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Research Article

Fixed Points for Multivalued Suzuki Type $(\theta, \mathscr{R})$-Contraction Mapping with Applications

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In this paper, we will introduce the concept of Suzuki type multivalued $(\theta, \mathscr{R})$-contraction and we will prove some fixed point results in the setting of a metric space equipped with a binary relation. Our results generalize and extend various comparable results in the existing literature. Examples are provided to support the results proved here. As an application of our results, we obtain a homotopy result, proving the existence of a solution for a second-order differential equation and for a first-order fractional differential equation.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping on X. An element $x \in X$ is called a fixed point of T if it remains invariant under the action of T; that is, $x = Tx$. A mapping T on a metric space X is said to be a Banach contraction if

$$d(Tx, Ty) \leq kd(x, y) \tag{1}$$

holds for all $x, y \in X$, where $0 \leq k < 1$. A Banach contraction mapping defined on a complete metric space has a unique fixed point. This result is known as Banach contraction principle. Several authors have extended and generalized Banach contraction principle in different directions.

Jleli and Samet [1] suggested a modification in the contraction condition and introduced a ϑ-contraction mapping. Consistent with [1], the following notations, definitions, and results will be needed in the sequel.

Suppose that

$$\Omega = \{\vartheta : (0, \infty) \longrightarrow (1, \infty) \text{ satisfy } (\vartheta_1), (\vartheta_2) \text{ and } (\vartheta_3)\} \tag{2}$$

where

(ϑ_1) ϑ is nondecreasing;

(ϑ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\longrightarrow\infty}\vartheta(t_n) = 1$ if and only if $\lim_{n\longrightarrow\infty}t_n = 0^+$;

(ϑ_3) there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\longrightarrow 0^+}((\vartheta(t) - 1)/t^r) = l$.

Example 1. Define $\eta_i : (0, \infty) \longrightarrow (1, \infty)$ for $i = 1, 2, 3$ by

$$\eta_1(t) = e^{\sqrt{t}}$$
$$\eta_2(t) = 5^{\sqrt{t}} \text{ and} \tag{3}$$
$$\eta_3(t) = e^{\sqrt{te^t}}.$$

Then, $\eta_1, \eta_2, \eta_3 \in \Omega$.

Let (X, d) be a metric space and $\vartheta \in \Omega$. A mapping $T : X \longrightarrow X$ is called a ϑ-contraction if for any $x, y \in X$, we have

$$\vartheta(d(Tx, Ty)) \leq [\vartheta(d(x, y))]^\alpha \tag{4}$$

whenever $d(Tx, Ty) > 0$ and $0 \leq \alpha < 1$. Jleli and Samet [1] proved the following fixed point theorem in the framework of a generalized metric space in the sense of Branciar; i.e., the triangle inequality is replaced by the inequality $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$, for all pairwise distinct points $x, y, u, v \in X$.

Theorem 2. *Let (X,d) be a complete generalized metric space in the sense of Branciari and $T : X \longrightarrow X$ be a ϑ-contraction. Then T has a unique fixed point.*

Hussain et al. [2] considered the following class of mappings:

$$\Theta = \{\theta : [0,\infty) \longrightarrow [1,\infty) : \theta \text{ satisfies } (\theta_1) - (\theta_5)\} \quad (5)$$

where

(θ_1) θ is nondecreasing;

(θ_2) $\theta(t) = 1$ if and only if $t = 0$;

(θ_3) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\longrightarrow\infty}\theta(t_n) = 1$ if and only if $\lim_{n\longrightarrow\infty} t_n = 0^+$;

(θ_4) there exists $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\longrightarrow 0^+}((\theta(t) - 1)/t^r) = l$;

(θ_5) $\theta(t_1 + t_2) \leq \theta(t_1)\theta(t_2)$.

Example 3. Let $\theta_1, \theta_2 : [0,\infty) \longrightarrow [1,\infty)$ be defined by $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = 5^{\sqrt{t}}$, respectively, then θ_1 and $\theta_2 \in \Theta$.

Hussain et al. [2] proved the following result.

Theorem 4. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a continuous mapping. Suppose there exist $\theta \in \Theta$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ with $0 \leq \alpha + \beta + \gamma + 2\delta < 1$, such that for any $x, y \in X$, we have*

$$\begin{aligned} \theta(d(Tx,Ty)) \leq{} & [\theta(d(x,y))]^{\alpha} [\theta(d(x,Tx))]^{\beta} \\ & \cdot [\theta(d(y,Ty))]^{\gamma} \\ & \cdot [\theta(d(x,Ty) + d(y,Tx))]^{\delta}. \end{aligned} \quad (6)$$

Then T has a unique fixed point.

For other results in this direction, we refer to [3, 4] and references mentioned therein.

Let (X,d) be a metric space and $CB(X)$ (respectively $K(X)$) be the family of all nonempty closed and bounded (nonempty compact, respectively) subsets of X. For $A, B \in CB(X)$ and $x \in X$, define

$$\begin{aligned} d(x,A) &:= \inf_{a\in A} d(x,a), \\ H(A,B) &:= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}. \end{aligned} \quad (7)$$

H is the Hausdorff-Pompeiu metric on $CB(X)$ (or on $K(X)$) induced by d. Let $T : X \longrightarrow CB(X)$ be a given multivalued mapping. An element $x \in X$ is called a fixed point of T if $x \in Tx$. A mapping $T : X \longrightarrow CB(X)$ is said to be a Nadler contraction if there exists $k \in (0,1)$ such that

$$H(Tx,Ty) \leq kd(x,y), \quad (8)$$

for any $x, y \in X$. Nadler [5] obtained the following multivalued version of Banach contraction principle.

Theorem 5. *Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a Nadler contraction. Then T has at least one fixed point.*

Later on, many researchers have obtained fixed point results for multivalued mappings satisfying generalized contraction type conditions. For example, recently, Hançer et al. [6] proved the following fixed point result for multivalued θ-contractions.

Theorem 6. *Let (X,d) be a complete metric space and $T : X \longrightarrow K(X)$ be a multivalued mapping. Suppose that there exist $\theta \in \Theta$ and $0 \leq \alpha < 1$ such that*

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^{\alpha}, \quad (9)$$

for any $x, y \in X$, provided that $H(Tx,Ty) > 0$. Then T has at least one fixed point.

Durmaz [7] introduced a new type of generalized multivalued θ-contraction and proved some interesting fixed point results (see also [8]). Kikkawa and Suzuki [9] refined Nadler' result by proving the following theorem.

Theorem 7. *Let $\beta : [0,1) \longrightarrow (1/2,1]$ be defined as $\beta(b) = 1/(1+b)$. Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$. Assume there exists $b \in (0,1)$ such that*

$$\begin{gathered} x, y \in X \\ \text{with } \beta(b)\, d(x,Tx) \leq d(x,y) \\ \text{implies } H(Tx,Ty) \leq bd(x,y). \end{gathered} \quad (10)$$

Then T has at least one fixed point.

We denote $\mathbb{R}^+ := [0,\infty)$ and define the following class of mappings, which was considered in [10].

$$\Phi = \left\{\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R} \text{ and } \varphi(s,t) \leq \frac{1}{2}s - t\right\}. \quad (11)$$

Example 8. Let $\varphi_1 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined by $\varphi(s,t) = v(s) - u(t)$ where $v, s : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are given by $v(s) = s/2$ and $u(t) = t$. Obviously $\varphi_1 \in \Phi$.

Example 9. Let $\varphi_2 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined by $\varphi(s,t) = s/2 - (v(s,t)/u(s,t))t$ where $v, s : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are defined by $v(s,t) = st$ and $u(s,t) = st + t$ for all $s, t > 0$. Note that $\varphi_2 \in \Phi$.

Many results, dealing with existence of fixed points of mappings satisfying certain contraction type conditions in the framework of complete metric spaces endowed with a partial ordering, have appeared in the last decade. Ran and Reurings [11] proved an analogue of Banach's fixed point theorem in a metric space endowed with partial ordering and gave an application of their results to solve matrix equations. Alam and Imdad [12] proved another variant of

Banach's fixed point theorem in a metric space equipped with a binary relation which generalized many comparable results, including Ran and Reuring's result in [11]. Senapati and Dey [13] proved Banach's fixed point theorem in metric spaces equipped with an arbitrary binary relation using ω-distance. They employed their results to prove the existence of solutions of nonlinear fractional differential equations and fractional thermostat model involving the Caputo fractional derivative. A very nice Ph.D. thesis was written on the same subject; see Dobrican [14].

Let us first recall the following definitions.

Definition 10. Let X be a nonempty set and $\mathscr{R}$ be a binary relation defined on $X \times X$. Then, x is $\mathscr{R}$-related to y if and only if $(x, y) \in \mathscr{R}$.

We denote $\mathbb{N}_0 := \{0, 1, 2, \ldots, \}$ and $\mathbb{N} := \{1, 2, \ldots, \}$.

Definition 11. Let X be a nonempty set and $\mathscr{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathscr{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathscr{R} \quad \forall n \in \mathbb{N}_0. \tag{12}$$

Definition 12. Let (X, d) be a metric space. A binary relation $\mathscr{R}$ defined on X is called d-self closed if whenever $\{x_n\}$ is an $\mathscr{R}$-preserving sequence and x_n converges to x, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with either $(x_{n_k}, x) \in \mathscr{R}$ or $(x, x_{n_k}) \in \mathscr{R}$ for all $k \in \mathbb{N}_0$.

Definition 13. Let (X, d) be a metric space and $\mathscr{R}$ a binary relation defined on X. A mapping $T : X \longrightarrow P(X)$ is $\mathscr{R}$-closed if for any $x, y \in X$, $(x, y) \in \mathscr{R}$ implies that $(u, v) \in \mathscr{R}$ for any $u \in Tx$ and $v \in Ty$.

If $T : X \longrightarrow P(X)$ is a multivalued map, then we set

$$X(T; \mathscr{R}) = \{x \in X : (x, y) \in \mathscr{R} \text{ for some } y \in Tx\}. \tag{13}$$

In particular, if T is single-valued, then we denote

$$X'(T; \mathscr{R}) = \{x \in X : (x, Tx) \in \mathscr{R}\}. \tag{14}$$

Motivated by the results in [2, 10, 12], we introduce the concept of a Suzuki type multivalued $(\theta, \mathscr{R})$ -contraction mapping and present some fixed point results in metric spaces equipped with a binary relation. Our results extend and generalize several results given in [2, 15–19]. We also provide applications of our results to homotopy theory proving the existence of a solution of second-order differential equations and first-order fractional differential equations.

2. Multivalued Suzuki Type $(\theta,\mathscr{R})$-Contraction

In this section, we obtain a fixed point result for multivalued Suzuki type $(\theta, \mathscr{R})$-contraction in a metric space equipped with a binary relation $\mathscr{R}$.

Throughout this paper $\theta \in \Theta$ satisfies the following additional property:

$$(\theta_6) : \theta(\inf A) = \inf_{a \in A} \theta(a) \tag{15}$$

where $A \subset [0, \infty)$.

We will denote

$$\Theta^* = \{\theta : [0, \infty) \longrightarrow [1, \infty) : \theta \text{ satisfies } (\theta_1) - (\theta_6)\} \tag{16}$$

We start with the following definition.

Definition 14. Let (X, d) be a metric space and $\mathscr{R}$ a binary relation on X. Assume that $\varphi \in \Phi$ and $\theta \in \Theta^*$. A mapping $T : X \longrightarrow P(X)$ is a multivalued Suzuki type $(\theta, \mathscr{R})$-contraction if for any $x, y \in X$ with $(x, y) \in \mathscr{R}$

$$\begin{aligned} &\varphi(d(x, Tx), d(x, y)) < 0 \Longrightarrow \\ &\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{\alpha} [\theta(d(x, Tx))]^{\beta} \\ &\quad \cdot [\theta(d(y, Ty))]^{\gamma} [\theta(d(y, Tx) + d(x, Ty))]^{\delta}, \end{aligned} \tag{17}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ with $0 \leq \alpha + \beta + \gamma + 2\delta < 1$.

Our first main result is the following.

Theorem 15. *Let (X, d) be a complete metric space, $\mathscr{R}$ a binary relation on X, and $T : X \longrightarrow CB(X)$ a multivalued Suzuki type $(\theta, \mathscr{R})$-contraction. Suppose that following conditions hold:*

(1) *$X(T; \mathscr{R})$ is nonempty,*

(2) *T is $\mathscr{R}$-closed,*

(3) *$\mathscr{R}$ is d-self closed or T has closed graph.*

Then T has at least one fixed point.

Proof. Since $X(T; \mathscr{R})$ is nonempty, if we choose $x_0 \in X(T; \mathscr{R})$, then there exists some $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathscr{R}$. If $x_0 = x_1$, the result follows. Assume that $x_0 \neq x_1$. As $(x_0, x_1) \in \mathscr{R}$ and

$$\begin{aligned} &\varphi(d(x_0, Tx_0), d(x_0, x_1)) \\ &\quad \leq \frac{1}{2} d(x_0, Tx_0) - d(x_0, x_1) \\ &\quad < d(x_0, Tx_0) - d(x_0, x_1) \leq d(x_0, x_1) - d(x_0, x_1) \\ &\quad = 0, \end{aligned} \tag{18}$$

we have

$$\begin{aligned} \theta(d(x_1, Tx_1)) &\leq \theta(H(Tx_0, Tx_1)) \\ &\leq [\theta(d(x_0, x_1))]^{\alpha} [\theta(d(x_0, Tx_0))]^{\beta} \\ &\quad \cdot [\theta(d(x_1, Tx_1))]^{\gamma} \\ &\quad \cdot [\theta(d(x_1, Tx_0) + d(x_0, Tx_1))]^{\delta}. \end{aligned} \tag{19}$$

Now $d(x_1, Tx_0) = 0$ and $d(x_0, Tx_0) \leq d(x_0, x_1)$ imply that

$$\begin{aligned} \theta(d(x_1, Tx_1)) &\leq [\theta(d(x_0, x_1))]^{\alpha+\beta} \\ &\quad \cdot [\theta(d(x_1, Tx_1))]^{\gamma} \\ &\quad \cdot [\theta(d(x_0, Tx_1)]^{\delta}. \end{aligned} \tag{20}$$

Then by $d(x_0, Tx_1) \le d(x_0, x_1) + d(x_1, Tx_1)$ and (θ_5), it follows that

$$\theta(d(x_0, Tx_1)) \le \theta(d(x_0, x_1))\,\theta(d(x_1, Tx_1)) \tag{21}$$

and hence

$$\begin{aligned}&\theta(d(x_1, Tx_1))\\ &\quad\le [\theta(d(x_0, x_1))]^{\alpha+\beta+\delta}[\theta(d(x_1, Tx_1))]^{\gamma+\delta}.\end{aligned} \tag{22}$$

Hence, we obtain

$$[\theta(d(x_1, Tx_1))]^{1-\gamma-\delta} \le [\theta(d(x_0, x_1))]^{\alpha+\beta+\delta} \tag{23}$$

and, in conclusion, we get that

$$\theta(d(x_1, Tx_1)) \le [\theta(d(x_0, x_1))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{24}$$

By (θ_6) we have

$$\theta(d(x_1, Tx_1)) = \inf_{y\in Tx_1}\theta(d(x_1, y)). \tag{25}$$

Thus,

$$\inf_{y\in Tx_1}\theta(d(x_1, y)) \le [\theta(d(x_0, x_1))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{26}$$

We can choose $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \le [\theta(d(x_0, x_1))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{27}$$

As $(x_0, x_1) \in \mathscr{R}$, $x_1 \in Tx_0$, $x_2 \in Tx_1$ and T is $\mathscr{R}$- closed, we have that $(x_1, x_2) \in \mathscr{R}$. If $x_1 = x_2$, the result follows. Assume that $x_1 \ne x_2$. Also,

$$\begin{aligned}&\varphi(d(x_1, Tx_1), d(x_1, x_2))\\ &\quad\le \frac{1}{2}d(x_1, Tx_1) - d(x_1, x_2)\\ &\quad< d(x_1, Tx_1) - d(x_1, x_2) < 0.\end{aligned} \tag{28}$$

Hence,

$$\begin{aligned}&\theta(d(x_2, Tx_2)) \le \theta(H(Tx_1, Tx_2))\\ &\quad\le [\theta(d(x_1, x_2))]^{\alpha}[\theta(d(x_1, Tx_1))]^{\beta}\\ &\quad\cdot [\theta(d(x_2, Tx_2))]^{\gamma}\\ &\quad\cdot [\theta(d(x_2, Tx_1) + d(x_1, Tx_2))]^{\delta}.\end{aligned} \tag{29}$$

By $d(x_1, Tx_2) \le d(x_1, x_2) + d(x_2, Tx_2)$ and (θ_5), we have

$$\theta(d(x_1, Tx_2)) \le \theta(d(x_1, x_2))\,\theta(d(x_2, Tx_2)). \tag{30}$$

As $d(x_2, Tx_1) = 0$ and $d(x_1, Tx_1) \le d(x_1, x_2)$, we obtain that

$$[\theta(d(x_2, Tx_2))]^{1-\gamma-\delta} \le [\theta(d(x_1, x_2))]^{\alpha+\beta+\delta}. \tag{31}$$

Hence,

$$\theta(d(x_2, Tx_2)) \le [\theta(d(x_1, x_2))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{32}$$

By (θ_6), we have

$$\theta(d(x_2, Tx_2)) = \inf_{y\in Tx_2}\theta(d(x_2, y)). \tag{33}$$

Hence,

$$\inf_{y\in Tx_2}\theta(d(x_2, y)) \le [\theta(d(x_1, x_2))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)} \tag{34}$$

We can choose $x_3 \in Tx_2$ such that

$$\theta(d(x_2, x_3)) \le [\theta(d(x_1, x_2))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{35}$$

By (27), we get

$$\theta(d(x_2, x_3)) \le [\theta(d(x_0, x_1))]^{((\alpha+\beta+\delta)/(1-\gamma-\delta))^2}. \tag{36}$$

Since $(x_1, x_2) \in \mathscr{R}$, and T is $\mathscr{R}$-closed, we have $(x_2, x_3) \in \mathscr{R}$.

Continuing this way, we can obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and $\{x_n\}$ is $\mathscr{R}$-preserving. Obviously, we have $d(x_n, Tx_n) \le d(x_n, x_{n+1})$, for all natural numbers $n \ge 0$. Hence,

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\le [\theta(d(x_{n-1}, x_n))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}\\ &\le \cdots\\ &\le [\theta(d(x_0, x_1))]^{((\alpha+\beta+\delta)/(1-\gamma-\delta))^n}.\end{aligned} \tag{37}$$

Letting $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty}\theta(d(x_n, x_{n+1})) = 1. \tag{38}$$

It follows from (θ_3) that

$$\lim_{n\longrightarrow\infty}d(x_n, x_{n+1}) = 0. \tag{39}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If we set $t_n = d(x_n, x_{n+1})$, then from (37), we obtain

$$\begin{aligned}1 &< \theta(t_n) \le (\theta(t_{n-1}))^{(\alpha+\beta+\delta)/(1-\gamma-\delta)} \le \cdots\\ &\le (\theta(t_1))^{((\alpha+\beta+\delta)/(1-\gamma-\delta))^n}.\end{aligned} \tag{40}$$

Further, from (θ_4), there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\theta(t_n) - 1}{(t_n)^r} = l. \tag{41}$$

Suppose $l < \infty$. Let $B = l/2 > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we obtain that

$$\left|\frac{\theta(t_n) - 1}{(t_n)^r} - l\right| \le B. \tag{42}$$

Hence, for all $n \ge n_0$, we have

$$\frac{\theta(t_n) - 1}{(t_n)^r} - l \ge -B. \tag{43}$$

This implies that

$$n(t_n)^r \le An[\theta(t_n)-1] \quad \forall n \ge n_0 \text{ and for } A=\frac{1}{B}. \tag{44}$$

If $l=\infty$, then for $B>0$ there exists n_0 such that for all $n \ge n_0$, we have

$$\frac{\theta(t_n)-1}{(t_n)^r} \ge B \tag{45}$$

which implies that

$$n(t_n)^r \le An[\theta(t_n)-1] \quad \forall n \ge n_0, \text{ where } A=\frac{1}{B}. \tag{46}$$

Hence, for each case, we obtain that

$$n(t_n)^r \le An[\theta(t_n)-1] \quad \forall n \ge n_0. \tag{47}$$

Thus, using (40) we have

$$n(t_n)^r \le An\left[(\theta(t_1))^{((\alpha+\beta+\delta)/(1-\gamma-\delta))^n}-1\right] \quad \forall n \ge n_0. \tag{48}$$

Therefore,

$$\lim_{n\longrightarrow\infty} n(t_n)^r = 0. \tag{49}$$

So, there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$, we have $0 < n(t_n)^r < 1$ which implies that $t_n < (1/n)^{1/r}$. Let $m > n > n_1 \in \mathbb{N}$. Then

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots \\ &\quad + d(x_{m-1},x_m) = t_n + t_{n+1} + \cdots + t_{m-1} \\ &\le \sum_{i=n}^{\infty} t_i < \sum_{i=n}^{\infty}\left(\frac{1}{i}\right)^{1/r}. \end{aligned} \tag{50}$$

By the convergence of the series $\sum_{i=1}^{\infty}(1/i)^{1/r}$ we get $d(x_n,x_m) \longrightarrow 0$ as $m,n \longrightarrow \infty$. Hence, $\{x_n\}$ is Cauchy. Since (X,d) is complete, there exists x^* in X such that $x_n \longrightarrow x^*$. We show that T has a fixed point. Assume on the contrary that T does not have a fixed point. Then, $d(x_n,Tx_n) > 0$ for all natural numbers $n \ge 0$. As $(x_n,x_{n+1}) \in \mathcal{R}$, we have

$$\frac{1}{2}d(x_n,Tx_n) < d(x_n,Tx_n) \le d(x_n,x_{n+1}), \tag{51}$$

which implies that

$$\varphi(d(x_n,Tx_n), d(x_n,x_{n+1})) < 0. \tag{52}$$

Furthermore,

$$d(x_{n+1},Tx_{n+1}) \le H(Tx_n,Tx_{n+1}) \tag{53}$$

gives that

$$\begin{aligned} \theta(d(x_{n+1},Tx_{n+1})) &\le \theta(H(Tx_n,Tx_{n+1})) \\ &\le [\theta(d(x_n,x_{n+1}))]^{\alpha}\cdot[\theta(d(x_n,Tx_n))]^{\beta} \\ &\quad \cdot [\theta(d(x_{n+1},Tx_{n+1}))]^{\gamma} \\ &\quad \cdot [\theta(d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1}))]^{\delta}. \end{aligned} \tag{54}$$

Thus

$$[\theta(d(x_{n+1},Tx_{n+1}))]^{1-\gamma-\delta} \le [\theta(d(x_n,x_{n+1}))]^{\alpha+\beta+\delta}, \tag{55}$$

which further implies that

$$\begin{aligned} \theta(d(x_{n+1},Tx_{n+1})) &\le [\theta(d(x_n,x_{n+1}))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)} \\ &< \theta(d(x_n,x_{n+1})). \end{aligned} \tag{56}$$

Hence

$$d(x_{n+1},Tx_{n+1}) < d(x_n,x_{n+1}), \quad \forall n \in \mathbb{N}_0. \tag{57}$$

If $\mathcal{R}$ is d-self closed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that either $(x_{n_k},x^*) \in \mathcal{R}$ or $(x^*,x_{n_k}) \in \mathcal{R}$. Assume that $(x_{n_k},x^*) \in \mathcal{R}$. If $\varphi(d(x_{n_k},Tx_{n_k}), d(x_{n_k},x^*)) \ge 0$, then we have

$$\frac{1}{2}d\left(x_{n_k},Tx_{n_k}\right) \ge d\left(x_{n_k},x^*\right). \tag{58}$$

From (57), we obtain that

$$\begin{aligned} d\left(x_{n_k},x_{n_{k+1}}\right) &\le d\left(x_{n_k},x^*\right) + d\left(x^*,x_{n_{k+1}}\right) \\ &\le \frac{1}{2}d\left(x_{n_k},Tx_{n_k}\right) + \frac{1}{2}d\left(x_{n_{k+1}},Tx_{n_{k+1}}\right) \\ &< \frac{1}{2}d\left(x_{n_k},x_{n_{k+1}}\right) + \frac{1}{2}d\left(x_{n_k},x_{n_{k+1}}\right) \\ &= d\left(x_{n_k},x_{n_{k+1}}\right), \end{aligned} \tag{59}$$

a contradiction. Hence, $\varphi(d(x_{n_k},Tx_{n_{k+1}}), d(x_{n_k},x^*)) < 0$ for all $k \in \mathbb{N}_0$.

By our assumption $x^* \notin Tx^*$. Thus

$$\begin{aligned} d(x^*,Tx^*) &\le d\left(x^*,x_{n_k}\right) + d\left(x_{n_k},Tx^*\right) \\ &\le d\left(x^*,x_{n_k}\right) + H\left(Tx_{n_k},Tx^*\right), \end{aligned} \tag{60}$$

which implies that

$$\theta(d(x^*,Tx^*)) \le \theta\left(d\left(x^*,x_{n_k}\right)\right)\theta\left(H\left(Tx_{n_k},Tx^*\right)\right). \tag{61}$$

Consequently,

$$\begin{aligned} &\theta\left(d\left(x^*,x_{n_k}\right)\right)\theta\left(H\left(Tx_{n_k},Tx^*\right)\right) \le \theta\left(d\left(x^*,x_{n_k}\right)\right) \\ &\quad \cdot \left[\theta\left(d\left(x_{n_k},x^*\right)\right)\right]^{\alpha}\left[\theta\left(d\left(x_{n_k},Tx_{n_k}\right)\right)\right]^{\beta} \\ &\quad \cdot [\theta(d(x^*,Tx^*))]^{\gamma} \\ &\quad \cdot \left[\theta\left(d\left(x^*,Tx_{n_k}\right)+d\left(x_{n_k},Tx^*\right)\right)\right]^{\delta}. \end{aligned} \tag{62}$$

Also,

$$d\left(x_{n_k},Tx^*\right) \le d\left(x_{n_k},x^*\right) + d(x^*,Tx^*). \tag{63}$$

From (61), it follows that

$$\begin{aligned}\theta\left(d\left(x^*,Tx^*\right)\right) \leq &\left[\theta\left(d\left(x^*,x_{n_k}\right)\right)\right]^{1+\alpha+\delta} \\ &\cdot\left[\theta\left(d\left(x^*,Tx^*\right)\right)\right]^{\delta+\gamma} \\ &\cdot\left[\theta\left(d\left(x_{n_k},x_{n_{k+1}}\right)\right)\right]^{\beta} \\ &\cdot\left[\theta\left(d\left(x^*,x_{n_{k+1}}\right)\right)\right]^{\delta}.\end{aligned} \tag{64}$$

Letting $k \longrightarrow \infty$ in (64) we obtain that

$$\begin{aligned}\theta\left(d\left(x^*,Tx^*\right)\right) &\leq \left[\theta\left(d\left(x^*,Tx^*\right)\right)\right]^{\delta+\gamma} \\ &< \theta\left(d\left(x^*,Tx^*\right)\right),\end{aligned} \tag{65}$$

a contradiction. Hence, $x^* \in Tx^*$.

If T has closed graph, since $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$ and $\lim_{n\longrightarrow\infty} x_n = x^*$, we get that $x^* \in Tx^*$. □

If we take $\beta, \gamma, \delta = 0$ in Theorem 15, we obtain a Suzuki type generalization of the result in [6] in the framework of a complete metric space equipped with a binary relation $\mathcal{R}$.

Corollary 16. *Let (X,d) be a complete metric space, $\mathcal{R}$ a binary relation on X and $T : X \longrightarrow CB(X)$. Assume that $\varphi \in \Phi$ and $\theta \in \Theta^*$. Suppose that there exists $0 \leq \alpha < 1$ such that for any $x, y \in X$ with $(x, y) \in \mathcal{R}$,*

$$\varphi\left(d\left(x,Tx\right),d\left(x,y\right)\right) < 0 \tag{66}$$

implies that

$$\theta\left(H\left(Tx,Ty\right)\right) \leq \left(\theta\left(d\left(x,y\right)\right)\right)^{\alpha}. \tag{67}$$

If conditions (1)-(3) in Theorem 15 are satisfied, then T has a fixed point.

If we take $\theta(t) = e^{\sqrt{t}}$n, Theorem 15, then we have the following multivalued extension of Ćirić result in [16].

Corollary 17. *Let (X,d) be a complete metric space, $\mathcal{R}$ a binary relation on X, and $T : X \longrightarrow CB(X)$. Suppose that $\varphi \in \Phi$ and there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ with $0 \leq \alpha+\beta+\gamma+2\delta < 1$, such that for any $x, y \in X$ with $(x, y) \in \mathcal{R}$,*

$$\varphi\left(d\left(x,Tx\right),d\left(x,y\right)\right) < 0 \tag{68}$$

implies that

$$\begin{aligned}\sqrt{H\left(Tx,Ty\right)} \leq\ &\alpha\sqrt{d\left(x,y\right)} + \beta\sqrt{d\left(x,Tx\right)} \\ &+ \gamma\sqrt{d\left(y,Ty\right)} \\ &+ \delta\sqrt{\left(d\left(x,Ty\right) + d\left(y,Tx\right)\right)}.\end{aligned} \tag{69}$$

Assume that conditions (1)-(3) in Theorem 15 are satisfied. Then T has a fixed point.

Remark 18. Note that the conclusion of Corollary 17 can be written as

$$\begin{aligned}&H\left(Tx,Ty\right) \\ &\leq \alpha^2 d\left(x,y\right) + \beta^2 d\left(x,Tx\right) + \gamma^2 d\left(y,Ty\right) \\ &\quad + \delta^2\left\{d\left(x,Ty\right) + d\left(Tx,y\right)\right\} \\ &\quad + 2\alpha\beta\sqrt{d\left(x,y\right)d\left(x,Tx\right)} \\ &\quad + 2\alpha\gamma\sqrt{d\left(x,y\right)d\left(y,Ty\right)} \\ &\quad + 2\alpha\delta\sqrt{d\left(x,y\right)\left(d\left(x,Ty\right) + d\left(y,Tx\right)\right)} \\ &\quad + 2\beta\gamma\sqrt{d\left(x,Tx\right)d\left(y,Ty\right)} \\ &\quad + 2\beta\delta\sqrt{d\left(y,Ty\right)\left(\left(d\left(x,Ty\right) + d\left(y,Tx\right)\right)\right)} \\ &\quad + 2\gamma\delta\sqrt{d\left(y,Ty\right)\left(d\left(x,Ty\right) + d\left(y,Tx\right)\right)}\end{aligned} \tag{70}$$

Notice that if we take $\alpha, \beta, \gamma = 0$ in Theorem 15, using Remark 18, we obtain the following multivalued Suzuki type generalization of Chatterjea's result in [15].

Corollary 19. *Let (X,d) be a complete metric space, $\mathcal{R}$ a binary relation on X, and $T : X \longrightarrow CB(X)$. Suppose that $\varphi \in \Phi$ and there exists $\delta \in [0, 1/2)$ such that for any $x, y \in X$ with $(x, y) \in \mathcal{R}$,*

$$\varphi\left(d\left(x,Tx\right),d\left(x,y\right)\right) < 0 \tag{71}$$

implies that

$$H\left(Tx,Ty\right) \leq \delta^2\left\{d\left(x,Ty\right) + d\left(y,Tx\right)\right\} \tag{72}$$

Assume that conditions (1)-(3) in Theorem 15 are satisfied, then T has a fixed point.

If we take $\alpha = \delta = 0$ in Theorem 15, using Remark 18, we obtain the following multivalued Kannan type result in [17].

Corollary 20. *Let (X,d) be a complete metric space, $\mathcal{R}$ a binary relation on X, and $T : X \longrightarrow CB(X)$. Suppose that $\varphi \in \Phi$ and there exist $\beta, \gamma \in \mathbb{R}^+$ with $0 \leq \beta + \gamma < 1$ such that for any $x, y \in X$ with $(x, y) \in \mathcal{R}$,*

$$\varphi\left(d\left(x,Tx\right),d\left(x,y\right)\right) < 0 \tag{73}$$

implies that

$$\begin{aligned}H\left(Tx,Ty\right) \leq\ &\beta^2 d\left(x,Tx\right) + \gamma^2 d\left(y,Ty\right) \\ &+ 2\beta\gamma\sqrt{d\left(x,Tx\right)d\left(y,Ty\right)}.\end{aligned} \tag{74}$$

Assume that conditions (1)-(3) in Theorem 15 are satisfied. Then T has a fixed point.

Taking $\delta = 0$ in Theorem 15, we have a multivalued extension and generalization of Reich's result in [20].

Corollary 21. *Let (X,d) be a complete metric space, $\mathscr{R}$ a binary relation on X, and $T : X \longrightarrow CB(X)$. Suppose that $\varphi \in \Phi$ and there exist $\alpha, \beta, \gamma \in \mathbb{R}^+$ with $0 \le \alpha+\beta+\gamma < 1$, such that for any $x, y \in X$ with $(x,y) \in \mathscr{R}$, the following implication is true*

$$\varphi(d(x,Tx), d(x,y)) < 0 \tag{75}$$

implying that

$$\begin{aligned} H(Tx,Ty) &\le \alpha^2 d(x,y) + \beta^2 d(x,Tx) + \gamma^2 d(y,Ty) \\ &\quad + 2\alpha\beta\sqrt{d(x,y)\,d(x,Tx)} \\ &\quad + 2\alpha\gamma\sqrt{d(x,y)\,d(y,Ty)}2\beta\gamma\sqrt{d(x,Tx)\,d(y,Ty)}. \end{aligned} \tag{76}$$

Assume that conditions (1)-(3) in Theorem 15 are satisfied. Then T has a fixed point.

Similarly, if we take $\theta(t) = e^{\sqrt[n]{t}}$, then we obtain the following corollary.

Corollary 22. *Let (X,d) be a complete metric space, $\mathscr{R}$ a binary relation on X, and $T : X \longrightarrow CB(X)$. Suppose that $\varphi \in \Phi$ and there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ with $0 \le \alpha+\beta+\gamma+2\delta < 1$, such that for any $x, y \in X$ with $(x,y) \in \mathscr{R}$,*

$$\varphi(d(x,Tx), d(x,y)) < 0 \tag{77}$$

implies that

$$\begin{aligned} \sqrt[n]{H(Tx,Ty)} &\le \alpha\sqrt[n]{d(x,y)} + \beta\sqrt[n]{d(x,Tx)} \\ &\quad + \gamma\sqrt[n]{d(y,Ty)} \\ &\quad + \delta\sqrt[n]{(d(x,Ty)+d(y,Tx))}. \end{aligned} \tag{78}$$

Assume that conditions (1)-(3) in Theorem 15 are satisfied. Then T has a fixed point.

We now give an example of a multivalued Suzuki type $(\theta,\mathscr{R})$-contraction which is neither a multivalued Banach contraction nor a multivalued $(\theta,\mathscr{R})$-contraction.

Example 23. Let $X = \{x_1, x_2, x_3\}$. Define the binary relation on X as follows:

$$\begin{aligned} \mathscr{R} = \{&(x_1,x_1),(x_1,x_2),(x_2,x_1),(x_2,x_3),(x_2,x_2), \\ &(x_3,x_2),(x_3,x_3)\} \end{aligned} \tag{79}$$

Let $d : X \times X \longrightarrow \mathbb{R}^+$ be defined by

$$\begin{aligned} &d(x_1,x_2) = 4, \\ &d(x_2,x_3) = 1, \\ &d(x_1,x_3) = 3 \end{aligned} \tag{80}$$

and $d(x,y) = d(y,x), d(x,x) = 0, \quad$ for $x, y \in X$.

Define the mapping $T : X \longrightarrow CB(X)$ by

$$Tx = \begin{cases} \{x_1,x_2\}, & \text{whenever } x \in \{x_1,x_2\} \\ \{x_1\}, & \text{whenever } x = x_3. \end{cases} \tag{81}$$

Clearly, T is $\mathscr{R}$-closed and $X(T;\mathscr{R})$ is nonempty. Indeed, if $x = x_1$, $Tx_1 = \{x_1,x_2\}$, then $(x_1,x_2) \in \mathscr{R}$. Take an $\mathscr{R}$-preserving sequence $\{x_n\}$ such that $\{x_n\}$ converges to x and $(x_n,x_{n+1}) \in \mathscr{R}$ for all $n \in \mathbb{N}_0$. Then, $(x_n,x_{n+1}) \in \{(x_1,x_1),(x_1,x_2),(x_2,x_1),(x_2,x_3),(x_2,x_2),(x_3,x_2),(x_3,x_3)\}$ for all $n \in \mathbb{N}_0$. Thus, $\{x_n\} \subset \{x_1,x_2\}$ or $\{x_2,x_3\}$. Since both $\{x_1,x_2\}$ and $\{x_2,x_3\}$ are closed, either $(x_n,x) \in \mathscr{R}$ or $(x,x_n) \in \mathscr{R}$.

Let $\varphi(r,s) = r/2 - s$ and $\theta(t) = e^{\sqrt{t}}$. We consider following cases:

(1) If $(x,y) = (x_3,x_2)$, then $(1/2)d(x_3,Tx_3) - d(x_3,x_2) = 3/2 - 1 > 0$.

(2) If $(x,y) = (x_2,x_3)$, then clearly $(1/2)d(x_2,Tx_2) - d(x_2,x_3) < 0$. For $\alpha, \beta, \gamma = 0$ and $\delta = 0.9$ we have

$$\begin{aligned} \theta(H(Tx_2,Tx_3)) &\le (\theta(d(x_2,x_3)))^{\alpha} \\ &\quad \cdot (\theta(d(x_2,Tx_2)))^{\beta}(\theta(d(x_3,Tx_3)))^{\gamma} \\ &\quad \cdot (\theta(d(x_2,Tx_3)+d(x_3,Tx_2)))^{\delta} \end{aligned} \tag{82}$$

which implies that $e^{\sqrt{4}} < (e^{\sqrt{5}})^{0.9}$.

All other cases are trivial. Thus all the conditions of Theorem 15 are satisfied. Moreover, T has two fixed points x_1 and x_2.

Note that, for $(x_3,x_2) \in \mathscr{R}$,

$$H(Tx_3,Tx_2) = 4 > (k)1 = kd(x_3,x_2) \tag{83}$$

for any $0 < k < 1$. Thus T is not a Nadler's contraction. Also,

$$\theta(H(Tx_3,Tx_2)) = e^{\sqrt{4}} > \left(e^{\sqrt{1}}\right)^k = (\theta(d(x_3,x_2)))^k \tag{84}$$

for any $0 < k < 1$. Thus, T is not multivalued $(\theta,\mathscr{R})$-contraction.

If $T : X \longrightarrow X$ is a single-valued map, then we have the following result.

Theorem 24. *Let (X,d) be a complete metric space, $\mathscr{R}$ a binary relation on X, and $T : X \longrightarrow X$. Suppose that $\varphi \in \Phi$, $\theta \in \Theta^*$ and there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ with $0 \le \alpha+\beta+\gamma+2\delta < 1$, such that for any $x, y \in X$ with $(x,y) \in \mathscr{R}$,*

$$\varphi(d(x,Tx), d(x,y)) < 0 \tag{85}$$

implies that

$$\begin{aligned} \theta(d(Tx,Ty)) &\le \theta(d(x,y))^{\alpha}\,\theta(d(x,Tx))^{\beta} \\ &\quad \cdot \theta(d(y,Ty))^{\gamma} \\ &\quad \cdot \theta((d(x,Ty)+d(y,Tx)))^{\delta}. \end{aligned} \tag{86}$$

In addition, assume that the following conditions also hold:

(1) *$X'(T;\mathcal{R})$ is nonempty,*

(2) *if $x, y \in X$ such that $(x, y) \in \mathcal{R}$, then $(Tx, Ty) \in \mathcal{R}$,*

(3) *$\mathcal{R}$ is d-self closed or T has closed graph.*

Then T has at least one fixed point.

Remark 25. The above theorem generalizes various existing results in literature. If in Theorem 24 we take $\varphi(s,t) = (1/2)s - t$, then we have the following:

(1) We obtain a Suzuki type generalization of θ-contraction result in [2] in the setup of metric spaces endowed with binary relation. If $\theta(t) = e^{\sqrt{t}}$, the above theorem also generalizes the result in [16].

(2) If $\theta(t) = e^{\sqrt{t}}$, $\beta, \gamma, \delta = 0$, and $\alpha < 1$, we obtain a Suzuki type result in the setting of metric spaces endowed with binary relation (see [19]).

(3) If $\theta(t) = e^{\sqrt{t}}$, $\alpha, \beta, \gamma = 0$, and $\delta < 1/2$, we obtain a Suzuki type version of Chatterjea result [15] in the setup of metric spaces endowed with binary relation.

(4) If $\theta(t) = e^{\sqrt{t}}$, $\alpha, \delta = 0$, and $\beta + \gamma < 1$, we obtain a Suzuki type result for generalized Kannan mappings [17] in the setup of metric spaces endowed with binary relation.

Example 26. Let $X = [0, 3] \cup [5, 9]$ be the usual metric space with binary relation defined as follows:

$$\mathcal{R} = \{(1,1), (1,3), (3,1), (3,3), (1,5), (5,5), (5,9), (9,5), (9,9)\}. \quad (87)$$

Define the mapping $T : X \longrightarrow X$ by

$$Tx = \begin{cases} 2x + 3, & \text{whenever } 1 \le x \le 3 \\ 9, & \text{whenever } 5 \le x \le 9. \end{cases} \quad (88)$$

Let $\varphi(s,t) = (1/2)s - t$ and $\theta(t) = 5^{\sqrt{t}}$. Note that T is not a $(\theta, \mathcal{R})$-contraction because

$$\theta(d(T1, T3)) = \theta(d(1,5)) = 5^{\sqrt{4}} > (\theta(d(1,5)))^k = \left(5^{\sqrt{4}}\right)^k. \quad (89)$$

Consider the following cases:

(1) If $(x, y) \in \{(1,1), (1,3), (3,1), (3,3)\}$, then $(1/2)d(x, Tx) - d(x, y) < 0$ does not hold.

(2) If $(x, y) \in \{(5,5), (5,9), (9,5), (9,9)\}$, then clearly T is a Suzuki type $(\theta, \mathcal{R})$-contraction.

(3) If $(x, y) = (1, 5)$, then we have

$$\frac{1}{2}(d(1, T1)) - d(1,5) = \frac{1}{2}d(1,5) - (d(1,5)) < 0 \quad (90)$$

For $\alpha, \beta, \gamma = 0.1$ and $\delta = 0.4$, we obtain that

$$\theta(d(T1, T5)) < (\theta(d(1,5)))^{\alpha} (\theta(d(1, T1)))^{\beta} \cdot (\theta(d(5, T5)))^{\gamma} \cdot (\theta(d(1, T5)) + d(5, T1))^{\delta} \quad (91)$$

which implies that

$$5^{\sqrt{4}} < \left(5^{\sqrt{4}}\right)^{0.1} \left(5^{\sqrt{4}}\right)^{0.1} \left(5^{\sqrt{4}}\right)^{0.1} \left(5^{\sqrt{8}}\right)^{0.4}. \quad (92)$$

Clearly, T is $\mathcal{R}$-closed and $X'(T;\mathcal{R})$ is nonempty, since $(5,9) \in \mathcal{R}$. Take an $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $\{x_n\}$ converges to x and we have $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$. Note that $(x_n, x_{n+1}) \notin \{(1,5)\}$. Then, $(x_n, x_{n+1}) \in \{(1,1), (1,3), (3,1), (3,3), (5,5), (5,9), (9,5), (9,9)\}$ for $n \in \mathbb{N}_0$. Thus, $\{x_n\}$ is a subset of $\{1,3\}$, $\{3,5\}$, or $\{5,9\}$. Since all of these sets are closed, either $(x_n, x) \in \mathcal{R}$ or $(x, x_n) \in \mathcal{R}$. Thus all the conditions of Theorem 24 are satisfied. Moreover, $x = 9$ is a fixed point of T.

3. Application to Homotopy Results

In this section, as an application of our above fixed point result for Suzuki type $(\theta, \mathcal{R})$-contractions, we obtain a homotopy result for this class of multivalued mappings. For the beginning, we give a local fixed point result for multivalued Suzuki type $(\theta, \mathcal{R})$-contraction. Let $B(x_0, r)$ be an open ball and $\overline{B(x_0, r)}$ the closure of $B(x_0, r)$.

Theorem 27. *Let (X, d) be a complete metric space, $\mathcal{R}$ a binary relation on X, $x_0 \in X$ and $r > 0$. Suppose that $T : \overline{B(x_0, r)} \longrightarrow CB(X)$ be a multivalued Suzuki type $(\theta, \mathcal{R})$-contraction with $\theta(d(x_0, Tx_0)) < (\theta(r))^{1-k}$ where $0 < k < 1$. Assume that the following conditions are also satisfied:*

(1) *$x_0 \in X(T;\mathcal{R})$,*

(2) *T is $\mathcal{R}$-closed,*

(3) *T has closed graph or $\mathcal{R}$ is d-self closed.*

Then T has a fixed point in $\overline{B(x_0, r)}$.

Proof. Since $x_0 \in X(T;\mathcal{R})$, there exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathcal{R}$. If $x_0 = x_1$, the result follows. Assume that $x_0 \neq x_1$. Let $0 < s < r$ be such that $B(x_0, s) \subset \overline{B(x_0, r)}$ and $\theta(d(x_0, Tx_0)) < (\theta(s))^{1-k} < (\theta(r))^{1-k}$. Clearly, $d(x_1, Tx_1) \le H(Tx_0, Tx_1)$. Now

$$\varphi(d(x_0, Tx_0), d(x_0, x_1)) \le \frac{1}{2}d(x_0, Tx_0) - d(x_0, x_1) < d(x_0, Tx_0) - d(x_0, x_1) < 0 \quad (93)$$

implies that

$$\begin{aligned}\theta(d(x_1,Tx_1)) &\leq \theta(H(Tx_0,Tx_1)) \\ &\leq [\theta(d(x_0,x_1))]^{\alpha}[\theta(d(x_0,Tx_0))]^{\beta} \\ &\cdot [\theta(d(x_1,Tx_1))]^{\gamma} \\ &\cdot [\theta(d(x_0,Tx_1)+d(x_1,Tx_0))]^{\delta}.\end{aligned} \tag{94}$$

Thus, we have

$$\begin{aligned}&\theta(d(x_1,Tx_1)) \\ &\quad\leq [\theta(d(x_0,x_1))]^{\alpha+\beta+\delta}[\theta(d(Tx_1,x_1))]^{\gamma+\delta}.\end{aligned} \tag{95}$$

Hence

$$[\theta(d(x_1,Tx_1))]^{1-\gamma-\delta} \leq [\theta(d(x_0,x_1))]^{\alpha+\beta+\delta}, \tag{96}$$

which implies that

$$\theta(d(x_1,Tx_1)) \leq [\theta(d(x_0,x_1))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{97}$$

By the condition (θ_6), we get

$$\inf_{y\in Tx_1}\theta(d(x_1,y)) \leq [\theta(d(x_0,x_1))]^{(\alpha+\beta+\delta)/(1-\gamma-\delta)}. \tag{98}$$

We may now choose $x_2 \in Tx_1$ such that $(x_1,x_2) \in \mathscr{R}$ and

$$\theta(d(x_1,x_2)) \leq [\theta(d(x_0,x_1))]^{k} \tag{99}$$

where $k := (\alpha+\beta+\delta)/(1-\gamma-\delta) < 1$. Furthermore,

$$\theta(d(x_1,x_2)) \leq [\theta(d(x_0,x_1))]^{k} < ((\theta(s))^{1-k})^{k}. \tag{100}$$

Note that

$$\begin{aligned}\theta(d(x_0,x_2)) &\leq \theta(d(x_0,x_1))\,\theta(d(x_1,x_2)) \\ &< (\theta(s))^{1-k}(\theta(s))^{k(1-k)} < (\theta(s))^{1-k^2} \\ &< \theta(s).\end{aligned} \tag{101}$$

Thus $x_2 \in B(x_0,s)$ as $d(x_0,x_2) < s$. Continuing this way, we obtain a sequence $\{x_n\}$ with following properties:

(1) $x_n \in B(x_0,s)$,

(2) $(x_n,x_{n+1}) \in \mathscr{R}$,

(3) $x_{n+1} \in Tx_n$ $\forall n \in \mathbb{N}_0$,

(4) $\theta(d(x_n,x_{n+1})) < (\theta(s))^{k^n(1-k)}$.

By similar arguments to those in the proof of Theorem 15, we obtain that $\{x_n\}$ is a Cauchy sequence which converges to $u \in \overline{B(x_0,r)}$ and $u \in Tu$. □

Now we present the following homotopy result.

Theorem 28. *Let (X,d) be a complete metric space, U an open set of X, $\mathscr{R}$ a d-self closed binary relation on X and $\Lambda : \overline{U} \times [0,1] \longrightarrow P(X)$ be a multivalued mapping with closed values. Suppose that for each $t \in [0,1]$, $\Lambda(\cdot,t) : \overline{U} \longrightarrow CB(X)$ satisfies the conditions (1)-(2) in Theorem 27. Assume that the following conditions are also satisfied:*

(1) *$x \notin \Lambda(x,t)$, for each $x \in \partial U$ and each $t \in [0,1]$;*

(2) *$\Lambda(\cdot,t) : \overline{U} \longrightarrow CB(X)$ is a multivalued Suzuki type $(\theta,\mathscr{R})$-contraction with closed graph, for each $t \in [0,1]$;*

(3) *there exists a continuous increasing function $\eta : [0,1] \longrightarrow \mathbb{R}$ such that*

$$\theta(H(\Lambda(x,t),\Lambda(x,s)) \leq \theta(|\eta(t)-\eta(s)|) \tag{102}$$

for all $t,s \in [0,1]$ and each $x \in \overline{U}$.

Then $\Lambda(\cdot,0)$ has a fixed point if and only if $\Lambda(\cdot,1)$ has a fixed point.

Proof. Suppose that $\Lambda(\cdot,0)$ has a fixed point z, then by (1) $z \in U$. Define

$$Q = \{((x,t) \in U \times [0,1] : x \in \Lambda(x,t)\} \tag{103}$$

As $(z,0) \in Q$, so $Q \neq \emptyset$. Define partial order on Q by

$$(x,t) \preccurlyeq (y,s) \text{ if and only if } t \leq s \tag{104}$$

and

$$\begin{aligned}&\theta(d(x,y)) \leq (\theta(\eta(t)-\eta(s)))^{2/(1-k)}, \\ &\qquad\qquad\text{where } k = \frac{\alpha+\beta+\delta}{1-\gamma-\delta}.\end{aligned} \tag{105}$$

Let A be a totally ordered set in Q and $t_* = \sup\{t : (x,t) \in A\}$. Consider a sequence $\{(x_n,t_n)\}$ in A such that $(x_n,t_n) \preccurlyeq (x_{n+1},t_{n+1})$ and $t_n \longrightarrow t_*$ as $n \longrightarrow \infty$. Then, we have

$$\theta(d(x_m,x_n)) \leq (\theta(|\eta(t_m)-\eta(t_n))|)^{2/(1-k)}. \tag{106}$$

As

$$\lim_{m,n\longrightarrow\infty}|\eta(t_m)-\eta(t_n)| = 0, \tag{107}$$

by (θ_3), we have

$$\lim_{m,n\longrightarrow\infty}\theta(|\eta(t_m)-\eta(t_n)|) = 1. \tag{108}$$

Thus (106) gives that

$$1 \leq \lim_{m,n\longrightarrow\infty}\theta(d(x_m,x_n)) \leq 1 \tag{109}$$

and

$$\lim_{m,n\longrightarrow\infty}d(x_m,x_n) = 0. \tag{110}$$

Hence $\{x_n\}$ is Cauchy sequence which converges to an element $x_* \in \overline{U}$. As $x_n \in \Lambda(x_n,t_n)$ for $n \in \mathbb{N}$ and Λ is closed, so $x_* \in \Lambda(x_*, t_*)$. Also, from (1) we have $x_* \in U$. Hence, $(x_*, t_*) \in Q$. Since A is totally ordered, therefore $(x, t) \leq (x_*, t_*)$ for each $(x, t) \in A$. That is, (x_*, t_*) is an upper bound of A. By Zorn's Lemma, the set Q admits a maximal element $(x_0, t_0) \in Q$. We claim that $t_0 = 1$. Assume, on the contrary, that $t_0 < 1$. Choose $r > 0$ and $t \in (t_0, 1]$ such that $B(x_0, r) \subset U$ and

$$\theta(r) = \left[\theta(\left|\eta(t) - \eta(t_0)\right|)\right]^{2/(1-k)}. \quad (111)$$

Note that

$$\begin{aligned} &\theta(d(x_0, \Lambda(x_0, t))) \\ &\quad \leq \theta(d(x_0, \Lambda(x_0, t_0)) + H(\Lambda(x_0, t_0), \Lambda(x_0, t))) \\ &\quad \leq \theta(\left|\eta(t) - \eta(t_0)\right|) = (\theta(r))^{(1-k)/2} < (\theta(r))^{1-k}. \end{aligned} \quad (112)$$

Thus $\Lambda(\cdot, t) : \overline{B(x_0, r)} \longrightarrow CB(X)$ satisfies all assumptions of Theorem 27. Hence, for all $t \in [0, 1]$, there exists $x \in \overline{B(x_0, r)}$ such that $x \in \Lambda(x, t)$. Hence $(x, t) \in Q$. Now, $d(x_0, x) < r$ implies that

$$\theta(d(x_0, x)) < \theta(r) = \left[\theta(\left|\eta(t) - \eta(t_0)\right|)\right]^{2/(1-k)}, \quad (113)$$

which further implies that $(x_0, t_0) < (x, t)$, a contradiction to the fact that (x_0, t_0) is a maximal element.

Conversely, if $\Lambda(\cdot, 1)$ has a fixed point, then by similar arguments to those given before, we obtain that $\Lambda(., 0)$ has a fixed point. □

4. Existence of a Solution of Second-Order Differential Equation

In this section, we study the existence of solution of a two-point boundary value problem associated with a second-order differential equation. Let $X = C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$. The metric on X is given by

$$d(x, y) = \|x - y\|_\infty = \max_{t \in [0,1]} \left|x(t) - y(t)\right|. \quad (114)$$

Define the binary relation on X by

$$\begin{aligned} &(x, y) \in \mathscr{R} \quad \text{if and only if } x \leq y \\ &(\text{i.e., } x(t) \leq y(t),\ t \in [0, 1]). \end{aligned} \quad (115)$$

Note that the space $X = (C[0, 1], d)$ is complete metric space. We consider the following two-point boundary value problem:

$$\begin{aligned} -x''(t) &= f(t, x(t)), \quad t \in [0, 1] \\ x(0) &= x(1) = 0, \end{aligned} \quad (116)$$

where $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Then, the problem (116) can be written in the following integral form:

$$x(t) = \int_0^1 G(t, s) f(s, x(s))\, ds, \quad (117)$$

where the associated Green function is

$$G(t, s) = \begin{cases} t(1 - s); & 0 \leq t \leq s \leq 1 \\ s(1 - t); & 0 \leq s \leq t \leq 1, \end{cases} \quad (118)$$

see [21] for details.

Theorem 29. *Assume that the following conditions are satisfied:*

(1) $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous,*

(2) $f(s, .) : \mathbb{R} \longrightarrow \mathbb{R}$ *is increasing for all* $s \in [0, 1]$,

(3) *there exists* $\tau \in [1, \infty)$ *such that the following condition holds for all* $x, y \in X$, *with* $x \leq y$,

$$\left|f(s, x) - f(s, y)\right| \leq 8e^{-\tau}\left|x(s) - y(s)\right|, \quad \forall s \in [0, 1], \quad (119)$$

(4) *there exists* $x_0 \in X$ *such that for all* $t \in [0, 1]$, *we have*

$$x_0(t) \leq \int_0^1 G(t, s) f(s, x_0(s))\, ds. \quad (120)$$

Then the problem (116) has a solution in X.

Proof. If we define the mapping $T : X \longrightarrow X$ by

$$T(x)(t) = \int_0^1 G(t, s) f(s, x(s))\, ds, \quad (121)$$

then our problem can be written as a fixed point equation $x = Tx$.

Obviously, T is continuous. As $f(s, .)$ is increasing, for any $x, y \in X$ with $x(t) \leq y(t)$ for all $t \in [0, 1]$, we obtain that

$$\int_0^1 G(t, s) f(s, x(s))\, ds \leq \int_0^1 G(t, s) f(s, y(s))\, ds, \quad \text{i.e., } T(x)(t) \leq T(y)(t). \quad (122)$$

Thus, T is $\mathscr{R}$-closed. If $x, y \in X$ such that $x(t) \leq y(t)$, then we have

$$\begin{aligned} \left|T(x)(t) - T(y)(t)\right| &= \left|\int_0^1 G(t, s) f(s, x(s))\, ds - \int_0^1 G(t, s) f(s, y(s))\, ds\right| \\ &\leq \int_0^1 G(t, s)\left|f(s, x(s)) - f(s, y(s))\right| ds \\ &\leq \int_0^1 8e^{-\tau}\left|x(s) - y(s)\right| G(t, s)\, ds \\ &\leq 8e^{-\tau} d(x, y) \int_0^1 G(t, s)\, ds. \end{aligned} \quad (123)$$

As $\int_0^1 G(t,s)ds = t/2 - t^2/2$, we get that $\sup_{t\in[0,1]}(\int_0^1 G(t,s)ds) = 1/8$. The inequality (123) becomes

$$\begin{aligned} \left|T(x)(t) - T(y)(t)\right| &\leq 8e^{-\tau} d(x,y)\frac{1}{8} \\ &\leq e^{-\tau} d(x,y). \end{aligned} \tag{124}$$

Hence, we have

$$\max_{t\in[0,1]} \left|T(x)(t) - T(y)(t)\right| \leq e^{-\tau} d(x,y), \tag{125}$$

which implies that

$$d(T(x), T(y)) \leq e^{-\tau} d(x,y). \tag{126}$$

Taking square root on both sides and passing through exponential function, we obtain that

$$e^{\sqrt{d(T(x),T(y))}} \leq e^{\sqrt{e^{-\tau}d(x,y)}} = \left[e^{\sqrt{d(x,y)}}\right]^{\sqrt{e^{-\tau}}}, \tag{127}$$

where $e^{-\tau} < 1$ as $\tau \geq 1$. Hence,

$$e^{\sqrt{d(T(x),T(y))}} \leq \left[e^{\sqrt{d(x,y)}}\right]^{\alpha} \quad \text{with } \alpha = \sqrt{e^{-\tau}} \in (0,1). \tag{128}$$

Thus,

$$\theta(d(Tx,Ty)) \leq (\theta(d(x,y)))^{\alpha}, \tag{129}$$

where $\theta(t) = e^{\sqrt{t}}$ and $(x,y) \in \mathscr{R}$.

As above inequality is true for any $x, y \in X$ with $x(t) \leq y(t)$, so is for any $\varphi \in \Phi$, with $\varphi(d(x,Tx), d(x,y)) < 0$. Thus

$$\theta(d(Tx,Ty)) \leq (\theta(d(x,y)))^{\alpha}. \tag{130}$$

By Theorem 24 we get that (116) has a solution in X. □

5. Existence of a Solution of Fractional Boundary Value Problem

In this section, we investigate the existence of solutions of a nonlinear fractional differential equation. Let the space X and the metric d be defined as in the above section.

Consider the following fractional differential equation

$$^{C}D^{\beta} x(t) = f(t, x(t)); \quad 0 < t < 1,\ 1 < \beta \leq 2, \tag{131}$$

with boundary conditions

$$\begin{aligned} x(0) &= 0, \\ Ix(1) &= x'(0). \end{aligned} \tag{132}$$

Here $^{C}D^{\beta}$ stands for the Caputo fractional derivative of order β, defined by

$$^{C}D^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^n(s)\,ds, \tag{133}$$

(where we consider $n - 1 < \beta < n$ and $n = [\beta] + 1$) and $I^{\beta} f$ denotes the Riemann-Liouville fractional integral of order β of a continuous function f, given by

$$I^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,ds, \quad \text{with } \beta > 0. \tag{134}$$

Senapati and Dey ([13]) showed that the problem (131)+(132) can be written in the following integral form:

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))\,ds \\ & + \frac{2t}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r, x(r))\,dr\,ds. \end{aligned} \tag{135}$$

Theorem 30. *Suppose that following conditions are satisfied:*

(1) $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function,*

(2) $f(s,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ *is an increasing function,*

(3) *for every* $x, y \in X$ *with* $x \leq y$*, the following condition holds:*

$$\begin{aligned} \left|f(s,x) - f(s,y)\right| \leq \frac{e^{-\tau}\Gamma(\beta+1)}{4} \left|x(s) - y(s)\right|, \\ \forall s \in [0,1], \end{aligned} \tag{136}$$

where $\tau \in [1,\infty)$,

(4) *there exists* $x_0 \in X$ *such that, for all* $t \in [0,1]$*, we have*

$$\begin{aligned} x_0(t) \leq {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_0(s))\,ds \\ & + \frac{2t}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r, x_0(r))\,dr\,ds. \end{aligned} \tag{137}$$

Then, (131)+(132) has at least one solution in $C[0,1]$.

Proof. Define the mapping $T : X \longrightarrow X$ by

$$\begin{aligned} & T(x)(t) \\ & = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))\,ds \\ & \quad + \frac{2t}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r, x(r))\,dr\,ds. \end{aligned} \tag{138}$$

Then (135) can be written as a fixed point equation for T; i.e., $x = Tx$. Consider on $X := C[0,1]$ the binary relation defined as follows

$$x \mathscr{R} y \text{ if and only if } x(t) \leq y(t), \quad \forall t \in [0,1]. \tag{139}$$

By the given assumption (4), $X'(T;\mathscr{R})$ is nonempty. If $x, y \in X$ are such that $x(t) \leq y(t)$ for every t, then by assumption that $f(s,\cdot)$ is increasing we have

$$\begin{aligned}&\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))\,ds \\ &\quad + \frac{2t}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1} f(r,x(r))\,drds \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))\,ds \\ &\quad + \frac{2t}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1} f(r,y(r))\,dr\,ds,\end{aligned} \tag{140}$$

which implies that $T(x)(t) \leq T(y)(t)$. Therefore, T is $\mathscr{R}$-closed. Nieto and López [22] have shown that if there exists a sequence $\{x_n\}$ in X such that $x_n(t) \leq x_{n+1}(t)$ and x_n converges to some x, then $x_n(t) \leq x(t)$. Hence, $\mathscr{R}$ is d-self closed. If $x, y \in X$ such that $x(t) \leq y(t)$, then

$$\begin{aligned}&|T(x)(t) - T(y)(t)| \\ &= \left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))\,ds\right. \\ &- \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))\,ds \\ &+ \frac{2t}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1} f(r,x(r))\,drds \\ &\left.- \frac{2t}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1} f(r,y(r))\,drds\right| \\ &\leq \frac{1}{\Gamma(\beta)} \\ &\cdot \int_0^t (t-s)^{\beta-1} |f(s,x(s))\,ds - f(s,y(s))|\,ds \\ &+ \frac{2}{\Gamma(\beta)} \\ &\cdot \int_0^1\int_0^s (s-r)^{\beta-1} |f(r,x(r)) - f(r,y(r))|\,dr\,ds \\ &\leq e^{-\tau}\frac{\Gamma(\beta+1)}{4\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)-y(s)|\,ds \\ &+ 2e^{-\tau}\frac{\Gamma(\beta+1)}{4\Gamma(\beta)} \\ &\cdot \int_0^1\int_0^s (s-r)^{\beta-1}|x(r)-y(r)|\,dr\,ds \leq e^{-\tau} \\ &\cdot \frac{\Gamma(\beta+1)}{4\Gamma(\beta)} d(x,y)\int_0^t (t-s)^{\beta-1}\,ds + 2e^{-\tau} \\ &\cdot \frac{\Gamma(\beta+1)}{4\Gamma(\beta)} d(x,y)\int_0^1\int_0^s (s-r)^{\beta-1}\,dr\,ds \leq e^{-\tau} \\ &\cdot \frac{\Gamma(\beta)\Gamma(\beta+1)}{4\Gamma(\beta)\Gamma(\beta+1)} d(x,y) + 2e^{-\tau}B(\beta+1,1) \\ &\cdot \frac{\Gamma(\beta)\Gamma(\beta+1)}{4\Gamma(\beta)\Gamma(\beta+1)} d(x,y) \leq \frac{e^{-\tau}}{4} d(x,y) \\ &+ \frac{e^{-\tau}}{2} d(x,y),\end{aligned} \tag{141}$$

where B is the beta function. From the above inequality, we obtain that

$$d(Tx,Ty) \leq e^{-\tau} d(x,y). \tag{142}$$

Taking square root on both sides and passing through exponential function, we have

$$e^{\sqrt{d(Tx,Ty)}} \leq e^{\sqrt{e^{-\tau}d(x,y)}}, \tag{143}$$

that is,

$$e^{\sqrt{d(Tx,Ty)}} \leq \left(e^{\sqrt{d(x,y)}}\right)^{\alpha}, \tag{144}$$

where $\alpha = \sqrt{e^{-\tau}} < 1$. Since the above inequality holds for any $x, y \in X$ such that $x(t) \leq y(t)$ so is true for any $\varphi \in \Phi$, with

$$\varphi(d(x,Tx), d(x,y)) < 0. \tag{145}$$

Hence we have

$$\theta(d(Tx,Ty)) \leq (\theta(d(x,y)))^{\alpha}. \tag{146}$$

Thus T is a Suzuki type $(\theta,\mathscr{R})$-contractive mapping. Since all the conditions of Theorem 24 are satisfied, the problem (131) has at least one solution. □

Data Availability

The authors declare that the data supporting the findings of this study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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