Selected deterministic models for lot sizing of growing items inventory

by

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Abstract

One of the more recent advances in inventory management is the modelling of inventory systems consisting of items which are capable of growing during the course of the replenishment cycle. These items, such as livestock, are a vital part of life because most of them serve as saleable food items downstream in supply chains.

In the context of this study, growth is defined as achieving an increase in weight. This increase in weight is what differentiates growing items from conventional items. A typical inventory system for growing items has two distinct periods, namely growing and consumption periods. The growth period starts when a shipment of live newborn arrives. The live items are fed so that they can grow. All the items in each lot are assumed to grow at the same rate. Once the weight of the items reaches a specific target, they are slaughtered. This marks the end of the growth period and thus the start of the consumption period. The slaughtered items are kept in stock and consumed continuously at a given demand rate. At the instant that the consumption period ends, items in the next cycle would have completed their growth period and they will be ready for slaughter and consumption. A feeding cost is incurred for feeding the live items during the growth period whereas holding costs are incurred for keeping the slaughtered items in stock during the consumption period.

This study is aimed at developing lot sizing models for growing items under three different conditions which might occur in food supply chains. These selected conditions are used to develop three Economic Order Quantity (EOQ) models for growing items. In addition to item growth, these three models assume, respectively, that a certain fraction of the items is of imperfect quality due to errors in one of the processing stages; the available growing and storage facilities have limited capacities; and the vendor of the items offers incremental quantity discounts. These models are aimed at answering two of the most important questions facing inventory managers, namely "how much to order?" and "when to place order?". A third question, which is specific to growing items, arises, namely "when should the items be slaughtered?".

In the imperfect quality model, it is assumed that the poor quality items are also sold, but at a discounted price. Furthermore, there is a screening process, conducted on all the items before they are sold, to separate the poor quality items from those of good quality. For the limited capacity model, it is assumed that if the order quantity exceeds the available capacity, additional growing and storage capacities are rented from an external service provider, but this comes at a cost as the rented warehouse has higher

holding costs. The final model assumes that the supplier of the newborn items offers the purchasing company incremental quantity discounts.

For all three model presented in this study, the proposed inventory systems are given vivid descriptions which are used to formulate corresponding mathematical models. Solution procedures for solving the proposed mathematical models are also presented. Numerical examples are provided to demonstrate the solution procedures and to conduct sensitivity analyses on the major input parameters.

The presence of poor quality items means that more items need to be ordered in order to meet a specific demand for good quality items. The effect worsens as the fraction of imperfect quality items increases. Having capacity constraints on the growing and storage facilities increases total costs mainly because of the higher holding costs in the rented facility. As the capacity increases, the total costs decrease, but increasing capacity is capital intensive and poses financial risks if market conditions change for the worst. Quantity discounts were shown to reduce the purchasing cost of the newborn items, however ordering very large quantities has downsides as well. The biggest downsides are the risk of running out of storage capacity, the increased holding costs and item deterioration since larger order quantities result in increased cycle times. Through sensitivity analyses conducted for all three models, the target slaughter weight was shown to have the greatest effect on the EOQ than any other input parameter.

The inventory models presented in this study can be used by procurement and inventory managers, working in industries which stock growing items, as a guideline when making purchasing decisions. This can result in sizable reductions in inventory-related costs. Seeing that growing items are an integral part of food supply chains, the resulting cost savings can be used to cushion consumers against rising food prices or from a financial stand point, the savings can be used to boost profit margins.

Contents

List of Figures

List of Tables

List of Abbreviations and Acronyms

AM-GM	Arithmetic Mean-Geometric Mean
DEL	Dynamic Economic Lot
EOQ	Economic Order Quantity
EPQ	Economic Production Quantity
FIFO	First-In-First-Out
LIFO	Last-In-Last-Out
OM	Operations Management
OR	Operations Research
VAT	Value-Added Tax
VMI	Vendor-Managed Inventory
WIP	Work-In-Process
ZAR	South African Rand

Chapter 1

Introduction

1.1 Background

Inventories are an important part of organisations, regardless of whether they are for-profit or non-profit. Having the right levels and mix of inventories ensures that operations run smoothly and customer demands are satisfied. Furthermore, inventories represent a sizable portion of assets on the balance sheet. High inventory levels might reduce the possibility of stock-outs and contribute to higher customer satisfaction, but they increase operating costs. On the other hand, low inventory levels reduce the operating costs and increase the possibility of running out of stock. Inventory management is concerned with balancing these contradictory objectives (Stevenson, 2018). Inventory management affects multiple functions or departments (such as production, marketing and sales, finance and procurement) within the primary organisation and across the entire supply chain.

Inventory management is a well-studied area that has been researched for over a century. Since the publication of the seminal model by Harris (1913), various researches have modified and extended the model in order to represent different and more realistic inventory systems. An area that seems to be receiving attention lately is the modelling of growing items inventory. The first work dealing with inventory management for growing items was presented by Rezaei (2014) and there seems to still be opportunities for further research in this area. Growing items can be considered as a distinct class of items within inventory theory, similar to perishable, ameliorating or repairable items, to name a few. These items, unlike conventional items, experience an increase in weight during the inventory planning period as a result of being fed. These items are fed (or raised) for a certain period of time and after growing to a specific weight, they are slaughtered and then sold to market.

The aim of this study is to extend the basic inventory model for growing items by relaxing three implicit assumptions made in Rezaei (2014)'s model. The first assumption is that all the ordered inventory is of acceptable quality. This is not true for all situations because prior to selling the items, they need to be prepared (in the form of slaughtering, cutting and packaging, among other processes) for sale. Like most production systems, the preparation processes are subject to variation and human error and thus it is highly unlikely that all the items produced from the process will be of perfect quality. The second assumption is that all the ordered items are grown in one facility and stored in one facility regardless of the size of the order. This implies that the capacity of the growing and storage facilities are unlimited. In reality, these facilities, just like all assets, have limited capacities. The third assumption is that quantity discounts are not permitted. Sometimes suppliers offer customers discounts for purchasing larger quantities as a means of reducing their transportation costs (i.e. by incentivising customers to buy higher volumes of stock, the supplier has to make fewer shipments of larger order quantities) or to stimulate demand in response to difficult market conditions or competitor pricing.

1.2 Motivation and significance

The research presented in this dissertation is motivated by perceived lack of sufficient research in the area of inventory management for growing items. While growing items (which includes livestock and fish, among others) are only recently receiving attention in inventory management research, they are a very important part of daily life. Most growing inventory items serve as food downstream in most supply chains, and as such having a better understanding of this class of inventory items will enrich the knowledge currently available in this relatively new and important subarea of inventory theory.

Growing items are an important part of food supply chains. Typical food supply chains have producers, processors, distributors, retailers and consumers. The producers are typically farmers who are involved in the production of fruits, vegetables, grains or livestock, among others. Processors transform the items from the producers into various saleable food items which meet consumer needs. Logistics providers ensure that the saleable food items reach consumers, who typically purchase these food items through retailers, in an acceptable condition (Dani, 2015). The business of food producers is typically the production (in the form of rearing and eventually slaughtering) of growing items. Food supply chains are faced with a number of issues, three of which are the subject of this study. These are quality, outsourcing due to capacity limits and cost reduction through better procurement practices.

There a number of processes in food supply chains that can compromise the quality of the items. Growing items are often processed into various customer-preferred variants, for example chickens reach consumers in various forms such as fillets, burgers, sausages, etc., and these processing stages, like most production systems, seldom produce perfect quality items. This reason makes quality screening a vital part of inventory management in food supply chains. Slaughtering is an important aspect of a growing items inventory system and errors might occurs during this stage, resulting in improperly slaughtered pieces which might not be appreciated by aesthetically-conscious customers. For this reason, the imperfect quality items are removed from the lot and sold as a single batch at a reduced price to a bulk-purchasing and value-oriented customer. Imperfect quality items are not necessarily defective or unhealthy, they just don't meet all the aspects of a quality inspection process and hence, they are sold as a single batch to a different type of customer.

Capacity planning is another important decision which affects inventory management in supply chains. When facilities are set up, companies need to make decisions about the quantity of items their facilities can accommodate. Changing macro-economic conditions or the departure of a competitor, among other reasons, might increase customer demand and the result is that the capacity of the company-owned facilities can not keep up with demand. In this situation, it might be necessary for the company to outsource some of its activities, such as rearing and warehousing in the case of growing items, from an external service provider.

Suppliers often offer quantity discounts as a means of encouraging customers to purchase higher volumes of stock. In an effort to reduce costs, procurement managers often buy larger quantities of stock in order to take advantage of quantity discounts. However, the right quantity of stock needs to be ordered because if too much is ordered then the holding costs will increase and this might negate the effects of quantity discounts.

The role that growing items play in the food supply chain, which is mainly to serve as saleable food items, makes them important because food consumption is an essential part of daily life. Likewise, the management of inventoried growing items is significant. Recently, South Africa has experienced a number of events which have led to increases in the price of food. These events include increases in the Value-Added Tax (VAT) and the fuel levy as well as unfavourable weather conditions (BusinessTech, 2018), among others. South Africa, has high levels of poverty which can be attributed to, among other things, the high unemployment rate. Consequently, basic food items are unaffordable to most people. This makes effective inventory management in food supply chains even more important because through effective inventory management, food producers can achieve significant cost savings which they can use to either absorb some of the increases in the price of food (and thus making food more affordable to a larger customer base) or from a financial standpoint, to improve their profit margins.

1.3 Problem statement, objectives and research questions.

1.3.1 Problem statement

Certain inventory items are living organisms and hence they are capable of growing during the course of the replenishment cycle. These items are purchased soon after birth as as newborn items. The items are fed and this leads to growth. In the context of this dissertation, growth is quantified only through an increase in weight. Feeding continues until the items reach a certain target weight, after which they are slaughtered and consumed (i.e. put on sale). A number of issues might arise in such inventory systems, three of which are considered in this dissertation. Firstly, between the growth and consumption period, the quality of some of the items might deteriorate and at the time of consumption it is discovered that a certain fraction of the items is of unacceptable quality. Secondly, the space used for raising the live items and the space used for stocking the slaughtered items might have limited capacities. Lastly, the supplier of the newborn items might offer discounts for purchasing larger quantities.

1.3.2 Objectives

Based on the problem statement, the following objectives are identified:

- To develop an Economic Order Quantity (EOQ) model for an inventory system consisting of growing items when a certain fraction of the items is of poor quality.
- To develop an EOQ model for growing items reared (or grown) and stored in two facilities because the first facilities, which are company-owned, have limited capacities.

- To formulate an EOQ model for growing items when incremental discounts are granted by the supplier of the newborn items.

1.3.3 Research questions

The following research questions, which apply to all three models, are formulated:

How many newborn items should be ordered at the beginning of a growing cycle?

Given a target slaughter weight for the items, when (i.e. how long after receiving them as newborn items) should they be slaughtered?

How much time should elapse between successive order replenishments?

1.4 Scope of the study

The common themes among the three models presented in this study are item growth and deterministic demand. The first theme implies that the live inventory items purchased at the beginning of a replenishment cycle experience an increase in weight for a certain period of time during the course of the replenishment cycle. The second theme implies that customer demand rate for the slaughtered inventory items is assumed to be known with certainty.

With regards to the first theme, a number of additional assumptions are made. Firstly, a uniform growth rate is assumed for all the ordered items in each lot. This means that all the live items received grow at the same rate and consequently, they complete their growth period at the same time and they are slaughtered at the same time since they would have reached the target weight at the same time. Secondly, sickness and mortality as a result of sickness are not accounted for in the models presented. Nonetheless, these two factors are important and they present an opportunity for further development of the models presnted in this work.

1.5 Research methodology

Bertrand and Fransoo (2002) proposed various methodologies for conducting quantitative research in Operations Management (OM) and Operations Research (OR). The following methodology, which is adapted from Bertrand and Fransoo (2002), was used when conducting the research presented in this dissertation:

Theoretical model of the problem or process: Vivid descriptions of the proposed inventory systems are provided. The descriptions are based on real life situations that might arise in firms stocking growing items.

Scientific model of the problem or process: Mathematical models are formulated based on the worded problem definitions. Since the mathematical models are intended to mimic real life situations as closely are possible, it is necessary to make a number of assumptions which aid in bridging the gap between reality and the mathematical formulations of the problems.

Solution to the scientific model: Solution procedures for solving the proposed mathematical models are presented. The solution algorithms are then applied to numerical examples in order to illustrate their use in finding solutions to the proposed inventory systems.

Proof of the solution: It is then shown that unique solutions to the mathematical models representing the proposed inventory systems exist.

Observations on the theoretical model: Sensitivity analyses are conducted to determine the most significant inputs. The results are used to make recommendations on real life inventory systems for growing items.

1.6 Dissertation outline

The remainder of this dissertation is structured as follows:

Chapter 2 documents a review of literature related to the classic EOQ model as well as EOQ models for growing items, items with imperfect quality, items with two storage facilities and items purchased under incremental quantity discounts.

Chapter 3 addresses one the three main objectives of the work presented in this dissertation, which is to develop an EOQ model for growing items with imperfect quality. The proposed inventory system considers a situation where a company orders a certain number of items which are capable of growing during the course of the inventory replenishment cycle. Growth is facilitated by the company through feeding the items. These items are then slaughtered after growing to a certain weight. A certain fraction of the items is not of acceptable quality. Prior to being sold, the items are screened to separate the good quality items from the poor quality items. Good quality items are sold at a given price while poor quality items are salvaged as a single batch at the end of the screening period.

An EOQ model for growing items with two growing and storage facilities is presented in Chapter 4. The inventory system under study considers a company which purchases newborn items. The company's growing and storage facilities have limited capacities. If the order size exceeds the capacities of the company-owned facilities, the company leases a second growing facility and a second storage facility. These facilities can raise or store as many items as possible, but they charge higher holding costs. For this reason, the items in the rented facility are consumed first.

In Chapter 5, incremental quantity discounts are incorporated to the basic EOQ model for growing items. Incremental quantity discounts are one of two quantity discounts often offered by suppliers as a way of encouraging customers to purchase larger volumes of stock, the other being all-units quantity discounts. The difference between them is that incremental quantity discounts result in reduced purchasing cost for the entire order if larger volumes are ordered whereas if incremental quantity discounts are offered, the reduced purchasing cost only applies to items bought above a certain quantity. The inventory system studied in this chapter considers a company whose supplier offers discounts for purchasing larger quantities of newborn items which are capable of growing during the replenishment cycle.

Chapter 6 concludes the dissertation by providing a summary of the findings, contributions, limitations and possible areas for future research in the area of inventory management for growing items.

Chapter 2

Literature Review

2.1 Introduction

Any resource stocked or stored by an entity, with the intention of using the resource, is termed inventory (Jacobs and Chase, 2018). Entities hold inventories for a variety of reasons, Simchi-Levi et al. (2008) stated four major reasons. The first reason is to hedge against unanticipated changes in customer demands. Secondly, inventory acts as a buffer which cushions the entity against the effects of various forms of uncertainty. Another reason for keeping inventory is to take advantage of economies of scale brought by purchasing large quantities or incentives given by transportation companies. Lastly, holding inventory safeguards against the possibility of stock-outs during supply lead times. While maintaining high inventory levels has business benefits, such as high customer service levels and smooth running operations, there are potential disadvantages. The biggest of which is the excessive costs associated with holding inventory (Mentzer et al., 2007; Simchi-Levi et al., 2008; Jaber et al., 2008; Muckstadt and Sapra, 2010; Stevenson, 2018). These include all costs incurred as a result of storage, handling, taxes, insurance, depreciation, deterioration and obsolescence, among others, and most importantly interest or the opportunity cost of capital had the money been invested elsewhere (Jacobs and Chase, 2018).

Inventory can be categorised into one of five groups depending on the reason for keeping it (Muckstadt and Sapra, 2010), namely cycle, safety, pipeline, decoupling and anticipation inventories. Cycle stock is held for the purpose of satisfying customer demand between replenishment periods. Companies hold safety stocks as a buffer to absorb the effects of uncertainties in lead time, supply and demand. Items in transit, or in the case of manufacturing systems Work-In-Process (WIP) inventories, are classified as pipeline inventories. Decoupling stock is a specific type of safety stock in manufacturing systems which ensures that manufacturing operations run smoothly in the presence of variation in setup times or breakdowns. Anticipation inventories are held by companies in anticipation of certain events which might occur in the future such as price increases or building up inventory prior to a selling period in the case of seasonal products or new product introductions.

Successful management of inventory not only ensures that operations run smoothly, but it also improves cash flow and profitability in the case of commercial entities. For these reasons, inventory management is one of the most important activities in production and operations management (Stevenson, 2018). Poor inventory management results in over- or under-stocking of resources, both of which are undesirable and can have major cost repercussions across the entire supply chain. On the other hand, good inventory management not only reduces the possibility of stock-outs or overstocking but it also results in improved customer satisfaction while keeping inventory costs as low as possible. The two major decisions in inventory management are the order quantity and the cycle time, i.e. how much to order and when to place an order (Stevenson, 2018). These decisions are addressed through lot sizing models of which the EOQ is the classic model.

2.2 The classic EOQ model and major extensions

The EOQ model works out the optimal order size which minimises certain inventory related costs. These costs vary with the order size and the frequency of ordering. In the EOQ's most basic form, these costs are those associated with placing an order and keeping the order in storage. While the model was made famous by Wilson (1934), its foundations lie in Harris (1913)'s work. Harris (1913) proposed a formulation for determining the order quantity while balancing the purchasing cost, setup cost and inventory holding cost. The resulting formulation, which would later morph into what is now referred to as the classic EOQ model, makes a number of unrealistic assumptions. These assumptions limit the model's application to most real-life inventory systems. In attempts to make the model more realistic, various researchers have made extensions to the model by considering new assumptions, in addition to relaxing some of the assumptions made when the original model was formulated (Andriolo et al., 2014).

2.2.1 The classic EOQ model

The classic EOQ model is the simplest inventory control model. It is used to determine a fixed order order quantity which minimises the sum of the inventory holding costs and the ordering costs. The purchasing cost can also be included, but it is often excluded because it has no effect on the optimal order quantity except when quantity discounts are taken into account. In essence, the EOQ model finds a balance between the holding and ordering costs because as the order quantity increases the holding costs increase and the ordering costs decrease. The reverse is also true. These trade-offs are shown in Figures 2.1 and 2.2 which demonstrate how the total costs change with order quantity and the changes to the inventory level with time respectively.

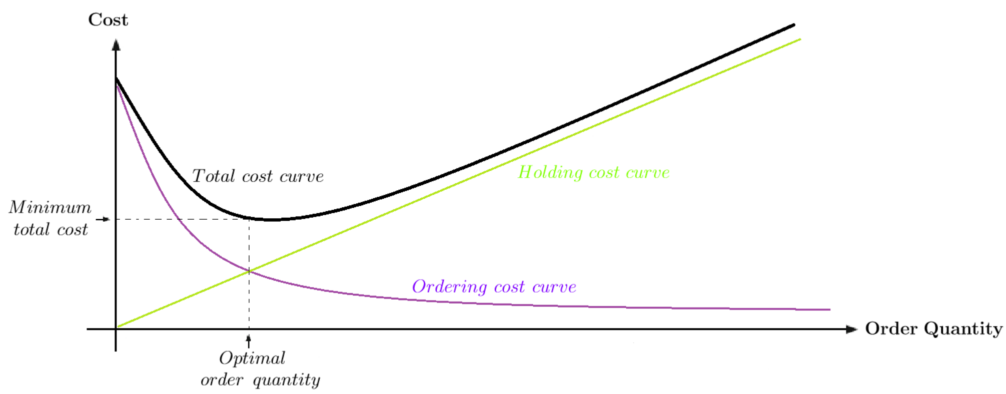
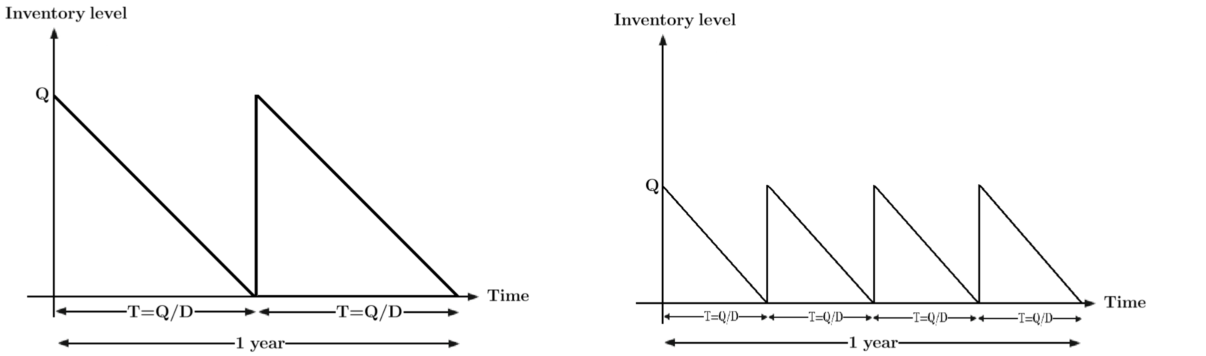


Figure 2.1: Holding cost, ordering cost and total cost as functions of order quantity



(a) Fewer large orders result in higher inventory holding costs and lower setup costs

(b) Numerous small orders result in lower inventory holding costs and higher setup costs

Figure 2.2: Typical inventory system behaviour for the classic EOQ model

The inventory system depicted in Figure 2.2 considers only one type of item and it is assumed that there is no lead time. An order for Q items is received, in one shipment, at the beginning of each inventory planning cycle. Each time an order for Q items is placed, a ordering cost of K is incurred. The items are consumed at a constant annual rate D until they are used up at the end of period T. A new order for Q items is received at the instant the previous order finishes up. The items are kept in stock at an annual holding cost of h per item. Additional assumptions made when developing the model are that quantity discounts and shortages are not permitted. The total cost per unit time, TCU, is given by

$$TCU = h\left(\frac{Q}{2}\right) + K\left(\frac{D}{Q}\right). \tag{2.1}$$

The value of Q which minimises Equation (2.1), denoted by Q^* and referred to as the EOQ, is determined using differential calculus as

$$Q^* = \sqrt{\frac{2KD}{h}}. \tag{2.2}$$

2.2.2 Significant extensions made to the classic EOQ model

Following the publication of Harris (1913)'s paper, a multitude of inventory models have been developed based on that particular model (Andriolo et al., 2014). Some of the most significant extensions made to Harris' model are depicted in Figure 2.3, which also shows how research in the area of inventory theory has evolved since the publication of Harris' pioneering work over a century ago.

The first major extension to the classic EOQ model, courtesy of Taft (1918), came in the form of what is now widely known as the Economic Production Quantity (EPQ) model. In the EPQ model, two major events take place simultaneously, namely continuous consumption and periodic production. The consumption rate of items is assumed to be less than the capacity to produce the items, i.e. the production rate exceeds the demand rate.

Inventory models presented inn the first half of the 20th century made the assumption that the demand rate is static. Wagner and Whitin (1958) relaxed this assumption through the development of the Dynamic Economic Lot (DEL) model.The difference between the DEL model and earlier lot sizing models is that the demand rate is assumed to be changing over a set number of periods.

As a way of persuading customers to order larger quantities, a supplier might offer price reductions for large orders (Stevenson, 2018). Quantity discounts were taken into consideration when modelling inventory systems for the first time by Hadley and Whitin (1963). Two types of quantity discounts were modelled, namely all unit and incremental quantity discounts. The models considered different price breaks which are a result of step changes in the prices of predetermined order quantities.

Subsequently, Hadley and Whitin (1963) studied inventory systems where shortages are permitted. When shortages are taken into account in inventory theory, they are either fully or partially backordered. Full backordering of shortages means that all customers are willing to wait until the next shipment arrives. When shortages are partially backordered, some customers are willing to wait while others are not, resulting in lost sales.

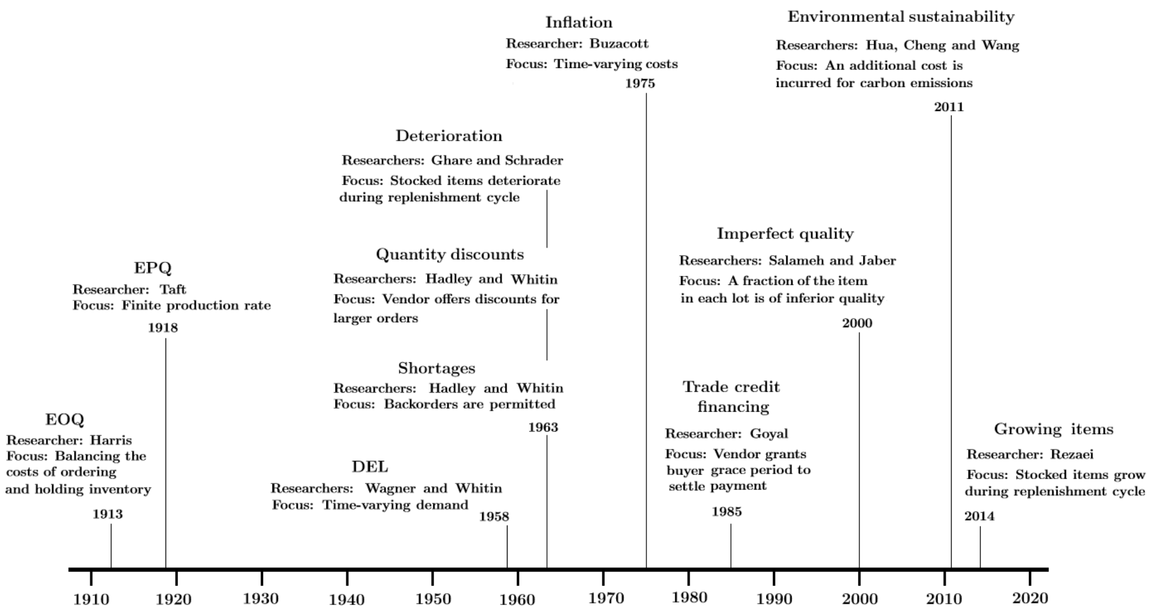


Figure 2.3: Timeline showing some of the major developments in inventory theory

Harris (1913)' model assumes that items can be stored indefinitely without a change to the item's integrity, utility or value. However, this is not true for certain items like fruits, vegetables, dairy products and eggs, to name a few. Such items are called perishable or deteriorating items. Deterioration is an umbrella term which emcompasses any form of damage, pilferage, evaporation, loss of usefulness or spoilage of the item (Palanivel and Uthayakumar, 2016). Ghare and Schrader (1963) presented an inventory model for items which deteriorate while in stock. The items' deterioration rate was modelled by a decaying exponential function.

An implicit assumption of most inventory models is that costs remain constant over time. However, this is not true in practice as inflation and the time value of money influence costs. Buzacott (1975) investigated the impact that cost increases, brought by inflation, had on the order quantity by taking into account a rate of inflation on the various cost components. Gurnani (1983) studied, under discounted cash flow analysis, a few inventory models in order to determine the effect that the time value of money had on the EOQ and total costs.

Goyal (1985) presented an EOQ model in which delays in payments (from the buyer to the vendor) are allowed. The motivation behind the model was that in most situations ordered items are not paid for at the instant the vendor delivers them to the customer. The delay in payment can be because the buyer wants to inspect the order after receipt to ensure that it is in the agreed upon state, or it can be due to the supplier giving buyers a grace period to settle their bills.

Salameh and Jaber (2000) extended the basic EOQ model by relaxing the assumption that all items are of good quality. They studied an inventory system where a certain fraction of items is assumed to be of imperfect quality. Both perfect and imperfect quality items are then sold, but the imperfect quality items are sold at a discount.

Stringent environmental policies legislated in some jurisdictions around the world as a means of reducing carbon emissions, such as the cap and trade emissions trading scheme in the European Union, motivated Hua et al. (2011) to incorporate carbon emissions to the classic EOQ model. They assumed that a company incurs a cost associated with its emissions, released as a result of its production and logistics activities, based on the cap and trade emissions trading scheme.

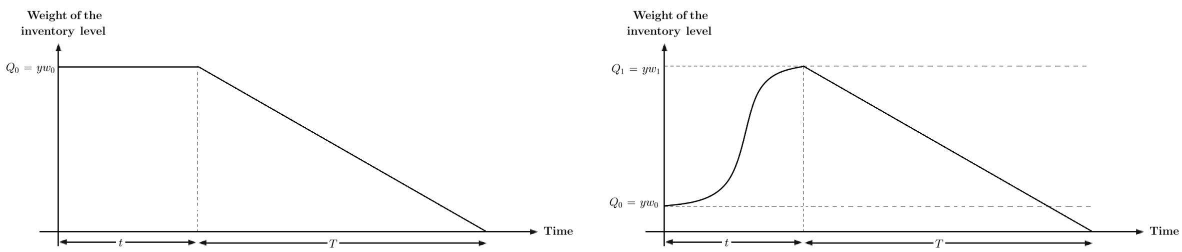
Rezaei (2014) used the growth rate and feeding functions of broiler chickens to develop an EOQ model for a new class of inventory items called growing items. These are items which grow, through gaining weight, during the course of a replenishment cycle.

In order to model specific and realistic inventory systems, some of the features of classic EOQ model and its extensions are combined, either together or with new assumptions. This study is concerned with the management of growing inventory items under a set of realistic conditions, namely, imperfect quality, limited capacity and quantity discounts, which might arise in food supply chains.

2.3 Growing inventory

2.3.1 EOQ model for growing items

The classic EOQ model makes the assumption that the weight of the inventory remains constant throughout the replenishment cycle. This is not true for items like livestock which grow, and thus their weight varies over the course of an inventory planning cycle. Item growth in inventory theory was first addressed by Rezaei (2014). Growing items are defined as items whose weight increases over time. This does not happen with conventional items. Figures 2.4a and 2.4b depict typical inventory system behaviour for conventional items and growing items respectively.



(a) Conventional items *(b) Growing items*

Figure 2.4: Behaviour of typical inventory systems for conventional and growing items

The weight of conventional inventory items, such as books or car parts, remains constant over time if they are not consumed (i.e. period t in Figure 2.4a). A change to the weight of the inventory level is only brought by consumption (i.e. period T in Figure 2.4a). However, the weight of growing items, such as poultry or livestock, increases over time as they are fed (i.e. period t in Figure 2.4b) until the point of slaughter. The weight of the items then decreases continuously as a result of consumption as indicated by period T. Although the model developed by Rezaei (2014) is generalised, it depends on the growth rate of the items under study because different growing items have different growth rates. Rezaei considered an inventory situation in which newborn animals, of a certain initial weight, are purchased. The animals are then fed (or raised) until they reach a specific weight. Following this, the animals are slaughtered and sold to the market. Rezaei developed a model that determined the optimal quantity of newborn animals to order and the optimal day to slaughter them following the growth period. The model could not be solved in closed form. The optimal slaughter age was determined using a bisection method.

The series of events taking place in the inventory system studied by Rezaei (2014) can be summarized as follows:

- A company orders a lot size of y newborn animals at a cost of p each at the beginning of each cycle. At the time of receiving the order, the newborn animals each weigh w_0.
- It costs the company K to setup for a new growth cycle. This is a fixed cost and it is incurred at the beginning of each cycle.
- In order for the animals to grow, they are fed at a unit cost c. However, the animals consume different quantities of feed stock based on their age. The relationship between the animals' age and the quantity of feed stock they consume is given by the feeding function $f(t)$.
- After growing to a target weight w_t, the animals are slaughtered and kept in storage. The company pays a annual holding cost of h per weight unit.
- The slaughtered animals are sold at a price of s per weight unit. The annual demand for the slaughtered animals is a deterministic constant D.

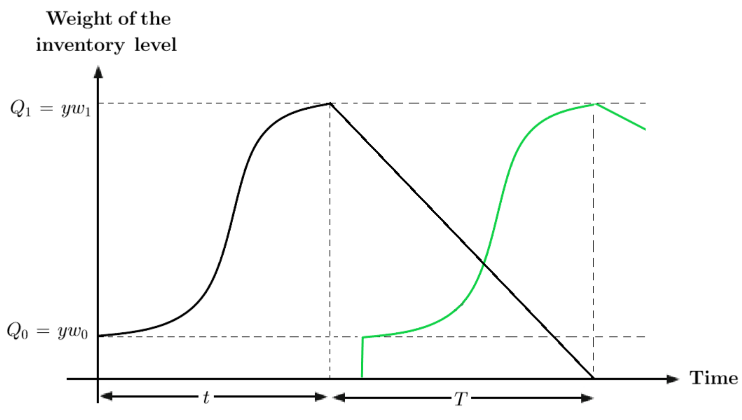


Figure 2.5: Inventory system behaviour for growing items

The behaviour of the inventory system over time is depicted in Figure 2.5, where t is the slaughter age and T is the duration of consumption period or the cycle time. The total profit per cycle is defined as the difference between the revenue from the sales of the slaughtered animals minus the total costs. The total costs is the sum of the purchase, setup, production (i.e. feeding) and holding costs. The total profit per cycle, denoted by TP, is given by:

$$TP = syw_t - pyw_0 - cy\int_o^t f(t)dt - hT\left(\frac{yw_t}{2}\right) - K \tag{2.3}$$

An expression for the total profit per unit time, TPU, is determined by dividing TP by the length of the sales period, T, which equals yw_t/D. In order to make the model specific, an assumption was made that the animals in the inventory system under study are broiler chickens. Hence, the feeding function, $f(t)$, and the growth function, $f(w_t|w_0)$, were replaced by functions developed for broiler chickens and proposed by (Goliomytis et al., 2003) and Richards (1959) respectively. The feeding and the growth functions are given by the following expressions respectively:

$$f(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 \tag{2.4}$$

$$w_t = A\left(1 + le^{-gt}\right)^{-1/n} \tag{2.5}$$

The order quantity is determined by equating the derivative of the expression for the total profit per unit time, TPU, with respect to y to zero and the result is

$$y = \sqrt{\frac{2KD}{h\left[A^2\left(1 + le^{-gt}\right)^{-2/n}\right]}} \tag{2.6}$$

The model's decision variables, namely the optimal order quantity (y^*) and optimal slaughter age (t^*), could not be determined in closed form. Rezaei (2014) used a bisection method, in particular the Newton-Raphson method, to work out the optimal value of t^*. This value was then used to determine the EOQ from Equation (2.6).

2.3.2 Extensions made to the EOQ model for growing items

Zhang et al. (2016) formulated an inventory model for growing items in a carbon-constrained environment. Their model used the same basic assumptions, including the growth and feeding functions, as Rezaei (2014)'s model. They extended Rezaei's model by assuming that the company under study operates in a country where carbon taxes are legislated. The carbon tax is based on the amount of emissions released into the atmosphere as a result of the company's inventory holding, ordering and transportation activities. As was the case with Rezaei's model, the total profit was the objective function and the decision variables were the optimal order quantity and optimal slaughter age. The optimal solution to the problem was determined through an algorithm.

Nobil et al. (2018) developed an EOQ model for growing items with shortages. The model presented by Nobil et al. (2018) differed from Rezaei (2014)'s model in two ways. Firstly, in the former model shortages are allowed and fully backordered and secondly, the growth function of the items was approximated by a linear function in former model as opposed to using Richards (1959)'s growth curve as was the case in the latter model. Earlier inventory models for growing items, namely those by Rezaei (2014) and Zhang et al. (2016), utilised the feeding function proposed by Goliomytis et al. (2003), which increased the computational complexity when determining optimal solutions. By assuming a linear growth function, Nobil et al. (2018) drastically reduced the complexity of computing the feeding cost. Consequently, the optimal solution to the problem was determined using a relatively simple heuristic.

2.4 Imperfect quality inventory

2.4.1 EOQ model for items with imperfect quality

Item quality was incorporated to inventory management research by Salameh and Jaber (2000). They relaxed the assumption that all the items received in each order are of good quality. They proposed an inventory situation in which a certain fraction of the ordered items is of poor quality. Before the items are sold, they are all subjected to a screening process so as to separate the items of good quality from those of poor quality. Both good and poor items are then sold. While good quality items are sold continuously during the inventory replenishment cycle, poor quality items are salvaged (i.e. sold at a price lower than that charged for good quality items) as a single batch at the end of the screening process. The model's optimal solution was determined in closed form.

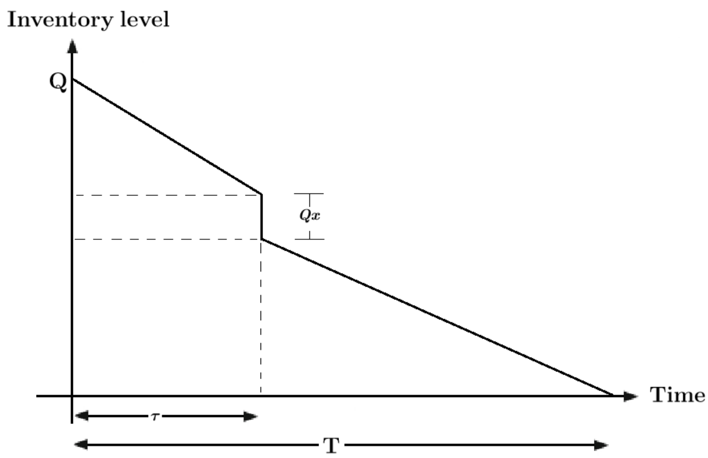


Figure 2.6: Inventory system behaviour for items with imperfect quality

The series of events taking place in inventory system for items with imperfect quality, as proposed by Salameh and Jaber (2000) and depicted in Figure 2.6, can be summarised as follows:

- A lot size of Q items is ordered by a company at cost of p per item.
- Each time the company places an order, an ordering cost of K is charged.
- Not all of the items received in each lot are of good quality. A fraction of the items, x, is of poor quality. The fraction of poor quality items, x, is random and has a known probability density function $g(x)$.
- Before selling the items, they are subjected to a 100% screening process, which separates the good quality items from those of poor quality. Unit screening cost

is z and the items are screened at a rate r per unit time. The screening process occurs for the duration τ.

- The company keeps all the items in stock. The cost associated with keeping one unit in stock for a year is h.
- Good quality items are sold at a price of s per unit and are demanded at an annual rate of D. Poor quality items are sold as a single batch at a price v per unit, which is lower than the price charged for good quality items, at the end of the screening period.

The total profit per cycle is given by

$$TP = sy(1-x) + vyx - py - K - zy - h\Big(\frac{y(1-x)^2}{2D} + \frac{xy^2}{r}\Big). \tag{2.7}$$

Since x is a random variable with a known probability density function, the expected value of the total profit per unit time, $E[TPU]$, is computed by dividing Equation (2.7) by the expected value of the cycle time. This approach was suggested by Maddah and Jaber (2008) when they presented a corrected version of Salameh and Jaber (2000)'s model. The EOQ is determined, through the use of differential calculus, as

$$Q^* = \sqrt{\frac{2KD}{h\Big[E\big[(1-x)^2\big] + 2E[x]D/r\Big]}}. \tag{2.8}$$

2.4.2 Extensions made to the EOQ model for items with imperfect quality

Goyal and Cardenas-Barron (2002) proposed a simpler approach for arriving at the optimal solution to Salameh and Jaber (2000). They achieved this by changing the expression for the expected value of the total profit per unit time. They determined the expected values of the revenue and total cost separately, unlike Salameh and Jaber who first defined an expression for the total profit before finding an expression for the expected value of that quantity. The result was that Goyal and Cardenas-Barron's approach required less computational effort compared to Salameh and Jaber's approach. Like Salameh and Jaber, they found a solution to the problem in closed form. The difference between their model and Salameh and Jaber's model was negligible (i.e. in the region of 0.0002%) (Goyal and Cardenas-Barron, 2002). While Goyal and Cardenas-Barron made no new major contributions to the literature on inventory models for imperfect quality items, they found a simpler method which yielded almost the same results.

Huang (2002) extended Salameh and Jaber (2000)'s model by considering a vendor-buyer cooperative supply chain relationship. Instead of only optimising the cost on the buyer's side, Huang optimised the total costs incurred by both the buyer and the vendor. In the model, the vendor supplies items to the buyer who screens them before putting them up for sale. The buyer only sells good quality items and the vendor incurs a warranty cost for the fraction of poor quality items they supply to the buyer.

Chan et al. (2003) developed an inventory model for a situation where there might be lower pricing, rework and reject of some of the items. The biggest difference between their work and Salameh and Jaber (2000)'s work is that in their model the screening process separates the items into three distinct groups as opposed to two in the latter. These groups are good quality items (which are sold at the regular price), poor quality items (which are either sold at a price lower than the regular price or reworked) and defective items (which are rejected and not sold).

Chang (2004) applied fuzzy sets theory to the basic EOQ model with imperfect quality items. The objective was to determine the optimal order quantity which maximises total profit provided that some of the inputs to the model exhibited fuzzy behaviour. In applying fuzzy theory to the model, Chang considered two cases. In the first case, the fraction of imperfect quality items was considered to be a fuzzy variable. The second case assumed that both the fraction of imperfect quality items and the demand rate rate were fuzzy variables.

Yu et al. (2005) extended Salameh and Jaber (2000)'s work along two dimensions, namely the incorporation of item deterioration and partial backordering. They assumed that the items deteriorate while in stock and that shortages are partially backordered for customers who are willing to wait. A lost sales charge was incurred for customers who are not willing to wait for the stock on backorder.

Wee et al. (2007) relaxed the assumption in Salameh and Jaber (2000)'s model that shortages are not allowed. They developed an inventory model, for items with a fraction of imperfect quality items, where shortages are allowed and fully backordered. This meant that the company did not incur a lost sales cost because all the customers are willing to wait for the backordered stock (i.e. the company only incurred a backorder cost).

Jaber et al. (2008) extended the imperfect quality EOQ model by taking into account learning effects. The only difference between their model and Salameh and Jaber (2000)'s was that they assumed that the fraction of imperfect quality items decreases according to a learning curve. The logic behind the assumption is that in most repetitive operations the cost to produce a single item decreases by a certain fixed percentage as the quantity of items produced increases two fold (Jaber, 2006). They assumed that the learning effects could be modelled by the S-shaped logistic learning curve.

Salameh and Jaber (2000)'s model is based on the classic EOQ model and thus, it assumes that all inventory items are kept in a single warehouse with unlimited capacity. Chung et al. (2009) relaxed this assumption by considered a situation where the items are kept in two warehouses. The first warehouse is owned by the company and has limited capacity. The second warehouse is rented and is assumed to have unlimited capacity. Keeping one item in stock in the second warehouse costs more than keeping the same item in the first warehouse, and for this reason the items in the second warehouse are sold first. The items are screened in both warehouses, to separate the good and poor quality items, before being put up for sale.

Learning effects were the subject of another extension, this time by Khan et al. (2010). The model further assumed that a lost sales cost is incurred as a result of learning effects and shortages are fully backordered. They developed an EOQ model for three different learning scenarios. The scenarios studied were full, partial and no transfer of learning, which correspond to situations where the inspector has full, some and no screening experience respectively.

Chang and Ho (2010) presented a new version of the EOQ model for imperfect quality items with shortage. While this was not the first model to consider shortages, the major difference between this model and earlier inventory models for imperfect quality items with shortages is that Chang and Ho did not use differential calculus to solve for the optimal order and backorder quantities. They solved their model algebraically using the

Arithmetic Mean-Geometric Mean inequality (AM-GM) theorem.

Chen and Kang (2010) studied a vendor-buyer inventory system for items with imperfect quality. In addition, they assumed that the vendor grants the buyer trade credit financing by allowing the buyer to receive stock and only pay for it at a later stage. The buyer incurs an interest charge for this type of transaction, which is paid to the vendor. The objective function in their model was the total system costs for both the buyer and the vendor.

Lin (2010) developed an inventory model for items with imperfect quality and an influential buyer. It was assumed that the buyer had negotiating and buying powers with the supplier. As a result, the buyer was often offered discounts. These discounts were only offered to the influential buyer and not all of the supplier's customers. The discounts offered to the buyer were structured based on the quantity ordered.

Maddah et al. (2010) introduced random supply to the EOQ model for items with imperfect quality. In their model, they assumed that the supplier's production process follows a two-state Markov process. Furthermore, their model studied two scenarios relating to the shipping of the imperfect quality items. In the first scenario, imperfect quality items are removed from the stock and no additional cost is incurred. In the second scenario, they are consolidated into a single batch and sold at a lower price after a specific amount of time.

In inventory theory, shortages are sometimes allowed and in such cases they are either fully or partially backordered. However, the occurrence of shortages might negatively affect customer service levels. In order to counter this and maintain service levels, Maddah et al. (2010) suggested overlapping orders in an inventory system with imperfect quality items. This is achieved by meeting demand during potential shortages and screening time with items ordered in the previous cycle. The previous order was assumed to be larger than a regular order, which is an order placed without overlapping. The model resulted in shortages being completely avoided and customer services levels were maintained. However, this model resulted in higher overall costs than an equivalent model without order overlapping.

Hu et al. (2010) extended the EOQ model for items with imperfect quality along a few dimensions, the most significant being the introduction of fuzzy variables. In addition to assuming that the demand rate and the fraction of imperfect quality items were fuzzy, they assumed that shortages were allowed and fully backordered. Furthermore, customer service levels were incorporated into the model. Service levels were linked to the percentage of customer demand being met by backordered items.

Wahab et al. (2011) extended the theory behind the Salameh and Jaber (2000)'s model to a vendor-buyer supply chain model. While such models had been presented before, this was different because it was studied under three realistic situations. In the fist situation, both the buyer and the vendor are in the same country, this was an implicit assumption of all other vendor-buyer inventory models for items with imperfect quality developed thus far. In the second situation, the vendor and the buyer are in different countries. For this case, they assumed that the exchange rate between the two countries was stochastic. In the third situation, the model was studied under the assumption that the vendor and the buyer are in different countries and carbon emission costs are charged for the production and logistics activities that occur when fulfilling orders.

An implicit assumption of Salameh and Jaber (2000)'s model as well as its multiple extensions is that the screening process is perfect. This implies that the screening process is capable of separating all poor quality items from good quality items. Kahn et al. (2011)

relaxed this assumption by developed an inventory model which considered errors in the screening process. The probability of the inspector committing an error (i.e. classifying a poor quality item as a one of good quality and vice versa) was assumed to be known.

In the EOQ model for imperfect quality items, it is assumed that the screening of the items occurs at the retailer despite the fact that the it is the vendor who is supplying the imperfect quality items. Rezaei and Salimi (2012) presented an inventory model where the responsibility for screening switches from the retailer to the vendor.

Yassine et al. (2012) presented an EPQ version of Salameh and Jaber (2000)'s model which considered two scenarios relating to the shipment of poor quality items, namely aggregation and disaggregation. In the aggregation scenario, poor quality items are consolidated over a number of production runs and shipped as a single batch. In the disaggregation scenario, poor quality items are assumed to be sold during each production cycle.

Pricing and marketing plans are rarely taken into account when developing inventory models, despite the fact that they have a significant impact on the demand rate (Lee and Kim, 1993). This motivated Sadjadi et al. (2012) to develop an EPQ model for items with imperfect quality which accounted for the effects of a company's marketing plans. The model further assumed that a number of variables were capacitated, namely the budget allocated for marketing, the maintenance costs for each production run, the warehouse capacity for storing the items and the machine hours available for each production run.

Yadav et al. (2012) developed an inventory model for items with imperfect quality with fuzzy demand and full backordering of shortages. Furthermore, the demand rate was assumed to be dependent on the amount of money spent on advertising and the screening process was assumed to follow a learning curve.

Some retailers allow customers to return products if they are not completely satisfied with them. Naturally, the presence of imperfect quality items in a retailer's order would result in a higher probability of customers returning the items. Hsu and Hsu (2013) proposed an inventory model for items with imperfect quality items and sales returns. In their model, they further assumed that the screening process had errors, which also increased the probability of customers returning the items. A sales return resulted in the retailer incurring additional charges. These charges include the cost of the item, the charges incurred when refunding the customer and the costs associated with the reverse logistics of the item.

Khan et al. (2016) incorporated one of the most recent trends in supply chain management to the inventory model for items with imperfect quality, namely Vendor-Managed Inventory (VMI). Their model considered a vendor-buyer inventory system, where the vendor supplies the buyer with items which are not all of good quality. In a VMI agreement, the vendor owns the stock which is kept at the buyer's warehouse or store. Management of the stock is the responsibility of the buyer. The model was motivated by the growing number of manufacturers and retailers who have VMI agreements in place.

De et al. (2018) presented an EPQ model for items with imperfect quality, where some of the imperfect quality items are reworked and the rest are sold at a discounted price. Furthermore, environmental regulations were also factored into the model by making the assumption that a carbon tax is charged if the manufacturer's production processes produce a specified quantity of carbon emissions.

Carbon emissions were the focus of another recent paper, by Tiwari et al. (2018), on inventory models for items with imperfect quality. They studied a vendor-buyer inventory system for deteriorating items with imperfect quality. The objective of the model was to minimize the total costs incurred by both the buyer and the vendor. Carbon emissions costs, which resulted from the production, logistics and warehousing activities, were also included in the total costs function.

2.5 Inventory with two levels of storage

2.5.1 EOQ model for items with two levels of storage

Harris (1913)'s model makes the assumption that all the ordered items are stored at the company's own storage facility, regardless of the order size. Simply put, it is assumed that the company has unlimited storage space. This is not true in most real life situations because storage space, like any asset capacity, has limits. Hartley (1976) proposed an inventory system where a company's warehouse has limited storage capacity. If the company orders more units than can be kept at its warehouse, it rents a second warehouse from another party. The cost of holding inventory in the rented warehouse is assumed to be more than the cost of holding the items in the company's own warehouse. Inventory in the rented warehouse is cleared before inventory in the company's own warehouse because of the higher holding costs in the rented warehouse. The optimal order quantity was determined in closed form.

The series of events taking place in inventory system for items with two levels of storage, as proposed by Hartley (1976) and depicted in Figure 2.7, can be summarised as follows:

- A company places an order for Q items.
- An ordering cost of K is incurred each time the company places an order.
- The company's own warehouse has a capacity of storing M items.
- The company rents a warehouse to keep the excess, $(Q - M)$, items.
- The rented warehouse charges the company h_r annually for keeping one unit in stock, which is greater than the annual unit holding cost in the company's own warehouse (h_m).
- The items in the rented warehouse are consumed before those in the company's own warehouse.
- Demand for the items is constant at D units annually.

The total costs incurred per unit time, TCU, is given by

$$TCU = \frac{KD}{Q} + h_r\left[\frac{(Q-M)^2}{2Q}\right] + h_m\left[\frac{M^2 + 2M(Q-M)}{2Q}\right] \tag{2.9}$$

The value of Q which minimises Equation (2.9) is the EOQ given by

$$Q^* = \sqrt{\frac{2KD + M^2(h_r - h_m)}{h_r}}. \tag{2.10}$$

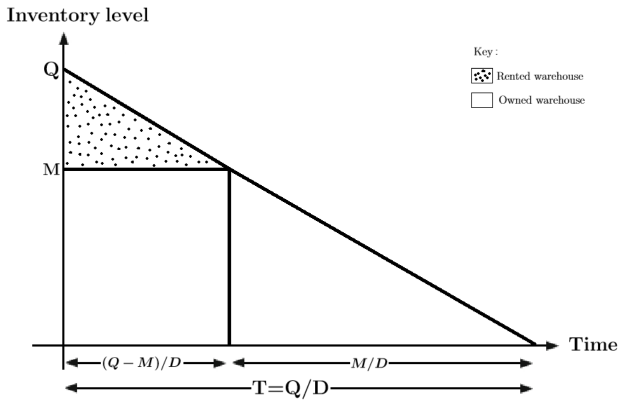


Figure 2.7: Inventory system behaviour for items with two levels of storage

2.5.2 Extensions made to the two-warehouse EOQ model

Sarma (1987) developed a two-warehouse inventory model for a deteriorating item. In the model, it was assumed that the retailer owns one warehouse with limited capacity and rents a second warehouse if the lot size exceeds the capacity of the owned warehouse. The item under study was assumed to deteriorate while being kept in storage. The annual unit holding costs in the rented warehouse were assumed to be higher than those in the owned warehouse because the rented warehouse offered better preservation facilities (i.e. the deterioration rates in the two warehouses were assumed to be different).

Pakkala and Achary (1991) relaxed three assumptions in Hartley (1976)'s model. These assumptions related to shortages, demand and deterioration. In their model, shortages were allowed, the items deteriorated with an exponentially distributed deterioration rate and demand was assumed to be probabilistic. Furthermore, they assumed different deterioration rates and holding costs in the two warehouses. An algorithm was used to determine the optimal order quantity.

Two-warehouse EOQ models developed up until 1992 considered the demand rate to either be deterministic or probabilistic. Goswami and Chaudhuri (1992) were the first researchers to have introduced the two-warehouse inventory model studied under the assumption that the demand rate has a linear positive trend.

Another extension which assumed a linear trend in demand was presented by Bhunia and Maiti (1998). The major difference between this work and Goswami and Chaudhuri (1992)'s is that Bhunia and Maiti further assumed that the items under study deteriorate during the replenishment cycle.

Ray et al. (1998) modified the basic two-warehouse inventory model by assuming that the demand rate is dependent on the inventory level and that a transportation cost is incurred for transporting the items from the rented warehouse to the owned warehouse.

Hariga (1998) presented a stochastic version of the two-warehouse inventory model. With exception to the demand, which was assumed to not be known with certainty, all of the basic two-warehouse EOQ model's assumptions (namely no shortages, deterioration or quantity discounts) applied to the model.

Kar et al. (2000) developed an EOQ model for an item with a positive linear trend in demand, two storage facilities, shortages and a fixed planning period. Furthermore, they assumed that consecutive cycle times were in arithmetic progression. Their model was intended to mimic demand for fresh produce during consecutive harvest cycles for small scale farmers in developing countries.

Zhou (2003) presented a version of Hartley (1976)'s model which considered multiple warehouses, instead of just two. In the model, Zhou assumed that a company owns a single warehouse, with fixed capacity, and it rents a certain number of warehouses which also have fixed capacities. It was also assumed that the demand rate is time dependent and shortages are allowed and backordered.

Yang (2004) incorporated deterioration, shortages and the effects of inflation into the two-warehouse inventory model. Two versions of the model were presented. The first version considered a case where shortages occur at the end of a replenishment cycle and in the second version shortages occur at the beginning of a replenishment cycle.

Zhou and Yang (2005) proposed a new EOQ model for items with two levels of storage and inventory level-dependent demand. Although this class of models had been developed before, this model was different because it assumed that the demand rate was a polynomial function of the current stock level and that the transfer of inventory from the rented warehouse to the owned warehouse occurred through a bulk release pattern. In addition, the costs associated with transferring stock from the rented warehouse to the owned warehouse were included in the model.

Mahapatra et al. (2005) applied the theory behind two-warehouse inventory models to the wholesaler-retailer problem. In their model, they assumed that there is one wholesaler and two retailers with access to two warehouses. The first warehouse had limited storage capacity and was owned by the retailers, and the second warehouse had unlimited capacity and was rented. Demand at both retailers was assumed to be dependent on the stock level.

Lee (2006) formulated an EOQ model with two storage facilities for deteriorating items which are dispatched through a FIFO (First-In-First-Out) policy. The motivation behind the model was that most inventory items, especially perishable or deteriorating items, are dispatched through FIFO policies. For comparison, the model was formulated for two different dispatching policies, namely FIFO and LIFO (Last-In-First-Out) so as to determine which of the two dispatching policies resulted in lower total costs.

Two-warehouse inventory models had been studied since 1976, nonetheless all of the models developed considered a single item. This changed when Maiti et al. (2006) developed a multi-item model for items stored in two warehouses. In addition, quantity discounts were incorporated into their model. The model was formulated as a mixed integer non-linear programming model.

Trade credit financing was introduced to the two-warehouse inventory model by Chung and Huang (2007). Not only did they make the assumption that the supplier permits a delay in payments from the retailer, they also assumed that the retailer grants trade credit finance to customers as a way of increasing demand. Interest charged by the supplier to the retailer was assumed to be different from the interest charged by the retailer to the customer, with the supplier's interest charge being higher.

Rong et al. (2008) introduced fuzzy lead time to the two-warehouse inventory model. In addition to fuzzy lead time, they also assumed that the ordering cost was a function of lead time. The model was studied under two shortages conditions, firstly they were fully backordered and in the second case they were partially backordered. The problem was formulated as a multi-objective non-linear integer programming model.

The price charged for an item is one of the most important factors affecting its demand rate. Nonetheless, it is rarely accounted for in inventory management. Jaggi and Verma (2008) studied a two-warehouse inventory system for an item with price-dependent demand. Their model further accounted for the transportation cost between the two warehouses and it was assumed that shortages are permitted and partially backordered.

Lee and Hsu (2009) developed an EPQ model for a deteriorating item with time-dependent demand. Unlike most inventory models which assume that the cycle times are equal, this model was formulated under the assumption that the production cycle times vary over a finite planning horizon.

Liao et al. (2012) presented an EOQ model for items with two storage facilities under conditions of permissible delay in payment. While this was not the first model to consider both trade credit financing and two storage facilities, the main difference between this model and earlier works is that this model assumed that trade credit financing is granted based on the order quantity. This means that a certain minimum number of unit need to be ordered in order to qualify for trade credit financing, otherwise payment is due immediately.

Sett et al. (2012) studied an inventory system for an item with a quadratically-increasing demand rate, time-dependent deterioration and two storage facilities. In addition, they assumed that the cycle times are not of equal duration.

Ghiami et al. (2013) formulated a two-echelon supply chain model which incorporated features of the two-warehouse EOQ model. The supply chain was assumed to have one wholesaler and one retailer who owns a warehouse with a limited capacity. If the order size exceeds the capacity, the retailer rents a second warehouse.

Panda et al. (2014) developed an inventory model for an item stored in two warehouses with shortages and fuzzy demands. The total demand was represented by triangular fuzzy numbers and the demand during lead time was assumed to be variable and fuzzy. Furthermore, a constraint on the available budget was included in the model.

Lin and Srivastava (2015) formulated a two-warehouse inventory model for item with a production process which, at times, goes into an out-of-control state and produces imperfect quality items. In order to get the process back into an in-control state, certain maintenance actions are taken for a certain amount of time at a specific cost. In addition, it was assumed that the purchasing cost of the items is based on the all-units quantity discount structure.

Tiwari et al. (2017) investigated the effects of inflation, deterioration, time value of money and inventory level-dependent demand on the two-warehouse inventory model. In addition, shortages were backordered at a variable rate which was assumed to be dependent on the waiting time between successive order replenishments.

Features of the two-warehouse inventory model were applied to a location routing problem for a disaster logistics network by Vahdhani et al. (2018). They formulated a multi-objective location routing problem for a humanitarian supply chain with three levels, namely multiple suppliers, distribution centres and disaster points. As is the case with the two-warehouse inventory model, the warehouses at the distribution centres and the disaster points had limited capacities.

Bhunia et al. (2018) applied particle swarm optimisation to a version of the two-warehouse inventory model which was formulated as a non-linear constrained optimisation problem. Their model further assumed that shortages and delays in payment are allowed and the items deteriorate while in stock. The level of shortage backordering was assumed to be dependent on the waiting time between the arrival of orders.

2.6 Inventory purchased under incremental quantity discounts

2.6.1 EOQ model with incremental quantity discounts

The classic EOQ model is formulated under the assumption that the cost of all the ordered items is the same despite the fact that vendors sometimes offer discounts for purchasing large quantities. This is usually done to reduce their unit production setup costs. Hadley and Whitin (1963) took this fact into account when they proposed the lot sizing formula under two different types of common quantity discounts, namely all-units and incremental quantity discounts. Under the all-units discounts cost structure, the unit purchasing cost decreases on all units purchased as the order quantity decreases. With incremental quantity discounts, the unit purchasing cost decreases incrementally on additional units purchased as the order quantity decreases (Muckstadt and Sapra, 2010). The EOQ model under incremental quantity discounts differs from the classic EOQ model with respect to the computation of the purchasing cost. The holding cost may also be different depending on whether it is assumed to be charged as a fixed percentage of the purchase cost or if it is constant. This study investigates incremental quantity discounts under the assumption that the holding cost is charged as a fixed percentage of the purchasing cost. The most critical aspect of the basic EOQ model with incremental quantity discounts is the computation of the average purchasing cost. This cost is determined by first defining $q_1 = 0, q_2, ..., q_j, q_{j+1}, ..., q_m$ as the order quantities at which the purchase cost per weight unit changes. In such an instance, there are m price breaks. When a supplier offers incremental quantity discounts, the purchasing cost per weight unit, p_j, is the same for all Q values in $[q_j, q_{j+1})$. The purchasing cost per weight unit decreases from one price break to the next (i.e. $p_1 > p_2 > ... > p_j > p_{j+1} > ... > p_m$). The average purchasing cost, determined by dividing the purchasing cost for Q units in each price break by Q, is

$$\frac{p(Q)}{Q} = \frac{R_j}{Q} + p_j - p_j\left(\frac{q_j}{Q}\right), \tag{2.11}$$

where $R_j = p1(q_2 - q_1) + p2(q_3 - q_2) + ... + p_{j-1}(q_j - q_{j-1})$ for $j \geq 2$ and $R_1 = 0$. The average total cost per unit time, ATCU, is thus

$$ATCU = \left[\frac{R_j}{Q} + p_j - p_j\left(\frac{q_j}{Q}\right)\right]D + \frac{KD}{Q} + \frac{i\left[R_j + p_j(Q - q_j)\right]}{2}, \tag{2.12}$$

where i represents the holding cost rate per unit time. Through rearrangement of the terms and the use of differential calculus, the order quantity is

$$Q_j^* = \sqrt{\frac{2\left(R_j - p_j q_j + K\right)D}{ip_j}}. \tag{2.13}$$

The EOQ is solved through a heuristic which calculates the order quantity for each price break and then checks if the calculated order quantity falls within the range of that particular price break. If an order quantity does not meet the criteria, it is infeasible. A corresponding average total cost is calculated for the feasible order quantities. The EOQ is selected based on the feasible order quantity which has the lowest average total cost per unit time.

2.6.2 Extensions made to the EOQ model with incremental quantity discounts

Lal and Staelin (1984) studied an integrated vendor-buyer inventory system taking into account incremental quantity discounts and pricing policies. The aim of their model was to determine the buyer's order quantity and the optimal price the vendor should charge in order to minimise total system costs.

Most inventory model which consider incremental quantity discounts assume that demand is deterministic. This is seldom true in most real life inventory systems. This prompted Abad (1988) to develop inventory models with incremental quantity discounts under two non-constant demand patterns, namely constant-price elasticity and linear demand functions.

Guder et al. (1994) relaxed two assumptions in the basic EOQ model with incremental quantity discounts to create a new model. As opposed to the basic model which considers only one type of item, their model considered multiple items. Secondly, they assumed that there is a capacity limit on the number of items that can be ordered.

When discounts are taken into account when modelling inventory systems, they are considered to be offered only on the basis of purchasing larger quantities of stock. This changed when Tersine et al. (1995) developed an inventory model which considered both quantity and freight volume discounts. They studied an inventory system which considers a company which is offered incremental and all-units discounts based on the quantity of stock ordered. Furthermore, the company's logistics provider offers freight discounts based on the amount of stock transported from the supplier to the company.

Rubin and Benton (2003) studied an integrated vendor-buyer inventory system with multiple items, incremental quantity discounts and full backordering of shortages. In addition, constraints on the available storage space and budget were also taken into account.

Burke at al. (2006) developed a heuristic for a situation where a company purchases the same item from multiple suppliers offering different types of quantity discounts, including incremental. Based on anticipated customer demand and the purchasing cost structures of the different suppliers, the heuristic could determine the optimal quantity to order from each supplier.

Haksever and Moussourakis (2008) formulated a multi-item inventory model with incremental quantity discounts taking into account a number of constraints, including maximum purchasing cost of each item, maximum number of items of each type that can be ordered and maximum number of items that can be purchased in each price break. The model also investigated whether a common cycle time for all the different types of items could reduce the total costs.

Lee et al. (2013) studied an integrated vendor-buyer inventory system considering both all-units and incremental quantity discounts. This model was different from vendor-buyer inventory model with quantity discounts because it also took into account supplier selection. The model was aimed at evaluating a number of suppliers who offer different quantity discounts to the buyer and the EOQ is based on the supplier who offers the best deal (i.e. results in the lowest total system costs).

Taleizadeh et al. (2015) developed an inventory model with incremental quantity discounts under two different shortage conditions. In the first case, shortages are considered to be fully backordered (i.e. all the customers are willing to wait for the backordered stock to arrive) and in the second case (partial backordering) it was assumed that some of the customer are not willing to wait for the backordered stock and in this case a lost sales cost is taken into account.

Tamjidzad and Mirohammadi (2015) incorporated stochastic demand and budget constraints to the basic inventory model with incremental quantity discounts. It was assumed that the demand rate follows a Poisson distribution and that the budget allocated to the purchasing stock is limited.

Bohner and Minner (2017) formulated a vendor-buyer inventory model with supplier selection. It was assumed that the buyer has a number of potential supplier offering various types of discounts, including all-units and incremental, for purchasing larger volumes of stock with different purchasing cost structures. Furthermore, all the suppliers had an associated failure risk (i.e. likelihood of not delivering order as promised) and multiple inventory items were considered.

Mohammadivojdan and Geunes (2018) studied the newsvendor problem (i.e. single period inventory model) with multiple vendors and various types of discounts. The model assumed that the vendors offered incremental and all-units discounts for purchasing the items and carload discounts for transporting the items to the buyer. In addition, the capacity of each of the suppliers was assumed to be limited.

2.7 Conclusion

A great deal of research has been conducted in the area of inventory management. Nonetheless, not all areas have been thoroughly researched. Research into inventory management for growing items is sparse and the area has only recently started receiving attention, following the publication of Rezaei (2014)'s work. Some extensions have been developed, but none of them take into account imperfect quality, capacity limits on the growing and storage facilities and quantity discounts. Owing to these gaps in the current body of knowledge, new extensions are proposed and formulated in the next three chapters. When formulating mathematical models for the proposed inventory systems, two models (i.e. general and specific models) are presented for each extension, as was the case in Rezaei (2014)'s work. However, when formulating the specific model, the assumption of a linear growth function, as used in the model developed by Nobil et al. (2018), is adapted because it decreases the computational complexity in determining the feeding cost and likewise, the model's optimal solution.

Chapter 3

Economic order quantity model for growing items with imperfect quality*

3.1 Introduction

Harris (1913)'s work on the classic Economic Order Quantity (EOQ) model has laid the groundwork for modern inventory management. However, the classic EOQ model makes a number of unrealistic assumptions. In attempts to model more realistic inventory situations, numerous researchers have relaxed these assumptions to create new models. Two such assumptions are relaxed to create a new EOQ model presented in this chapter. The two assumptions are that the items do not grow and that all the items are of good quality.

The above assumptions are not true for all situations. Certain items, such as poultry or beef, are capable of growing during the course of a planning period (Rezaei, 2014). Preparation of these items for sale usually involves some degree of processing, and item quality is seldom perfect in most production processes (Salameh and Jaber, 2000).

Research into inventory modelling for growing items is relatively sparse and new, with the first paper published by Rezaei (2014). The major difference between growing items, for example livestock, and conventional items, for example books, is that the total weight of growing items increases during the course of an inventory cycle without ordering extra items. Item growth was the major differentiator between the model by Rezaei (2014) and the classic EOQ model. Growth of items implies feeding, and so, there was the inclusion of feeding costs in Rezaei (2014)'s model because otherwise the items will not grow if they are not being fed. Researchers have started to extend the classic work of Rezaei in growing items inventory into diverse other areas. For instance, Zhang et al. (2016) incorporated environmental sustainability to Rezaei (2014)'s work by developing an EOQ model for growing items in a carbon constrained environment. Nobil et al. (2018) extended the growing items inventory model by relaxing the assumption that shortages are not allowed and approximating the growth of the items by a linear function.

Imperfection of quality of items is another area that has recently been included in research efforts because not every single item manufactured or procured is of perfect quality. Salameh and Jaber (2000) were the the first researchers to incorporate imperfect

*A modified version of this chapter has been submitted to *Operations Research Perspectives* for review and a revision was requested. The revised manuscript has been resubmitted.

quality into the classic EOQ model. They formulated an inventory model for a situation where a certain fraction of the items received in each lot is of poor quality. Over the years, this model has been improved in several ways. Cardenas-Barron (2000) and Maddah and Jaber (2008) corrected computational errors made in the expressions for the EOQ and expected total profit respectively. Goyal and Cardenas-Barron (2002) proposed a simpler method for computing the EOQ. Salameh and Jaber (2000)'s work has also been extended in several ways. Huang (2002) studied a vendor-buyer inventory system for items with imperfect quality. Chang (2004) presented a model in which the fraction of imperfect quality items and the demand rate were assumed to be fuzzy variables. Building upon Salameh and Jaber (2000), Yu et al. (2005) incorporated partial backordering and deterioration. Wee et al. (2007) extended Salameh and Jaber's work by relaxing the assumption that shortages are not allowed. Jaber et al. (2008) presented an EOQ model for imperfect quality item subject to learning effects. Chung et al. (2009) developed a two-warehouse inventory model for items with imperfect quality. Chang and Ho (2010) derived an EOQ model for items with imperfect quality and shortages without the use of differential calculus. Chen and Kang (2010) studied an integrated vendor-buyer inventory system for items with imperfect quality under conditions of permissible delay in payments. The effects of a company's pricing and marketing plans were accounted for in an inventory model for items with imperfect quality presented by Sadjadi et al. (2012). Hsu and Hsu (2013) extended the work of Salameh and Jaber (2000) by incorporating sales returns in to the model. Khan et al. (2016) studied a vendor-buyer inventory system for items with imperfect quality where the vendor and the buyer have a vendor-managed inventory (VMI) agreement in place. Jaggi et al. (2017) presented an EOQ model for deteriorating items with imperfect quality taking into account trade credit financing and the use of an additional rented warehouse. Tiwari et al. (2018) developed a vendor-buyer inventory model for deteriorating items with imperfect quality and carbon emissions cost.

Table 3.1: Gap analysis of related works in literature

References	**Characteristics of the inventory system**					**Solution technique**	
	Conventional items	Growing items	Imperfect quality	Carbon tax	Shortage	Closed form	Heuristic
Harris (1913)	✓					✓	
Salameh and Jaber (2000)	✓		✓			✓	
Rezaei (2014)		✓					✓
Zhang et al. (2016)		✓		✓			✓
Nobil et al. (2018)		✓			✓		✓
This chapter		✓	✓				✓

A review of current literature seems to suggest that there is no work published on inventory modelling which considered the assumptions of growing and imperfect quality items simultaneously. In this paper, an attempt is made to develop an inventory which considers growing items with imperfect quality. The inventory model presented is more practical than the classic EOQ model and it serves as an extension to models presented by Salameh and Jaber (2000) and Rezaei (2014). A comparison of the proposed inventory system and previously published relevant inventory models in the literature is provided in Table 3.1, which shows the contributions made by various research papers in the existing literature as well as what this chapter adds to inventory theory research for growing items.

3.2 Problem definition

The inventory system under study considers a situation where a company orders a certain number of items which are capable of growing over time, such as chickens. Figure 3.1 represents the typical behaviour of such an inventory system. The growth process is facilitated by the company through feeding the items. The company incurs a cost for feeding and raising the items. At the end of the growth period (i.e. after having grown to a certain weight), the items are slaughtered and sold. A certain fraction of the slaughtered items is not of acceptable quality. Prior selling the items, the company screens them to separate the good quality items from the poor quality items. Good quality items are sold at a given price throughout the sales cycle while the poor quality items are salvaged (i.e. sold at a price which is lower than the price of the good quality items) as a single batch after the screening process. Each inventory cycle can be divided into two distinct periods, namely the growth and the consumption periods. During the growth period (i.e. period t_1 in Figure 3.1), ordered newborn items are fed and raised until they grow to a certain target weight. This marks the end of the growth period and the items are slaughtered. During the consumption period (i.e. period T in Figure 3.1), the slaughtered items are kept in stock and sold to consumers following a screening process (i.e. period t_2 in Figure 3.1) which separates the items of good quality from the ones of poor quality. Figure 3.1 also shows the relationship between one inventory cycle and the next. The inventory level of one complete inventory cycle finishes up at time T. At this time, the items in the next inventory cycle would have completed their growth cycle. This means that the items in the next cycle will be ready for consumption (i.e. they have grown to the target weight) at the instant the items in the previous inventory cycle are used up. The company wants to determine the number of items to order when a growing cycle begins in order to maximize its total profit. The total profit is defined as the difference between total revenue and total costs. Total revenue includes the revenue from the sale of both good and poor quality items. The total cost is the sum of the purchasing, feeding, holding, setup and screening costs.

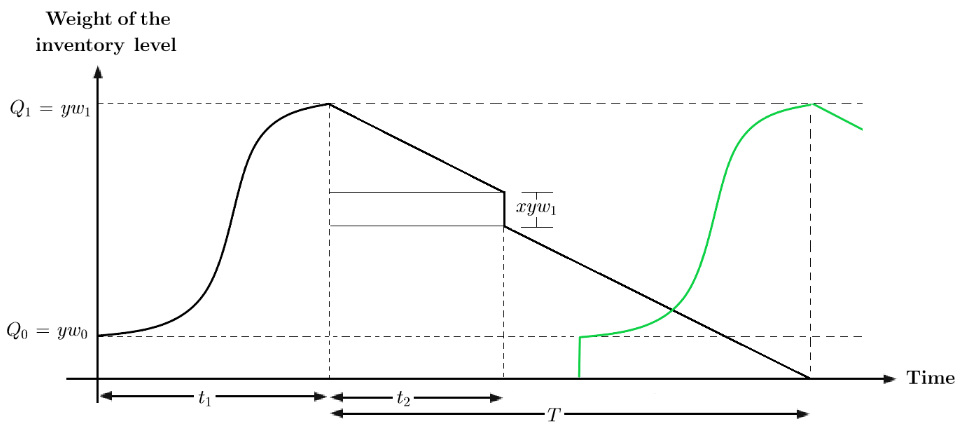


Figure 3.1: Inventory system behaviour for growing items with imperfect quality

The inventory system is studied through a mathematical model in order to address the two primary concerns of inventory management, namely, how much to order and when to place an order. The objective function of the mathematical model is the expected total profit and the jointly-determined decision variables are the cycle time and the order quantity.

3.3 Notations and assumptions

3.3.1 Notations

The following notations are used when developing the mathematical model:

Table 3.2: Notations used in the formulation of the mathematical model

Symbol	Description
y	Number of ordered newborn items per cycle
T	Cycle time
w_0	Approximated weight of each newborn item
w_1	Approximated weight of each grown item at the time of slaughtering
Q_t	Total weight of all the ordered inventory items at time t
$f(t)$	Feeding function for each item
b	Growth rate per item in weight units per unit time
p	Purchasing cost per weight unit
s	Selling price of good quality item per weight unit
v	Selling price of poor quality item per weight unit
h	Holding cost per weight unit per unit time
K	Setup cost per cycle
D	Demand for good quality items in weight units per unit time
c	Feeding cost per weight unit per unit time
x	Percentage of slaughtered items that are of poor quality
z	Screening cost per weight unit
r	Screening rate
t_1	Duration of the growing period
t_2	Screening time
t_s	Setup time
$E(.)$	Expected value of any variable

3.3.2 Assumptions

Some assumptions were made in order to formulate the mathematical model. These include:

- The ordered items are capable of growing prior to being slaughtered.
- A single type of item is considered.
- A cost is incurred for feeding and growing the items.
- The cost of feeding the items is proportional to the weight gained by the items.
- Holding costs are incurred for the duration of the consumption period.
- A random fraction of the slaughtered items is of poor quality.
- All poor quality items can be sold.
- Poor quality items are salvaged as a single batch after the screening period.

- The selling price of good quality items is greater than that of the poor quality items.
- There is no rework or replacement of poor quality items.
- Demand is a deterministic constant.

3.4 Model development

3.4.1 General mathematical model

At the beginning of a growing cycle, a company purchases y newborn items which are capable of growing, such as livestock. At the time of receiving the order, each newborn item weighs w_0. The total weight of the inventory at this point, Q_0, is determined by multiplying the weight of each of the items by the number of items ordered (i.e. $Q_0 = yw_0$). The items are fed and they grow to a final weight w_1 which is a function of time. After reaching the target weight of w_1, they are slaughtered. The weight of each fully grown item, w_1, is a function of time. The total weight of the inventory at the time of slaughter is $Q_1 = yw_1$. The items are screened for the period t_2 at a rate of r. The proposed inventory system is depicted by Figure 3.1. A fraction, x, of the slaughtered items is of poor quality. This fraction is assumed to be random. At the end of the screening period, all the poor quality items are sold as a single batch at a discounted price. The good quality items are sold throughout the consumption period, T, at a demand rate of D unit weights per unit time.

The objective of the proposed inventory model is to maximize the company's total profit (TP), which is the company's total costs subtracted from its total revenue (TR). The total cost per cycle is made up of five components, namely purchasing, setup, screening, feeding and holding costs, denoted by PC, SC , ZC, FC and HC respectively. The company's total profit function per cycle is given by

$$TP = TR - PC - SC - HC - FC - ZC. \tag{3.1}$$

Since the fraction of poor quality items, x, is assumed to be a random variable with a known expectation given by $E(x)$, the expected value of the total profit per cycle is thus

$$E[TP] = E[TR] - PC - SC - E[HC] - FC - ZC. \tag{3.2}$$

In order to meet an annual demand of D weight units of good quality items, the company needs to setup growth facilities $D/[yw_1(1-E[x])]$ times a year. The inverse of the number of times the company should setup growing facilities gives the expected cycle duration of the consumption period. Hence,

$$E[T] = \frac{yw_1\big(1 - E[x]\big)}{D}. \tag{3.3}$$

Since all the slaughtered inventory is subjected to screening prior to being sold, the total weight of slaughtered inventory, yw_1, and the screening rate, r, are used to compute the duration of the screening period, t_2, as

$$t_2 = \frac{yw_1}{r}. \tag{3.4}$$

3.4.1.1 Expected revenue per cycle

Since the company sells both good and poor quality items, the total revenue includes income from sales of both good and poor quality items. Good quality items are sold continuously at a price of s per weight unit. At the end of the screening process, the poor quality items are salvaged as a single batch at a discounted price of v per weight unit. Hence, the expected value of the revenue per cycle is

$$E[TR] = syw_1\big(1 - E[x]\big) + vyw_1E[x]. \tag{3.5}$$

3.4.1.2 Purchasing cost per cycle

At the start of each cycle, the company purchases y newborn items, each weighing w_0, at a cost of p per weight unit. Hence, the purchasing cost per cycle is

$$PC = pyw_0. \tag{3.6}$$

3.4.1.3 Setup cost per cycle

At the beginning of each cycle, a fixed setup cost of K is incurred by the company and thus the setup cost per cycle is given by

$$SC = K. \tag{3.7}$$

3.4.1.4 Feeding cost per cycle

Growth of the items is facilitated by the company through feeding the items for the period t_1. Feeding the items costs the company c per weight unit per unit time. The amount of food consumed by the items is given by the function $f(t)$. These three quantities, along with the amount of items that need to be fed (*i.e.* number of ordered items), are used to determine the feeding cost per cycle as

$$FC = cy\int_0^{t_1} f(t)\,dt. \tag{3.8}$$

3.4.1.5 Screening cost per cycle

A fraction of the items, x, is of poor quality. A screening process is conducted for the duration t_2 to separate the items of good quality from those of poor quality. It costs the company z to screen a single weight unit of the slaughtered items. The cost of screening all the items in each cycle is given by

$$ZC = zyw_1. \tag{3.9}$$

3.4.1.6 Expected holding cost per cycle

Following the growth period, the items are slaughtered after having grown to a target weight of w_1. The holding cost component is essentially the costs associated with keeping the fully grown slaughtered items in storage. Therefore, the company pays the holding cost for the period T. It costs the company h to keep a single weight unit of the slaughtered items in storage for a year. The expected value of the holding cost per cycle, computed by multiplying the holding cost per weight unit per unit time (h) and the area under the consumption period in Figure 3.1, is given by

$$E[HC] = h\left[\frac{y^2w_1^2\big(1-E[x]\big)^2}{2D} + \frac{y^2w_1^2E[x]}{r}\right]. \tag{3.10}$$

3.4.1.7 Expected total profit function

The expression for the expected total profit per cycle is determined by substituting Equations (3.5) through (3.10) into Equation (3.2). The expected value of the total profit per cycle is thus

$$E[TP] = syw_1\big(1-E[x]\big) + vyw_1E[x] - pyw_0 - K - cy\int_0^{t_1} f(t)\,dt - zyw_1 - h\left[\frac{y^2w_1^2\big(1-E[x]\big)^2}{2D} + \frac{y^2w_1^2E[x]}{r}\right]. \tag{3.11}$$

The expected total profit per unit time, $E[TPU]$, is determined by dividing Equation (3.11) by Equation (3.3) which results in

$$E[TPU] = \frac{E[TP]}{E[T]} = sD + \frac{vDE[x]}{\big(1-E[x]\big)} - \frac{pDw_0}{w_1\big(1-E[x]\big)} - \frac{KD}{yw_1\big(1-E[x]\big)} - \frac{zD}{\big(1-E[x]\big)} - \frac{cD}{w_1\big(1-E[x]\big)}\int_0^{t_1} f(t)\,dt - h\left[\frac{yw_1\big(1-E[x]\big)}{2} + \frac{yw_1DE[x]}{r\big(1-E[x]\big)}\right]. \tag{3.12}$$

Equation (3.12) is a general function for the expected total profit per unit time and it is applicable to a variety of growing items. Consequently, it has two general expressions (which vary per growing item) that are functions of time. These are the feeding function, $f(t)$, and the growth function, $f(w_1|w_0)$. As a result of these two functions being different for different growing items, it is necessary to specify these variables in order to reduce the computational complexity in solving the model. Rezaei (2014), who presented the first EOQ model for growing items, used growth and feeding functions for broiler chickens, which were developed by Richards (1959) and Goliomytis et al. (2003) respectively. Due to using those particular growth and feeding functions, Rezaei (2014)'s model had the slaughter time, in addition to the cycle time and order quantity, as another decision variable. For that reason, Rezaei (2014) used a heuristic which makes use of the Newton-Raphson method to solve the model. Nobil et al. (2018) solved an EOQ model for growing items with shortages using a relatively simpler heuristic by assuming that the growth function of the items is linear. Furthermore, this assumption made the processes of determining the feeding cost straightforward. The assumption of linear growth function, as used by Nobil et al. (2018), is adapted when formulating the specific mathematical model because it reduces the complexity in determining the feeding cost and solving the model.

3.4.2 Specific mathematical model: Case of items with a linear growth function

Consider a case where the growth function of each item, $f(w_1|w_0)$, is linear. In such a case, each item's weight increases (i.e. grows) at a constant rate of b weight units per unit time. Growth occurs for the duration t_1 and when the newborn items are received, they each weigh w_0. This means that the slaughter weight of the items is a linear function with gradient b and y-intercept w_0. Mathematically, the slaughter weight is defined as

$$w_1 = w_0 + bt_1. \tag{3.13}$$

The growth function for all the ordered items can be obtained by multiplying every term by y. Thus,

$$yw_1 = yw_0 + ybt_1, \tag{3.14}$$

from which Figure 3.2, depicting the behaviour of the inventory system over time, is constructed. The duration of the growth period is computed from Equation (3.13) as

$$t_1 = \frac{w_1 - w_0}{b}. \tag{3.15}$$

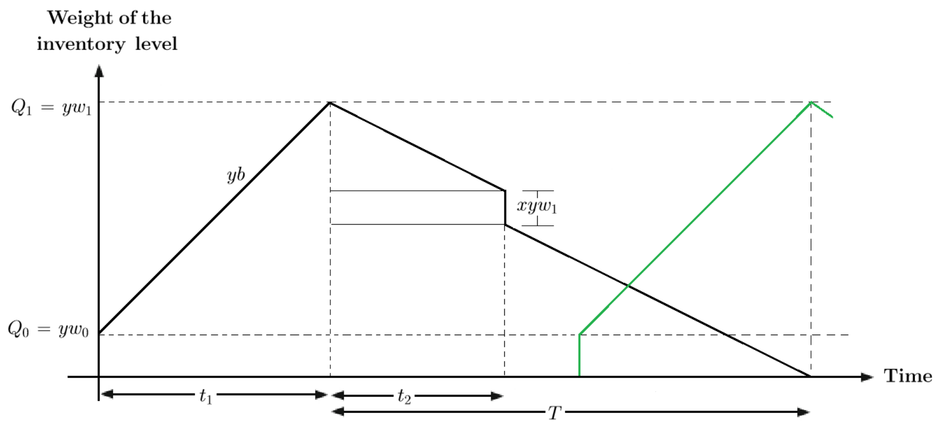


Figure 3.2: Inventory system behaviour for growing items with imperfect quality under the assumption of linear growth function

3.4.2.1 Feeding cost per cycle

The feeding cost is determined using Figure 3.2 as the product of the feeding cost per weight unit per unit time (c), the duration of the feeding period (which is equal to the growth period t_1), the difference in the weight of the items from the moment they start consuming food (w_0) until when they stop consuming food at the time of slaughter (w_1) and the number of ordered items (y) which all need to be fed. The feeding cost, computed by multiplying the feeding cost per weight unit per unit time (c) by the are under the growth period in Figure 3.2, is given by

$$FC = c\left[\frac{t_1\left(yw_1 - yw_0\right)}{2}\right]. \tag{3.16}$$

Equation (3.15) is substituted into Equation (3.16), resulting in

$$FC = c\left[\frac{y\left(w_1 - w_0\right)^2}{2b}\right]. \tag{3.17}$$

3.4.2.2 Expected total profit function

The expected revenue, purchasing cost, setup cost, screening cost and holding cost per cycle remain the same as those in Equations (3.5), (3.6), (3.7), (3.9) and (3.10) respectively. The expected profit per cycle is thus

$$E[TP] = syw_1\big(1-E[x]\big) + vyw_1E[x] - pyw_0 - K - c\left[\frac{y\big(w_1-w_0\big)^2}{2b}\right] - h\left[\frac{y^2w_1^2\big(1-E[x]\big)^2}{2D} + \frac{y^2w_1^2E[x]}{r}\right]. \quad (3.18)$$

To further simplify Equation (3.18), an expression for y is determined from Equation (3.3) as

$$y = \frac{DE[T]}{w_1\big(1-E[x]\big)}. \quad (3.19)$$

Equation (3.19) is substituted into Equation (3.18) to yield an expression for the expected total profit per cycle, $E[TP]$, in terms of $E[T]$. Thus,

$$E[TP] = sDE[T] + \frac{vDE[T]E[x]}{\big(1-E[x]\big)} - \frac{pDE[T]w_0}{w_1\big(1-E[x]\big)} - K - \frac{zDE[T]}{\big(1-E[x]\big)} - \frac{cD\big(w_1-w_0\big)^2E[T]}{2bw_1\big(1-E[x]\big)} - h\left[\frac{DE[T^2]}{2} + \frac{D^2E[T^2]E[x]}{r\big(1-E[x]\big)^2}\right]. \quad (3.20)$$

The expected total profit per unit time, $E[TPU]$, is computed by dividing the new $E[TP]$ function by the expected cycle time and the result is

$$E[TPU] = \frac{E[TP]}{E[T]} = sD + \frac{vDE[x]}{\big(1-E[x]\big)} - \frac{pDw_0}{w_1\big(1-E[x]\big)} - \frac{K}{E[T]} - \frac{zD}{\big(1-E[x]\big)} - \frac{cD\big(w_1-w_0\big)^2}{2bw_1\big(1-E[x]\big)} - h\left[\frac{DE[T]}{2} + \frac{D^2E[T]E[x]}{r\big(1-E[x]\big)^2}\right]. \quad (3.21)$$

3.4.2.3 Model constraints

Two governing constraints may be important for the feasibility of the proposed inventory system. The first constraint is to ensure that the items are ready for consumption at the required time, while the second one ensures that shortages are avoided during the screening period.

Constraint 1: In order to ensure that the slaughtered items are ready for consumption during period, the setup time and the length of the growth period should be less than or equal to the consumption period. A restriction on the expected value of the consumption period, $E[T]$, is formulated as

$$t_1 + t_s \leq E[T]. \quad (3.22)$$

By substituting t_1 from Equation (3.15), Equation (3.22) becomes

$$E[T] \geq \left\{\frac{w_1-w_0}{b} + t_s = T_{min}\right\}. \quad (3.23)$$

Constraint 2: Define $N(yw_1, x)$ as the number of good quality slaughtered items minus the number of poor quality slaughtered items per cycle. This can be represented, mathematically, as

$$N(yw_1, x) = yw_1 - xyw_1 = (1-x)yw_1. \tag{3.24}$$

One of the assumptions made is that shortages are not allowed. In order to avoid shortages, the number of good quality items must be at least equal to the demand during the screening time t_2. It follows that

$$N(yw_1, x) \geq Dt_2. \tag{3.25}$$

Substituting Equation (3.24) and the value of t_2 into Equation (3.25), a restriction on x is formulated as

$$x \leq \left\{1 - \frac{D}{r} = x_{res}\right\}. \tag{3.26}$$

3.4.2.4 Mathematical formulation of the EOQ model for growing items with imperfect quality

Using the objective function in Equation (3.21) and the constraints, the mathematical formulation for the the proposed inventory system is given by

$$Max \quad \left\{E[TPU] = sD + \frac{vDE[x]}{\left(1-E[x]\right)} - \frac{pDw_0}{w_1\left(1-E[x]\right)} - \frac{K}{E[T]} - h\left[\frac{DE[T]}{2} + \frac{D^2E[T]E[x]}{r\left(1-E[x]\right)^2}\right] - \frac{cD\left(w_1-w_0\right)^2}{2bw_1} - \frac{zD}{\left(1-E[x]\right)}\right\} \tag{3.27}$$

$$\begin{aligned} s.t. \quad & E[T] \geq T_{min} \\ & E[T] \geq 0 \\ & x \leq 1 - D/r. \end{aligned}$$

3.4.3 Solution

3.4.3.1 Determination of the decision variables

The optimal solution to the proposed inventory system, which has $E[TPU]$ as the objective function and $E[T]$ as a decision variable, is determined by setting the partial derivative of $E[TPU]$ with respect to $E[T]$ to zero and the result is

$$\begin{aligned} \frac{\partial E[TPU]}{\partial E[T]} &= \frac{K}{E[T^2]} - h\left[\frac{D}{2} + \frac{D^2E[x]}{r\left(1-E[x]\right)^2}\right] \\ &= 0 \rightarrow E[T] = \sqrt{\frac{2KD}{hD^2\left[1 + \frac{2DE[x]}{r\left(1-E[x]\right)^2}\right]}}. \end{aligned} \tag{3.28}$$

By substituting Equation (3.28) into Equation (3.19), the order quantity which maximizes the expected value of the total profit per unit time is

$$y = \sqrt{\frac{2KD}{hw_1^2\left[\left(1-E[x]\right)^2 + \frac{2DE[x]}{r}\right]}}. \tag{3.29}$$

3.4.3.2 Proof of concavity of the objective function

In order to show that there exists a unique solution for Equation (3.21) and that the value at the point actually maximizes the objective function, it suffices to calculate the grad of the function to identify the optimum point, and to show that the Hessian is negative (semi) definite. Equation (3.30) shows the location of the optimum point from the grad.

$$\frac{\partial E[TPU]}{\partial E[T]} = \frac{K}{E[T^2]} - h\left[\frac{D}{2} + \frac{D^2 E[x]}{r\left(1 - E[x]\right)^2}\right] \tag{3.30}$$

The Hessian matrix of the objective function, given by

$$\begin{bmatrix} \frac{\partial^2 E[TPU]}{\partial E[T^2]} & \frac{\partial^2 E[TPU]}{\partial E[T]\partial y} \\ \frac{\partial^2 E[TPU]}{\partial E[T]\partial y} & \frac{\partial^2 E[TPU]}{\partial y^2} \end{bmatrix}, \tag{3.31}$$

is shown to be negative semi-definite in Equation (3.32).

$$\begin{bmatrix} -\frac{K}{E[T^3]} & -\frac{K}{E[T^3]} \\ -\frac{K}{E[T^3]} & 0 \end{bmatrix} \tag{3.32}$$

The quadratic form of the objective function is determined from the Hessian matrix as

$$\begin{bmatrix} E[T] & y \end{bmatrix} \begin{bmatrix} -\frac{K}{E[T^3]} & -\frac{K}{E[T^3]} \\ -\frac{K}{E[T^3]} & 0 \end{bmatrix} \begin{bmatrix} E[T] \\ y \end{bmatrix} = -\frac{2K}{E[T]}\left(1 + \frac{y^2}{E[T^2]}\right) \leq 0. \tag{3.33}$$

From Equation (3.33), the quadratic form of the objective function is shown to be negative which implies that the objective function is concave.

3.4.3.3 Computational algorithm

The following optimization algorithm is proposed for determining the solution to the EOQ model for growing items with imperfect quality:

Step 1 Compute T_{min} using Equation (3.23).

Step 2 Check the problem's feasibility. The problem is feasible provided that $T_{min} \geq 0$. If it is feasible proceed to Step 3, otherwise proceed to Step 8.

Step 3 Compute x_{res} using Equation (3.26).

Step 4 If $x \leq x_{res}$, then problem is feasible and proceed to Step 5. Otherwise the problem is infeasible and proceed to Step 8.

Step 5 Compute $E[T]$ using Equation (3.28).

Step 6 $T^* = E[T]$ provided that $E[T] \geq T_{min}$, otherwise $T^* = T_{min}$.

Step 7 Compute y^* and $E[TPU^*]$ using Equations (3.29) and (3.21) respectively considering the T^* value.

Step 8 End.

3.5 Numerical results

3.5.1 Numerical example

The proposed inventory system is applied to a numerical example which considers a company which purchases day-old chicks, feeds/grows them until they reach a targeted weight and then puts them on sale after screening for quality. The following parameters, mostly adapted from a study by Nobil et al. (2018), are utilised to analyse the proposed inventory system:

Demand rate, D =1 000 000 g/year

Setup cost, K =1 000 ZAR/cycle

Holding cost, h =0.04 ZAR/g/year

Feeding cost, c =0.08 ZAR/g/year

Approximated weight of newborn (one day old) chick, w_0 =53 g/chick

Approximated weight of chicken at the time of slaughtering, w_1 =1 267 g/chicken

Growth rate, b =15 330 g/chick/year

Setup time, t_s =0.01 year

Purchase cost, p =0.025 ZAR/g

Selling price of good quality item, s =0.05 ZAR/g

Selling price of poor quality item, v =0.02 ZAR/g

Fraction of poor quality items, $E[x]$ =0.02

Screening cost, z =0.00025 ZAR/g

Screening rate, r =10 g/minute

It is assumed that the inventory operation runs 24 hours/day for 365 days, then the annual screening rate, r, = 10 g/minute x 1 440 minutes/day x 365 days/year = 5 256 000 g/year.

Solution procedure application: The proposed solution algorithm is illustrated by applying it to the numerical example. The procedure is outlined as follows:

Step 1 Compute T_{min} using Equation (3.23).

$$T_{min} = \frac{1267 - 53}{15330} = 0.0891$$

Step 2 The problem is feasible since $T_{min} \geq 0$, proceed to Step 3.

Step 3 Compute x_{res} using Equation (3.26).

$$x_{res} = 1 - \frac{1000000}{5256000} = 0.8097$$

Step 4 The problem is feasible since $x \leq x_{res}$, proceed to Step 5.

Step 5 Compute $E[T]$ using Equation (3.28).

$$E[T] = \sqrt{\frac{2 \times 1000 \times 1000000}{(0.04 \times 1000000^2)\left[1 + \frac{2 \times 1000000 \times 0.98}{5256000 \times 0.9605}\right]}} = 0.2227$$

Step 6 Since $T \geq T_{min}$, $T^* = T = 0.2227$

Step 7 Compute $E[TPU^*]$ and y^*.

$$y^* = \sqrt{\frac{2 \times 1000 \times 1000000}{(0.04 \times 1267^2)\left[0.9605 + \frac{2 \times 1000000 \times 0.98]}{5256000}\right]}} = 179.3654$$

$$\begin{aligned} E[TPU]^* = 0.05 \times 1000000 + \frac{0.02 \times 1000000 \times 0.02}{0.98} - \frac{0.025 \times 1000000 \times 53}{1267 \times 0.98} \\ - \frac{1000}{0.2227} - 0.04\left[\frac{1000000 \times 0.2227}{2} + \frac{1000000^2 0.2227 \times 0.02}{5256000\big(0.98\big)^2}\right] \\ - \frac{0.08 \times 1000000\big(1267 - 53\big)^2}{2 \times 15330 \times 1267} - \frac{0.00025 \times 1000000}{\big(0.98\big)} = 37009.23 \end{aligned}$$

Step 8 End.

A summary of the results from the numerical example, particularly the decision variables and duration of the growth and screening periods, are given in Table 3.3.

Variable	Units	Quantity
t_1	year	0.0792
t_2	year	0.0432
T^*	year	0.2227
y^*	items	180
$E[TPU]^*$	ZAR/year	37 009.23

Table 3.3: Summary of the results from the numerical example

From the results of the numerical example, the company should order 180 newborn items at the beginning of each cycle. The newborn items should be grown for a period of 0.0792 years (29 days) and the consumption period lasts for a period of 0.2227 years (81 days). An order should be placed every 0.2227 years (81 days) and the company should expect to make a yearly profit of 37 009.23 ZAR. Screening for quality should start immediately as the consumption begins and it should happen for a period of 0.0432 years (16 days), after which the imperfect quality items should be sold as a single batch.

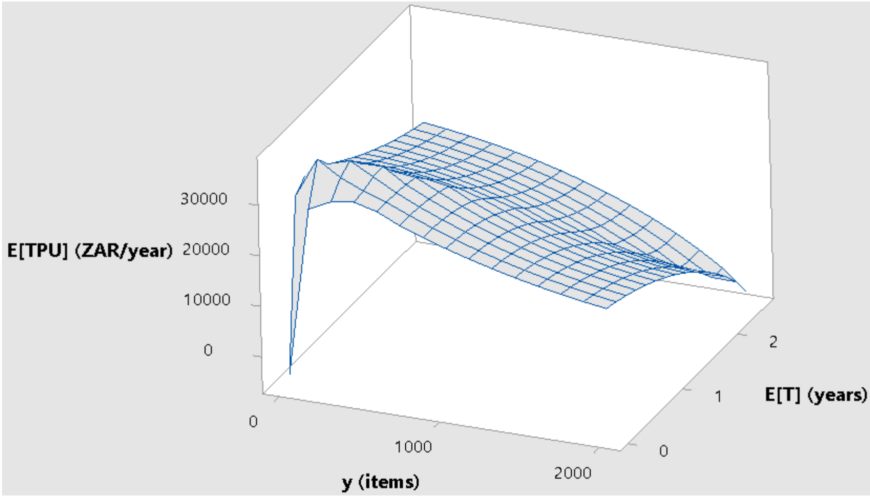


Figure 3.3: Graph of expected total profit per unit time, cycle time and order quantity

The response of the objective function to the changes in the decision variables are graphed in Figure 3.3, which also shows that the objective function is concave. The expected profit per unit time increases with the expected cycle time until it reaches maximum at the optimal cycle. This maximum points also corresponds to the optimal order quantity (i.e. the EOQ). After this, the expected total profit per unit time decreases with an increase in the cycle time and order quantity.

3.5.2 The effect of poor quality on the lot size

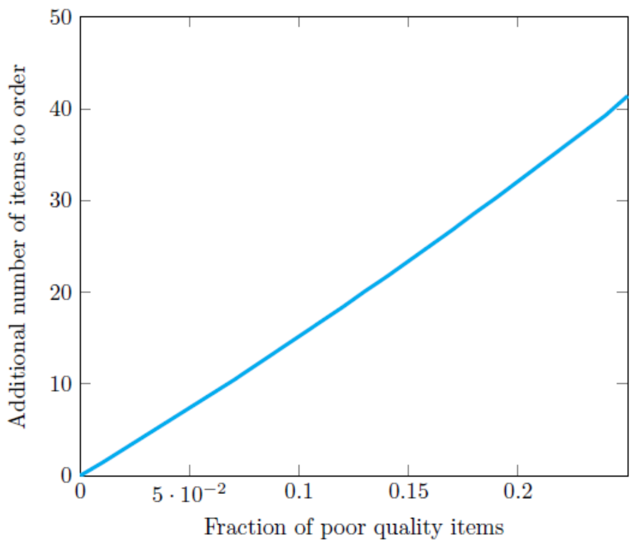


Figure 3.4: The impact of the presence of poor quality items on the order quantity

The effect of poor quality on the order quantity is investigated by varying the expected fraction of poor quality items and the results are illustrated through Figure 3.4. In order to test the effect of imperfect quality, the order quantity when all the items are of good quality is first determined. This serves as a base for comparison. Following this, the fraction of poor quality items was increased gradually and the new order quantities required to satisfy the demand for good quality items were recorded at various fractions of poor quality items. When all the items are of good quality, no additional items are required in order to meet the annual demand for good quality items. As the fraction of poor quality items increases, more additional items need to be ordered. This highlights the potential repercussions of the presence of poor quality items as the need to order additional items means increased inventory related costs.

3.5.3 Sensitivity analysis

A sensitivity analysis is conducted on select parameters in order to investigate the effects that changes in those parameters have on the expected total profit per unit time and the EOQ and to draw some managerial insights about managing inventory for a supply chain with growing items.

Table 3.4: Changes to y^* and $E[TPU]^*$ due to changes in c

% change in feeding cost	**EOQ**		**E[TPU]**	
	items	% change	ZAR/year	% change
-50	180	0	38 563	4.2
-37.5	180	0	38 156	3.1
-25	180	0	37 786	2.1
-12.5	180	0	37 379	1
0	180	0	37 009	0
12.5	180	0	36 639	-1
25	180	0	36 232	-2.1
37.5	180	0	35 862	-3.1
50	180	0	35 455	-4.2

The EOQ is not sensitive to the feeding cost (i.e. changing the feeding cost has no effect), as shown in Table 3.4. However, increasing the feeding cost reduces the expected total profit per unit time. By reducing feeding costs, either through better procurement practices or reducing the items' rations, managers can save significant amounts of money without increasing the risk of stock-outs or overstocking.

Table 3.5: Changes to y^* and $E[TPU]^*$ due to changes in K

% change in setup cost	**EOQ**		**E[TPU]**	
	items	% change	ZAR/year	% change
-50	127	-29.3	39 637	7.1
-37.5	142	-20.9	38 896	5.1
-25	156	-13.4	38 230	3.3
-12.5	168	-6.5	37 601	1.6
0	180	0	37 009	0
12.5	191	6.1	36 454	-1.5
25	201	11.8	35 936	-2.9
37.5	211	17.3	35 455	-4.2
50	221	22.5	34 974	-5.5

The setup cost has significant effects on both the EOQ and the expected total profit per unit time as shown in Table 3.5. Decreasing the setup cost increases the expected profit. Managers should pay close attention to the setup cost because trying to reduce this cost by ordering less frequently leads to an increase in the size of the order. Larger order sizes translate to higher holding costs which will erode the expected profits.

Table 3.6: Changes to y^* and $E[TPU]^*$ due to changes in h

% change in holding cost	**EOQ** items	% change	**E[TPU]** ZAR/year	% change
-50	255	41.4	39 637	7.1
-37.5	228	26.5	38 896	5.1
-25	210	15.5	38 230	3.3
-12.5	192	6.9	37 601	1.6
0	180	0	37 009	0
12.5	170	-5.7	36 454	-1.5
25	161	-10.6	35 936	-2.9
37.5	154	-14.7	35 455	-4.2
50	147	-18.3	34 974	-5.5

From Table 3.6, it is evident that increasing the holding cost increases the EOQ and reduces the expected profit. When trying to optimise inventory control, managers should keep the holding cost as low as possible because this increases their margins and enables them to order and keep more items at a lower cost, which in turn increases customer satisfaction levels by reducing the possibility of running out of stock.

Table 3.7: Changes to y^* and $E[TPU]^*$ due to changes in w_1

% change in slaughter weight	**EOQ** items	% change	**E[TPU]** ZAR/year	% change
-50	360	100	37 639	1.7
-37.5	288	60	37 639	1.7
-25	240	33.3	37 490	1.3
-12.5	206	14.3	37 268	0.7
0	180	0	37 009	0
12.5	160	-11.1	36 712	-0.8
25	144	-20	36 380	-1.7
37.5	131	-27.3	36 047	-2.6
50	120	-33.3	35 677	-3.6

Table 3.7 shows the response of the EOQ and the expected profit to changes in the slaughter weight. An increase in the slaughter weight results in decreases in both the EOQ and the profit. This is because if the items are slaughtered at larger weights, the company would need to order fewer newborn items to meet the same demand rate. However, this leads to a slight decrease in the expected profit most likely due to increased holding cost since larger items (i.e. more weight units of items) need to be held in stock.

3.6 Conclusion

In this chapter, an economic order quantity model for growing items with imperfect quality is developed. A general mathematical model, which can be applied to a variety of growing items, is presented first. This particular model has two general functions which increase the complexity in solving the model. To counter this, a specific model which considers a linear growth function is presented as well. The specific model is solved using a relatively simple heuristic and the proposed solution procedure is applied to a numerical example.

The major contribution made by the research presented in this chapter is the incorporation of imperfect quality into the EOQ model for growing items. The presence of imperfect quality items has a significant impact on the order quantity. This finding should motivate production and operations managers to pay attention to quality checks and ensure that the percentage of imperfect quality items is kept as low as possible.

The model presented in this chapter can be extended by incorporating some of the popular extensions to the classic EOQ model, such as time inflation, trade credits, partial backordering of shortages and quantity discounts, among others. Furthermore, the proposed inventory system assumed that the screening process is 100% effective at separating good and poor quality items, this along with the inclusion of learning effects in the screening process are other possible areas for further development.

Chapter 4

Economic order quantity model for growing items with two growing and storage facilities†

4.1 Introduction

The classic Economic Order Quantity (EOQ) model has been the workhorse of the inventory management system, but its application has been limited by the number of assumptions that may not be quite realistic. This has been a critical weakness of this model. As a result, the model has been extended in numerous ways to reflect some more realistic situations. In this chapter, an attempt is made to create another model by relaxing two implicit assumptions of the classic EOQ: first is the assumption that the ordered items do not grow (i.e. they do not experience an increase in weight); and the other is the assumption that all the inventoried items are raised in a single growing facility and stored in a single storage facility (or warehouse) with unlimited capacity.

While Harris (1913)'s and many subsequent authors' models assume that ordered items do not undergo any physical changes, this assumption is not true for many items. Certain inventory items undergo changes during the course of a planning horizon such as amelioration, deterioration or growth. Numerous inventory models which considered or relaxed these assumptions, either solely or in combination with various other assumptions have been proposed. However, these two assumptions do not seem to have been considered together in any known article. The individual assumptions of item growth and two storage facilities were first incorporated to inventory theory through the works of Rezaei (2014) and Hartley (1976) respectively.

Rezaei (2014) studied an inventory system for items which experience an increase in weight (i.e. grow) during an inventory planning cycle. In addition to the usual holding and setup costs included in the classic EOQ model, this model also included a cost associated with feeding and raising the items so that they can grow. Zhang et al. (2016) studied an inventory system for growing items in an environment where carbon emissions from a company's operations are taxed. Nobil et al. (2018) developed an inventory model for growing items with shortages.

The use of a second (and rented) warehouse is another area in inventory theory which has received considerable interest in recent years. The classic EOQ model implicitly as-

†A modified version of this chapter has been submitted to *Opsearch* for review.

sumes that all the inventory is kept in a single storage facility. As such, it is implied that the storage facility has unlimited capacity. However, this is not realistic in practice because there is always limitation on the available storage space in a warehouse. Furthermore, in some practical situations such as temporary or bulk discounts at the supplier, or for economies of scale, the company might order more units than they can keep in their own warehouse. In such situations, the excess inventory may be kept at a rented warehouse with abundant storage capacity. The unit holding costs in the rented warehouse are higher than those in the company-owned warehouse because of the additional costs of maintenance, materials handling and transportation. Consequently, the company clears the stock in the rented warehouse before consuming stock in its own warehouse. Hartley (1976) proposed the first inventory system with two storage facilities. Sarma (1987) presented a two-warehouse inventory model for a perishable item. Pakkala and Achary (1991) developed an EOQ model for an item with exponential deterioration, probabilistic demand and shortages. Goswami and Chaudhuri (1992) investigated the effect of a linear trend in demand on the two-warehouse inventory model. Bhunia and Maiti (1998) presented an EOQ model for a deteriorating item with a linear trend in demand and two storage facilities. Ray et al. (1998) developed a two-warehouse inventory for items with stock-level-dependent demand rate. Lee and Ma (2000) studied an inventory system for deteriorating items with two storage facilities and time-dependent demand under a continuous release rule. Kar et al. (2000) developed a two-warehouse inventory model for an item with a linearly-increasing demand rate and shortages under the assumption that consecutive cycle times are in arithmetic progression. Based on the logic behind Hartley (1976)'s two-warehouse model, Zhou (2003) developed a multi-warehouse inventory model with shortages and time-dependent demand. Yang (2004) presented an EOQ model for an item with two levels of storage and shortages which occur at the beginning of a replenishment cycle under inflationary conditions. Zhou and Yang (2005) studied an inventory system for an item with inventory level-dependent demand and two storage facilities. Lee (2006) presented an EOQ model for items with two storage facilities, deterioration and a First-In-First-Out (FIFO) dispatching policy. Chung and Huang (2007) studied a two-warehouse inventory system under conditions of permissible delay in payments. Rong et al. (2008) developed a two-warehouse inventory model with shortages, deterioration and fuzzy lead time. Chung et al. (2009) studied the two-warehouse inventory system under the assumption that a certain fraction of the ordered items is of imperfect quality. Lioa and Huang (2010) developed an inventory model for a perishable item two storage facilities under the assumption that the supplier permits delayed payments through trade credit financing. Yadav et al. (2012) applied a genetic algorithm to solve a two-warehouse inventory model with inventory-level-dependent demand and deterioration under a fuzzy environment. Yang and Chang (2013) developed a two-warehouse inventory model for deteriorating items with partial backordering of shortages and trade credit financing under inflation. Ouyang et al. (2015) studied a vendor-buyer inventory system for an item with two storage facilities under conditions of permissible delay in payments. Palanivel and Uthayakumar (2016) studied a finite-horizon inventory system for a deteriorating item with partially backordered demand and two storage facilities under the effects of inflation and the time value of money. Jaggi et al. (2017) presented an EOQ model for deteriorating items with imperfect quality taking into account trade credit financing and the use of an additional rented warehouse. Palanivel et al. (2018) presented a two-warehouse inventory model for a perishable item with shortages and a demand rate which depends on promotional initiatives aimed at increasing sales.

Table 4.1: Gap analysis of related works in literature

References	**Characteristics of the inventory system under study**						**Solution technique**	
	Regular items	Growing items	Two growing facilities	Two storage facilities	Carbon tax	Shortage	Closed form	Heuristic
Harris (1913)	✓						✓	
Hartley (1976)	✓			✓			✓	
Rezaei (2014)		✓						✓
Zhang et al. (2016)		✓			✓			✓
Nobil et al. (2018)		✓				✓		✓
This chapter		✓	✓	✓				✓

Previous studies on inventory management for growing items had not considered the effect of capacity limits on the growing facility (used for rearing the live inventory) and the storage facility (used for holding the slaughtered inventory). This chapter addresses this gap in literature, as shown in Table 4.1, by formulating an EOQ model for growing with two growing and storage facilities.

4.2 Problem definition

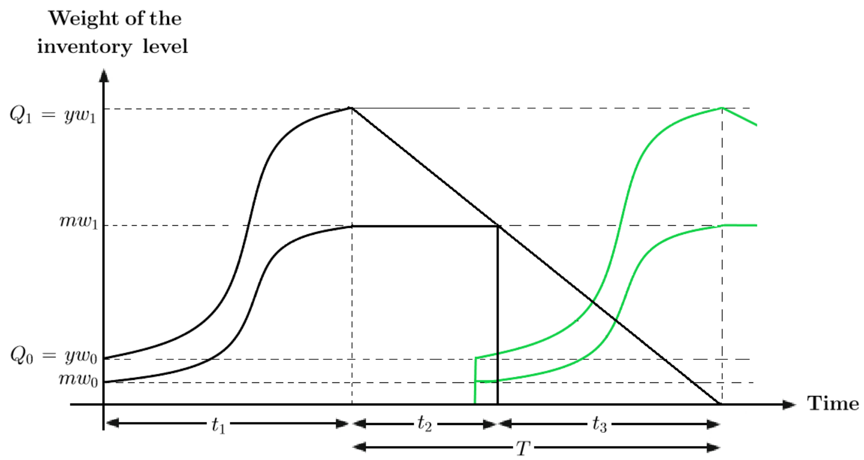


Figure 4.1: Inventory system behaviour for growing items with two growing and storage facilities

The proposed inventory system considers a situation where a company orders a certain number of live newborn growing items. Growing inventory items are items which experience an increase in weight during the inventory planning period (Rezaei, 2014). The proposed inventory system is illustrated in Figure 4.1. Each complete inventory cycle has two distinct regions. The first region is the growth period, where the company raises the items by feeding them, and the second region is the consumption period where the slaughtered items are sold. The growth and feeding regions are represented by periods t_1 and T in Figure 4.1. respectively. The consumption period of one cycle and the growth period of the next cycle end at the same time. This means that by the time all the slaughtered inventory is used up, the items in the next item would be fully grown and ready for sale. This is in keeping with the zero inventory property of an inventory system if possible. After receiving the ordered live items, the company feeds the items so that they grow to a specific weight after which they are slaughtered. However, the company's growing facilities can only accommodate a limited number (shown as m in Figure 4.1) of live items. This prompts the company to rent a growing facility from a third party in case the order quantity exceeds the available space. The same capacity limitation also applies to the company's warehouse or storage facility which is used for keeping the slaughtered items. The cost of the items' food in both facilities is assumed to be the same, however, the rented facility's inventory holding costs are higher than those at the company's owned facility.

The company needs to determine the optimal number of live newborn items to order at the beginning of a growing cycle and the frequency of placing orders which minimises the total cost, the sum of the setup, purchasing, feeding and holding costs. The proposed inventory system is studied as a cost minimisation problem, with the total cost as the objective function, and the cycle time (or the order quantity) as the decision variable(s) - since both are jointly determined.

4.3 Notations and assumptions

4.3.1 Notations

The following notations are employed when formulating the mathematical model:

Table 4.2: Notations used in the formulation of the mathematical model

Symbol	Description
y	Number of ordered items per cycle
T	Cycle time
m	Capacity of the company-owned facility
w_0	Approximated weight of each newborn item
w_1	Approximated weight of each grown item at the time of slaughtering
Q_t	Total weight of inventory at time t
$f(t)$	Feeding function for each item
b	Growth rate per item in weight units per unit time
p	Purchasing cost per weight unit
h_m	Holding cost per weight unit per unit time in the owned warehouse
h_r	Holding cost per weight unit per unit time in the rented warehouse
K	Setup cost per cycle
D	Demand in weight units per unit time
c	Feeding cost per weight unit per unit time
t_1	Growing period
t_2	Consumption period in the rented warehouse
t_3	Consumption period in the owned warehouse
t_s	Setup time

4.3.2 Assumptions

The following assumptions were made when formulating the mathematical model:

- The ordered items are capable of growing prior to being slaughtered.
- A single type of item is considered.
- A cost is incurred for feeding and growing the items.
- The cost of feeding the items is proportional to the weight gained by the items.
- Holding costs are incurred for the duration of the consumption period.
- Items arrive at both facilities at the same time.
- The company-owned growing facility and warehouse have limited capacities of m units.
- The rented growing facility and warehouse have unlimited capacity.
- The holding cost in the rented facility are higher than those in the owned warehouse.
- Items in the company owned warehouse are only sold after those in the rented warehouse are sold out.
- Demand is a deterministic constant.

4.4 Model formulation

4.4.1 General mathematical model

Figure 4.1 depicts the proposed inventory system. At the beginning of a growing cycle, a company purchases y newborn items, each weighing w_0, which are capable of growing. However, the company's growing and storing facilities only have the capacity to grow and store m items. The company makes use of rented facilities to grow and store the excess $(y - m)$ items. This means that at the beginning of a growing cycle, the company owned growing facility receives m newborn items and at this point the total weight of the inventory is mw_0. Likewise the total weight of the inventory in the rented facility at this point is $(y-m)w_0$. The total weight of the inventory in both facilities at this point, Q_0, is the sum of the total weight of the inventory in both facilities (*i.e.* $Q_0 = yw_0$). The items are then fed and they grow for the period t_1. The weight of each fully grown item w_1 is a function of time. The amount of food consumed by the items during the growth period is modelled by the function $f(t)$. At the end of the growth period, each item would have grown to a target weight of w_1. At this point, the total weight of the inventory in both facilities is $Q_1 = yw_1$. At the end of the growing period, the items are slaughtered. The total weight of the inventory at the time of slaughter (or end of the growing period) in the owned and the rented facilities are mw_1 and $(y - m)w_1$ respectively.

As a result of higher holding costs in the rented warehouse, inventory kept in that particular warehouse is cleared prior to consuming inventory in the company-owned warehouse. The consumption period in the rented warehouse, t_2, is determined from the annual demand, D, and the weight of the inventory in the rented warehouse, $(y - m)w_1$, as

$$t_2 = \frac{(y - m)w_1}{D}. \tag{4.1}$$

Likewise, the weight of the inventory in the company-owned warehouse, mw_1, and the demand rate, D, are utilized in determining an expression for the length of the consumption period in the owned warehouse, t_3, as

$$t_3 = \frac{mw_1}{D}. \tag{4.2}$$

The cycle length, T, is the sum of the consumption periods in both facilities. It is given by

$$T = \frac{yw_1}{D}. \tag{4.3}$$

An expression for the order quantity, y, is determined from Equation (4.3) as

$$y = \frac{DT}{w_1}. \tag{4.4}$$

The objective of the proposed inventory model is to determine the optimal order quantity which minimizes total cost (TC), which is the sum of the purchasing (PC), setup (SC), feeding (FC) and holding (HC) costs. The total cost per cycle is thus

$$TC = PC + SC + FC + HC. \tag{4.5}$$

4.4.1.1 Purchasing cost per cycle

At beginning of each cycle, the company purchases y newborn items at a cost of p per weight unit. At the time of receiving the order, each item weighs w_0 . The purchasing cost per cycle is determined as

$$PC = pyw_0. \tag{4.6}$$

4.4.1.2 Setup cost per cycle

A cost of K is incurred for setting up the growing/feeding facilities at the beginning of each cycle. Hence,

$$SC = K. \tag{4.7}$$

4.4.1.3 Holding cost per cycle

The holding cost per cycle is determined from Figure 4.1 by multiplying the holding cost per unit weight per unit time in the respective warehouses, h_r and h_m, by the relevant area under the consumption period in the graph. The holding costs in the rented and company-owned warehouses are given, respectively, by

$$HC_r = h_r \left[\frac{(y-m)^2 w_1^2}{2D} \right] \tag{4.8}$$

$$HC_m = h_m \left[\frac{m(y-m)w_1^2}{D} + \frac{m^2 w_1^2}{2D} \right]. \tag{4.9}$$

The total holding cost per cycle is the sum of the holding costs in the rented and owned warehouses. It is given by

$$HC = h_r \left[\frac{(y-m)^2 w_1^2}{2D} \right] + h_m \left[\frac{m(y-m)w_1^2}{D} + \frac{m^2 w_1^2}{2D} \right]. \tag{4.10}$$

4.4.1.4 Food procurement cost per cycle

Since the items in both company-owned and rented facilities have similar growth rates, it is assumed that they are fed the same type of food which is procured from the same supplier. Hence, the cost of items' food, c, is the same regardless of the facility. The cost of the items food (c), the general feeding function $f(t)$, the duration of the feeding period (which equals the growing period, i.e. t_1) and the quantity of items in the rented facilities, $(y-m)$, are used to determine the feeding cost in the rented facility as

$$FC_r = c(y-m)\int_0^{t_1} f(t)\,dt. \tag{4.11}$$

Considering the amount of items grown and fed in the company-owned facility, m, the feeding cost in the owned facility is determined as

$$FC_m = cm\int_0^{t_1} f(t)\,dt. \tag{4.12}$$

The summation of Equations (4.11) and (4.12) yields an expression for the feeding cost per cycle in both facilities as

$$FC = cy\int_0^{t_1} f(t)\,dt. \tag{4.13}$$

4.4.1.5 Total cost function

Equations (4.6), (4.7), (4.10) and (4.13) are substituted into Equation (4.5) to yield an expression for the total cost as

$$TC = pyw_0 + K + h_r\left[\frac{(y-m)^2 w_1^2}{2D}\right] + h_m\left[\frac{m(y-m)w_1^2}{D} + \frac{m^2 w_1^2}{2D}\right] + cy\int_0^{t_1} f(t)\,dt. \tag{4.14}$$

The total cost per unit time, TCU, is computed using Equations (4.14) and (4.3) as

$$TCU = \frac{TC}{T} = \frac{pw_0 D}{w_1} + \frac{KD}{yw_1} + h_r\left[\frac{(y-m)^2 w_1}{2y}\right] + h_m\left[\frac{m(y-m)w_1}{y} + \frac{m^2 w_1}{2y}\right] + \frac{cD}{w_1}\int_0^{t_1} f(t)\,dt. \tag{4.15}$$

Equation (4.15) is a general expression for the total cost per unit time which can be used for different types of growing items. It has two general functions, namely, the feeding ($f(t)$) and the growth ($f(w_1|w_0)$) functions. Because these functions are different for different growing items, it is very difficult to solve the model for a general case which works for all growing items. Getting around this problem involves setting up a specific model which makes use of specific growth and feeding functions. In the available literature on EOQ models for growing items, there are two approaches to solving the model for a specific case. Rezaei (2014) and Zhang et al. (2016) used growth and feeding functions for broiler chickens, which were first developed by Richards (1959) and Goliomytis et al. (2003) respectively. These functions were complex and consequently both those models were solved through numerical techniques, with Rezaei (2014) resorting to a heuristic which applies the Newton-Raphson method. Nobil et al. (2018) assumed a linear growth rate for the items, which meant that the feeding function (which is assumed to be dependent on the weight gained by the items) could easily be determined from the inventory behaviour graph (in a manner similar to determining the holding cost). The model presented by Nobil et al. (2018) was solved using a relatively simpler heuristic as compared to the heuristic employed in the works by Rezaei (2014) and Zhang et al. (2016). Because of the reduced computational complexity in the model by Nobil et al. (2018), their assumption of a linear growth rate for the items is adapted to set up the specific mathematical model.

4.4.2 Specific mathematical model: Case of a linear growth function

Suppose that the growth function of each item, w_1, is modelled by a linear function. This means that each item grows at a constant rate of b weight units per unit time. Noting that the initial weight of the newborn items at the time of order receipt is w_0 and the period of growth is t_1, an expression for each item's growth function can be formulated as

$$w_1 = w_0 + bt_1. \tag{4.16}$$

When considering all the ordered items, Equation (4.16) becomes

$$yw_1 = yw_0 + ybt_1, \tag{4.17}$$

from which Figure 4.2, depicting the behaviour of the inventory system over time, is constructed. An expression for the duration of the growth period is determined from Equation (4.16) as

$$t_1 = \frac{w_1 - w_0}{b}. \tag{4.18}$$

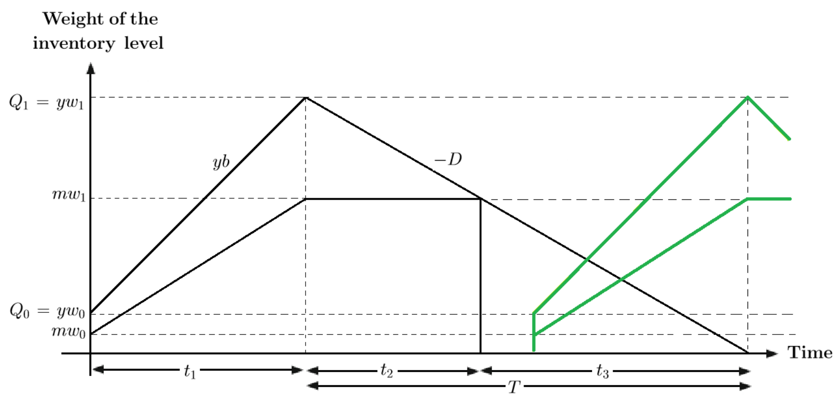


Figure 4.2: Inventory system behaviour for growing items with two growing and storage facilities and a linear growth function

4.4.2.1 Food procurement cost per cycle

The cost of procuring food in the rented facility is determined from Figure 4.2 through multiplying the cost of food per weight unit per unit time, c, with the duration of the feeding (*i.e.* growing) period, t_1, and the difference in weight of the inventory at the beginning (yw_0) and at the end (yw_1) of the growing period.

$$FC_r = c\left[\frac{t_1\left[(y-m)w_1 - (y-m)w_0\right]}{2}\right] \tag{4.19}$$

By substituting Equation (4.18), Equation (4.19) becomes

$$FC_r = c\left[\frac{(w_1 - w_0)\left[(y-m)w_1 - (y-m)w_0\right]}{2b}\right]. \tag{4.20}$$

Likewise, the food procurement cost in the owned facility is

$$FC_m = c\left[\frac{t_1(mw_1 - mw_0)}{2}\right] = c\left[\frac{(w_1 - w_0)(mw_1 - mw_0)}{2b}\right]. \tag{4.21}$$

The cost of feeding the items in both facilities per cycle is determined by adding Equations (4.20) and (4.21). Hence,

$$FC = c\left[\frac{y(w_1 - w_0)^2}{2b}\right]. \tag{4.22}$$

4.4.2.2 Total cost function

The other cost components (i.e. the purchasing, setup and holding costs) are not affected by the assumption of a linear growth function and thus they remain the same as those in Equations (4.6), (4.7) and (4.10). These three cost components along with the new food procurement cost, given in Equation (4.22), are summed to give an expression for the total cost per cycle as

$$TC = pyw_0 + K + h_r\left[\frac{(y-m)^2 w_1^2}{2D}\right] + h_m\left[\frac{m(y-m)w_1^2}{D} + \frac{m^2 w_1^2}{2D}\right] + c\left[\frac{y(w_1 - w_0)^2}{2b}\right]. \tag{4.23}$$

For computational convenience, Equation (4.4) is substituted into Equation (4.23) and the result is

$$TC = \frac{pDTw_0}{w_1} + K + h_r\left(\frac{DT^2}{2}\right) + \left(h_m - h_r\right)\left(\frac{2DTmw_1 - m^2 w_1^2}{2D}\right) + \frac{cDT\left(w_1 - w_0\right)^2}{2bw_1}. \tag{4.24}$$

The total cost per unit time (TCU) is computed, by dividing by Equation (4.24) the cycle time, as

$$TCU = \frac{TC}{T} = \frac{pDw_0}{w_1} + \frac{K}{T} + h_r\left(\frac{DT}{2}\right) + \left(h_m - h_r\right)\left(mw_1 - \frac{m^2 w_1^2}{2DT}\right) + \frac{cD\left(w_1 - w_0\right)^2}{2bw_1}. \tag{4.25}$$

4.4.2.3 Model constraint

The proposed inventory system is controlled by the following constraint which ensures that the slaughtered items are ready for consumption on time:

Constraint: In order to ensure that the slaughtered items are ready for sale on time (i.e. at the beginning of the consumption period T), the setup time and the length of the growth period should be less than or equal to the consumption period. A restriction on the consumption period, T, is formulated as

$$t_1 + t_s \leq T. \tag{4.26}$$

Through the substitution of the expression for t_1 from Equation (4.18), Equation (4.26) becomes

$$T \geq \left\{ \frac{w_1 - w_0}{b} + t_s = T_{min} \right\}. \tag{4.27}$$

4.4.2.4 Mathematical formulation of the EOQ model for growing items with two growing and storage facilities

The mathematical formulation for the the proposed inventory system is formulated by using the objective function in Equation (4.25) and the constraint in Equation (4.27) as

$$Min \quad \left\{ TCU = \frac{pDw_0}{w_1} + \frac{K}{T} + h_r\left(\frac{DT}{2}\right) + \left(h_m - h_r\right)\left(mw_1 - \frac{m^2w_1^2}{2DT}\right) + \frac{cD\left(w_1 - w_0\right)^2}{2bw_1} \right\} \tag{4.28}$$

$$s.t. \quad T \geq T_{min}$$
$$T \geq 0.$$

4.4.3 Solution

4.4.3.1 Determination of the decision variables

The value of the cycle time, T, which minimises total costs per unit time, TCU, is determined as

$$\frac{\partial TCU}{\partial T} = -\frac{K}{T^2} + \frac{h_rD}{2} + \left(h_m - h_r\right)\left(\frac{m^2w_1^2}{DT^2}\right) = 0 \rightarrow T = \sqrt{\frac{2KD + m^2w_1^2\left(h_r - h_m\right)}{h_rD^2}}. \tag{4.29}$$

The order quantity which minimises the total costs per unit time, computed by substituting T from Equation (4.29) into Equation (4.4), is given by

$$y = \sqrt{\frac{2KD + m^2w_1^2\left(h_r - h_m\right)}{h_rw_1^2}}. \tag{4.30}$$

4.4.3.2 Proof of convexity of the objective function

So as to show that the objective function, given in Equation (4.25), has a unique solution, the grad of that function is calculated in order to identify the optimum point and the Hessian matrix of the function is shown to be positive. The location of the optimum point from the grad is shown by

$$\frac{\partial TCU}{\partial T} = -\frac{K}{T^2} + \frac{h_r D}{2} + \left(h_m - h_r\right)\left(\frac{m^2 w_1^2}{DT^2}\right). \tag{4.31}$$

The Hessian matrix of the objective function, given by

$$\begin{bmatrix} \frac{\partial^2 TCU}{\partial T^2} & \frac{\partial^2 TCU}{\partial T \partial y} \\ \\ \frac{\partial^2 TCU}{\partial T \partial y} & \frac{\partial^2 TCU}{\partial y^2} \end{bmatrix}, \tag{4.32}$$

is shown, in Equation (4.33), to be positive semi definite.

$$\begin{bmatrix} \frac{K}{T^3} + \left(h_r - h_m\right)\left(\frac{m^2 w_1^2}{DT^3}\right) & \frac{K}{T^3} + \left(h_r - h_m\right)\left(\frac{m^2 w_1^2}{DT^3}\right) \\ \\ \frac{K}{T^3} + \left(h_r - h_m\right)\left(\frac{m^2 w_1^2}{DT^3}\right) & 0 \end{bmatrix} \tag{4.33}$$

The quadratic form of the objective function is determined from the Hessian matrix as

$$\begin{bmatrix} T & y \end{bmatrix} \begin{bmatrix} \frac{K}{T^3} + \left(h_r - h_m\right)\left(\frac{m^2 w_1^2}{DT^3}\right) & \frac{K}{T^3} + \left(h_r - h_m\right)\left(\frac{m^2 w_1^2}{DT^3}\right) \\ \\ \frac{K}{T^3} + \left(h_r - h_m\right)\left(\frac{m^2 w_1^2}{DT^3}\right) & 0 \end{bmatrix} \begin{bmatrix} T \\ y \end{bmatrix}$$
$$= \frac{K}{T} + 2\left[\frac{Ky}{T^2} + \left(h_r - h_m\right)\frac{y m^2 {w_1}^2}{DT^2}\right] \geq 0 \tag{4.34}$$

The quadratic form of the objective function is shown to be positive in Equation (4.34) which indicates that the objective function is convex.

4.4.3.3 Computational algorithm

The following optimization algorithm is proposed for determining a solution to the proposed inventory system:

Step 1 Compute T_{min} using Equation (4.27).

Step 2 Check the problem's feasibility. The problem is feasible provided that $T_{min} \geq 0$. If it is feasible proceed to Step 3, otherwise proceed to Step 6.

Step 3 Compute T using Equation (4.29).

Step 4 $T^* = T$ provided that $T \geq T_{min}$, otherwise $T^* = T_{min}$.

Step 5 Compute y^* and TCU^* using Equations (4.30) and (4.25) respectively considering the value of T^*.

Step 6 End.

4.5 Numerical results

4.5.1 Numerical example

The proposed inventory system is analysed through a numerical example utilising the following parameters, mostly adapted from a study by Nobil et al. (2018):

Demand rate, D =1 000 000 g/year

Setup cost, K =1 000 ZAR/cycle

Holding cost in the owned warehouse, h_m =0.04 ZAR/g/year

Holding cost in the rented warehouse, h_r =0.06 ZAR/g/year

Capacity of the company-owned facility, m =100

Approximated weight of newborn (one day old) chick, w_0 =53 g/chick

Approximated weight of chicken at the time of slaughtering, w_1 =1 267 g/chicken

Growth rate, b =15 330 g/chick/year

Setup time, t_s =0.01 year

Purchase cost, p =0.025 ZAR/g

Feeding cost, c =0.08 ZAR/g/year

Solution procedure application: The proposed solution procedure is illustrated by applying it to the numerical example. The procedure is outlined as follows:

Step 1 Compute T_{min} using Equation (4.27).

$$T_{min} = \frac{1267 - 53}{15330} + 0.01 = 0.0891$$

Step 2 The problem is feasible since $T_{min} \geq 0$, proceed to Step 3.

Step 3 Compute T using Equation (4.29).

$$T = \sqrt{\frac{2 \times 1000 \times 1000000 + \left(100^2 \times 1000000^2\right)\left(0.06 - 0.04\right)}{\left(0.06 \times 1000000^2\right)}} = 0.1967$$

Step 4 Since $T \geq T_{min}$, $T^* = T = 0.1967$

Step 5 Compute y^* and TCU^*.

$$y^* = \sqrt{\frac{2 \times 1000 \times 1000000 + \left(100^2 \times 1000000^2\right)\left(0.06 - 0.04\right)}{\left(0.06 \times 1267^2\right)}} = 155.2354$$

$$TCU^* = \frac{0.025 \times 1000000 \times 53}{1267} + \frac{1000}{13347.91} + 0.06\left(\frac{1000000 \times 13347.91}{2}\right) + \left(0.04 - 0.06\right)\left(100 \times 1267 - \frac{100^2 \times 1267^2}{2 \times 1000000 \times 13347.91}\right) + \frac{0.08 \times 1000000\left(1267 - 53\right)^2}{2 \times 15330 \times 1267} = 13347.91$$

Step 6 End.

Table 4.3: Summary of the results from the numerical example

Variable	Units	Quantity
t_1	year	0.0792
t_2	year	0.0700
t_3	year	0.1267
T^*	year	0.1967
y^*	items	156
TCU^*	R/year	13 347.91

Based on the results of the numerical example, some of which are summarised in Table 4.3, the company should place an order for 156 day-old chicks at the beginning of each cycle. Since the owned facility can only grow and store 100 items, 56 items will be grown, slaughtered and stored at a rented facility. The items should be grown for 0.0792 years (29 days). All the slaughtered items in the rented warehouse will be consumed after 0.0700 years (26 days) and those in the owned warehouse will be sold out after 0.1267 years (46 days). Orders should be replenished every 0.1967 years (72 days). Following this optimal inventory policy, the company will incur total costs of 13 347.91 ZAR per year.

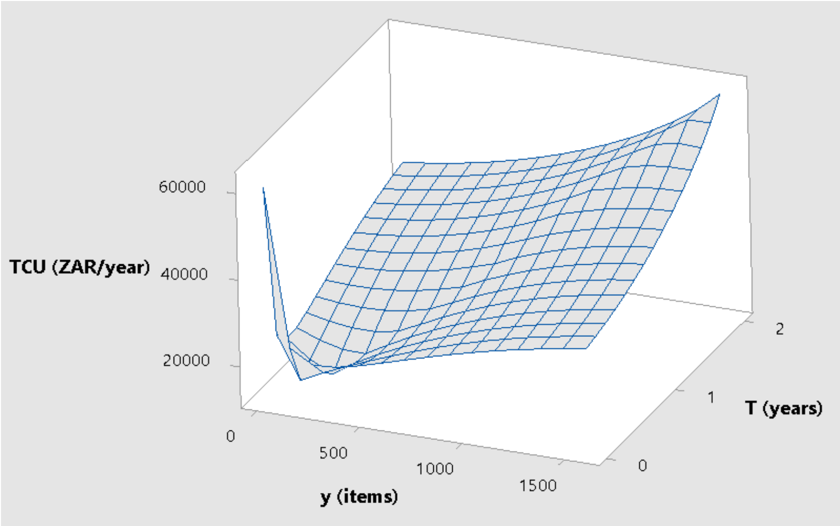


Figure 4.3: Graph of total cost per unit time, cycle time and order quantity

The changes to the objective function as a result of changes in the decision variables are illustrated in Figure 4.3, which also shows that the objective function is convex. The total cost per unit time decreases with the cycle time until it reaches minimum at the optimal cycle time and order quantity (i.e. the EOQ).

4.5.2 The effect of limited growing and storage capacity of the owned facility

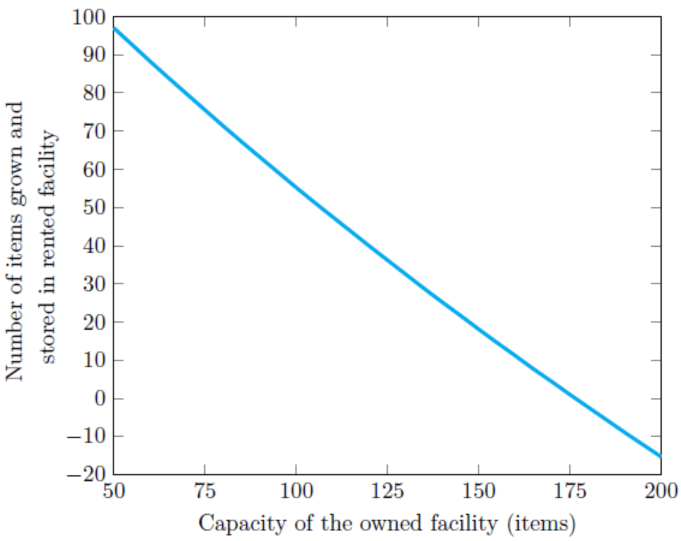


Figure 4.4: The effect of limited growing and storage capacity on the lot size

The effect of the limited capacity of the company-owned facility on the order quantity is investigated by comparing the EOQ under the assumption that there is no capacity limit on the company-owned growing and storage facilities with the EOQ from the proposed inventory system. In order to determine the base EOQ for growing items (i.e. the EOQ under the assumption that there is no capacity limit on the company owned growing and storage facilities), m is set to zero and h_r is set to h_m. The capacity of the company-owned facility is increased gradually and the EOQ is determined at various capacities. The EOQ's at various capacities are compared to the base EOQ for growing items by determining the number of number of items that need to be grown and stored at the rented facility. As the capacity of the owned facility increases, the number of items grown and stored in the rented facility increases until the capacity of the owned facility equals the base EOQ for growing items (shown by the negative amount of items after the 176 items mark in Figure 4.4). After this point, the owned facility has excess growth and storage space which is not utilized. While the company does not need to rent additional capacity, it has unused capacity in its owned facility which also has a negative cost implication.

4.5.3 Sensitivity analysis

Sensitivity analyses are performed on the major input parameters in order to investigate their impacts on the decision variables and to provide managerial insights for improving inventory management.

Table 4.4: Changes to y^* and TCU^* due to changes in c

% change in	**EOQ**		**TCU**	
feeding cost	items	% change	ZAR/year	% change
-50	156	0	11 826	-11.4
-37.5	156	0	12 213	-8.5
-25	156	0	12 587	-5.7
-12.5	156	0	12 974	-2.8
0	156	0	13 348	0
12.5	156	0	13 722	2.8
25	156	0	14 109	5.7
37.5	156	0	14 496	8.6
50	156	0	14 870	11.4

Table 4.4 shows the results of the sensitivity analysis of the EOQ and the total costs to the feeding cost. With regards to the order quantity, increasing the feeding cost does not change the EOQ at all. From a cost perspective, increasing the feeding cost has a negative effect on inventory management because it leads to increased total costs. The reverse is also true and this presents inventory managers with an opportunity to reduce total costs without the need to hold additional stock. By reducing feeding costs, either through better procurement practices or reducing the items' rations, managers can save significant amounts of money without increasing the risk of stock-outs.

Table 4.5: Changes to y^* and TCU^* due to changes in K

% change in setup cost	**EOQ** items	**EOQ** % change	**TCU** ZAR/year	**TCU** % change
-50	118	-24.6	10 451	-21.7
-37.5	128	-17.7	11 252	-15.7
-25	138	-11.4	12 000	-10.1
-12.5	147	-5.5	12 694	-4.9
0	156	0	13 348	0
12.5	164	5.2	13 962	4.6
25	172	10.2	14 563	9.1
37.5	179	15	15 123	13.3
50	187	19.6	15 657	17.3

The setup cost has significant impacts on both the EOQ and the total inventory costs, as shown in Table 4.5. Decreasing the setup cost decreases the total costs. Managers should pay close attention to the setup cost because trying to reduce this cost by ordering less frequently leads to an increase in the order size. Larger order sizes mean higher holding costs and the need to store more items in the rented facility which further escalates the costs.

Table 4.6: Changes to y^* and TCU^* due to changes in h_m

% change in owned facility holding cost	**EOQ** items	**EOQ** % change	**TCU** ZAR/year	**TCU** % change
-50	166	6.7	11 599	-13.1
-37.5	164	5.1	12 040	-9.8
-25	161	3.4	12 480	-6.5
-12.5	159	1.7	12 291	-3.2
0	156	0	13 348	0
12.5	153	-1.7	13 775	3.2
25	151	-3.4	14 216	6.5
37.5	148	-5.1	14 656	9.8
50	146	-6.7	15 097	13.1

Table 4.6 shows that as the holding cost in the owned facility increases, the EOQ decreases and the total total costs increase. The increase in total costs is due to the fact that holding costs are one of the main components of the total cost function. In order to reduce costs, management should reduce the holding costs in their facility.

Table 4.7: Changes to y^* and TCU^* due to changes in h_r

% change in rented facility holding cost	**EOQ**		**TCU**	
	items	% change	ZAR/year	% change
-50	196	25.9	12 774	-4.3
-37.5	181	16.2	12 974	-2.8
-25	171	9.3	13 121	-1.7
-12.5	162	4.1	13 241	-0.8
0	156	0	13 348	0
12.5	151	-3.3	13 428	0.6
25	147	-6	13 508	1.2
37.5	143	-8.3	13 575	1.7
50	140	-10.3	13 628	2.1

Table 4.7 shows that as the rented facility holding cost increases the total costs increase slightly. The increases in the total costs are not as dramatic as those caused by similar percentage increases in the owned facility holding cost despite the fact that the rented facility charges more than the owned facility. This is because the amount of items kept in the rented warehouse is small and those items sell out within a short space of time. If the items kept in the rented facility increases, the increase in the total costs will be much greater. In order to reduce total costs, management should keep as little inventory as possible in the rented facility. While better results can be achieved by not keeping any inventory in the rented warehouse, exceptional circumstances might necessitate renting space and in such instances, small amounts of amounts should be kept in the rented space and it is very important for management to start clear this stock ahead of the stock in their own facility.

Table 4.8: Changes to y^* and TCU^* due to changes in m

% change in capacity of owned facility	**EOQ**		**TCU**	
	items	% change	ZAR/year	% change
-50	148	-5.3	13 989	4.8
-37.5	149	-4.3	13 788	3.3
-25	151	-3.1	13 615	2
-12.5	154	-1.6	13 468	0.9
0	156	0	13 348	0
12.5	159	1.8	13 241	-0.8
25	162	3.8	13 161	-1.4
37.5	165	6	13 108	-1.8
50	169	8.3	13 054	-2.2

From Table 4.8, it is evident that an increase in the capacity of the owned facility results in an increase in the EOQ and a decrease in the total costs. Managers should invest in expanding their own facilities' capacities because of the potential costs savings. While increasing capacity requires significant amounts of capital investment initially, the benefits in the long term are that the company will not incur high holding costs charged by third parties. The downside to increasing capacity is that if market conditions change negatively and the company's demand reduces significantly, the company will have ample unutilised capacity. However, if market conditions change for the better the company will be better positioned since they would have invested in extra capacity.

Table 4.9: Changes to y^* and TCU^* due to changes in w_1

% change in slaughter weight	**EOQ**		**TCU**	
	items	% change	ZAR/year	% change
-50	295	89.3	13 388	0.3
-37.5	239	53.1	13 188	-1.2
-25	202	29.2	13 148	-1.5
-12.5	175	12.4	13 215	-1
0	156	0	13 348	0
12.5	141	-9.5	13 548	1.5
25	130	-16.9	13 775	3.2
37.5	120	-22.9	14 055	5.3
50	113	-27.8	14 362	7.6

Table 4.9 shows that as the slaughter weight increases, the EOQ decreases significantly and the total cost increases. The EOQ decreases because the company slaughters the items at larger weights and thus fewer items are required to meet annual demand. While it might be tempting to meet demand with slightly fewer items, this comes at a cost because total costs increase with the slaughter weight. The increase in total costs is most probably due to the fact that if items are slaughtered at larger weights, they have to grown for a longer period of time or fed significantly more. In order to keep costs down, managers should opt to slaughter the items at smaller weights because they can reduce their costs slightly.

4.6 Conclusion

This chapter incorporates the concepts of the basic two-warehouse inventory model and the EOQ model for growing items to create a new EOQ model for growing items with two growing and storage facilities. The proposed EOQ model is studied through a mathematical model which has cycle time (and order quantity, since they are jointly determined) as the decision variable(s) and the total costs as the objective function. The biggest contribution to the literature on inventory modelling for growing items is the inclusion of a limited capacity for the growing and storage facility. The capacity of the owned facility has a significant effect on the EOQ and the total inventory costs because of the difference in the holding costs charged by both facilities. As the capacity increases, the total costs decrease and the EOQ increases, but increasing capacity is capital intensive and poses financial risks if market conditions change for the worst. On the other hand, if market conditions change for the best and the company has invested in additional capacity, it will be better positioned to deal with increased customer demand without the need to outsource additional growing and storage capacities.

The proposed EOQ model only considered item growth and two growing and storage facilities. It can be extended by considering multiple growing and storage facilities. This model can also be extended by assuming that the growth rates of the items in the owned and rented facilities are different. Furthermore, popular extensions to the basic EOQ model, such as shortages, deterioration, quantity discounts and permissible delay in payments, can be incorporated to the model.

Chapter 5

Economic order quantity model for growing items with incremental quantity discounts‡

5.1 Introduction

While the basic Economic Order Quantity (EOQ) model, developed by Harris (1913), has found some practical applications, it makes a number of assumptions which do not reflect most real life inventory systems. In order to model more realistic systems, various researchers have formulated new EOQ models by amending the model assumptions in some ways. In an attempt to create a new variation of the EOQ model, this chapter proposes an inventory system where the ordered items, purchased from a supplier who offers incremental quantity discounts, are capable of growing during the course of the inventory replenishment cycle.

Rezaei (2014) was the first researcher to incorporate item growth into inventory theory by developing an EOQ model for growing items. Rezaei (2014)'s proposed inventory system had two distinct periods, namely, growing and consumption. During the growing period, the ordered live items are fed and raised until they reach an acceptable weight for sale. The items are then slaughtered, and put on sale during the consumption period. The increase in weight experienced by growing items during the growth period is what differentiates them from conventional items, whose weight does not change if they are not consumed or more items are added to the system (i.e. growth in this context is quantified only through a weight increase). Zhang et al. (2016) developed a carbon-constrained version of the EOQ model for growing items. Building on Rezaei (2014)'s work, Nobil et al. (2018) studied an inventory system for growing items where shortages are allowed and fully backordered.

EOQ models with quantity discounts were first proposed by Hadley and Whitin (1963). Quantity discounts are usually offered by suppliers as a means of encouraging buyers to purchase larger volumes. In inventory theory, suppliers usually offer one of two types of quantity discounts. These are all-units quantity discounts, which result in reduced purchasing cost for the entire order if larger volumes are ordered; and incremental quantity discounts where the reduced purchasing cost only applies to items

‡A modified version of this chapter has been submitted to *Journal of Industrial Engineering International* for review.

bought above a certain quantity. Lal and Staelin (1984) studied an integrated vendor-buyer inventory where the vendor offers incremental quantity discounts to the buyer. Dada and Srikanth (1987) developed a lot sizing model with quantity discounts and price-dependent holding cost. Abad (1988) presented an EOQ model with incremental quantity discounts and a linear demand function. Federgruen and Lee (1990) developed the Dynamic Economic Lot (DEL) version of the basic EOQ model with both all-units and incremental quantity discounts. Guder et al. (1994) studied a capacitated multi-item inventory model with incremental quantity discounts. Tersine et al. (1995) developed an inventory model with incremental quantity discounts and freight discounts offered on the transportation cost. Rubin and Benton (2003) presented a multi-item inventory model with incremental quantity discounts and space constraints. Mahmood and Kindi (2006) presented an inventory model for an item with shortages considering both all-units and incremental quantity discounts. Haksever and Moussourakis (2008) formulated a multi-item model with incremental quantity discounts and budget constraints. Mendoza and Ventura (2008) incorporated transportation costs into the EOQ model under both all-units and incremental quantity discount structures. Bai (2011) formulated an inventory model considering quantity discounts and multiple suppliers. Lee (2013) studied an integrated supplier-vendor inventory system for supplier selection considering both all-units and incremental quantity discounts. Archetti et al. (2014) modelled an inventory system where incremental and all-units discounts are offered simultaneously, the former being offered by a supplier for the ordered items while the latter are offered by the transportation company for transporting the items. Taleizadeh et al. (2015) formulated an EOQ model with incremental quantity discounts under the assumption that shortages are partially backordered. Bohner and Minner (2017) developed a supply chain model with supplier selection, multiple items and quantity discounts with an associated failure risk. Mohammadivojdan and Geunes (2018) studied the single -period inventory model with space constraints under both all-units and incremental quantity discounts. Benton and Park (1996) and Pereira and Costa (2015) presented extensive literature reviews, covering research from the periods 1963 to 1994 and 1995 to 2013 respectively, for lot sizing models under quantity discounts.

Table 5.1: Gap analysis of related works in literature

References	**Characteristics of the inventory system**				
	Conventional items	Growing items	Incremental quantity discounts	Carbon tax	Shortage
Harris (1913)	✓				
Hadley and Whitin (1963)	✓		✓		
Rezaei (2014)		✓			
Zhang et al. (2016)		✓		✓	
Nobil et al. (2018)		✓			✓
This chapter		✓	✓		

A review of literature on inventory management for growing items suggests incremental quantity discounts have not been incorporated into the EOQ model for growing items. This chapter aims to address this gap in literature by developing an EOQ model for growing items with incremental quantity discounts. A comparison of the proposed inventory system and related published works in literature is provided in Table 5.1, which also shows the contribution of this chapter to inventory management research for growing items.

5.2 Problem definition

The proposed inventory system considers a situation where a company orders a certain number of items which are capable of growing during the course of the inventory planning cycle, for example livestock. The supplier of the newborn items offers the purchasing company incremental quantity discounts over fixed price breaks. Under the incremental quantity discounts pricing structure, the discounted purchasing costs only apply to the incremental quantity. Figure 5.1 represents the typical behaviour of an inventory system for growing items. In order for growth to occur the company needs to feed the items. Every replenishment cycle can be divided into two periods, namely the growth and the consumption periods. During the growth period (shown as period t in Figure 5.1), ordered newborn items are fed and raised until they grow to a certain target weight. Once the weight of items reach the target weight, the growth period ends and the items are slaughtered. During the consumption period (shown as period T in Figure 5.1), the slaughtered items are kept in stock and sold to market. The company incurs feeding cost during the growth period and it incurs holding costs for keeping the slaughtered items in stock. The inventory level of one complete inventory cycle finishes up at time T and at that point the items in the next inventory cycle would have completed their growth cycle (i.e. the items in the next cycle will have grown to the target weight and are ready for sale). The company wants to determine the optimal number of newborn items to order at the beginning of the growth cycle in order to minimise total inventory costs (i.e. the sum of the purchasing, setup, feeding and holding costs).

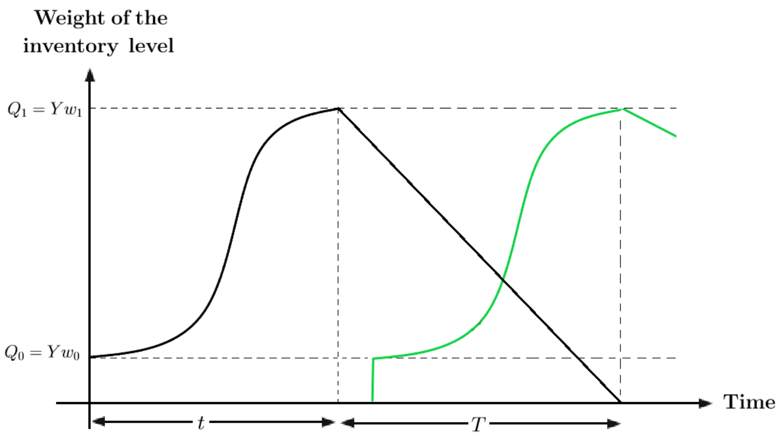


Figure 5.1: Behaviour of an inventory system for growing items

The proposed inventory system is studied through a mathematical model, aimed at determining the optimal order quantity, frequency of order replenishments and slaughter time, which minimise total inventory costs.

5.3 Notations and assumptions

5.3.1 Notations

The following notations are employed when formulating the mathematical model:

Table 5.2: Notations used in the formulation of the mathematical model

Symbol	Description
Y	Number of ordered newborn items per cycle
w_0	Approximated weight of each newborn item
w_1	Approximated weight of each grown item at the time of slaughtering
Q_t	Total weight of inventory at time t
$f(t)$	Feeding function for each item
b	Growth rate per item in weight units per unit time
p_j	Purchasing cost per weight unit at the jth break point
m	Number of break points
y_j	Lower bound for the order quantity for price j
h	Holding cost per weight unit per unit time
K	Setup cost per cycle
D	Demand in weight units per unit time
c	Feeding cost per weight unit per unit time
t	Growing period
T	Cycle length
t_s	Setup time

5.3.2 Assumptions

The following assumptions were made when formulating the mathematical model:

- The ordered items are capable of growing prior to being slaughtered.
- A single type of item is considered.
- A cost is incurred for feeding and growing the items. This cost is proportional to the weight gained by the items.
- Holding costs are incurred for the duration of the consumption period.
- The holding cost per weight unit per unit time is charged as a fixed percentage of the purchasing cost per weight unit.
- The supplier of the live newborn items offers incremental quantity discounts.
- Demand is a deterministic constant.

5.4 Model development

5.4.1 General mathematical model

When a new growing cycle begins, a company purchases Y newborn items, each weighing w_0, which are capable of growing during the replenishment cycle. The total weight of the inventory at this point, Q_0, is determined by multiplying the weight of each of the items by the number of items ordered (i.e. $Q_0 = Yw_0$). The company feeds the items and they grow to a target weight of w_1. This marks the end of the growth period and at this point the items are slaughtered. The total weight of the inventory at the time of slaughter is $Q_1 = Yw_1$. The behaviour of the inventory system over time is depicted in Figure 5.1. As a way of encouraging larger order sizes, the company's supplier offers incremental quantity discounts. The discount cost structure is

$$p_j = \begin{cases} p_1 & y_1 = 0 \leq Y < y_2 \\ p_2 & y_2 \leq Y < y_3 \\ \vdots & \vdots \\ p_m & y_m \leq Y \end{cases}.$$

Growth and consumption occur over the periods t and T respectively and hence the company incurs feeding and holding over those respective time periods. The demand rate, D, and the weight of the inventory level at the beginning of the consumption period, $Q_1 = Yw_1$, are utilised to determine the cycle time as

$$T = \frac{Yw_1}{D}. \tag{5.1}$$

Define $y_1 = 0, y_2, ..., y_j, y_{j+1}, ..., y_m$ as the order quantities at which the purchase cost per weight unit changes. In such an instance, there are m price breaks. When a supplier offers incremental quantity discounts, the purchasing cost per weight unit, p_j, is the same for all Y values in $[y_j, y_{j+1})$. The purchasing cost per weight unit decreases from one price break to the next (i.e. $p_1 > p_2 > ... > p_j > p_{j+1} > ... > p_m$).

5.4.1.1 Average purchasing cost per unit time

Let Y be in the jth price break (i.e. $y_j \leq Y < y_{j+1}$). The purchasing cost per cycle for Y items, each weighing w_0, in this price break is given by

$$p(Yw_0) = p_1(y_2 - y_1)w_0 + p_2(y_3 - y_2)w_0 + ... + p_{j-1}(y_j - y_{j-1})w_0 + p_j(Y - y_j)w_0. \tag{5.2}$$

Define R_j as the sum of the terms in Equation (5.2) which are independent of Y, and thus

$$R_j = \begin{cases} p_1(y_2 - y_1)w_0 + p_2(y_3 - y_2)w_0 + ... + p_{j-1}(y_j - y_{j-1})w_0, & j \geq 2 \\ 0, & j = 1 \end{cases}. \tag{5.3}$$

Equation (5.2) can be rewritten as

$$p(Yw_0) = R_j + p_j w_0(Y - y_j). \tag{5.4}$$

The average unit purchasing cost per unit weight, $p(Yw_0)/Yw_0$, is equal to

$$\frac{p(Yw_0)}{Yw_0} = \frac{R_j}{Yw_0} + p_j - \frac{p_j y_j}{Y}. \tag{5.5}$$

Multiplying Equation (5.5) by the demand in weight units per unit time, D, yields an expression for the average unit purchasing cost per unit time (denoted by $APCU$), as

$$APCU = D\left[\frac{R_j}{Yw_0} + p_j - \frac{p_j y_j}{Y}\right]. \tag{5.6}$$

5.4.1.2 Food procurement cost per unit time

The food procurement cost per cycle, FC, is determined from the feeding cost per weight unit per unit time, c, the number of items to be fed (i.e. the order quantity), Y, the feeding duration, t, and the feeding function for each item, $f(t)$, as

$$FC = cY\int_0^t f(t)\,dt. \tag{5.7}$$

Dividing Equation (5.7) by the cycle time yields an expression for the feeding cost per unit time, FCU, given by

$$FCU = \frac{cD}{w_1}\int_0^t f(t)\,dt. \tag{5.8}$$

5.4.1.3 Setup cost per unit time

Every time the company places an order for live newborn items, it incurs a cost of K for setting up the growth and feeding facilities. The setup cost per cycle is thus

$$SC = K. \tag{5.9}$$

The setup cost per unit time, SCU, is determined by dividing the setup cost per cycle by the cycle time as

$$SCU = \frac{KD}{Yw_1}. \tag{5.10}$$

5.4.1.4 Average holding cost per unit time

The holding cost per cycle, HC, is computed from Figure 5.1 using the area under the consumption period (since the holding cost is incurred for the slaughtered inventory) and thus

$$HC = h\left[\frac{Y^2 {w_1}^2}{2D}\right]. \tag{5.11}$$

The holding cost per unit time, HCU, is computed by dividing Equation (5.11) by Equation (5.1), hence

$$HCU = h\left[\frac{Yw_1}{2}\right]. \tag{5.12}$$

It is assumed that the holding cost per weight unit per unit time, h, is charged as a fixed percentage of the purchasing cost per weight unit p. Defining i as the holding cost rate per unit time, it follows that the holding cost per weight unit per unit time is

$$h = ip. \tag{5.13}$$

When considering the average unit purchasing cost per weight unit, $p(Yw_0)/Yw_0$, from Equation (5.5), Equation (5.13) becomes

$$h = i\left(\frac{R_j}{Yw_0} + p_j - \frac{p_j y_j}{Y}\right). \tag{5.14}$$

Substituting Equation (5.14) into Equation (5.12) yields an expression for the average holding cost per unit time, $AHCU$, as

$$AHCU = \frac{i}{2}\left[\frac{R_j w_1}{w_0} + p_j w_1\big(Y - y_j\big)\right]. \tag{5.15}$$

5.4.1.5 Average total cost per unit time

The average total cost per unit time, $ATCU$, is computed by summing Equations (5.6), (5.8), (5.10) and (5.15) and therefore

$$ATCU = D\left[\frac{R_j}{Yw_0} + p_j - \frac{p_j y_j}{Y}\right] + \frac{cD}{w_1}\int_0^{t_1} f(t)\,dt + \frac{KD}{Yw_1} + \frac{i}{2}\left[\frac{R_j w_1}{w_0} + p_j w_1\big(Y - y_j\big)\right]. \tag{5.16}$$

Through rearranging the terms, Equation (5.16) becomes

$$ATCU = p_j D + \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y} + \frac{ip_j Y w_1}{2} + \frac{i\big(\frac{R_j w_1}{w_0} - p_j y_j w_1\big)}{2} + \frac{cD}{w_1}\int_0^{t} f(t)\,dt. \tag{5.17}$$

Equation (5.17) is a general function for the average total cost per unit time which can be applied to different growing items. As a result, it has two general expressions which are functions of time. These are the feeding function, $f(t)$, and the growth function, $f(w_1|w_0)$. In order to solve the model in a relatively easy manner, it is necessary to consider specific items (i.e. with specific growth and feeding functions). Rezaei (2014), who presented the first EOQ model for growing items, used growth function for broiler chickens, developed by Richards (1959), and feeding function for broiler chickens, developed by Goliomytis et al. (2003). Consequently, Rezaei (2014)'s model had the slaughter time, in addition to the cycle time and order quantity, as another decision variable. For that reason, Rezaei (2014) used a relatively complex heuristic to solve the model. Nobil et al. (2018) developed and solved an EOQ model for growing items with shortages in using a relatively simpler heuristic by assuming that the growth function of the items is linear. By assuming a linear growth function, the feeding cost in Nobil et al. (2018)'s model could be determined in the same way as the holding cost is determined. Due to Nobil et al. (2018)'s model requiring less computations to solve because of the assumption of a linear growth function, that assumption is adapted when formulating the specific mathematical model for the proposed inventory system.

5.4.2 Specific mathematical model: Case of a linear growth function

Suppose that the growth function of each item, w_1, is linear. This implies that each item's weight increases at a constant rate of b weight units per unit time. An expression for the growth function of each item can be formulated considering the fact that growth occurs for the duration t and keeping in mind that each item weighs w_0 at the beginning of the cycle. This means that the slaughter weight of the items is a linear function with gradient b and y-intercept w_0, as shown in Figure 5.2 which shows the behaviour of the inventory system over time, and hence

$$w_1 = w_0 + bt. \tag{5.18}$$

If Y items are ordered, the growth function for all the ordered items is

$$Yw_1 = Yw_0 + Ybt. \tag{5.19}$$

The duration of the growth period, t, is computed from Equation (5.18) as

$$t = \frac{w_1 - w_0}{b}. \tag{5.20}$$

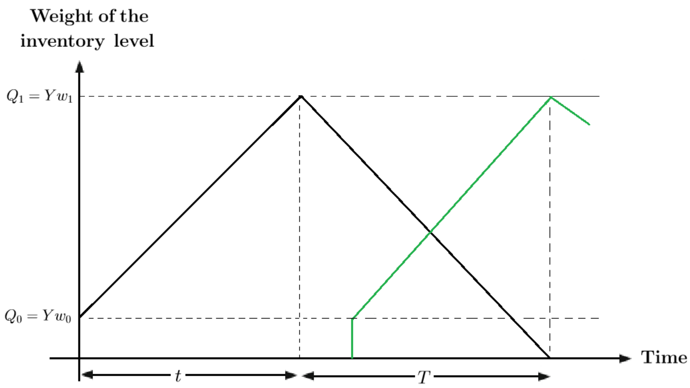


Figure 5.2: Inventory system behaviour for growing items under the assumption of linear growth function

5.4.2.1 Feeding cost per unit time

The feeding cost per cycle is computed using Figure 5.2 as the product of the feeding cost per weight unit per unit time (c) and the area under the growth period. Hence

$$FC = c\left[\frac{t\left(Yw_1 - Yw_0\right)}{2}\right]. \tag{5.21}$$

Substituting Equation (5.20) into Equation (5.21) results in

$$FC = c\left[\frac{Y\left(w_1 - w_0\right)^2}{2b}\right]. \tag{5.22}$$

The feeding cost per unit time (FCU), computed by dividing Equation (5.21) by the cycle time, is

$$FCU = \frac{cD\left(w_1 - w_0\right)^2}{2bw_1}. \tag{5.23}$$

5.4.2.2 Average total cost per unit time

The average purchasing, setup and holding costs per unit time are not affected by the assumption of a linear growth function and thus they remain the same as those given in Equations (5.6), (5.10) and (5.15). The average total cost is thus

$$ATCU = p_jD + \left(\frac{R_j}{w_0} - p_jy_j + \frac{K}{w_1}\right)\frac{D}{Y} + \frac{ip_jYw_1}{2} + \frac{i\left(\frac{R_jw_1}{w_0} - p_jy_jw_1\right)}{2} + \frac{cD\left(w_1 - w_0\right)^2}{2bw_1}. \quad (5.24)$$

5.4.2.3 Model constraints

Two constraints ensure the feasibility of the proposed inventory system. The first constraint ensures that the items are ready for consumption at the required time. The second constraint ensures the feasibility of the optimal order quantity, meaning that the optimal quantity Y_j is acceptable for each price break region and must fall between the the price breaks y_j and y_{j+1}.

Constraint 1: In order to ensure that the slaughtered items are ready for sale during the consumption period, the setup time and the duration of the growth period should be at least equal to the consumption period. This results in a constraint (on the duration of the consumption period) being formulated as

$$t + t_s \leq T. \quad (5.25)$$

Through substituting t from Equation (5.20), Equation (5.25) becomes

$$T \geq \left\{\frac{w_1 - w_0}{b} + t_s = T_{min}\right\}. \quad (5.26)$$

Constraint 2: In order to ensure that the order quantity determined falls within the range of the given price break, a constraint on the order quantity is formulated as

$$y_j \leq Y < y_{j+1}. \quad (5.27)$$

5.4.2.4 Mathematical formulation of the EOQ model for growing items with incremental quantity discounts

The objective function, given in Equation (5.24), and the constraints are utilised to formulate a mathematical model for the proposed inventory system. Hence,

$$Min \quad \left\{ATCU = p_jD + \left(\frac{R_j}{w_0} - p_jy_j + \frac{K}{w_1}\right)\frac{D}{Y} + \frac{ip_jYw_1}{2} + \frac{i\left(\frac{R_jw_1}{w_0} - p_jy_jw_1\right)}{2} + \frac{cD\left(w_1 - w_0\right)^2}{2bw_1}\right\} \quad (5.28)$$

$$\begin{aligned} s.t. \quad & y_j \leq Y < y_{j+1} \\ & T \geq T_{min} \\ & T \geq 0. \end{aligned}$$

5.4.3 Solution

5.4.3.1 Determination of the decision variables

The optimal solution to the proposed inventory system is determined by setting the objective function to zero resulting in

$$\begin{aligned}\frac{\partial ATCU}{\partial Y} &= -\left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^2} + \frac{ip_j w_1}{2} \\ &= 0 \rightarrow Y = \sqrt{\frac{2\left(\frac{R_j w_1}{w_0} - p_j y_j w_1 + K\right)D}{ip_j {w_1}^2}}.\end{aligned} \tag{5.29}$$

The optimal cycle time is computed by substituting Equation (5.29) into Equation (5.1) and the result is

$$T = \sqrt{\frac{2\left(\frac{R_j w_1}{w_0} - p_j y_j w_1 + K\right)D}{ip_j D^2}}. \tag{5.30}$$

5.4.3.2 Proof of convexity of the objective function

In order to show that the objective function, given in Equation (5.24), has a unique solution, the grad of that function is calculated so as to identify the optimum point and the Hessian matrix of the function is shown to be positive. The location of the optimum point from the grad is shown by:

$$\frac{\partial ATCU}{\partial Y} = -\left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^2} + \frac{ip_j w_1}{2} \tag{5.31}$$

The Hessian matrix of the objective function, given by

$$\begin{bmatrix} \frac{\partial^2 ATCU}{\partial Y^2} & \frac{\partial^2 ATCU}{\partial Y \partial T} \\ \frac{\partial^2 ATCU}{\partial Y \partial T} & \frac{\partial^2 ATCU}{\partial T^2} \end{bmatrix}, \tag{5.32}$$

is shown, in Equation (5.33), to be positive semi-definite.

$$\begin{bmatrix} \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^3} & \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^3} \\ \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^3} & 0 \end{bmatrix} \tag{5.33}$$

The quadratic form of the objective function is determined from the Hessian matrix as

$$\begin{aligned}&\begin{bmatrix} Y & T \end{bmatrix} \begin{bmatrix} \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^3} & \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^3} \\ \left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right)\frac{D}{Y^3} & 0 \end{bmatrix} \begin{bmatrix} Y \\ T \end{bmatrix} \\ &= \frac{2D}{Y}\left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right) + \frac{DT^2}{Y^3}\left(\frac{R_j}{w_0} - p_j y_j + \frac{K}{w_1}\right) \geq 0.\end{aligned} \tag{5.34}$$

From Equation (5.34), the quadratic form of the objective function is shown to be positive which implies that the objective function is convex.

5.4.3.3 Computational algorithm

The solution to the EOQ model for growing items with incremental quantity discounts is determined using the following optimisation algorithm:

Step 1 Compute Y for each j using Equation (5.29). Denote this as Y_j.

Step 2 Check each Y_j's feasibility. They are feasible if $y_j \leq Y < y_{j+1}$. Infeasible Y_j's are disregarded and only the feasible ones proceed to Step 3.

Step 3 For each feasible Y, compute the corresponding T using Equation (5.30).

Step 4 Check the feasibility of each computed Y_j with regards to the cycle time. Each Y_j is feasible if $T \geq T_{min}$. Infeasible Y_j's are disregarded and only the feasible ones proceed to Step 5.

Step 5 Compute $ATCU$ using Equation (5.24) for all the feasible Y_j's. The Y_j value which results in the lowest $ATCU$ is the EOQ.

Step 6 End.

5.5 Numerical results

5.5.1 Numerical example

The proposed inventory system is applied to a numerical example which considers a company which purchases day-old chicks, feeds/grows them until they reach a targeted weight and then puts them on sale after screening for quality. The following parameters, mostly adapted from a study by Nobil et al. (2018), are utilised to analyse the proposed inventory system:

Demand rate, D =250 000 g/year

Setup cost, K =400 ZAR/cycle

Holding cost rate, i =0.4

Feeding cost, c =0.08 ZAR/g/year

Approximated weight of newborn (one day old) chick, w_0 =53 g/chick

Approximated weight of chicken at the time of slaughtering, w_1 =1 267 g/chicken

Growth rate, b =15 330 g/chick/year (42 g/chick/day)

Setup time, t_s =0.01 year

The purchasing cost structure is given in Table 5.3.

Table 5.3: Purchase cost structure under incremental quantity discounts

Quantity purchased	Price per weight unit (ZAR/g)
0-100	0.025
101-200	0.023
201-300	0.021
301+	0.019

Solution procedure application: The proposed solution algorithm is illustrated by applying it to the numerical example. The procedure is outlined as follows:

Step 1 Compute Y for each j using Equation (5.29).

$$R_1 = 0$$

$$R_2 = 0.025(101 - 0) = 133.8$$

$$R_3 = 133.8 + 0.023(201 - 101) = 255.7$$

$$R_4 = 255.7 + 0.021(301 - 201) = 367.0$$

$$Y_1 = \sqrt{\frac{2\left(\frac{0\times1267}{53} - 0.025 \times 0 \times 1267 + 400\right)250000}{0.4 \times 0.025 \times 1267^2}} = 111.6$$

$$Y_2 = \sqrt{\frac{2\left(\frac{133.8\times1267}{53} - 0.023 \times 101 \times 1267 + 400\right)250000}{0.4 \times 0.023 \times 1267^2}} = 149.0$$

$$Y_3 = \sqrt{\frac{2\left(\frac{255.7\times1267}{53} - 0.021 \times 201 \times 1267 + 400\right)250000}{0.4 \times 0.021 \times 1267^2}} = 207.9$$

$$Y_4 = \sqrt{\frac{2\left(\frac{367.0\times1267}{53} - 0.019 \times 301 \times 1267 + 400\right)250000}{0.4 \times 0.019 \times 1267^2}} = 281.1$$

Step 2 Check each Y_j's feasibility. They are feasible if $y_j \leq Y < y_{j+1}$. Infeasible Y_j's are disregarded and only the feasible ones proceed to Step 3.

$$0 \leq Y_1 = 111.6 \not< 101$$

$$101 \leq Y_2 = 149.0 < 201$$

$$201 \leq Y_3 = 207.9 < 301$$

$$301 \not\leq Y_4 = 281.1$$

Thus Y_2 and Y_3 are feasible.

Step 3 For each feasible Y, compute the corresponding T using Equation (5.30).

$$T_2 = \sqrt{\frac{2\left(\frac{133.8\times1267}{53} - 0.023 \times 101 \times 1267 + 400\right)250000}{0.4 \times 0.023 \times 250000^2}} = 0.7552$$

$$T_3 = \sqrt{\frac{2\left(\frac{255.7\times1267}{53} - 0.021 \times 201 \times 1267 + 400\right)250000}{0.4 \times 0.021 \times 250000^2}} = 1.0535$$

Step 4 Check the feasibility of each computed Y_j with regards to the cycle time. Each Y_j is feasible if $T \geq T_{min}$. Infeasible Y_j's are disregarded and only the feasible ones proceed to Step 5.

$$T_{min} = \frac{1267 - 53}{15330} + 0.01 = 0.0891$$

Thus Y_2 and Y_3 are feasible since $T_2 \geq T_{min}$ and $T_3 \geq T_{min}$.

Step 5 Compute $ATCU$ using Equation (5.24) for all the feasible Y_j's. The Y_j value which results in the lowest $ATCU$ is the EOQ.

$$ATCU_2 = 0.023 \times 250000 + \left(\frac{133.8}{53} - 0.023 \times 101 + \frac{400}{1267}\right)\frac{250000}{149.0} + \frac{0.4 \times 0.023 \times 149.0 \times 1267}{2} + \frac{0.4\left(\frac{133.8 \times 1267}{53} - 0.023 \times 101 \times 1267\right)}{2} + \frac{0.08 \times 250000\left(1267 - 53\right)^2}{2 \times 15330 \times 1267} = 8297.01$$

$$ATCU_3 = 0.021 \times 250000 + \left(\frac{255.7}{53} - 0.021 \times 201 + \frac{400}{1267}\right)\frac{250000}{207.9} + \frac{0.4 \times 0.021 \times 207.9 \times 1267}{2} + \frac{0.4\left(\frac{255.7 \times 1267}{53} - 0.021 \times 101 \times 1267\right)}{2} + \frac{0.08 \times 250000\left(1267 - 53\right)^2}{2 \times 15330 \times 1267} = 8374.11$$

$Y^* = Y_2$ since $ATCU_2 < ATCU_3$.

Step 6 End.

Table 5.4: Summary of the results from the numerical example

Variable	Units	Quantity
t	year	0.0792
T^*	year	0.7552
Y^*	items	149
$ATCU^*$	R/year	8 297.01

Based on the results of the numerical example, some of which are summarised in Table 5.4, the company should place an order for 149 day-old chicks at the beginning of each cycle. This order quantity lies in the 101-200 price-break and therefore the company will pay 0.023 ZAR per gram. Based on the targeted slaughter weight, the items should be grown for 0.0792 years (29 days). Orders should be replenished every 0.7552 years (275 days). Following this optimal inventory policy, the company will incur an average total cost of 8 297.01 ZAR per year.

Figure 5.3 shows the response of the the average total cost function to different order quantities. The are four curves corresponding to the four price breaks. Each curve is valid for a given order quantity interval. The valid interval for each curve is represented by a solid line.

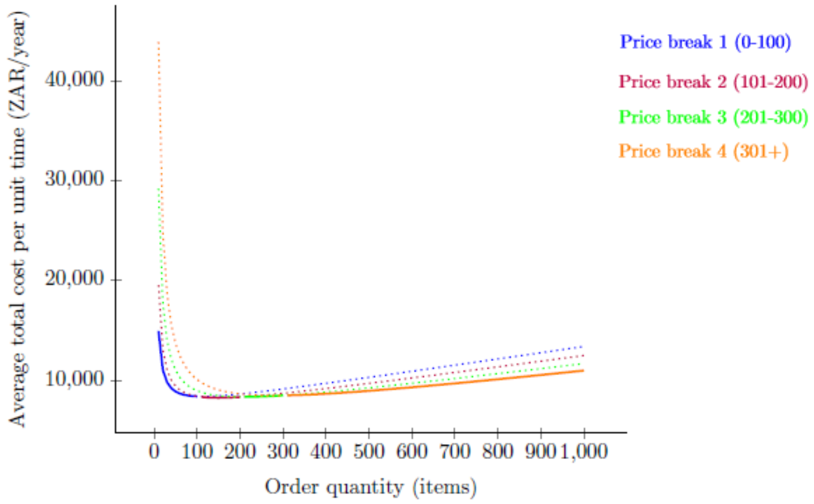


Figure 5.3: Average total cost under incremental quantity discounts

5.5.2 Sensitivity analysis

Sensitivity analyses are performed on the major input parameters in order to investigate their impacts on the decision variables and to provide managerial insights for improving inventory management.

Table 5.5: Changes to Y^* and $ATCU^*$ due to changes in K

% change in setup cost	**EOQ** items	% change	**ATCU** ZAR/year	% change
-50	124	-16.8	8 008	-3.5
-37.5	130	-12.8	8 086	-2.5
-25	137	-8.1	8 159	-1.7
-12.5	143	-4.0	8 229	-0.8
0	149	0	8 297	0
12.5	154	3.4	8 362	0.8
25	159	6.7	8 425	1.5
37.5	165	10.7	8 485	2.3
50	170	14.1	8 544	3.0

Increasing the setup cost increases both the EOQ and the average total costs, as shown in Table 5.5. Nonetheless, changes to the setup cost did not shift the EOQ into a different price break. Managers can offset the increase in the average total cost by purchasing larger quantities (i.e. placing orders less frequently). However, this should be done in moderation because if too much stock is ordered the holding cost will increase and this will lead to an increase in total costs.

Table 5.6: Changes to Y^* and $ATCU^*$ due to changes in i

% change in holding cost rate	**EOQ**		**ATCU**	
	items	% change	ZAR/year	% change
-50	397	166.4	7 576	-8.7
-37.5	262	75.8	7 853	-5.3
-25	240	61.1	8 039	-3.1
-12.5	159	6.7	8 178	-1.4
0	149	0	8 297	0
12.5	140	-6.0	8 409	1.3
25	133	-10.7	8 515	2.6
37.5	127	-14.8	8 616	3.8
50	121	-18.8	8 713	5.0

Increasing the holding cost rate increases the average total cost and reduces the EOQ, as shown in Table 5.6. The effect of decreasing the holding cost rate on the EOQ is substantial because the EOQ shifts into different price breaks. As a result of the shift in price breaks, the average cost reduces because of the lower purchasing cost per unit weight in the new price break. This shift into a lower price breaks offsets the increased holding costs (due to the increase in the EOQ as a result of ordering larger quantities).

Table 5.7: Changes to Y^* and $ATCU^*$ due to changes in c

% change in feeding cost	**EOQ**		**ATCU**	
	items	% change	ZAR/year	% change
-50	149	0	7 918	-4.6
-37.5	149	0	8 012	-3.4
-25	149	0	8 107	-2.3
-12.5	149	0	8 202	-1.1
0	149	0	8 297	0
12.5	149	0	8 392	1.1
25	149	0	8 487	2.3
37.5	149	0	8 582	3.4
50	149	0	8 676	4.6

Table 5.7 shows that the average total costs increase with increasing feeding costs whereas the EOQ is not affected by changes in the feeding cost. The feeding is essentially the cost of procuring feedstock for the items, and it is very difficult for managers to reduce this cost since it is set by the feedstock suppliers. Nonetheless, managers can reduce this cost through procuring larger volumes (of the items' feedstock) which normally have discounted pricing.

Table 5.8: Changes to Y^* and $ATCU^*$ due to changes in w_1

% change in slaughter weight	**EOQ** items	**EOQ** % change	**ATCU** ZAR/year	**ATCU** % change
-50	436	192.6	7 353	-11.4
-37.5	377	153.0	7 660	-7.7
-25	253	69.8	7 940	-4.3
-12.5	166	11.4	8 145	-1.8
0	149	0	8 297	0
12.5	135	-9.4	8 448	1.8
25	124	-16.8	8 599	3.6
37.5	81	-45.6	8 733	5.2
50	74	-50.3	8 836	6.5

Table 5.8 shows that increasing the slaughter weight decreases the EOQ and increases the average total cost. The EOQ decreases because the company slaughters the items at larger weights and therefore fewer items are required to meet annual demand. The average total costs increase because if the company slaughters the items at larger weights, the items are allowed to grow for longer periods of time and this means that the more feedstock is required not only because the items' growth period is longer but also because as the items get larger they feeding requirements increase as well. In order to keep costs down, managers should opt to slaughter the items at slightly smaller weights because they can reduce their costs slightly, however this might not always be possible when taking into account customer preferences.

Table 5.9: Changes to Y^* and $ATCU^*$ due to changes in y_j

% change in lower bound for order quantity for price j	**EOQ** items	**EOQ** % change	**ATCU** ZAR/year	**ATCU** % change
-50	133	-10.7	8 093	-2.5
-37.5	137	-8.1	8 146	-1.8
-25	141	-5.4	8 197	-1.2
-12.5	145	-2.7	8 248	-0.6
0	149	0	8 297	0
12.5	152	2.0	8 345	0.6
25	156	4.7	8 393	1.2
37.5	159	6.7	8 439	1.7
50	162	8.7	8 485	2.3

Increasing the lower bound for the order quantity in each price break increases both the EOQ and the average total costs, as shown in Table 5.9. The changes to the average costs are minimal most likely because the EOQ remains in the same price break. Managers don't have much control over the discount quantity structure as it is determined by the supplier, but if it happens that the suppliers reduce the lower bounds on the order quantities in each price break it is beneficial for managers to order less items.

Table 5.10: Changes to Y^* and $ATCU^*$ due to changes in p_j

% change in purchasing cost at jth price break	**EOQ** items	**EOQ** % change	**ATCU** ZAR/year	**ATCU** % change
-50	240	61.1	4 742	-42.8
-37.5	174	16.8	5 653	-31.9
-25	163	9.4	6 539	-21.2
-12.5	155	4.0	7 420	-10.6
0	149	0	8 297	0
12.5	143	-4.0	9 172	10.5
25	139	-6.7	10 045	21.1
37.5	136	-8.7	10 916	31.6
50	91	-38.9	11 866	43.0

Table 5.10 shows that increasing the purchasing cost decreases the EOQ and increases the average total costs. In fact the EOQ shifts into different price breaks and consequently the effect on the average cost is significant as well. By reducing purchasing costs, managers can save significantly on their average total costs by ordering larger quantities. While this will result in an increase in the holding cost, the savings which result from lower purchasing cost more outweigh the impact of the increased holding cost.

5.6 Conclusion

The major contribution made by this chapter to the literature on inventory modelling for growing items is the incorporation of incremental quantity discounts. This addition to literature is important because suppliers often offer discounts for purchasing larger volumes of stock. The cost structure , in terms of both the purchasing cost in each price break and the lower bounds for the order quantities in each price break, was shown to have a significant impact on the order quantity and the average total cost of managing inventory. This indicates that incremental quantity discounts have considerable impact on inventory management and this presents operations managers with opportunities to reduce costs through better procurement practices. However, certain factors need to be considered when purchasing larger volumes, namely the available storage space, deterioration and the available procurement budget.

The proposed inventory system did not take into account issues like deterioration, growing and storage facility capacity and budget constraints. These factors are important when purchasing larger quantities because in certain instances, management might be forced to lease extra capacity if they purchased more items than can be grown and stored in their owned facilities. This will certainly increase costs and negate the benefit of purchasing larger quantities. The model presented in this chapter can be extended to include capacity limits, budget limits, among other popular EOQ extensions. Deterioration becomes increasingly important if larger quantities are purchased, extensions which account for deterioration during the consumption period are another possible area for future research as they represent more realistic inventory systems.

Chapter 6

Conclusion

6.1 Summary of findings

The aim of this dissertation was to develop three lot sizing models for growing inventory items. The models were developed by extending the basic EOQ model for growing items along three dimensions, namely quality, limited capacity and incremental quantity discounts.

The first model proposed an inventory system in which newborn items are ordered and fed/grown until they reach a specific weight. At this point, the items are slaughtered and kept in stock. Before the slaughtered items are sold, they are subjected to a screening process which separates good quality items from poor quality items. Both item qualities are sold, but the poor quality items are sold at a much lower price. The inventory system was formulated mathematically to determine the optimal order quantity, slaughter weight and cycle time. The presence of poor quality items means that more items need to be ordered in order to meet a specific demand for good quality items. This effect worsens as the fraction of imperfect quality items increases. This finding should motivate production and operations managers to pay attention to quality checks and ensure that the percentage of imperfect quality items is kept as low as possible.

The second model considered an inventory situation whereby a company orders newborn items and rears them until they are of acceptable weight for slaughter. However, the company-owned growth and storage facilities have limited capacities. In the event that the order quantity exceeds the capacity, the company rents secondary growing and storage facilities. It is assumed that the holding costs in the rented storage facility are higher than those in the company-owned facility. For this reason, stock kept in the rented warehouse is consumed first. The capacity constraints on the growing and storage facilities increased total costs mainly because of the higher holding costs in the rented facility. As the capacity increases, the total costs decrease, but increasing capacity is capital intensive and poses financial risks if market conditions change for the worst.

The last model proposed an inventory system in which a company orders newborn items and rears them until they reach a specific target weight, after which they are slaughtered. The supplier of the live newborn grants the company incremental quantity discounts. The cost structure, in terms of both the purchasing cost in each price break and the lower bounds for the order quantities in each price break, was shown to have a significant impact on the order quantity and the average total cost of managing inventory. This indicates that incremental quantity discounts have considerable impact on inventory management and this presents operations managers with opportunities to reduce costs through better procurement practices.

6.2 Possible practical applications of findings

Practical applications of the classic EOQ model are limited because of the assumptions made in the model which do not reflect most realistic inventory systems. For this reason, the model has been extended numerous times in order to model more realistic inventory systems, as was the case in this study.

The models presented in this study can be applied to a variety of growing inventory systems. While the focus of the examples was on livestock production, with some minor modifications, the models can be applied to crop production as well. Examples of modifications which can be done to the models so that they are suited to crop production include consideration of harvest periods, irrigation costs and weather conditions.

Possible beneficiaries of the results from this study are inventory managers in enterprises involved in livestock and crop production. These enterprises face a variety issues, including, but not limited to, quality, limited storage capacity and quantity discounts. The three models presented in this study deal with those issues and as a result they can assist those responsible for managing inventory and making purchasing decisions in livestock- and crop-producing enterprises.

6.3 Contributions

The contributions made by this study to the literature on inventory theory is mainly the development of economic order quantity models for growing items under the following set of conditions:

- A certain fraction of the items fails a quality screening check and is classified as being of poor quality.
- Two growing and storage facilities are required because the growth facility for raising the live items and the storage facility for stocking the slaughtered inventory have limited capacities.
- The supplier of the newborn items offers incremental quantity discounts.

6.4 Possible areas for future research

In addition to extending the imperfect quality model to account for popular extensions such as shortages, permissible delay in payment, among others, the model can be extended by relaxing the assumption that the screening process is 100% effective at separating good and poor quality items, this along with the inclusion of learning effects in the screening process are other possible areas for further development.

The model with two growing and storage facilities can be extended to include multiple facilities. The assumption that the rented facility has abundant capacity is not realistic. Furthermore, the model can be extended to include other popular EOQ extensions such as shortages, imperfect quality and trade credit financing, to name a few.

The incremental quantity discount model did not take into account issues like deterioration, growing and storage facility capacity and budget constraints. These factors are important when purchasing larger quantities because in certain instances, management might be forced to lease extra capacity if they purchased more items than can be grown and stored in their owned facilities. Therefore, the model presented in this chapter can be extended to include capacity limits, budget limits, among other popular EOQ extensions. Deterioration becomes increasingly important if larger quantities are purchased because if larger volumes are purchased the cycle time increases as well meaning that the items spend longer periods in stock. Extensions which account for deterioration during the consumption period are another possible area for future research as they represent more realistic inventory systems.

All three models presented in this dissertation made the assumption that the demand rate is deterministic. This assumption can be relaxed to account for stochastic demand and the resulting models will more closely represent real life inventory systems.

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