numbers", which includes whole numbers, decimals, fractions, ratio and percentages; the "ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion"; and "the means to communicate this effectively in a variety of ways", and concisely define multiplicative reasoning as "the capacity to work flexibly with the concept, techniques and representations of multiplication (and division) as they occur in a wide range of contexts".

This definition, which I appropriated for my own purposes as I feel that it complements current methodologies that largely focused on procedural fluency, acknowledges three components of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001), namely conceptual understanding, procedural fluency and strategic competence. I combined the definition of multiplicative reasoning and three components of mathematical proficiency to formulate my research questions (section 1.5 of this chapter). The National Research Council (NRC) changed the way mathematical proficiency was defined (Kilpatrick et al., 2001). Before the reconceptualisation of proficiency in 2001, the emphasis was on subject knowledge. However, the emphasis changed from being one-dimensional, with the focus on subject knowledge, to being multi-dimensional, which incorporates different types of skills together with subject knowledge (Kilpatrick et al., 2001; Schoenfeld, 2007). According to Kilpatrick et al. (2001) the multi-dimensional view of mathematical proficiency includes five components, namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Table 1.1 provides a parallel between the components of mathematical proficiency, the definition of multiplicative reasoning and my study.

Table 1.1: Parallel between mathematical proficiency, multiplicative reasoning and my study

Definition of multiplicative reasoning (Siemon et al., 2010)	Components of mathematical proficiency (Kilpatrick et al., 2001)	My study
• Identifies concept (multiplication) in a wide range of contexts • Flexible use of representations	• Conceptual understanding – knowledge of concepts, operations, and relations	• Conceptual understanding – abstract, semi-concrete, concrete representations – misrepresentations – misconceptions
• Flexible use of calculation techniques	• Procedural fluency – skill to use procedures flexibly, correctly, and efficiently	• Procedural fluency – calculation technique levels – calculation errors
• Work flexibly with representations and calculation techniques in a specific context	• Strategic competence – being able to represent and solve mathematical problems effectively and efficiently	• Strategic competence – effective and efficient use of abstract, semi-concrete and concrete representations – calculation technique types

According to Kilpatrick et al. (2001), the first component, i.e. conceptual understanding, emphasises knowledge of concepts, operations and relations. Multiplicative reasoning includes the concepts of both multiplication and division. However, my study focused only on learners' misconceptions and misrepresentations regarding the concept of multiplication. This links to Siemon et al.'s (2010) definition of the ability to identify the concept of multiplication in different real-life contexts, as well as the ability to present multiplication in different ways to demonstrate understanding.

For the purpose of this study, I required a better understanding of learners' conceptual understanding by using concrete, semi-concrete and abstract representations. Therefore, I also focused on learners' reasoning by asking for explanations and explored the reasons for their misconceptions of multiplication with whole numbers.

The second component, procedural fluency, links with the flexible use of calculation techniques, as stated in the definition of multiplicative reasoning. Procedural fluency is the skill that enables us to use calculation techniques to solve mathematical problems flexibly, correctly and efficiently (Kilpatrick et al., 2001). For the purpose of this study, procedural fluency included the calculation technique levels, as well as the calculation errors that they made.

Kilpatrick et al. (2001) define strategic competence, the third proficiency component, as the ability to represent and solve mathematical problems effectively and efficiently. This links with the definition of multiplicative reasoning, which requires learners to work flexibly with representations and calculation techniques in a specific context. For my study, I inferred strategic competence in how effectively and efficiently learners used the three types of representations (abstract, semi-concrete and concrete) to represent a specific multiplication problem concept in context. Furthermore, I inferred strategic competence in how effectively and efficiently learners used calculation technique types to solve multiplication problems.

The last two components of mathematical competence are adaptive reasoning and productive disposition (Kilpatrick et al., 2001). Adaptive reasoning is the ability to think logically, reflect, explain and justify your reasoning. In my study, I did not consider adaptive reasoning as a separate component but integrated it under the conceptual understanding component.

From experience, I have observed that many learners with learning difficulties have problems with language and with expressing their thoughts. I used learners' explanations and reasoning in conjunction with their representations to get a better understanding of their conceptual understanding. Productive disposition refers to learners' positive attitude towards mathematics and their confidence in their ability to solve mathematical problems. This component falls outside the scope of my study as I explored learners' cognitive processes, and not their attitudes towards mathematics. The connectedness of these components and the fact that they interlink with one another to create a multi-dimensional view of proficiency are the indicators that can determine proficiency in multiplicative reasoning.

An assessment of the proficiency of learners requires more than merely evaluating the answers that they provide to given multiplication problems (the product); it requires an exploration of the cognitive thinking processes of learners, which are multi-dimensional and complex in nature. Vergnaud (2013a) suggests that in order to assess the proficiency of learners, one should focus on the cognitive activity (i.e. the cognitive processes) and not only on the answer (the product). The various cognitive thinking processes that encompass proficiency and were the focus of this study are: First, the conceptual understanding of multiplication, for which I explored the abstract, semi-concrete and concrete representations as a multi-dimensional approach to evaluate learners' conceptual understanding of multiplication and their misconceptions and misrepresentations; hence, how they used these external representations, as well as how they reasoned while solving multiplication problems. For the second, procedural fluency, I explored the level of calculation techniques that learners used to solve the multiplication problems and the errors they made. Finally, I looked at strategic competence, which relates to how effectively and efficiently learners solved multiplication problems within a specific real-life context. I explored the efficiency of their external representations, which represented their conceptual understanding within the given context. Furthermore, I determined which calculation techniques learners used and how effective and efficient they were.

1.3 Rationale of the study

In my experience we only require learners in the South African school context to show their procedural fluency of mathematical calculation techniques. We seldom focus on the development of conceptual understanding. By focusing on understanding their learners' thinking processes, or the lack thereof, and on the interplay between internal and external representations when they solve mathematical problems, teachers can meaningfully enhance the quality of teaching and learning. Studying the interplay between internal and external representations might serve as a bridge between research on learning and on teaching (Lamon, 1994). For this reason, this study focused on the multiplicative reasoning proficiency of Grade 6 learners with learning difficulties by using multiple external representations. An attempt to determine these learners' use of external representations in a real-life context, and the reasons for misconceptions and calculation errors, can further shed light on their multiplicative reasoning proficiency.

I chose Grade 6 learners for my study because the Intermediate Phase (Grades 4 to 6) is important as learners are required to understand more abstract concepts than those dealt with in the Foundation Phase (Grades R to 3). The Foundation Phase focuses, for example, on addition and subtraction. In the Intermediate Phase they are expected to think more abstractly when they learn to multiply and divide. During the early 1980s, Vergnaud (1989) showed how mathematical concepts become more abstract because of the developmental nature of mathematical content. He developed a cognitive developmental theory about conceptual fields of which multiplicative reasoning (multiplication and division) and additive reasoning (addition and subtraction) are two examples (Vergnaud, 1989; 2013a). Other authors who subsequently conducted research on multiplicative reasoning agree on the importance of multiplicative reasoning during the primary school years, especially during the Intermediate Phase (Baturo, 1997; Brown et al., 2010; Carrier, 2014; Clark & Kamii, 1996; Hurst & Hurrell, 2015; McClintock, Tzur, Xin, & Si, 2011; Tzur et al., 2010; 2013).

Multiplicative reasoning as a field of study is important for the following reasons: First, researchers agree that multiplicative reasoning form the foundation and is a pre-requisite for much of the mathematics that is learned in primary and secondary school, such as fractions, proportional reasoning and algebraic reasoning (Brown et al., 2010; Hurst & Hurrell, 2014; 2015; 2016; Vergnaud, 1983). Second, a major conceptual shift takes place when learners progress from additive to multiplicative reasoning, with a reconceptualisation that needs to take place (Hurst, 2015; Hurst & Hurrell, 2014; Tzur et al., 2013).

Since multiplicative reasoning is such an important conceptual field in mathematics, it is interesting to note that the first research on multiplicative reasoning was only conducted as late as in the 1980s. Vergnaud, a French mathematician, philosopher, educator and psychologist, developed the idea of conceptual fields in mathematics, of which multiplicative reasoning is one (Grenier, 2007; Vergnaud, 1982). Most of the research on multiplicative reasoning was conducted in the USA (Caddle & Brizuela, 2011; Carrier, 2014; Clark & Kamii, 1996; Empson & Turner, 2006; Kouba, 1989; McClintock, et al., 2011; Tzur et al., 2010; Tzur, Johnson, McClintock, Kenny, Xin, Si, Woordward, Hord, & Jin, 2013; Zhang, Xin, & Si, 2011) and Australia (Baturo, 1997; Hurst & Hurrell, 2015, 2016; Mulligan, 1992; Siemon et al., 2010). Similar studies were also conducted in England (Brown et al., 2010; Nunes et al., 2008), the Netherlands (Bakker, Van den Heuvel-Panhuizen, & Robitzsch, 2014) and Canada (Agostino et al., 2010). Only two research studies in this field (for a master's and a PhD degree) could be found in South Africa (Mofu, 2014; Long, 2011). Of a total of 21 studies found on the Eric database, sixteen had been conducted in mainstream schools. Only five of the 21 studies conducted in the USA focused on learners with special needs (McClintock et al., 2011; Nunes et al., 2008; Zhang et al., 2011; Tzur et al., 2010, 2013), and only four required learners to explain or justify their thinking (Baturo, 1997; Hurst & Hurrell, 2015, 2016; Siemon et al., 2010). One study required learners to show their understanding by making use of calculations, for which they could use bundling sticks to help them solve the problem (Hurst & Hurrell, 2016). However, none of those studies investigated the level of understanding in multi-representational forms, such as with the use of abstract symbols, the drawing of pictures and using concrete objects, thus a multi-dimensional approach. This study represents an attempt to address this gap in the literature.

1.4 Problem statement

For many learners the transition from additive to multiplicative reasoning is one of the key hurdles to be overcome (Ell, 2001; Tzur et al., 2010). Multiplicative reasoning is a major feat for learners in the Intermediate Phase, which is when the transition from additive reasoning should take place (Long & Dunne, 2014), and even more so for learners with learning difficulties (McClintock et al., 2011; Tzur et al., 2010).

Even though authors agree on the importance for studying multiplicative reasoning, research studies on multiplicative reasoning are one-dimensional in nature. In six of the studies that were reviewed, the focus was on the calculation techniques used by learners to solve multiplicative problems (Carrier, 2014; Clark & Kamii, 1996; Jacob & Willis, 2003; Kouba, 1989; Mulligan, 1992; Zhang et al., 2011), while a further seven investigated the reasons why learners struggle to think multiplicatively (Agostino, Johnson, & Pascual-Leone, 2010; Bakker et al., 2014; Brown et al., 2010; McClintock et al., 2011; Nunes, Bryant, Burman, Bell, Evans, & Hallett, 2008; Tzur et al., 2010; 2013). I believe that this one-dimensional focus does not give a complete picture of the proficiency of learners, especially those experiencing difficulties with multiplicative reasoning. For this reason, I used a multi-dimensional approach to explore the multiplicative reasoning proficiency of Grade 6 learners with learning difficulties. Learners with learning difficulties struggle more with abstract thinking than other learners (Allsopp et al., 2007; Dednam, 2011; Miller & Mercer, 1997), and if research is one-dimensional it will give an inaccurate view of their proficiency in multiplicative reasoning. A multi-dimensional approach can shed light on where the problem lies in their thinking process, which is the focus of my study.

1.5 Research questions

Since the proficiency in multiplicative reasoning is multi-dimensional, I followed a multi-dimensional approach. As explained in section 1.2 of this chapter, I formulated my research question by using three of the strands of mathematical proficiency, as identified by Kilpatrick et al. (2001), in conjunction with Siemon et al.'s (2010) definition.

I interviewed Grade 6 learners with learning difficulties. I chose Grade 6 learners as they are in the last grade of the Intermediate Phase which, as explained, is an important phase for the development of multiplicative reasoning.

Primary research question

How proficient are Grade 6 learners with learning difficulties in multiplicative reasoning?

Secondary research questions

All the questions required multiplication of whole numbers. The secondary research questions were:

1. What is the status of the learners' conceptual understanding of multiplication?
2. What is the level of the learners' procedural fluency related to multiplication?
3. What is the nature of the learners' strategic competence when solving multiplication problems?

1.6 Purpose of my study

Research in the field of multiplicative reasoning only started in the 1980s, with authors using various research methods. A search on the Eric database produced 21 research studies dealing with multiplicative reasoning. Twelve of the 21 studies were qualitative in nature and used interviews as research method; six were quantitative using tests; three were mixed-method studies using both tests and interviews for data collection. The number of participants varied greatly, with the number involved in quantitative studies ranging from 155 to 7000 (Agostino et al., 2010; Siemon et al., 2010), while the qualitative studies involved between one and 336 participants (Clark & Kamii, 1996; McClintock et al., 2011). Only four of the 21 research studies focused on learners with special needs. Three of those studies used quantitative research methods and involved between one and twelve participants (McClintock et al., 2011; Tzur et al., 2010; Zhang et al., 2011). Furthermore, the participants were primary school learners (Grades 1 to 7), except in the case of the studies by Long (2011) and Brown et al. (2010), which focused on the Senior Phase (Grades 7 to 9). Just over half of the studies (thirteen) included participants from different grades. To date, no studies have been undertaken in LSEN schools in South Africa to determine learners' thinking process during multiplicative reasoning. This is what I attempted to do with this study.

Learners with learning difficulties often do not make the conceptual shift from additive to multiplicative reasoning. To make this shift learners need to be proficient in multiplicative reasoning. Therefore, the purpose of this study was to determine the multiplicative reasoning proficiency of Grade 6 learners with learning difficulties through a multi-dimensional approach that included various cognitive thinking processes with abstract, semi-concrete and concrete representations. This study, consequently, explored the cognitive thinking processes of conceptual understanding (conceptions, misconceptions and misrepresentations), procedural fluency (calculation technique levels and calculation errors) and the strategic competence (conceptions and calculation technique types) when solving multiplication problems.

1.7 Methodological considerations

As stated above, my study focused on the multiplicative reasoning proficiency of Grade 6 learners with learning difficulties. I chose a convenient sample of fifteen Grade 6 learners with learning difficulties (numbered from Learner 3 to Learner 17) from three learners with special educational needs (LSEN) public schools in Pretoria. The first two learners (Learner 1 and Learner 2) formed part of my pilot study. I followed a qualitative research approach, with my study grounded in the critical realism philosophy, which focuses strongly on ontology to identify interactions with our reality. I chose a single-case study design to obtain an in-depth understanding of Grade 6 learners' proficiency in multiplicative reasoning and collected the data through one-on-one task-based interviews. During the task-based interviews, I asked each participant to solve ten multiplication problems within a specific context. These problems were based on specific classes of multiplication problems (see Chapter 2). Furthermore, I required participants to solve each of the problems using three types of representations: Using symbols (abstract), then drawing a picture (semi-concrete), and finally using 3D material (concrete). During these task-based interviews, I asked participants to explain the decisions they had made during problem solving.

I deductive and inductive reasoning to analyse my data according to the task-based questions. I did this to determine the level of abstractness at which they were most proficient. Under each representation I chose categories with my secondary research questions in mind, namely operation type, operation concept, misconceptions, misrepresentations, verbal explanations, calculation technique levels and types, and calculation errors. Thereafter, I used these categories as headings and using my transcribed and picture data, I placed the relevant data under appropriate headings. I then used deductive and inductive reasoning to highlight phrases from the data that I could use as indicators to identify subcategories under each of the chosen categories. Since my study was qualitative in nature, I used tables to summarise and compare the data and rich description to analyse, describe and interpret the data.

1.8 Possible contribution of my study

I hope that my research will make a contribution by empowering mathematics teachers, tutors and learning-support therapists who teach learners with learning difficulties to help their learners to make the transition from additive reasoning to multiplicative reasoning so as to enable them to reach their full potential in mathematics. Furthermore, I hope that it will shed light on how learners with learning difficulties reason when solving multiplication problems, their conceptions and misconceptions, the calculation techniques they use to solve them, and common calculation errors.

1.9 Structure of my thesis

The thesis consists of six chapters. Chapter 1 contains the introduction and a background to the study, which explains what is meant by multiplicative reasoning proficiency. It furthermore presents the rationale for this study, the problem statement, research questions and the purpose of this study. The methodology applied is explained, the important concepts are clarified and the possible contribution of the study is discussed. Chapter 2 includes the literature review and an in-depth exposition of the cognitive thinking processes, the interplay between internal and external representations, and how this relates to multiplicative reasoning. Furthermore, it offers an in-depth discussion of how multiplicative reasoning develops and an explanation and discussion of each component, which includes the external representations, the classes of multiplication problems and the calculation techniques. I set out my conceptual framework and explained the relationship between the components. Chapter 3 contains the methodological considerations, an explanation of why my study was underpinned by critical realism, and a discussion of the research design and how the participants were selected. It also offers an explanation of how I collected the data by conducting one-on-one task-based interviews. I explain how I analysed the data and what quality measures I took to ensure that my study would be credible and reliable. I end with details of the ethical principles I considered to ensure the anonymity and confidentiality of the participants. Chapter 4 consists of the analysis of the data. The analysis of each question is dealt with separately. Under each question I analysed all three representations together under pre-determined subcategories. This is followed by the interpretation and discussion of the data. Chapter 5 contains a discussion and interpretation of the data to answer each of my three secondary research questions, as well as the conclusions and the implications of my study.

Moreover, it includes a reflection on this study, a discussion of its limitations and recommendations for further related research.

CHAPTER 2

LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

2.1 Introduction

Long and Dunne (2014) believe that mathematical knowledge can be classified either by topics or by conceptual fields. The South African Curriculum and Assessment Policy Statement (CAPS) follows a topic approach, with the result that learners study topics in isolation. For instance, multiplication and division of whole numbers, fractions, decimal numbers, area and volume are different topics that learners study separately, with the result that there is limited integration between the different topics. In contrast, conceptual fields group similar concepts together, so that learners develop one big idea, or schema, connecting different topics. For example, multiplication and division of whole numbers, decimal numbers and fractions, as well as area and volume are grouped under multiplicative reasoning; thus making connections between different topics but using the same concept of multiplication and division (Siemon et al., 2010; Vergnaud, 1983; 1992). As a result, learners will develop a better conceptual understanding as they will form more and a stronger network of concepts, schemas and schemes.

This chapter starts with a discussion about conceptual fields as a field of study and the internal and external representations that form part of conceptual fields. I then discuss in-depth multiplicative reasoning as a conceptual field and how it develops. Furthermore, I explain what the classes of multiplication problems entails, how calculation techniques develop and what the different levels under each calculation technique are. Lastly, I discuss the conceptual framework and how the above-mentioned aspects relate to one another.

2.2 Conceptual fields as a field of study

In the early 1980s, Vergnaud did a study of cognitive development and the learning of mathematical knowledge and coined the term conceptual fields (Grenier, 2007;

Vergnaud, 1989; 2011; 2013b). Therefore, the study of a specific conceptual field, such as multiplicative reasoning, can be used to explore how proficient learners are in that specific field. At first, Vergnaud (1982, p. 36; 1983; 1986) defined a conceptual field as “a set of situations the mastering of which requires a variety of concepts, procedures and symbolic representations tightly connected with one another”. Moreover, he believed that a conceptual field is a set of situations, rather than a set of concepts (Vergnaud, 1982). Later in his career, Vergnaud (1992) came to believe that one can view conceptual fields from two complementary perspectives, namely as a set of situations and as a set of concepts, or a set of situations for which the “progressive mastery calls for a variety of interconnected concepts, schemes and symbolic representations” and “a set of concepts, whose meaning and explanatory power stem from their joint intervention in the same situations and schemes” (Vergnaud, 1992, p. 289; 2013b). Hence, a conceptual field is by definition a set of concepts (schemas and conceptualisations) through which adaptation operates, and at the same time a structured set of situations (Vergnaud, 2007; 2013a). Vergnaud’s (2013b) latter view of conceptual fields implies that a set of concepts contribute to the mastery of a set of situations, just as the mastery of a set of situations requires a variety of concepts, schemes and representations that are closely connected. The various concepts that make up a conceptual field form a hierachical structure of which the organisation is progressive, becoming increasingly more complex throughout a persons’s life. In order to study a conceptual field over a short period, a study has to be defined in narrower terms, for instance as a specific activity, which involves the risk of losing the organisational process of the development of proficiency (Vergnaud, 2013b). This study focused on the developmental aspect of one concept, namely the multiplication of whole numbers, within a specific time during the cognitive development of Grade 6 learners.

As a set of situations and a set of concepts, conceptual fields have two aims. The first aim is to describe and analyse the progressive complexity of concepts in a specific conceptual field. This process of describing and analysing the progressive complexity of concepts differs from Piaget’s cognitive developmental levels which he logically ordered in the following stages according to learners’ ages: Sensorimotor (birth to two years), preoperational (two to seven years), concrete operational (seven to eleven years) and formal operational (eleven to fifteen years) (Miller, 2011). According to

Vergnaud (2009; 2013a) Piaget reduced the progressive complexity of cognitive development to logical processes involving the age of learners. In contrast, the theory of conceptual fields focuses on conceptual processes, as conceptualisation is always taking place in cognitive development and integrates the organisation of any activity, regardless of the learners' ages (Vergnaud, 2013a; 2013b).

The second aim is to establish better connections between the operational and predicative forms of knowledge (Vergnaud, 2009). The operational form of knowledge focuses on actions, i.e. on what you do in a situation to show what you know. The predicative form of knowledge is your explanation of what you know, i.e. what you say or your linguistic and symbolic expressions of knowledge (Vergnaud, 2009; 2013b). Your actions (operational knowledge) and linguistic and symbolic explanations (predicative knowledge) are therefore external representations to show what you know internally.

2.2.1 Internal representations

The term representation can be defined as a configuration of signs, characters, or objects that as a whole, or part by part, correspond to, stand for, symbolise, mean, refer to, embody, resemble, represent, or stand in place of something else (Godino & Font, 2010; Goldin & Kaput, 1996; Pape & Tchoshanov, 2001). Representations can be either internal (which means that they are in your mind and are non-observable) or external (they can be observed or heard). Internal representations may or may not share structural similarities with external representations (Godino & Font, 2010; Goldin & Kaput, 1996). To explore what learners know (internal representations) about a particular concept such as multiplication, teachers can at best make inferences based on learners' verbal and written (external) representations.

Panasuk (2010) loosely defines internal representations as mental images, while other authors define them as internal abstractions of complex cognitive schemas to establish an internal network through experience (Godino & Font, 2010; Lesser & Tchoshanov, 2005; Panasuk, 2010; Pape & Tchoshanov, 2001). From a mathematical perspecitive, Ayub, Ghazali, and Othman (2013) define internal representations as a cognitive configuration that is inferred from learners' verbal or written responses, which are the result of mathematical thinking and problem solving. Internal representations can only be inferred through external representations as they are mentally configured and therefore cannot be directly observed (Ayub et al., 2013; Debrenti, 2013; Godino & Font, 2010; Goldin & Kaput, 1996). Learners' pre-existing mental images, language, ability to solve problems and attitude toward mathematics influence the way they form their internal representations (Lesser & Tchoshanov, 2005; Panasuk, 2010; Pape & Tchoshanov, 2001).

According to the theory of conceptual fields, internal representations consist of pre-existing concepts, schemas and schemes. Learners use internal representations to organise mathematical concepts into schemas and to solve problems, and pre-existing schemes to form new schemes (Pape & Tchoshanov, 2001). Learners who use the various types of internal representations, gradually build more complex mental images of concepts (Pape & Tchoshanov, 2001). Cognitive development occurs through the use of different internal representations that create strong interrelated networks or schemas (Goldin & Kaput, 1996). Learners therefore solve problems by using existing schemes, transforming old schemes, or forming new ones.

This study focused on the conceptual field of multiplicative reasoning. Examples of concepts in the multiplicative conceptual field include the multiplication and division of whole numbers, fractions and decimal numbers, area and volume. Vergnaud (1982; 2011; 2013b) provides an operational definition of a concept. He defines a concept as three distinct interdependent sets. First, a concept is a set of situations that promote an understanding of that concept. Second, it is a set of operational invariants (schemes) that relies on the organisation of activities (see this chapter, section 2.2.2). Last, it is a set of symbolic forms and language, which allows us to represent the concepts and their relationship to the action, and as a result also the situation and the schemes (operational thoughts) they evoke. A concept is, therefore, a product of various experiences (Vergnaud, 2013b). Concepts are internal representations, constructed by internalising actions and perceptions.

Related concepts clustered together can form a schema. Feldman (2007) explains that according to Piaget, schemas contain stabilised information about the features or characteristics of objects, while Siraj-Blatchford (2002) maintains that schemas are figurative thinking. Schemas are essentially building blocks of knowledge that form high-level internal (mental) representations. Schemas are therefore the result of prior experiences which can facilitate the interpretation of new information in relation to prior knowledge (Colman, 2015; McLeod, 2015; Skemp, 1987). Since concepts and schemas are internal representations, the only way that teachers can evaluate learners' knowledge of a conceptual field is through schemes, which represent operational thought (Siraj-Blatchford, 2002). This points to an interplay between internal and external cognitive processes.

2.2.2 Interplay between internal and external cognitive processing

Representations do not exist in isolation, since they belong to highly structured schemas and schemes within a particular context (Goldin & Kaput, 1996). Both internal and external representations are fundamental to mathematics as mathematical thinking is abstract and hence requires external representations to communicate and learn mathematical concepts, and to internalise and create mental abstractions of those concepts (Lesser & Tchoshanov, 2005; Panasuk, 2010; Panasuk & Beyranevand, 2011). External representations can influence internal representations, and vice versa. Since they are connected and interrelated, inferences can be made about internal representations by studying external representations (Ayub et al., 2013; Debrenti, 2013; Godino & Font, 2010; Pape & Tchoshanov, 2001). Since schemes and schemas change whenever one is asked to solve a problem, a mental transformation takes place when such an interplay occurs between internal and external representations. This interplay is important as it develops conceptual understanding in mathematics and is seen as a broad definition of mathematics learning (Ayub et al., 2013; Lingefjard & Ghosh, 2016; Panasuk & Beyranevand, 2011; Pape & Tchoshanov, 2001). Although Lingefjard and Ghosh (2016) are of the opinion that it is still unclear how mental transformation takes place, Vergnaud maintains that this process takes place by means of schemes.

When learners have to solve a mathematical problem, they make use of both internal and external mental processes. The internal mental process, as discussed earlier, consists of consulting existing internal representations (mental images) of concepts and schemas. The decision about how to solve a given mathematical problem requires that learners consult pre-existing schemes, change a pre-existing scheme, or form a new scheme (internal representations). The external process takes place when learners represent their thinking externally by verbally expressing their thoughts and/or using symbols or drawings, or manipulating 3D material. Schemes are operational thought or thoughts in action and connect the internal and external cognitive processes. Therefore, when teachers want to evaluate learners' understanding of a mathematical concept, they should evaluate their schemes, i.e. their external representations.

2.2.2.1 Schemes

Some authors use schemes and schemas interchangeably and Siraj-Blatchford (2002) even suggests that a distinction between schemes and schemas is unnecessary as schemas determines schemes. However, I do not agree with this view since in my opinion schemes, which consist of operational thought, make schemas visible, which are the building blocks of knowledge. Schemes and schemas are therefore different and have different functions.

Some authors, including Piaget (1970) and Vergnaud (1994; 2013a), distinguish between schemas and schemes. Piaget (1970) suggests that schemes are operational thought that is repeatable and generalisable in an activity. Furthermore, we use a scheme as a mental tool to gather and interpret information on known and new situations (Siraj-Blatchford, 2002; Steffe, 1994; Vergnaud, 1994). Vergnaud (1994; 2013a) generally agrees with Piaget and defines schemes as a fixed organisation of action to solve a specific set of situations in context by means of sensory-motor skills and intellectual skills. Schemes therefore generate various actions which depend on the context and as the context changes (Vergnaud, 2013b). This implies that each time learners are asked to solve problems in new situations they need to either utilise or adapt old schemes, or form a new scheme for the new situation.

However Vergnaud, who was a PhD student supervised by Piaget, not only defined schemes in general terms, but elaborated on Piaget's definition of schemes to use it in his research (Vergnaud, 2011; 2013a). He further defines a scheme as "a mapping from a multi-dimensional space of information variables onto a multi-dimensional space of action variables” (Vergnaud, 2013a, p. 47). For example, by mapping classes of multiplication problems onto the possible sequence of steps that can lead to their solutions. In his most elaborate definition of a scheme, Vergnaud (1998; 2013a; 2013b) states that it contains operational invariants (concepts-in-action and theorems-in-action). Operational invariants play an important role in problem solving as learners use them as mental tools to select relevant information, to infer the consequences of the selected action taken and to select subsequent information (Vergnaud, 2007; 2013b). Operational invariants consist of two interrelated components, namely concepts-in-action and theorems-in-action, each of which has an internal and an external component.

Internal representations, as previously discussed, consist of concepts, schemas and pre-formed schemes that learners formed when learning about a specific concept such as multiplication, and when previously asked to solve for example multiplication problems. However, we cannot research concepts and theorems as they are not observable. For example, we cannot observe whether learners understand the concept multiplication as it exists only inside their heads, or whether they can solve a multiplication problem. It is for this reason that Vergnaud (1998; 2013a; 2013b) coined the terms concepts-in-action and theorems-in-action, which we can observe, for example, when learners demonstrate their understanding of multiplication by using external representations, such as drawing pictures or explaining their methods when solving multiplication problems. Concepts-in-action are the building blocks for theorems-in-action as they help with the identification of the relevant information from the context and the selection or forming of theorems-in-action, from which procedures can be organised to calculate the answer (Vergnaud, 2013b). One concept-in-action always has more than one theorem-in-action, which learners gain and learn through experience in solving many different problems in various contexts, and throughout their lives as cognitive development takes place (Vergnaud, 2007). In turn, the theorems-in-action leads to external representations for learners to demonstrate their knowledge of a concept.

Since schemes generate both mental and external actions, they can be regarded as the foundation of external representations. Schemes which include concepts-in-action and theorems-in-action, play an important role in the analysis of operational knowledge, which demonstrates what learners know when they solve problems (Vergnaud, 2009). Any action that learners take to solve a problem is possible only because of schemes that enable them to analyse information and then decide what to do with it (Vergnaud, 1994). Whenever learners are required to solve a mathematical problem in a new context, they adapt existing schemes or create new ones. This creates a scheme-context pair, which is central to learning and cognitive development (Vergnaud, 2013a). Learners use schemes to make schemas explicit. They do this by selecting relevant information through concepts-in-action, which in turn select relevant theorems-in-action and are then made explicit through verbal explanation, writing of symbols, drawing, or manipulating 3D material (Vergnaud, 1994).

2.2.3 External representations

Actions that are physically observable actions, such as words, pictures, graphs, numerals, equations, tables, diagrams and charts are external representations of internal mental concepts (Barmby, Harries, Higgins, & Suggate, 2007; Goldin & Kaput, 1996; Panasuk, 2010; Pape & Tchoshanov, 2001). The various external representations can be categorised in different ways, as shown in Table 2.1.

Table 2.1: Summary of the divisions of external representations

Allsopp et al. (2007); Van de Walle et al. (2015)	Bruner (1963)	Skemp (1987)	Panasuk (2010); Panasuk & Beyranevand (2011)	Ayub et al. (2013); Lesh et al. (1987)	Barmby et al. (2007); Hiebert & Carpenter (1992); Pape & Tchoshanov (2001)
Concrete	Enactive (knowing by doing)			Manipulatives	Physical objects
Semi-concrete / representational	Iconic (images / pictures)	Visual symbols (diagrams)	Visual (diagrams, pictures, graphs)	Pictures or diagrams	Pictures
Abstract	Symbolic (words and mathematical symbols)	Verbal symbols (spoken, written language and mathematical symbols)	Verbal (written and spoken language)	Spoken symbols	Verbal / spoken symbols
			Symbolic (numbers and letters)	Written symbols	Written symbol (words or numbers)
				Real-world situation or experience-based 'scripts'	

Bruner (1963) was the first to categorise external representations. His categories consisted of enactive (e.g. physical actions and objects), iconic (e.g. images or pictures) and symbolic (e.g. words or mathematical symbols) representations. Some researchers (Allsopp et al., 2007; Van de Walle, Karp, & Bay-Williams, 2015) agree with this division and attach the same meaning to each category, however they label enactive representations as concrete, iconic representations as semi-concrete or representational, and symbolic representations as abstract (see Table 2.1).

In contrast, Skemp (1987) postulates that, from a cognitive point of view, external representations should be divided into two rather than three main categories, namely verbal and visual symbols (Skemp, 1987). Verbal symbols include both written and spoken words and mathematical symbols, whereas visual symbols include diagrams. Skemp (1987) does not include concrete or semi-concrete representations under any of his categories. Panasuk (2010), like Skemp (1987), also includes visual symbols and exclude concrete representation. Panasuk (2010) divided verbal symbols into two categories, namely verbal and symbolic representations, which include abstract representations. In my opinion this is unnecessary as written and spoken language can both be grouped under verbal or abstract representations as suggested by both Skemp (1987) and Allsopp et al. (2007).

Some researchers (Barmby et al., 2007; Hiebert & Carpenter, 1992; Pape & Tchoshanov, 2001) classify representations into four categories, namely physical objects, pictures, verbal/spoken and written symbols. These categories are the same as those proposed by Lesh, Post, and Behr (1987), who added a fifth category, namely real-world situations or experience-based ‘scripts’. Knowledge is organised around real-world situations that serve as the context for solving other similar problem situations. Hiebert and Carpenter’s (1992) first two categories of representations are the same as those of Bruner (1963) and Allsopp et al. (2007), namely objects/manipulatives and pictures. However, like Panasuk (2010), they added the extra category of real-world situations and split Bruner (1963) and Allsopp et al.’s (2007) abstract/symbolic category into verbal/spoken symbols and written symbols.

For my study, I adopted Allsopp et al.’s (2007) and Van de Walle et al.’s (2015) division of external representations, namely concrete, semi-concrete and abstract representations as the other divisions can all be categorised under these main categories.

2.2.3.1 Concrete, semi-concrete and abstract representations

Concrete, semi-concrete, abstract (CSA) sequencing as a teaching technique is well documented in literature (Allsopp et al., 2007; Bruner, 1963; Debrenti, 2013; Hoong, Kin, & Pien, 2015; Hui et al., 2017; Lesser & Tchoshanov, 2005; Pape & Tchoshanov, 2001; Post, 1981), but very little information are available on the use of concrete, semi-concrete and abstract representations as an assessment tool (Allsopp et al., 2007; Ayub et al., 2013; Barmby et al., 2007; Panasuk, 2010). Mercer et al. (2014) and Van de Walle et al. (2015) view CSA sequencing as effective in teaching learners with learning difficulties, which should inadvertently imply that teachers need to use various representations to evaluate and develop the conceptual understanding of learners with learning difficulties. Teachers should not rely on only a single response to a task (such as giving the answer) when assessing the extent of learners’ conceptual understanding of a mathematical concept, but should rather infer understanding from learners’ use of multiple representations (Ayub et al., 2013; Barmby et al., 2007;

Lesser & Tchoshanov, 2005; Panasuk, 2010; Panasuk & Beyranevand, 2011; Pape & Tchoshanov, 2001). According to Panasuk (2010), each representation (CSA) provides a specific meaning to a mathematical concept (Panasuk & Beyranevand, 2011). For example, a picture or the use of 3D material could shed light on the conceptual understanding of a concept, for example multiplication, whereas the use of abstract symbols shed light on the procedural fluency of such a concept. Furthermore, when learners can use the different external representations for a concept in a flexible and fluent manner, it indicates deep conceptual understanding (Ayub et al., 2013; Panasuk & Beyranevand, 2011). If learners use only one type of representation, it indicates that they have made only limited connections and probably memorised the technique, rather than understood the concept (Ayub et al., 2013). Since understanding is a complex network of schemas, a combination of these external representations is necessary to evaluate learners' level of conceptual understanding.

Vergnaud (2009; 2013b) states that expressing your understanding is an essential part to conceptualise a concept and that many people have difficulty explaining their understanding. Furthermore, learners with learning difficulties struggle with abstract thinking (Allsopp et al., 2007), which include expressing themselves with language and symbols as shown in Table 2.1. Thus, one of the ways teachers could evaluate how well learners with learning difficulties understand a certain concept in mathematics, is to ask them to represent their thinking in various ways, for example by making use of concrete objects, semi-concrete representations (drawings or pictures) and abstract symbols (Allsopp, Kyger, Lovin, Gerretson, Carson, & Ray, 2008). It is possible for learners to have a limited understanding of a concept that is restricted to one representation. It may also appear as if learners have knowledge of a particular concept when they use abstract representations, when in fact they may be demonstrating only procedural fluency. The opposite may also be true, in other words, they may know how to solve a problem with semi-concrete representations, but not by using abstract representations (Allsopp, Kyger, & Ingram, n.d.). Therefore, requiring learners to solve a problem with concrete, semi-concrete and abstract representations can shed light on their schemas (Allsopp et al., 2007). Learners' understanding of a mathematical concept is determined by the amount and strength of the connections between the different types of representation (Hiebert & Carpenter, 1992). The more connections there are, the better the understanding. Teachers often require learners to only solve a problem on the abstract level, which may not provide a true reflection of their understanding. Concrete representations are especially useful in the lower grades, and much is written on the importance of using concrete material to develop and represent mathematical concepts (Goldsby, 2009; Lerner & Johns, 2012; Marshall & Swan, 2008; Mercer et al., 2014; Van de Walle et al., 2015).

i. Concrete representations

Concrete representations are defined as objects that appeal to different senses that learners can touch, move, arrange, or rearrange (Goldsby, 2009). Concrete objects, which are three-dimensional, viewed by Allsopp et al. (2007) as the most basic level and also the most crucial level for developing conceptual understanding. Concrete representations support and enhances learners' understanding of the essence of a concept (Debrenti, 2013). Concrete objects are useful when solving mathematical problems as learners can manipulate physical objects by moving and touching them, which involves more than one sensory organ. However, for the use of concrete objects to be beneficial, the manipulatives should represent the conceptual understanding behind the mathematical problem (Pennsylvania Department of Education, 2017).

Allsopp et al. (2007) classify concrete representations into two classes, namely discrete and continuous representations. Discrete concrete representations can be counted individually, for example base-ten or pattern blocks, unifix cubes, beans, marbles, plastic pieces, poker chips, place-glue sticks, people, cookies, and so forth (Allsopp et al., 2007; Lerner & Kline, 2006; Pennsylvania Department of Education, 2017). Continuous concrete representations are not used for counting but are used when solving problems of measurement and can include items such as rulers, weight scales and measuring cups (Allsopp et al., 2007).

ii. Semi-concrete representations

Semi-concrete representations are two-dimensional and are seen as an intermediate step between concrete and abstract representations (Allsopp et al., 2007). Semi-concrete means that instead of manipulating real objects to solve a problem, learners draw pictures, dots, tallies, lines, or circles that represent the actual objects (Allsopp et al., 2007; Lerner & Johns, 2012; Pennsylvania Department of Education, 2017). As concrete representations, semi-concrete representations support and enhance learners' understanding of mathematical concepts (Debrenti, 2013).

iii. Abstract representations

Symbolic or abstract representations are the most compact and abstract representations of a concept (Debrenti, 2013). Mathematical content includes a highly complex process of abstraction (Panasuk, 2010). The term reflective abstraction is used to explain the process of developing conceptual understanding (Panasuk, 2010). Through abstract representations, learners utilise numbers, notation and mathematical symbols to solve mathematical problems without using real objects or drawing pictures (Allsopp et al., 2007; Lerner & Johns, 2012; Pennsylvania Department of Education, 2017). However, learners with learning difficulties experience problems with abstract thinking and might benefit from using concrete objects and/or semi-concrete representations to understand and solve mathematical problems (Allsopp et al., 2007).

My study focused on the conceptual field of multiplicative reasoning (discussed in the next section), and specifically on the multiplication of whole numbers. Each conceptual field consists of many concepts, schemas and schemes, a set of operational invariants and a set of symbolic and language representations. Multiplication is classified into various classes of problems (discussed in section 2.3.2, of this chapter), calculation techniques to solve those problems (discussed in section 2.4 of this chapter), and different representations, as discussed above. The schemas formed by these different concepts can be explored by means of schemes, which consist of operational invariants (concepts-in-action and theorems-in-action). Learners were required to represents the various classes of multiplication problems by using different representations and a calculation technique to solve a problem.

2.3 Multiplicative reasoning as a conceptual field

Brown et al. (2010) acknowledge that the delineation of multiplicative reasoning is complex, since mathematical content is developmental (concepts become more abstract and complex) (Agosino et al., 2010; Geary, 2004). Multiplicative reasoning as a conceptual field consists of numerous interconnected concepts represented in various forms. Siemon et al. (2010, p. 2) identify three characteristics of multiplicative reasoning. First, they suggest that it is the “capacity to work flexibly and efficiently with an extended range of numbers”, which includes whole numbers, decimals, fractions, ratios and percentages. Second, it is the “ability to recognise and solve a range of problems involving multiplication or division, including direct and indirect proportion”. Third, it is “the means to communicate effectively in a variety of ways”. The various ways of communication that they suggest are through external representations and include the use of words, writing, diagrams and symbolic expressions. Siemon et al. (2010, p. 2) summarise this as the “capacity to work flexibly with the concept, techniques and representations of multiplication (and division) as they occur in a wide range of contexts”.

2.3.1 Developmental nature of conceptual fields

Mathematical content is developmental, since additive reasoning requires lower-order reasoning, whereas multiplicative reasoning requires higher-order reasoning and proportional reasoning requires even more complex reasoning that combines numerous ideas and techniques (Ernst, 2004; Hart, 1981). Multiplicative reasoning is also a developmental theory as it deals with the progression in the complexity of mathematical knowledge and calculation techniques over an extended period (Vergnaud, 1982; 2009; 2013a). For example, according to the CAPS followed in South Africa, learners solve multiplication problems with whole numbers from Grade 2 onwards (DBE 2011a). The multiplication of fractions and decimal numbers is introduced in Grade 5 and 6 respectively (DBE, 2011b). Consequently, the understanding of the multiplication of decimals is more complex than the understanding of the previously learnt concepts (multiplication of whole numbers and fractions).

The developmental nature of mathematical content is evident in the difference in complexity of additive, multiplicative and proportional reasoning. Since learners need to make a cognitive shift from additive to multiplicative and then to proportional reasoning, it is important to understand the difference between these conceptual fields. Table 2.2 contains a summary of the differences between the various types of reasoning, which are subsequenlty explained.

Table 2.2: The difference between additive, multiplicative and proportional reasoning

Additive reasoning	Multiplicative reasoning	Proportional reasoning
Numerical / single-unit reasoning	Quantitative / equal-group reasoning (composite structure)	Comparative / Part-whole and part-part reasoning
Preserving composition	Transforming composition	Transforming composition
Absolute reasoning	Relative reasoning	Relative reasoning
Part-part-whole reasoning / additive partitioning	Factor-factor-product reasoning	Factor-factor-product reasoning
Lower-order reasoning	Higher-order of abstract reasoning	A complex form of reasoning

To indicate the differences between these three types of conceptual fields, additive, multiplicative and proportional reasoning are explained in the next section.

i. Additive reasoning

Additive reasoning includes the operations addition and subtraction, and learners are required to think of numbers as a unit (e.g. 1, 2, 3 each is a unit on its own), which involves lower-order reasoning (Tobias & Andreason, 2013; Tzur et al., 2013). Additive reasoning requires part-part-whole reasoning, which implies adding or subtracting numbers of the same unit, in other words, the composition is preserved (see Table 2.2) (Hurst & Hurrell, 2014; Siemon et al., 2010; Tzur et al., 2010; 2013). For example, if there are five apples and you add three apples you have eight apples in total. Hence, the answer is given in the unit that you have added (apples, in this case). Furthermore, when reasoning additively you reason in an absolute manner, which implies that you make comparisons in additive terms. For example, if Sue has five apples and John has seven apples, John will have two more apples than Sue (Reynolds, 2013b).

ii. Multiplicative reasoning

Multiplicative reasoning consists of multiplication and division and calculations are done in equal groups (composite structures). This is also known as quantitative reasoning. For example, to calculate 4×2, learners count in 2s four times: 2, 4, 6, 8

(Ell, 2001; Siemon et al., 2010). They need to count in groups and keep track of how many times they need to count in a particular group, which demands a higher order of abstract reasoning than that required for additive reasoning (see Table 2.2) (Clark & Kamii, 1996). Multiplicative reasoning requires factor-factor-product reasoning, rather than the part-part-whole reasoning used as in additive reasoning. For example, 5 dogs × 4 legs per dog = 20 legs altogether (Siemon et al., 2010; Tzur et al., 2010). Therefore, when multiplying with different units (dog × legs per dog = legs), the composition is transformed and requires relative reasoning (Tzur et al., 2010). The legs per dog is relative to how many dogs there are, therefore, multiplicative reasoning requires learners to recognise the relationships between quantities (Luneta, 2013; Tobias & Andreason, 2013; Zhang et al., 2011).

What makes multiplicative reasoning more abstract than additive reasoning is the ability to unitise, in other words, to construct a reference or composite unit, and then use it to reinterpret a problem in terms of that unit (e.g. thinking in terms of 2s or 3s, instead of one object at a time) (Lamon, 1994). Moreover, learners are required to combine two magnitudes with different units of measurements to produce a quantity whose unit of measurement is different from those that are combined (e.g. h × km/h = km), or to produce an intensive quantity that is a new unit of measurement (e.g. km ÷ h = km/h) (Lamon, 1994). Learners therefore have to make a shift to more abstract thinking in order to reconceptualise their thinking and form a new schema. This conceptual shift needs to take place during the Intermediate Phase (Grades 4 to 6) to enable them to understand more complex and even more abstract mathematical content such as proportional and algebraic reasoning during the Senior Phase (Grades 7 to 9).

iii. Proportional reasoning

As shown in Table 2.2 and discussed under the heading multiplicative reasoning, proportional reasoning also requires relative reasoning and the transformation of the composition of a problem. This contrasts with absolute reasoning, in additive reasoning, where the composition is preserved. Proportional reasoning is an extension of, and is more abstract than multiplicative reasoning in that it demands a complex form of reasoning including many interconnected ideas and techniques (Hurst & Hurrell, 2014; Luneta, 2013). In the case of proportional reasoning, learners no longer need to recognise relationships between quantities as in multiplicative reasoning, but have to recognise relationships between relationships (Luneta, 2013). For example, the ratio of apples to cost is 1:3. If I buy ten apples, how much will they cost? Moreover, learners need to compare part to whole (e.g. fractions, percentages) or part to part (e.g. ratios) in proportional reasoning (Tobias & Andreason, 2013). In other words, when learners reason proportionally, they can calculate the multiplicative relationship between a base ratio (e.g. double, half, four times greater/smaller) and the proportional context to which it applies (Reynolds, 2013a; 2013c). For example, if the ratio of boys to girls is 2:4, the number of girls is double that of the boys, or the number of boys is half that of the girls. The base ratio in this instance is then either double or half.

To summarise, it can be said that additive, multiplicative and proportional reasoning demonstrate the developmemental nature of mathematical content. Each of the three types of reasoning requires more complex and more abstract reasoning than the previous one, and an understanding of proportional reasoning depends to an extent on multiplicative reasoning, which in turn depends to an extent on an understanding of additive reasoning. Additive, multiplicative and proportional reasoning are all examples of conceptual fields. Problems that require multiplicative reasoning, on which my study focused, include different situations or contexts that clarify multiplication. For the purpose of this study, these different situations or contexts are called classes of multiplication problems, which will now be discussed.

2.3.2 Classes of multiplication problems

Multiplication problems can be categorised into various classes based on the different levels of abstraction. These classes of problems are developmental as the level of abstraction differs from one problem to the next (Greer, 1992; Vergnaud, 1983). Different authors categorise the classes of multiplication problems differently, however all the authors that are discussed use Vergnaud’s categorisation as their basis for identifying the different classes of multiplication problems, as summarised in Table 2.3.

Table 2.3: Classes of multiplication problems for whole numbers according to different authors

Vergnaud (1983)		Siemon (2005), Siemon et al. (2010)	Greer (1992)	Mulligan (1992)
Isomorphism of measures / simple proportion	- Equal sharing - constant price - uniform speed - constant density on a line	- groups of - partitioning / sharing	- equal groups	- equivalent groups
			- multiplicative comparison	- comparison
Product of measures	- area - volume	- arrays / regions - area	- rectangular area or arrays	- arrays
	- Cartesian product	- Cartesian product / for each	- Cartesian product	- Cartesian product
Multiple proportion	- consumption - production - expense			

Vergnaud (1982; 1983) points out the importance of the various classes of problems relating to multiplication and division and categorises multiplication problems into three broad classes: Isomorphism of measures/simple proportion; product of measures; and multiple proportion. The first class of multiplication problems is isomorphism of measures, which includes simple, direct proportion. The word isomorphism refers to a one-to-one correspondence where the relationship between the two numbers remains the same (Hosch, 2016). This one-to-one correspondence can be represented by a linear function of one variable with a constant. For example, 5 groups × 4 boys = 20 boys, with 5 being the variable and 4 boys the constant coefficient (Vergnaud, 2009). The second class of multiplication problems, product of measures, includes the Cartesian product. It is the product of two variables that changes into a third variable with a different unit (e.g. 1 m × 1 m = 1 m^2). The last class of multiplication problems is multiple proportion, where the product is proportional to the two variables (Vergnaud, 1983). However, in the case of multiple proportion the variables cannot be reduced to a product of the others, for example: persons × weeks = expense.

Vergnaud (1983) further refines the above-mentioned classes of problems by using specific situations where they are likely to be frequently found. During my literature search, I found that Greer (1992), Mulligan (1992), Mulligan and Mitchelmore (1997), and Siemon et al. (2010) based their categorisation of the classes of multiplication problem on Vergnaud's (1983) work, with slight differences (see Table 2.3). According to Siemon et al. (2010), the classes of multiplication problems are developmental in nature, which in this context, means that the various classes of multiplication problems have different degrees of difficulty or complexity and some require more abstract thinking than others. Greer (1992), Mulligan (1992), Mulligan and Mitchelmore (1997), and Siemon et al.'s (2010) categorisation of the classes of multiplication problems is less comprehensive than that of Vergnaud (1983). Greer (1992), Mulligan (1992) and Mulligan and Mitchelmore (1997) added multiplicative comparison and arrays, but did not include multiplicative proportion. The various classes of multiplication problems will now be discussed in more detail by using Vergnaud's (1983) main classes of multiplication problems.

2.3.2.1 Isomorphism of measures

Many multiplication problems encountered in everyday life can be categorised as *isomorphism of measures*. According to Vergnaud (1983), *isomorphism of measures* has a structure that can be seen as direct proportion between two measure-spaces, M_1 (measure-space 1) and M_2 (measure-space 2), and can be represented by a simple correspondence table (see Table 2.4). For example: If one sweet costs R7, how much will nine sweets cost? In this case the correspondence table will look as follows.

Table 2.4: Example of isomorphism of measures

M_1 (Sweets)	M_2 (Rand)
1	R7
9	?

In general, these multiplication problems have two numbers (e.g. 9 sweets × R7 per sweet) that must be multiplied. However, these two numbers have different roles. The one number is the multiplier (9 sweets) and the other the multiplicand (R7), with the multiplier operating on the multiplicand (Greer, 1992). Vergnaud (1983) refines this idea by explaining that learners can conceptualise these problems in two different ways: Seeing it as either a binary law of composition or as a unary operation. If, on the one hand, they conceptualise it as a binary law of composition, learners view the two numbers that need to be multiplied as plain numbers with no magnitude connected to them (e.g. 9 × 7 or 7 × 9). On the other hand, they could conceptualise it as a unary operation, which implies that they will utilise one number as a scalar operator with no magnitude (see Table 2.5).

Table 2.5: Using a scalar operator

M_1 (Sweets)	M_2 (Rand)
1	R7
× 9	× 9
9	?

Alternately, learners may use it as a function operator, which implies a coefficient of a linear function from M_1 to M_2. Its unit is the quotient of two other units (e.g. rand per sweet) (see Table 2.6).

Table 2.6: Using a function operator

M_1 (Sweets)	M_2 (Rand)
1 →	R7
× 7	
9 →	?
× 7	

Vergnaud (1983) proposes four classes of multiplication problems under *isomorphism of measures*, namely *equal sharing/groups, constant price, uniform speed* and *constant density on a line* (see Table 2.3). Like Vergnaud (1983), Greer (1992), Mulligan (1992), Mulligan and Mitchelmore (1997), and Siemon et al., (2010) also include *equal sharing/groups,* but exclude *constant price, uniform speed and constant density on a line*. Mulligan (1992), Mulligan and Mitchelmore (1997), and Greer (1992) add *multiplicative comparison* to Vergnaud's (1983) classification (see Table 2.3). Each of the five classes of multiplication problems will now be discussed.

i. Equal groups

Equal groups are multiplication problems that learners encounter in their first years of school (Greer, 1992). Multiplication problems of this class normally include several groups, each consisting of the same number of persons and objects (Greer, 1992; Mulligan, 1992; Mulligan & Mitchelmore, 1997; Vergnaud, 1983). For example: Four boys each have ten marbles. How many marbles do they have altogether? Or: There are five tables with two learners seated at each table. How many learners are there altogether? Greer (1992) postulates that the *equal groups* idea can be seen in different situations, for example in cases of natural replication (e.g. 4 girls each have 5 fingers), repetition of a sequence of actions (e.g. taking 5 steps 8 times) and human practices (e.g. I give 3 learners 4 sweets each). Siemon (2005) and Siemon et al. (2010) also include *equal groups* (counting in, for example, 3s: One 3 = 3, two 3s = 6, three 3s = 9, etc.), which they propose lead to *partitioning/sharing* (e.g. 12 is two 6s, three 4s or four 3s).

ii. Constant price

The second class of multiplication problems is *constant price,* which includes goods and cost and represent simple proportions with no comparisons (e.g. Mary buys 8 chocolates at R9 each. How much does she have to pay?) (Vergnaud, 1983). The mathematical idea that applies here is the same as the *equal groups* idea in that it represents groups that consist of the same number of people/objects. However, even though problems in this class are mathematically equivalent to the *equal groups* problems, they have different semantic structures, which can evoke a different scheme. In turn, these different schemes could lead to different calculation techniques and learners could perceive them as more difficult than the *equal groups* problems (Mulligan, 1992; Mulligan & Mitchelmore, 1997).

iii. Uniform speed and constant density on a line

The third and fourth classes of multiplication problems are *uniform speed* and *constant density on a line*. *Uniform speed* includes duration and distance (e.g. Eric drives 80 km/hour on the highway. How far will he drive in 5 hours?), and *constant density on a line* includes, for example, trees and distances (e.g. There are 5 trees per km with equal distances between them. How many trees will there be over a distance of 8 km?) (Vergnaud, 1983). These classes of multiplication problems are mathematically the same as the *equal groups* and *constant price* classes, but again the semantic structures are different. While learners in South African schools solve money mathematical problems in the Intermediate Phase (Grades 4 to 6) (DBE, 2011b), however they are only confronted with *uniform speed* in the Senior Phase (Grades 7 to 9) and *constant density on a line* problems are extremely rare, if done at all (DBE, 2011c). For South African learners, *uniform speed* and *constant density on a line* problems are therefore expected to be more difficult than *equal groups* and *constant price* problems.

iv. Multiplicative comparison

Greer (1992), Mulligan (1992), and Mulligan and Mitchelmore (1997) identify *multiplicative comparison* as another isomorphism of measures. Kouba (1989) refers to problems of this kind as scalar problems. Although Vergnaud (1983) does not explicitly identify this as a separate class of problems, he does explain it as one way in which learners can solve *isomorphism of measures* problems (see Table 2.5). An example of *multiplicative comparison* is: Susan has 4 pens. If Pete has two times as many pens as Susan, how many pens does Pete have? Or: There are 8 boys in a class and three times as many girls in a class. How many girls are there? Problems of this type can be identified by the phrase (keywords) *times as many,* which, if understood, can be a link to understanding ratio (Greer, 1992; Hurst, 2015). Greer (1992) agrees with Vergnaud (1983) that problems of this type require a different kind of thinking than the previous problems. They can be viewed in two ways, i.e. the scalar operator can either be seen as the multiplier (e.g. 2 × 4 pens, where 2 is the scalar operator/multiplier), or can be seen in terms of many-to-one correspondence (e.g. Pete had 2 pens and Susan has 1 pen), where the 2 pens are the multiplier, thus making use of a ratio or proportional reasoning. Consequently, Greer (1992), Mulligan (1992) and Mulligan and Mitchelmore (1997) make use explicitly of a semantic structure that has a scalar operator (*times as many*), whereas Vergnaud (1983) expects learners to imply it without using the explicit semantic structure.

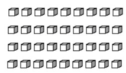
To summarise, the following five classes of multiplication problems are classified under the isomorphism of measures: *Equal groups, constant price, uniform speed, constant density* and *multiplicative comparison*. Although all five classes of multiplication problems are seen as mathematically equivalent, they have different semantic structures, which can make the multiplication problems less or more complex. However, *multiplicative comparison* problems differ slightly from the other classes and can be viewed either as having scalar operators or as ratio problems (Greer, 1992). In the South African school context, *uniform speed* and *constant density on a line* multiplication problems would be more difficult as learners in Grade 6 have not yet been exposed to them.

2.3.2.2 Product of measures

Product of measures multiplication problems include two measure-spaces, M_1 and M_2, which change into a third, M_3 (Vergnaud, 1983). These problems have at least three variables and can be represented by a double-correspondence table (see Table 2.7). In these problems the two numbers that are multiplied play equivalent roles, in other words, there is no distinction between the multiplier and the multiplicand (Greer, 1992). For example: The length of a floor is 5 metres and the breadth is 3 metres. What is the area of the floor?

Table 2.7: Double correspondence table for product of measures problems

Breadth (metres)	**Length (metres)**				
	1	2	3	4	5
1					
2					
3					15 m^2 (area)

The first class of problems under the *product of measures* are *arrays*. While Vergnaud (1983) does not include *arrays*, Hurst (2015), Jacob and Mulligan (2014), and Siemon et al. (2010) agree on their importance and that they differ from *area*. Arrays make use of discrete objects and the conception is additive (e.g. ). A grid, as seen in Table 2.7 above, is regarded as more powerful than *arrays* as it links with the *area* idea, promotes the idea of a composite unit or an entity, promotes an understanding of the distributive and commutative property of multiplication, and represents Cartesian product problems (Hurst, 2015; Jacob & Mulligan, 2014). Siemon et al. (2010) suggest that the *arrays/region* idea embodies the *groups of* idea and leads to the *area* idea where one can work with bigger numbers. Furthermore, they view grids as the basis for understanding fraction diagrams, which can help learners to appreciate the two-dimensional (semi-concrete) property of multiplication (Young-Loveridge, 2005).

The second and third classes of multiplication problems under *product of measures* are *area* (e.g. The length of a floor is 5 metres and the breadth is 3 metres. What is the area of the floor?) and *volume* (e.g. If a pipe is 100 cm long and has an area of 13 cm^2, what will the volume be?). As discussed above, *area* has the same benefits as *arrays*, with the difference that larger numbers can be used in *area* problems. *Area* provides a useful way to represent binary operations, such as commutativity (e.g. 3 × 4 or 4 × 3) (Greer, 1992). The same arguments apply to *volume*, which has only one added dimension, i.e. depth, which is conceptually more abstract than *area*.

The last class of problems under the *product of measures* is the *Cartesian product* (see Table 2.3). For example: Six girls and three boys are at a dance. Each boy wants to dance with each girl, and vice versa. How many different girl-boy couples are possible? Vergnaud (1983) views this type of problem as more difficult to conceptualise than the other types of multiplicative context problems since it involves double proportions. In France, this type of problem was introduced in the second and third grades; however, learners were unable to conceptualise it as they first needed to understand simple-proportion problems before they could grasp double-proportion problems, of which the *Cartesian product* is an example. Greer (1992) agrees with Vergnaud and defines it as a sophisticated way of defining multiplication. The *Cartesian product* idea can also be seen in how the place-value system is structured. For example, for each ten there are 10-ones, for each one there are 10-tenths, and so forth (Siemon et al., 2010). In the South African school context, the *Cartesian product* is introduced with probability in the FET (Further Education and Training) Phase (Grades 10 to 12).

To summarise, *product of measures* includes four classes of multiplication problems: *Arrays, area, volume* and the *Cartesian product*. The first three *products of measure* seem similar in respect of the calculation technique that learners can employ to solve them. However, whereas in the case of *arrays* learners may make use of additive reasoning, they can make use of only multiplicative reasoning for calculations of *area* and *volume*. Since it involves multiple proportion, the *Cartesian product* requires different reasoning and learners usually struggle to understand the reasoning behind it.

2.3.2.3 Multiple proportion

Although the structure of *multiple proportion* problems is similar to that of the *product of measures* with regard to mathematical relationships, it differs in respect of the third measure-space. These problems consist of two different and independent measure-spaces, namely M_1 and M_2, with M_3 being proportional to M_1 and M_2 (Vergnaud, 1983). Time is often present as a magnitude in problems of this type, with each of the magnitudes having its own meaning, which means that it cannot be reduced to a product of the others. For example: A family of three people wants to spend ten days at a resort. The cost per person is R400 per day. How much will the holiday cost? The calculation will be: 3 people × 10 days × R400 = R12 000 (Vergnaud, 1983). The classes of multiplication problems that Vergnaud (1983) uses to represent *multiple proportion* are *consumption, production,* and *expense*.

The first class of multiplication problems that involves *multiple proportion* is *consumption.* For example: At a camp each child eats 500 grams of cereal per day. How much cereal will 150 children eat over a 10-day period? The second class of problems is *production*. For example: One cow produces an average of 15 litres of milk in six days. How much milk will seven cows produce over a period of 120 days? The third class of problems is *expense* (Vergnaud, 1983). For example: A family of five goes to a resort for a holiday. The cost per person is R350 per day. What will it cost the family for seven days? *Multiple proportion* problems are more difficult to solve as the context consists of three variables, instead of two as with the other problems. These examples have more than one unknown and one unknown has to be worked out to determine the other unknown. In South Africa, problems of this type are only done in secondary school, if at all.

2.4 The developmental nature of calculation techniques

When solving addition and multiplication problems, learners use various calculation techniques that provide an indication of the level on which they are solving those problems, and whether they reason multiplicatively or additively. While Ell (2001), Siemon et al. (2010) and Zhang et al. (2011) are uncertain about how learners advance from additive to multiplicative reasoning, Clark and Kamii (1996), Hurst and Hurrell (2014), and Tobias and Andreasen (2013) believe that multiplicative calculation techniques develop from additive calculation techniques, which are based on counting. In contrast, Confrey (1994) postulates that multiplication calculation techniques should rather develop from splitting (e.g. dividing a piece of paper in two halves, four quarters, eight eighths, etc.). When learners count, the focus is on the concept of unit numbers, which leads to additive reasoning, whereas splitting focuses on the relationship between numbers and is a precursor of understanding exponents (Confrey, 1994; Vergnaud, 2009). Splitting avoids additive reasoning as it forces learners to reason multiplicatively and proportionally (Confrey, 1994). A study done by Empson and Turner (2005), which involved 30 learners in Grades 1, 3 and 5 in the USA, found that by teaching splitting, learners are encouraged to think multiplicatively. Confrey (1994) suggests that counting and splitting should be taught simmultaneously as both are necessary and of equal importance.

2.4.1 Other research studies on calculation techniques

Numerous counting techniques exist that lead to multiplication calculation techniques. Various authors identify calculation techniques that might indicate learners' progression from additive to multiplicative reasoning. However, they suggest different categorisation of the calculation techniques. While Hulbert and Laird (2013) created a framework of calculation techniques based on a long-term project, the calculation techniques proposed by Jacob and Willis (2003) are a summary of other research studies, and those suggested by Carrier (2014), Kouba (1989), Mulligan (1992) and Zhang et al. (2011) are based on empirical research. The calculation techniques suggested by the abovementioned authors are summarised in Table 2.8 below.

Table 2.8: Summary of calculation techniques identified by different authors

Mulligan (1992, p. 34)	Jacob & Willis (2003)	Kouba (1989, p. 151); Zhang et al. (2011, pp. 56, 58)	Hulbert & Laird (2013)	Carrier (2014, p. 91)
				Non-quantifier
			Guess	Spontaneous guesser
				Keyword finder
Counting all	One-to-one counting	Unitary counting	Counting by ones	Counter
			Inconsistent groupings	
Skip counting		Skip counting	Skip counting	
Repeated addition	Additive composition	Repeated addition	Repeated addition	Repeated adder
Additive doubling				
Known addition fact				
			Doubling	
	Many-to-one counting	Double counting		
			Algorithms	
	Multiplicative relations		Distributive properties	
Derived multiplication fact			Derived fact	
Known multiplication fact	Operating on the operator	Direct retrieval	Known fact	

While conducting a longitudinal study in Australia, Mulligan (1992) interviewed four times 70 female participants in Grades 2 and 3. She asked them to solve five multiplication and five division word problems with whole numbers, using either mental calculation or 3D cubes. The aim was to analyse the calculation techniques that those participants used and to establish how they had developed those techniques. The questions included the following five classes of multiplication problems: Equivalent groups, rate, comparison, array and Cartesian product. Only correct answers were analysed. Based on the analysis, she identified seven multiplication calculation techniques, which are shown in Table 2.8. The participants used all seven multiplication calculation techniques for all the questions. With each interview the participants' performance improved. They performed better doing the problems with small numbers than those with bigger numbers and continued struggling with Cartesian and factor problems.

In the USA, Zhang et al. (2011) conducted a teaching experiment (using a pre- and a post-test) with three Grade 5 learners with learning difficulties. The questions included multiplication, partitive, and division problems. The participants were taught how to double count by using 3D cubes. When answering the questions, they had to explain how they would solve each problem. The aim of this study was to determine which intuitive calculation techniques the participants used, and whether the teaching of double counting was likely to develop their multiplicative reasoning. This particular study was based on another study conducted in the USA by Kouba (1989), who interviewed 128 participants in Grades 1 to 3 who had been asked to solve 12 multiplication, division, addition and subtraction problems (which included only two multiplication problems). These participants, who were allowed to use 3D material to solve the problem, used four intuitive calculation techniques, namely, unitary counting, skip counting and repeated addition. In their study, Zhang et al. (2011) also found that three of these four techniques were used intuitively. The participants were then taught the fourth calculation technique, i.e. double counting. In the pre-test, participants mostly used unitary counting, while in the post-test they used double counting more often than unitary counting. All three participants showed significant improvement after the intervention as they were able to advance from intuitive calculation techniques to using more advanced techniques for calculation. However, it took more than six times attempts before they started consistently using double counting. Zhang et al.'s (2011) study further showed that learners with learning difficulties had problems with both conceptual understanding and information retrieval, and used fewer calculation techniques than those without learning difficulties.

Carrier (2014) conducted a study in the USA for which fourteen Grade 4 participants were interviewed while they answered ten computer-based questions that focused on proportion. Participants were allowed to use 3D material to help them solve the problems and had to answer the questions on a computer, but also had to write them down and explain their calculations, or use 3D material to demonstrate how they had arrived at the solutions. The purpose of this study was to identify indicators of multiplicative reasoning. Carrier (2014) identified twelve calculation technique levels. These twelve levels are: Non-quantifier, spontaneous guesser, keyword finder, counter, adder, quantifier, measurer, repeated adder, coordinator, multiplier, splitter and predictor. The twelve levels are an expansion of Clark and Kamii's (1996) five levels on which Carrier (2014) had based his study (i.e. no serial correspondence or serial correspondence with qualitative quantification, additive thinking with a numeral sequence of +1 or +2, additive thinking involving +2 and +3, multiplicative thinking, but not with immediate success, and multiplicative thinking with immediate success). Clark and Kamii (1996) interviewed 336 participants in Grades 1 to 5 and asked them three questions that focused on proportion. Since the questions asked were specific to the participants' levels, the findings cannot be used for my study, but it is interesting to note that they found that multiplicative reasoning starts in the early grades and develops slowly. Carrier's (2014) twelve calculation technique levels show that learners make use of various calculation techniques that could serve as learning trajectories which could assist teachers in helping learners along to ultimately use multiplicative calculation techniques. Apart from *guesser*, also identified by Hulbert and Laird (2013), none of his other calculation techniques were identified by other authors.

Hubert and Laird (2013), Jacob and Willis (2003) and Mulligan (1992) propose different categorisation of the calculation techniques. For my study, I chose the developmental approach to the categorisation of calculation techniques to determine whether learners use non-calculation, additive, or multiplicative calculation techniques. My categorisation is based on and adapted from Hulbert and Laird's (2013) classification model. Each of the calculation techniques proposed by these authors (as listed in Table 2.8) will be discussed in the next section.

2.4.2 Non-calculation techniques

Carrier (2014) identifies three *non-calculation techniques*, namely *non-quantifier, guesser,* and *keyword finder.* The only one of these that Hulbert and Laird (2013) and I agree with is *guesser*. Learners that use the calculation technique of *guesser*, will guess an answer and not use additive or multiplicative calculation techniques to solve the problem. *Non-quantifier* and the use of *keywords* are not calculation techniques, as *non-quantifiers* cannot preserve quantity; therefore, they do not see eight as a specific quantity, but choose any number to solve a problem. These learners will, for instance, when asked to calculate 3 × 2, calculate 3 × 4 as they do not yet understand the values of numbers (Carrier, 2014). While using incorrect numbers to calculate the answer, they have to use another calculation technique that would probably be additive in nature. In the same way, *keyword finders* decide on an operation by identifying keywords in a problem (Carrier, 2014). They can use either valid or invalid keywords to decide which operation to use. For example, they may wrongly think that the keyword 'altogether' or 'total' means multiplication, or they may rightly think that 'times' means multiplication. Having chosen which operation to use based on a keyword they can then choose any calculation technique to solve the problem.

2.4.3 Additive calculation techniques

When learners use *additive calculation techniques*, they think additively when solving multiplication problems. The six *additive calculation techniques* (summarised in Table 2.8) are: *Unitary counting, inconsistent groupings, skip counting, repeated addition, additive doubling* and *known addition fact*. When learners start to count, they count objects one at a time. This is called *unitary counting*. All the authors included this calculation technique, but use different names, such as *counting by ones* (Hulbert & Laird, 2013), *counter* (Carrier, 2014), *counting all* (Mulligan, 1992; Zhang et al., 2011), and *one-to-one counting* (Jacob & Willis, 2003) (see Table 2.8). Instead of objects, learners can use their fingers or tallies to count. Second, *inconsistent groupings*, suggested only by Hulbert and Laird (2013), entail that learners make groupings that are different from those indicated by the problem. For example, to calculate 3 × 4, learners count in groups of two (2, 4, 6, 8, 10, 12), instead of in groups of four (4, 8, 12).

Third, Hulbert and Laird (2013), Mulligan (1992) and Zhang et al. (2011) identify *skip counting* as a calculation technique. Ell (2001), Jacob and Willis (2003) and Zhang et al. (2011) believe that *skip counting* usually follows *unitary counting*, but Hulbert and Laird (2013) disagree and place *inconsistent groupings* between *direct counting* and *skip counting.* When *skip counting,* learners count in multiples of numbers such as 3, 6 and 9, but do not know when to stop counting. For example, if they are asked to give the answer to 2 × 3, they will count in threes, but will not know when to stop (e.g. 3, 6, 9, 12, 15), which indicates that they are, not yet able to coordinate two quantities, keep track when counting in threes and grasp how many times they need to count in threes (Zhang et al., 2011).

Fourth, Carrier (2014), Hulbert and Laird (2013), Mulligan (1992) and Zhang et al. (2011) all include *repeated addition*, or *additive composition*, as it is called by Jacob and Willis (2003) (see Table 2.8). Learners use *repeated addition* to solve a multiplication problem by adding the same number repeatedly (e.g. they calculate 3 × 4 as 4 + 4 + 4 = 12). These learners still think in additive terms. When working with whole numbers, multiplication is often seen as repeated addition and even though the answer may be correct, it is not obtainded by way of multiplicative reasoning as learners cannot use this thinking process to solve problems involving fractions and decimal numbers (Kouba, 1989).

Only Mulligan (1992) includes the last two calculation techniques, namely *additive doubling* and *known addition fact*. Additive doubling occurs when learners calculating, for example 2 × 4, will calculate it as 2 + 2 is 4, and 4 + 4 is 8. In the case of *known addition fact,* learners know an addition fact, for example 2 + 4 is 6.

2.4.4 Multiplicative calculation techniques

Multiplicative calculation techniques develop from *additive calculation techniques* and require higher-order multiplicative reasoning (Clark & Kamii, 1996). *Multiplicative calculation techniques* include *doubling, double counting*, *algorithms, distributive properties, derived and known multiplicative facts*. Hulbert and Laird (2013) are the only authors who identify *doubling* as a *multiplicative calculation technique.* This means that learners double a number to solve a problem (e.g. 6 × 4, can be calculated as 6 doubled is 12 and 12 doubled is 24). *Double counting* (Zhang et al., 2011) or *many-to-one counting* (Jacob & Willis, 2003) means that learners are able to keep track of two quantities at the same time. For example, if they are asked what 4 × 5 is, they will put up a finger each time they count in 5s until they have counted it four times. Therefore, one 5 is 5, two 5s are 10, three 5s are 15, four 5s are 20.

Hulbert and Laird (2013) are the only author who identify *algorithms* as a *multiplicative calculation technique* that entails the use of the column method to solve a multiplication problem. *Distributive properties* (Hulbert & Laird, 2013), also called *multiplicative relations* by Jacob and Willis (2003), occur where one or both numbers are 'broken up' in their place values to facilitate multiplication (e.g. 12 × 5, can be calculated as 10 × 5 + 2 × 5 = 50 + 10 = 60). Mulligan (1992) identifies *derived multiplicative fact,* which is called *derived fact* by Hulbert and Laird (2013). This is when, for example, learners calculate 13 × 6 as 12 × 6 is 72 + 6 is 78. The last calculation technique, which is identified by all the authors with the exception of Carrier (2014), is *known multiplicative fact* (Mulligan, 1992), *operating on the operator* (Jacob & Willis, 2003), *direct retrieval* (Zhang et al., 2011), or *known fact* (Hulbert & Laird, 2013). Recall of learnt multiplicative facts is the highest-ranking calculation technique learners can use (Zhang et al., 2011). For example, learners will immediately give the answer to 4 × 5 as 20. Even though direct recall of multiplication facts is seen as the highest calculation technique, it is not by itself an indicator of multiplicative reasoning and thus not an indicator of conceptual understanding (Hurst & Hurrell, 2016).

Thus the use *non-calculation techniques* indicate that learners are not using any additive or multiplicative calculation technique, which is usually due to a lack of good number sense. When solving multiplication problems, *additive calculation techniques* are less efficient than *multiplicative calculation techniques*. If *multiplicative calculation techniques* are learnt after *additive calculation techniques,* they are developmental, which means that the mastering of *multiplicative calculation techniques* require thinking of a higher order than *additive calculation techniques*. How these calculation techniques link to internal and external representations is explained in the conceptual framework, which will be discussed below.

2.5 Conceptual framework

My study was based on the conceptual fields theory of Vergnaud (1982; 2009; 2013a), which builds on Piaget's theory of cognitive development. Vergnaud (2013a) acknowledges the work of Piaget in his conceptual fields theory, but points out shortcomings in Piaget's theory regarding cognitive development. According to Vergnaud's (2013a) view of progressive conceptualisation, conceptualisation continuously takes place in cognitive development during specific activities. Furthermore, Vergnaud (2013b) postulates that Piaget's idea of cognitive development through adaptation is too general, biological and social to study and that Piaget fails to explain who adapts and to what. However, Piaget (1953) does explain that people adapt to their environments, and that intelligence structures the environment it encounters. Piaget (1953, p. 4) also proposes that it is "the relationship of thought to things" that adapts. Although in cognitive development the main function of thought remains invariant, the structures in the mind are variant (Piaget, 1953). The two invariant operations mentioned by Piaget (1953) namely organisation and adaptation, link well with what Vergnaud (2009; 2013b) call schemes. According to Piaget (1953), the process of adaptation takes place when a person is transformed by the environment (Piaget 1953). With his conceptual fields theory, Vergnaud (2013b) clarifies this idea and postulates that learners adapt to situations, and it is in fact the schemes that adapt when learners are confronted with new situations. Vergnaud (2009; 2013b) proposes that this situation-scheme pair replaces the stimulus-response, which is well known in psychology. The situation-scheme pair is the root of cognitive development. The activity (external representations) that is generated when learners are confronted by classes of problems (situations) is the focus of my study, as summarised in Figure 2.1. Proficiency in this activity (external representations) for solving specific mathematical problems is essential if learners are to master mathematical content.

2.5.1 The interplay between internal thinking processes and external representations

In any interaction between a researcher and learners, the researcher provides external representations to the learners. These representations are the classes of mathematical problems, the material that they should use to solve the problem and the sequence in which they are required to solve a multiplication problem. Learners need to interpret the given information and decide, based on that information and their prior knowledge (internal representations), how to solve the given multiplication problem. This decision-making process involves internal cognitive processes, which include the transformation or creation of new schemes and schemas. Once the learners have decided how to solve the given problem, they then give the solution by using observable and measurable external representations (see Figure 2.1). This process will be discussed in the next section.

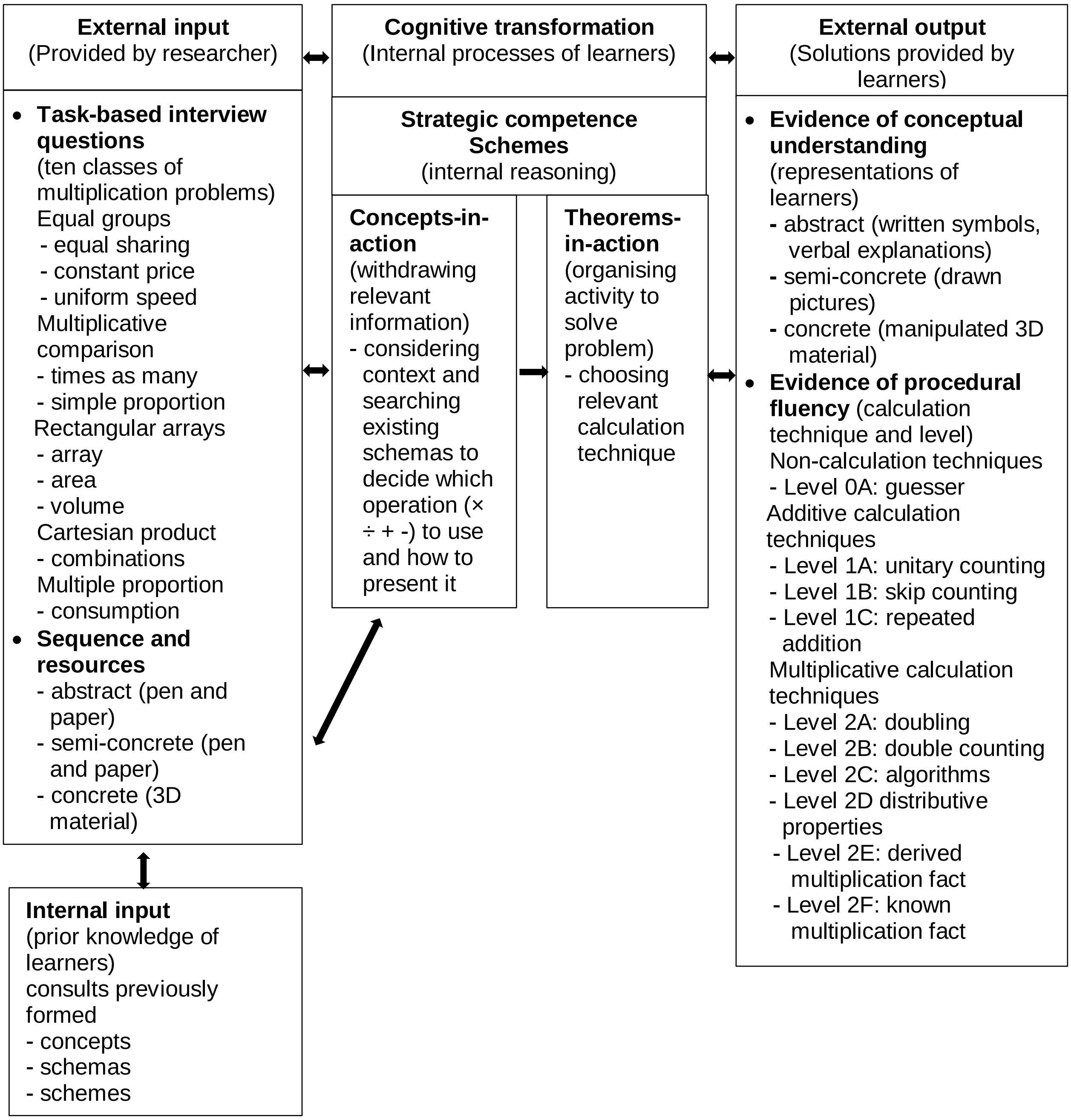


Figure 2.1: The interplay between internal thinking processes and external representations

As a researcher, I sought to gather measurable and observable data (external representations). In order to accomplish this, I needed instruments and methods that could produce measurable and observable results. Furthermore, I was interested in determining which processes learners followed to solve multiplication problems, and whether their cognitive processes were proficient. For this reason I used Vergnaud's (2009; 2013a; 2013b) scheme theory, for which he provided a comprehensive definition (see section 2.2.2 of this chapter), with the two important components concepts-in-action and theorems-in-action. However, the transformation process is not visible or measurable. It is only through external representations that I could infer how learners used schemes to solve problems.

2.5.1.1 External and internal input

I provided the materials and question to serve as external input, which included the ten multiplication problems, the sequence in which I wanted them to solve the problems and the material that could be used for that purpose. Learners consulted their prior knowledge of the concept of multiplication, their schemas, and their existing schemes for solving multiplication problems. These served as internal input for my study (see Figure 2.1).

i. Internal input

Conceptual understanding implies an integrated and functional grasp of related mathematical ideas (Kilpatrick et al., 2001). It furthermore implies that learners with conceptual understanding are able to grasp the full meaning of a mathematical concept and can discern, interpret and compare ideas in various situations (Panasuk, 2010). This idea corresponds to what Skemp (1976) refers to as relational understanding, which he defines as the knowledge of what to do and why you do it. Learners should have knowledge of the multiplication concept which should be more than isolated facts. In mathematics, conceptual understanding is a highly complex process of abstraction as learners' understanding is determined by the number and strength of the connections that they make when confronted with new concepts (Barmby et al., 2007; Panasuk, 2010). The more connections are made, the better the conceptual understanding. The idea to group related concepts together has led to the forming of conceptual fields, which include multiplicative reasoning. The multiplicative conceptual field is made up of schemas that consist of different concepts and competences. It is important to have a conceptual understanding of the multiplicative conceptual field as it helps learners to build a schema that consists of a network of different representations of a specific concept, and consequently to make connections between mental representations (Barmby et al., 2007). It is, therefore, possible for learners to have conceptual understanding before they are able to verbalise it (Kilpatrick et al., 2001). When learners are asked to solve a mathematical problem, they do it by using their internal representations together with given external representations.

ii. External input

I provided external input, which included multiplication problems, the sequence in which I wanted the learners to present their answers, and the material needed to solve the given problems (see Table 2.1). To evaluate the participants' proficiency in solving whole-number multiplication problems, I selected ten classes of multiplication problems from the relevant literature. These can be grouped together into five categories of multiplication. The five categories of multiplication on which the majority of authors agree and their classes of multiplication problems were discussed in depth in section 2.3.2 of this chapter and for the purpose of this study, included: Equal groups (*equal sharing, constant price, unit speed*), multiplicative comparison (*times as many, simple proportion*), rectangular arrays (*arrays, area, volume*), Cartesian product (*combinations*), and multiple proportion (*consumption).* In order to evaluate the conceptual understanding of the various classes of multiplication problems among learners with learning difficulties, learners were asked to externally represent their schemes by using abstract, semi-concrete and concrete representations (see section 2.2.3 of this chapter for discussion) and different materials.

a. Proposed classes of multiplication problems for my study

Twelve classes of multiplication problems were discussed (see section 2.3.2 of this chapter) under three main categories. These classes of multiplication problems and main categories can be summarised as follows:

- *Isomorphism of measures,* which includes the classes of multiplication problems known as *equal sharing, constant price, unit speed, constant density on a line* and *multiplicative comparison;*
- *Product of measures,* which includes the classes of multiplication problems known as *arrays, area, volume* and *Cartesian product;*
- *Multiple proportion,* which includes the classes of multiplication problems known as *consumption, production* and *expense.*

For my study, I chose five categories, which are a combination of Greer (1992), Mulligan (1992), Mulligan and Mitchelmore (1997), and Vergnaud's (1983) classification, and included *equal groups, multiplicative comparison, rectangular arrays, Cartesian product* and *multiple proportion* (see Figure 2.1). Under *multiplicative comparison,* I included another class of problems, namely *simple proportion*, which was not included by Greer (1992), Mulligan (1992), Mulligan and Mitchelmore (1997) and Siemon et al. (2010), but was implied by Vergnaud (1983). I included it under *multiplicative comparison* and not under *equal groups* or *multiple proportion,* since the problems in the category *equal groups* have two variables and do not count as comparative problems. *Multiple proportion* has six variables, of which two are unknown. *Simple proportion* has four variables, one of which is known. The category *multiplicative comparison* includes the idea of *times as many,* which compares two units of the same kind, and because of its comparative nature, *simple proportion* naturally fits in here even though it is somewhat more complex than *times as many*.

Simple proportion is a comparison of two “between” or of two “within” ratios that covary (Lamon, 1994; Van de Walle et al., 2015). Vergnaud (1994) also refers to these types of proportions, but names them functional and scalar ratios respectively. “Between” ratio problems have two variables in different contexts, whereas “within” ratio problems have two variables in the same context. Proportion problems of this kind include four variables and one unknown, unlike double proportions that have six variables and two unknowns (Vergnaud, 1983). For example: If 2 boxes fruit juice cost R20, how much would 8 boxes of fruit juice cost? If this is conceptualised it as a “within” or scalar ratio, the ratio of the original number of fruit juices (2) in comparison to the number of fruit juices in the second situation (8) will be considered. When it is conceptualised as a “between” or functional ratio, the ratio of fruit juices (2) to money (R20) will be considered (Lamon, 1994; Vergnaud, 1983). Personally I know many learners that struggle with proportional reasoning of this type, as teachers do not teach it as proportional reasoning but teach it rather in a procedural way. Learners are taught to calculate the cost of one box of fruit juice and then to work out what eight will cost.

The five categories of multiplication for this study are therefore *equal groups, multiplicative comparison, rectangular arrays, Cartesian product,* and *multiple proportion*. First, under equal groups I included the following three classes of multiplication problems: *Equal sharing, constant price* and *uniform speed*. The reason for the inclusion of *equal sharing* and *constant price* is that they are the most recognisable problems that learners have been asked to solve since the Foundation Phase (Grades R to 3) and form the foundation of multiplication (DBE, 2011a). I also included *uniform speed* because it is unfamiliar to learners as they will only be doing these problems later in their Grade 9 year (DBE, 2011c), however according to the mathematics textbooks they are introduced to the concept in Grade 4 (Bowie, Gleeson-Blaird, Jones, Morgan, Morrison, and Smallbones, 2012a). I believe that it is important to see how learners reason when confronted with classes of multiplication problems with which they are not familiar. I have left out *constant density on a line* as learners are not confronted with problems of this type in the South African school curriculum.

Second, under *multiplicative comparison,* I included two classes of multiplication problems, namely *times as many* and *simple proportion,* which I separated from *equal groups* as they require thinking that is different from that required for *equal groups* (DBE, 2011a; Greer, 1992). The first class of multiplication problems has a specific identifying phrase, namely *times as many,* and functions as a bridge to understanding ratios (Greer, 1992; Hurst, 2015). *Simple proportion* is a new class of problem that I wanted to include as it compares two situations and problems of this kind are included in both mathematics and mathematical literacy in the South African curriculum (DBE, 2011a).

Third, under *rectangular arrays* I included three classes of multiplication problems, namely *arrays*, *area* and *volume*. I decided to include *arrays* and *area* because learners are familiar with both. *Arrays* are not included in the Intermediate Phase (Grades 4 to 6), but are dealt with it in the Foundation Phase (Grades R to 3) (DBE, 2011a; 2011b). I also included *volume*, and even though the reasoning is very similar to that for *area*, *volume* is a 3D problem whereas the other two are 2D problems. In my experience, learners with learning difficulties find 3D thinking more difficult than 2D thinking.

Fourth, under *Cartesian product* I included only *combinations* as a class of problems. Like Greer (1992), Mulligan (1992), Mulligan and Mitchelmore (1997) and Siemon et al. (2010), I separated the *Cartesian product* from *rectangular arrays*. Although *combinations* are only introduced in the FET Phase in the South African school curriculum (DBE, 2011d), I decided that I would like to explore how participants reason to solve problems in this class, which they have not previously encountered.

Last, under *multiple proportion* I included the class of problems of *consumption,* which is explained by Vergnaud (1983). *Consumption* problems are the most difficult problems, as they include two unknowns and they are not covered in the South African school curriculum. Proportional reasoning is the essence of multiplicative reasoning its inclusion in a study of multiplicative reasoning is therefore essential. I did not include *expense* as a class of problem as I did include *consumption,* which is already a difficult problem for learners. Thus, both familiar and unfamiliar classes of multiplication problems were included in my study. This was done to enable me to explore the conceptions that learners with learning difficulties have of both familiar and unfamiliar classes of multiplication problems. How learners solve multiplication problems in these classes is discussed next.

2.5.1.2 Transformation processes

When asked to solve a multiplication problem, learners have to use a scheme. They need to either create a new scheme or (more often) adapt an existing scheme to solve the multiplication problem. Learners need to consider the external input given to them, as well as internal input, i.e. their prior knowledge of the concept, and then withdraw relevant information from both internal and external input. This process, known as concepts-in-action is part of a scheme (thought in action) that learners activate to decide how to solve the problem. Using the available information, they need to decide what type of problem it is: Multiplication, addition, subtraction, or division. Furthermore, once they have decided what type of operation to use, they must decide on the appropriate calculation technique to be used. This forms part of the second component of a scheme, called theorems-in-action (see section 2.2.2.1 of this chapter for an in-depth discussion of schemes). The role of theorems-in-action is to organise mental activity to solve a problem and thus to choose a relevant calculation technique based on their prior knowledge of calculation techniques. The act of using a scheme to solve the given multiplication problem transforms the thinking and memory of learners and the adapted scheme becomes part of their memory and therefore transforms their prior knowledge. Once learners have decided which operation and calculation technique they can use, they can externally represent their thinking and the answer.

2.5.1.3 External output

When learners are asked to solve a multiplication problem, they first need to decide how the problem should be represented, and then what calculation technique to use. Even though a multiplication problem has only one correct answer, the answer can be calculated in multitude ways. When researchers want to explore the proficiency of learners, they need to follow a multi-dimensional approach. Considering only the answer is not enough as it does not tell you how learners think (Ayub et al., 2013; Barmby et al., 2007; Lesser & Tchoshanov, 2005; Panasuk, 2010; Panasuk & Beyranevand, 2011; Pape & Tchoshanov, 2001). The use of different representations (abstract, semi-concrete and concrete) can shed light on learners' understanding of a concept (Panasuk, 2010; Panasuk & Beyranevand, 2011). This said, an evaluation of the calculation techniques that learners use reveals the level of procedural fluency they have for solving a problem and makes it possible to evaluate the proficiency of their schemes.

Learners could use various materials, which served as their output, to solve multiplication problems and demonstrate their understanding. These materials include pen and paper, drawings and the use of 3D materials, which they could manipulate to solve the problem and demonstrate their conceptual understanding. The solving of a problem also requires calculation techniques that allow learners to count on their fingers, use symbols and explain in words how they arrived at the answer. To decide which calculation technique to use, learners have to consider the internal and external input and then choose the best calculation technique for solving the specific problem.

In order to help learners to make the switch to multiplicative reasoning it is important for teachers and researchers to be able to determine whether learners reason additively or multiplicatively (Jacob & Willis, 2003). Carrier (2014), Hubert and Laird (2013), Jacob and Willis (2003), and Mulligan (1992) propose different categorising models. For the purpose of this study, I used levels, which I adapted from those proposed by Carrier (2014), and selected categories to group the calculation techniques, namely non-calculation, additive, and multiplicative calculation techniques, to reveal their developmental structures. These categories, which were based on and adapted from those proposed by Hulbert and Laird (2013), as well as their corresponding calculation techniques and levels, are summarised in Table 2.9.

The various calculation techniques that learners were the most likely to use were selected based on the information in the relevant literature. They are discussed in detail in section 2.4 of this chapter and include *non-calculation techniques* (*guesser*), *additive calculation techniques* (*unitary counting, skip counting,* and *repeated addition*), and *multiplicative calculation techniques* (*doubling, double counting, algorithms, distributive properties, derived*, and *known multiplication fact*). The calculation techniques were arranged into levels, which can be used to indicate how proficient learners are in solving a given multiplication problem.

i. Proposed calculation techniques for solving multiplicative problems

Since this study focused on learners with learning difficulties and some of the learners are behind their grade level in understanding mathematics, it was expected that the range of calculation techniques used would show considerable variation. Section 2.4 of this chapter, contains a discussion of all the calculation techniques that I could find that apply to multiplicative reasoning. They are:

- *Non-calculation techniques,* including *non-quantifier, keyword finder* and *guesser*
- *Additive calculation techniques,* including *unitary counting, inconsistent groupings, skip counting, repeated addition, additive doubling* and *known addition fact*
- *Multiplicative calculation techniques,* including *doubling, double counting, algorithms, distributive properties, derived multiplicative fact* and *known multiplicative fact*

For my study, I chose not to use the calculation techniques *non-quantifier, keyword finder*, *inconsistent groupings* and *additive doubling*. I do not agree with Carrier (2014) that *non-quantifier* and *keyword finder* are calculation techniques. Using different numbers than what is given is not a calculation technique, but rather an error.

Learners still have to use a Level 1 or Level 2 calculation technique to solve the problem. Learners that use keywords only use it to identify what operation to use and they also need to use a Level 1 or Level 2 calculation technique for finding an answer. It, therefore, always needs to be used in conjunction with another calculation technique. I also did not include *inconsistent groupings* and *additive doubling* as they are calculation techniques that are used when using 3D material to calculate an answer. I evaluated the calculation techniques used by learners when they solved the multiplication problem by using abstract symbols. Furthermore, I do not agree with Mulligan's (1992) opinion that *known addition fact* as a calculation technique can be used for solving multiplication problems. This calculation technique implies that you add, and that you see the problem as an addition and not as a multiplication problem. The chosen eleven calculation techniques for my study are summarised in Table 2.9.

Table 2.9: Summary of calculation techniques and levels related to the development of multiplicative reasoning

Calculation level	Calculation technique	Description
Level 0	**Non-calculation techniques**	
Level 0A	Guesser	Guesses the answer with no understanding of the problem
Level 1	**Additive calculation techniques**	
Level 1A	Unitary counting	Uses fingers or tallies to calculate answer counting each unit separately, e.g. 2 + 3 is 1, 2, 3, 4, 5
Level 1B	Skip counting	Counts in multiples, such as 3, 6, 9, but does not know when to stop counting
Level 1C	Repeated addition	Uses repeated addition, e.g. 3 + 3 + 3 = 9
Level 2	**Multiplicative calculation techniques**	
Level 2A	Doubling	Uses doubling, e.g. 6 × 4, is 6 doubled is 12, 12 doubled is 24
Level 2B	Double counting	Counts in multiples while keeping track of how many groups have been counted, e.g. 5 × 3, is calculated as one 5 is 5, two 5s are 10, three 5s are 15
Level 2C	Algorithms	Uses an algorithm, e.g. the column method
Level 2D	Distributive properties	For example, 12 × 5 is calculated as 10 × 5 + 2 × 5 = 50 + 10 = 60)
Level 2E	Derived multiplication fact	For example, 13 × 5 is calculated as 12 × 5 is 60 + 5 is 65
Level 2F	Known multiplication fact	For example, 3 × 4 is 12

The calculation technique levels specify the development of proficiency, in other words, the higher the level, the more proficient learners' calculation techniques are. The first calculation technique category consists of the *non-calculation techniques,* which means that learners are not using any counting techniques as they guess the answer. There is only one *non-calculation technique*, Level 0A, as the technique is not additive or multiplicative yet. The second category of calculation consists of the *additive calculation techniques,* which include *unitary counting*, *skip counting* and *repeated addition* and labelled as Levels 1A to 1C, as they imply that learners reason additively and have not yet made the switch to multiplicative reasoning. The third calculation technique category was labelled Levels 2A to 2F, which means that learners could coordinate two quantities at the same time and had therefore made the conceptual shift to multiplicative reasoning to solve multiplication problems (Zhang et al., 2011). The use of multiplication facts demonstrates abstract thinking and is seen as the highest cognitive developmental calculation technique (Hurst & Hurrell, 2014; Zhang et al., 2011). However, knowing multiplication facts is not an indicator of conceptual understanding, but only of procedural fluency. This was found in a study conducted by Hurst and Hurrell (2016) in Australia, where they interviewed sixteen Grade 6 participants. Their aim was to determine whether participants could first solve problems mentally, using pen and paper, if not they were asked to show their calculations concretely. They found that half of the learners could neither use an algorithm to calculate the answer, nor use 3D material to display the problem.

2.6 Summary

Multiplicative reasoning is a conceptual field that is developmental in nature and should develop over time with the transition from additive reasoning to multiplicative reasoning taking place during the Intermediate Phase (Grades 4 to 6). To evaluate whether learners have made the transition, the nature of the calculation techniques that they use to solve multiplication problems can be explored. Although there are four levels of calculation techniques, only Level 3 calculation techniques are indicative of learners' ability to operate on a multiplicative reasoning level. If the transition from additive to multiplicative reasoning is not made in the Intermediate Phase, learners will struggle with, for instance, proportional reasoning and algebraic reasoning, which build on multiplicative reasoning. Moreover, to evaluate learners' conceptions and misconceptions of multiplication problems I explored how Grade 6 learners with learning difficulties represented each class of problem abstractly, semi-concretely and concretely. These three levels of abstractness could give insight into learners' understanding of multiplicative reasoning.

From the literature review it emerged that learners struggle, and those with learning problems seems to struggle more, with the transition from additive to multiplicative reasoning. This study attempted to explore the reasons for this failure in transitioning from additive to multiplicative reasoning by exploring learners' proficiency in multiplicative reasoning, which includes conceptions and misconceptions in the abstract, semi-concrete and concrete understanding of multiplication and the calculation techniques, or lack of calculation techniques, used and the errors they made.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

Ell (2001), Vergnaud (1982) and Zhang et al. (2011) claim that interviews offer the most powerful research method as it can shed light on learners' thinking processes. I therefore decided to conduct a qualitative case study with task-based interviews as my aim was to explore the proficiency of learners' multiplicative reasoning.

This chapter starts with a discussion of the paradigmatic perspective that underpins this study, which is critical realism. This is followed by an explanation of my research design, which included the selection of participants and how I collected the data. I then discuss the instrument used for data collection, which consisted of task-based interviews. Thereafter, I explain how I analysed the data, followed by a discussion of the quality measures and ethical considerations.

3.2 Paradigmatic perspective

The philosophical paradigm that underpinned this study was critical realism. Bhaskar coined the term critical realism during the 1970s as an alternative to the positivist and the interpretivist paradigms (Fletcher, 2017; Wynn & Williams, 2012). One of the main beliefs subscribed to by critical realism is that we cannot reduce the nature of reality (ontology) to our knowledge of reality (epistemology). Critical realism therefore places the focus on ontology, rather than on epistemology, and seeks to identify the components and interactions within a reality (Wynn & Williams, 2012).

3.2.1 Ontological assumptions

I discuss the ontological assumptions first as they lead to epistemology, and since the focus of critical realism focus is on ontology, rather than on epistemology (Cohen, Manion, & Morrison, 2011; Wynn & Williams, 2012). Cohen et al. (2011) define ontology as how we think about the nature of reality or the nature of things. According to Wynn and Williams (2012), critical realism has four ontological assumptions. The first of these is that reality exists outside of people and can only be partially known

(Easton, 2010). In this study, it is assumed that knowledge of multiplicative reasoning is developmental and exists independently of participants.

The second ontological assumption is unique to critical realism as it views reality as stratified into three levels: Real, actual and empirical (Easton, 2010; Fletcher, 2017). First, the real level of reality consists of structures inherent in objects that act as causal mechanisms that produce events at the empirical level. In this study, the real level included the classes of multiplication problems, the calculation techniques and the external representations that I required learners to use to demonstrate their conceptual understanding and procedural fluency of multiplicative reasoning. Each of the multiplication problems represented various classes of multiplication with different levels of complexity (structures of mathematics). Furthermore, I asked participants to demonstrate their conceptual understanding of the various classes of multiplication problems by means of abstract, semi-concrete and concrete representations (structures of understanding). I then required them use calculation techniques to solve the multiplication problem (structures of calculation). The classes of multiplication problems, the external representations and the calculation techniques act as causal mechanisms that required participants to produce a spoken answer that I could hear and one that I could observe in what they wrote, drew, or constructed by using 3D material (empirical level). What the participants did depended on their prior knowledge, or internal representations, which they filtered through their experiences. When as the researcher I interpreted their responses and representations on the empirical level, I interpreted them by applying the theory and inevitably my interpretation was filtered through my experiences and what I saw, heard and observed (Godino & Font, 2010). Second, the actual level of reality is a subset of the real level where events occur, regardless of whether they are observed or not. The actual level of reality is usually different from what we observe at the empirical level as it is not filtered through human experience. The actual level did not form part of my study as I could not observe it and it could therefore not be empirically measured (Easton, 2010; Fletcher, 2017). This refers to events that occurred during the thinking process while participants were deciding how to solve the problem. Since these events were not verbalised, I could only make inferences and draw conclusions based on what I observed or heard from the external representations, and could not speculate on what I had not observed or heard. Finally, the empirical level of reality is a subset of the actual level and is an event that we experience, measure or observe, and which influences our interpretation of it (Easton, 2010; Fletcher, 2017). This study focused on the empirical level of reality, i.e. the external representations of participants, which I could observe or hear.

The third ontological assumption is that of emergence. This implies that the components of the different structures will emerge as they interact with one another and should not be seen in isolation (Wynn & Williams, 2012). As the participants in this study solved the multiplication problem by using representations and calculation techniques, their conceptual understanding and procedural fluency of the multiplication problem and their strategic competence emerged, which provided an indication of how proficient they were in solving various classes of multiplication problems.

The last ontological assumption of critical realism is the fact that reality is an open system that cannot be directly controlled. This implies that as a mechanism enacts in a system and may change that system, it cannot be assumed that it will produce the same event in the future. Because of this, the focus is on identifying tendencies (Wynn & Williams, 2012). This study was seen as an open system and I admit that it is possible that mechanisms (classes of multiplication problems, order of representations and calculation techniques as well as my presence) might have influenced participants while they were busy with the task. This could have had an influence on what emerged, as discussed previously, which was a limitation of this study. For this reason, the focus was on identifying trends and tendencies, and not on predicting the future.

3.2.2 Epistemological assumptions

Various authors define epistemology as how we come to know reality. It determines how we develop knowledge claims, how we evaluate the truth and the validity of these claims, and how we measure them against existing knowledge (Cohen et al., 2011; Wynn & Williams, 2012). The ontological assumptions of critical realism are linked with the epistemological assumptions. Wynn and Williams (2012) posit five epistemological assumptions. The first of these is that of mediated knowledge.

Knowledge of the intransitive structures situated in the real level, where reality is independent, forms at the empirical level of reality, which is transitive. This implies that knowledge, for critical realists, is theoretically informed (Wynn & Williams, 2012). In this study it is assumed that knowledge of the classes of multiplication problems, external representations and calculation techniques can only be known by looking at literature and what others and I have observed.

The second epistemological assumption is that the goal of critical realism is to explain why events have happened, rather than make predictions about future events. We cannot predict the future when we assume an open system, since there are too many factors that may influence events (Wynn & Williams, 2012). The focus of this study was on exploring how participants with learning difficulties reasoned when solving multiplication problems, and not to predict how they would reason in the future.

The third epistemological assumption is that we should describe events via mechanisms. Because of the open-system ontological assumption, the descriptions should be based on theories and should focus on the causal relationships between the mechanisms for a specific event (Wynn & Williams, 2012). In this study, I used mechanisms to describe the proficiency of participants when solving multiplication problems. I could only describe the proficiency of participants in multiplication and causality if I had examined various classes of multiplication problems by using multiple external representations and evaluating the calculation techniques that participants used when solving the problems (which are the mechanisms for this study).

The fourth epistemological assumption was the unobservability of mechanisms. Knowledge of reality is not always based only on what we can observe, but also on what we do. We can therefore not always observe the mechanisms directly, but can observe only how they manifest (Wynn & Williams, 2012). In this study, the manifestation of the identified mechanisms was the verbal and non-verbal responses from participants (for example using their fingers to count), as well as how they represented the problems with 3D material and how they drew them. The combination of all these provided me with a multi-dimensional picture of the mechanisms.

Owing to the open-system ontological assumption, the final epistemological assumption of critical realism is multiple possible descriptions of what had caused an event. The description that is selected is the one that is the most likely, given the mechanisms involved, and that most accurately represents the real world, given what we know now (Wynn & Williams, 2012). In this study, the descriptions that I selected were based on what I found in the literature and were given from a multi-dimensional perspective of different external representations.

3.2.3 Methodological assumptions

Ontological and epistemological assumptions determine the methodological assumptions (Cohen et al., 2011). Wynn and Williams (2012) identify five methodological assumptions. The first of those is the explication of events. This means that, for my study, I needed to identify the different aspects of the event that I studied. Stratified ontology and mediated knowledge epistemology determine this assumption (Wynn & Williams, 2012). The different aspects that I studied were set out and explained in the conceptual framework (see section 2.5 of Chapter 2). The order of the task-based interviews will be discussed in section 3.4.

The second methodological assumption is the explication of structure and context. This means identifying what it was about the structures that have produced the results. To do that, it is necessary to follow a process of abstraction, which requires re-describing the components of structure and their relationships in terms of existing theories (Wynn & Williams, 2012). The structures of the classes of multiplication problems, different external representations, and calculation techniques were discussed in Chapter 2. In Chapter 4 I explained how these were observed.

The third methodological assumption is that of retroduction, which refers to the identification of possible causal mechanisms that may link the structure to the results (Wynn & Williams, 2012). It shifts the focus from the results to the relationships that produced the results (Fletcher, 2017). In this study, I used retroduction to draw inferences by using the external representations and calculation techniques that the participants used for various classes of multiplication problems to describe how the different representations were linked to one another and which calculation techniques were used.

The fourth methodological assumption relates to empirical corroboration, which means that there should be sufficient depth in describing the relationships (Wynn & Williams, 2012). In this study, the different external representations (CSA) and the calculation techniques used to solve the different classes of multiplication problems gave the depth required to identify tendencies and causal relationships between the external representations and the calculation techniques used.

The final methodological assumption is triangulation, which refers to the use of various data-collection methods (Wynn & Williams, 2012). I used only task-based interviews, but within those task-based interviews I used three different external representations, namely concrete, semi-concrete and abstract representations, calculation techniques and participants' verbal responses. All these external representations triangulated to provide a holistic picture of participants' proficiency in multiplicative reasoning.

3.3 Research design

Research design is a logical plan that guides researchers from asking questions to where they can draw conclusions about the information obtained from the responses (Yin, 2014). I decided to use a case study research design for my study. The case study research design helped me to answer the research questions in a meaningful way as this study aimed to explore and describe multiplicative reasoning in a group of fifteen participants with learning difficulties. There are different ways that case studies can be categorised and I found the categories of Yin (2014) the most appropriate. He divided case study research designs into singe-case and multiple-case design of which I chose the single-case study design with the group of fifteen participants as the unit of analysis (Zainal, 2007; Yin, 2014). My single-case study included fifteen Grade 6 participants from three LSEN public schools in Pretoria. The method used to select my participants and the data collection procedures will now be discussed.

3.3.1 Selection of participants and sampling

I chose to use a convenient sampling method to select the participants for the task-based interviews. The reason for this was that there are only four LSEN public schools for learners with learning difficulties in Pretoria, and consequently a limited number of participants to choose from. Seventeen Grade 6 learners from three of these schools were willing to participate. My reason for selecting Grade 6 learners was that, according to the relevant literature, the switch from additive reasoning to multiplicative reasoning should take place in the Intermediate Phase (Grades 4 to 6) (Long & Dunne, 2014; McClintock, et al., 2011; Tzur et al., 2010). Since Grade 6 is the last year of the Intermediate Phase, it is assumed that learners in this grade would already have made this switch. At the beginning of Grade 6, when I conducted the task-based interviews, the participants had not yet learned about decimal numbers and percentages, and had a limited understanding of multiplication with fractions. I decided that asking multiplication questions with only whole numbers would give me an indication of the extent to which they had made the switch from additive reasoning to multiplicative reasoning, which should prepare them to better understand fractions, decimal numbers and percentages. I chose LSEN participants as at the time of writing no studies involving LSEN participants had yet been undertaken in South Africa. In fact, I could find only one South African study on multiplicative reasoning, which involved high school participants (Long, 2011).

3.3.2 Data collection procedures

I conducted one-on-one task-based interviews with altogether seventeen participants, but decided to use the first two participants for a pilot study. After collecting their data and analysing their video recordings, I realised that I needed to change certain aspects of how I conducted the task-based interviews. The first problem I noticed was that I dominated the conversation, with Learner 1 barely saying a word. I also did not ask enough questions. I changed that during the second interview and gave the participant time to explain. Second, I noticed that I kept fidgeting with the plastic bags that contained the 3D blocks, sweets and money, which could distract the participant. From the second interview I avoided the fidgeting, and waited to unpack the material and only put it away after the participant finished solving the problem. Last, I noticed that after the participants had abstractly presented their solutions, I did not remove that page when they drew their picture, with the result that they looked at what they had done before and have the same answer. I wanted them to provide a semi-concrete representation of the problem without having access to the previous representation. From the third interview, I removed the page on which the participants had done their calculations and made sure that they did not have access to any previous representations.

After the pilot study and for the next three months, I continued with the one-on-one task-based interviews with fifteen Grade 6 learners with learning difficulties. I used the literature study and conceptual framework to compile ten multiplication problems (see Chapter 2). I made video-recordings of the tasked-based interviews, which I then transcribed to be able to revisit the interviews and code them to ensure the credibility of the data. Three types of data were transcribed, namely the conversation between the learner and I, any non-verbal communication observed (such as counting on fingers), and what the participants said, what they wrote and drew, and how they used the 3D material. As the researcher my role was to ask the questions and to guide learners in the order that they needed to represent their answers, as well as to ask further clarifying questions on what they had done and why they had done it, and to answer any questions that they had, which made me part of the research.

As the task-based interviews continued, I made further adjustments when necessary. During the third interview, the participant fixated on the different colours of the sweets wrappings that I used in three of the questions. After that interview, I started phasing out the different colours and used sweets wrapped in the same colour. Furthermore, I did not finish all the questions in an hour. I had previously decided to limit the task-based interviews to one hour, if possible, as participants with learning difficulties struggle to concentrate for long and also because the interviews were conducted after school hours. I therefore had to decide which questions to omit once I realised that we would not finish within an hour. This resulted in a limitation of the study as I was unable to select the questions scientifically. I decide that all the participants should solve the first four task-based questions as they had already come across those questions in previous grades. However, the criterion that I applied to choose which other questions to ask was to make sure that I asked them at the same rate. During each of the following interviews I started by asking the first four questions, after which

I determined which questions I had not put to the previous participants and asked those first before continuing with the rest. In the end, all fifteen the participants answered Questions 1 to 4, eleven answered Questions 5 to 7 and nine answered Questions 8 to 10.

3.4 Instrument for data collection

Allsopp et al. (2007) suggest that when conducting interviews with participants who can verbalise their thinking when solving mathematical problems, it is possible to form an idea of learners' conceptual understanding of a specific topic. For this reason, the instrument that I used to collect my data was task-based interviews. However, since learners with learning difficulties often struggle to express their thoughts, I included other external representations, namely semi-concrete and concrete representations, to obtain a clearer picture of their conceptual understanding.

Since researchers view interviews as the preferred and most powerful method for studying multiplicative reasoning (Ell, 2001; Vergnaud, 1982; Zhang et al., 2011), I conducted one-on-one task-based interviews with seventeen participants over a period of three months. Interviews enable researchers to explore the thinking of participants in a way that is not possible with normal paper-and-pencil tests. To date I have not found any trace of research conducted in mainstream or LSEN schools in South Africa to determine the conceptions and misconceptions of learners' multiplicative reasoning.

The task-based interviews consisted of ten classes of multiplication problems, as suggested in the relevant literature and discussed in my literature review (see section 2.3.2 in Chapter 2). These questions covered different ideas about multiplication which, when answered, provided a holistic picture of the participants' proficiency in multiplication with whole numbers. While choosing the context and the numbers to be used in the questions, I considered the following criteria:

- I chose multiplication of whole numbers, since Grade 6 learners only start working with decimal numbers and percentages and the multiplication of fractions towards the end of the school year (DBE, 2011b).

- My decision to choose the classes of multiplication problems, and the order in which I asked them, were based on a study of the literature and I included easy, moderate and difficult classes of problems (see section 2.3.2 of Chapter 2). Questions that were regarded as lower than, on a par with and above the learners' levels of development were necessary to give an accurate picture of their proficiency in multiplicative reasoning (Way, 1994).
- I chose the numbers in the questions with a specific purpose. The numbers are either numbers below fifteen or numbers with which participants should be familiar and can easily work with, such as 20, 30 and 50. Moreover, I selected all the numbers below ten at least once and in such a way that the numbers that the participants had to multiply were not repeated. One aim of the study was to find out whether the learners used additive of multiplicative calculation techniques to solve multiplication problems.
- Since some participants struggled with reading, each question given to them was printed on an index card and I also read it to them as many times as they wanted me to do so.
- I wrote all the numbers used in the questions as numbers and not in words (for example, 9 and not nine) since participants with reading difficulties might have been unable to pick out numbers in the question if they were written as words.

I wanted the participants to represent the multiplication problems in a specific order. After first asking them to use abstract representations to solve a problem, I asked them to solve it with semi-concrete representations, and finally with concrete representations. There were two reasons for this. First, I did not want the use of semi-concrete and concrete representations to influence their abstract thinking, since abstract representations are the most difficult. Second, when teachers assess learners at school, they need to show their calculations by using abstract symbols only. Teachers decide whether learners understand a concept by assessing only their abstract answers. By asking the participants to represent the problems in different ways using various external representations, I could explore the depth of their conceptual understanding, the extent of their schemas, and the extent of the connections between their internal representations. The order of the external representations and what I expected of participants is explained below.

- I started by giving the participants the question on an index card. I then read it to them, gave them a pen and paper and asked them to solve the problem by means of abstract representations, like they do at school.
- Next, to explore their reasoning, I asked them to explain their thinking while they were solving the problem. This also gave me insight into what they had written in their abstract representation and they had gone about solving the problem.
- Once they had finished explaining and solving the problem abstractly, I asked them to draw a picture to semi-concretely represent the problem and give the answer again.
- Finally, I asked them to use 3D material, such as 3D blocks, sweets, money, cans, bottles and pens, to concretely demonstrate their understanding of the problem.

I repeated this order for each question. After the participants had completed each representation I asked them, where necessary, to further clarify their answers. My questions, adapted from Tapper (2012, p. 98), included the following: Why did you decide to multiply / add / divide? How did you calculate the answer? Explain what you have drawn / built there. Why did you change your mind?

Table 3.1 provides a summary of the ten tasked-based interview questions. The class to which each multiplication problem belongs and the degree of complexity of each question are indicated in the second and third columns. This is based on the literature review and is discussed in section 2.3.2 of Chapter 2. My choice of the multiplication questions (Column 4) was influenced by the relevant literature (international) and my study of South African mathematics textbooks (Column 5) undertaken to determine in which grade these concepts are introduced in an abstract form (Column 6). In the case of material taken from an international source, I changed the language and scenarios to fit the South African context and language use. I also changed the numbers used in the questions to fit the criteria, as previously discussed.

Table 3.1: Summary of tasked-based interview questions

Question number	Class of multiplication problem	Complexity of question (Vergnaud, 1982)	Question	Authors (International, CAPS and textbook)	CAPS Curriculum (Abstract introduction of concept)
1	Equal sharing	Isomorphism of measures (one variable with constant)	There are 5 orange trees in a park. Each tree has 13 oranges on them. How many oranges are there altogether?	Adapted from Vergnaud (1983), DBE (2011a, p. 190), Mostert (2011, p. 66)	CAPS Grade 1 (Found in Grade 1 textbook, Term 3)
2	Constant price	Isomorphism of measures (one variable with constant)	If 1 sweet costs R7, how much will 9 sweets cost?	Adapted from Vergnaud (1983), DBE (2011a, p. 318), Bowie et al. (2012a, p. 48)	CAPS Grade 2 (Found in Grade 4 textbook, Term 1)
3	Uniform speed	Isomorphism of measures (one variable with constant)	Thabo rides his bicycle at a speed of 50 metres in 1 minute. How far will he ride his bicycle in 3 minutes?	Adapted from Vergnaud (1983), DBE (2011c, p. 120), Bowie et al. (2012a, p. 151)	CAPS Grade 9 (Found in a Grade 4 textbook, Term 3)
4	Times as many	Isomorphism of measures (one variable with constant)	Paul has 4 coloured pens. If Sarah has 8 times as many coloured pens as Paul, how many coloured pens does Sarah have?	Adapted from Mulligan (1992), Mulligan & Mitchelmore (1997), DBE (2011a, p. 222), Bowie et al. (2012a, p. 85)	CAPS Grade 2 (Found in a Grade 4 textbook, Term 2)
5	Simple proportion	Isomorphism of measures (one variable with constant)	If 4 sweets cost R10, how much will 12 sweets cost?	Adapted from Van de Walle et al. (2015), DBE (2011b, p. 120), Bowie et al. (2012a, p. 151)	CAPS Grade 4 (Found in a Grade 4 textbook, Term 3)
6	Array	Product of measures (two variables, answer third different unit	There are 8 rows of chairs in the school hall. There are 8 chairs in each row. How many chairs are there altogether?	Adapted from Mulligan (1992), Mulligan & Mitchelmore (1997), DBE (2011a, p. 190), Mostert (2011, p. 67)	CAPS Grade 1 (Found in textbook for Grade 1, Term 3)

7	Area	Isomorphism of measures (one variable with constant)	A piece of paper is 30 cm long and 20 cm wide. What is the area of the paper?	Adapted from Vergnaud (1983), DBE (2011c, p. 106), Bowie et al. (2013, p. 122)	CAPS Grade 8 (Found in a Grade 7 textbook, Term 2)
8	Volume	Product of measures (two variables, answer third different unit	What volume of water is needed to fill up a rectangular fish tank if the fish tank is 6 metres long, 2 metres wide and 4 metres high?	Adapted from Vergnaud (1993), DBE (2011c, p. 57), Bowie et al. (2012c, p. 203)	CAPS Grade 7 (Found in textbook for Grade 6, Term 4)
9	Combinations	Cartesian product	There are 3 brands of cool drinks (Coke, Pepsi and Sprite) which are available in both cans and bottles. If you want to buy one cooldrink, how many different possibilities are there?	Adapted from Mulligan (1992), Mulligan & Mitchelmore (1997), DBE (2011c, p. 152)	CAPS Grade 9
10	Consumption	Multiple proportion	A mother gives each of her 5 children 3 sweets per day. How many sweets will the children eat over a 3-day period?	Adapted from Vergnaud (1983), DBE (2011a, p. 287)	CAPS Grade 2 (Not found in any primary school textbooks)

The multiplication problems included problems of several difficulty levels and from various classes of multiplication. The participants had already learned to solve some of the multiplication problems in previous grades, whereas others had not yet been dealt with. I asked familiar and unfamiliar questions to explore their conceptual understanding and procedural fluency and to obtain a more in-depth and holistic picture of their proficiency in multiplicative reasoning.

3.5 Data analysis and interpretation

According to Cohen et al. (2011), data can be organised and presented in seven ways, namely as groups, individuals, issue or themes, research questions, instruments, cases, or narratives. For this study, I chose to organise and present the data as a group (the fifteen participants) so as to explore and describe the proficiency of the participants with their diverse learning difficulties. I discussed the data by posing task-based questions and discussed each question according to the order in which I asked participants to give their answers by using three representations, namely abstract (the most abstract), semi-concrete and concrete representations (the least abstract) – in that order. I kept this order when I analysed the data to explore at which level of abstractness the participants were the most proficient.

Clarke and Braun (2013) describe six steps for analysing qualitative data. The first step was to familiarise yourself with the data. I familiarised myself with the data by transcribing the video data myself, which helped me to become intimately familiar with what each participant had said and done. The second step was to code the data, which I called categorisation (Clarke & Braun, 2013). I did not code all the data, but only that which was relevant to the categories that I had decided on beforehand. Under each representation (abstract, semi-concrete and concrete), I structured the data into categories by using deductive reasoning based on my conceptual framework. I had chosen those categories with my secondary research questions in mind, which helped me to these questions. The categories were type of operation, operation concept, misconceptions, misrepresentations, calculation technique levels, calculation technique types and calculation errors, as summarised in Table 3.2. I used an Excel spreadsheet to organise my transcribed data from the video recordings and the photos that I took.

Table 3.2: Categories used for the analysis of my data

Abstract representations	Semi-concrete representations	Concrete representations
Operation type	Drawing operation concept	3D material operation concept
Misconception and/or misrepresentation from operation type	Misrepresentation from drawing	Misrepresentation from 3D material
Calculation technique level		
Calculation technique type		
Calculation error	Calculation error	

The third step in analysing data, according to Clarke and Braun (2013), is to search for themes, which I called subcategories. Under each category, I used both deductive and inductive reasoning to identify subcategories, as summarised in Table 3.2. When I used deductive reasoning to identify subcategories, I accepted that new subcategories might emerge. I used different colours to highlight phrases, which I used as indicators for similar subcategories. The fourth step is to review the subcategories to determine whether they sufficiently represent the participants' levels of understanding. According to Clarke and Braun (2013), it is necessary to consider the relationship between the subcategories. When doing this, I asked myself which subcategories contained the answers to which secondary research questions and then discussed those subcategories together. As a fifth step, I defined and named the subcategories (Clarke & Braun, 2013). Some subcategories, for instance those found under the calculation technique levels, had already named for my conceptual framework. Other subcategories, found under misconceptions and calculation errors, were named by identifying the essence of each one. The final step in the data analysis is to write everything up (Clarke & Braun, 2013). The data analysis and extracts of the data can be found in Chapter 4. I also interpret and discuss my research in the context of other studies included in my literature review. I used the secondary research questions to discuss and interpret the data analysis. Table 3.3 shows how the data analysis answered the secondary research questions. The combined answers to the three secondary research questions indicate learners' proficiency in solving multiplication problems.

Table 3.3: Parallel between secondary research questions, components of proficiency and analysis

Secondary research questions	Components of proficiency	Analysis
1. What is the status of learners' conceptual understanding of multiplication?	Conceptual understanding	Abstract, semi-concrete and concrete representations and explanations; misconceptions and misrepresentations
2. What is the level of the learners' procedural fluency related to multiplication?	Procedural fluency	Calculation technique levels; calculation errors
3. What is the nature of learners' strategic competence when solving multiplication problems?	Strategic competence	Effective representations and calculation technique types

When I analysed the representations (abstract, semi-concrete, and concrete) and determined the operation type indicated by each representation, together with the participants' explanations and the identified conceptions, misconceptions and misrepresentations, I could answer the first secondary research question relating to conceptual understanding. Analysing the calculation technique levels and the calculation errors made by participants, answered the second secondary research question, which related to their procedural fluency. The third secondary research question, which required information on the participants' strategic competence, was answered by analysing the efficiency of participants' representations and the calculation technique types they used. I was therefore able to answer the primary research questions about the learners' proficiency in multiplicative reasoning.

3.6 Quality measures

Participants solved ten selected multiplication problems as part of task-based interviews. These ten multiplication problems had to be credible and dependable. Since this study is qualitative in nature, I chose these two terms, rather than valid and reliable, which are more often used in quantitative research (Cohen et al., 2011; Creswell, 2007). Credibility, in my context, means that these multiplication problems should have tested what they were supposed to test (Cohen et al., 2011). Various types of credibility applied to this study, namely construct, content, concurrent and cultural credibility.

First, construct credibility means that the task-based interviews should measure the specific concept that it is supposed to measure (Cohen et al., 2011; Gareis & Grant, 2015; Kubiszyn & Borich, 2013). For the purpose of this study, I studied the concept of multiplication of whole numbers and asked the participants to solve ten classes of multiplication problem, which they had to represent in three different ways, namely abstractly, semi-concretely and concretely. Both the classes of problems and the different representations improved and deepened my understanding of the participants' conceptual understanding of the multiplication of whole numbers.

Second, content credibility means that adequate samples from the content that should be asked are included in the task-based interviews (Gareis & Grant, 2015; Kubiszyn & Borich, 2013). For this study, all ten multiplication questions were based on classes of multiplication problems that were identified in the literature consulted and about which authors agree that they are credible for multiplicative reasoning and cover the various classes of multiplication problems.

Third, concurrent credibility means that the types of problems that were asked correlated with similar questions in the field (Cohen et al., 2011). All the questions were adapted from the work of experts in the field of multiplicative reasoning and are indicated as such in Table 3.1. I also verified when each class of multiplication problem was introduced in the South African school curriculum and how each class of multiplication problem is presented in South African textbooks (see Table 3.1).

The last is cultural credibility, which required that the language and culture of the participants were taken into consideration when questions are formulated (Cohen et al., 2011). All the cultural groups in South Africa are familiar with the names and situations used in the questions. I asked colleagues who belong to cultures other than mine to look at the multiplication questions to make sure that they contained no cultural bias, that the reading level was appropriate and that there are no grammatical errors (Gareis & Grant, 2015).

Dependability is another important quality measure when conducting qualitative research. Dependability is concerned with the research situation and the factors affecting the researcher and the participants (Cohen et al., 2011). One way to ensure dependability in qualitative research is to ensure inter-rater reliability. This implies that someone who is not involved in the research should also be asked to analyse and code the recorded data. Inter-rater reliability is calculated by using the formula: The number of times two observers agree, divided by the number of possible opportunities to agree, calculated as a precentage (Cohen et al., 2011). The higher the level of agreement, which should be more than 90 per cent, the more dependable the study is (Cohen et al., 2011). For this study, I asked a colleague to evaluate the categories and subcategories of the transcribed data in conjunction with the tabels where I summarised my categories and subcategories. She evaluated about 25% of my data, which included the categorising of the data for fifteen participants' first two questions they solved and half of the participants for Question 3. There were 241 possible opportunities to agree and 229 times we did agree, which gives 95 per cent agreement.

3.7 Ethical considerations

Permission to conduct my research was obtained from the Ethics Committee of the University of Pretoria and the Gauteng Department of Education, and informed consent for the task-based interviews and video-recordings was obtained from the principals of the three LSEN schools, the teachers and the participants' parents. I also obtained informed assent and permission to make video-recordings of the interviews form the participants. Before each interview, I took time to explain to the participants that they were free to withdraw their participation at any time. Even though these learners had various learning problems, they understood the concept of free participation and I made it easy for them to stop the interview at any time.

When conducting research, anonymity, privacy and confidentiality of the participants are important. Anonymity and privacy were accomplished by giving each participant a pseudonym, e.g. Learner 1, Learner 2, etc. The schools are referred to as School A, School B and School C. The video-recordings of the task-based interviews were done in such a way that if at all possible, the participants' faces were not visible. Where this was not possible, the videos were password protected to maintain anonymity. Furthermore, I did not use the names of the participants during the interview. Confidentiality was maintained by providing an envelope into which they could put their replies. This ensured that only I knew who had given permission for the research. Furthermore, since my participants were learners with special needs I made sure that they were not harmed my being sensitive towards the emotional feedback that they gave me. If participants asked that they did not want to continue, I stopped the interview, or when they stuggled to answer a question I moved on and did not force them to give me an answer.

3.8 Summary

In this chapter I explained why critical realism, as a paradigm, underpinned my study. I discussed why the focus was on ontology rather than epistemology. I used a qualitative research approach with a single-case study design and selected a convenient sample of fifteen participants with learning difficulties from three LSEN schools in Pretoria, with the fifteen participants forming a group as the unit of analysis. I collected the data through one-on-one task-based interviews, which consisted of ten multiplication questions. I asked the participants to solve each problem by using abstract, semi-concrete and concrete representations. After the participants had solved the problems by using different representations, I asked them to explain their reasoning. Thereafter, I categorised and analysed their task-based interviews in predetermined categories in order to answer my secondary research questions. Under each category I either chose subcategories (deductive), or coded (inductive) those parts that did not have pre-determined categories. Finally, I described the quality criteria and the ethical considerations of this study. In the next chapter I will present, analyse and discuss the data.

CHAPTER 4

DATA PRESENTATION AND ANALYSIS

4.1 Introduction

This study explores the multiplicative reasoning proficiency of Grade 6 learners with learning difficulties. In Chapter 1, I established that, for the purpose of this study, multiplicative reasoning proficiency could be measured by a multi-dimensional approach involving investigating participants' conceptual understanding, their procedural fluency and their strategic competence. This multi-dimensional approach was discussed in my conceptual framework in Chapter 2, where I illustrated the interplay between participants' internal and external interpretations. In Chapter 3, I explained that this research was grounded in critical realism, which means that only that which is observable can be measured. In order to have enough measurable data, I interviewed seventeen participants (labelled Learner 1 to Learner 17). Learners 1 and 2 were used for my pilot study. As explained in Chapter 3, I made certain changes before I continued interviewing the rest of the participants. The data of fifteen participants (Learners 3 to 17) was therefore analysed and discussed. Data were collected by way of one-on-one task-based interviews and deductive and inductive reasoning were used to analyse my data for each of the ten questions the participants had to answer. I analysed each question separately and under each question presented the data in the order of the representations I had asked the participants to use, namely abstract, semi-concrete and concrete representations. I selected the following categories: conceptions, misconceptions, misrepresentations, calculation technique levels, calculation technique types and calculation errors, and used indicators to identify subcategories through both inductive and deductive reasoning in each category. The subcategories are summarised in tables throughout the presentation and discussion of the data.

4.2 Participants' backgrounds

The fifteen participants attended three different special needs schools in Pretoria. Five of the participants turned twelve, which is the usual age for Grade 6, seven turned thirteen, two turned fourteen and one turned fifteen during the year in which I conducted my research. The group included seven boys and eight girls. Their achievements in mathematics during their first term in Grade 6 ranged from 35% to 96%. All the participants experienced some form of learning difficulty. The teachers indicated, among others, the following learning difficulties: severe learning disorder, possibility of mild intellectual disorder, attention-deficit hyperactivity disorder (ADHD), attention-deficit disorder (ADD), difficulties with learning abstract content, language problems, such as struggling to read and understand word problems, damage to the language centres of the brain due to near drowning, dyslexia, possibility of dyscalculia, Tourette's syndrome, Asperger's disorder, severe behavioural problems, emotional problems, epilepsy, premature birth and slow learning. Twelve of the participants were Afrikaans speaking.

4.3 Question 1: Combined categorisation, analysis and discussion

The first task-based question was: *There are 5 orange trees in a park. Each tree has 13 oranges on them. How many oranges are there altogether?* Question 1 represented the 'equal sharing' class of multiplication problems (Greer, 1992; Mulligan, 1992; Vergnaud, 1983) and, together with Questions 2 and 3, was categorised under 'equal groups' (see section 2.3.2.1 in Chapter 2 for a full explanation). All the participants answered Question 1. Since it was the first question asked, I had explained in greater depth than for the subsequent questions what I expected of them. With all the questions I asked participants to start by solving the multiplication problem using abstract representations, i.e. numbers and symbols, as they are required to do at school. I then asked them to draw a picture, and then to use 3D blocks to solve the problem (see section 3.3.2 in Chapter 3 for a full discussion). I expected the participants to illustrate their conceptual understanding by drawing five trees with thirteen oranges on each and using the 3D blocks to make five groups of thirteen blocks each to represent the trees and oranges.

The ten task-based questions were developmental in nature. Conceptually, Question 1 was the least complex and therefore the least abstract, and should have been the easiest to visualise (trees and oranges on the trees). The abstract reasoning required to solve problems of this class is introduced in Grade 1 (Mostert, 2011; DBE, 2011a);

therefore participants should have been familiar with solving such problems. When participants were unsure of what to draw by using semi-concrete representations, I told them to draw trees and oranges to illustrate the problem. When they were required to concretely represent the problem, I gave them 3D blocks to represent the oranges. I provided prompts for Question 1 only, after which they understood what was expected of them.

Table 4.1 summarises the categories and subcategories of the data for all the participants' representations for Question 1. I colour coded similar subcategories for easier recognition. For all the questions discussed, I added pictures of the participants' work. I also included examples of the fifteen participants' explanations (in italics). Even though twelve of the participants were Afrikaans speakers, the explanations were given in English.

The data under each heading in Table 4.1 was analysed, starting with the participants' conceptions, then their misconceptions and misrepresentations. The levels and types of calculation techniques used, and the calculation errors were also analysed. Finally, the analysis of all the data for this question was discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels, as derived from my conceptual framework.

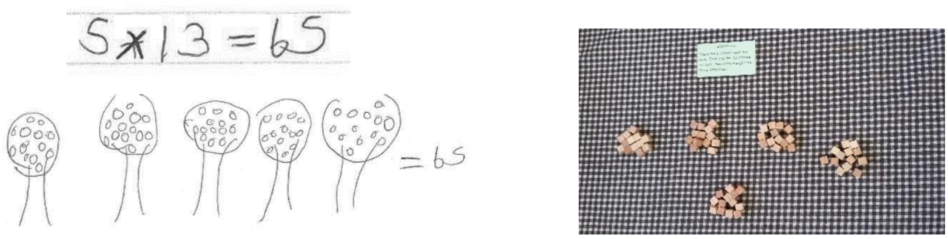
Table 4.1: Summary of the categories and subcategories of Question 1 for all participants and representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner[1]	**Written operation and answer**	**Verbal misconception**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 3	× (65)		Level 2B Double counting	Counted in		× Equal sharing (65)			× Equal sharing	
Learner 4	+ (18)		Level 1A Unitary counting	Counted on		Addition + (18)	Combination of measures		Intention was +	Abstract numbers
Learner 5	× (65)		Level 2C Algorithm	Column method		× Equal sharing (56)		Memory error (Answer 56)	× Equal sharing	
Learner 6	× (65)	Keyword (altogether)	Level 2C Algorithm	Column method		× Equal sharing (65)			× Equal sharing	
Learner 7	+ (18)	Added different units	Level 1A Unitary counting	Counted from one		Intention was equal sharing (36)		Counting error (Answer 36) Memory error (3 trees)	× Equal sharing	
Learner 8	+ (18)	Added different units	Level 1A Unitary counting	Counted on		× Equal sharing (18)		Disconnect between abstract and drawing (Answer 18)	Addition +	Combination of measures
Learner 9	× (65)		Level 2B Double counting	Counted in		× Equal sharing (65)			× Equal sharing	

[1] Learners 1 and 2 eventually became the subjects for my pilot study; therefore the fifteen participants reported on were numbered Learners 3 to 17. Fifteen participants answered Questions 1 to 4.

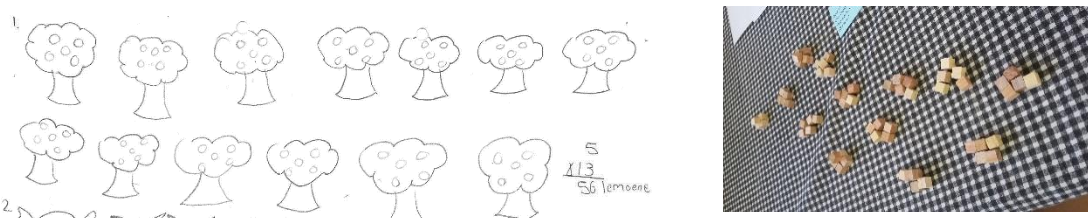
Learner 10	× (65)		Level 2C Algorithm	Column method		Intention was × (65)	Abstract numbers		Intention was ×	Abstract numbers
Learner 11	÷ (3)		Level 0A Guesser	Guessed		× Equal sharing (68)		Counting error (Answer 68)	× Equal sharing	
Learner 12	× (65)		Level 2B Double counting	Counted in		× Equal sharing (65)			× Equal sharing	
Learner 13	+ (18)	Keyword (altogether)	Level 1A Unitary counting	Counted on		× Equal sharing (18)		Disconnect between abstract and drawing (Answer 18)	× Equal sharing	
Learner 14	× (Say 65)		Level 2B Double counting	Counted in	Writing error (writes 56)	× Equal sharing (say 65)		Writing error (writes 56)	× Equal sharing	
Learner 15	× (65)		Level 2C Algorithm	Column method		× Equal sharing (65)			× Equal sharing	
Learner 16	+ (18)	Keyword (altogether)	Level 1A Unitary counting	Counted on		× Equal sharing (18)		Disconnect between abstract and drawing (Answer 18)	Addition +	Combination of measures
Learner 17	× (65)		Level 2E Derived multiplication fact	Times table and addition		× Equal sharing (65)			× Equal sharing	

Eight of the fifteen participants (Learners 3, 5, 6, 9, 12, 14, 15 and 17) were able to solve this multiplication problem by using all three types of representation. Those eight learners, whose names are marked in dark blue on Table 4.1, multiplied to solve this problem by using abstract representations. Moreover, they drew oranges and trees and the way they placed the 3D blocks was indicative of 'equal sharing'. Picture 4.1 shows Learner 9's drawing and 3D block representations of 'equal sharing', as well as his equation indicating multiplication. Learners 3, 6, 9, 14, 15 and 17 all drew similar pictures and arranged 3D blocks to represent five groups of thirteen, as required by the question.



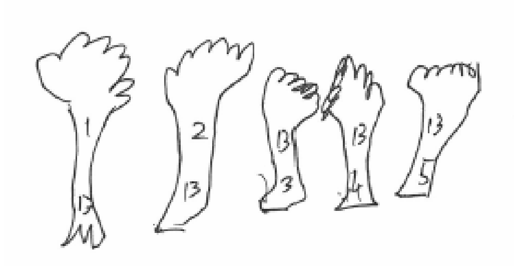
Picture 4.1: Question 1: Learner 9's equation, conceptual drawing and 3D blocks representing 'equal sharing'

Learner 5, whose response is illustrated in Picture 4.2 below, drew 13 trees with five oranges on each and placed the 3D blocks in 13 groups of five. Although this also demonstrated a conceptual understanding of 'equal sharing', it was not an exact representation of the question.



Picture 4.2: Question 1: Learner 5's conceptual drawing and 3D block representation

Picture 4.3 below shows Learner 12's semi-concrete drawing, which does not contain any semi-concrete representations.



Picture 4.3: Question 1: Learner 12's semi-concrete drawing

Even though Learner 12 also showed a conceptual understanding of the problem as one of 'equal sharing', his drawing was a combination of semi-concrete and abstract representations.

Learners 11 and 13, whose names are marked in light blue in Table 4.1, could solve the problem by using semi-concrete and concrete representations, but could not do so with abstract representations. Their drawings and concrete representations included five groups of thirteen 3D blocks, which are all indicative of 'equal sharing'. Learners 8 and 16's drawings represented 'equal sharing', while Learner 7's abstract use of the 3D blocks represented 'equal sharing' and Learner 10 could solve the problem with abstract representations only. (These learners' names are marked in purple in Table 4.1).

4.3.1 Question 1: Misconceptions and misrepresentations

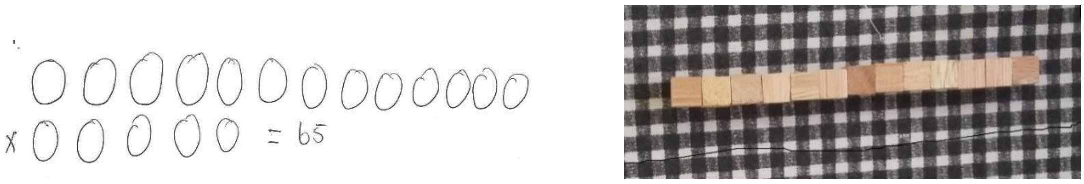
Misconceptions and misrepresentations were identified and categorised by way of inductive reasoning. For this question, two misconceptions were identified and categorised, namely 'added different units' and 'keyword' (see Table 4.1). 'Added different units' was categorised as a misconception as participants explained that they had added different units together, which could indicate a misinformed concept of addition. Moreover, I categorised 'keyword' as a misconception, since the participants focused on specific words in the problem to help them determine whether addition or multiplication was required. This could indicate that the participants had an incorrect schema and therefore a misconception that specific words in word problems could help them decide what to do.

Learners 7 and 8 explained that it was possible to add, in this case oranges and trees, together, which I categorised as a misconception, i.e. 'added different units' (see Table

4.1). Learner 7, for instance, explained: *I added the trees and the oranges*. Three participants (Learners 6, 13 and 16) explained that they could use the word ‘altogether’ in the problem to determine whether the problem was an addition or multiplication problem, which led to the categorisation of the misconception ‘keyword’. Learners 13 and 16 both thought that the word ‘altogether’ meant that they had to add, whereas Learner 6 thought it meant that he had to multiply.

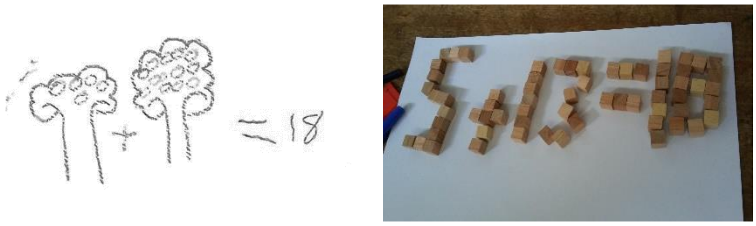
Learner 4 added, while Learner 11 divided to solve the problem with abstract representations. However, I was unable to identify any misconceptions based on their verbal explanations. Neither appeared to have any abstract understanding of the problem. For example, Learner 11 incoherently explained why she had added: *Because, if it is thirteen … because you want each to … There are five trees and each tree carry five oranges, and then you have to divide how many oranges, how many oranges is part of the tree … on the five trees.*

In both the semi-concrete and concrete representations, two misrepresentations were identified and categorised for this question, namely ‘abstract numbers’ and ‘combination of measures’. Participants who drew or replaced the numbers in their equations with circles or 3D blocks to represent the numbers in the equation were categorised as ‘abstract numbers’ misrepresentations. Learner 4’s concrete representation and Learner 10’s semi-concrete and concrete representations were categorised as a misrepresentation of ‘abstract numbers’. Learner 10 wrote a multiplication equation and it seemed that he had this equation in mind when he drew the problem and arranged the 3D blocks. He simply replaced the numbers in the equation with circles, as shown in Picture 4.4. This learner also struggled to represent the problem using the concrete 3D blocks. He packed out only thirteen 3D blocks to represent the oranges on one tree and did not add five 3D blocks to represent the trees.



Picture 4.4: Question 1: Learner 10's misrepresentation of 'abstract numbers'

Learner 4 was the only participant who understood the problem as addition in all three forms of representation. Her misrepresentation with the 3D blocks was also categorised as 'abstract numbers', as it seemed that she had had her abstract equation in mind and had used the 3D blocks to reproduce that equation (see Picture 4.5).



Picture 4.5: Question 1: Learner 4's misrepresentations of 'combination of measures' and 'abstract numbers'

Picture 4.5 also shows her semi-concrete drawing, which was categorised as 'combination of measures', using the term coined by Vergnaud (1982; 1992) to classify addition of this kind in which two numbers are combined (added) to calculate the answer. In the case of this study it is seen as a misrepresentation since all the problems required multiplication and not addition. Learners 8 and 16's concrete representations were also categorised as 'combination of measures'. Since both their equations involved addition, one could infer that the two separate rows of thirteen and five indicated addition. Learner 8's misrepresentation is illustrated in Picture 4.6. This participant placed fourteen 3D blocks in the first row, instead of thirteen.

Picture 4.6: Question 1: Learner 8's misrepresentation of 'combination of measures'

For this question, two misconceptions ('added different units' and 'keyword') and two misrepresentations ('abstract numbers' and 'combination of measures') could be identified and categorised. The levels and types of calculation techniques will be discussed next.

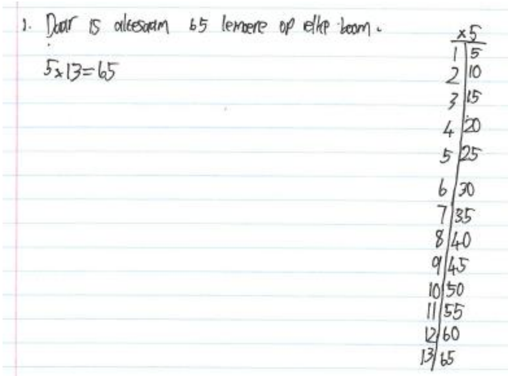
4.3.2 Question 1: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2) and the calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink respectively in Table 4.1). The calculation technique levels that were identified were: Level 0A (guesser), Level 1A (unitary counting), Level 2B (double counting), Level 2C (algorithms) and Level 2E (derived multiplication fact). The calculation technique types identified for solving Question 1 were: 'guesser', 'counted from one', 'counted on', 'counted in', 'column method' and 'times table and addition' (see Table 4.1).

Level 0 is ineffective as participants rely on non-calculation techniques to find the answer. Learner 11 was categorised on Level 0A (guesser), since she tried to divide five into thirteen and said: *I think it is about three.* She could not calculate the answer and simply guessed what it should be.

Level 1 calculation techniques are additive in nature and five participants (Learners 4, 7, 8, 13 and 16) added to calculate the answer and were therefore categorised on Level 1A (unitary counting). Two calculation technique types were identified and categorised as 'counted from one' and 'counted on'. Learner 7's calculation technique type was categorised as 'counted from one', since he made marks on the paper and then started counting the marks from one to calculate the answer. Learners 4, 8, 13 and 16's calculation technique type was categorised as 'counted on', since they started counting from 13 and then counted on another five.

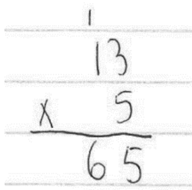
The Level 2 calculation techniques are considered to be multiplicative, which means that the participants were able to think multiplicatively when solving the problem (Carrier, 2014). Nine participants used multiplicative calculation techniques to solve this problem. Of these nine, four (Learners 3, 9, 12 and 14) were categorised on Level 2B (double counting). These participants' calculation technique type was categorised as 'counted in', since they used their fingers to count in 5s. The only exception was Learner 12, who used a paper-based calculation technique that his teacher had taught them (see Picture 4.7).



Picture 4.7: Question 1: Learner 12's paper-based multiplication method

Learner 14 started out using the 'column method', but forgot how to calculate using this method and then changed to the 'counted in' type of calculation technique. I therefore categorised her method as 'counted in' and not as 'column method'.

Four of the other nine participants (Learners 5, 6, 10 and 15) were categorised on Level 2C (algorithms) and their calculation technique type was categorised as the 'column method'. Picture 4.8 shows how Learner 15 calculated her answer by using the 'column method'.



Picture 4.8: Question 1: Learner 15's use of the 'column method'

The last of the nine participants (Learner 17) was categorised on Level 2E (derived multiplication facts), which is considered one of the highest calculation technique levels (Hurst & Hurrell, 2014; Zhang et al., 2011). Her calculation technique type was categorised as 'times table and addition', since she used her knowledge of the five times table and then added another five. She explained: *I knew that … uhm… that 5 times 12 is 60 and then added 5.*

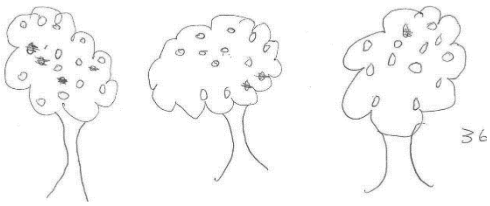
Nine participants used multiplicative calculation techniques, while five used additive calculation techniques to calculate the answer to this problem. One participant guessed the answer. The calculation errors will be discussed next.

4.3.3 Question 1: Calculation errors

Calculation errors were considered only in respect of the abstract and semi-concrete representations. I did not report on any calculation errors made in the concrete representations as I had not considered the answers of all the participants and since most of their answers had been influenced by their answers obtained by using abstract and semi-concrete representations, which would not have provided valid data. Inductive reasoning was used to identify and categorise calculation errors.

Four calculation errors were identified and categorised for Question 1, namely ‘writing error’, ‘memory error’, ‘counting error’ and ‘disconnect between abstract and drawing’ (see Table 4.1). The first calculation error that was identified and categorised was a ‘writing error’. Although Learner 14 stated that the answer was 65, she had written it as 56 in both the abstract and semi-concrete representations.

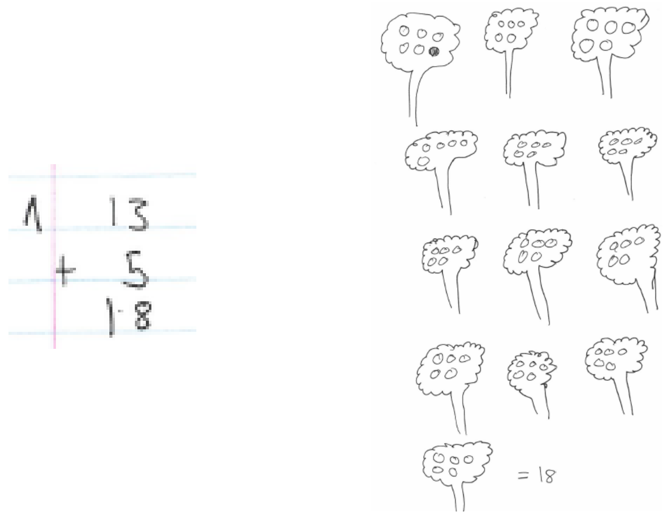
The second calculation error was categorised as a ‘memory error’. Although Learner 5’s abstract representation gave the answer as 65, she wrote it as 56 after she had drawn her picture of the problem (see Picture 4.2). Learner 7 drew only three trees, in spite of the fact that the card containing the question was in front of him. He was convinced that there should be three trees and not five (see Picture 4.9). I asked him: *Why do you have three groups, three trees?* He answered: *I have three trees, because the question asks that there are three trees, and each tree there were thirteen oranges.*



Picture 4.9: Question 1: Learner 7's 'memory error' and 'counting error'

The third calculation error was categorised as a 'counting error'. Learners 7 and 11 both made 'counting error'. Learner 7 drew only three trees and also struggled with counting. Picture 4.9 shows that he had scratched through some oranges. He recounted the oranges several times. In spite of that, his trees still did not have the same number of oranges on them, i.e. there were 14, 11 and 12 oranges on the three trees respectively. Learner 11 also made a 'counting error' when she counted 68 instead of 65 oranges on the trees.

The last calculation error, categorised as a 'disconnect between abstract and drawing', was evident in the calculations of Learners 8, 13 and 16. This calculation error implies that while the participants had drawn their pictures using semi-concrete representations, their answers reflected that they had been calculated based on the abstract representations. Picture 4.10 shows Learner 13's drawing, in which the answer 18 does not correspond with the number of oranges in his picture, but rather with the answer provided in his abstract representation. These three participants appeared to be unaware of the fact that what they had drawn did not reflect their answers. They added using their abstract representations and illustrated 'equal sharing' with their semi-concrete representations.



Picture 4.10: Question 1: Learner 13's calculation error of 'disconnect between abstract and drawing'

The four calculation errors that were identified and categorised were 'counting error', 'writing error', 'memory error' and 'disconnect between abstract and drawing'. I will discuss the analysis of the data next.

4.3.4 Question 1: Discussion of the analysis

Question 1 was the least conceptually complex in the category 'equal groups', which included Questions 1 to 3. This question should have been easy to visualise as the participants had been solving this class of problem since Grade 1 (DBE, 2011a; Mostert, 2011). My analysis revealed that eight of the 15 participants (Learners 3, 5, 6, 9, 12, 14, 15 and 17) could solve the problem of 'equal sharing' without any difficulty in all the representational forms. Their semi-concrete and concrete representations of five groups of thirteen made it possible to infer that they had a conceptual understanding of 'equal sharing'. Moreover, their procedural fluency could be inferred as they used multiplicative calculation techniques to solve their multiplication equations. According to Hiebert and Carpenter (1992), the more connections there are between different representations, the better the learners' understanding. It could therefore be concluded that the abovementioned eight participants had a clear understanding of the connections between the different types of representations as all their representations indicated the 'equal sharing' class of problem, and therefore a good schema and scheme of 'equal sharing'. Since these participants had a good interconnected schema of the 'equal sharing' between the three representations, they were able to consider the correct multiplicative concept-in-action, which allowed them to choose their most effective theorem-in-action to solve the problem (Vergnaud, 1998; 2013a; 2013b). This in turn could indicate good procedural fluency (see conceptual framework, Figure 2.1 in Chapter 2).

A further two participants (Learners 11 and 13) could represent 'equal sharing' of five groups of thirteen with both their semi-concrete drawings and the 3D blocks, but did not use multiplication for their equations. It could therefore be inferred that they understood 'equal sharing' conceptually, but not in its abstract form. One could conclude that their schema of 'equal sharing' was limited, since the abstract schema was lacking and they used additive and division schemes to solve the problem.

Although Learners 8 and 16 could illustrate the 'equal sharing' of five groups of thirteen, their abstract and concrete representations were indicative of addition, whereas only Learner 7's concrete representation, and not his abstract and semi-concrete representations, indicated 'equal sharing'. According to Ayub et al. (2013), if only one type of representation is correct, it could be an indication that limited connections were made. One could therefore conclude that Learners 8 and 16 had only a semi-concrete schema, while Learner 8 had only a concrete schema of this problem, which could have led to the use of an incorrect scheme to solve the problem – in this case addition. Their conceptual understanding was limited to either semi-concrete or concrete representation. One could infer that while these five participants (Learners 7, 8, 11, 13 and 16) did have limited conceptual understanding, they lacked procedural fluency. It is possible that they used the incorrect scheme because I had asked them to solve this problem with abstract representations first. When they considered their options by relying on their concept-in-action, they considered only their abstract schemas and not their semi-concrete schemas, which may have led them to make an incorrect choice between addition and division. This could indicate that they lacked the necessary strategic competence.

One participant (Learner 10) could solve this problem only by using abstract representations, which could indicate limited connections between the different representations and only an abstract schema. This could further indicate that he had memorised a procedure, but lacked conceptual understanding (Ayub et al., 2013).

Only one participant (Learner 4) could not solve this problem as she had used addition in all her representations. It could be concluded that she did not have an ‘equal groups’ schema in any of the representations and no scheme, and therefore lacked the procedural fluency or strategic competence required to solve this problem.

Eight of the fifteen participants (Learners 3, 5, 6, 9, 12, 14, 15 and 17) could solve the problem with all the representations. Learners 11 and 13 could solve it with both semi-concrete and concrete representations, Learner 7 could solve it only with concrete representations, Learners 8 and 16 could solve it with semi-concrete representations and Learner 10 could solve it with abstract representations only. In total, fourteen of the fifteen participants could solve the problem with at least one of the representations. Only Learner 4 could not solve this problem with any of the representations.

4.3.4.1 Question 1: Discussion of misconceptions and misrepresentations

Two misconceptions were identified and categorised, namely ‘keyword’ and ‘added different units’. Three participants (Learners 6, 13 and 16) thought the word ‘altogether’ in the question could help them decide what operation to use, while Learners 7 and 8 thought that it was possible to add two different units together (i.e. trees and oranges). Both of these misconceptions could indicate a problem with how they were taught or it could be their own method that they think might indicate to them what operation to use. Furthermore, teachers might neglect to emphasise the fact that different units (e.g. oranges and trees) cannot be added together.

Two misrepresentations were identified and categorised, namely ‘abstract numbers’ and ‘combination of measures’. The semi-concrete and/or concrete misrepresentations included the answer given by Learner 10, who had replaced his abstract equation of numbers with circles, and Learner 4, who had used the 3D blocks to represent her abstract equation (she used the 3D blocks to form the numbers 5 and 13). This misrepresentation of ‘abstract numbers’ could possibly indicate that those participants were unable to visualise the actual trees and the oranges on the trees, and had only an abstract equation schema and no other way to visualise this problem. The visualisation of abstract concepts is characteristic of learners with learning difficulties (Allsopp et al., 2007).

Three participants (Learners 4, 8 and 16) drew pictures or arranged the 3D blocks to reflect the misrepresentation of 'combination of measures'. This means that the two numbers in the problem were added together. This misrepresentation of 'combination of measures' could indicate that those participants may not have any schema and scheme for the 'equal sharing' class of problem, which may explain why they used addition instead of multiplication. This misrepresentation suggests the total absence of any conceptual understanding of 'times as many' and a lack of procedural fluency.

4.3.4.2 Question 1: Discussion of the calculation technique levels and types

Nine of the fifteen participants (Learners 3, 5, 6, 9, 10, 12, 14, 15 and 16) were categorised on Level 2, whereas four (Learners 3, 9, 12 and 14) were categorised on Level 2B (double counting) and had used the calculation technique type categorised as 'counted in'. Four participants (Learners 5, 6, 10 and 15) were categorised on Level 2C (algorithms) and the calculation technique type that they had used was categorised as the 'column method'. Learner 17 was categorised on Level 2E (derived multiplication fact) and the calculation technique type she had used was categorised as 'times table and addition'. One could infer that the abovementioned participants had an abstract schema and a multiplicative scheme for this 'equal sharing' class of problem. It can be concluded that since their abstract concept-in-action was multiplicative, they were able to choose an appropriate multiplicative theorem-in-action to solve this problem, thus demonstrating good procedural fluency. However, the majority of the nine participants who had used multiplicative schemes used calculation technique types that were less efficient than the more efficient calculation technique type used by, for example, Learner 17. The lower the level of the calculation technique, the more effective the technique will be, with Level 2F being the highest and most efficient level (see the conceptual framework, Figure 2.1 in Chapter 2). One could therefore infer that the majority of the participants were less strategically competent, since they did not use the most effective calculation technique type to solve this problem.

Five of the fifteen participants (Learners 4, 7, 8, 13 and 16) were categorised on Level 1A (unitary counting), and except for Learner 7, who 'counted from one', their calculation technique type was categorised as 'counted on'. These participants therefore used additive calculation technique types to solve this problem and also understood the problem to be an additive problem. It could be inferred that they lacked a correct schema and scheme of ‘equal sharing’ since their schema and scheme were additive and could therefore not be used to solve this problem. Because their concept-in-action was incorrect, they chose an incorrect theorem-in-action. Learner 11 guessed the answer, which suggests that she did not have any schema and scheme for solving this problem. These five learners had no procedural fluency and lacked the strategic competence required to solve this problem.

4.3.4.3 Question 1: Discussion of calculation errors

Four calculation errors were identified and categorised, namely ‘writing error’, ‘memory error’, ‘counting error’ and ‘disconnect between abstract and drawing’. It is very difficult to reliably determine the reasons for these calculation errors. One possible reason could be the learning difficulties experienced by the participants. Learner 14 wrote the answer as 56 in both her abstract and semi-concrete representations, but verbally indicated that the answer was 65. This ‘writing error’ could possibly be ascribed to dyslexia, which could explain why the written numbers were inverted. The ‘memory error’ made by Learner 5, who wrote 65 in her abstract representations, but 56 when she drew her picture could possibly be ascribed to either ADHD or dyslexia. Learner 7’s ‘memory error’ (thinking that there were three trees instead of five) might also be ascribed to ADHD, dyslexia or a problem in remembering information due to his learning difficulties. The ‘counting errors’ made by Learners 7 and 11, who repeatedly recounted the circles they had drawn, could possibly be ascribed to organisational or visual / spatial problems, since they recounted their circles over and over, each time either missing circles or counting some of them twice. The ‘disconnect between the abstract and drawing’ calculation error (Learner 8) could have been a result of the fact that she was not used to having to draw pictures of word problems. She did not seem to make a connection between what she drew and the answer she gave. She repeated the answer derived from the abstract representations and did not refer to her picture for verification. When learners are asked to illustrate a problem, teachers could point out to them that the number of oranges drawn, for example, provides an indication of what the answer should be.

4.4 Question 2: Combined categorisation, analysis and discussion

The second task-based question, which all the participants were required to answer, was: *If 1 sweet costs R7, how much will 9 sweets cost?* Question 2 represented the 'constant price' class of multiplication problems (Vergnaud, 1983). Since the participants now understood what was expected of them, there was no need for lengthy explanations. As for Question 1, the participants had to solve the problem using abstract symbols first, followed by semi-concrete drawings and finally 3D material. I expected the participants to illustrate their understanding by drawing nine sweets and writing R7 at each sweet. The concrete objects I gave them were sweets and money and I expected them to lay out nine sweets and place one R5 and one R2 coin with each sweet.

Question 2 was conceptually more complex that Question 1, since money was one of the units. Participants were expected to know that R7 is made up of R5 and R2. Although the abstract reasoning required to solve problems in this class should be introduced in Grade 2 (DBE, 2011a), I scrutinised several textbooks and could only find this type of problem in a Grade 4 textbook (Bowie et al., 2012a). The participants should therefore have previously encountered problems of this kind. Money problems play an important role in the South African school curriculum and are dealt with as a separate topic every year during the primary school phase.

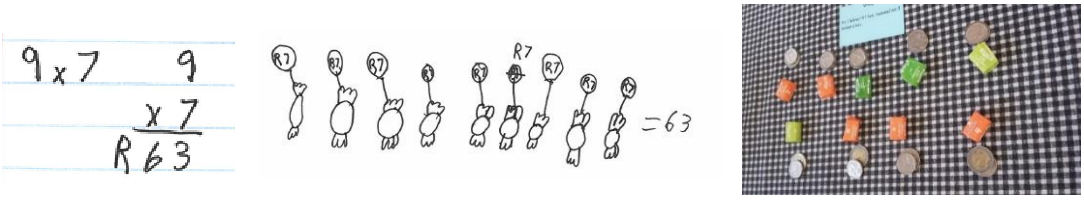
Table 4.2 contains a summary of the categories and subcategories of data obtained from all the participants' representations in their answers to Question 1. Similar subcategories have been colour coded for easier recognition. The data analysed under each heading in the table starts with the participants' conceptions, misconceptions and misrepresentations, after which the levels and types of calculation techniques and calculation errors are analysed. Finally, the analysis of all the data for this question is discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

Table 4.2: Summary of the categories and subcategories of Question 2 for all participants and all the representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner	**Written operation and answer**	**Verbal misconception**	**Level of calculation technique and name**	**Type of calculation technique used**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 3	× (63)		Level 2A Doubling Level 1A Unitary counting	Doubling Counted on		× Constant price (61)		Counting error (Answer 61)	× Constant price	
Learner 4	+ (15)		Level 1A Unitary counting	Counted on	Wrong number (wrote 9 + 6)	Addition + (15)	Combination of measures	Wrong number (wrote 9 + 6)	Intention was +	Abstract numbers
Learner 5	× (63)		Level 2B Double counting Level 1C Repeated addition	Counted in Counted on		× Constant price (63)			Intention was ×	Answer
Learner 6	× (63)		Level 2E Derived multiplication fact	Times table and subtraction		× Constant price (63)			× Constant price	
Learner 7	+ (17)	Added different units	Level 1A Unitary counting	Counted on		Intention was × (58)	Abstract numbers	Counting error (Answer 58)	× Constant price	
Learner 8	× (70)		Level 1C Repeated addition	Counted on	Tracking error (Answer 70)	Intention was × (70)	Abstract numbers	Counting error (Answer 70)	Intention was ×	Answer
Learner 9	+ (64)		Level 1C Repeated addition	Counted on	Counting error (Answer 64)	× Constant price (63)			× Constant price	

Learner 10	× (63)		Level 2F Known multiplication fact	Times table		Intention was × (63)	Abstract numbers		Intention was ×	Abstract numbers
Learner 11	× (63)		Level 1C Repeated addition	Counted on		× Constant price (63)			[Unable to determine]	
Learner 12	× (63)		Level 1C Repeated addition	Counted on		Intention was × (69)	Abstract numbers	Counting error (Answer 69)	Intention was ×	Answer
Learner 13	+ (17)	Keyword (How much)	Level 1A Unitary counting	Counted on	Counting error (Answer 17)	Intention was + (17)	Answer		× Constant price	
Learner 14	× (63)		Level 2B Double counting	Counted in		× Constant price (63)			× Constant price	
Learner 15	× (63)		Level 2B Double counting	Counted in		Intention was × (63)	Abstract numbers		Intention was × (63)	Abstract numbers
Learner 16	+ (16)		Level 1A Unitary counting	Counted on		× Constant price (16)		Disconnect between abstract and drawing (Answer 16)	× Constant price	
Learner 17	× (63)		Level 2E Derived multiplication fact	Times table and subtraction		× Constant price (63)			× Constant price	

Four of the fifteen participants (Learners 3, 6, 14 and 17, whose names are marked in dark blue in Table 4.2) could solve this multiplication problem using all three forms of representation. To solve the problem, they multiplied by using abstract representations. They placed and then drew nine sweets and allocated R7 to each sweet, which is indicative of ‘constant price’ (see Picture 4.11).



Picture 4.11: Question 2: Learner 6’s equation, conceptual drawing and 3D material representing ‘constant price’

The example of Learner 6’s work, shown in Picture 4.11, demonstrates his representation of ‘constant price’ in the semi-concrete drawing and using the 3D material. It also shows the abstract equation that he used to solve the problem.

Learners 9 and 16 (whose names are marked in light blue in Table 4.2) could solve the problem by using both semi-concrete and concrete representations, but could not solve it by using abstract representations. Their drawings and their use of the 3D material indicated an understanding of ‘constant price’ as R7 had been allocated to each of the nine sweets. Seven of the fifteen participants could solve this problem by using either semi-concrete or abstract representations, or both. Of these seven participants, whose names are marked in purple in Table 4.2, two (Learners 5 and 11) could solve the problem by using both semi-concrete and abstract representations; two (Learners 7 and 13) used only concrete representations; and three (Learners 10, 12 and 15) could solve it only by using abstract representations.

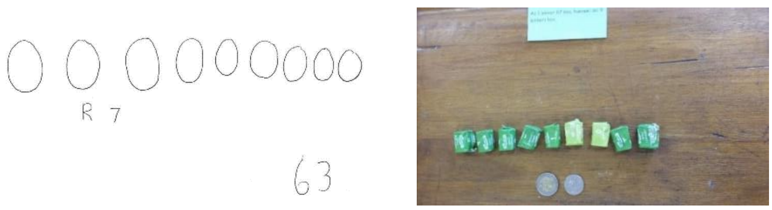
4.4.1 Question 2: Misconceptions and misrepresentations

As in the responses to Question 1, two misconceptions were identified and categorised, namely ‘added different units’ and ‘keyword’ (see Table 4.2). Learner 7 explained that it was possible to add, in this instance sweets and money, together, which I categorised as a misconception and described as ‘added different units’. He explained his view as follows: *Because they said, if 1 sweet costs R7, how much will*

9 cost? So, I added 1 and 7 and 9 altogether. Learner 13 explained that the phrase 'how much' in the problem indicated addition, *because it says, if 1 sweet costs R7, how much will 9 sweets cost? So, how much?* This was categorised as a 'keyword' misconception.

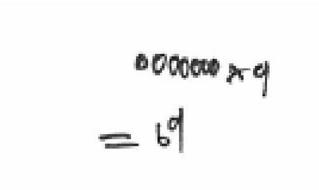
Learners 4 and 16 added the numbers together to solve the problem using abstract representations and I could not identify any misconceptions in their explanations. Both seemed to lack an abstract understanding of the problem. For example, Learner 16 explained his reasoning as follows: *I thought it was going to be a subtraction sum, and then I tried the minus sum. And now I think it is an addition sum.*

In the responses to this question, three misrepresentations were identified and categorised, namely 'abstract numbers', 'answer' and 'combination of measures' (see Table 4.2). Six participants' semi-concrete and/or concrete representations were categorised as 'abstract numbers'. Two (Learners 10 and 15) of the six participants' semi-concrete and concrete representations were categorised as a misrepresentation of 'abstract numbers', since their pictures and their 3D materials could indicate that they had their equation in mind when they solved the problem using abstract numbers. They simply replaced the numbers of the multiplication equation with sweets and money (see Picture 4.12).



Picture 4.12: Question 2: Learner 15's misrepresentation of 'abstract numbers'

The semi-concrete drawings of another three of these six participants' (Learners 7, 8 and 12) indicated a misrepresentation of 'abstract numbers' as Learners 7 and 8 each drew nine sweets and Learner 12 drew seven sweets and multiplied the number of sweets by 9, inversing the units, which still represent his equation and not the multiplication concept of 'constant price' (see Picture 4.13).



Picture 4.13: Question 2: Learner 12's misrepresentation of 'abstract numbers'

The concrete representation of the last of the seven participants (Learner 4) was a replication of the numbers in her equation and therefore also a misrepresentation of 'abstract numbers' (Picture 4.14).

Picture 4.14: Question 2: Learner 4's misrepresentation of 'abstract numbers'

Learner 13's drawings and Learners 5, 8 and 12's 3D material were categorised as a misrepresentation of the word 'answer', since they represented the answer to the problem instead of the problem. For his abstract presentation, Learner 13 drew seventeen circles. Learners 5, 8 and 12 calculated an amount of R63, which was the answer to the question, instead of placing R7 with each sweet. Learner 12 mistakenly counted out twelve sweets instead of nine. Picture 4.15 illustrates Learner 5's misrepresentation of the word 'answer'.

Picture 4.15: Question 2: Learner 5's misrepresentation of 'answer'

Learner 4's semi-concrete representation was categorised as a misrepresentation of a 'combination of measures', as she had drawn nine and then seven sweets separately.

It was not possible to identify and categorise Learner 11's concrete representation, as she had used an incorrect amount of money and made two groupings of the sweets and money, instead of placing nine sweets and R7 with each sweet. Picture 4.16 shows her misrepresentation when using the 3D material, when she placed four sweets with R4 and five sweets with R5.

Picture 4.16: Question 2: Learner 11's misrepresentation

For this question, two misconceptions ('added different units' and 'keyword') and three misrepresentations ('abstract numbers', 'answer' and 'combination of measures') were identified and categorised. The calculation technique levels and types will be discussed next.

4.4.2 Question 2: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2). The calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.2). The following calculation technique levels were identified and categorised: Level 1A (unitary counting), Level 1C (repeated addition), Level 2A (doubling), Level 2B (double counting), Level 2E (derived multiplication fact) and Level 2F (known multiplication fact). The calculation technique types identified for solving Question 2 were: 'counted on', 'doubling', 'counted in', 'times table and subtraction' and 'times table' (see Table 4.2).

Level 1 calculation techniques are additive and eight participants were categorised on Level 1, since they added to calculate the answer. Four of those eight participants (Learners 4, 7, 13 and 16) were categorised on Level 1A (unitary counting) as their calculation technique type was categorised as 'counted on'. They 'counted on' starting with either nine or seven and then added the other number on. Learners 8, 9, 11 and 12 were categorised on Level 1C (repeated addition). Their calculation technique type was also categorised as ‘counted on’. Counting on their fingers, they added seven or nine each time while keeping track of how many times they had added the seven or nine.

Learners 3 and 5 used a combination of additive and multiplicative calculation technique types and their calculation techniques were categorised as Level 1A (doubling) and Level 2 (unitary counting). Learner 3’s calculation technique types were categorised as ‘doubling’ and ‘counted on’, since she first doubled fourteen and then counted on her fingers while keeping track of how many she had counted on. She counted out loud, saying: *7, 14, 14 + 14 is 28… 31, 32… is 42 plus 14 is 56, 57, 58, 59, 60, 61, 62, 63. So it is 63*. Learner 5’s calculation technique levels were categorised on Level 2B (double counting) and Level 1C (repeated addition) respectively, since her calculation technique types were categorised as ‘counted in’ and ‘counted on’. She counted in 7s as she wrote 7, 21, 28 and ‘counted on’ 7 each time.

Level 2 calculation techniques are considered to be multiplicative, meaning participants were able to think multiplicatively when solving the problem (Carrier, 2014). Five participants used multiplicative calculation techniques to solve this problem. Learners 14 and 15’s calculation techniques were categorised on Level 2B (double counting). Their calculation technique type was categorised as ‘counted in’. Learners 14 and 15 counted in 9s to calculate their answers. Learners 6 and 17’s calculation techniques were categorised on Level 2E (derived multiplication facts), since their calculation technique type was categorised as ‘times table and subtraction’. They both multiplied ten by seven and then subtracted seven. Learner 17 explained: *So, I multiplied 7 with 10, it is 70. And then if you minus one 7, it is 63.* Learner 10’s calculation technique was categorised on Level 2F (known multiplication facts), and since he knew that nine times seven is 63 his calculation technique type was categorised as ‘times table’.

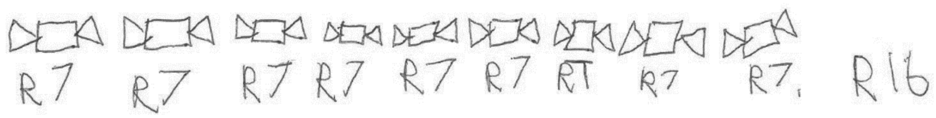
To calculate the answer to this problem, five participants used multiplicative calculation techniques, eight used additive calculation techniques and two used a combination of additive and multiplicative calculation techniques. The calculation errors will be discussed next.

4.4.3 Question 2: Calculation errors

Four calculation errors were identified and categorised, namely ‘counting error’, ‘wrong number’, ‘tracking error’ and ‘disconnect between abstract and drawing’ (see Table 4.2). Six participants made errors that were categorised as ‘counting errors’. Learners 9 and 13 made ‘counting errors’ when attempting to solve the problem using abstract representations and Learner 9 incorrectly added seven every time and finally calculated a total of 64, whereas Learner 13 added 9 and 7 and calculated a total of 17. Learners 3, 7, 8 and 12 counted incorrectly when they referred to their drawings to recalculate their answers. Learner 3 calculated R61, Learner 7 R58 and Learner 8 R70.

The second calculation error identified and categorised was ‘wrong number’. Learner 4 had used an incorrect number when adding in both her abstract and semi-concrete representations. She had added together 9 and 6 instead of 9 and 7. The third calculation error that was categorised was a ‘tracking error’. Learner 8 struggled to keep track of the number of times she had added 7 and added ten instead of nine 7s.

The last calculation error identified and categorised was a ‘disconnect between the abstract and drawing’. Learner 16 gave the answer as R16 when he drew his picture using semi-concrete representations, which represented his answer with the abstract representations. However, his drawing did illustrate the ‘constant price’ concept (see Picture 4.17).



Picture 4.17: Question 2: Learner 16’s misrepresentation of ‘disconnect between abstract and drawing’

The four calculation errors that were identified and categorised were ‘counting error’, ‘wrong number’, ‘tracking error’ and ‘disconnect between abstract and drawing’. A discussion of the analysis of the data follows below.

4.4.4 Question 2: Discussion of the analysis

Mathematically Question 2 is similar to Question 1, but it is slightly more complex, as one of the units to be calculated was money and the participants had to understand that two coins – a R2 and a R5 coin – had to be allocated to each of the nine sweets. Ideally the abstract reasoning required to solve problems of this type should be introduced in Grade 2 (DBE, 2011a). However, in the textbooks that I perused I only found such problems in a Grade 4 textbook (Bowie et al., 2012a). The participants should therefore have gained experience in solving problems of this nature. My analysis revealed that only four of the fifteen participants (Learners 3, 6, 14 and 17) could solve the problem of 'constant price' without any difficulty in all the representational forms. From their semi-concrete and concrete representations of nine sweets, with R7 placed with each sweet, one could infer their conceptual understanding of 'constant price'. Moreover, their procedural fluency could be inferred from their use of multiplicative calculation techniques to solve their multiplication equations. The one exception was Learner 3, who used both multiplicative and additive calculation techniques. This could indicate good connections between their different types of representations, and more connections indicate better understanding (Hiebert & Carpenter, 1992). Good connections between different types of representations could be inferred as all these learners' representations indicated the 'constant price' type of problem, and therefore a good schema and scheme of 'constant price'. Since these participants demonstrated a good interconnected schema of 'constant price' between the three representations, they were able to consider the correct multiplicative concept-in-action, which allowed them to choose their most effective theorem-in-action to solve the problem (Vergnaud, 1998; 2013a; 2013b). This in turn could indicate that they had good procedural fluency (see conceptual framework, Figure 2.1 in Chapter 2).

A further two participants (Learners 9 and 16) could use semi-concrete drawings and 3D material by placing R7 next to each of the nine sweets, but they did not use multiplication for their equations. Learner 16's abstract representations indicated that the problem had been perceived as being additive, while Learner 9 used repeated addition to calculate his answer. When asked about his abstract representation, it was clear that he had understood the problem to be additive. This is in line with what Kouba (1989) found in his study and explains why repeated addition was classified as an additive, and not as a multiplicative calculation technique. It could therefore be inferred that while the learners understood the 'constant price' conceptually, they did not understand it in the abstract. One could conclude that their schema of 'constant price' was limited, since the abstract schema was lacking as they used additive schemes to solve the problem.

Four more participants (Learners 5, 7, 11 and 13) could either illustrate constant price of nine sweets by placing R7 next to each sweet, or could use 3D material to represent the sweets and the money, but could not solve the problem by using abstract representations. When only one type of representation is correct, it could indicate that limited connections were made (Ayub et al., 2013). One could conclude that Learners 5 and 11 had only semi-concrete schemas, while Learners 7 and 13 had only a concrete schema of constant price, which could have led to their use of incorrect schemes to solve the problem. Their conceptual understanding was limited to either the semi-concrete or concrete representations, which suggests limited conceptual understanding and no procedural fluency. One possible reason for the incorrect choice of a scheme could be that when I asked them to solve the problem by using abstract representations, they only tried to access the abstract schema, and since it was not available, they accessed the incorrect concept-in-action, which led to an incorrect theorem-in-action being chosen. This could indicate a lack of strategic competence.

Four participants (Learners 8, 10, 12 and 15) could solve this problem using only abstract representations. This could suggest that they saw only limited connections between the representations, which could indicate that they had only an abstract schema of the problem and not a semi-concrete or a concrete schema of 'constant price'. It is possible that they simply memorised the procedure without conceptually understanding it (Ayub et al., 2013). Only one participant (Learner 4) could not solve this problem as all her representations indicated addition. One might therefore conclude that she did not have a 'constant price' schema in any of the three representations, and no scheme, and therefore lacked the procedural fluency or strategic competence needed to solve this problem.

Four of the fifteen participants (Learners 3, 6, 14 and 17) could solve this problem of constant price by using all the representational forms. Learners 9 and 16 could solve it by using both semi-concrete and concrete representations; Learners 5 and 11 could solve it by using only the semi-concrete representations; and Learners 7 and 13 could only solve it by using concrete representations. Four participants (Learners 8, 10, 12 and 15) could only solve this problem by using abstract representations. In total, fourteen of the fifteen participants could solve it by using at least one of the representations. Only Learner 4 was unable to solve it by using any of the representations.

4.4.4.1 Question 2: Discussion of misconceptions and misrepresentations

The misconceptions identified and categorised were the same as for Question 1, namely, 'keyword' and 'added different units'. Learner 13 thought that the words 'how much' in the question indicated the operation to be used, while Learner 7 thought that different units could be added together (i.e. sweets and money). As explained in the discussion of the answers given to Question 1, this could possibly be indicative of teachers' failure to adequately emphasise the fact that different units cannot be added together and a possible over-emphasis of the use of keywords as indicators of the operations to be used.

Three misrepresentations were identified and categorised, namely 'abstract numbers', 'answer' and 'combination of measures'. Learners 10 and 15 replaced their abstract equations with sweets and money when they did their semi-concrete and/or concrete representations, while Learner 4 arranged the sweets to form numbers to represent her abstract equation. These misrepresentations could indicate that the above-mentioned participants struggled to visualise 'constant price' and could visualise only their abstract equations, which they then used to choose their concept-in-action. Four participants (Learners 7, 8, 10 and 12) drew sweets or blocks representing sweets, which represented only one part of the problem (i.e. the sweets). The misrepresentation of 'abstract numbers' could indicate that these participants had only a schema of sweets, and not one of money, as sweets are more concrete and easier to visualise than money. Therefore, because they struggled to visualise the money, they only drew what they knew best, namely sweets, even though the amount of money should have constituted the answer. It could be inferred that these learners struggled with abstract thinking, which is characteristic of learners with learning difficulties (Allsopp et al., 2007).

Three participants (Learners 5, 8, 12) arranged the money not to represent 'constant price', but to represent the answer to the problem, and one participant (Learner 13) drew sweets to give the answer. The reasons for this misrepresentation of the 'answer' could be twofold: either they were influenced by what they did for the abstract representations (they already had an answer in mind and then counted out the answer when they had to solve the problem again), or they struggled to visualise the problem either semi-concretely or concretely as 'constant price', which means that they did not have a very strong schema of problems of this nature. It could also be a combination of the two reasons: because they did not have a strong visual schema, they were easily influenced by what they had already done.

Learner 4 drew a combination of two set of sweets, indicating addition. She added when she had to represent the problem by using abstract representations. This misrepresentation of 'combination of measures' could indicate that she did not have a schema and scheme of the 'constant price' class of problem, and therefore chose an additive schema and scheme. She also gave the answer in sweets instead of in money, which could indicate a problem with abstract thinking, as money is more abstract to visualise than sweets. This misrepresentation suggests that she had no conceptual understanding of 'times as many' and no procedural fluency.

4.4.4.2 Question 2: Discussion of the calculation technique levels and types

Even though ten of the fifteen participants (Learners 3, 5, 6, 8, 10, 11, 12, 14, 15 and 17) used equations that indicated multiplication, only Learners 6, 10, 14, 15 and 17 used multiplicative calculation technique types to solve the problem and were categorised on Level 2. Learners 14 and 15 were categorised on Level 2B (double counting) and the calculation technique was categorised as 'counted in'. Learners 6 and 17 were categorised on Level 2E (derived multiplication facts) and the calculation technique was categorised as 'times table and subtraction'. This could indicate that only a third of all the participants had an abstract schema and multiplicative schemes of this 'constant price' class of problem, which in turn could indicate that they possessed good procedural fluency. Learners 6 and 17 were the only ones for whom good strategic competence could be inferred, since they were categorised on Level 2E, which demonstrates abstract thinking and is therefore seen as one of the highest cognitive developmental calculation techniques (Hurst & Hurrell, 2014; Zhang et al., 2011).

Learners 3 and 5 used a combination of multiplicative and additive calculation techniques. Learner 3 was categorised on Level 2A (doubling) and Level 1A (unitary counting), and her calculation technique types were categorised as 'doubling' and 'counted on', while Learner 5 was categorised on Level 2B (double counting) and Level 1C (repeated addition), and her calculation technique types as 'counted in' and 'counted on'. The use of a mixture of calculation technique types could indicate that these participants lacked a scheme to effectively solve this problem. It might also indicate that their multiplicative schemes were underdeveloped, which necessitated the use of multiplicative and additive calculation techniques. One could conclude that their procedural fluency and strategic competence were limited.

Eight participants (Learners 4, 7, 8, 9, 11, 12, 13 and 16) used additive calculation techniques types. Four of them (Learners 8, 9, 11 and 12) understood the problem to be multiplicative, but used additive calculation technique types to calculate the answer and were categorised on Level 1C (repeated addition). Their calculation technique type was categorised as 'counted on', which might indicate that even though they knew that it was a multiplication problem, they were unable to use multiplicative calculation techniques and had to resort to addition to solve it. The other four participants (Learners 4, 7, 13 and 16), who understood the problem to be additive, added the numbers together and were categorised on Level 1. They were categorised on Level 1A (unitary counting) and their calculation technique type was categorised as 'counted on'. One could infer that they lacked a correct schema and scheme of 'constant price', but had an additive schema and scheme and could therefore not solve this problem. They lacked the necessary procedural fluency and strategic competence to solve this problem and since their concept-in-action was incorrect, they chose an incorrect theorem-in-action.

4.4.4.3 Question 2: Discussion of calculation errors

Four calculation errors were identified and categorised, namely 'counting errors', 'wrong number', 'tracking error' and 'disconnect between abstract and drawing'. It is very difficult to give decisive reasons why participants made these calculation errors, which might have been due to their learning difficulties. Learner 4 added 6 and 9, instead of 7 and 9, which could indicate difficulties with remembering information. Six participants (Learners 3, 7, 8, 9, 12 and 13) found it difficult to add the big numbers together and easily lost their places, which resulted in 'counting errors'. Learner 8 knew that she had to add nine 7s together, but lost track of how many 7s she had already added. This 'counting error' could be indicative of organisational or visual-spatial problems experienced due to their learning difficulties. The 'disconnect between the abstract and drawing' calculation error that was identified for Learner 16, could have occurred because he was not used to drawing pictures of word problems. In my opinion, teachers who allow learners to make drawings of problems should explain to them how their pictures can be used to verify their answers. In the case of this problem, for which sweets and money had to be drawn, it was the amounts of money next to the sweets that had to be added together to arrive at the answer.

4.5 Question 3: Combined categorising, analysis and discussion

The third task-based question was: *Thabo rides his bicycle at a speed of 50 metres in one minute, how far will he ride his bicycle in 3 minutes?* This question represented the 'uniform speed' class of multiplication problems (Vergnaud, 1983) which, together with the two classes of problems addressed in Questions 1 and 2, are classified under 'equal groups' (see section 2.3.2.1 of Chapter 2 for full explanation). The learners were asked to answer this question by using first abstract, then semi-concrete and finally concrete representations. I expected them to illustrate their conceptual understanding by drawing a line (indicating distance) divided into three parts (each indicating one minute). The only concrete material I could suggest for participants to demonstrate their understanding of distance and time was a standard ruler. I expected them to point to the numbers 50, 100 and 150 marked on the ruler.

Since Question 3 included the abstract concepts distance and time, it was conceptually more complex than the previous two questions and therefore more difficult to visualise. Problems in this class are gradually introduced to learners from Grade 4 (Bowie et al., 2012a), and their abstract introduction follows in Grade 9, when they are taught as problems involving distance, speed and time (DBE, 2011c).

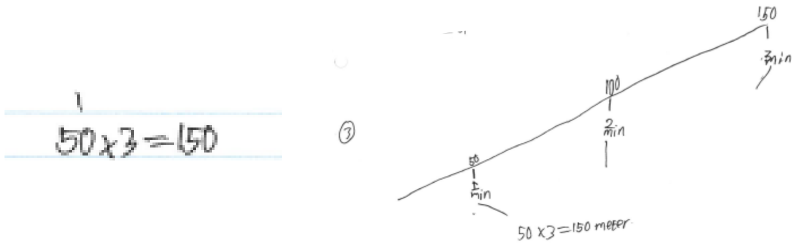
Table 4.3 provides a summary for all the participants of the categories and subcategories of the data and representations dealt with in Question 3. I colour coded similar subcategories for easier recognition and analysed the data under each heading in Table 4.3 by starting with the participants' conceptions, followed by their misconceptions and misrepresentations. Subsequently the levels and types of calculation techniques and calculation errors were analysed. A discussion of the analysis of all the data for this question, as well as Questions 1 to 3 together, followed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels as derived from my conceptual framework.

Table 4.3: Summary of the categories and subcategories of Question 3 for all the participants and representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner	Written operation and answer	Verbal misconception	Level of calculation technique and name	Type of calculation technique used	Calculation error with abstract representations	Drawing conception and answer	Drawing misrepresentation	Calculation error with semi-concrete representations	3D material conception	3D material misrepresentation
Learner 3	× (150)		Level 1C Repeated addition	Repeated addition		Intention was + (150)	Abstract numbers Wrong unit		Intention was ×	Answer
Learner 4	+ (33)		Level 1A Unitary counting	Counted on	Wrong number (wrote 30 + 3)	Addition + (33)	Combination of measures Wrong unit	Wrong number (wrote 30 + 3)	Addition +	Combination of measures
Learner 5	× (150)		Level 1C Repeated addition	Repeated addition		× Uniform speed (150)			[Unable to determine]	
Learner 6	+ (150)		Level 1C Repeated addition	Repeated addition		× Uniform speed (150)			× Uniform speed	
Learner 7	+ (150)		Level 1 Addition algorithm	Column method		[Unable to determine] (No answer)	Wrong unit		× Uniform speed	
Learner 8	+ (150)		Level 2B Double counting	Counted in	Wrong number (wrote 50 + 30)	[Unable to draw] (No answer)			× Uniform speed	
Learner 9	× (150)		Level 2B Double counting	Counted in		Intention was × (150)	Wrong unit		[Unable to do]	
Learner 10	× (150)		Level 2C Algorithms	Column method		Intention was × (150)	Abstract numbers Wrong unit		× Uniform speed	

Learner 11	× (150)		Level 2B Double counting	Counted in		Intention was × (150)	Abstract numbers Wrong unit		× Uniform speed	
Learner 12	× (150)		Level 2B Double counting	Counted in		× Uniform speed (150)			× Uniform speed	
Learner 13	+ (53)		Level 1A Unitary counting	Counted on		Intention was + (53)	Wrong unit		Addition +	Combination of measures
Learner 14	× (150)		Level 2B Double counting	Counted in		Intention was × (150)	Abstract numbers		× Uniform speed	
Learner 15	× (150)		Level 2C Algorithms	Column method		Intention was × (150)	Wrong unit		Intention was ×	Abstract numbers
Learner 16	× (140)		Level 2B Double counting	Counted in	Counting error (Answer 140)	Intention was × (140)	Abstract numbers Wrong unit		[Unable to do]	
Learner 17	× (150)		Level 2B Double counting	Counted in		× Uniform speed (150)			Intention was ×	Abstract numbers

Only one of the fifteen participants (Learner 12, whose name is marked in dark blue in Table 4.3) multiplied to solve the problem by using abstract representations and could solve this problem by using all three representations. He drew distance and time and could also indicate distance and time on a ruler, which indicated ‘uniform speed’. Picture 4.18 shows how Learner 12 represented ‘uniform speed’ by using abstract and semi-concrete representations. Referring to the concrete representation, he explained his understanding: Pointing to the 50 on the ruler: *It is one minute*, *it is the first minute.* Pointing to the 100 on the ruler: *It is the second minute ...* Pointing to the 150 on the ruler: *and the third minute.*

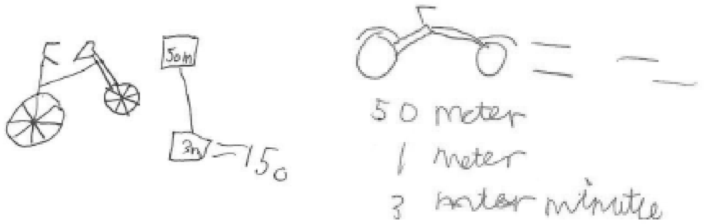


Picture 4.18: Question 3: Learner 12’s abstract and semi-concrete drawing representing uniform speed

Learner 6 (whose name is marked in light blue in Table 4.3) could demonstrate his understanding of ‘uniform speed’ with both the semi-concrete and concrete representations, but not with the abstract representations, while Learners 5 and 17 could demonstrate an understanding of ‘uniform speed’ with both abstract and semi-concrete representations, and Learners 10, 11 and 14 could demonstrate their understanding of ‘uniform speed’ with both abstract and concrete representations. The way in which Learners 7 and 8 pointed out distances on the ruler indicated their understanding of ‘uniform speed’ in concrete representations, and Learners 3, 9, 15 and 16 (whose names are marked in purple in Table 4.3) could only solve the problem with abstract representations, using multiplicative equations.

4.5.1 Question 3: Misconceptions and misrepresentations

Learners 4 and 13 added the two numbers together, but their explanations did not reveal any verbal misconceptions. I could identify and categorise four misrepresentations for this question, namely ‘abstract numbers’, ‘answer’, ‘combination of measures’ and ‘wrong unit’ (see Table 4.3). Learners 3, 10, 11, 14 and 16's semi-concrete representations were categorised as 'abstract numbers', since they used abstract numbers and/or drew bicycles that represented their equations. Picture 4.19 shows how Learners 11 and 16 misrepresented this problem.



Picture 4.19: Question 3: Learners 11 and 16's misrepresentations of 'abstract numbers'

Learners 15 and 17's misrepresentation of 'abstract numbers' was with the use of the concrete material. They showed the multiplication of two numbers on the ruler.

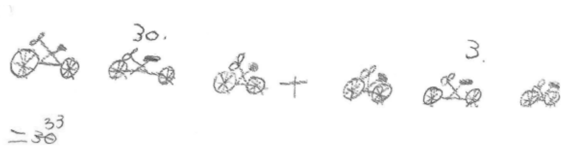
Learner 3's misrepresentation when using the 3D material was categorised as 'answer', since she gave the answer to the problem, but failed to show how it had been calculated. She actually used 3D blocks with the ruler to show the answer (see Picture 4.20).

Picture 4.20: Question 3: Learner 3's misrepresentation of the answer

This learner had placed fifteen blocks alongside the ruler until she reached the 150 mark on the ruler. She explained her representation as follows: *What if I take a block and say that each block is a bicycle?* I responded: *OK, but if I give you the ruler, how will you use the ruler to show it to me?* Then she said: *So, each block is 10 cm and each 10 cm equals 10 metres*.

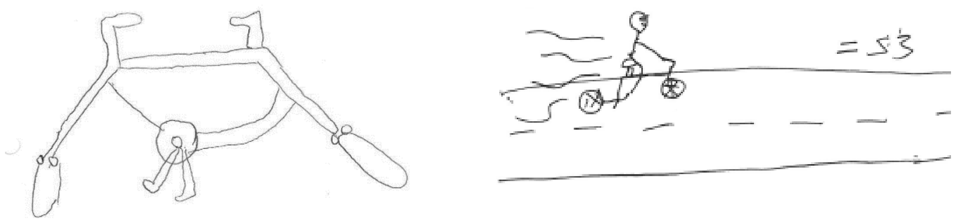
Learners 4 and 13 added the two numbers together when they used the 3D material and Learner 4 used addition as she added the bicycles together. This misrepresentation was categorised as 'combination of measures'. Picture 4.21 shows

Learner 4's semi-concrete drawing of the 'combination of measures' misrepresentation.



Picture 4.21: Question 3: Learner 4's misrepresentation of 'combination of measures'

The context of this problem was Thabo riding his bicycle, and nine of the participants (Learners 3, 4, 7, 9, 10, 11, 13, 15 and 16) drew bicycles, which did not represent one of the units in the problem (which were distance and time). I categorised this misrepresentation as 'wrong unit'. Picture 4.22 shows Learners 7 and 13's misrepresentations in which each drew only a bicycle and none of the real units that had to be used to calculate the answer.



Picture 4.22: Question 3: Learners 7 and 13's misrepresentation of the 'wrong unit'

Learners 5, 9 and 16 were unable to use the 3D material (a ruler) to solve the problem. They simply pointed randomly at the ruler without showing any understanding. Only Learner 8 did not attempt to make a drawing to answer the question.

No misconceptions were identified for this question, but four misrepresentations ('abstract numbers', 'answer', 'combination of measures' and 'wrong unit') were identified and categorised. The levels and types of calculation techniques will be discussed next.

4.5.2 Question 3: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2) and the calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.3). The following calculation technique levels were identified: Level 1A (unitary counting); Level 1C (repeated addition); Level 2B (double counting); and Level 2C (algorithms). Another additive level, not included in my conceptual framework, was identified and categorised, namely addition algorithm. This is a valid method taught in South African schools when adding larger numbers together (Bowie, Gleeson-Baird, Jones, Morgan, Morrison, & Smallbones, 2012b). This calculation technique was not mentioned in my conceptual framework as none of the research studies I had consulted about multiplicative reasoning mentioned it as a calculation technique, and it is more likely to be found in addition than in multiplicative problems. The calculation technique types identified for solving Question 3 were: 'counted on', 'repeated addition', 'counted in' and the 'column method' (see Table 4.3).

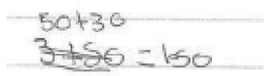
Level 1 calculation techniques are additive in nature and six participants (Learners 3, 4, 5, 6, 7 and 13) were categorised on Level 1, even though multiplication was used for the equations of two participants (Learners 3 and 5). Learner 7 was categorised on the new level, namely Level 1 (addition algorithm), as he had added the three 50s together using the 'column method' as calculation technique type. Learners 4 and 13 were categorised on Level 1A (unitary counting), since they had understood the problem as addition and had added the numbers together. Their calculation technique type was categorised as 'counted on', since they had started counting from one number and continued by adding the other. The three participants (Learners 3, 5 and 6) who had added the three 50s together were categorised on Level 1C (repeated addition). Their calculation technique type was categorised as 'repeated addition', since they had first added two 50s together to get 100 and had then added the other 50. For example, although Learner 3's algorithm was indicative of multiplication, when she explained how she had calculated the answer, she said: *Because 50 + 50 is 100, and 100 + 50 is 150*.

The Level 2 calculation techniques are considered to be multiplicative, which means that the participants were able to think multiplicatively when solving the problem (Carrier, 2014). Eight participants (Learners 9, 10, 11, 12, 14, 15, 16 and 17) used multiplicative calculation techniques to solve the problem and although Learner 8 used equation for her addition, she used a multiplicative calculation technique. Seven of these nine participants (Learners 8, 9, 11, 12, 14, 16 and 17) were categorised on Level 2B (double counting). The calculation technique type they used was categorised as 'counted in' as they all 'counted in' 50s, with the exception of Learner 17, who counted in 3s. Learners 10 and 15 were categorised on Level 2C (algorithms). Their calculation technique type was categorised as the 'column method', since they wrote the numbers below one another and multiplied each number separately.

Eight participants used multiplicative calculation techniques and six used additive calculation techniques to calculate the answer to this problem. The calculation errors will be discussed next.

4.5.3 Question 3: Calculation errors

Two calculation errors were identified and categorised, namely 'wrong number' and 'counting error' (see Table 4.3). Learners 4 and 8 both used incorrect numbers to calculate their answers, and this calculation error was categorised as 'wrong number'. Learner 4 wrote '30 + 3' instead of '50 + 3' in both the abstract and the semi-concrete representations, while Learner 8 wrote '50 + 30' instead of '50 × 3' in the abstract representation (see Picture 4.23).



Picture 4.23: Question 3: Learner 8's calculation error of 'wrong number'

When I asked Learner 8 to explain how he had arrived at 150, he explained: *I added. I took the 50 metres … I counted in 50s. I counted in 50s until I got to … wait, I said it is 150. I counted in 50s three times.* When I asked him to write his calculation down, he first wrote 3 + 50 and then said: *No, 50. I will just say 50 + 30 equals 150*. There was therefore was no correspondence between what he said and what he wrote down.

The other calculation error that was identified and categorised was a ‘counting error’. Learner 16 had struggled to calculate the answer. He had decided to count in 3s, but could not keep track of counting in 3s 50 times. To help him, I gave him a 100 chart, but that seemed to confuse him even more. During our conversation that followed, he explained: *You must multiply 3 with … you must multiply 3 with 50.* He then pointed to every third number on the hundred chart and said: *It will be … it will be 130.* When I asked him: *Did you keep counting in 3s all the time or what?* He explained: *Yes, 20, 40, 60, 80, then 100. Wait it is not … it is actually 140.*

4.5.4 Question 3: Discussion of the analysis

Question 3 was more conceptually complex than the previous two questions as the concepts distance and time are more abstract than trees, oranges, money and sweets. Although learners are occasionally exposed to problems of this kind from Grade 4 (Bowie et al., 2012a), they are only frequently encountered from Grade 9 (DBE, 2011c). The analysis showed that only Learner 12 could solve this problem of ‘uniform speed’ without any difficulty in all the representational forms. He had drawn a line with three intervals marked on it to indicate distance and time and could show three 50s on the ruler, which indicated a conceptual understanding of ‘uniform speed’. Procedural fluency was suggested by his use of multiplicative calculation techniques to solve this problem. This could indicate good connections between his different representations, which in turn indicates better understanding (Hiebert & Carpenter, 1992). One could therefore infer that he had good connections between his different representations, which indicated ‘uniform speed’, and therefore a good schema and scheme for ‘uniform speed’. Since he had a good interconnected schema for ‘uniform speed’ between the three representations, he was able to consider the correct multiplicative concept-in-action. This allowed him to choose the most effective theorem-in-action to solve the problem (Vergnaud, 1998; 2013a; 2013b), which could indicate good procedural fluency (see the conceptual framework, Figure 2.1 in Chapter 2).

Learner 6 could represent ‘uniform speed’ in his semi-concrete drawings by drawing a line and three intervals, and also when using 3D material. However, since his equation was additive, it could be inferred that he thought the problem was additive when he used the abstract representations to calculate the answer. One could therefore infer that he understood ‘uniform speed’ conceptually, but not as an abstract concept, and conclude that his schema for ‘uniform speed’ was limited, since the abstract schema was lacking as he had used an additive scheme to solve the problem.

Seven more participants could solve this problem by using either semi-concrete representations (Learners 5 and 17) or concrete representations (Learners 7, 8, 10, 11 and 14). All these participants could solve this problem in the abstract form, but used addition to calculate the answer. One could conclude that Learners 5 and 17 had semi-concrete schemas and addition schemes, while Learners 7, 8, 10, 11 and 14 had concrete schemas for ‘uniform speed’ and addition schemes. Their schema for ‘uniform speed’ was not yet well integrated between the representations. Limited conceptual understanding with additive procedural fluency could therefore be inferred.

Four participants (Learners 3, 9, 15 and 16) could solve this problem through multiplication by using only abstract representations, which could indicate that they had only limited connections between the representations, and therefore that they had only an abstract schema for the problem. This could also indicate that these learners had memorised a procedure without developing any conceptual understanding (Ayub et al., 2013). Learners 4 and 13 could not solve this problem as all their representations indicated that they had used addition. Therefore, one could possibly conclude that they did not have a ‘uniform speed’ schema in any of the representations and no scheme, and thus lacked the procedural fluency or strategic competence needed to solve this problem.

Learner 12 could solve the problem of ‘uniform speed’ in all the representational forms, while Learner 6 could solve it with both semi-concrete and concrete representations and the use of addition with the abstract representation. Learners 5 and 17 used semi-concrete representations to solve this problem, while Learners 7, 8, 10, 11 and 14 solved it with concrete representations and could solve it with addition in abstract representations. Learners 3, 9, 15 and 16 solved this problem through multiplication using only abstract representations. In total, thirteen of the fifteen participants could solve this problem in at least one of the representations. Learners 4 and 13 could not solve it in any of the representations.

4.5.4.1 Question 3: Discussion of misconceptions and misrepresentations

While the participants' explanations revealed no misconceptions, four misrepresentations could be identified and categorised, namely 'abstract numbers', 'answer', 'combination of measures' and 'wrong unit'. Nine participants (Learners 3, 4, 7, 9, 10, 11, 13, 15 and 16) found the context of riding a bicycle confusing and their misrepresentation was categorised as 'wrong unit'. Their semi-concrete pictures showed misrepresentations as they had drawn one or more bicycles without any indication of distance and/or time. This could indicate that they did not know how to represent time and distance, which in turn could be an indication that they did not have semi-concrete schemas for distance and time. A possible conclusion could therefore be that they used the only schema they had (the bicycle), which is more concrete (a visible object) than distance and time, to represent the problem. The bicycle was their only point of reference and the only object with which they were familiar.

Seven participants (Learners 3, 10, 11, 14, 15, 16 and 17) tried to replace their abstract equations with bicycles or a combination of bicycles and numbers in their semi-concrete representations, which resulted in a misrepresentation of 'abstract numbers'. This could possibly indicate that they were unable to visualise distance and time and could only visualise their abstract equations and the bicycles. The visualisation of abstract concepts is characteristic of learners with learning difficulties (Allsopp et al., 2007).

Learner 3 used a concrete representation to demonstrate the answer instead of showing how she had calculated the answer. There are two possible reasons for this misrepresentation of 'answer': either she was influenced by her abstract representation and already had an answer in mind when she tried to solve the problem using a concrete representation, or she found it difficult to visualise the problem as 'uniform speed'. It could also be a combination of the abovementioned possibilities, as she may not have had a strong visual schema and was easily influenced by what she had already done. This participant could have had a poor conceptual understanding of, and consequently an inadequate schema for 'uniform speed'.

The two participants (Learners 4 and 13) who thought that the question required addition misrepresented either the semi-concrete or the concrete representation, or both. One can infer an understanding of ‘combination of measures’ as they simply added the two units (time and distance) together. This could indicate that they had no schema and scheme for ‘uniform speed’, but only an additive schema for problems of this class. Because of this misrepresentation, one could infer that they had no conceptual understanding of ‘times as many’ and no procedural fluency.

I believe that teachers have a responsibility to not only teach learners procedural fluency, but to also help them to develop conceptual understanding. Teachers are encouraged to teach by using the CSA sequencing, which means that they have to first use concrete, then semi-concrete and finally abstract representations (Allsopp et al., 2007; Bruner, 1963; Debrenti, 2013; Hoong et al., 2015; Hui et al., 2017; Lesser & Tchoshanov, 2005; Pape & Tchoshanov, 2001; Post, 1981). If learners were taught to demonstrate their understanding in the form of concrete, semi-concrete and abstract (CSA) representations, those with learning difficulties might have a more thoroughly integrated schema for distance and time.

4.5.4.2 Question 3: Discussion of calculation technique levels and types

Nine of the fifteen participants (Learners 8, 9, 10, 11, 12, 14, 15, 16 and 17) were categorised on Level 2. Seven (Learners 8, 9, 11, 12, 14, 16 and 17) were categorised on Level 2B (double counting) and the calculation technique type that they had used was categorised as ‘counted in’. Learners 10 and 15 were categorised on Level 2C (algorithms) and their calculation type was categorised as the ‘column method’. One could infer that they had an abstract and a multiplicative schema for this class of problem, namely ‘uniform speed’. A possible conclusion is that since their concept-in-action was multiplicative, they were able to choose an appropriate multiplicative theorem-in-action to solve the problem. Although they possessed procedural fluency, none of them used very efficient calculation technique types, since the lower the calculation technique level, the more efficient the calculation technique. It could be inferred that these learners were not really strategically competent, since they were categorised on Levels 2B and 2C, which are not among the highest cognitively developmental calculation techniques (Hurst & Hurrell, 2014; Zhang et al., 2011).

Six participants (Learners 3, 4, 5, 6, 7 and 13) were categorised on Level 1, while Learners 4 and 13 were categorised on Level 1A (unitary counting) and the calculation technique type was categorised as ‘counted on’. Learners 3, 5 and 6 were categorised on Level 1C (repeated addition), the calculation technique type was categorised as ‘repeated addition’ and Learner 7 was categorised on a new Level 1 (addition algorithm). The calculation technique type was categorised as ‘column method’. These participants also thought that this was an addition problem and therefore used additive calculation technique types to solve it. It could be inferred that they did not have a correct schema for ‘uniform speed’ and could not solve the problem because their schema and scheme were additive. They lacked the procedural fluency and strategic competence required to solve this problem. Since their concept-in-action was incorrect, they chose an incorrect theorem-in action.

4.5.4.3 Question 3: Discussion of calculation errors

The calculation errors identified and categorised were ‘wrong number’ and ‘counting error’. As mentioned in the discussion of the first two questions, it is very difficult to explain why the participants had made these calculation errors. Learner 4 added 30 and 3 together, instead of 50 and 3, and Learner 8 added 50 and 30 together, which led to a categorisation as ‘wrong number’. This could indicate that these learners struggled to remember the information, which could be as a result of their learning difficulties. Learner 16 had made a ‘counting error’ when he tried to add three 50 times. His calculation did not make any sense to me as he gave up counting in 3s and continued counting in 20s. I could therefore not find any pattern in what he was doing. He might have found adding three 50 times overwhelming, which points to a lack of strategic competence.

4.6 Question 4: Combined categorising, analysis and discussion

The fourth task-based question, which represented the ‘times as many’ class of multiplication problems (Greer, 1992; Mulligan, 1992), was: *Paul has 4 coloured pens. If Sarah has 8 times as many coloured pens as Paul, how many coloured pens does Sarah have?* The classes of problems presented in Questions 4 and 5 are categorised as ‘multiplicative comparison’, which is more conceptually complex than ‘equal groups’ (Questions 1 to 3) (see section 2.3.2.2 in Chapter 2 for a full explanation). All the participants were required to first answer this question with abstract representations, then with semi-concrete and lastly with concrete representations. I provided the learners with coloured pens as 3D material and expected them to illustrate their conceptual understanding by drawing eight groups of four pens each.

According to the CAPS, problems of this class should be introduced in Grade 2 (DBE, 2011a), but the first example I found was in a Grade 4 textbook (Bowie et al., 2012a). 'Times as many' problems are the least complex as in the multiplicative category as they involve only one unit (coloured pens).

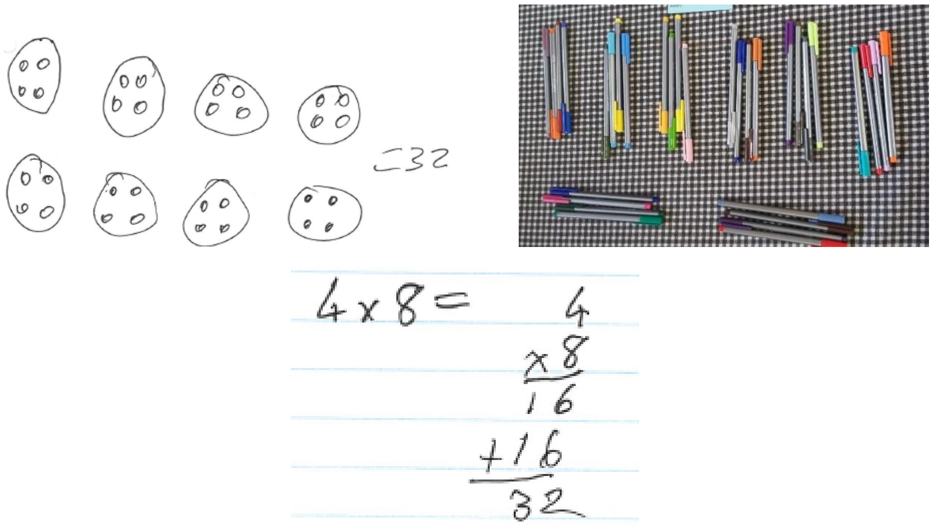
Table 4.4 contains a summary of the categories and subcategories of the data for all the representations of all the participants. Similar subcategories are colour coded for easier recognition. I analysed the data under each of the headings in Table 4.4, starting with the participants' conceptions, then their misconceptions and misrepresentations. This is followed by an analysis of the levels and types of calculation techniques and calculation errors. Finally, the analysis of all the data for this question is discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels, as derived from my conceptual framework.

Table 4.4: Summary of the categories and subcategories of Question 4 for all participants and all their representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner	**Written operation and answer**	**Verbal misconception**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 3	× (32)		Level 2B Double counting Level 1A Unitary counting	Counted in Counted on		× Times as many (32)			× Times as many	
Learner 4	- (4)		Level 1 Unitary subtraction	Counted down		Subtraction - (4)	Take away		Subtraction -	Abstract numbers
Learner 5	× (32)		Level 2B Double counting Level 1C Repeated addition	Counted in Counted on		Intention was × (32)	Abstract numbers		Intention was ×	Abstract numbers
Learner 6	× (32)		Level 2E Derived multiplication fact	Split multiplication and addition		× Times as many (32)			× Times as many	
Learner 7	+ (16)		Level 2A Doubling	Doubling	Wrong number (wrote 4)	× Times as many (16)		Wrong number (wrote 4)	× Times as many	
Learner 8	× (16)		Level 2B Double counting	Counted in	Wrong number (wrote 4)	Intention was × (16)	Answer		× Times as many	
Learner 9	× (32)		Level 2B Double counting	Counted in		× Times as many (32)			× Times as many	

Learner 10	+ (12)	Keyword (times as many)	Level 1 Known addition fact	Addition fact		Addition + (12)	Combination of measures		Addition +	Combination of measures
Learner 11	× (32)		Level 2B Double counting Level 1A Unitary counting	Counted in Counted on		Intention was × (32)	Abstract numbers		Intention was ×	Abstract numbers
Learner 12	× (32)		Level 1C Repeated addition	Counted on		Intention was × (32)	Abstract numbers		Intention was ×	Abstract numbers
Learner 13	× (32)		Level 1C Repeated addition	Counted on		Intention was × (32)	Answer		Intention was ×	Answer
Learner 14	+ (12)		Level 1A Unitary counting	Counted on		Intention was + (12)	Answer		Addition +	Combination of measures
Learner 15	× (32)		Level 2B Double counting	Counted in		Intention was × (32)	Abstract numbers		Intention was ×	Abstract numbers
Learner 16	- (4)		Level 1 Unitary subtraction	Counted down		Subtraction - (4)	Take away		Intention was -	Take away
Learner 17	× (32)		Level 2E Derived multiplication fact	Times table and subtraction		× Times as many (32)			× Times as many	

Four of the fifteen participants (Learners 3, 6, 9 and 17, whose names are marked in dark blue in Table 4.4) could solve this problem with all three representations after first multiplying to solve it with abstract representations. They drew eight groups of four pens each, which is indicative of the 'times as many' concept (see Picture 4.24).



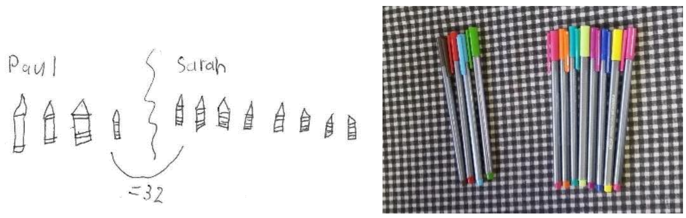
Picture 4.24: Question 4: Learner 6's equation, conceptual drawing and coloured pens representing 'times as many'

Learner 7 (whose name is marked in light blue in Table 4.4) produced semi-concrete and concrete drawings that were indicative of 'times of many', but drew only four instead of eight groups of four coloured pens each. He could not solve the problem with the abstract representations. Learner 8's concrete representation was also indicative of 'times as many', whereas the abstract representations of Learners 5, 11, 12, 13 and 15, whose names are marked in purple in Table 4.4, were multiplicative. Even though Learner 8's concrete representation was categorised as showing the 'times as many' class of problem, she had placed the coloured pens in eight groups of two each, which although indicative of multiplication, was not the grouping required by the question. Furthermore, six of the fifteen participants (Learners 5, 6, 9, 11, 12 and 13) knew that the keywords 'times as many' indicate multiplication and therefore multiplied to solve the problem. When asked: *Is there something in the question that tells you that it is a multiplication sum?* Learner 11 answered: *Times as many.* However, there were some misconceptions regarding these keywords, which will be discussed next.

4.6.1 Question 4: Misconceptions and misrepresentations

Only one misconception was identified and categorised, namely ‘keyword’. Learner 10 explained that the phrase ‘times as many’ in the problem indicated addition (see Table 4.4). Although ‘times as many’ is used to indicate multiplication, this participant associated it with addition and it was therefore categorised as a misconception of ‘keyword’. He explained: *Because Paul had four and Sarah had eight times as many, so I added.* Learners 4 and 16 subtracted, and Learner 14 added the two numbers together, but I could not identify any misconceptions in their explanations.

Four misrepresentations were identified and categorised, namely ‘abstract numbers’, ‘answer’, ‘combination of measures’ and ‘take away’ (see Table 4.4). Five participants’ semi-concrete and concrete representations were categorised as misconceptions of ‘abstract numbers’. The semi-concrete and concrete representations of four of the five participants (Learners 5, 11, 12 and 15) were categorised as ‘abstract numbers’, since they simply replaced the numbers of their equations with coloured pens, as shown in Learner 11’s work in Picture 4.25. Learner 4 recreated her equation by using the coloured pens to replace the numbers.



Picture 4.25: Question 4: Learner 11’s misrepresentations of ‘abstract numbers’

Learners 8, 13 and 14 drew the answer and Learner 13 placed the coloured pens together to represent the answer to the problem instead of the problem itself. Their misrepresentations were categorised as ‘answer’. Picture 4.26 shows Learner 13’s semi-concrete and concrete misrepresentation of the ‘answer’.

Picture 4.26: Question 4: Learner 13's misrepresentation of the 'answer'

Learner 10's drawing and concrete representation and Learner 14's concrete representation were categorised as misrepresentations of a 'combination of measures' as they simply drew and placed eight and four pens together, which indicates addition. While the two abovementioned learners thought that addition was required, Learners 4 and 16 thought that the problem should be solved by subtraction. To categorise this misconception, I used Vergnaud's (1982) classification of subtraction of this kind, i.e. 'take away'. In the case of this study it should be seen as a misrepresentation since all the problems required multiplication and not subtraction. The learners subtracted eight from four to calculate the answer, and Learner 4 used the coloured pens to form a four and an eight, which inferred a misrepresentation of 'abstract numbers'.

For this question, one misconception ('keyword') and four misrepresentations ('abstract number', 'answer', 'combination of measures' and 'take away') could be identified and categorised. The levels and types of calculation techniques will be discussed next.

4.6.2 Question 4: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2) and the calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.4). The following calculation technique levels were identified and categorised: Level 1A (unitary counting), Level 1C (repeated addition), Level 2A (doubling), Level 2B (double counting), and Level 2E (derived multiplication fact). Two other additive levels, categorised on Level 1, were identified, namely 'known addition fact' and 'unitary subtraction', but were not mentioned in my conceptual framework since my study focused on multiplication and these two calculation technique levels can only be used when adding and subtracting. The following calculation technique types were identified for calculating the answer to Question 4: ‘counted on’, ‘addition fact’, ‘counted down’, ‘doubling’, ‘counted in’, ‘split multiplication and addition’ and ‘times table and subtraction’.

Level 1 calculation techniques are additive in nature and six participants (Learners 4, 10, 12, 13, 14 and 16) were categorised on this level. Learner 14 was categorised on Level 1A (unitary counting), since her calculation technique type was categorised as ‘counted on’ as she had arrived at 12 by starting from eight and counting on four. Learners 12 and 13 were categorised on Level 1C (repeated addition) and their calculation technique type was also categorised as ‘counted on’ as Learner 12 had counted on four each time and Learner 13 had counted on eight each time to arrive at 32. Learners 4, 10 and 16’s calculation technique levels were not part of my conceptual framework as they are not multiplicative in nature. Learner 10 understood the problem as addition and was categorised on Level 1 (known addition fact). His calculation technique type was categorised as ‘addition fact’, since he knew that 8 + 4 = 12. Learners 4 and 16 thought that it was a subtraction problem and were categorised on Level 1 (unitary subtraction) as their calculation technique type was categorised as ‘counted down’. They had subtracted four from eight to calculate the answer.

Learners 3, 5 and 11 used a combination of additive and multiplicative calculation technique types, which were categorised on Levels 1A (unitary counting), 1C (repeated addition) and 2B (double counting). The calculation technique type with which these learners had started out was categorised as ‘counted in’, since Learners 3 and 5 had counted in 4s and Learner 11 had counted in 8s. When they did not know how to continue, Learners 3 and 11 counted on the last four and eight respectively and their calculation technique type was categorised as ‘counted on’ and Level 1A (unitary counting). Learner 5 was categorised on Level 1C (repeated addition) as she had repeatedly counted on four to find the answer, and her calculation technique type was categorised as ‘counted on’.

The Level 2 calculation techniques are considered multiplicative as the participants were able to think multiplicatively when solving the problem (Carrier, 2014). Six participants were categorised on Level 2 as they had used multiplication calculation techniques to calculate their answers. Learner 7 was categorised on Level 2A, since his calculation technique type was categorised as 'doubling'. He doubled four to get eight and then doubled eight to get 16, which was not the correct answer. Three of the six participants (Learners 8, 9 and 15) were categorised on Level 2B (double counting) since their calculation technique type was categorised as 'counted in'. They had all counted in 4s to calculate the answer. Learners 6 and 17 were categorised on Level 2E (derived multiplication fact). Learner 6's calculation technique type was categorised as 'split multiplication and addition' and he explained it as follows: *I took the 8, I divided the 4 in two 2-2, then I multiplied 8 by 2, which gave me 16. Then I did it again, which gave me 16. Then I added it together to get 32*. Learner 17's calculation technique type was categorised as 'times table and subtraction'. She explained her calculation as follows: *4 × 10 is 40 and then you subtract only 4, 8, because you subtract only two times four*.

Six participants used multiplicative calculation techniques and another six used additive calculation techniques to solve this problem. Three participants used a combination of additive and multiplicative calculation techniques. The calculation errors will be discussed next.

4.6.3 Question 4: Calculation errors

Only one calculation error was identified and categorised, namely 'wrong number' (see Table 4.4). Learners 7 and 8 both wrote 4 x 4 instead of 4 x 8. Learner 7 used the 'wrong number' in the abstract and semi-concrete representations, while Learner 8 used the 'wrong number' in the abstract representations. Their explanations provided no indication as to their reasons for using four instead of eight.

4.6.4 Question 4: Discussion of the analysis

Question 4 was the least conceptually complex problem in the category 'multiplicative comparison', which also included Question 5 as the participants only needed to visualise eight groups of four coloured pens. Although according to the CAPS learners should be introduced to this concept in Grade 2 (DBE, 2011a), the first examples I could find were in a Grade 4 textbook (Bowie et al., 2012a). My analysis showed that only four of the fifteen participants (Learners 3, 6, 9 and 17) could solve this 'times as many' problem without any difficulty in all the representational forms, and from their semi-concrete and concrete representations of eight groups of four it could be inferred that they had a conceptual understanding of 'times as many'. Moreover, procedural fluency could be inferred as they used multiplicative calculation techniques to solve their multiplication equations. According to Hiebert and Carpenter (1992), the more connections there are between the different types of representations, the better a person's understanding will be. One could therefore infer that these four participants had good connections between their different types of representations as all their representations indicated the 'times as many' class of problem, and therefore that they had a good schema and scheme for 'times as many'. Their good interconnected schema for 'times as many' between the three types of representations enabled them to consider the correct multiplicative concept-in-action, which allowed them to choose the most effective theorem-in-action to solve this problem, which in turn made it possible to infer that they had good procedural fluency (see conceptual framework, Figure 2.1 in Chapter 2).

Learner 7 could represent 'times as many', but only with four groups of four sweets each in the semi-concrete and concrete representations, but he did not use multiplication for his equation. Although he did not draw eight, but only four groups of four, conceptual understanding could nevertheless be inferred, except in the abstract representation. One could conclude that his schema was limited, since the abstract schema was lacking. Although Learner 8 placed the 3D material in eight instead of four groups, I did infer limited connections between the representations and limited conceptual understanding in the case of the concrete representations as she had made eight groups. However, she lacked procedural fluency.

Learners 5, 11, 12, 13 and 15 could solve the problem only with abstract representations, which could indicate that they had only limited connections between the different representations and only an abstract schema. This could further indicate that they memorised a procedure without gaining conceptual understanding (Ayub et al., 2013). Four participants (Learners 4, 10, 14 and 16) had trouble solving this problem in any of the representations. Learners 4 and 16 thought that the problem could be solved by subtracting, while Learners 10 and 14 thought that it could be solved by addition. It could therefore be concluded that they did not have a 'times as many' schema for any of the representations and no scheme, and therefore lacked the necessary procedural fluency or strategic competence to solve this problem.

Four of the fifteen participants (Learners 3, 6, 9 and 17) could solve this 'times as many' problem in all the representational forms. Learner 7 could represent 'times as many' with both semi-concrete and concrete representations, while Learner 8 could only place the 3D material in eight groups of two. Five of the participants (Learners 5, 11, 12, 13 and 15) could only solve the problem with abstract representations. In total, eleven of the fifteen participants could solve this problem with at least one of the representations and four (Learners 4, 10, 14 and 16) could not solve it with any of the representations.

4.6.4.1 Question 4: Discussion of misconceptions and misrepresentations

One misconception could be identified and categorised as 'keyword'. Learner 10 thought that the phrase 'times as many' in the question implied addition, which could possibly indicate a problem with how the participant had been taught or it might be his own method that he thinks may help him to choose the operation to solve the problem.

Four misrepresentations were identified and categorised, namely 'abstract numbers', 'answer', 'combination of measures' and 'take away'. Learners 5, 11, 12 and 15 replaced their abstract equations with four and eight pens, while Learner 4 used the pens to form an eight and a four. Learners 12 and 15 used a combination of abstract numbers (i.e. 8) and drew four pens for their semi-concrete representations. All these misrepresentations of 'abstract numbers' could possibly indicate that these participants struggled to visualise 'times as many' and could only visualise their abstract equations, which they used to choose their concepts-in-action. Learners with learning difficulties typically visualise abstract concepts (Allsopp et al., 2007).

Learners 8, 13 and 14 misrepresented the problem by drawing or displaying the coloured pens as the 'answer', instead of arranging them in eight groups of four pens each. Learner 8 drew 16 pens, while Learner 13 drew 32 pens and Learner 14 drew 12. Two possible reasons for this could be that they were either influenced by their answer in their abstract representations, or could have found it difficult to visualise the problem as 'times as many', which points to the absence of a strong schema for problems of this class. Alternately it could be a combination of the abovementioned possibilities, as they may not have had strong visual schemas and were easily influenced by what they had already done. The abovementioned participants could have had a poor conceptual understanding of 'uniform speed' and therefore inadequate schemas.

Learners 10 and 14's representations were categorised as a misrepresentation of 'combination of measures' as they had added the two numbers together. This misrepresentation could indicate that they lacked schemas and schemes for problems classified as 'times as many', which might explain why they had used a 'combination of measures' concept-in-action, which had led to their choice of addition as the theorem-in-action. Learners 4 and 16 misrepresented the problem as subtraction and deducted eight from four. This misrepresentation, categorised as 'take away', could indicate that they also did not have a correct schema and scheme for this class of problem, but used a 'take away' concept-in-action that led them to choose a subtraction theorem-in-action. In both of the above misrepresentations one could infer a total lack of conceptual understanding of 'times as many' and of procedural fluency.

4.6.4.2 Question 4: Discussion of the calculation technique levels and types

Even though ten of the fifteen participants (Learners 3, 5, 6, 8, 9, 11, 12, 13, 15 and 17) used equations that indicated multiplication, only five of them (Learners 6, 8, 9, 15 and 17) used multiplicative calculation technique types to solve the problem. Learner 7 used addition for his equation, but used a multiplicative calculation technique type to solve the problem. Six participants (Learners 6, 7, 8, 9, 15 and 17) were categorised on Level 2. Learner 7 was categorised on Level 2A (doubling) and his calculation technique type was categorised as 'doubling' since he had first doubled 4 and then 8, but had not doubled 8 again. Learners 8, 9 and 15 were categorised on Level 2B (double counting) and their calculation technique type was categorised as 'counted in'. Learners 6 and 17 were categorised on Level 2E (derived multiplication fact) and their calculation technique types were categorised as 'split multiplication and addition' and 'times table and subtraction' respectively. This could indicate that less than half of the participants had abstract schemas and multiplicative schemes for the 'times as many' class of problem and had used multiplicative calculation techniques, which in turn could indicate that they had good procedural fluency. However, Learners 6 and 17 were the only ones about whom it could be inferred that they had good strategic competence as they were categorised on Level 2E, which is one of the highest developmental calculation technique levels and demonstrates abstract thinking (Hurst & Hurrell, 2014; Zhang et al., 2011).

Learners 3, 5 and 11 used a combination of multiplicative and additive calculation techniques. Learners 3 and 11 were categorised on Level 2B (double counting) and Level 1A (unitary counting), and their calculation technique types were categorised as 'counted in' and 'counted on'. Learner 5 was categorised on Level 2B (double counting) and Level 1C (repeated addition), and the calculation technique types used were categorised as 'counted in' and 'counted on'. This could indicate that these three participants lacked an efficient scheme to solve the problem. Moreover, their multiplicative schemes might not yet have been well developed and therefore they had to make use of multiplicative and additive calculation techniques. One could conclude that they had limited procedural fluency and poor strategic competence.

Six participants (Learners 4, 10, 12, 13, 12 and 16) used additive calculation technique types to solve the problem and were therefore categorised on Level 1. Learners 4 and 16 subtracted and were categorised on a new Level 1 (unitary subtraction) while their calculation technique type was categorised as 'counted down'. Learners 10 and 14 understood the problem as addition and were categorised on a new Level 1 (known addition fact) and Level 1A (unitary counting), and the calculation technique types were categorised as 'addition fact' and 'counted on'. One could infer that they did not have correct schemas and schemes for 'times as many' and that they could not solve the problem as their schemas and schemes were additive. They had neither procedural fluency nor strategic competence. One could conclude that because their concepts-in-action were incorrect, they chose incorrect theorems-in-action. Learners 12 and 13 understood the problem as multiplication, but used additive calculation techniques and were categorised on Level 1C (repeated addition). Their calculation technique type was categorised as 'counted on'. One could infer that their schemas were correct, but that they had additive schemes to solve the problem. They had additive procedural fluency, but poor strategic competence. Their concepts-in-action were correct; however their theorems-in-action were not very efficient at solving the problem.

This could possibly indicate that even though they knew that it was a multiplication problem, they were unable to solve it by using a multiplicative calculation technique and had to resort to using only addition or a combination of multiplication and addition calculation techniques. In the case of those who had used a combination of additive and multiplicative calculation techniques one could only infer a degree of strategic competence with limited procedural fluency.

4.6.4.3 Question 4: Discussion of calculation errors

Only one calculation error was identified and categorised, namely 'wrong number'. Learners 7 and 8 used the number four instead of eight to solve their problem. The reason for this error is unclear, but it could indicate difficulties with remembering information.

4.7 Question 5: Combined categorising, analysis and discussion

The fifth task-based question was: *If 4 sweets cost R10, how much will 12 sweets cost?* This question represented multiplication problems belonging to the class 'simple proportion' (Lamon, 1994; Vergnaud, 1983). Due to the one-hour time limit placed on the interviews, only eleven of the fifteen participants answered this question – first with abstract, then with semi-concrete and lastly with concrete representations. I expected them to illustrate their conceptual understanding by drawing three groups of four sweets, with a R10 note next to each group. I gave the participants sweets and R10 notes as 3D material and expected them to make three groups of four sweets each and place a R10 note with each group.

According to the CAPS, problems of this class should be first introduced in Grade 4 (Bowie et al., 2012a; DBE, 2011b). In the 'multiplicative comparison' category under which Questions 4 and 5 fall, Question 5 was more conceptually complex than Question 4 (see section 2.3.2.2 in Chapter 2 for full explanation) as it required participants to think both proportionally and multiplicatively.

Table 4.5 contains a summary of the categories and subcategories of the data for the eleven participants and all their representations. I colour coded similar subcategories for easier recognition. I analysed the data under the different headings in Table 4.5, e.g. the participants' conceptions and their verbal, drawn and 3D material misrepresentations. This is followed by an analysis of the levels and types of calculation techniques and calculation errors. Finally, the analysis of all the data for this question is discussed together with the data obtained through the analysis of Questions 4 and 5. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

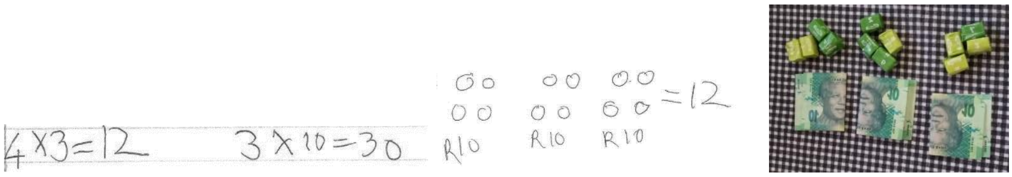
Table 4.5: Summary of the categories and subcategories of Question 5 for eleven participants and all the representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner[2]	**Written operation and answer**	**Written misrepresentation**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 3	÷ (3)		[Division, none of my Levels]	[Incorrect division]		× Simple proportion (40)			× Simple proportion	
Learner 5	× (120 or 48)	Non-consideration of proportion	Level 2C Algorithms	Column method		[Unable to draw]			× Simple proportion	
Learner 6	× (480)	Non-consideration of proportion	Level 2C Algorithms	Column method		Multiplication × (120)	Equal sharing Wrong unit	Disconnect between abstract and drawing (Answer 120)	× Simple proportion	
Learner 7	× (120)	Non-consideration of proportion	Level 2B Double counting	Counted in		Intention was × (120)	Answer		Intention was ×	Constant price
Learner 8	× (No final answer)	Non-consideration of proportion	[Unable to determine]	[Unable to determine]		[Unable to draw]			[Unable to do]	
Learner 9	× (30)		Level 2B Double counting	Counted in		× Simple proportion (12)		Disconnect between abstract and drawing (Answer 12)	× Simple proportion	
Learner 10	× (48)	Non-consideration of proportion	Level 2C Algorithms	Column method		Intention was × (48)	Abstract numbers		Intention was ×	Abstract numbers
Learner 11	× (48)	Non-consideration of proportion	Level 2C Algorithms	Column method	Counting error	Intention was ×	Answer Wrong unit		Intention was ×	Answer

[2] Only eleven of the fifteen participants answered Questions 5 to 7.

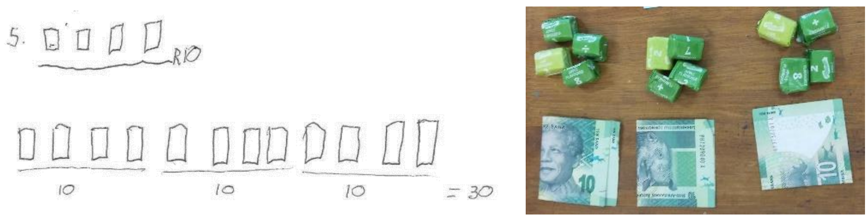
					(Answer 48)	(48)				
Learner 14	× (30)	Equation inconsistent with answer	Level 2B Double counting	Counted in		× Simple proportion (30)			× Simple proportion	
Learner 15	× (30)	Equation inconsistent with answer	[Unable to determine]	[Unable to determine]		Intention was × (30)	Repeated addition		Intention was ×	Repeated addition
Learner 17	[Unable to do]					× Simple proportion (30)			× Simple proportion	

Of the eleven participants, only Learner 9 (whose name is marked in dark blue in Table 4.5) could solve this problem with all three representations, since he multiplied to solve the problem with abstract representations. Furthermore, he drew three groups of four sweets and a R10 note next to each group, which is indicative of ‘simple proportion’ (see Picture 4.27).



Picture 4.27: Question 5: Learner 9’s equation, conceptual drawing and 3D material representing ‘simple proportion’

Three participants (Learners 3, 14 and 17, whose names are marked in light blue in Table 4.5) drew three groups of four sweets and a R10 note, and used the 3D material to place four sweets in three groups, adding a R10 note to each, from which an understanding of ‘simple proportion’ could be inferred. However, they had trouble solving the problem using abstract representations. The semi-concrete and concrete representations of one of these learners can be seen in Picture 4.28 below.



Picture 4.28: Question 5: Learner 17’s semi-concrete and concrete representations of ‘simple proportion’

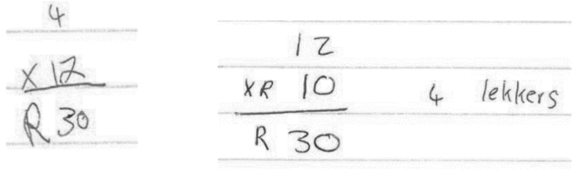
Learners 14 and 15 could also solve the problem by reasoning it out, but could not write an equation to show how they had calculated the answer with their abstract representations. Learners 5 and 6 (whose names are marked in dark blue in Table 4.5) could solve the problem with the concrete representations only. Although Learner 8, whose name is indicated in dark red in Table 4.5, attempted to solve the problem with the abstract representations, she gave up and did not attempt to solve the problem with the other representations.

4.7.1 Question 5: Misconceptions and misrepresentations

Learner 3 divided to get the answer and no verbal misconception could be identified from her explanations. However, since written misrepresentations were identified in the abstract representations, I changed the heading for Table 4.5 from 'verbal misconception' to 'written misrepresentation' in order to report on those misrepresentations. The two written misrepresentations that were identified and categorised were 'non-consideration of proportion' and 'equation inconsistent with answer'.

Six participants (Learners 5, 6, 7, 8, 10 and 11) could not solve the problem with abstract representations and their equations were categorised as a misrepresentation of 'non-consideration of proportion' (see Table 4.5). Learners 6 and 11 attempted to solve the problem by multiplying all three numbers with one another (i.e. 12, 4 and 10) and gave the answer as 480. Learner 7 multiplied 10 by 20 and calculated the answer to be 120, and Learner 10 multiplied 4 by 8 and gave 48 as the answer. Learner 5 calculated the answer by multiplying 10 by 20 and the answer she arrived at was 120, and also multiplied 4 by 12 twelve to calculate the answer, which was 48. However, she could not decide which answer was correct. Learner 8 multiplied 10 by 4, but became confused and tried different calculation techniques and never committed to a final answer. All these calculations were categorised as 'non-consideration of proportion', since the participants used different combinations of the numbers given in the question without considering how they related to one another or realising that they should not simply be multiplied.

Learners 14 and 15 knew that 30 was the correct answer, but struggled to explain their reasoning in writing. Their equations were inconsistent with the answers they gave and these misrepresentations were categorised as 'equation inconsistent with answer'. Picture 4.29 shows how they wrote their equations.

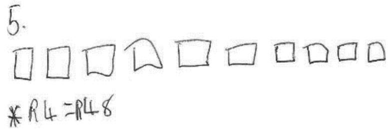


Picture 4.29: Question 5: Learners 14 and 15's misrepresentation of 'equations inconsistent with answer'

Their explanations corroborated their answers, which both gave as R30. Learner 14 explained: *Each fourth sweets costs R10. Four cost R10, eight cost R20 and 12 cost R30*. Learner 15's explanation was: *R10 is for four sweets. Then it is 4 × 3; 4 × 3 is 12, then it is R10 for 4, then it is R30.*

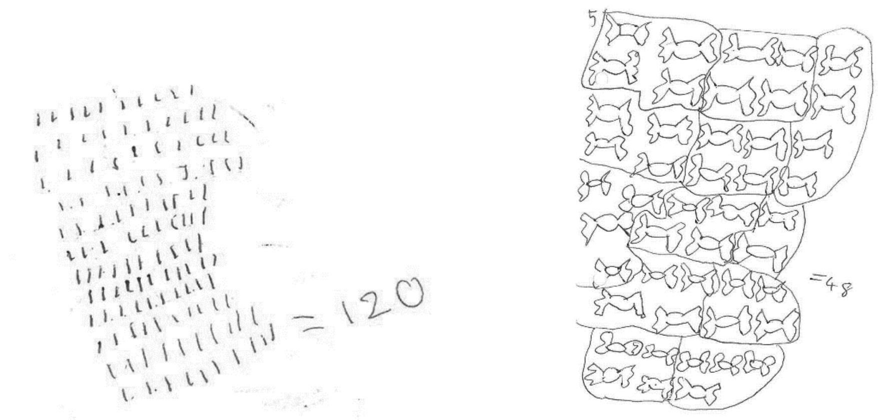
Learner 3 wanted to calculate the cost of one sweet by dividing four by ten, but she struggled to calculate the answer and eventually gave up. In this case the cost of one sweet was a decimal number and she was unable to work that out.

Six misrepresentations with the semi-concrete and concrete representations were identified and categorised, namely 'abstract numbers', 'answer', 'equal sharing', 'constant price', 'wrong unit' and 'repeated addition' (see Table 4.5). Learner 10's semi-concrete and concrete representations were categorised as 'abstract numbers' as he had replicated his equation with sweets and money in his drawing and also with the 3D material. His equation involved the multiplication of 12 by 4, which was incorrect, but was replicated with the 3D material and incorrectly replicated in his drawing (which depicted 10 sweets instead of 12) (see Picture 4.30).



Picture 4.30: Question 5: Learner 10's misrepresentation of 'abstract numbers'

Learner 7's drawing and Learner 11's semi-concrete and concrete representations were categorised as misrepresentations of 'answer'. Learner 7 made 120 marks on the paper, while Learner 11 drew 48 sweets and then circled groups of four sweets each (see Picture 4.31). She then counted out 48 sweets, and placed groups of four and five sweets on the 11 R10 notes. These participants' misrepresentations were categorised as 'answer'.



Picture 4.31: Question 5: Learners 7 and 11’s misrepresentations of ‘answer’

Learner 6’s drawing was categorised as a misrepresentation of ‘equal sharing’. He had drawn three groups of four sweets each, which is indicative of the ‘equal sharing’ class of problem and not of ‘simple proportion’. Learner 7’s concrete representation was categorised as a misrepresentation of ‘constant price’. He paired each R10 note with one sweet, which is indicative of the ‘constant price’ class of problems and not of ‘simple proportion’. Picture 4.32 shows these two participants’ misrepresentations of ‘equal sharing’ and ‘constant price’.

Picture 4.32: Question 5: Learners 6 and 7’s misrepresentation of ‘equal sharing’ and ‘constant price’

Learners 6 and 11’s drawings represented only the unit sweets (see Pictures 4.32 and 4.33). The question required them to calculate the cost of the sweets. The unit in which they drew their pictures was not correct and I categorised this misrepresentation as ‘wrong unit’. Learner 15’s drawing and concrete representation illustrated repeated addition as she had placed R10 with the first group of sweets, R20 with the next group and R30 with the last group (see Picture 3.33). This is categorised as a misrepresentation of ‘repeated addition’, since she added the previous amount to the new amount and represented the new amount each time.

Picture 4.33: Question 5: Learner 15’s misrepresentation of ‘repeated addition’

Learner 8 was unable to draw a picture to illustrate the problem and did not know how to use the money and sweets for a concrete representation of the problem, and Learner 5 could not illustrate the problem by using semi-concrete representations.

While no misconceptions could be identified for this problem, two written misrepresentations (‘non-consideration of proportion’ and ‘equation inconsistent with answer’) were identified and categorised, as well as six misrepresentations (‘abstract numbers’, ‘answer’, ‘equal sharing’, ‘constant price’, wrong unit’ and ‘repeated addition’) in the semi-concrete and concrete representations.

4.7.2 Question 5: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2). The calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.5). The following calculation technique levels were identified and categorised: Level 2B (double counting) and Level 2C (Algorithms). The calculation technique types identified for Question 5 were ‘counted in’ and ‘column method’ (see Table 4.5).

Learners 3, 8, 15 and 17’s calculation techniques could not be categorised as it was difficult to determine what they had done. Learner 3 tried to divide, but could not find an answer, while Learners 8 and 17 could not decide on how to calculate the answer. Leaner 15’s equation did not show how she had calculated the answer as she had not used the numbers she had written down for her calculation.

Learners 5, 6, 7, 9, 10, 11 and 14 were categorised on Level 2. Level 2 calculation technique levels are considered to be multiplicative, which means that the participants were able to think multiplicatively when solving the problem. Three of these seven participants (Learners 7, 9 and 14) were categorised on Level 2B (double counting) and their calculation technique type was categorised as ‘counted in’. Learner 7 counted in 10s, whereas Learners 9 and 14 counted in 4s to solve the problem. Four of the seven participants (Learners 5, 6, 10 and 11) were categorised on Level 2C (algorithms) and their calculation technique type was categorised as ‘column method’. They wrote the numbers below each other in a column and multiplied with one digit at a time to calculate the answer.

Seven participants used multiplicative calculation techniques, while the calculation techniques of four participants could not be determined and categorised. The calculation errors will be discussed next.

4.7.3 Question 5: Calculation errors

Two calculation errors were identified and categorised, namely ‘counting error’ and ‘a disconnect between abstract and drawing’ (see Table 4.5). For her abstract representation, Learner 11 multiplied 48 by 10, but forgot to add the zero of the 10 with which she had multiplied and gave the answer as 48. This calculation error was categorised as a ‘counting error’. Learners 6 and 9’s semi-concrete drawings with their answers were categorised as a ‘disconnect between abstract and drawing’. Learner 6 drew three groups of four and wrote the answer as R120, which did not correspond with what he had drawn (see Picture 4.34).

Picture 4.34: Question 5: Learner 6’s calculation error, ‘disconnect between abstract and drawing’

Learner 9 also drew three groups of four and wrote R10 next to each group, but gave the answer as 12, which was the number of sweets and not the amount in rand.

4.7.4 Question 5: Discussion of the analysis

Question 5 was conceptually more complex than Question 4, which falls under the category 'multiplicative comparison'. The reason for this is that even though problems of this class are introduced in Grade 4 (Bowie et al., 2012a; DBE, 2011b), experience has shown that learners in South African primary schools are rarely presented with them, which explained why the participants were not confident when attempting to solve this kind of problem. As a result of the one-hour limit I had placed on the interviews, only eleven of the fifteen participants were asked to solve this problem. The analysis showed that only Learner 9 could solve this problem of 'simple proportion' in all the representational forms without any difficulty. With his semi-concrete and concrete representations of three groups of four sweets and a R10 note next to each group, one could infer his conceptual understanding of 'simple proportion'. Moreover, his procedural fluency could be inferred from his use of a multiplicative calculation technique to solve the problem. This could indicate good integrated connections between the different representations, since more connections indicate better understanding (Hiebert & Carpenter, 1992). His good interconnected schema of 'simple proportion' between the three representations enabled him to consider the correct multiplicative concept-in-action, which allowed him to choose the most effective theorem-in-action to solve the problem (Vergnaud, 1998; 2013a; 2013b). This in turn could indicate that he had good strategic competence (see conceptual framework, Figure 2.1 in Chapter 2).

A further three participants (Learners 3, 14 and 17) could represent 'simple proportion' by showing three groups of four sweets each with a R10 note next to each group in both their semi-concrete drawings and their arrangements of the 3D material. However, Learner 3 used division for the equation and Learner 17 could not solve the problem by using abstract representations. Learner 14 used multiplication, but the answer was incorrect. One could therefore infer that they had a conceptual, but not an abstract understanding of 'simple proportion'. The abstract schema was clearly lacking since their abstract equations did not show an understanding of 'simple proportion'. It could therefore be concluded that their schema of 'simple proportion' was limited.

Learners 5 and 6 used the concrete material to make three groups of four sweets with a R10 note next to each group, which was indicative of ‘simple proportion’. When only one type of representation is correct, it could be an indication that only limited connections were made (Ayub et al., 2013). One could conclude that their conceptual understanding of ‘simple proportion’ was limited to concrete representations. Furthermore, one could infer limited conceptual understanding, with no procedural fluency, which meant that they lacked the strategic competence needed to solve the problem.

Learner 15 could verbally give the correct answer, but could not write the equation. Not a single participant could solve this problem with only abstract representations. One could possibly infer that none of the participants, with the exception of Learner 9, had any abstract schema and scheme for solving problems of this class. Learner 8 was the only participant who could not solve the problem with any of the representations. It could be that the participants had no abstract conceptual understanding and because their concepts-in-action was lacking, they had no theorems-in-action to solve the problem.

Only Learner 9 could solve this problem of ‘simple proportion’ in all the representational forms. Three participants (Learners 3, 14 and 17) were able to solve it with semi-concrete and concrete representations, while Learners 5 and 6 could solve it in the concrete representation only. In total, six of the eleven participants could solve this problem with at least one of the representations and only Learner 8 could not solve it with any of the representations.

4.7.4.1 Question 5: Discussion of misconceptions and misrepresentations

No misconceptions could be identified from the participants’ explanations. However, I could identify and categorise written misrepresentations in their written equations. The two written misrepresentations that were identified are ‘non-consideration of proportion’ and ‘equation inconsistent with answer’. Six of the eleven participants (Learners 5, 6, 7, 8, 10 and 11) did not consider the proportionality of this problem by first determining the number by which they needed to multiply, therefore the misrepresentation was categorised as ‘non-consideration of proportion’. They simply multiplied by all three numbers, or by a combination of two of the three numbers given in the problem, which could indicate that they did not understand proportionality. It could therefore be inferred that they lacked an abstract schema of ‘simple proportion’ and a scheme to solve the problem.

Learners 14 and 15’s equations were inconsistent with their answers. They could mentally calculate the answer correctly, but the numbers used for their multiplication did not match their answers. This was categorised as ‘equation inconsistent with answer’ and could indicate that the participants had some type of scheme for solving the problem, but could not explain their reasoning in writing. Their lack of experience with problems of this class could explain why they struggled to present their reasoning in abstract equations.

In the semi-concrete and concrete representations, six other misrepresentations could be identified and categorised, namely ‘wrong unit’, ‘equal sharing’, ‘answer’, ‘constant price’, ‘abstract numbers’ and ‘repeated addition’. Learners 6 and 11 drew only sweets, even though the question was about the cost of the sweets. It is possible that they did not realise that their pictures did not reflect the question, and that they drew only sweets because they were less abstract than money and therefore easier to visualise and represent.

One participant (Learner 10) simply replaced his abstract equations with sweets and money in both the semi-concrete and concrete representations. This misrepresentation was categorised as ‘abstract numbers’. He probably recalled his abstract equation, which could indicate that he had only an abstract schema of the problem and that he struggled to visualise the problem in any other way. Learners 7 and 11 drew the answer and/or arranged 3D material to illustrate it, but did not include the problem. This misrepresentation was categorised as ‘answer’ and could be an indication that these participants also found it difficult to visualise the problem due to a poor semi-concrete and concrete schema of ‘simple proportion’. Learners with learning difficulties generally struggle to visualise abstract concepts (Allsopp et al., 2007).

Learner 6 drew three groups of four sweets, which was categorised as ‘equal sharing’. Learner 7 took twelve sweets and placed a R10 note with each sweet, which was categorised as ‘constant price’. Both these participants were probably either thinking of problems of other classes that they had previously solved and became confused, or did not have conceptual schemas of ‘simple proportion’ and used schemas that they thought resembled the question.

Learner 15 misrepresented semi-concrete and concrete representations as she placed a R10 note with the first group of four sweets, R20 with the second group of sweets, and R30 with the third group of sweets, instead of one R10 note with each group. This misrepresentation was categorised as ‘repeated addition’. One could infer that although she had a conceptual understanding of ‘simple proportion’, she was unable to demonstrate it correctly. She might have been thinking of her scheme of repeated addition, and not of the problem, when she represented the latter.

4.7.4.2 Question 5: Discussion of calculation technique levels and types

Seven of the eleven participants (Learners 5, 6, 7, 9, 10, 11 and 14) used multiplication for their equations and were categorised on Level 2. However, only Learner 9 could solve the problem by providing an answer that corresponded with his equation. The other participants knew that they had to multiply, but multiplied by the wrong numbers. Learners 7, 9 and 14 were categorised on Level 2B (double counting) and the calculation technique type was categorised as ‘counted in’. The other four participants (Learners 5, 6, 10 and 11) were categorised on Level 2C (algorithms) and their calculation technique type was categorised as the ‘column method’. This could indicate that although these participants, with the exception of Learner 9, knew that they had to multiply, they probably did not have a correct abstract schema of ‘simple proportion’ and could therefore not solve the problem. Learner 9, one could infer, had an abstract schema of ‘simple proportion’ and could choose a scheme that could abstractly represent and solve the problem. Furthermore, one could infer that Learner 9 was the only one who had procedural fluency and some strategic competence, since he did not use one of the highest calculation technique types to solve the problem.

Learner 3 divided and was not categorised on any calculation technique level, and I was unable to determine Learners 8 and 15’s calculation technique levels. Learner 17 did not know how to solve the problem with an abstract representation. One could therefore infer that these participants did not have any abstract schema of ‘simple proportion' and therefore no scheme for solving the problem. Because their concepts-in-action were lacking, they could not choose theorems-in-action to solve the problem.

4.7.4.3 Question 5: Discussion of calculation errors

Only two calculation errors were identified and categorised, namely 'counting error' and 'disconnect between abstract and drawing'. It is difficult to give decisive reasons for these calculation errors, which could be related to learning difficulties. Learner 11 forgot to add the zero when he multiplied by ten. This error, categorised as a 'counting error', could indicate that the participant did not know how to multiply by ten, or that he simply forgot to add the zero to the answer, which could be due to a lack of concentration. Learners 6 and 9's drawings did not correspond with their answers, which were categorised as a 'disconnect between abstract and drawing'. This could possibly be because although they could recall the answers they had given in their abstract representations, they did not realise that their drawings did not correctly illustrate those answers. This could possibly indicate that they did not know that they could use their drawings to verify their answers. It is also possible that they were not familiar with drawing pictures of word problems. In my view, teachers who allow learners to make drawings of problems should explain to them how they can use their pictures to verify their answers. In Question 5, which required the learners to draw sweets and money, it was the total cost of the sweets (the amount of money) that had to be calculated.

4.8 Question 6: Combined categorising, analysis and discussion

The sixth task-based question was: *A piece of paper is 30 cm long and 20 cm wide. What is the area of the paper?* Question 6 represented multiplicative problems of the class 'area' (Vergnaud, 1983) and, together with Questions 7 and 8, falls under the category 'rectangular array'. These questions are conceptually more complex than the previous ones (see section 2.3.2.2 in Chapter 2 for a full explanation). Only eleven of the fifteen participants answered this question because of the one-hour time limit I had placed on the interview. As with the other questions, they were asked to answer this question by first using abstract representations, then semi-concrete and finally concrete representations. I expected them to illustrate their conceptual understanding by drawing a rectangle and to write 30 cm along the long edge and 20 cm along the short edge. For the concrete representations, I used a clean sheet of paper and asked them to show me its length, breadth and area.

When I prepared the questions for the task-based interviews, I accidentally swopped Questions 6 and 7 and only realised my mistake after I had collected the data. Developmentally the order should be: Question 7, then Question 6 and then Question 8. I will discuss the answers in the order in which the questions were asked to maintain a logical sequence. Although, according to the CAPS, problems of this class should only be introduced abstractly in Grade 8, I found examples in a Grade 7 textbook (Bowie et al., 2013; DBE, 2011c). The participants in this study had already been introduced to area, but had only been required to count the number of blocks inside a fixed space to determine area, and not to use a formula.

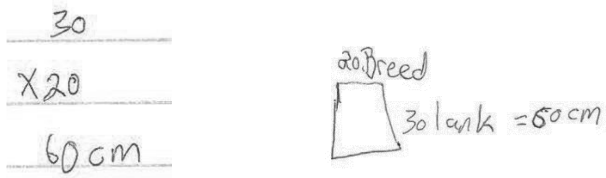
Table 4.6 contains a summary of the categories and subcategories of the data for the eleven participants' representations. I colour coded similar subcategories for easy recognition and analysed the data under each heading, starting with the participants' conceptions, then their misconceptions and misrepresentations. This is followed by an analysis of the levels and types of calculation techniques and errors. Finally, the analysis of all the data for this question is discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

Table 4.6: Summary of the categories and subcategories of Question 6 for eleven participants' representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE REPRESENTATIONS	
Learner	**Written operation and answer**	**Written misrepresentation**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 3	[Unable to do]					Addition + (142)	Perimeter		Length, breadth, area	
Learner 5	× (60 and 40)	Perimeter	Level 2F Known multiplication fact	Times table		[Unable to draw]			Length, breadth, area	
Learner 6	× (60)		Level 2F Known multiplication fact	Times table	Counting error (Answer 60)	× Area (60)		Counting error (Answer 60)	Length, breadth, area	
Learner 8	[Unable to do]					[Unable to draw]			[Unable to identify it]	
Learner 9	+ (50)		Level 1 Known addition fact	Addition fact		Intention was + (50)	Intention was Answer		Length, breadth, area	
Learner 10	+ (100)	Perimeter	Level 1 Addition algorithm	Column method		Addition + (no answer)	Perimeter		Length, breadth	
Learner 11	[Unable to do]					[Unable to draw]			[Unable to identify it]	
Learner 13	- (10)		[Unable to determine]	[Unable to determine]		Intention was - (10)			Length, breadth, area	
Learner 14	× (60)		Level 2B Double counting	Counted in	Counting error (Answer 60)	× Area (60)		Counting error (Answer 60)	Length, breadth, area	
Learner 15	+ (100)	Perimeter	Level 1	2× addition		Addition +	Perimeter		Length, breadth, area	

			Derived addition fact			(100)				
Learner 17	+ (100)	Perimeter	Level 1 Derived addition fact	2× addition		Addition + (142)	Perimeter		Length, breadth, area	

Two of the eleven participants (Learners 6 and 14, whose names are marked in dark blue in Table 4.6) multiplied length by breadth for their abstract representations and could solve this problem with all three representations. Their semi-concrete pictures were rectangles and they wrote the measurements in the right places. They could also indicate the length, breadth and area of a blank sheet of paper. In spite of the fact that the equation was written correctly, they calculated the answer incorrectly (to be discussed in section 4.8.3). Picture 4.35 shows Learner 14's representation of 'area' with abstract and semi-concrete representations.



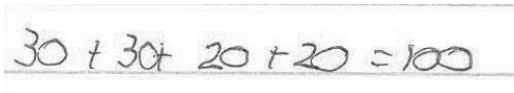
Picture 4.35: Question 6: Learner 14's equation and conceptual drawing representing 'area'

Six of the eleven participants (Learners 3, 5, 9, 13, 15 and 17, whose names are marked in purple in Table 4.6) could only indicate the length, breadth and area of a sheet of paper shown to them. They were unable to solve the problem with abstract or semi-concrete representations. Learners 8 and 11 (whose names are marked in dark red in Table 4.6) did not attempt to solve the problem as they did not know what area meant.

4.8.1 Question 6: Misconceptions and misrepresentations

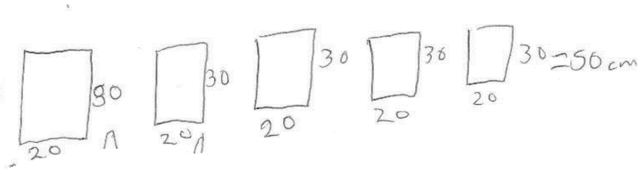
No verbal misconceptions could be identified in the explanations given by Learner 9, who added the two numbers together, and Learner 13 who subtracted. However, a written misrepresentation was identified in the abstract representations and I therefore changed the heading in Table 4.6 from verbal misconception to written misrepresentation in order to report on this misrepresentation, namely 'perimeter'. 'Perimeter' was also identified and categorised under misrepresentations of semi-concrete representations. The other misrepresentation that was identified and categorised was that of 'answer'.

Learners 10, 15 and 17's written equations and semi-concrete representations were categorised as a misrepresentations of 'perimeter', whereas Learner 3's semi-concrete representation and Learner 5's written equation were categorised as misrepresentations of 'perimeter', as they had written the measurements of all four sides of the rectangle that they had drawn and their answers showed that they had added all the sides together. Although this is an 'area' problem, these participants all added the four sides together to calculate the answer, which is indicative of a misrepresentation of 'perimeter'. Had the questions asked for the perimeter, their method would have been correct perimeter. Picture 4.36 shows how Learner 17 added the lengths of all the sides together.



Picture 4.36: Question 6: Learner 17's equation indicative of the misrepresentation of 'perimeter'

Learner 9's semi-concrete representation was categorised as a misrepresentation of 'answer'. He drew five rectangles and his answer was 50, as can be seen in Picture 4.37. When I asked him why he had drawn five rectangles, he answered: *Because the total of this is 50*. Although it could be inferred from his explanation that he had attempted to draw the answer, the same cannot be inferred from his picture.



Picture 4.37: Question 6: Learner 9's misrepresentation of 'answer'

No verbal misconception could be identified for this question, but two misrepresentations ('perimeter' and 'answer') were identified and classified. The levels and types of calculation techniques will be discussed next.

4.8.2 Question 6: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2) and the calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.6). The additive calculation technique levels identified on Level 1 were additional levels that could be used only when adding, and included ‘addition algorithm’, ‘derived additional fact’ and ‘known additional fact’. The multiplicative calculation technique levels identified were Level 2B (double counting) and Level 2F (known multiplication fact). The calculation technique types identified for solving Question 6 were: ‘addition fact’, ‘column method’, ‘2x addition’, ‘counted in’ and ‘times table’ (see Table 4.6).

Learners 3, 8 and 11 were unable to solve this problem as they did not understand what area meant. Learner 13 subtracted, but explained that he had divided, therefore I could not determine exactly what he had intended to do. Four participants (Learners 9, 10, 15 and 17) were categorised on Level 1.

Level 1 calculation techniques are additive in nature and four participants (Learner 9, 10, 15 and 17) all used calculation techniques that were not included in my conceptual framework. They thought that it was an addition problem and used calculation techniques that are used only when adding. Learner 9 knew that 20 and 30 would add up to 50 and the calculation technique type used to solve the problem was categorised as ‘addition fact’. Learner 10’s calculation technique type was categorised as the ‘column method’, as he had written the numbers below one another and had added each digit separately. Learners 15 and 17’s calculation technique type was categorised as ‘2x addition’ as they first added two sides, then the other two sides, after which they added the two answers together. Learner 17 explained her calculation as follows: *I said 30 + 30… 60. Plus 20 plus 20 is 40, and 60 + 40 is 100.*

The Level 2 calculation techniques are multiplicative, which means that the participants should think multiplicatively when solving problems (Carrier, 2014). Three participants (Learners 5, 6 and 14) used multiplicative calculation techniques to solve this problem. Learner 14 was categorised on Level 2B (double counting). Her calculation technique type was categorised as ‘counted in’ as she had counted in 2s to calculate the answer. Learners 5 and 6 were categorised on Level 2F (known multiplication fact) and their calculation technique type was categorised as ‘times table’. Learner 5 multiplied 30 x 2 and 20 x 2, while Learner 6 incorrectly ‘knew’ that 30 x 20 = 60.

Three participants used multiplicative calculation techniques, while four used additive calculation techniques to calculate the answer to this problem. Three made no attempt to solve the problem and the calculation technique used by one could not be identified. The calculation errors will be discussed next.

4.8.3 Question 6: Calculation errors

The only calculation error identified and categorised was a 'counting error'. Both Learners 6 and 14 multiplied 20 by 30 in both their abstract and semi-concrete representations and calculated the answer to be 60 instead of 600. This calculation error was categorised as a 'counting error'.

4.8.4 Question 6: Discussion of the analysis

As mentioned in section 4.8, this question is developmentally more complex than Question 7. When I collected my data I inadvertently swapped Questions 6 and 7, with the result that the more complex question was asked first. From a developmental perspective, Question 6 belongs between Questions 7 and 8, as Question 7 deals with 'arrays', which have discrete elements and can be counted. Questions 6 and 8 have measurements which are not discrete and cannot be counted and cognitively more complex than Question 6. These three questions are all in the category 'rectangular arrays'. Although the learners had worked with area before, it was in a context where they had to cover an area with blocks and count the blocks to calculate the answer. According to the CAPS and school textbooks, the learners had no previous experience of abstract calculations that required the multiplication of length by breadth (DBE, 2011c).

My analysis showed that two of the eleven participants (Learners 6 and 14) could write an equation for which they multiplied the length by the breadth. Their drawings were indicative of the 'area' class of problems as they had indicated the length and breadth correctly in writing. Using 3D material, they could also indicate where the length, breadth and area were. According to Hiebert and Carpenter (1992), the more connections there are between the representations, the better the understanding will be. One can therefore infer that these learners had a conceptual understanding of 'area', which could be indicative of a schema of 'area', and that there were good connections between the different representations. Moreover, their procedural fluency could be inferred from their use of multiplicative calculation techniques to solve the problem.

Six of the participants (Learners 3, 5, 9, 13, 15 and 17) could only indicate the length, breadth and area of the sheet of paper when working with 3D material. Where only one type of representation was correct, it is possible that only limited connections were made (Ayub et al., 2013). One could therefore infer that these participants had only a concrete schema of the 'area' class of problems, which could have led to the use of incorrect schemes to solve the problem. It could further be inferred that they had an incomplete schema of this problem, limited to concrete representations, with no procedural fluency. One possible reason for having chosen the incorrect scheme is that when I asked them to solve the problem using abstract representations, they only tried to access the abstract schema, and in its absence, they accessed the incorrect concept-in-action, which led them to choosing an incorrect theorem-in-action. This could indicate a lack of strategic competence.

Learner 10 thought that the problem should be solved through addition and could not solve it in any of the representational forms. It could be inferred that this participant has a schema of 'perimeter', but not of 'area' and therefore used an additive scheme to solve the problem. Learners 8 and 11 did not know what 'area' meant and did not attempt to solve this problem in any of the representational forms. It is possible that they had no abstract conceptual understanding, and because their concepts-in-action were lacking they had no theorems-in-action to solve the problem. This could indicate that they had no schemas of 'area' and therefore no schemes to solve it. Since these participants had no concepts-in-action available, they also did not have theorems-in-action that could be used to solve the problem.

Learners 6 and 14 demonstrated a conceptual understanding of 'area' in all representational forms, while six participants (Learners 3, 5, 9, 13, 15 and 17) demonstrated their conceptual understanding when working with the concrete material. Eight of the eleven participants could show conceptual understanding of this problem with at least one of their representations. Learner 10 could not solve this problem with any of the representations, while Learners 8 and 11 did not know what 'area' meant and made no attempt to solve the problem.

4.8.4.1 Question 6: Discussion of misconceptions and misrepresentations

Although I identified no misconceptions, I did identify and categorise a written misrepresentation that was also identified in the semi-concrete representations, namely 'perimeter'. Four participants (Learners 5, 10, 15 and 17) misrepresented this problem in their abstract equations when they added all the sides together. Learners 3, 10, 15 and 17 wrote the length and breadth on all the sides, which I categorised as 'perimeter' as their answers indicated perimeter. It is possible that they either mistook area for perimeter, or lacked exposure to the abstract calculation of 'area' (DBE, 2011c). A possible inference is that they chose the only schema available to them, i.e. 'perimeter', and used an additive scheme to solve the problem.

Learner 9 attempted to base his drawing on the abstract representation by drawing five rectangles, thus categorising it as 'answer'. This was a strange way of representing the problem as each rectangle could be interpreted as a discrete entity with its own area. I am convinced that he did not have an 'area' schema as he added the two numbers instead of multiplying them and therefore used an additive scheme to solve the problem. It is possible that he had an abstract equation in mind when he tried to use semi-concrete representations to solve the problem.

4.8.4.2 Question 6: Discussion of calculation technique levels and types

Three of the eleven participants (Learners 5, 6 and 14) were categorised on Level 2. However, Learner 5 did not solve the problem as she multiplied the length twice and the breadth twice. She was categorised on Level 2E (known multiplication fact) and the calculation technique level was categorised as 'times table'. Learner 14 was categorised on Level 2B (double counting) and the calculation technique type was categorised as 'counted in'. Learner 5 was categorised on Level 2F (known multiplication fact) and the calculation technique was categorised as 'times table'. Since from this one could infer that even though these learners thought multiplicatively, none of them could actually solve the problem and no procedural fluency could be inferred for this question, which led to the conclusion that not one of them was strategically competent.

Four participants (Learners 9, 10, 15 and 17) were categorised on new Level 1 as these calculation techniques can be used only for addition. Learner 9 was categorised on Level 1 (known addition fact) and the calculation technique type was categorised as 'addition fact'. Learner 10 was categorised on Level 1 (addition algorithm) and the calculation technique type was categorised as the 'column method'. Both Learners 15 and 17 were categorised on Level 1 (derived addition fact) and the calculation technique type was categorised as '2× addition', which could indicate that these participants did not have multiplicative schemes to solve this problem. Because their concepts-in-action were incorrect, they chose incorrect theorems-in-action to solve the problem.

4.8.4.3 Question 6: Discussion of calculation errors

One calculation error could be identified and categorised, namely 'counting error'. Learners 6 and 14 made 'counting errors' in both their abstract and semi-concrete representations, since they both struggled to multiply 30 by 20. Their answer was 60 instead of 600. As always, it is difficult to give reasons for their calculation errors. A possible explanation is that they had not yet learnt to multiply with two zeros, or that, due to their learning difficulties and their possible struggle with mental calculation, they had simply forgotten to add the second zero.

4.9 Question 7: Combined categorising, analysis and discussion

The seventh task-based question was: *There are 8 rows of chairs in the school hall. There are 8 chairs in each row. How many chairs are there altogether?* This question represented the 'array' class of multiplication problems (Hurst, 2015; Jacob & Mulligan, 2014; Simon et al., 2010). Questions 6, 7 and 8 fall in the category 'rectangular arrays' (see conceptual framework, Figure 2.1 in Chapter 2). Only eleven of the fifteen participants had time to answer this question due to the one-hour time limit I had placed on the interview. They were again asked to start with abstract representations, followed by semi-concrete and finally concrete representations. I expected them to illustrate their conceptual understanding by drawing eight rows of eight chairs each. When working with the 3D blocks, I expected them to make eight rows of eight blocks each.

As explained in section 4.8, Questions 7 was asked before Question 6. Question 7 was the least conceptually abstract and complex question in the category ‘rectangular arrays’ as discrete chairs are easier to visualise than measurements such as ‘area’ and ‘volume’. Furthermore, although problems of this class are introduced from Grade 1 (DBE, 2011a; Mostert, 2011) and are solved throughout the Foundation Phase (Grades 1 to 3), they are rarely dealt with really during the Intermediate Phase (Grades 4 to 6), when grids are introduced as the basis for calculating area.

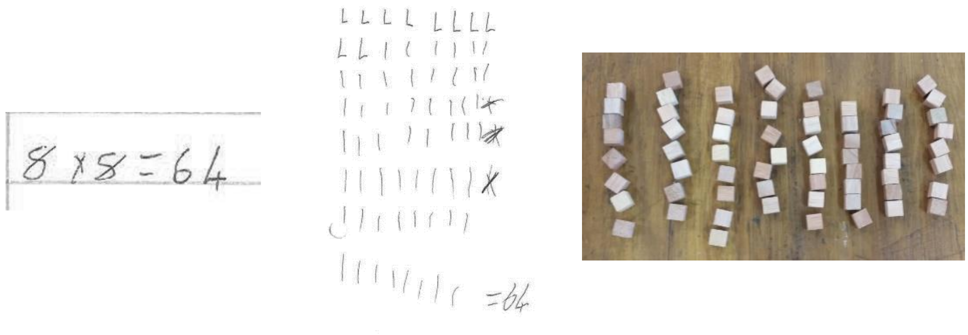
Table 4.7 contains a summary of the categories and subcategories of the data for all the participants’ representations for Question 7. Similar subcategories are colour coded for easier recognition. The participants’ conceptions, misconceptions and misrepresentations are analysed under the various headings. This is followed by an analysis of the levels and types of calculation techniques and calculation errors. Finally, the analysis of all the data for this question is discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

Table 4.7: Summary of the Question 7 categories and subcategories for eleven participants and all the representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner	**Written operation and answer**	**Written misrepresentation**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 4	+ (16)		Level 1 Known addition fact	Addition fact		Addition + (16)	Combination of measures		Intention was +	Abstract numbers
Learner 5	× (64)		Level 2B Double counting Level 1C Repeated addition	Counted in Counted on		× Array (No answer)			Multiplication ×	Equal sharing
Learner 6	× (64)		Level 2E Derived multiplication fact	Split multiplication and addition		Intention was × (64)	Abstract numbers		× Array	
Learner 7	+ (16)		Level 1A Unitary counting	Counted on		× Array (63)		Counting error (Answer 63)	× Array	
Learner 8	+ (12)		Level 1C Repeated addition	Repeated addition	Counting error (Answer 12)	Addition + (8)	Combination of measures	Writing error (write 8, say 16)	Addition +	Combination of measures
Learner 9	× (64)		Level 1C Repeated addition	Counted on		Multiplication × (64)	Equal sharing		Multiplication ×	Equal sharing
Learner 10	× (128)	Perimeter	Level 2F Known multiplication fact Level 1 Known addition fact	Times table Addition fact		Intention was × (128)	Abstract numbers		Intention was ×	Perimeter

Learner 13	+ (63)		Level 1C Repeated addition	Counted on	Counting error (Answer 63)	× Array (72)			× Array	
Learner 14	× (64)		Level 1C Repeated addition	Counted on		× Array (No answer)			Multiplication ×	Equal sharing
Learner 15	× (64)		Level 2B Double counting Level 1C Repeated addition	Counted in Counted on		× Array (No answer)			Multiplication ×	Equal sharing
Learner 17	× (64)		Level 2F Known multiplication fact	Times table		× Array (64)			× Array	

Only one of the eleven participants (Learner 17, whose name is marked in dark blue in Table 4.7) multiplied to solve the problem with her abstract representation and could solve it with all three representations. Furthermore, she drew eight rows of eight chairs each and placed the 3D blocks in eight rows of eight blocks each, which is indicative of the ‘array’ concept (see Picture 4.38).



Picture 4.38: Question 7: Learner 17’s equation, conceptual drawing and 3D material representing the ‘array’ concept

Learners 7 and 13 (whose names are indicated in light blue in Table 4.7) could draw the problem correctly and placed the 3D blocks in an array, demonstrating their understanding of an ‘array’ with their semi-concrete and concrete representations, but could not solve the problem with abstract representations. However, Learner 13 added an extra row to his array, which resulted in an incorrect answer. Learner 6 could solve the problem with abstract and concrete representations, and Learners 14 and 15 could solve it with abstract and semi-concrete representations. Learners 5 and 9 (whose names are marked in purple in Table 4.7) used a multiplicative equation and could solve the problem with abstract representations only.

4.9.1 Question 7: Misconceptions and misrepresentations

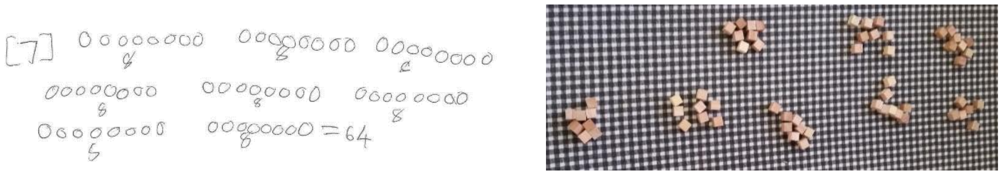
I could not identify any verbal misconceptions in the explanations given by Learners 4, 7 and 8, who had added the two numbers together. A written misrepresentation was identified in the abstract representations and I therefore changed the heading in Table 4.7 from verbal misconception to written misrepresentation in order to report on this misrepresentation, namely ‘perimeter’. ‘Perimeter’ was also identified and categorised under misrepresentations of semi-concrete representations. Another three misrepresentations could be identified and categorised, namely ‘abstract numbers’, ‘equal sharing’ and ‘combination of measures’ (see Table 4.7).

Learner 10 used the method followed for calculating perimeter and added all four sides together to calculate his answer. However, he multiplied one length by one breadth and then the other length and breadth and finally added the two answers together. He used the 3D blocks to form the outline of a square with eight blocks on each side. He did not fill in the inside of the square with 3D blocks, even though he said that that should be done.

Learners 6 and 10’s drawings and Learner 4’s concrete representation were categorised as misrepresentations of ‘abstract numbers’. Learner 10 replaced the abstract numbers with chairs to calculate his answer, as shown in Picture 4.39, and Learner 6 used a combination of pictures and numbers when he drew eight chairs and then wrote the number eight, also replacing his abstract equation with chairs and numbers. Learner 4 arranged the 3D blocks to represent the numbers used in her abstract equation, as shown in Picture 4.39.

Picture 4.39: Question 7: Learner 4 and Learner 10’s misrepresentation of ‘abstract numbers’

Another misrepresentation that was identified and categorised was ‘equal sharing’. Learner 9’s drawing and concrete representation were categorised as misrepresentations of ‘equal sharing’, while only the concrete representations of Learners 5, 14 and 15 were categorised as misrepresentation of ‘equal sharing’. They had made eight groups of eight by either bundling the groups or arranging them in rows, but not in an array form. Picture 4.40 shows Learner 9’s misrepresentation of ‘equal sharing’.



Picture 4.40: Question 7: Learner 9's misrepresentation of 'equal sharing'

Learner 8's semi-concrete and concrete representations and Learners 4's semi-concrete drawing were categorised as misrepresentations of a 'combination of measures' as both had added the numbers together. Picture 4.41 shows Learner 8's misrepresentation of 'combination of measures'.

Picture 4.41: Question 7: Learner 8's misrepresentation of 'combination of measures'

While no verbal misconception could be identified for this question, I did identify and categorise four misrepresentations ('perimeter', 'abstract numbers', 'equal sharing' and 'combination of measures'). The levels and types of calculation techniques will be discussed next.

4.9.2 Question 7: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2). The calculation technique types were inductively identified and grouped into similar categories (colour coded in green and dark pink in Table 4.7). The calculation technique levels that were identified were the following: Level 1A (unitary counting), Level 1C (repeated addition), Level 1 (known addition fact), Level 2B (double counting), Level 2E (derived multiplication fact) and Level 2F (known multiplication fact). The calculation technique types that were identified for solving Question 7 were: 'counted on', 'addition fact', 'repeated addition', 'counted in', 'split multiplication and addition', and 'times table' (see Table 4.7).

Level 1 calculation techniques are additive in nature and six participants (Learners 4, 7, 8, 9, 13 and 14) were categorised on this level. Learners 7, 9, 13 and 14 were categorised on Level 1A (unitary counting) and their calculation technique type was categorised as ‘counted on’. They counted on eight each time. Four participants (Learners 8, 9, 13 and 14) were categorised on Level 1C (repeated addition). Learner 8 added four each time, calculating the answer and adding four again. This calculation technique type was categorised as ‘repeated addition’. Learner 13 explained: *I plussed eight sixteen times*. Learner 4 was categorised on Level 1 (known addition fact) and her calculation technique type was also categorised on Level 1 (known addition fact) as she had added eight and eight together.

Three participants (Learners 5, 10 and 15) used a combination of additive and multiplicative calculation techniques types. Learners 5 and 15 were categorised on Level 2B (double counting) and Level 1C (repeated addition). Their calculation technique types were categorised as ‘counted in’ and ‘counted on’. They both started to count in 8s, but they switched over to counting on eight each time until they arrived at the answer. Learner 10 was categorised on Level 2F (known multiplication fact) and Level 1 (known addition fact), and his calculation technique types were categorised as ‘times table’ and ‘addition fact’. He knew that eight times eight is 64 and that 64 plus 64 is 128.

The Level 2 calculation techniques are multiplicative, therefore participants who were categorised on this level were able to think multiplicatively when solving the problem (Carrier, 2014). Learners 6 and 17 used multiplicative calculation techniques to solve the problem. Learner 6’s calculation technique was categorised on Level 2E (derived multiplication fact), since his calculation technique type was categorised as ‘split multiplication and addition’. He explained his calculation as follows: *I took the one 8 and I divided it up in 4. Then I got 32 at each… 2 groups. Then I added it together, then I got 64.* Learner 17’s calculation technique was categorised on Level 2F (known multiplication fact), since her calculation technique type was categorised as ‘times table’. She knew that eight times eight is 64.

Two participants used multiplicative calculation techniques, six used additive calculation techniques and three used a combination of additive and multiplicative calculation techniques to calculate the answer to this problem. The calculation errors will be discussed next.

4.9.3 Question 7: Calculation errors

Two types of calculation errors were identified and categorised, namely 'counting errors' and 'writing errors' (see Table 4.7). Three participants' calculation errors were categorised as 'counting errors'. Learner 7 added the marks he had made in his picture incorrectly and gave 63 as his answer, and Learners 8 and 13's 'counting errors' were identified in their abstract representations. These two participants had used their fingers to count and had given the answers as 12 instead of 16 and 63 instead of 64 respectively. Learner 8 had also made another calculation error, categorised as a 'writing error', as she had written the answer as 8, but said that it was 16, which should have been the answer according to her picture.

4.9.4 Question 7: Discussion of the analysis

Question 7 was the least conceptually complex question in the 'rectangular arrays' category as it was easy to visualise discrete chairs. Problems of the class 'array' are introduced in Grade 1 (DBE, 2011a; Mostert, 2011); however, as soon as learners switch to learning about 'area' they no longer solve problems dealing with arrays. My analysis indicated that only Learner 17 could solve this 'array' problem in all the representational forms without any difficulty. Her semi-concrete and concrete representations represented an array of eight chairs in eight rows and could therefore infer a conceptual understanding of 'arrays'. Moreover, her use of multiplication calculation techniques to solve the problem inferred procedural understanding. Hiebert and Carpenter (1992) suggest that the more connections there are between different representations, the better the understanding. Learner 17 was able to choose the correct concept-in-action, which allowed her to choose her most effective theorem-in-action to solve the problem. One could therefore conclude that she had a good schema of problems in the 'array' class and an appropriate multiplicative scheme to solve them.

Two of the eleven participants (Learners 7 and 13) could solve the problem with both semi-concrete and concrete representations and drew and placed eight chairs in eight rows, which showed that they understood ‘array’ conceptually, but not in the abstract. However, since they had used addition to solve the problem one could infer that they lacked abstract schemas had only additive procedural fluency.

Four participants could solve the problem with their abstract representations and with either semi-concrete (Learners 5, 14 and 15) or concrete representations (Learner 6). From this one could infer a limited schema with limited connections between the different representations. One could also infer an abstract schema that allowed for an appropriate scheme to be used, thus procedural fluency. These learners were able to choose a concept-in-action that allowed them to choose the correct theorem-in-action (Vergnaud, 1998; 2013a; 2013b).

Learner 9 could solve the problem with the abstract representation. One could therefore infer an abstract schema of ‘array’. According to Ayub et al. (2013), if only one type of representation is correct, it could be an indication that limited connections were made. One could therefore infer a limited number of connections between the representations. It could further be inferred that Learner 9 had memorised a procedure without conceptual understanding (Ayub et al., 2013).

Learners 4 and 8 could not solve this problem in any of the representational forms. Since they added to solve the problem, one could infer that they had no ‘array’ scheme. Because they did not have an ‘array’ schema, they also lacked a scheme for solving the problem. Learner 10 knew that it was a multiplication problem, but multiplied four sides. He apparently tried to solve the problem by using a combination of ‘perimeter’ and ‘area’. This could be an indication that his schema for ‘array’ was not yet well established. It could be inferred that these participants had no abstract conceptual understanding and because their concepts-in-action were lacking, they had no theorems-in-action to solve the problem.

Learner 17 could solve this problem of ‘array’ using all the representational forms. Learners 7 and 13 could solve it with semi-concrete and concrete representations, Learners 5, 14 and 15 could solve it with semi-concrete representations, and Learner

6 with concrete representations. Learner 9 could solve it only with the abstract representation. Eight of the eleven participants could solve this problem with at least one form of representations and three (Learners 4, 8 and 10) could not solve it in any form.

4.9.4.1 Question 7: Discussion of misconceptions and misrepresentations

While no misconceptions were identified, I did identify and categorise a written misrepresentation that was also identified in the concrete representations, namely 'perimeter'. Learner 10 thought that all the sides had to be included to calculate the area, which could indicate that he mistook area for perimeter. He did know that the sides had to be multiplied, but multiplied all instead of only two. He probably had a schema of 'perimeter', but not of 'array'. It is also possible that he had no recent exposure solving 'array' problems or he mistook area for perimeter (DBE, 2011b).

Three other misrepresentations were identified and categorised, namely 'combination of measures', 'abstract numbers' and 'equal sharing'. Learners 4 and 8 added the numbers and drew two rows of eight chairs each, which was categorised as 'combination of measures'. They might have struggled to visualise the eight chairs and the rows and therefore had an addition schema of this problem, which prevented them from understanding it as a multiplicative problem. Learners 4, 6 and 10 replaced the numbers in their abstract equations with chairs. This misrepresentation, categorised as 'abstract numbers', could indicate that they also had difficulty to visualise the problem and had an abstract schema, but no semi-concrete or concrete schema of this problem. One could infer that these participants struggled with abstract thinking, which is characteristic of learners with learning difficulties (Allsopp et al., 2007).

Learners 5, 9, 14 and 15 drew eight groups of eight chairs each or placed the 3D blocks in eight groups of eight, instead of rows. Even though this gave them the correct answer, it was not conceptually correctly presented. This misrepresentation was categorised as 'equal sharing'. Problems of these two classes are conceptually different. Having a schema of an 'array' is important as it is the precursor to forming an area schema (Simon, 2005; Simon et al., 2010). Having an array schema promotes learners' understanding of more abstract problems, such as 'area' problems, at a later stage. In my opinion teachers should make sure that learners have a schema of arrays to help them understand area, and eventually also volume.

4.9.4.2 Question 7: Discussion of calculation technique levels and types

Two of the eleven participants (Learners 6 and 17) were categorised on Level 2 as they used multiplication for their equations. Learner 6 was categorised on Level 2E (derived multiplication fact) and his calculation technique type as ‘split multiplication and addition’, and Learner 17 was categorised on Level 2F (known multiplication fact) and her calculation technique type as ‘times table’. This could indicate that these two participants had an abstract schema of problems of the ‘array’ class and multiplicative schemes for solving them, which in turn could indicate that they had good procedural fluency. Both used efficient calculation technique types to solve the problem, which indicates good strategic competence. Level 2E is one of the highest cognitive levels as Level 2F is the highest cognitive developmental calculation technique (Hurst & Hurrell, 2014; Zhang et al., 2011).

Learners 5, 10 and 15 used a combination of multiplicative and additive calculation techniques. Learners 5 and 15 were categorised on Level 2B (double counting) and Level 1C (repeated addition), and their calculation technique types were categorised as ‘counted in’ and ‘counted on’ respectively. Learner 10 was categorised on Level 2F (known multiplication fact) and Level 1 (known addition fact). The use of a combination of additive and multiplicative calculation techniques could indicate that these participants did not have a scheme to successfully solve the problem. Moreover, their multiplicative schemes might not have been well developed and therefore they had to make use of multiplicative and additive calculation techniques. One could conclude that they had limited procedural fluency and strategic competence.

Five participants (Learners 7, 8, 9, 13 and 14) used additive calculation technique types to solve the problem and were categorised on Level 1. Learner 7 was categorised on Level 1A (unitary counting) and his calculation technique type was categorised as ‘counted on’. Learners 8, 9, 13 and 14 were categorised on Level 1C (repeated addition). Learner 8’s calculation technique type was categorised as ‘repeated addition’ and the other three participants’ calculation technique type was categorised as ‘counted on’. Learners 7 and 8 thought that it was an addition problem, from which one could infer that they had no conceptual understanding regarding abstract representations of an 'array' and lacked the procedural fluency needed to solve the problem, which in turn indicates a lack of strategic competence. It can therefore be concluded that they did not have a schema of 'array' and no scheme to solve the problem. However, Learners 9, 13 and 14 had conceptual understanding of an 'array' as they understood it to be eight times eight. Since their schemes were additive, they had the necessary additive procedural fluency to solve this problem, but no strategic competence.

4.9.4.3 Question 7: Discussion of calculation errors

Two calculation errors were identified and categorised. Three participants (Learners 7, 8 and 13) could not keep track of their counting and were categorised as 'counting error'. When Learners 8 and 13 calculated their answers, they used their fingers and Learner 7 counted the marks on the paper. The reason for this could either be that they had difficulty remembering while counting, or that they experienced visual-spatial problems and had to try to visually keep track of where they were, which could both be ascribed to their learning difficulties. The other calculation error identified was a 'writing error'. Learner 8 drew sixteen chairs, but used her fingers to count and although she had counted sixteen, she wrote eight on the paper. It is difficult to say for sure why she made this error; however it could indicate that she was unable to see a connection between her drawing and her answer.

4.10 Question 8: Combined categorising, analysis and discussion

The eighth task-based question was: *What volume of water is needed to fill a rectangular fish tank if the fish tank is 6 metres long, 2 metres wide and 4 metres high?* and represented multiplication problems of the class 'volume' (Vergnaud, 1983). The three classes of problems dealt with in Questions 6, 7 and 8 are categorised under 'rectangular arrays' (see section 2.3.2.2 of Chapter 2 for the full explanation). Only ten of the fifteen participants answered this question due to the time limit of one-hour I had placed on the interviews. They were required to solve the problem first using an abstract representation, and then semi-concrete and concrete representations. I expected them to illustrate their conceptual understanding by drawing a rectangular prism and writing the words length, breadth and height along the correct sides. With the concrete representation (e.g. a plastic container), I expected them to show me where length, breath, height and volume are measured.

Question 8 was more conceptually complex than Questions 6 and 7 as it included the added dimension of height, making it three-dimensional. Multiplication problems of this class are introduced in an abstract format in the fourth term in Grade 6 (Seelinger & Mouton, 2012) even though, according to the CAPS, they should be introduced in Grade 7 (DBE, 2011a). This means that the participants would not have been familiar with this kind of problem in the form in which it was asked.

Table 4.8 contains a summary of the categories and subcategories of the data for all the representations of the ten participants who answered Question 8. I colour coded similar subcategories for easier recognition. I analysed the data under the different headings in Table 4.8, i.e. participants' conceptions, misconceptions and misrepresentations. This is followed by an analysis of the levels and types of calculation techniques and errors, and finally a discussion of the analysis of all the data for this question. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

Table 4.8: Summary of the Question 8 categories and subcategories for ten participants and all the representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner[3]	**Written operation and answer**	**Verbal misconception**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 4	+ (12)		Level 1A Unitary counting	Counted on		Addition + (12)	Combination of measures Discrete		None correct	Sides confusion
Learner 5	[Unable to do]					[Unable to determine]	Wrong figure		[Unable to do]	
Learner 6	× (48)		Level 2F Known multiplication fact	Times table		× Intention was volume (48)	Sides confusion		Length, breadth, height, volume	
Learner 7	+ (12)		Level 1A Unitary counting	Counted on		Intention was volume (None given)	Sides confusion		Length, height	Sides confusion
Learner 8	+ (12)		Level 1A Unitary counting	Counted on		Addition + (12)	Combination of measures Discrete		Height and volume	Sides confusion
Learner 9	+ (14)		Level 1A Unitary counting	Counted on	Counting error (Answer 14)	[Unable to draw]			Height and volume	Sides confusion
Learner 10	× (24)	Unit conversion	Level 2F Known multiplication fact	Times table		[Unable to draw]			Height and volume	Sides confusion
Learner 13	+ (12)	Keyword (needed)	Level 1A Unitary counting	Counted on		+ Intention was volume	Sides confusion		Height and volume	Sides confusion

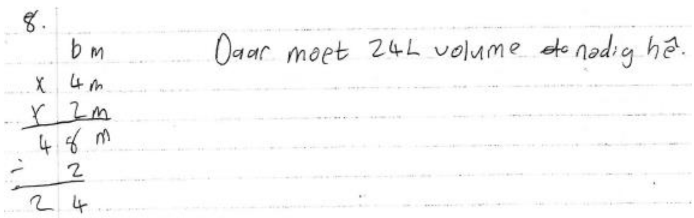
[3] Only ten of the fifteen participants answered Questions 8 to 10

						(12)				
Learner 14	× (48)		Level 2B Double counting	Counted in		× Intention was volume (48)	Wrong figure		Height and volume	Sides confusion
Learner 15	+ (12)		Level 1A Unitary counting	Counted on		+ Circle (12)	Wrong figure		Length, breadth, height, volume	

None of the ten participants could solve this problem using all three forms of representation. Learner 6 could solve it with his abstract representation and correctly indicated length, breadth, height and volume. Learner 14 could only solve the problem with the abstract representation and Learner 12 could only correctly indicate the length, breadth, height and volume (the names of these participants are marked in purple in Table 4.8). Learner 5 (whose name is marked in dark red in Table 4.8) did not know what volume was and was the only participant who did not attempt to solve this problem.

4.10.1 Question 8: Misconceptions and misrepresentations

Two verbal misconceptions were identified and categorised, namely ‘unit conversion’ and ‘keyword’ (see Table 4.8). In his written equation, Learner 10 correctly multiplied the three numbers, but then divided his answer by two (see Picture 4.42). The explanation written in Afrikaans next to his answer (see Picture 4.42), translated into English, is: *24L volume are needed.* When I asked him why he had divided, he explained: *Ma’am I have 48 there. Ma’am*, *I am going to take the half of 48.* Researcher: *OK, why?* Participant: *Because it has to be volume and not litres … full as in capacity*. *So, it must be a little bit less.* Researcher: *To get it into litres?* Participant: *Yes.* Based on his explanation the misconception can be categorised as ‘unit conversion’ as he had incorrectly assumed that to convert units you need to divide by two.

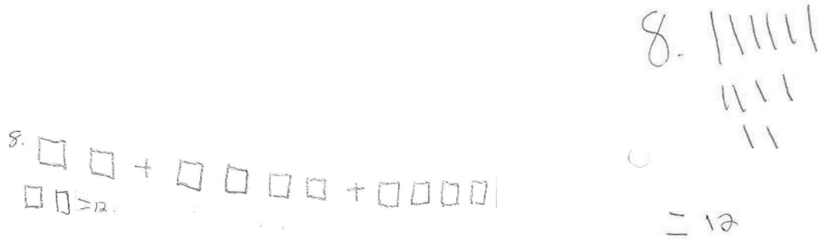


Picture 4.42: Question 8: Learner 10’s misconception of ‘unit conversion’

Learner 13’s explanation that the word ‘needed’ in the question indicated to him that he had to add. One can therefore categorise his misconception as ‘keyword’. Four other participants (Learners 4, 7, 8 and 15) also used addition for their abstract representations; however, I could not identify any verbal misconceptions.

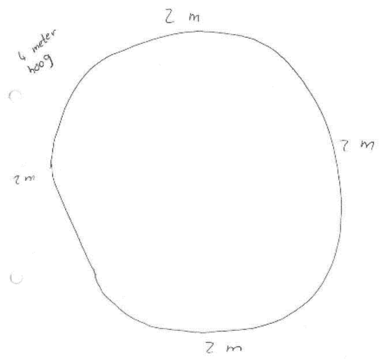
Four misrepresentations could be identified and categorised, namely ‘sides confusion’, discrete’, ‘combination of measures’ and ‘wrong figure’ (see Table 4.8). The semi-concrete and/or concrete representations of eight participants were categorised as a misrepresentation of ‘sides confusion’. Learners 7 and 13’s semi-concrete and concrete representations were categorised as a misrepresentation of ‘sides confusion’ because the measurement they wrote on their picture did not correspond with the sides along which they were written and they could not correctly indicate all the sides of the plastic container that represented the fish tank. Learner 6 could not write the correct measurements on the picture, and Learner 4 was unable to identify any of the sides correctly. Four participants (Learners 8, 9, 10 and 14) could not correctly indicate the length and breadth of the plastic container.

The semi-concrete drawings made by Learners 4 and 8 were categorised as both ‘discrete’ and a ‘combination of measures’. The latter categorisation was based on the fact that they added the three numbers together to calculate their answers. Furthermore, both these participants did not draw a 3D fish tank, but drew two, six and four discrete units, therefore this misrepresentation was categorised as ‘discrete’ (see Picture 4.43).



Picture 4.43: Question 8: Learners 4 and 8’s misrepresentations of ‘combination of measures’ and ‘discrete’

Three participants drew the incorrect figure and their misrepresentations were categorised as ‘wrong figure’. Learner 5 drew a cup, which represented a cylinder, Learner 14 drew a rectangle instead of a 3D fish tank and Learner 15 drew a circle, which were all categorised as ‘wrong figure’. Learner 15’s misrepresentation of ‘wrong figure’ can be seen in Picture 4.44. Her picture is confusing as she tried to show sides on the circle. She also ignored the 6 metres given as the length of the fish tank and did not add it to her drawing.



Picture 4.44: Question 8: Learner 15's misrepresentation of 'wrong figure'

For this question, two verbal misconceptions ('unit conversion' and 'keyword') and four misrepresentations ('perimeter', 'abstract numbers', 'equal sharing' and 'combination of measures') were identified and categorised. The levels and types of calculation techniques will be discussed next.

4.10.2 Question 8: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2) and the calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.8). The following calculation technique levels were identified: Level 1A (unitary counting), Level 2B (double counting) and Level 2F (known multiplication fact). The calculation technique types identified for solving Question 8 were 'counted on', 'counted in' and 'times table' (see Table 4.8).

Learner 5 was the only participant who did not attempt to solve this problem. Level 1 calculation techniques are additive in nature and the participants who were categorised on this level all added to solve this problem. Six participants (Learners 4, 7, 8, 9, 13 and 15) were categorised on Level 1A (unitary counting) since their calculation technique type was categorised as 'counted on'. They started at six, then added on four and then two.

Level 2 calculation techniques are multiplicative, therefore participants categorised on this level were able to think multiplicatively when solving the problem (Carrier, 2014).

Learners 6, 10 and 14 used multiplication to solve this problem. Learner 14 was categorised on Level 2B (double counting) and her calculation technique type was categorised as ‘counted in’. She first counted in 2s and then in 12s to calculate the answer. She calculated it out loud by saying: *6, 12. OK, then 12 x 4 is … 12, 24, 36, 48.* Learners 6 and 10 were categorised on Level 2F (known multiplication fact) and their calculation technique type was categorised as ‘times table’. They used their knowledge of the times tables to calculate their answers.

Three participants used multiplicative calculation techniques, while six used additive calculation techniques to calculate the answer to this problem. One participant did not know how to calculate the answer.

4.10.3 Question 8: Calculation errors

Only one calculation error was identified and categorised, namely a ‘counting error’ (see Table 4.8). Learner 9 added six, four and two and calculated it to be fourteen instead of twelve.

4.10.4 Question 8: Discussion of the analysis

Question 8, which represented multiplication problems belonging to the class ‘volume’ and had the added dimension of height, was the most conceptually complex of the questions in the category ‘rectangular arrays’, which also included Questions 6 and 7. The participants had not previously done any abstract calculations of multiplication problems in this class (Bowie et al., 2012c; DBE, 2011c). My analysis showed that none of the ten participants could solve this problem (volume) using the three different representations. Learner 6 was the only participant whose abstract and concrete representations were correct, from which it could be inferred that a schema had not yet been well formed and that only limited connections could be made between the different representations. One could assume that abstract and concrete schemas existed that allowed for an appropriate scheme to be used to solve the problem. It could therefore be concluded that although his conceptual understanding was limited, he had procedural fluency. He could draw a 3D fish tank, but was unable to connect the correct numbers to the correct sides on his drawing. He was able to choose a concept-in-action that allowed him to choose the correct theorem-in-action (Vergnaud, 1998; 2013a; 2013b).

Learner 15 could only indicate length, breadth, height and volume when using the 3D material, while Learner 14 could solve the problem only with the abstract representation. One could possibly infer that some conceptual understanding existed, but that these participants had only either a concrete or an abstract schema of problems belonging to this class. When only one type of representation is correct, it could indicate that only limited connections were made between the representations (Ayub et al., 2013). Learner 14 had procedural fluency, which Learner 15 lacked. Furthermore, since Learner 14 could not solve the problem using the other representations, one could perhaps infer that this participant had limited conceptual understanding of 'volume' and might have memorised a procedure without conceptual understanding (Ayub et al., 2013).

Seven participants (Learners 4, 7, 8, 9, 10, 13 and 15) could not solve the problem with any of the forms of representation as they had added to find the answer, except for Learner 10, who multiplied but calculated incorrectly. Learners 8, 9 and 15 struggled to illustrate 'volume' or did not attempt it at all. Only Learner 5 made no effort to solve the problem. One could infer that these participants did not have a 'volume' schema and therefore no conceptual understanding, which leads to the conclusion that since they did not have a 'volume' schema, they also did not have an appropriate scheme for solving the problem, and thus no procedural fluency. These participants might have had difficulty with this question because they had previously seen 3D figures made up of blocks indicating volume, which they had to count (Bowie et al., 2012b). Learners of this age are not expected to draw 3D figures and match the correct numbers to the correct sides. However, I think that exposure to problems of this kind is important to develop their conceptual understanding of the different sides of a 3D figure and where volume is located. If they are then asked to solve such a problem, they will be able to identify the sides represented on the 3D figure and understand why the three sides have to be multiplied. One could finally conclude that the majority of these participants associated 'volume' problems with addition, rather than with multiplication.

Learner 6 could solve this problem with abstract and concrete representations. Learner 15 could indicate only length, breadth, height and volume when using the 3D material, while Learner 14 could solve the problem with the abstract representation only. Three participants could solve this problem using at least one form of representation and seven participants (Learners 4, 7, 8, 9, 10, 13 and 15) could not solve it at all.

4.10.4.1 Question 8: Discussion of misconceptions and misrepresentations

Two misconceptions were identified and categorised, namely ‘unit conversion’ and ‘keyword’. The first verbal misconception was ‘unit conversion’. Learner 10 incorrectly thought that if he divided his answer by two it would change the unit from litre to volume. The reason for this misconception cannot be explained as I unfortunately neglected to ask him to explain his reasoning. In my opinion this participant had an abstract conceptual understanding of ‘volume’, but no procedural fluency. The second misconception identified was ‘keyword’. Learner 13 thought that the word ‘needed’ in the question indicated addition, possibly because this participant had been taught to look for keywords or it might be his own method that he thought would help him to decide which operation to use. However, this is not the best approach as very few words used in word problems reliably indicate the appropriate calculation to be applied.

The misrepresentations identified were ‘combination of measures’, ‘discrete’, ‘wrong figure’ and ‘sides confusion’. The first two, categorised as ‘combination of measures’ and ‘discrete’, were identified when Learners 4 and 8 added three discrete numbers together to calculate ‘volume’, most likely because they had no ‘volume’ schema and the problem was too abstract to visualise. Allsopp et al. (2007) do point out that learners with learning difficulties may struggle with abstract thinking, as seen here. The difference between this question and the question involving ‘array’ was that volume measurements are continuous, while ‘arrays’ consist of discrete objects. Discrete items can be counted, whereas continuous measurements cannot be counted, which makes calculation more cognitively complex.

The third misrepresentation, ‘wrong figure’, was identified when Learner 15 drew a circle instead of 3D fish tank, Learner 5 drew a cup and Learner 14 attempted to draw a fish tank, but drew a rectangle. It is possible that these learners had no conceptual schema of 'volume', or simply did not known how to draw a 3D fish tank. The last and most significant misrepresentation, evident in the work of eight of the ten participants (Learners 4, 6, 7, 8, 9, 10, 13 and 14), was categorised as 'sides confusion'. These participants struggled mostly with identifying the length and breadth of the 3D figure, which led to the assumption that their schemas of both 'area' and 'volume' had not yet been properly formed. They seemed to be unable to understand that 'volume' is simply an extra dimension (height) added to 'area', and that the length and breadth of an area remain the same when height is added. Since they did not have a good schema of 'volume', they were unable to choose an appropriate scheme, which could explain why they could not solve the problem abstractly.

4.10.4.2 Question 8: Discussion of the calculation technique levels and types

Learners 6, 10 and 14 multiplied to calculate the answer and were categorised on Level 2. Learner 14 was categorised on Level 2B (double counting) and the calculation technique type used was categorised as 'counted in'. Learners 6 and 10 were categorised on Level 2F (known multiplication fact) and their calculation technique type was categorised as 'times table'. This could indicate that they had an abstract schema and multiplication schemes for 'volume' problems, which in turn could indicate that they had procedural fluency. However, this does not apply to Learner 10, who in the end divided for 'unit conversion'. Learner 6 appeared to have good strategic competence, since he had used the calculation technique type 'times table' on Level 2F, which demonstrates abstract thinking and is seen as the highest cognitive developmental calculation technique (Hurst & Hurrell, 2014; Zhang et al., 2011).

Six participants (Learners 4, 7, 8, 9, 13 and 15) added to solve the problem and were categorised on Level 1. They were categorised on Level 1A and their calculation technique type was categorised as 'counted on', which could indicate that they lacked a correct schema or scheme of 'volume' and that their schemes were additive. They therefore lacked the conceptual understanding and procedural fluency, and also the strategic competence required to solve the problem. Their concepts-in-action were incorrect and therefore they chose incorrect theorems-in-action.

4.10.4.3 Question 8: Discussion of calculation errors

Only one calculation error was identified and categorised, namely 'counting error'. Learner 9 thought that he needed to add to solve the problem and added the numbers incorrectly. This 'counting error' might have occurred because he could not keep count on his fingers, but I cannot explain it as I did not ask him how he had arrived at fourteen.

4.11 Question 9: Combined categorising, analysis and discussion

The ninth task-based question was: *There are 3 brands of cool drinks (Coke, Pepsi and Sprite), which are available in both cans and bottles. If you want to buy one cool drink, how many different possibilities are there?* Since participants had trouble understanding this question, I repeated the last part by saying: *If you want to buy one cool drink, how many cool drinks can you choose from?* Question 9 represented multiplication problems of the class 'combinations' (Vergnaud, 1983) under the category 'Cartesian product' (see section 2.3.2.3 and Figure 2.1 in Chapter 2). Ten of the fifteen participants answered this question. I expected them to show their conceptual understanding by drawing three cans and three bottles of each brand and to place three cans and three bottles of each brand on the table.

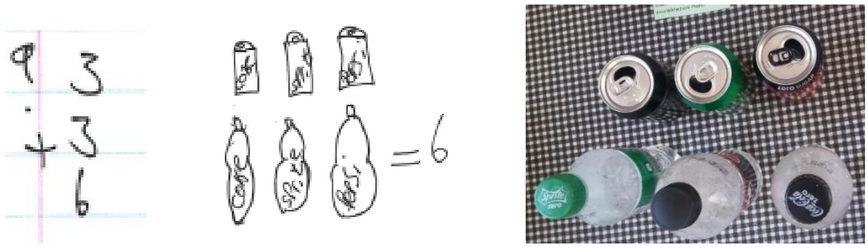
This question is conceptually more complex than all the previous questions as the participants had not seen questions of this type before and therefore found it difficult to understand and visualise. Problems of this kind are never taught in primary school and are only introduced in Grade 9 (DBE, 2011c) under the topic probability.

The categories and subcategories of all the participants' data for Question 9 are summarised in Table 4.9. Colour coding was used for easier recognition. I analysed the data under each heading in Table 4.9, starting with the participants' conceptions, then their misconceptions and misrepresentations. This is followed by an analysis of the levels and types of calculation techniques and calculation errors. Finally, the analysis of all the data for this question is discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

Table 4.9: Summary of the Question 9 categories and subcategories for ten participants and all the representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner	**Written operation and answer**	**Verbal misconception**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete repreentations**	**3D material conception**	**3D material misrepresentation**
Learner 6	+ (6)		Level 1C Repeated addition	Counted in		× Combinations (6)			× Combinations	
Learner 8	[Unable to do]					[Unable to draw]			[Unable to do]	
Learner 9	× (3)	Non-consideration of all units	Level 2B Double counting	Counted in		Addition + (3)	One unit		Addition +	One unit
Learner 10	[Unable to do]					Addition + (3)	One unit		Intention was Combinations ×	One unit
Learner 11	None (2)	Non-consideration of all units				× Combinations (6)			× Combinations	
Learner 13	+ (6)		Level 1 Known addition fact	Addition fact		× Combinations (6)			× Combinations	
Learner 14	None (3)	Non-consideration of all units				Addition + (3)	One unit		× Combinations	
Learner 15	None (3)	Non-consideration of all units				Addition + (3)	One unit		Addition +	One unit
Learner 16	+ (18)		Level 1C Repeated addition	Counted on		Multiplication ×	Equal sharing		Multiplication ×	Equal sharing
Learner 17	None (3)	Non-consideration of all units				Addition + (1)	One unit		[Unable to determine]	

As was the case with the previous question, none of the ten participants could solve this problem using all three representations. Three of the participants (Learners 6, 11 and 13, whose names are marked in light blue in Table 4.9) could solve the problem with semi-concrete and concrete representations, but their equations in their abstract representations were not indicative of multiplication. Even though they had used addition for their equations, they could draw pictures that showed three bottles and three cans of cool drink, and placed three bottles of each brand and three cans of cool drink on the table, which indicated an understanding of 'combinations'. Picture 4.45 shows Learner 13's understanding of 'combinations'.



Picture 4.45: Question 9: Learner 13's the equation, conceptual drawing and 3D material representing 'combinations'

Learner 14 (whose name is marked in purple in Table 4.9) could solve this problem only with a concrete representation. Learner 8 (whose name is marked in dark red in Table 4.9) was the only participant who did not attempt to solve it as she did not understand the question.

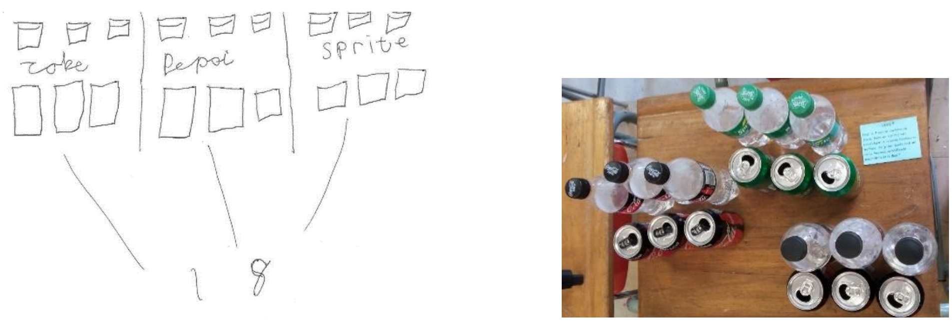
4.11.1 Question 9: Misconceptions and misrepresentations

One verbal misconception was identified and categorised as 'non-consideration of all units' (see Table 4.9). Five participants (Learners 9, 11, 14, 15 and 17) were categorised with this misconception. Learners 9, 14 and 17 considered only the different brands of cool drinks, ignoring the type of containers in which they were sold. Learner 9, for example, explained: *Ma'am, because there are three brands of cool drinks,* while Learners 11 and 15 considered only the type of containers and Learner 11 explained: *Because you can buy either a bottle or a can*, ignoring the different brands. Learner 16 used addition in the abstract representations and I could not identify any verbal misconception.

In the semi-concrete and/or concrete representations, two misrepresentations were identified and categorised, namely ‘one unit’ and ‘equal sharing’ (see Table 4.9). Three participants’ semi-concrete and concrete representations misrepresented only ‘one unit’, namely bottles (Learners 9 and 15) and Pepsi (Learner 10). Picture 4.46 shows Learner 15’s ‘one unit’ misrepresentation in the semi-concrete and concrete representations. Learners 14 and 17’s semi-concrete drawings misrepresented ‘one unit’, namely bottles.

Picture 4.46: Question 9: Learner 15’s misrepresentation of ‘one unit’

The other misrepresentation that was identified and categorised was ‘equal sharing’. Learner 16 drew and packed out three groups of cans and three groups of bottles. Picture 4.47 shows this misrepresentation of ‘equal sharing’ in the semi-concrete and concrete representations.



Picture 4.47: Question 9: Learner 16’s misrepresentation of ‘equal sharing’

Learner 17 placed three bottles and one can of cool drink on the table for her 3D representation. This could not be categorised as it did not represent anything specific. I did not ask her to explain why she had represented the problem in that way.

For this question, one verbal misconception (‘non-consideration of all units’) and two misrepresentations (‘one unit’ and ‘equal sharing’) could be identified and categorised. The levels and types of calculation techniques will be discussed next.

4.11.2 Question 9: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2). The calculation technique types were inductively identified and grouped into similar categories (colour coded in green and pink in Table 4.9). The following calculation technique levels were identified and categorised: Level 1C (repeated addition), Level 1 (known addition fact) and Level 2B (double counting). The calculation technique types identified for solving Question 9 were 'counted on', 'addition fact' and 'counted in' (see Table 4.9).

Learners 8 and 10 did not know how to calculate the answer to this problem and four other participants (Learners 11, 14, 15 and 17) gave an answer but did not do any calculations. No calculation techniques levels and types were therefore identified.

Level 1 calculation techniques are additive and three of the participants (Learners 6, 13 and 16) added when calculating their answers. Learners 6 and 16 were categorised on Level 1C (repeated addition), and their calculation technique types were categorised as 'counted in' and 'counted on' respectively. Learner 6 counted in 2s and Learner 16 counted on six each time. Learner 13 was categorised on Level 1 (known addition fact) and since he added three and three together his calculation technique type was categorised as 'addition fact'.

The Level 2 calculation techniques are considered multiplicative, meaning that the participants were able to think multiplicatively when solving the problem (Carrier, 2014). Learner 9 was categorised on Level 2B (double counting) and his calculation technique types were categorised as 'counted in'. To solve the problem, he wrote three times one (3 x 1).

One participant used multiplicative calculation techniques, while three used additive calculation techniques to calculate the answer to this problem. Two could not solve the problem and four gave answers, but did not do any calculations.

4.11.3 Question 9: Calculation errors

None of the participants made any calculation errors when calculating the answer to this question. Four of the ten participants did not do calculations because they thought the answer was two or three and considered calculations to be unnecessary. The numbers used for this problem were two and three and the calculation was so simple that there were no calculation errors.

4.11.4 Question 9: Discussion of the analysis

This is the only question in the category ‘Cartesian product’. It is conceptually complex and the participants had not previously been asked to solve problems from this class, which are actually only introduced under the topic probability at the end of the Senior Phase (Grade 9) (DBE, 2011c). My analysis showed that none of the ten participants could solve this problem in all the representational forms. This was expected, since they had never encountered problems of this class before. One could infer that the participants lacked a well-integrated schema of this problem and an appropriate scheme for solving it.

Three of the ten participants (Learners 6, 11 and 13) could draw three cans and three bottles of different brands and arranged the actual objects in the same way, but Learners 6 and 13 used addition for their equations and Learner 11 did not write out any type of equation. From this one can infer a conceptual understanding of ‘combinations’, but not in the abstract form. However, since the participants had added and not multiplied to calculate their answers, I could only infer additive procedural fluency. It could be concluded that their schemas of ‘combinations’ were limited since their abstract schemas were lacking, as demonstrated by their use of an additive scheme or no calculation scheme at all to solve the problem.

Learner 14 could solve this problem only when she used 3D material. It could be inferred that her schema, and therefore her understanding, was limited to the concrete as there were no connections with the other types of representation (Ayub et al., 2012; Hiebert & Carpenter, 1992). The fact that her conceptual understanding was limited to the concrete could have led to the use of an incorrect scheme to solve the problem and a lack of procedural fluency. She struggled to solve the problem as she had chosen the incorrect concept-in-action, which led her to choose the incorrect theorem-in-action.

Five participants (Learners 9, 10, 15, 16 and 17) could not solve the problem in any of the representational forms and Leaner 8 made no attempt at all to solve it. One could possibly conclude that they could not solve the problem because they did not have a ‘combinations’ schema in any of the representational forms and no scheme, and therefore no procedural fluency. It could further be assumed that they had no abstract conceptual understanding, and because their concepts-in-action were lacking they also had no theorems-in-action to solve the problem.

No participant could solve this problem in all the forms of representation. Learners 6, 11 and 13 could solve the problem of ‘combination’ with semi-concrete and concrete representations, while Learner 14 could solve it only with the concrete representation. Four of the ten participants could solve it with at least one of the representations and five (Learners 9, 10, 15, 16 and 17) could not solve it in any of the representational forms. Leaner 8 made no attempt at all to solve the problem.

4.11.4.1 Question 9: Discussion of misconceptions and misrepresentations

One misconception could be identified and categorised, namely 'non-consideration of all units’. Five participants (Learners 9, 11, 14, 15 and 17) considered either only the cool drink brands or only the fact that there were bottles and cans, which was categorised as ‘non-consideration of all units’. One reason for this misconception could be that there was simply too much information to be considered. The participants had never done any problems of this class and therefore had no concept-in-action for ‘combinations’. This might have caused them to focus on one part of the question and ignore the other part.

Two misrepresentations were identified and categorised, namely ‘one unit’ and ‘equal sharing’. Five participants (Learners 9, 10, 14, 15 and 17) drew only one of the units mentioned in the question, which implies that they considered only one of the units (i.e. either the brands, or cans and bottles) when they tried to solve the problem. The reason for this misrepresentation could be that the problem was too complex and that they had not been taught strategies for solving unfamiliar problems.

The other misrepresentation was categorised as ‘equal sharing’. Learner 16 placed the bottles and cans into three groups. The reason for this misrepresentation is difficult to explain, but it is possible that this participant did not know what to do and used another schema (i.e. equal sharing) to solve the problem.

4.11.4.2 Question 9: Discussion of calculation technique levels and type

Only four participants (Learners 6, 9, 13 and 16) did any type of calculation and with the exception of Learner 9 none of them multiplied to solve the problem. Learner 9 multiplied three by one (3 x 1), which was not the correct calculation, and was categorised on Level 2B (double counting), while the calculation technique type was categorised as ‘counted in’. Even though he did multiply, it was not the correct calculation for solving the problem and he lacked procedural fluency and therefore also strategic competence.

The other three participants (Learners 6, 13 and 16) used additive calculation techniques and were categorised on Level 1. Learners 6 and 16 were categorised on Level 1C (repeated addition) and their respective calculation technique types were categorised as ‘counted in’ and ‘counted on’. Learner 13 was categorised on Level 1 (known addition fact) and his calculation technique type was categorised as ‘addition fact’. Learners 6 and 13 were the only participants who could solve this problem in the abstract form by using additive calculation techniques. For them one could infer no procedural fluency and no strategic competence.

4.11.4.3 Question 9: Discussion of calculation errors

The participants used addition or no calculation at all to solve this problem and no calculation errors were identified. There could be two reasons for this: The first is that the numbers they worked with were small numbers (i.e. two and three) and the second is that no calculation errors were made because of the misconception of ‘non-consideration of all units’. They did not have to do any calculations to solve this problem as the answer was clear enough.

4.12 Question 10: Combined categorising, analysis and discussion

The last task-based question was: *A mother gives each of her 5 children 3 sweets per day. How many sweets will the children eat over a 3-day period?* When I asked this question, I noticed that some participants did not know what the word period meant. I had asked the question as it appeared on the card, but then repeated the last part saying: *How many sweets will the children eat in 3 days or over 3 days?* Question 10 represented multiplication problems belonging to the class 'consumption' (Vergnaud, 1983) under the category 'multiple proportion' (see section 2.3.2.3 and Figure 2.1 in Chapter 2). Ten of the fifteen participants answered this question by first using abstract representations, then semi-concrete and lastly concrete representations (see section 3.3.2 in Chapter 3 for a full discussion). I expected them to draw a picture with five children and three groups of three sweets each for each child, and to make the same arrangement using the 3D material. They could use 3D blocks to represent the children and place three groups of three sweets each next to each block.

Although according to the CAPS, problems of this kind should be introduced concretely in Grade 2 (DBE, 2011a), I could not find any such problems in the textbooks that the schools used where the participants attended. The fact that there were more than two numbers to multiply made this problem more conceptually complex than the previous questions. Experience has taught me that some learners with learning difficulties and those struggling with mathematics find it difficult to solve two-step problems.

Table 4.10 contains a summary of the categories and subcategories of the data for this question for all the ten participants' representations. Similar subcategories are colour coded for easier recognition. I analysed the data under each heading in Table 4.10, starting with the participants' conceptions, misconceptions and misrepresentations. I then analysed the levels and types of calculation techniques and calculation errors. Finally, the analysis of all the data for this question and the analysis of the last two questions is discussed. I used inductive reasoning to analyse the misconceptions, misrepresentations, calculation technique types and calculation errors, and deductive reasoning to analyse the calculation technique levels derived from my conceptual framework.

Table 4.10: Summary of the Question 10 categories and subcategories for ten participants and all the representations

	ABSTRACT REPRESENTATIONS					SEMI-CONCRETE REPRESENTATIONS			CONCRETE PREPRSENTATIONS	
Learner	**Written operation and answer**	**Verbal misconception**	**Level of calculation technique and name**	**Type of calculation technique**	**Calculation error with abstract representations**	**Drawing conception and answer**	**Drawing misrepresentation**	**Calculation error with semi-concrete representations**	**3D material conception**	**3D material misrepresentation**
Learner 5	× (45)		Level 2B Double counting	Counted in		[Unable to draw]			Multiplication ×	Equal sharing
Learner 6	× (45)		Level: 2F Known multiplication fact Level 2E Derived multiplication fact	Times table Times table and addition		× Consumption (no answer)			Multiplication ×	Equal sharing
Learner 7	+ (27)		Level 1C Repeated addition	Repeated addition		Intention was + (27)	Abstract numbers	Disconnect between abstract and drawing (Answer 27)	Multiplication ×	Equal sharing
Learner 8	+ (11)		Level 1A Unitary counting	Counted on		Addition + (11)	Combination of measures		Addition +	Combination of measures
Learner 9	× (45)		Level 2B Double counting	Counted in		Multiplication × (45)	Equal sharing	Disconnect between abstract and drawing (Answer 45)	Multiplication ×	Equal sharing
Learner 10	+ (45)		Level 2B Double counting Level 1 Addition algorithm	Counted in Column method		Intention was × (45)	Abstract numbers		Intention was ×	Abstract numbers
Learner 11	× (24)		Level 2B Double counting	Counted in	Counting error (Answer 24)	[Unable to determine]	Answer		Multiplication ×	Equal sharing
Learner 14	× (15)		Level 2B Double counting	Counted in		Multiplication × (Says 45)	Equal sharing	Writing error (writes 54)	Multiplication ×	Equal sharing

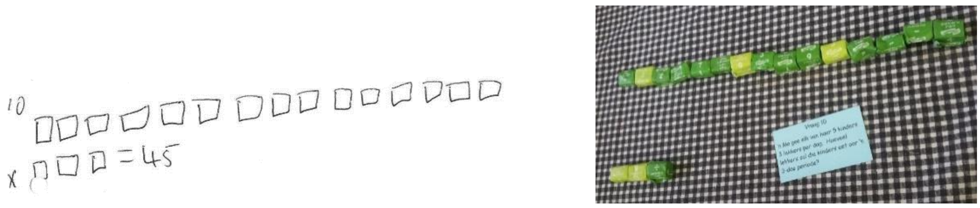
Learner 15	× and ÷ (1)		Level 2B Double counting	Counted in		Intention was × and ÷ (1)	Abstract numbers	Disconnect between abstract and drawing (Answer 1)	Multiplication ×	Equal sharing
Learner 16	+ (20)		Level 2B Double counting	Counted in		Multiplication × (20)	Equal sharing	Disconnect between abstract and drawing (Answer 20)	Intention was ×	Abstract numbers

As with the previous two questions, none of the ten participants could solve this problem with all three representations. Learner 6 could solve it with an abstract representation (multiplication) and his semi-concrete representation showed an understanding of 'consumption' as he had drawn three groups of three sweets each for each child. Learners 5 and 9 (whose names are marked in purple in Table 4.10) could solve this problem with abstract representations only and used multiplication for their equations.

4.12.1 Question 10: Misconceptions and misrepresentations

No verbal misconceptions could be identified, even though three participants added to calculate their answers. Learner 8 simply added the three numbers together, while Learners 7 and 16 added numbers other than those given. Learner 15 multiplied and divided to calculate her answer. It is possible that they did not understand the question.

Four misrepresentations could be identified and categorised, namely 'abstract numbers', 'answer', 'equal sharing' and 'combination of measures' (see Table 4.10). Four participants' drawings and/or concrete representations were identified and categorised as misrepresentations of 'abstract numbers'. Learner 10 misrepresented this problem in the semi-concrete and concrete representations. This participant drew and placed fifteen sweets and three sweets to represent the numbers in his abstract equation (see Picture 4.48).

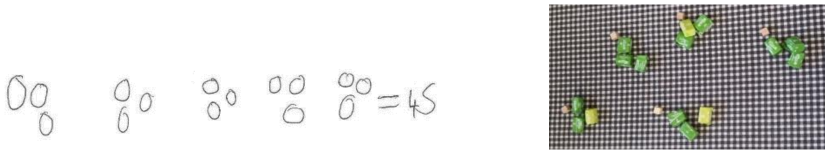


Picture 4.48: Question 10: Learner 10's misrepresentation of 'abstract numbers'

Learners 7 and 15's semi-concrete representations were also categorised as 'abstract numbers' as they had used both pictures and numbers to represent the problem, while Learner 16's concrete representation was categorised as 'abstract numbers' as he had arranged the 3D material to reflect his abstract representation.

The second misrepresentation, identified and categorised as ‘answer’, was difficult to categorise as Learner 11 drew only three sweets as her answer.

The third misrepresentation was identified and categorised as ‘equal sharing’. Eight participants misrepresented this problem in their semi-concrete and/or concrete representations. In the semi-concrete and concrete representations, Learner 9 grouped three sweets together, ignoring the fact that the children each received three sweets daily for three days (see Picture 4.49). Learner 14 also grouped three sweets together and wrote ‘3x’ in her picture, and in her 3D representation grouped fifteen sweets together, instead of nine sweets per child. This was categorised as ‘equal sharing’ as it represented this class of problem rather than ‘consumption’.



Picture 4.49: Question 10: Learner 9’s misrepresentation of ‘equal sharing’

Five of the eight participants’ (Learners 5, 6, 7, 11 and 15) misrepresentations in their concrete representations were categorised as ‘equal sharing’. Like Learner 9, Learners 5, 11 and 15 made five groups of three sweets (see Picture 4.49) for the concrete representations, while Learners 6 and 7 arranged five groups of five 3D blocks each (to represent the five children) and added nine sweets to each of the groups of five 3D blocks. Picture 4.50 shows Learner 6’s misrepresentation of ‘equal sharing’.

Picture 4.50: Question 10: Learner 6’s misrepresentation of ‘equal sharing’

Learner 16's semi-concrete representation was categorised as 'equal sharing', since he drew five sweets for each child, thus five groups of five.

Learner 8's misrepresentation in her semi-concrete and concrete representations was categorised as 'combination of measures', since she had added the three numbers together (see Picture 4.51).

Picture 4.51: Question 10: Learner 8's misrepresentation of 'combination of measures'

For this question no verbal misconceptions could be identified; however, four misrepresentations ('abstract numbers', 'answer', 'equal sharing' and 'combination of measures') were identified and categorised. The levels and types of calculation techniques will be discussed next.

4.12.2 Question 10: Levels and types of calculation techniques

The calculation technique levels were categorised according to the conceptual framework (see section 2.5.1 and Figure 2.1 in Chapter 2). The calculation technique types were identified and grouped into similar categories (colour coded in green and pink in Table 4.10). The following calculation technique levels were identified and categorised: Level 1A (unitary counting), Level 1C (repeated addition), Level 2B (double counting), Level 2C (algorithms), Level 2E (derived multiplication fact) and Level 2F (known multiplication fact). The calculation technique types that I identified for solving Question 10 were: 'counted on', 'repeated addition', 'counted in', 'column method', 'times table and addition' and 'times table' (see Table 4.10).

Only two of the ten participants used additive calculation techniques. Learner 8, whose calculation technique type was categorised as 'counted on', which was categorised on Level 1A (unitary counting), had added the three numbers in the problem together, starting with five, then three and then another three. Learner 7 wrote his equation in the form of repeated addition and was therefore categorised on Level 1C (repeated addition). His calculation technique type was also indicative of ‘repeated addition’.

Level 1 calculation techniques are additive in nature. Two participants had added to calculate the answer and were categorised on Level 1. Learner 8 was categorised on Level 1A (unitary counting) and her calculation technique type was categorised as ‘counted on’. She started from five and, counting on her fingers, added three and then two to calculate the answer. Learner 7, who had added nine each time, was categorised on Level 1C (repeated addition) and his calculation technique type was categorised as ‘repeated addition’.

Learner 10 used a combination of additive and multiplicative calculation technique types. His calculation technique was categorised as Level 2B (double counting) and Level 1 (addition algorithm), and the calculation technique types he used to solve the problem were categorised as ‘counted in’ and the ‘column method’. He had counted in 3s first, after which he had written three fifteens below one another before adding all the numbers together.

Level 2 calculation techniques are multiplicative, meaning that participants categorised on this level were able to think multiplicatively when solving the problem (Carrier, 2014). Eight participants used multiplication to solve this problem. Learners 5, 9, 11, 14, 15 and 16’s calculation techniques were categorised on Level 2B (double counting) and the calculation technique type was categorised as ‘counted in’. Learner 5 first counted in 3s and then in 5s to solve the problem. Learner 9 counted in 15s, while Learners 11 and 15 counted in 3s. Learner 14 counted in 5s and Learner 16 in 4s. Learner 6 used two different multiplicative calculation techniques to solve this two-step problem. Learner 6’s calculation techniques were categorised on Level 2F (known multiplication fact) and Level 2E (derived multiplication fact) and his calculation technique types were categorised as ‘times table’ and ‘times table and addition’. He first calculated 5 x 3 = 15 and then calculated 15 x 3. He knew that 12 x 3 = 36, and added 9 to get the answer, which was 45.

Seven participants used multiplicative calculation techniques, two used additive calculation techniques and one used a combination of multiplicative and additive calculation techniques to calculate the answer to this problem. The calculation errors will be discussed next.

4.12.3 Question 10: Calculation errors

Three calculation errors were identified and categorised, namely ‘counting error’, ‘writing error’ and ‘disconnect between abstract representation and drawing’ (Table 4.10). The first calculation error that was identified and categorised was the ‘counting error’ made by Learner 11, who had multiplied five by three and gave the answer as 24. When she reached 12, she doubled it instead of adding the last three. The second calculation error, categorised as a ‘writing error’, had been made by Learner 14, who had written 54 but said that the answer was 45. The last calculation error was categorised as a ‘disconnect between abstract and drawing’. Four participants (Learners 7, 9, 15 and 16) had made this error. Learner 9 had drawn five groups of three and gave his answer as 45, which was his answer in his abstract representation (see Picture 4.49). Learners 7, 15 and 16 also gave answers that did not correspond with their pictures.

4.12.4 Question 10: Discussion of the analysis

Question 10, which is the only question in the category ‘multiple proportion’, is conceptually complex as the calculation involves more than two numbers. Although according to the CAPS the problems belonging to the class ‘consumption’ should be concretely introduced in Grade 2 (DBE, 2011a), I could not find any such problems in the textbooks that the schools used where the participants attended. Since the participants had not encountered this type of problem before, it was to be expected that they would struggle to solve it. An unforeseen problem that was identified was the use of the word period in the question as some participants understood it to refer to a school period. Consequently, I had to rephrase the question as follows: *How many sweets will they eat in three days, or over three days.* This word confusion might have contributed to their misunderstanding of the problem and incorrect calculations.

None of the participants could solve this problem in all the representational forms. Learner 6 could solve it with abstract and semi-concrete representations. Hiebert and Carpenter (1992) suggest that the more connections the better the understanding, which could indicate that this participant had some connection between the abstract and semi-concrete representations and thus some understanding of 'consumption' problems. Learner 6 had an abstract and semi-concrete schema of 'consumption' and therefore a scheme to solve the problem. One could infer that since he had some conceptual understanding of the problem, he could choose a correct concept-in-action that led to an appropriate theorem-in-action. Procedural fluency and strategic competence could therefore be inferred.

Learners 5 and 9 could solve this problem with multiplicative abstract representations only. When only one type of representation is correct, it could be an indication that limited connections were made (Ayub et al., 2013). It could therefore be inferred that they had limited connections and only an abstract conceptual schema of this type of problem, and consequently no real conceptual understanding. While it is possible that they memorised the procedure without conceptually understanding it (Ayub et al., 2013), procedural fluency could be inferred. Learner 10 could solve this problem with abstract representations, but used an additive calculation technique. One could infer that he had an abstract schema with only additive procedural fluency and no strategic competence.

Six participants (Learners 7, 8, 11, 14, 15 and 16) could not solve the problem in any of the representational forms. It could be inferred that they had no connections between the representations of multiplication problems belonging to the class 'consumption', and therefore no schema and scheme for this problem. It could further be concluded that they had no conceptual understanding, procedural fluency and strategic competence. Since they did not have a 'multiple proportion' schema, they did not have the correct concept-in-action and therefore chose the incorrect theorem-in-action.

Learner 6 could solve this problem with abstract and semi-concrete representations, while Learners 5 and 9 could solve it using multiplicative abstract representations. Three of the ten participants could solve this problem using at least one of the representations and six (Learners 7, 8, 11, 14, 15 and 16) could not solve it in any of the representational forms.

4.12.4.1 Question 10: Discussion of misconceptions and misrepresentations

While no misconceptions were identified, I could identify and categorise four misrepresentations, namely 'equal sharing', 'abstract numbers', 'combination of measures' and 'answer'. The most prevalent misrepresentation was that of 'equal sharing'. Eight of the participants' (Learners 5, 6, 7, 9, 11, 14, 15 and 16) pictures and/or representation using 3D material were indicative of 'equal sharing' and were categorised as such. The reason for this might be that since these participants had not previously dealt with problems of this type, they did not have a schema of the 'consumption' class of multiplication problem, but only of the 'equal sharing' class, and therefore solved the problem by using another concept-in-action that they had.

For their semi-concrete and/or concrete representations, four participants (Learners 7, 10, 15 and 16) simply replaced the numbers of their abstract equations with sweets and blocks. This was categorised as the misrepresentation of 'abstract numbers'. Learner 11 drew only three sweets to represent the answer to the problem. This was categorised as a misrepresentation of 'answer'. The reason for these two misrepresentations could be that they struggled to visualise the problem semi-concretely and/or concretely. One could infer that these participants struggled with abstract thinking, which is characteristic of learners with learning difficulties (Allsopp et al., 2007).

For her semi-concrete representation, Learner 8 drew five stripes. Underneath that she drew three stripes, with another three underneath the previous tree, which she added together for an answer of 11. For her concrete representation she placed three sweets, three blocks and then another five blocks in a line, which was indicative of addition. This misrepresentation, which was categorised as a 'combination of measures', could indicate that she did not have a schema and scheme for problems belonging to the class 'multiple proportion'. She might not have known how to distinguish between multiplication and addition problems.

4.12.4.2 Question 10: Discussion of calculation technique levels and types

Seven participants (Learners 5, 6, 9, 11, 14, 15 and 16) used multiplication calculation techniques and were categorised on Level 2. Six (Learners 5, 9, 11, 14, 15 and 16) were categorised on Level 2B (double counting) and their calculation technique type was categorised as ‘counted in’. Learner 6 was categorised on Level 2E (derived multiplication fact) and Level 2F (known multiplication fact) and his calculation technique types were categorised as ‘times table and addition’ and ‘times table’ respectively. This could indicate that these participants had an abstract schema and multiplicative schemes for ‘multiple proportion’ problems, which in turn could indicate that they had procedural fluency. However, one could infer that only Learner 6 had good strategic competence, since he was categorised on Levels 2E and 2F, which are the two highest cognitive developmental calculation techniques demonstrating abstract thinking (Hurst & Hurrell, 2014; Zhang et al., 2011).

Learner 10 used a combination of multiplicative and additive calculation techniques to solve the problem and was categorised on Level 2B (double counting) and Level 1 (addition algorithm). The respective calculation technique types used were categorised as ‘counted in’ and ‘column method’. This could indicate that he did not have an effective scheme to solve the problem. Moreover, his multiplicative scheme might not yet have been well developed and therefore he had to use multiplicative and additive calculation techniques. It could be concluded that he had limited procedural fluency and poor strategic competence.

Learners 7 and 8 used additive calculation techniques and were therefore categorised on Level 1. Learner 8 was categorised on Level 1A (unitary counting) and the calculation technique type was categorised as ‘counted on’, while Learner 7 was categorised on Level 1C (repeated addition) and the calculation technique type was also categorised as ‘counted on’. One could infer that these participants did not have correct schemas and schemes for ‘multiple proportion’. Their schemas and schemes were additive and therefore they could not solve the problem. They had no procedural fluency and no strategic competence to solve the problem. Their concepts-in-action were additive and therefore they chose incorrect theorems-in-action.

4.12.4.3 Question 10: Discussion of calculation errors

Three calculation errors were identified and categorised, namely ‘counting error’, ‘writing error’ and ‘disconnect between abstract and drawing’. Learner 11 made a ‘counting error’, when counting in 3s. It is difficult to say for certain why she made this error, but it could be due to concentration problems. Learner 14 made a ‘writing error’, as she wrote 54 and said 45. This participant might have been dyslexic, which would explain the inversion of numbers. The drawings made by four participants (Learners 7, 9, 15 and 16) were inconsistent with the answer they gave. The reason why their pictures did not reflect their answers could be that they were not used to drawing pictures of problems and did not realise that they should use their pictures to calculate the answer. These participants used the answers given in their abstract representations without realising that their pictures did not reflect those answers.

4.13 Summary

In this chapter I systematically presented and analysed the data obtained from each of the ten questions. For each question I started by analysing the conceptions, written misconceptions and misrepresentations, which was followed by an analysis of the levels and types of calculation techniques. Thereafter I analysed the calculation errors for each question and ended with a discussion of what had been analysed. Following the same systematic presentation, I first discussed the conceptions, then the misconceptions and misrepresentations, the levels and types of calculation techniques, and lastly the calculation errors. I discussed the analysis by referring back to my conceptual framework and my literature study, which will help me, in the next chapter, to answer my three secondary research questions, which relate to the status of the participants’ conceptual understanding, their procedural fluency and strategic competence. In Chapter 5 I will show how the findings regarding the conceptions, misconceptions, misrepresentations, levels and types of calculation techniques and calculation errors that were found in the learners’ responses to each of the ten multiplication problems helped me to answer the secondary research questions and thus the main research question.

CHAPTER 5

CONCLUSION AND IMPLICATIONS

5.1 Introduction

In the previous chapter I identified, analysed and categorised the conceptions and misconceptions of the fifteen participants, the levels and types of calculation techniques they used and the calculation errors they made. During individual interviews with the participants, I asked them to solve ten different classes of multiplication problems, as categorised in my conceptual framework (see Figure 2.1, in Chapter 2 and Table 3.1, in Chapter 3). In the previous chapter I discussed each of the ten multiplication questions separately. In this chapter I will discuss the findings in order to answer my three secondary research questions and to ultimately establish the multiplicative proficiency of Grade 6 learners with learning difficulties. I will start this chapter by giving an overview of the preceding chapters, after which I will discuss the findings in relation to each of my secondary research questions and explain how my findings fit into the current research literature. This will be followed by a critical reflection on and discussion of my theoretical, methodological and practical contribution to the current research literature, the implications and limitations of this study and recommendations for future research. Finally, I will reflect on my study as a whole.

5.2 Overview of previous chapters

In Chapter 1 I introduced my study by highlighting the importance of the conceptual shift from additive to multiplicative reasoning that learners are expected to make during the Intermediate Phase (Grades 4 to 6). The degree of success with which this conceptual shift is made will determine how well they will cope with more cognitively complex mathematics in later years. This study was undertaken to investigate the proficiency in multiplicative reasoning of Grade 6 learners with learning difficulties by answering three secondary research questions.

Chapter 2 contains an in-depth overview of multiplicative reasoning as a cognitive field and the subcomponents relevant to this study, as well as an overview of the first research done in this field by Vergnaud (1982–2014) and other researchers. This overview was used to establish a conceptual framework for this study, based on the interplay between internal and external representations as a measure for proficiency in multiplicative reasoning.

In Chapter 3 I summarised the methodologies used in previous research studies in this field. I situated my study in the critical realism paradigm, focusing more on ontology than on epistemology. I established my study as qualitative using a single-case study design and explained how I had chosen the fifteen participants, and how I had collected my data through task-based interviews. I also reported on the quality measures and the ethical considerations applied during this study.

In Chapter 4 I analysed and discussed, with evidence, the data collected from the individual participants for each of the ten multiplication questions. I identified and categorised conceptions, misconceptions, misrepresentations, levels and types of calculation techniques and calculation errors by using both inductive and deductive reasoning. The participants' conceptions, misconceptions and misrepresentations of each of the ten multiplication problems were discussed together in order to answer the first secondary research question about the participants' conceptual understanding. The calculation technique levels and the calculation errors were discussed together in order to answer the second secondary research question, which related to the participants' procedural fluency. The calculation technique types were discussed separately, but were also linked to the calculation technique levels in order to answer the third secondary research question relating to strategic competence.

5.3 Discussion and conclusions relating to the three secondary research questions

The primary research question, as stated in Chapter 1 of this study, was:

> *How proficient are Grade 6 learners with learning difficulties in multiplicative reasoning?*

In order to answer the primary research question, I formulated three secondary research questions, which will each be discussed separately. As discussed in Chapter

1, it is important for learners to make a conceptual shift from additive to multiplicative reasoning in order to become proficient in multiplicative reasoning. The way in which proficiency was measured for this study included a combination of conceptual understanding (Secondary research question 1), procedural fluency (Secondary research question 2) and strategic competence (Secondary research question 3), demonstrated by using abstract, semi-concrete and concrete representations. If participants could show their understanding in all three of these components, they were deemed proficient.

5.3.1 First secondary research question: Conceptual understanding

The first secondary research question was:

What is the status of the learners' conceptual understanding of multiplication?

In order to answer this question, participants' conceptions, misconceptions and misrepresentations of each of the ten task-based questions, as seen in their abstract, semi-concrete and concrete representations, were identified and categorised. As shown in the conceptual framework, participants were expected to also show their conceptual understanding through verbal explanations; however, the majority of the participants were unable to explain how they decided whether they should use multiplication or addition to solve a problem.

Moreover, the findings of this study revealed that only some of the participants demonstrated conceptual understanding of the ten classes of multiplication problems with both their semi-concrete and concrete representations. From the findings discussed in Chapter 4, one could conclude that in the case of the less cognitively complex questions (Questions 1, 2, 4 and 7), more participants showed conceptual understanding: ten showed their conceptual understanding for Question 1 (Learners 3, 5, 6, 9, 11, 12, 13, 14, 15 and 17), six for Question 2 (Learners 3, 6, 9, 14, 16 and 17), five for Question 4 (Learners 3, 6, 7, 9 and 17) and three for Question 7 (Learners 7, 13 and 17), while for the more cognitively complex questions (Questions 3, 5, 6, 8, 9 and 10), fewer participants showed conceptual understanding: two for Question 3 (Learners 6 and 12), four for Question 5 (Learners 3, 9, 14 and 17), two for Question 6 (Learners 6 and 14), none for Question 8, three for Question 9 (Learners 6, 11 and

13) and none for Question 10. From these findings one could further conclude that more participants had good interconnected semi-concrete and concrete schemas for the less cognitively complex classes of multiplication problems and could easily visualise them, while fewer participants could semi-concretely and concretely visualise the more cognitively complex classes of multiplication problems. The majority of participants lacked semi-concrete and concrete schemas for the more cognitively complex classes of multiplication problems. This is in line with Allsopp et al.'s (2007) view that learners with learning difficulties struggle with abstract thinking, and therefore with visualising more abstract concepts (cognitively complex). These findings indicate that most participants lacked the conceptual understanding and the semi-concrete and concrete conceptual schemas of multiplication needed to make the transition from additive to multiplicative reasoning. Possible reasons for this lack of semi-concrete and concrete conceptual schemas could be deduced from the misconceptions and misrepresentations identified in this study.

5.3.1.1 Misconceptions hindering conceptual understanding

Only four misconceptions could be identified and categorised due to the fact that the participants struggled to verbally explain and justify what they were doing (see Chapter 4). In my opinion, and based on my experience of working with learners with learning difficulties, a possible reason for their inability to explain and justify their reasoning is that although learners in South African schools are required, according to the curriculum (DBE, 2011b), to verbally explain and justify their solutions, many teachers teaching learners with learning difficulties do not follow this curriculum prescript due to the perception that learners with learning difficulties generally also experience language difficulties and have trouble expressing themselves.

The misconceptions identified were: the use of wrong 'keywords', 'added different units', 'unit conversion' and 'non-consideration of all units' (see Chapter 4 for more details). The participants who had these misconceptions appeared to have poor existing conceptual schemas, which could have been the result of inadequate explanations given by teachers. I believe that the more teachers ask learners to explain their reasoning and justify what they are doing, the easier it would be to identify learners' misconceptions, which in turn could help teachers to improve their instructional practice and eliminate the identified misconceptions.

5.3.1.2 Misrepresentations hindering conceptual understanding

A total of fourteen misrepresentations were identified and categorised, which is an indication that many of the participants were unable to properly represent the different classes of problems, namely ‘abstract numbers’, combination of measures’, ‘answer’, ‘wrong unit’, ‘take away’, ‘non-consideration of proportion’, ‘equal sharing’, ‘constant price’, ‘repeated addition’, ‘perimeter’, ‘discrete’, ‘wrong figure’, ‘sides confusion’ and ‘one unit’ (see Chapter 4 for more detail). A possible reason for these misrepresentations is that although the curriculum requires learners in the Intermediate Phase (Grades 4 to 6) to represent their understanding in different ways (DBE, 2011b), teachers do not always ensure the adequate exposure of learners to opportunities to make drawings or use 3D materials to represent their problems. Literature dealing with the teaching of learners with special needs encourages teachers teaching a new concept to first teach it concretely, then semi-concretely and then abstractly in order to help their learners to establish connections between the different representational forms (Allsopp et al., 2007; Bruner, 1963; Debrenti, 2013; Hoong et al., 2015; Hui et al., 2017; Lesser & Tchoshanov, 2005; Pape & Tchoshanov, 2001; Post, 1981). Ayub et al. (2012) and Hiebert and Carpenter (1992) suggest that limited connections between the representational forms could result in limited conceptual understanding, which was confirmed by the findings of this study. Based on the many and varied misrepresentations identified in the semi-concrete and concrete representations, and the small number of participants that showed conceptual understanding as discussed above, it could be concluded that the participants in this study showed limited conceptual understanding of multiplication and that their conceptual understanding was limited to the less conceptually complex classes of multiplication problems.

5.3.2 Second secondary research question: Procedural fluency

The second secondary research question was:

What is the level of the learners’ procedural fluency related to multiplication?

In order to answer this question, the participants’ calculation technique levels in their abstract calculations for each of the ten task-based questions, as well as the calculation errors they had made in their abstract and semi-concrete representations, were identified and categorised.

According to the findings of this study, the number of participants who could correctly solve the ten classes of multiplication problems with abstract representations using either additive or multiplicative calculation techniques were as follows: nine for Question 1 (Learners 3, 5, 6, 9, 10, 12, 14, 15 and 17); nine for Question 2 (Learners 3, 5, 6, 10, 11, 12, 14, 15 and 17); twelve for Question 3 (Learners 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15 and 17); nine for Question 4 (Learners 3, 5, 6, 9, 11, 12, 13, 15 and 17); three for Question 5 (Learners 9, 14 and 15); none for Question 6; six for Question 7 (Learners 5, 6, 9, 14, 15 and 17); two for Question 8 (Learners 6 and 14); two for Question 9 (Learners 6 and 13); and four for Question 10 (Learners 5, 6, 9 and 10). From these findings one could conclude that more of the participants were able to solve the multiplication problems (Questions 1, 2, 3, 4 and 7) of the classes that they had already learned to solve abstractly, while fewer participants could solve the multiplication problems (Questions 5, 6, 8, 9, 10) belonging to classes that they had not yet learned to solve. Since more participants could solve the ten multiplication problems using abstract representations than with the semi-concrete and concrete representations, one could conclude that, as Ayub et al. (2013) suggest, some participants might have memorised the calculation technique without conceptual understanding.

Only the participants who could solve the different classes of multiplication problems are mentioned here, since the others did not understand the problem and chose their calculation techniques based on their own interpretations of the problems. Although I did report on the other participants' calculation techniques in Chapter 4 in order to form a full picture, their procedural fluency could not be considered as they could not solve the problems. The calculation technique levels of the participants mentioned in the previous paragraph were categorised as follows: Level 1C (repeated addition) ten times (all the questions included), once on Level 1 (addition algorithm) and once on Level 1 (known addition fact). The participants were categorised nineteen times on Level 2B (double counting) (all the questions); six times on Level 2C (algorithms); six times on Level 2E (derived multiplication fact); and three times on Level 2F (known multiplication fact). Participants who used a combination of addition and multiplication calculation techniques were categorised altogether eight times on different Level 1 and Level 2s. The calculation techniques used by participants categorised on Level 2 are considered to be multiplicative calculation techniques as it means that they could coordinate two quantities at the same time and had therefore made the conceptual shift from additive to multiplicative reasoning to solve multiplication problems (Zhang et al., 2011). The findings indicate that most of the participants who could solve the problem were on Level 2 and could be considered procedurally fluent. One could therefore conclude that, for abstract representations, more participants could solve those kinds of problems that they had already learned to solve, while they struggled with unfamiliar types of problems. Procedural fluency is therefore limited to known classes of multiplication problems.

However, my analysis revealed that some of the participants who could not solve the ten multiplication problems with abstract representations were able to solve them with semi-concrete and/or concrete representations.

- Five of the six participants who could not solve Question 1 with abstract representations could solve it with semi-concrete and/or concrete representations.
- Four of the six participants who could not solve Question 2 with abstract representations could solve it with semi-concrete and/or concrete representations.
- None of the three participants who could not solve Question 3 with abstract representations could solve it with semi-concrete and/or concrete representations.
- Three of the six participants who could not solve Question 4 with abstract representations could solve it with semi-concrete and/or concrete representations.
- Four of the eight participants could not solve Question 5 with abstract representations could solve it with semi-concrete and/or concrete representations.

- Two of the eleven participants who could not solve Question 6 with abstract representations could solve it with semi-concrete and/or concrete representations.
- Three of the five participants who could not solve Question 7 with abstract representations could solve it with semi-concrete and/or concrete representations.
- None of the eight participants who could not solve Question 8 with abstract representations could solve it with semi-concrete and/or concrete representations.
- Two of the eight participants who could not solve Question 9 with abstract representations could solve it with semi-concrete and/or concrete representations.
- None of the six participants who could not solve Question 10 with abstract representations could solve it with semi-concrete and/or concrete representations.

As indicated by the above findings, it cannot be conclusively stated that those participants who could not solve the problems with abstract representations could solve them with semi-concrete and/or concrete representations. However, more of the participants could solve the problems that were less cognitively complex and easier to visualise when they were allowed to use semi-concrete and concrete representations. In other words, in the case of the more cognitively complex problems that were more difficult to visualise, fewer participants could solve them with either abstract and semi-concrete and/or concrete representations. Furthermore, it seems that when participants lacked a schema of the more cognitively complex problems, they were unable to solve them in any of the representations. However, the participants appeared to have some type of schema of the less cognitively complex problems as they could solve them by using the semi-concrete and concrete representations. For this reason I believe that learners should be given opportunities to demonstrate their conceptual understanding of problems by using representations other than abstract representations, since teachers who teach learners with learning difficulties are encouraged to teach using the CSA sequencing, which means that they have to teach by first using concrete, then semi-concrete and then abstract representations (Allsopp et al., 2007; Bruner, 1963; Debrenti, 2013; Hoong et al., 2015; Hui et al., 2017; Lesser & Tchoshanov, 2005; Pape & Tchoshanov, 2001; Post, 1981). If learners are taught to demonstrate their understanding by using concrete, semi-concrete and abstract (CSA) representations, learners with learning difficulties might be able to develop better-connected schemas, which could be helpful when they are asked to solve problems in new contexts. Calculation errors, which contributed to the lack of procedural fluency in some participants, will be discussed next.

5.3.2.1 Calculation errors hindering procedural fluency

Six calculation errors could be identified and categorised, namely 'writing error', 'memory error', 'counting error', 'disconnect between abstract and drawing', 'wrong number' and 'tracking error'. 'Writing error' was identified four times, 'memory error' twice, 'counting error' eighteen times, 'disconnect between abstract and drawing' ten times, 'wrong number' eight times and 'tracking error' once (see Chapter 4 for details). The calculation error that occurred most frequently was 'counting error', which is meaningful as this prevented participants from calculating the answer correctly and hindered procedural fluency. While I believe that more needs to be done to help learners to count correctly, I concede that this might be problematic in the case of learners with serious learning difficulties. The other calculation error that occurred often was 'disconnect between abstract and drawing', which was identified in using semi-concrete representations. It is my opinion that teachers should allow learners with learning difficulties to draw pictures of problems and teach them that their drawings could help them to calculate the answers, which is something many of the participants did not seem to realise. The use of 'wrong numbers' was also meaningful as some participants used numbers that had not been given. They did not verify the questions, which might have been because of a lack of concentration or because of their learning difficulties. Teachers should focus on these calculation errors and find ways to help learners to overcome them in order to improve their procedural fluency.

5.3.3 Third secondary research question: Strategic competence

The third secondary research question was:

> *What is the nature of the learners' strategic competence when solving multiplication problems?*

In order to answer this question, the calculation techniques used by the participants for their abstract representations were identified and categorised according to calculation technique types.

The following additive calculation technique types were identified and categorised: 'counted from one' was categorised once; 'counted on' twenty-nine times; 'repeated addition' five times; 'column method' twice; 'addition fact' three times; and '2× addition' twice. Eight participants used a combination of additive and multiplicative calculation technique types to calculate the answers. These participants were not strategically competent as they thought the problems required addition and therefore used additive calculation techniques to solve them (see Chapter 4 for details).

The following multiplicative calculation technique types were identified and categorised: 'counted in' which was categorised twenty-eight times; 'doubling' once; 'column method' ten times; 'split multiplication and addition' twice; 'times table and subtraction/addition' five times; and 'times table' seven times. The findings indicate that while the participants who had used multiplicative calculation techniques did demonstrate procedural fluency, not all of them demonstrated strategic competence. The most frequently identified calculation technique type was 'counted in' (28 times), which is considered the first step to multiplicative thinking. The calculation technique types 'times table and subtraction/addition' and 'times table' were identified only five and seven times respectively. These two calculation technique types are considered by Hurst and Hurrell (2014) and Zhang et al. (2011) as the two highest cognitive developmental calculation techniques and are categorised on Level 2E (derived multiplication fact) and Level 2F (known multiplication fact). Although there is nothing wrong with using the other calculation technique types, strategic competence implies using the most effective techniques to calculate the answer, which would be 'times table and subtraction/addition' and 'times table'. One can therefore conclude that although more participants showed procedural fluency, very few showed strategic competence. One of the reasons for this could be that most of the participants still relied heavily on using their fingers when calculating the answers. Another reason could be poor memory due to learning difficulties.

5.4 Final remarks concerning the main and secondary research questions

The different levels on which this study makes a contribution, namely the theoretical, methodological and instructional practice levels, will each be discussed separately.

5.5 Contribution of this study

The findings of this study indicate that the following numbers of participants could correctly solve the different classes of multiplication problems in all three representational forms and by using multiplicative calculation techniques:

- Eight (Learners 3, 5, 6, 9, 12, 14, 15 and 17) could solve Question 1, with only Learner 17 categorised on Level 2E.
- Three (Learners 6, 4 and 17) could solve Question 2, with Learners 6 and 17 categorised on Level 2E.
- One (Learner 12) could solve Question 3.
- Three (Learners 6, 9 and 17) could solve Question 4, with Learners 6 and 17 categorised on Level 2E.
- One (Learner 9) could solve Question 5.
- None of the participants could solve Question 6.
- One (Learner 17) could solve Question 7, with Learner 17 categorised on Level 2F.
- None of the participants could solve Questions 8 to 10.

From these findings one could conclude that very few participants had procedural fluency and a conceptual understanding of the different classes of multiplication problems. One of the findings of a study by Bakker et al. (2014) was that participants found it easier to solve problems from the class equal groups. Even though my findings suggest that more participants showed conceptual understanding and procedural fluency in respect of the least conceptually complex question, namely

Question 1 (part of ‘equal groups’), the same does not apply in the case of Questions 2 and 3, which were also categorised under ‘equal groups’ (see my conceptual framework, Figure 2.1). With regard to the rest of the questions, including Questions 2 and 3, which were conceptually more complex that Question 1, fewer participants showed conceptual understanding and procedural fluency.

These participants who could solve the problems without any problem with all the representations could be said to have made the transition from additive to multiplicative reasoning. These findings are in agreement with current research literature, according to which learners, especially those with learning difficulties, struggle to make this transition (Ell, 2001; McClintock et al., 2011; Tzur et al., 2010). Moreover, it can be concluded that some learners with learning difficulties are capable of making the necessary transition from additive to multiplicative reasoning, specifically for less cognitively complex and therefore more abstract multiplication problems. This confirms Allsopp et al. (2007), Dednam (2011) and Miller and Mercer’s (1997) finding that learners with learning difficulties struggle with abstract thinking.

Only Learners 6 and 17 demonstrated strategic competence, mostly when solving the less cognitively complex problems. It could therefore be concluded that even though the abovementioned participants had demonstrated conceptual understanding and procedural fluency in answering the questions mentioned, only Learner 17 showed proficiency in solving some of the less cognitively complex questions, such as Questions 1, 2, 4 and 7 (‘equal sharing’, ‘constant price’, times as many’ and ‘array’), while Learner 6 showed proficiency in answering Questions 2 and 4.

According to the findings of this study, the participants had limited conceptual understanding and lacked good interconnected schemes and schemas between the abstract, semi-concrete and concrete representations for the different classes of multiplication problems. Those who could solve the problems had good procedural fluency as they could make the conceptual shift from additive to multiplicative reasoning and used multiplicative calculation techniques to solve the problem. However, even though they had procedural fluency, they lacked strategic competence in using calculation techniques that calculated the answer correctly. Many of the participants did not have procedural fluency or strategic competence and could not solve the problems as they used additive calculation techniques, which indicates that they had not yet made the conceptual shift from additive to multiplicative reasoning. All three components (conceptual understanding, procedural fluency and strategic competence) are necessary for learners to be able to solve cognitively more complex problems in their later school years. It was encouraging to see that the majority of the participants who understood the problem as multiplication had already succeeded in making the conceptual shift from additive to multiplicative reasoning.

5.5.1 Theoretical contribution

This study makes a theoretical contribution in that it proposes a way to investigate multiplicative reasoning on multiple levels, including conceptual understanding (semi-concrete and concrete representations), procedural fluency (abstract representations) and strategic competence (concepts-in-action and theorems-in-action), by investigating the interplay between internal and external representations as set out in my conceptual framework. I believe that this conceptual framework can be used, improved or adapted to investigate proficiency in different mathematical conceptual fields. The conceptual framework for this study shows how the use of multiple external representations and Vergnaud's scheme theory (2009; 2013a; 2013b) could help us understand how learners access internal representations with concepts-in-action and theorems-in-action as the two components that connect internal and external representations (see Figure 2.1, in Chapter 2).

The reasons for the participants' lack of proficiency in multiplicative reasoning that emerged from this study were categorised as misconceptions, misrepresentations and calculation errors. I could find no previous studies that had specifically investigated these shortcomings as causes for learners' inability to solve different classes of multiplication problems. These error categories included:

- misconceptions: ‘keywords’, ‘added different units’, ‘unit conversion’ and ‘non-consideration of all units’;
- misrepresentations: ‘abstract numbers’, combination of measures’, ‘answer’, ‘wrong unit’, take away’, ‘non-consideration of proportion’, ‘equal sharing’, ‘constant price’, ‘repeated addition’, ‘perimeter’, ‘discrete’, ‘wrong figure’, ‘sides confusion’ and ‘one unit’; and
- calculation errors: ‘writing error’, ‘memory error’, ‘counting error’, ‘disconnect between abstract and drawing’, ‘wrong number’ and ‘tracking error’ (see Chapter 4 for more details).

5.5.2 Methodological contribution

The current research literature on multiplicative reasoning is one-dimensional, with studies focusing mainly on participants’ procedural fluency, and therefore on their abstract thinking (Carrier, 2014; Clark & Kamii, 1996; Jacob & Willis, 2003; Kouba 1989; Mulligan, 1992; Zhang et al., 2011). This study makes a contribution to the multi-dimensional methodological approach, as discussed in the conceptual framework. This multi-dimensional approach is dealt with on two levels. First, I explored the abstract reasoning of participants by investigating their procedural fluency and also their conceptual understanding and strategic competence. Second, I asked them to solve the problems by using different representational forms, namely abstract, semi-concrete and concrete representations. This multi-dimensional methodological approach ensured the emergence of a more complete picture of participants’ multiplicative reasoning, since the definition of multiplicative reasoning includes a multi-dimensional understanding, i.e. “… the capacity to work flexibly with the concept, techniques and representations of multiplication (and division) as they occur in a wide range of contexts” (Vergnaud, 2010, p. 2).

As explained in detail in Chapter 3, for the purpose of this study I deliberately asked the participants to answer the questions in a particular order, starting with the more difficult abstract representations before moving on to semi-concrete and concrete representations. In my opinion this was the best method for exploring proficiency. Forcing participants to access their internal concrete or semi-concrete schemas first, or allowing them to choose which representation they wanted to start with would have influenced their abstract thinking and the result would therefore not have been a true reflection of their abstract thinking ability.

5.5.3 Practical contribution

As explained in section 5.5.1, this study makes a theoretical contribution by giving three possible reasons why participants were not proficient in multiplicative reasoning, namely misconceptions, misrepresentations and calculation errors. However, it also makes a practical contribution. If teachers are aware of the types of misconceptions, misrepresentations and calculation errors they should expect when working with learners with learning disabilities, they will be able to improve their instructional practice.

Furthermore, I believe that this study shows that the key to helping learners with learning difficulties to make the conceptual shift from additive to multiplicative reasoning, is to strengthen their conceptual understanding of multiplication through the use of semi-concrete and concrete representations. Seeing the groupings will help them to understand the difference between multiplication and addition. This study clearly showed that the participants struggled to solve problems that were more conceptually complex, even with their semi-concrete and concrete representations. The reason for this could be the exclusive focus on abstract representations in the South African school context, and the assertion that learners with learning difficulties struggle with abstract thinking (Allsopp et al., 2007; Dednam, 2011; Miller & Mercer, 1997). In addition to using equations, learners should be explicitly taught different ways to solve problems in new contexts, for instance by drawing pictures to illustrate the problems. This will help learners with learning difficulties to develop better interconnected schemas and schemes for solving problems in new contexts.

Furthermore, since this study showed that the majority of the participants were unable to explain and justify their reasoning, teachers should pay more attention to the development of this ability. This will enable them to determine what and how learners think and to improve their instructional practice of a specific concept by addressing misconceptions that are identified.

5.6 Limitations of this study

The findings of this study and the generalisability of the results are subject to certain limitations. Since my research paradigm was critical realism, I could only report on what I observed and the inferences drawn on the basis of those observations. I therefore used external representations to report on what participants experienced internally. Fifteen randomly selected participants with different learning difficulties participated in this study. Although each participant was unique with regard to his or her specific type of learning difficulty, I analysed them as a group and not as individual participants. I limited the time for the task-based interviews to one hour each. Some of the learners were slower than others, so that not all the participants could complete all the questions, which was a limitation. All the participants answered the first four questions, while Questions 5 to 10 were asked randomly to make sure that all the questions were asked approximately the same number of times. Another limitation was that I neglected to ask some participants to explain their answers and to write their answers with their semi-concrete drawings. In spite of these limitations, I believe that the findings provide a general picture of the conceptions, misconceptions, misrepresentations, calculation technique levels and types, and calculation errors a teacher could expect when teaching multiplication to learners with learning difficulties.

5.7 Recommendations for further study

This study was undertaken in LSEN schools and involved an investigation of the conceptions, misconceptions, misrepresentations, calculation technique levels and types, and calculation errors commonly encountered when teaching learners with learning difficulties. First, further studies could focus on each of these aspects separately, especially the investigation of the misconceptions, as this could inform instructional practice. Second, a comparative study could be undertaken to compare the multiplicative reasoning proficiency of mainstream school learners and learners with special needs. Third, action research could also be done. This would be done by teachers who instruct learners using CSA sequencing and who investigate how learners' ability to solve multiplication problems is influenced if they use concrete representations first and then move on to semi-concrete and abstract representations. For the purpose of this study, I asked the participants to use abstract representations first; however, future studies could allow participants to use any representational form of their choice to determine which representational forms learners with learning difficulties prefer. Fourth, future studies could investigate how asking learners to explain and justify their reasoning could improve their schemas and schemes for solving multiplication problems. Finally, future studies could conduct a similar study as this one in other countries and results could be compared.

5.8 Implications and recommendations

The findings of this study indicate that more needs to be done to help learners with learning difficulties to make the conceptual shift from additive to multiplicative reasoning. This study shows that although some learners in special needs education have already made this conceptual shift, many more have not yet done so. The teachers who teach those learners are responsible for helping them to achieve this shift. One way in which this can be done is by helping them to develop good interconnected schemas and schemes by allowing them to use different representational forms, thus giving them more options for solving problems. This study clearly showed that those who had made the conceptual shift from additive to multiplicative reasoning had good interconnected schemas and schemes in all the representational forms, which helped them to solve problems, whereas the rest had poor interconnected schemas and schemes in the different representational forms. Learners with learning difficulties who struggle with abstract thinking need to be encouraged to use semi-concrete and concrete representations, which will hopefully serve as a bridge to abstract thinking.

When introducing new concepts, teachers in the Intermediate Phase (Grades 4 to 6) should initially focus less on abstract teaching and more on concrete and semi-concrete teaching. They should allow learners to first practise with semi-concrete pictures and concrete objects before they are asked to solve problems by using abstract representations. Furthermore, they should encourage learners to verbally explain and justify their reasoning while solving problems. In this way teachers will be able to identify misconceptions, which could then be addressed. In my opinion, teachers should aim to help those who struggle by applying a multi-disciplinary approach, i.e. their abstract teaching should be enhanced by interactive teaching to give learners opportunities to explain their reasoning and show their understanding in different representational forms. In South Africa, inclusive education has resulted in more learners who struggle academically attending mainstream schools, and teachers should be equipped to teach those who fall behind.

5.9 Final reflection

Learners with learning difficulties are very close to my heart and it is my desire to see them enjoying and doing well in mathematics. Once I had started researching the then still unfamiliar concept of multiplicative reasoning, I realised how important the development of this type of reasoning is for all learners who want to excel in high school mathematics. I discovered that few past research studies had focused on this topic and even fewer had investigated multiplicative reasoning in learners with learning difficulties. This and other studies have, however, clearly shown the importance of the conceptual shift from additive to multiplicative reasoning that learners need to make. Teachers need to be made aware of this and should implement strategies, as suggested in this study, to help learners with learning difficulties to become abstract thinkers. Only then will those learners be able to understand more abstract mathematical concepts.

I believe that this study sheds light on the interplay between internal and external representations and shows that when there is a lack of connections and well-formed schemes and schemas, what we receive from learners through external representations will not be sufficient. Mathematics teachers should focus on strengthening internal connections.

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APPENDICES

APPENDIX A: Letter of permission from the Gauteng Department of Education

For administrative use only:
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enquiries: 011 843 6503

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GDE RESEARCH APPROVAL LETTER

Date:	23 November 2016
Validity of Research Approval:	6 February 2017 to 29 September 2017
Name of Researcher:	Louw E.
Address of Researcher:	Postnet Suite #67; Private Bag X 20009; Garsfontein; 0042
Telephone / Fax Number/s:	072 266 0752
Email address:	louw.elizma@gmail.com
Research Topic:	Exploring the proficiency in multiplicative reasoning among learners with special needs.
Number and type of schools:	FOUR LSEN Schools
District/s/HO	Tshwane North and Tshwane South

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved. A separate copy of this letter must be presented to the Principal, SGB and the relevant District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted. However participation is VOLUNTARY.

The following conditions apply to GDE research. The researcher has agreed to and may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

CONDITIONS FOR CONDUCTING RESEARCH IN GDE

1. *The District/Head Office Senior Manager/s concerned, the Principal/s and the chairperson/s of the School Governing Body (SGB.) must be presented with a copy of this letter.*
2. *The Researcher will make every effort to obtain the goodwill and co-operation of the GDE District officials, principals, SGBs, teachers, parents and learners involved. Participation is voluntary and additional remuneration will not be paid;*
3. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal and/or Director must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*

2016/11/23

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management ER&KM)
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

4. *Research may only commence from the second week of February and must be concluded by the end of the THIRD quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.*
5. *Items 3 and 4 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
6. *It is the researcher's responsibility to obtain written consent from the SGB/s; principal/s, educator/s, parents and learners, as applicable, before commencing with research.*
7. *The researcher is responsible for supplying and utilizing his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institution/s, staff and/or the office/s visited for supplying such resources.*
8. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research title, report or summary.*
9. *On completion of the study the researcher must supply the Director: Education Research and Knowledge Management, with electronic copies of the Research Report, Thesis, Dissertation as well as a Research Summary (on the GDE Summary template). Failure to submit your Research Report, Thesis, Dissertation and Research Summary on completion of your studies / project – a month after graduation or project completion - may result in permission being withheld from you and your Supervisor in future.*
10. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned;*
11. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director/s and school/s concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

[signature]
....

Dr David Makhado

Director: Education Research and Knowledge Management

DATE: 2016/11/23

Office of the Director: Education Research and Knowledge Management ER&KM)
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

APPENDIX B: Letters of consent and assent

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

1 February 2017

Dear Sir/Madam/Doctor

Letter to the principal requesting permission

I am currently enrolled as a PhD student at the University of Pretoria. My research is aimed at exploring the proficiency in multiplicative reasoning among learners with special needs. In order to promote more effective teaching, it is important for teachers to be enabled to understand how learners reason and what strategies they use when solving multiplication and division problems. Relevant literature offers very little insight into how special needs learners reason when solving multiplication and division problems. This is what this study aims to achieve. I hereby request permission to conduct part of my research in your school. I would like to invite Grade 6 mathematics teachers and their learners to participate in this research. My research findings will be reported on in my PhD thesis, as well as in accredited academic journals and at conferences.

Before the commencement of the research I will approach all the Grade 6 mathematics teachers at your school with letters explaining the purpose of the study and what will be expected of them. Once the teachers have agreed to participate, letters of informed consent and assent will be sent to parents/guardians and learners respectively. Data collection will proceed only after informed consent and assent have been received from all the parties concerned.

Should you respond favourably to my request, the data will be collected this year in the following manner, during Terms 1 and/or 2 with the learners and Term 2 and/or 3 with the teachers:

- The learners who had agreed to participate will be asked to complete a multiplication and division test after school hours at a time that suits the school and the parents/guardians. Learners from different LSEN schools will complete this test. The teachers will be able to use the results to improve their teaching.
- At the same time of the test, interviews will be conducted concerning the answers to the test. The test and interview will not exceed one and a half hour and will take place after school hours at a time convenient to them, the parents/guardians and the school. I would like to video record the interviews if consent/assent can be obtained from all parties involved (principal, parents/guardians and learners). This will allow for a clear and accurate record. The files will be password protected and the faces of the learners will not be shown during the recording as the camera will focus on what they are doing during the test and interview. If for some reason their faces are recorded, they will be blurred out.
- Interviews will be conducted with the teachers after school hours and at a time that suits them. The interviews will be audio recorded.

All participation is voluntary and once committed to the research a teacher, learner or school may still withdraw at any time. Confidentiality and anonymity will be guaranteed at all times by using pseudonyms for the school, teachers and learners. Your school, teachers and the learners will,

Room 4-8, Natural Science Building, Groenkloof
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa
Tel: +27 (0)12 420 5551
Fax: +27 (0)12 420 5621
Email: elizma.louw@up.ac.za
www.up.ac.za

therefore, not be identifiable in the findings of my research. Data collected will be available to my supervisor and I and will also be available in the public domain. The video and audio recordings will be password protected.

After the successful completion of my PhD degree I will give feedback to the school in the form of a written report and, if the school is willing, I would like to present my findings to the mathematics teachers.

Should you have any questions before or during the research process, please feel free to contact me. If you are willing to allow members of your staff and learners to participate in this study, please sign this letter as a declaration of your consent.

Yours sincerely

____________________ ____________________

Researcher: Ms E Louw Date
Email: elizma.louw@up.ac.za
Contact number: 072 266 0752

____________________ ____________________

Supervisor: Prof. G Stols Date
Email: gerrit.stols@up.ac.za
Contact number: 012 420 5750

I, the undersigned, hereby grant consent to Ms E Louw to conduct her PhD research in this school and to video and audio record her interviews with learners and teachers.

School principal's name: _________________________

School principal's signature: _______________________ Date: _______________

School name: ___________________________________

Room 4-8, Natural Science Building, Groenkloof
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa
Tel: +27 (0)12 420 5551
Fax: +27 (0)12 420 5621
Email: elizma.louw@up.ac.za
www.up.ac.za

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
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Faculty of Education

1 February 2017

Dear Sir/Madam

Letter of consent for the Grade 6 mathematics teachers

I am currently enrolled as a PhD student at the University of Pretoria and would like to invite you to participate in a research project aimed at exploring the extent to which learners with special needs are proficient in multiplicative reasoning. In order to promote more effective teaching, it is important for teachers to be enabled to understand how learners reason and what strategies they use when solving multiplication and division problems. Relevant literature offers very little insight into how special needs learners reason when solving multiplication and division problems. This is what this study aims to achieve. This research will be reported on in my PhD thesis, as well as in accredited academic journals and at conferences.

Your participation in, and contribution to this research will be voluntary, anonymous and confidential. It is proposed that you form part of this study's data-collection phase by allowing your Grade 6 mathematics learners to complete a multiplication and division test after school hours at a time that will suit the school and the parents/guardians. I will also need to interview you to discuss your views on the origin(s) of learners' misconceptions regarding multiplication and division.

Should you agree to participate, the process will be as follows: during Terms 1 and/or 2, this year, you will be asked to allow me to conduct a test and interview with individual Grade 6 learners, for about one and half hours after school, which you do not have to attend, but help arrange. You will be given access to the test results, which may be of use to you during your teaching. Once the tests and interviews, with all the willing learners, have been completed, I will interview you at a time that suits you which will be audio recorded. This will likely take place during Term 2 or Term 3. You will not be required to do any preparation for the interview.

Should you declare yourself willing to participate in this research, you will form part of a group of about five teachers. Confidentiality and anonymity will be guaranteed at all times. This will be done by allocating pseudonyms to you, the school and the learners. You may decide to withdraw at any time without giving any reasons for doing so. You and your school will not be identifiable in the findings of my research. Data collected will be available to my supervisor and I and will also be available in the public domain. The audio recordings will be password protected.

After the successful completion of my PhD degree, I will give feedback to the school in the form of a written report and, if the school is willing, I would like to present my findings to all mathematics teachers at your school.

Room 4-8, Natural Science Building, Groenkloof
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa
Tel: +27 (0)12 420 5551
Fax: +27 (0)12 420 5621
Email: elizma.louw@up.ac.za
www.up.ac.za

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Lefapha la Thuto

If you are prepared to participate in this study, please sign this letter as a declaration of your consent, i.e. that you agree to participate voluntarily and that you understand that you may withdraw at any time.

Yours sincerely

____________________ ____________________

Researcher: Ms E Louw Date
Email: elizma.louw@up.ac.za
Contact number: 072 266 0752

____________________ ____________________

Supervisor: Prof. G Stols Date
Email: gerrit.stols@up.ac.za
Contact number: 012 420 5750

I, the undersigned, hereby grant consent to Ms E Louw to collect data, by means of a test and interview with my learners. I agree to participate in an interview that will be audio recorded. I understand that my participation is voluntary and that I may withdraw at any time.

Teacher's name: ____________________________

Teacher's signature: _________________________ Date: __________________

School: __________________________________

Room 4-8, Natural Science Building, Groenkloof
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa
Tel: +27 (0)12 420 5551
Fax: +27 (0)12 420 5621
Email: elizma.louw@up.ac.za
www.up.ac.za

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Faculty of Education

1 February 2017

Dear Parent/Guardian

Letter of consent to parents/guardians

I am currently enrolled as a PhD student at the University of Pretoria. My research is focused on exploring the proficiency in multiplication and division among learners with special needs. It is important for teachers to understand how learners reason and what strategies they use when solving multiplication and division problems, as this can enhance their teaching. Relevant literature offers very little insight into how special needs learners reason when solving multiplication and division problems. This study aims to shed light on these learners' understanding of such problems and on any misconceptions that may exist. In order to do this, a minimum of 50 Grade 6 learners from different special needs schools will be asked to complete a multiplication and division test during school hours. From these approximately 50 learners, 18 will then be selected for interviews, which will be conducted after school hours.

The school and your child's teacher have already given permission for the hour-long test to be written before or after school hours. Your child will not have to study for the test. No marks will be allocated as the aim is purely to investigate how learners reason and what strategies they use to solve multiplication and division problems. The teacher will have access to the results, which may be used to inform his/her teaching. Your child's name will not appear anywhere on the test. The form number will link your child with his/her test in order to protect your child's confidentiality and anonymity. Your child's teacher, my supervisor and I will have access to the results, which will also be available in the public domain. If your child is chosen for an interview, you will be requested to complete another letter of consent.

If you have any questions or concerns, please do not hesitate to contact me. If you agree to allow your child to complete the multiplication and division test, please sign this letter as a declaration of your consent.

Yours sincerely

____________________ ____________________

Researcher: Ms E Louw — Date
Email: elizma.louw@up.ac.za
Contact number: 072 266 0752

____________________ ____________________

Supervisor: Prof. G Stols — Date
Email: gerrit.stols@up.ac.za
Contact number: 012 420 5750

Room 4-8, Level 2, Natural Science Building, Groenkloof
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa
Tel +27 (0)12 420 5551
Fax +27 (0)12 420 5621
Email elizma.louw@up.ac.za
www.up.ac.za

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I, the undersigned, hereby grant consent / do not grant consent (circle the appropriate response) for my child to complete a multiplication and division test. I understand that my child's identity will be kept confidential at all times, that participation is voluntary and that if, for any reason, he/she wishes to withdraw from the research my child is free to do so.

Parent's/guardian's name: ______________________________

Parent's/guardian's signature: ___________________________ Date: _________________

Telephone number: ___________________________________

Email address: ______________________________________

Child's name: _______________________________________

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
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Faculty of Education

20 January 2017

Dear Learner

Letter to Grade 6 learners to obtain assent for the interviews

I am a student at the University of Pretoria. I am doing research and I need your help. Research helps us to learn new things, and with my research I hope to learn more about how you think when you have to solve mathematical problems.

I would like to chat to you and ask you some questions after school, for about an hour, on a day and at a time that will suit you and your parents or guardians. I will ask you to solve some mathematical problems for me. You will not have to worry about getting the right answers, because even if your answers are wrong you will be helping me. All I need is to find out how you solve the problem. I will ask you to tell me what you are thinking while solving it and why you do it in a particular way. If you decide to allow me to interview you, you will be one of many learners who will all be helping me with my studies.

Important things that you should know:

- ✓ You may decide whether you want to talk to me or not.
- ✓ No one will be upset if you do not want to talk to me.
- ✓ If you say 'yes', you can still change your mind later, even while we are talking.
- ✓ I will make a video recording of our conversation, but no one will be able to identify you. Your face will not be on it.

The results will help your teacher to teach you even better, and I will use them to complete my studies. Anyone else who looks at the video recording of the interview will not be able to recognise you and your name will not be written anywhere on the work you have done.

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If you are willing to talk to me, please write your name at the bottom of this letter and then give it back to your teacher. Do not write your name if you do not want to talk to me.

If you have any questions you may contact me or ask your parent/guardian to contact me at any time.

I agree to talk to Ms E Louw about how I solve mathematical problems. I understand that I do not have to talk to her if I do not want to. I also understand that even if I write my name now, I may later decide not to talk to Ms Louw.

Your name and surname: ______________________________

Date: ________________________

Kind regards

______________________ ______________

Researcher: Ms E Louw Date
Email: elizma.louw@up.ac.za
Contact number: 072 266 0752

______________________ ______________

Supervisor: Prof G Stols Date
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Email: elizma.louw@up.ac.za
www.up.ac.za

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APPENDIX C: Evidence indicators and categorising of the data

See attached CD