Classifying extremal spin-4 black holes in AdS_3

by

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I, Phillip Badenhorst, declare that the thesis, which I hereby submit for the degree MSc Physics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Abstract

The study of higher spin gravity theories has gathered much interest in recent years. In three dimensions, these theories are particularly tractable due to the fact that they can be defined in terms of a Chern-Simons theory. A higher spin black hole then corresponds to a flat connection with appropriate holonomy conditions in this framework. Building on this, we study what constitutes extremality of higher spin black holes. It has previously been proposed that extremal black holes can be defined in terms of the Jordan class of the holonomy around a non-contractable cycle. Extending the above studies, we then classify solutions of $sl(4|3) \oplus sl(4|3)$ Chern-Simons theory in terms of extremality and supersymmetry, and find non-extremal supersymmetric black hole solutions. One of the results, in this higher-spin context, indicates that supersymmetry and extremality do not imply each other. Finally, a review is made of hs[λ] theory, which is constructed using Vasiliev theory. This is another useful tool in describing higher spin theories, with the aim of extending hs[λ] to shs[λ], the supersymmetric analogue of which might have its own extremal conditions.

Contents

Chapter 1

Introduction

Black holes have long been an interesting field of study. In particular theories of gravity which are coupled to massless higher spin fields are very popular presently due to their appearance in the so-called AdS/CFT correspondence (more on that in the later chapters). One might wonder why higher spin theories are used, but this is simply because these theories provide a useful framework in which to test non-linear as well as non-local features that appear in quantum gravity. The construction of these theories, from the birth of general relativity to solutions to the appropriate equations which describe black hole mechanics, is rather involved and requires a certain level of prior knowledge, however to be concise we will cover the essentials of these topics in as much detail as is possible (within certain constraints).

1.1 General Relativity and Black Holes

1.1.1 Special Relativity and Flat Spacetime

Before starting to look at the construction and mechanics of black holes, it is necessary to review General Relativity. A fantastic review of relativity is given by [1]. Starting off with Special Relativity, which operates in a 4-dimensional space time called Minkowski space, one can define "distance", or rather spacetime intervals, on this manifold by defining the metric, which is a 4×4 matrix with two lowered indices:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.1}$$

however it is important to note that the metric can have an opposite sign, but is chosen as above in our case. Using this, one can then define the group of matrices that are invariant under rotations and boosts, also known as the Lorentz group. Adding the condition for translations creates the Poincaré group.

Let us also define a tensor as follows: Much the same as a dual vector is a linear map from vectors to $\mathbf{R}$, a tensor T of type/rank (k, l) is a multilinear map from a collection of dual vectors and vectors to $\mathbf{R}$,

$$T : T_p^* \times ... \times T_p^* \times T_p \times ... \times T_p \rightarrow \mathbf{R} \tag{1.2}$$

where T_p^* is multiplied k times and T_p is multiplied l times (with multiplied, we mean the Cartesian product).

A special kind of tensor is the Kronecker delta, δ^μ_ν, which is a (1,1) tensor. It can be used to relate the inverse metric, such that

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^\rho_\mu. \tag{1.3}$$

We can also use the metric to raise or lower indices on a tensor, and by doing such transform vectors into dual vectors and the reverse, for example

$$V_\mu = \eta_{\mu\nu}V^\nu \tag{1.4}$$
$$\omega^\mu = \eta^{\mu\nu}\omega_\nu. \tag{1.5}$$

It is also useful to consider the symmetries of a tensor. A tensor is symmetric in any of its indices if it is unchanged under any exchange of these indices. By the same logic, a tensor is anti-symmetric in its indices if it changes sign with any exchange of indices. If all of the indices are symmetric or anti-symmetric, then the tensor is known as "completely" symmetric or anti-symmetric.

We now move on to differential forms, which are a special class of tensor. A differential p-form is a $(0, p)$ tensor which is completely anti-symmetric. They are useful since they can be differentiated and integrated without the use of additional geometrical structures. We must also define the wedge-product, since it is extensively used in the actions. Given a p-form A and a q-form B, one can form a $(p+q)$-from using the wedge product $A \wedge B$, using the anti-symmetrized tensor product:

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]}. \tag{1.6}$$

Something that arises later concerning the form of the Chern-Simons action is the so-called exterior derivative. This operation "d" is used to differentiate p-from fields into $(p+1)$-form fields, and is also known as a normalized antisymmetric partial derivative:

$$(\mathrm{d}A)_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}. \tag{1.7}$$

We must also define what the weights of the tensor densities are. We begin by defining the flat-space Levi-Civita symbol,

$$\tilde{\epsilon}_{\mu_1\mu_2\dots\mu_n} = \begin{cases} +1 \text{ if } \mu_1\mu_2\dots\mu_n \text{ is an even permutation of } 0\ 1\dots(n-1), \\ -1 \text{ if } \mu_1\mu_2\dots\mu_n \text{ is an odd permutation of } 0\ 1\dots(n-1), \\ 0 \text{ otherwise.} \end{cases} \tag{1.8}$$

The following holds according to linear algebra, that for a determinant,

$$\tilde{\epsilon}_{\mu_1'\mu_2'\dots\mu_n'} = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| \tilde{\epsilon}_{\mu_1\mu_2\dots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \tag{1.9}$$

which is quite similar to the tensor transformation law. These objects are called tensor densities. They tend to have a certain weight associated to them. These are simply the power to which a Jacobian of the tensor is raised to. In general, one does not want to work with tensor densities, but tensors themselves. There is a simple method to achieve this, by simply multiplying with $|g|^{w/2}$, with w the weight of the density.

1.1.2 Manifolds

It is now important to note that the preceding work is all concerned with what we call "flat-space", i.e. that there is no curvature. Curvature of space is what Einstein came up with to generalize his theory for General Relativity, however before we can move on to that, we are required to define what a manifold is. The basic idea is that for any curved space that could also have a complicated topology, in local regions the space acts and looks like $\mathbf{R}^n$. With that we mean that the notions of open sets, functions and coordinates are the same in the analytical sense. We can then construct the entire manifold by smoothly sewing these local regions together. We will now take a quick but in-depth look into these manifolds, and that requires a few definitions.

We start with a map between two sets. For two sets M and N, a map $\phi : M \to N$ is a relationship that assigns, to each element of M, one element of N (a map can be seen as a generalization of a function). If there are two maps, $\phi : A \to B$ and $\varphi : B \to C$, then their composition $\varphi \circ \phi : A \to C$ can be given by the operation $(\varphi \circ \phi)(a) = \varphi(\phi(a))$. A map is then called one-to-one, or injective, if each element of N has at most one element of M mapped into it, and onto, or subjective, has at least one element of M mapped into it. We can refer to a map $\phi : \mathbf{R}^m \to \mathbf{R}^n$ as C^p if the function is continuous and p-times differentiable. We can then see that a map C^∞ can be called smooth, and if there exists a map $\phi : M \to N$ that is C^∞ with a C^∞ inverse $\phi^{-1} : N \to M$, then the two sets are called diffeomorphic and the map ϕ is called a diffeomorphism. From analysis, we know that an open set in $\mathbf{R}^n$ is a set made from an arbitrary union of open balls, where the open ball is the set of all points in $\mathbf{R}^n$ such that $|x - y| < r$ for some fixed $y \in \mathbf{R}^n, r \in \mathbf{R}$, with $|x - y| = [\sum_i (x^i - y^i)^2]^{\frac{1}{2}}$. Put simply, an open set is the interior of some $(n-1)$-dimensional closed surface. A chart consists of a subset U of set M, along with a one-to-one map $\phi : U \to \mathbf{R}^n$, such that the image $\phi(U)$ is open in $\mathbf{R}$. Finally, a C^∞ atlas is an indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$ which satisfies the following:

1. The union of the U_α is equal to M, such that the U_α covers M.

2. The charts are smoothly sewn together.

Using all the information above, we can finally define a manifold as the following: a C^∞ n-dimensional manifold is a set M along with a "maximal atlas", which is one that contains every possible compatible chart. It was necessary to be so precise with charts and overlaps, since most manifolds cannot be covered with a single chart.

Back to the metric, we must note that the naming of the metric as $\eta_{\mu\nu}$ is reserved specifically for the Minkowski metric. Instead, we construct a metric tensor which is non-degenerate and continuous. This tensor is named $g_{\mu\nu}$, and we can write it in its canonical form

$$g_{\mu\nu} = diag(-1, -1, ..., -1, +1, +1, ..., +1), \tag{1.10}$$

with *diag* being the diagonal elements of the matrix. If s is the number of $+1$'s and t is the number of -1's, then the signature of the metric is $s - t$, and the rank is $s + t$. If $t = 0$, then the metric is Euclidean or Riemannian, and if $t = 1$, then it is called Lorentzian or pseudo-Riemannian.

1.1.3 Curvature

We've seen so far that as soon as we define a manifold, we can set up tensors, derivatives of functions, etc. Once we introduce the metric it is possible to start defining more difficult concepts, like volume or path-length. To define curvature, we also use the metric, however we require some more geometric tools. Specifically, we require the connection. The need for this appears when we address the problem that the partial derivative is not an effective tensor operator. We then define the covariant derivative ∇. It does the same thing as the partial derivative, but in a way that is coordinate independent. It operates as such:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{1.11}$$

where the Γ matrices are the connection coefficients. Using this connection on the metric one can actually find what the connection coefficients are in terms of the metric, and doing so we find that

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{1.12}$$

This specific connection is known as the Christoffel connection, and will be used later. The connection vanishes in flat space (in Cartesian-type coordinate systems), which is why it is useful to describe the curvature using this connection. How we know the space is curved is due to parallel-transport, which relies on the connection. Parallel transport is the concept of moving a vector along a path while keeping this vector constant along this path, and in curved space, the result of parallel transport will depend on the path taken by the vector between points.

Using some of the knowledge known about parallel-transport, and defining what is known as the Riemann tensor $R^\rho_{\sigma\mu\nu}$, which is also known as the curvature tensor, we can define a formula for the curvature tensor in terms of the connection coefficients. The final result ends up looking like

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{1.13}$$

Hence from this we can see that General Relativity relies heavily on the Christoffel connection.

Using the Riemann tensor and contracting it, we can form the Ricci tensor, without even using the metric:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}. \tag{1.14}$$

If we then use the metric, and contract it further, we end up with the Ricci scalar:

$$R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu}. \tag{1.15}$$

We can then define the Einstein tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{1.16}$$

which will prove very useful in the future.

We would like to cover one more bit of formalism before describing gravitation, and that is the construction of a basis vector set within the tangent space which is not derived from any coordinate system. To this end we consider for each point in the manifold a set of basis vectors $\hat{e}_{(a)}$, which we choose to be orthonormal. If we take the canonical form of the metric to be η_{ab}, and choose basis vectors that are orthonormal appropriate to the signature of the manifold, then the inner product of the basis vectors are

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab} \tag{1.17}$$

where $g(\ ,\ )$ is the usual metric tensor[1]. The set of vectors that form an orthonormal basis is called a vielbein, which is German for "many legs". Using a linear combination of these basis vectors, we can express any vector. For example, our old basis vectors $\hat{e}_{(\mu)} = \partial_\mu$ can be expressed as

$$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)}. \tag{1.18}$$

We will refer to these e^a_μ as the vielbeins. We can use the inverse of the vielbeins to also describe the metric as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{1.19}$$

which leads to vielbeins sometimes being called the "square root" of the metric. We can also redefine the covariant derivative for a tensor in such a way that incorporates these non-coordinate basis. We shall replace the connection coefficients $\Gamma^\lambda_{\mu\nu}$ with the spin connection, $\omega_\mu{}^a{}_b$. This spin connection, in terms of the vielbeins and connection coefficient can be written as

$$\omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda \tag{1.20}$$

and will be used to redefine the covariant derivative as

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c. \tag{1.21}$$

1.1.4 Gravitation

With the advent of special relativity, it became clear that mass was simply a manifestation of energy and momentum, and as such the concept of the Weak Equivalence Principle, constructed around the time of Newton and Galileo to describe the laws of free-falling particles in a gravitational field and a uniformly accelerating frame, needed to be reworked. As such, the Einstein Equivalence Principle was created, which states that in small regions of spacetime, the laws of physics reduce to the laws of special relativity, and as such it is impossible to detect the existence of a gravitational field. This principle is what suggests that the action of gravity is attributed to the curvature of spacetime. Adding to this the idea that we can construct locally inertial frames in these regions, we can then construct Riemann normal coordinates anywhere on the manifold. These coordinates correspond to the metric taking on its canonical form and the vanishing of the Christoffel symbols. One can consider the appearance of gravitational redshift to reinforce the validity of the EEP. One can find many detailed explanations on this redshift, but the basics are thus: Consider two objects moving with constant acceleration and being a fixed distance apart, when one object emits a photon that will reach the other object after some time. While the photon is travelling, the objects will have picked up more velocity corresponding to the increase in acceleration, and hence the photon will have been redshifted much the same as the conventional Doppler effect. Accordingly, the EEP predicts the same effect to happen in the presence of a uniform gravitation field.

Following the work of Einstein, we can find the Einstein's field equations, either by following the methodology he himself used to describe how the metric responds to energy and momentum, or by starting with an appropriate action (Hilbert action), and then deriving the corresponding equations of motion. Regardless of method, the equations are given as follows:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.22}$$

$$= -\Lambda g_{\mu\nu}. \tag{1.23}$$

[1] There could be some confusion as to the definition of this tensor with respect to 1.10. The vielbein applies to metrics of any signature. In 1.10, we simply refer to the canonical form of the metric, while here we refer to the metric tensor itself

These equations can be thought of as second-order differential equations for the metric tensor field, and are extremely difficult to solve, even with the addition of certain simplifying assumptions. If one considers the right hand side as a kind of energy-momentum tensor, $T_{\mu\nu} = -\Lambda g_{\mu\nu}$, then Λ can be interpreted as the "energy density of the vacuum". This corresponds to certain aspects of quantum field theory. An interesting point in gravitation is the existence of singularities, which are points which are not in the manifold, but can be reached by travelling along a geodesic for a finite distance. These points occur when the curvature becomes infinite at a point, and is then no longer considered to be part of the spacetime. These points are interesting due to their association with black holes.

1.1.5 The Schwarzschild Solution

The Schwarzschild Solutions are solutions to Einstein's equations which describe spherically symmetric solutions. What we mean by spherically symmetric is that the solutions have the same symmetries as that of a S^2 sphere. The metric on a differentiable manifold is what is important to us, and as such we look for metrics with spherical symmetries. The characterization of symmetries of the metric is described by Killing vectors, and for S^2 there are three of these. Hence a spherically symmetric manifold has three Killing vector fields which act like those on S^2. By this we mean that the commutations of these vectors is the same in either case. For example, if we choose our Killing vectors to be $V^{(1)}, V^{(2)}$ and $V^{(3)}$, then the commutations between them are

$$[V^{(1)}, V^{(2)}] = V^{(3)} \tag{1.24}$$
$$[V^{(2)}, V^{(3)}] = V^{(1)} \tag{1.25}$$
$$[V^{(3)}, V^{(1)}] = V^{(2)}. \tag{1.26}$$

These are the same as the commutation relations for $SO(3)$, the group of rotations in three dimensions. The simplest case of spherical symmetry is simply $\mathbf{R}^3$, and spheres on $\mathbf{R}^3$. However we can also have spherical symmetry without an "origin", such as a so-called "wormhole". They have topology $\mathbf{R} \times S^2$, and if we suppress a dimension and draw the two-spheres as circles, best shown visually by Fig. (1.1).

We now follow a rigorous calculation for the Schwarzschild metric. Our submanifolds are two-spheres, with coordinates (θ, ϕ) with a metric of the form

$$d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2. \tag{1.27}$$

However, we are working in four dimensional spacetime, and as such we require two more coordinates. The following theorem will prove to be useful: If the submanifolds of a manifold are maximally symmetric spaces, then it is always possible to choose u-coordinates such that the metric on the entire manifold is of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{IJ}(v)dv^I dv^J + f(v)\gamma_{ij}(u)du^i du^j \tag{1.28}$$

where $\gamma_{ij}(u)$ is the metric on the submanifold. The proof of this theorem may be found in Chapter 13 of [3]. We can use this theorem to describe the metric on a spherically symmetric spacetime as

$$\begin{aligned} ds^2 &= g_{aa}(a,b)\mathrm{d}a^2 + g_{ab}(\mathrm{d}a\mathrm{d}b + \mathrm{d}b\mathrm{d}a) + g_{bb}(a,b)\mathrm{d}b^2 + r^2(a,b)d\Omega^2 \\ &= g_{aa}(a,r)\mathrm{d}a^2 + g_{ar}(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) + g_{rr}(a,r)\mathrm{d}r^2 + r^2 d\Omega^2. \end{aligned} \tag{1.29}$$

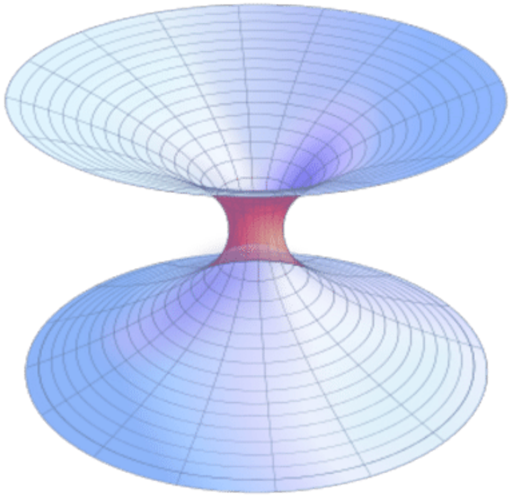

Figure 1.1: A Wormhole (also known as an Einstein-Rosen Bridge). This particular example [2] is one solution of the Schwarzschild metric, and is hence known as a Schwarschild wormhole.

We went from (a, b) to (a, r) by simply inverting $r(a, b)$. We now find a function $t(a, r)$ that in the (t, r) coordinate system has no $\mathrm{d}t\mathrm{d}r + \mathrm{d}r\mathrm{d}t$ cross terms, and by the chain rule

$$\mathrm{d}t^2 = (\frac{\partial t}{\partial a})^2\mathrm{d}a^2 + (\frac{\partial t}{\partial a})(\frac{\partial t}{\partial r})(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) + (\frac{\partial t}{\partial r})^2\mathrm{d}r^2. \tag{1.30}$$

If we set the requirements that $m(\frac{\partial t}{\partial a})^2 = g_{aa}$, $n + m(\frac{\partial t}{\partial t})^2 = g_{rr}$ and $m(\frac{\partial t}{\partial a})(\frac{\partial t}{\partial r}) = g_{ar}$ then we can rewrite the metric as

$$ds^2 = m(t, r)\mathrm{d}t^2 + n(t, r)\mathrm{d}r^2 + r^2 d\Omega^2. \tag{1.31}$$

Given some initial conditions for t, we can thus find solutions exactly. It is also rather similar to the metric for flat Minkowski space. However, we are looking for Lorentzian solutions, hence one of the components must be negative. We shall choose m, and replacing them with new functions α and β, we then have

$$ds^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2 d\Omega^2. \tag{1.32}$$

From this metric, one can work out the Christoffel symbols, followed by the Riemann tensor, and contracting to find the Ricci tensor. After this is done (the precise steps can be found within [1]), we find that $\beta = \beta(r)$ and $\alpha = f(r) + g(t)$, and choosing t in such a way that $g(t) = 0$, we can then have a spherically symmetric vacuum metric with a time-like Killing vector

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2 d\Omega^2. \tag{1.33}$$

Using the Ricci tensors R_{00} and R_{11} and using the proper coordinate scaling, it is implied that $\alpha = -\beta$, and $R_{22} = 0$ indicates that $e^{2\alpha} = 1 + \frac{\mu}{r}$ for some constant μ. These allow us to write our metric as

$$ds^2 = -(1 + \frac{\mu}{r})\mathrm{d}t^2 + (1 + \frac{\mu}{r})^{-1}\mathrm{d}r^2 + r^2 d\Omega^2. \tag{1.34}$$

It is possible to solve μ in the weak field limit of Einstein's field equations, as $r \to \infty$. A very brief description of this field limit entails that the nonlinear contributions of the spacetime metric are ignored. The metric tensor is treated as the sum of of an exact solution of the Einstein equations, often Minkoswki space, and a perturbation $g = \eta + h$, with η the nondynamical background metric being perturbed, and h the deviation from the true metric g from flat spacetime. Solutions are then found using perturbation theory. Now the above metric states that

$$g_{00}(r \to \infty) = -(1 + \frac{\mu}{r}) \tag{1.35}$$

$$g_{rr}(r \to \infty) = (1 - \frac{\mu}{r}), \tag{1.36}$$

however the weak field limit states that

$$g_{00} = -(1 + 2\Phi) \tag{1.37}$$

$$g_{rr} = (1 - 2\Phi) \tag{1.38}$$

with Φ being the potential $\Phi = -GM/r$. Hence if we set $\mu = -2GM$, we can finally obtain the Schwarzschild metric

$$ds^2 = -(1 - \frac{2GM}{r})\mathrm{d}t^2 + (1 - \frac{2GM}{r})^{-1}\mathrm{d}r^2 + r^2 d\Omega^2. \tag{1.39}$$

This metric holds true for any spherically symmetric vacuum solution to Einstein's equations. A few notes on this form. Note that M here denotes Newtonian mass, and that for $M \to 0$ we recover Minkowski space, as well as for $r \to \infty$. Analysis of the singularities is also greatly of interest. We see the metric becomes infinite when $r = 0$ and $r = 2GM$. Now, $r = 0$ is an honest singularity. It represents a point where the curvature becomes infinite, and it is not a point that is infinitely far away; it can be reached by travelling a finite distance along a curve. The point $r = 2GM$ is interesting however, since its curvature invariants remain normal, in that it doesn't blow up. It then leads us to believe that we have simply chosen a bad coordinate system, and can simply transform to a coordinate system that is more appropriate. The surface $r = 2GM$ is locally perfectly regular, however globally it acts as a point of no return, hence why it is known as the event horizon. Once a particle goes past this horizon, it can not come back. One can also interpret it as no event at $r \leq 2GM$ can influence any other event at $r > 2GM$. This is also where the name black hole arises, since it is technically impossible to see inside.

In the more astronomical sense of the word, black holes are formed from massive stars. A star is in constant turmoil from its own gravity as well as the outward push of pressure. Nuclear fusion occurs from the core, and the heat produced from this reaction causes the pressure. As the fuel is consumed, the temperature drops and the gravity of the star starts to take over. This process of shrinking is slowed and eventually stopped by the Pauli exclusion principle, which states that no two fermions can be in the same state. Hence the electrons resist the compression and the shrinking stops. These objects are then known as white dwarfs. If there is sufficient mass however, the electrons are not able to stop the gravitational pull, and the electrons merge with the protons, which then results in a neutron star. These stars are not terribly well understood, especially the cores, however it is reasonable to believe that given a sufficiently high mass, the neutron stars themselves won't be able to withstand the gravitational pull and continue to collapse. A fluid of neutrons is believed to be the densest material conceivable, and the collapse should result in a black hole.

One can also have charged black holes, where we start with our spherically symmetric solution, but since we are no longer in a vacuum, we must look at solutions with a non-zero electromagnetic field. The energy-momentum tensor for electromagnetism is given as

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\rho}F_{\nu}^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}), \tag{1.40}$$

which has field equations that are both Einstein's equations as well as Maxwell's equations. Using a similar approach as to that in the vacuum case, one can find the metric for a charged black hole, and it is known as the Reissner-Nordström metric

$$ds^2 = -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \tag{1.41}$$

with

$$\Delta = 1 - \frac{2GM}{r} + \frac{G(p^2+q^2)}{r^2}. \tag{1.42}$$

Here, q is the total electric charge, p is the total magnetic charge and M is the mass of the hole. For these black holes, there exist extremal cases, where the mass is equal to the charge, however these are not the same type of extremal black holes that we will consider later. For those interested however, [4] takes the units $c = G = \bar{h} = 1$ in the Reissner-Nordström metric, which leads the charge being expressible in terms of the mass, and in the extremal case, $|q| = M$.

A quick note now on black hole evaporation. Black holes do not evaporate in the traditional sense, however they do radiate energy as if they were a blackbody. The effect of this evaporation is called Hawking radiation, and it is loosely defined as such [5]: In quantum field theory there exists vacuum fluctuations, where particle and antiparticle pairs are spontaneously being created and annihilated in empty space. These are typically impossible to detect, since they average zero total energy, however if this event happens near to an event horizon, it may happen that as one of these pairs are created, one member might fall into the black hole, and the partner, unable to recombine and annihilate, escapes to infinity. This particle will have positive energy, however if the total energy needs to be conserved, then the black hole must lose some mass. These particles are then Hawking radiation. However this process also implies that information is lost which violates both quantum field theory as well as general relativity [6], [7]. A possible solution lies within string theory, but is outside the scope of this dissertation.

This rather rushed description of the foundation of general relativity and its consequences cannot be hoped to be encapsulated in such a short summary, and as such the reader is encouraged to refer to the source material for a more in-depth investigation. The material cited so far is mostly used only where applicable, and used as our own foundation for what is to come, where we start defining the specific space we are using as well as the framework where we will be describing our black holes.

1.2 AdS/CFT and Higher Spin Fields

We will now briefly review the basics of anti de Sitter space, and how it links to Conformal Field Theory. AdS/CFT, or Anti- de Sitter/ Conformal Field Theory, is a very useful field of study, with many relatively recent discoveries being made concerning it. One of these is the notion of a certain duality between theories of gravity in five dimensions and quantum field theories in four dimensions. This duality is incredivly useful, in that if the one theory is difficult to solve, for example the gravitational theory, then the QFT will be simple to solve, and the opposite is true as well. If we can consider general relativity to be eequivalent to QFT, then neither side is "deeper" than the other. We can then see why this is such an important conjecture, since if we have an AdS space which is then multiplied by a sphere, then these spaces arise as the near horizon geometry of extreme black holes and extreme p-branes, which is important in the understanding of M-theory [8]. This, however, is outside of our scope. What was noted is that on the AdS geometry, the near horizon region is related to the low energy CFT, which is described by the underlying branes. This appearance is the cause of Maldacena's AdS/CFT correspondence, which we will now very briefly review. Let us consider the Einstein-Hilbert action with a cosmological term

$$S = -s\frac{1}{16\pi G_D}\int d^D x\sqrt{|g|}(R+\Lambda) \tag{1.43}$$

where s indicates the choice of metric, with $s=-1$ being the Minkowski metric and $s=+1$ being the Euclidean signature. AdS as well as dS space are solutions of the empty space Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\Lambda g_{\mu\nu} \Rightarrow \tag{1.44}$$

$$R = \frac{D}{2-D}\Lambda \Rightarrow \tag{1.45}$$

$$R_{\mu\nu} = \frac{\Lambda}{2-D}g_{\mu\nu}. \tag{1.46}$$

These spaces have the property that the Ricci tensor is proportional to the metric tensor, which makes them Einstein spaces. We need a specific space that has maximal symmetry, hence we impose:

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}) \tag{1.47}$$

and examples of such spaces, for $R \neq 0$, are spheres (S^D), de Sitter spaces (dS_D) and anti de Sitter spaces (AdS_D). The difference between dS and AdS spaces is the sign of the cosmological constant, where AdS has negative Λ. This is of course mostly due to convention, however the literature concerning string theory tends to consider it to be negative.

We consider an $(n+1)$-dimensional AdS_{n+1} as a submanifold of a pseudo-Euclidean $(n+2)$-dimensional embedding space. It preserves the "Lorentz-like" group $SO(2,n)$. We then define AdS_{n+1} as the locus of $y^2 = b^2$ =const. where $(y^a) = (y^0, y^1, ..., y^n, y^{n+1})$ are the coordinates and the transformations of the group is $y^a \to y^{'a} = \Lambda^a_b y^b$. It can be shown then that quantum theories on AdS_{n+1} should have an $SO(2,n)$ invariance [9]. On the boundary of $SO(2,n)$, the group acts as the conformal group acting on Minkowski space[2].

If one considers an n-dimensional Euclidean space E^n, then its conformal group is $SO(1,n+1)$. One can check this by counting the number of generators between this group and the Poincaré group in n dimensions, and adding an extra conformal transformations for dilations and the special conformal transformation. One then requires that the action of $SO(1,n+1)$ on the boundary gives conformal transformations. We then have that the conformal algebra represents translations, Lorentz rotations, Dilations and Special Conformal Transformations. Knowing what we do now about AdS and the conformal group, Maldacena's conjecture then relates string theory in asymptotically AdS space to a conformal field theory on the boundary at spatial infinity. Also, concerning the AdS_3/CFT_2 correspondence, it is essentially stated that the partition functions on both sides are equal, and in the high energy regime the AdS side is dominated by an asymptotically AdS_3 black hole, hence the importance of the BTZ black hole.

[2]Proof of this can be found in [9]

On the CFT side, we must quickly cover the Virasoro algebra. Firstly, what the Virasoro algebra is and then secondly why it is important. The Virasoro algebra is a complex Lie algebra, which is spanned by generators L_N for the central charge c. They satisfy the commutation relations $[c, L_n] = 0$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{1.48}$$

If one is so inclined, one can find many sources pertaining to the derivation of the Virasoro algebra as that of the unique central extension of the Witt algebra, however it will not be covered in this dissertation.

It was found by [10] that the generators to the diffeomorphism transformations which preserve the asymptotic AdS form of the metric also form the Virasoro algebra, which will be viewed as the algebra that describes conformal symmetries in the CFT dual[3]. For the case of $sl(2, R)$ pure gravity, one obtains the generalization of the Virasoro algebra asymptotically, while $sl(N, R)$ generates the W_N algebra asymptotically [11]. On a side note, much like hs$[\lambda]$ theory, which we will discuss much later, the asymptotic symmetry is a particular W-algebra, named $W_\infty[\lambda]$.

The local Virasoro symmetry used to describe the CFT side of the theory is not available, however there exists the large N limit, which can be used as a mean field-like description. What we mean by the large N limit is that we use a technique to obtain solutions similar to that of the central limit theorem of the theory of probabilities. In a very vague sense, if a field has N components, then in the large N limit the scalar composite fields are made up of many terms and these terms may have some fluctuations if they are not explicitly correlated. As such, if we construct a field theory for these scalars, and integrate out the degrees of freedom, we are able to solve the theory in the large N limit as well as in a $1/N$ expansion [12]. This large N description of CFT includes gravity in an asymptotically AdS spacetime. This is linked to the existence of D-branes and string theory, which will not be covered in detail. Since we are working with an Einstein gravity dual, we move into a strong coupling. Quantizing strings in AdS spacetime complicates this regime.

Looking at the higher spin theories in AdS, the theories include an infinite number of massless interacting fields with spin $s \geq 2$, and these theories are possibly a description of the weak coupling limit of large N gauge theories, or at least a sector of it. Related to the large N limit, we define a 't Hooft limit

$$N, k \to \infty; \quad 0 \leq \lambda \equiv \frac{N}{k+N} \leq 1 \quad \text{fixed}. \tag{1.49}$$

These theories act as vector models, and the central charge $c_N(\lambda)$ scales as N [13]. There is a discrete set of CFT which coalesce into a line which is described by the 't Hooft coupling λ. A free theory of N complex fermions can be described with $\lambda = 0$, while $\lambda = 1$ best describes a type 'strong' coupling region. The coupling always remains of order one, which indicates an absence of a dual Einstein gravity regime. The existence of the continuous parameter λ also links the model to higher dimensional supersymmetric gauge theories.

The CFT's we look at now are the so-called $\mathcal{W}_N$ minimal models. These are described by the coset [14]

$$\frac{\mathfrak{g}_k \oplus \mathfrak{g}_1}{\mathfrak{g}_{k+1}} \tag{1.50}$$

[3] When we discuss the Chern-Simons formulation, these Virasoro generators manifest in terms of the Hamiltonian reduction of the gauge connection

where $\mathfrak{g} = su(N)$ for $\mathcal{W}_N$. After a bit of work, the central charge can be described by

$$c_N(p) = (N-1)[1 - \frac{N(N+1)}{p(p+1)}] \leq (N-1) \tag{1.51}$$

where we have introduced $p = k + N \geq (N+1)$. Some interesting choices to look at are for $N = 2$ we have the unitary series of Virasoro minimal models for $\mathfrak{g} = su(2)$, for $k = 1$ we have $c = \frac{2(N-1)}{N+2}$ which can be realized by a theory of Z_N parafermions [15]. Also for $p \to \infty$ where $c = (N-1)$ and the symmetry corresponds to the Casimir algebra of the $su(N)$ affine algebra.

We must now quickly touch on higher spin theories. In three dimensions, the massless higher spin fields do not contain propagating degrees of freedom. One can also truncate classically to a finite amount of them. Also there exists a Chern-Simons formulation[4] of the classical action to these theories. For spin N, the CS gauge group is $SL(N,R) \times SL(N,R)$ in the Lorentzian signature and $SL(N,C)$ in the Euclidean signature.

The massless spin s fields in three dimensions are completely symmetric tensors $\varphi_{\mu_1\mu_2\ldots\mu_s}$ which are subject to a double trace constraint

$$\varphi_{\mu_5\ldots\mu_s\alpha\lambda}{}^{\alpha\lambda} = 0. \tag{1.52}$$

This is of course only possible for $s > 4$. There is also a gauge invariance which leads to field configurations identification

$$\varphi_{\mu_1\mu_2\ldots\mu_s} \sim \varphi_{\mu_1\mu_2\ldots\mu_s} + \nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}. \tag{1.53}$$

That gauge parameter is traceless and a symmetric tensor of rank $(s-1)$, which is only traceless for $s \geq 3$. What makes AdS_3 special (and it was and shall again be discussed) is that for every $N \geq 2$, we have truncations to theories which have a spectrum of single massless fields for every spin $s = 2, \ldots, N$. This description of these fields was given by Fronsdal [16], but as we will see we will not require it as (in $3d$) we re-gauge them to Chern-Simons gauge fields.

The higher spin gauge fields can be expressed using a generalized vielbein and spin connection

$$e_\mu^{a_1\ldots a_{s-1}} \quad \omega_\mu^{a_1\ldots a_{s-1}} \tag{1.54}$$

with s the spin of the gauge field. Under a maximal spin N, the variables can be compacted into a single multiplet under the higher spin symmetry. The Chern-Simons action and level will be covered in higher detail in the next chapter, however relating to the $\mathcal{W}_N$, it was argued in [17] that using the Chern-Simons formulation, a maximal N theory has an asymptotic $\mathcal{W}_N$ symmetry algebra. On the classical level the central charge is

$$c = \frac{3l}{2G_N} \tag{1.55}$$

with l being the radius of curvature, and the result happens to be the same as the Brown-Henneaux result for Einstein gravity on AdS_3 [10].

The Fronsdal fields about an AdS background are related to the vielbeins as

$$\varphi_{\mu_1\mu_2\ldots\mu_s} = \frac{1}{s}\bar{e}^{a_1}_{(\mu_1}\ldots\bar{e}^{a_{s-1}}_{\mu_{s-1}}e_{\mu_s)a_1\ldots a_{s-1}}. \tag{1.56}$$

The $\bar{e}^a_\mu$ is the vielbein for the AdS_3 background metric. Generally, we describe the Chern-Simons combinations

$$j_\mu^{a_1\ldots a_{s-1}} = (\omega + \frac{e}{l})_\mu^{a_1\ldots a_{s-1}}, \quad \tilde{j}_\mu^{a_1\ldots a_{s-1}} = (\omega - \frac{e}{l})_\mu^{a_1\ldots a_{s-1}}, \tag{1.57}$$

[4]We will be discussing this in more detail in the following chapter.

and the gauge potentials

$$A = (j_\mu^a T_a + ... + j_\mu^{a_1...a_{N-1}} T_{a_1...a_{N-1}}) dx^\mu \tag{1.58}$$

$$\tilde{A} = (\tilde{j}_\mu^a T_a + ... + \tilde{j}_\mu^{a_1...a_{N-1}} T_{a_1...a_{N-1}}) dx^\mu. \tag{1.59}$$

Also, the T_a are $SL(2,R)$ generators with

$$T_{a_1...a_{s-1}} \sim T_{(a_1}...T_{a_{s-1})}. \tag{1.60}$$

From these equations, we can view $A, \tilde{A}$ as $SL(N,R)$ gauge fields. It was noticed in [18] that for $N=3$, the Fronsdal fields are expressed in terms of the $SL(3,R)$ Casimir generators. This should generalize to any N, which fits with the thought that the vacuum sector of CFT describes the pure higher spin field excitations, since it contains the $SU(N)$ Casimir algebra.

Back to the higher spin fields, in three dimensions, we have distinct matter multiplets which contains only scalar and/or fermion fields (in higher dimensions these matter fields lie in the same multiplet as the higher spin fields). The mass of these fields is related to a deformation parameter of the higher spin algebra

$$M^2 = \Delta(\Delta - 2). \tag{1.61}$$

This matter multiplet contains four scalars, the above two as well as two $M^2 = \Delta(\Delta + 2)$, which transform under a global symmetry group. This multiplet is typically truncated to just the first two scalars. For a generic Δ, one cannot truncate the massless fields to a maximal spin, hence a $N \to \infty$ limit is used when these fields are added.

We aim to describe the bulk gravity in a Chern-Simons formulation, and by doing this it is necessary to specify the boundary conditions. All the dynamics will be resultant from these conditions. For $N = 2$ AdS_3 gravity, we have the metric

$$ds^2 = l^2(1 + \frac{r^2}{l^2})dt^2 - (1 + \frac{r^2}{l^2})^{-1} dr^2 - r^2 d\phi^2. \tag{1.62}$$

The boundary in this case is a cylinder with a parametrization in t. We must add a boundary term or specify boundary conditions for the gauge field for the variational principle for the Chern-Simons action, so we add the boundary conditions of the cylinder (specifically on *its* boundary)

$$A_{\bar{w}} = 0, \quad \tilde{A}_w = 0. \tag{1.63}$$

This can be seen as the gauge fields $A, \tilde{A}$ having only left- and right-moving components on the boundary, respectively. Also, this indicates that the gauge fields are $SU(N)$ on the boundary. A note now, is that the boundary conditions are incomplete, since they do not yet guarantee an asymptotically AdS_3 geometry. For pure gravity, only the lowest-spin components in the algebra decomposition of the principal $sl(2,R)$ embedding is left after some additional conditions. As such, the Wess-Zumino-Witten (WZW) model [19] is gauged, and the resultant theory is a classical Drinfeld-Sokolov reduction [20]. Hence the affine Kac-Moody algebra is reduced to a $\mathcal{W}_N$-algebra, which establishes the symmetry generators [21].

On this note, it might seem prudent to investigate exactly what symmetry (and supersymmetry) entails, for the reader who has not had much experience in the field. This comes at the cost that it is an extremely large field and is actively being researched, and as such we will only be covering a very small portion of it. For completeness the topic of supersymmetry is covered once again in our very specific context much later, as to see how the addition of supersymmetry to our black hole solutions affect the type of solutions we obtain.

1.3 Supersymmetry

Symmetry and supersymmetry is fundamental to modern physics, and as such requires some time to review. We begin by defining simple symmetry, and though we talk of symmetries prior to this section, it will still be prudent to have a section dedicated to the study.

We begin by defining invariance, which is a property objects attain if they do not change under certain transformations. We also define conserved charges, by first referring to Noether's theorem. It states that for every continuous symmetry of an action, there is a corresponding conserved current. A conserved current satisfies the continuity equation

$$\partial_\mu j^\mu(x) = 0, \tag{1.64}$$

with $j^\mu(x)$ built up from fields in the lagrangian. There are also internal, as well as external symmetries, where external symmetries are tied to the invariance under the Poincaré group. Internal symmetries arise when the lagrangian has several fields, which appear in a symmetric way. More explicitly, external symmetries are related to spacetime transformations, which has continuous types, such as the Poincaré group; and discrete ones, such as parity, charge conjugation and time reversal. Internal symmetries appear when several particles of the same type are combined. Globally, they can appear as isospin and its generalization to flavour symmetry, or local (gauged), such as the $U(1)$ of electromagnetism, the $SU(2)$ of weak interactions or $SU(3)$ of the strong interaction.

Internal symmetries commute with the Poincaré group, and as such for some given internal symmetry generators R^a,

$$[R^a, R^b] = 2if^{abc}R^c \tag{1.65}$$

these commute with the Casimirs of the Poincaré group

$$[R^a, P^2] = 0, [R^a, W^2] = 0. \tag{1.66}$$

This indicates that particles that are related by internal symmetries have the same mass as well as spin.

One cannot mix internal and external symmetries in some non-trivial way due to the Coleman-Mandula theorem [22]. It assumed that all symmetries were Lie algebraic in nature, which appears to be a reasonable assumption since all the known continuous symmetries were of this type. We can then consider spinorial symmetries, which have half-integer spin generators, which are fermionic. This is due to the spin-statistic theorem, and due to it being fermionic, or anticommuting, it does not form a lie algebra. It was shown in [23] that the possibilities are only spin-$\frac{1}{2}$, and [24], [25] showed concrete realizations of this symmetry. This fermionic symmetry is the foundation of what is known as supersymmetry.

As such, supersymmetry is defined as the symmetry obtained by extending the Poincaré group with the addition of anticommuting spin-$\frac{1}{2}$ operators. Since the generators fall into spin-$\frac{1}{2}$ representations of the Lorentz group, they can be taken to be Weyl spinors Q_α.In 4 dimensions, they fill out a 4-component Dirac spinor

$$Q_D = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix} \tag{1.67}$$

which we will call supercharges. We can have more than one such generator, hence Q_D^A, which brings us to extended supersymmetry.

Within these extended supersymmetric theories, one has what is called Bogomol'nyi-Prasad-Sommerfield (BPS) states, which are states that only preserve supersymmetry partially. These

BPS states admit Killing spinors which satisfy the Killing spinor equations, and these are much simpler differential equations to solve than the equations of motion, this being due to the fact that they are first order equations. These are much simpler to solve than higher order equations. One can then look at specific sectors of the supersymmetric theory, for instance only looking at the bosonic sector's BPS states equates to states with minimized energy. One can actually impose fake supersymmetry on certain systems, which then obtain BPS states for non-supersymmetric (bosonic) models, and thus satisfies the lower order equations of motion [26], [27], [28].

Recall that generators of supersymmetry are fermionic, and as such each bosonic state requires a fermionic state due to this supersymmetry. As such, we must redefine a particle to include fermionic as well as bosonic components. These are then called superparticles and the individual components belong to the same supermultiplet.

We would like to know how supersymmetry acts on fields, since we have been considering supersymmetry only on states so far. The model we can consider as to how these act is the Wess-Zumino Lagrangian model [25], [24], which consists of a complex scalar (A), a Weyl fermion (ψ_α) and a complex auxiliary field (F) [23]. The full Lagrangian for the free theory is the sum of three terms,

$$\mathcal{L}_{\text{WZ}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{int}} \tag{1.68}$$

with kinetic term

$$\mathcal{L}_{\text{kin}} = \partial_\mu A^* \partial^\mu A + i\partial_\mu \psi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + F^* F, \tag{1.69}$$

mass term

$$\mathcal{L}_{\text{mass}} = -mAF + \frac{1}{2}m\psi^2 - mA^*F^* + \frac{1}{2}m\bar{\psi}^2 \tag{1.70}$$

and interaction terms

$$\mathcal{L}_{\text{int}} = -gAAF + gA\psi^\alpha\psi_\alpha - gA^*A^*F^* + gA^*\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}. \tag{1.71}$$

These terms also have the supersymmetry transformations

$$\delta_\xi A = \sqrt{2}\xi^\alpha\psi_\alpha, \tag{1.72}$$

$$\delta_\xi \psi_\alpha = \sqrt{2}\xi_\alpha F + i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu A, \tag{1.73}$$

$$\delta_\xi F = -\sqrt{2}i\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}. \tag{1.74}$$

Note that the WZ model we are currently investigating is in $4d$, and while the example we will be considering later on will be in $3d$ (i.e. the Chern-Simons model), the overall idea on supersymmetry is the same, with some details being different due to the change in dimension. It is also of interest to note that the supersymmetry parameter (ξ) is fermionic, or more specifically the parameter's conjugate is related to the variation of the fermion.

Now it is claimed that each term of the Lagrangian is invariant under a specific set of supersymmetry transformations. It is of course true, and hence the model is found to be supersymmetric.

Supersymmetry in the context of our extremal black holes will be reviewed much later when looking at our $sl(3|2)$ solutions, as well as the higher dimensional case. With symmetry and supersymmetry handled (at least for what we require of it), we can begin in earnest in describing Chern-Simons Theory.

Chapter 2

Chern-Simons Theory and Higher Spin Black Holes

The reason as to why we would like to investigate higher spin black holes is because of the appearance of gravity theories coupled to massless higher spin fields within the AdS/CFT correspondence. Even though they are more intricate than the usual supergravity approximation in AdS/CFT, they are more tractable than the full string theory in AdS, where an infinite tower of massive string excitations are used. We will now look at the three-dimensional higher spin theory.

The three dimensional case is considerably easier to work with than the higher dimensional cases, for the following reasons. Firstly, it proves to be consistent to truncate the tower of higher spin fields onto a finite set, specifically where the only spins that are included would be those where $s \leq N$. The second reason is that the action can be expressed as a Chern-Simons Theory. In the following example we will look at a simple case of a metric coupled to a spin-3 field and the associated equivalent of the BTZ black hole, which carries non-zero Virasoro zero-mode charges, which correspond to mass and angular momentum. We will discuss the precise definition of the BTZ black hole later on. Looking at the thermodynamics of the BTZ black hole, it can be seen as contributing a finite temperature partition function, and since the BTZ entropy takes the form of Cardy's formula, then the entropy can be matched to the asymptotic number of states of a dual boundary CFT [29]. Similarly, the spin-3 black hole we consider contributes to a partition function that includes the chemical potentials for the spin-3 charge.

2.1 Chern-Simons Action and the $sl(3, R)$ black hole

Witten discovered that gravity with a negative cosmological constant in $2+1$ dimensions can be reformulated as a $SL(2, R) \times SL(2, R)$ Chern-Simons theory [30]. We will briefly review it here. The Einstein-Hilbert action for gravity coupled to matter in $2+1$ dimensions is represented as

$$I = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g}(R - 2\Lambda) + I_{\text{matter}}. \tag{2.1}$$

The equations of motion for this action are

$$R_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}, \tag{2.2}$$

and these are diffeomorphism covariant. Now, the Riemann tensor and the Ricci tensor have an interesting property in this dimension in that they can both be expressed in terms of the other. This can be used to define the following:

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \tag{2.3}$$

This equation is traceless, which is to say that the Weyl curvature tensor is zero. This also implies that for a vacuum, the solutions to the Einstein equation are flat for $\Lambda = 0$, and have constant curvature for $\Lambda \neq 0$. For $\Lambda = 0$, we can show that the Chern-Simons action is equivalent to this form by setting a few conditions. The connection to the Einstein equations is made by defining the 1-Forms of the spin connection and the vielbeins as

$$A^a = \omega^a + \frac{i}{l}e^a, \quad \bar{A}^a = \omega^a - \frac{i}{l}e^a. \tag{2.4}$$

The curvature tensor is defined as

$$R_{ij}{}^a{}_b = \partial_i \omega^a_{j\,b} - \partial_j \omega^a_{i\,b} + [\omega_i, \omega_j]^a_b, \tag{2.5}$$

or equivalently $R = d\omega + \omega \wedge \omega$. Also, this will cause the Einstein-Hilbert action to be

$$I = \frac{1}{2}\int_M \epsilon^{ijkl}\epsilon_{abcd}(e^a_i e^b_j R_{kl}{}^{cd}), \tag{2.6}$$

where one can find by varying the connection in terms of ω, then it is seen that ω is torsion free, and if it is varied in terms of the vielbein e then we find that

$$e^k_a = R_{ik}{}^a{}_b = 0, \tag{2.7}$$

which corresponds to the vanishing of the Ricci tensor, similar to the Einstein equations in a vacuum.

Relatively recently, it was found that an $SL(N,R) \times SL(N,R)$ Chern-Simons theory corresponds to Einstein gravity coupled to $N-2$ symmetric tensor fields with spin $s = 3, 4, ...N$ [31].

We consider the case of $N = 3$, such that we have an $SL(3,R) \times SL(3,R)$ Chern-Simons Theory. This corresponded to spin-3 gravity in three dimensions with a negative cosmological constant. The following is based on [32] and will follow it closely.

The Chern-Simons action is of the following form:

$$S = S_{CS}[A] - S_{CS}[\overline{A}] \tag{2.8}$$

with

$$S_{CS}[A] = \frac{k}{4\pi}\int Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A), \tag{2.9}$$

and similarly for $\overline{A}$. These 1-forms A and $\overline{A}$ take values in the Lie algebra $SL(3,R)$. The Chern-Simons level k is linked to the Newton constant G and the AdS_3 radius l as

$$k = \frac{l}{4G}, \tag{2.10}$$

where we set $l = 1$.

The equations of motion for the action corresponds to the vanishing of the field strengths, i.e.

$$F = dA + A \wedge A = 0, \qquad \overline{F} = d\overline{A} + \overline{A} \wedge \overline{A} = 0. \tag{2.11}$$

It is required to relate these equations of motion to the spin-3 Einstein equations, and as such we use the vielbein e and the spin connection ω in such a form that

$$A = \omega + e, \qquad \overline{A} = \omega - e. \tag{2.12}$$

We then expand the vielbeins and spin connections in a basis of 1-forms dx^μ. We can then write the spacetime metric $g_{\mu\nu}$ and the spin-3 field $\varphi_{\mu\nu\gamma}$ as

$$g_{\mu\nu} = \frac{1}{2} tr(e_\mu e_\nu), \qquad \varphi_{\mu\nu\gamma} = \frac{1}{9\sqrt{-\sigma}} tr(e_{(\mu} e_\nu e_{\gamma)}). \tag{2.13}$$

The parameter σ is assumed to be negative, but it can be kept arbitrary for the moment. For the vanishing of the spin-3 fields, i.e. where $\varphi = 0$, the flatness condition is equivalent to Einstein's equations for the metric $g_{\mu\nu}$ with a torsion-free spin connection.

Upon acting on the metric and spin-3 field, the $SL(3,R) \times SL(3,R)$ gauge symmetries of the Chern-Simons theory turn into diffeomorphisms as well as the spin-3 gauge transformations. Under these diffeomorphisms, the metric and the spin-3 field will transform the same as the standard tensor transformation rules. The spin -3 gauge transformations however act non-trivially with the metric and spin-3 field. Ignoring the spin-3 gauge transformation will allow us to view the theory as a particular diffeomorphism invariant theory of a metric and rank-3 symmetric tensor field.

A quick look at the asymptotic AdS_3 boundary conditions shows that the metric of global AdS_3 in terms of the Fefferman-Graham coordinates [33] can be written as

$$ds^2_{AdS} = d\rho^2 - (e^\rho + \frac{1}{4} e^{-\rho})^2 dt^2 + (e^\rho - \frac{1}{4} e^{-\rho})^2 d\phi^2 \tag{2.14}$$

which is obtainable from the connections

$$A_{AdS} = (e^\rho L_1 + \frac{1}{4} e^{-\rho} L_{-1}) dx^+ + L_0 d\rho \tag{2.15}$$

$$\overline{A}_{AdS} = -(e^\rho L_{-1} + \frac{1}{4} e^{-\rho} L_1) dx^- - L_0 d\rho \tag{2.16}$$

with $x^\pm = t \pm \phi$, $\phi \cong \phi + 2\pi$ and the L_i's are the generators for $SL(3,R)$.

We will introduce $b(\rho) = e^{\rho L_0}$ such that the connection becomes

$$A_{AdS} = b^{-1}(L_1 + \frac{1}{4} L_{-1}) b dx^+ + b^{-1} \partial_\rho b d\rho \tag{2.17}$$

$$\overline{A}_{AdS} = -b(L_{-1} + \frac{1}{4} L_1) b^{-1} dx^- + b \partial_\rho b^{-1} d\rho. \tag{2.18}$$

The form of these connections can be said to be asymptotically AdS_3 if, accordingly to [21], the connection A obeys $A_- = 0$, $A_\rho = b^{-1}(\rho)\partial_\rho b(\rho)$, and

$$A = A_{AdS} \sim \mathcal{O}(1) \quad \text{as} \quad \rho \to \infty \tag{2.19}$$

and similarly for $\overline{A}$.

It is possible to make gauge transformations to essentially "gauge away" the radial dependence, and as such the asymptotic AdS_3 connections take the form

$$A = b^{-1} a(x^+) b + b^{-1} db, \qquad \overline{A} = b \overline{a}(x^-) b^{-1} + b db^{-1}, \tag{2.20}$$

where the a and $\overline{a}$ connections are given by

$$a(x^+) = (L_1 - \frac{2\pi}{k}\mathcal{L}(x^+)L_{-1} + \frac{\pi}{2k\sigma}\mathcal{W}(x^+)W_{-2})dx^+ \tag{2.21}$$

$$\overline{a}(x^-) = -(L_{-1} - \frac{2\pi}{k}\overline{\mathcal{L}}(x^-)L_1 + \frac{\pi}{2k\sigma}\overline{\mathcal{W}}(x^-)W_2)dx^-. \tag{2.22}$$

The convention differs between many sources as to the specific writing of these connections, such as the naming of the sources $\mathcal{L}$ and $\mathcal{W}$, and the form of the coefficients, etc. However in this case, the modes $\mathcal{L}$ and $\mathcal{W}$ are identified with the stress tensor and spin-3 current respectively.

Our discussions thus far have all been to be able to write out a general black hole using the framework we have established. However we must still cover an exceptionally important case of a black hole, named the BTZ black hole, due to its relevance in almost all of black hole physics since it is so well understood.

2.2 BTZ Black Hole in Higher Spin

The BTZ black hole, so named after the original creators, Bañados, Teitelboim and Zanelli, is a black hole in $(2+1)$ dimensions with negative cosmological constant [34]. It has a metric in Schwarzschild coordinates as

$$ds = -(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})dt^2 + (-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})^{-1}dr^2 + r^2(d\phi - \frac{J}{2r^2}dt)^2, \tag{2.23}$$

with M and J the mass and angular momentum parameters considered for a black hole. One can check that this metric satisfies the Einstein equation for $2+1$ dimensions, with a negative cosmological constant, which we can take as $\Lambda = -1/l^2$. The Einstein equations then look like

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{l^2}g_{\mu\nu}. \tag{2.24}$$

We also have an inner and outer horizon at $r = r_\pm$, corresponding to singular locations of the metric, at

$$r_\pm^2 = \frac{Ml^2}{2}\Big(1 \pm \sqrt{1 - \frac{J^2}{M^2l^2}}\Big). \tag{2.25}$$

We can then typically express the mass and angular momentum iin terms of these $r_\pm$,

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{2r_+r_-}{l}. \tag{2.26}$$

We choose the units in such a way that satisfies Newton's constant to be $8G = 1$. Also note that the $M = -1, J = 0$ metric results in three dimensional anti-de Sitter space (AdS_3).

For a BTZ black hole of mass M and angular momentum J, we have

$$\mathcal{L} = \frac{M - J}{4\pi}, \qquad \overline{\mathcal{L}} = \frac{M + J}{4\pi} \tag{2.27}$$

and $\mathcal{W} = \overline{\mathcal{W}} = 0$, which then produces the connections

$$A = (e^\rho L_1 - \frac{2\pi}{k}e^{-\rho}\mathcal{L}L_{-1})dx^+ + L_0 d\rho \tag{2.28}$$

$$\overline{A} = -(e^\rho L_{-1} - \frac{2\pi}{k}\overline{\mathcal{L}}e^{-\rho}L_1)dx^- - L_0 d\rho. \tag{2.29}$$

Using the equations we defined for relating the vielbeins and connections as well with the metric and spin-3 field, we can then write the BTZ metric as

$$ds^2 = d\rho^2 + \frac{2\pi}{k}(\mathcal{L}(dx^+)^2 + \overline{\mathcal{L}}(dx^-)^2) - (e^{2\rho} + (\frac{2\pi}{k})^2 \mathcal{L}\overline{\mathcal{L}} e^{-2\rho}) dx^+ dx^-. \tag{2.30}$$

The entropy is given by

$$S = \frac{A_H}{4G} = 2\pi(\sqrt{2\pi k \mathcal{L}} + \sqrt{2\pi k \overline{\mathcal{L}}}). \tag{2.31}$$

In terms of the Brown-Henneaux central charge,

$$c = \frac{3}{2G} = 6k, \tag{2.32}$$

we have the well known entropy result that takes the form of Cardy's formula [35].

If we move to the Euclidean signature, $x^+ \to z$ and $x^- \to -\overline{z}$, and demanding that there is the absence of a conical singularity at the horizon, we then have the periodicity conditions

$$(z, \overline{z}) \cong (z + 2\pi\tau, \overline{z} + 2\pi\overline{\tau}), \tag{2.33}$$

with the modular parameters

$$\tau = \frac{ik}{2}\frac{1}{\sqrt{2\pi k \mathcal{L}}}, \qquad \overline{\tau} = \frac{-ik}{2}\frac{1}{\sqrt{2\pi k \overline{\mathcal{L}}}}. \tag{2.34}$$

To cap off we define the Hawking temperature $T = 1/\beta$ and the angular velocity of the horizon Ω to be expressed similarly

$$\tau = \frac{i\beta + i\beta\Omega}{2\pi}, \qquad \overline{\tau} = \frac{-i\beta + i\beta\Omega}{2\pi}, \tag{2.35}$$

and looking at the framework of the AdS_3/CFT_2 correspondence, the BTZ black hole is contributing to the partition function

$$Z(\tau, \overline{\tau} = Tr_{AdS} e^{4\pi^2 i\tau\hat{\mathcal{L}} - 4\pi^2 i\overline{\tau}\hat{\overline{\mathcal{L}}}} = Tr_{CFT}(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}) \tag{2.36}$$

where $\hat{\mathcal{L}}$ and $\hat{\overline{\mathcal{L}}}$ denote the Virasoro zero mode operators: $2\pi\hat{\mathcal{L}} = L_0$, $2\pi\hat{\overline{\mathcal{L}}} = \overline{L}_0$.

With this finished, we can comfortably begin describing the next step in our journey: to review extremality, realise this does not cover our specific case and then redefine extremality to still be consistent.

Chapter 3

Extremal Higher Spin Black Holes

We aim towards finding black hole solutions for three-dimensional Chern-Simons theory, and as such we start off with its Euclidean action

$$I_{CS} = \frac{ik_{cs}}{4\pi} \int_M Tr[CS(A) - CS(\bar{A})] \tag{3.1}$$

with A and $\bar{A}$ valued in the same algebra. Also

$$CS(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A \tag{3.2}$$

is seen as the Chern-Simons form and Tr denotes the trace (or supertrace if sypersymmetry is involved) for the given representation. It is important to note that in the Euclidean signature the connections are usually complex, and as such we have

$$A^\dagger = -\bar{A}. \tag{3.3}$$

This is our reality condition for the action and any physical observables. If we work in the Lorentzian signature however, one works with two independent connections A and $\bar{A}$, each valued in the appropriate real form of its gauge algebra. In either case, charges and conjugate potentials are real in both sectors.

The Chern-Simons theory is useful in its own right due to its gauge freedoms. We will exploit these to essentially "gauge-away" the radial dependence on the connection, i.e:

$$A(\rho, z, \bar{z}) = b^{-1}(\rho)(a(z, \bar{z}) + d)b(\rho), \qquad \bar{A}(\rho, z, \bar{z}) = b(\rho)(\bar{a}(z, \bar{z}) + d)b^{-1}(\rho), \tag{3.4}$$

and from here we will focus on the boundary connections $a(z, \bar{z})$ and $\bar{a}(z, \bar{z})$. The reason we can gauge away this dependence, is due to the fact that in a purely 3d gravitation setting, the Fefferman-Graham expansion will truncate after a set amount of terms in its radial component [36].

3.1 Extremality

Extremality is typically tied to the confluence of two horizons, or at least this is the case with conventional gravitational theories. By taking a non-extremal black hole and considering the extremal limit, the standard static coordinates then become pathological [37], and that the region between the degenerating horizons remains of constant four-volume. If one considers regions which are a finite distance away from the horizon, and then take the extremal limit, one then has extremal black holes.

Black holes radiate and exhibit some formal similarity with thermodynamical systems, hence it is possible to associate a Bekenstein-Hawking entropy to the black hole, which is proportional to the area of the event horizon [7], [6], [5]. There have been many successes in reproducing the Bekenstein-Hawking entropy formula, however the theory for black hole entropy is still incomplete. An inconsistency arises for extremal black holes when considering solutions using semi-classical methods, which indicate that the entropy for these extremal black holes vanishes even if the event horizon area is non-zero. Despite this, extremal black holes were specifically used for string theory microstate counting in determining non-zero entropy [38].

A method to determine the entropy that is both complementary to the semi-classical methods as well as the string theory microstates counting, is referred to as "dual microstate counting". The main point of this approach is that for the near-horizon case, the geometry of an extremal black hole locally matches that of two-dimensional anti-de Sitter space times a two sphere ($AdS_2 \times S_2$). The symmetries of this geometry define a CFT (Conformal Field Theory), of which the entropy can be determined and then associated with a black hole [29]. This method has many successes and does not mention string theory, nor does it rely on supersymmetry. The discrepancy between the string theory and the semi-classical calculations remain elusive.

The following background closely follows [39], and as such we start off with four-dimensional, static, spherically symmetric solutions to the Einstein-Maxwell system with zero cosmological constant, with $G = 1$,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - \frac{F^2}{4}). \tag{3.5}$$

This action admits solutions of the Reissner-Nordström type, with a metric of the form

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega_2^2 \tag{3.6}$$

and a field strength of

$$F = \frac{Q_e}{r^2}dt \wedge dr + Q_m \sin\theta d\theta \wedge d\phi. \tag{3.7}$$

These coordinates do not cover the entire manifold, and event horizons are located at the coordinate singularities

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{3.8}$$

where

$$Q \equiv \sqrt{Q_e^2 + Q_m^2}. \tag{3.9}$$

We will always take this relation to be positive. We then look at extremality, where $M = Q$, and find that the event horizons coincide at the extremal radius:

$$\rho \equiv r_+ = r_- = M = Q. \tag{3.10}$$

We then have an extremal black hole metric as

$$ds^2 = -\frac{(r-\rho)^2}{r^2}dt^2 + \frac{r^2}{(r-\rho)^2}dr^2 + r^2 d\Omega_2^2. \tag{3.11}$$

We can also find a set of solutions to the Einstein-Maxwell system with the $AdS_2 \times S_2$ conditions, where the metric would look like

$$ds^2 = \frac{\rho^2}{\cos^2\theta}(-d\tau^2 + d\theta^2) + \rho^2 d\Omega_2^2, \tag{3.12}$$

with $\rho = Q$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $-\infty \leq \tau \leq \infty$. Both the extremal and the compactification solutions can be specified by the charges $\{Q_e, Q_m\}$, since the mass of the black hole is no longer an independent parameter. However, the charges alone are insufficient to specify the global properties of the solution. That being said, it is possible to find an equivalence for the solutions locally, when taking the near-horizon limit.

If one changes the spacelike radial coordinate to $\lambda = \frac{r-\rho}{\rho}$, then the metric becomes

$$ds^2 = -\frac{\lambda^2}{(1+\lambda)^2}dt^2 + \frac{(1+\lambda)^2}{\lambda^2}d\lambda^2 + \rho^2(1+\lambda)^2 d\Omega_2^2. \tag{3.13}$$

This extremal metric has the property that the proper distance along a $t =$ constant slice form a point λ to the horizon at $\lambda_0 \to 0$ which is logarithmically divergent

$$\int_{\lambda_0}^{\lambda} d\lambda \frac{1+\lambda}{\lambda} = \lambda - \lambda_0 + \log(\frac{\lambda}{\lambda_0}). \tag{3.14}$$

One then takes the near horizon limit of the extremal black hole ($\lambda \to 0$), then the metric becomes

$$ds^2 = -\lambda^2 dt^2 + \frac{\rho^2}{\lambda^2}d\lambda^2 + \rho^2 d\Omega_2^2, \tag{3.15}$$

which, after transforming the coordinates in the original metric, can be recognised as $AdS_2 \times S_2$

$$t = \frac{\rho \sin\tau}{\cos\tau - \sin\theta}, \qquad \lambda = \frac{\cos\tau - \sin\theta}{\cos\theta}. \tag{3.16}$$

From this, the horizon at $\lambda = 0$ is identified with $\theta = \pm\tau + \pi/2$. The extremal solution regions are locally approximated $AdS_2 \times S_2$ near the horizon, but the approximation is only exact at the horizon, which has infinite proper distance from any point outside of the black hole.

The feature of the confluence of two horizons usually implies zero Hawking temperature of the black hole. It would therefore make sense to declare that for higher spin theories a black hole solution is one where a solution has zero temperature. However, we would like a definition that only depends on the topological formulation of the theory, which incorporates the degeneration or confluence as well as yielding the condition of zero temperature as a result. This is a main condition on the thermodynamics of the system and is a required condition for extremality, which is difficult to encode into a general condition for extremality, as we shall see shortly.

Even the BTZ black hole, the simplest case for the study of black holes, can achieve an extremal state. The metric of a rotating BTZ black hole [40] is given by

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{R^2 r^2}dt^2 + \frac{R^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}dr^2 + r^2(d\phi - \frac{r_+ r_-}{R r^2}dt)^2 \tag{3.17}$$

where $r_\pm$ are the radii of the outer and inner horizons, and R is the radius of AdS space. In terms of the mass M, the angular momentum J and the Hawking temperature T_H, we can express them as

$$M = \frac{r_+^2 + r_-^2}{R^2}, \quad J = \frac{2r_+ r_-}{R}, \quad T_H = \frac{r_+^2 - r_-^2}{2\pi R^2 r_+}. \tag{3.18}$$

The extremal limit is realised when $MR = J$ which is achieved when $r_0 \equiv r_+ = r_-$ in the metric. This of course causes the Hawking temperature to vanish and reduces our metric to

$$ds^2 = -\frac{(r^2 - r_0^2)^2}{R^2 r^2}dt^2 + \frac{R^2 r^2}{(r^2 - r_0^2)^2}dr^2 + r^2(d\phi - \frac{r_0^2}{R r^2}dt)^2 \tag{3.19}$$

In the case that it might interest someone in studying the ultraviolet set of conformal theories, it will be necessary to map this to Poincaré form

$$ds^2 = R^2 \frac{dU^2 + d\omega_+ d\omega_-}{U^2} \tag{3.20}$$

which has coordinates [41]

$$\omega_- = \frac{R}{2r_0} e^{\frac{2r_0}{r}(\phi - t/R)}, \quad \omega_+ = \phi + \frac{t}{R} - \frac{Rr_0}{r^2 - r_0^2}, \quad U = \frac{r}{\sqrt{r^2 - r_0^2}} e^{\frac{r_0}{R}(\phi - t/R)}. \tag{3.21}$$

Seeing now that multiple different types of extremal black holes exist, one must wonder how to write an extremality condition since the thermodynamics of the black hole are not a simple matter. Or rather, we must try and define a new set of conditions that still encapsulate the zero temperature condition. It might be a point of concern as to why we do not use the already established conditions for extremality in the higher spin case. This is due to the fact that we are dealing with higher spin gauge transformations on top of coordinate transformations. Under these transformations not even the metric remains invariant. As such we must define new conditions which are higher spin gauge invariant, and holonomies of the connection are such conditions. As such one needs to express extremality in terms of properties of the connection without referring to the confluence of horizons, etc. which are concepts that do not translate well in higher spin theory.

3.1.1 Proposed Conditions for Extremality

Following [42], we propose that a $3d$ extremal higher spin black hole is a solution of Chern-Simons theory that corresponds to flat boundary connections a and $\bar{a}$ that satisfies the following:

1. They obey Drinfeld-Sokolov boundary conditions (which will be shown later),
2. They are stationary solutions, i.e. their components are constant,
3. They carry charges and chemical potentials, which are real in the Lorentzian signature,
4. The angular component of one (or both) of the connections a and $\bar{a}$ is non-diagonalizable.

All these conditions are important, however the most time and effort will be placed on the final condition, the fact that the angular component must be non-diagonalizable. The reasoning behind this is as follows: Suppose that both a_ϕ and $\bar{a}_\phi$ are diagonalizable. We have assumed that the boundary conditions are constant, since by the equations of motion, the Euclidean time components of the connection commute with the angular components, and can then be diagonalized with them simultaneously. By the smoothness condition for higher spin black holes [32]

$$Eigen(a_{contract}) = Eigen(\tau a_\omega + \bar{\tau} a_{\bar{\omega}}) = Eigen(iL_0) \tag{3.22}$$

where $a_{contract}$ denotes the connection along the cycle of the boundary torus which becomes contractible in the bulk. This can be solved to find a non-zero temperature and chemical potential. However, if at least one of a_ϕ and $\bar{a}_\phi$ is non-diagonalizable, then $a_{contract}$ will also be non-diagonalizable. By (3.22), both features are compatible if a zero temperature limit is taken, since the smoothness condition becomes degenerate as well. This seems to be consistent with the fact that the solid torus topology of a finite-temperature black hole changes at extremality.

We also add that for a general connection, the degeneration of eigenvalues does not necessarily imply non-diagonalizability, however the specific form of the flat connection from the Drinfeld-Sokolov boundary conditions guarantees that if two eigenvalues of a_ϕ are degenerate, then the connection will be non-diagonalizable [42]. Looking from this point of view, it is possible that equating the eigenvalues of a_ϕ is tied to the confluence of horizons for extremal black holes found in general relativity.

We now want to investigate the main section of this thesis, where we construct the $sl(4|3)$ black hole. This, as one might readily assume, is no simple matter, and as such we must first review the $sl(N|N-1)$ Racah basis, which is the algebra that the generators are constructed from.

Concerning extremality of black holes, what follows here is a review of the construction of the $sl(3|2)$ black hole, one which is consistent with our definition of extramality and which is the foundation that we will be using to construct our own solutions in a higher spin. The aim of this is that in understanding the construction of a higher spin case, one might be able to obtain insight into the generalization to a spin-N case.

3.2 $sl(3|2)$ Solutions

Here we quickly review the construction of solutions for $sl(3|2)$ black holes, in much the same style as [42], however it will be condensed since it is not the main study of the thesis. It is however the foundation of the methodology used to create the $sl(4|3)$ solutions. As such, these solutions will not be focussed on with as much depth as the following chapters. Note that the entire basis was constructed from the Racah basis, which will be covered in more detail in the next chapter, since its study and understanding is tantamount to the construction of any higher dimensional cases.

From the embedding of $sl(2|1)$ in $sl(3|2)$, the even graded sector of this superalgebra decomposes into $sl(2)$ generators (L_i), a spin 0 element (J), a spin 1 multiplet (A_i) and a spin 2 multiplet (W_M). The odd-graded elements decompose to two spin 1/2 multiplets (H_r and G_r) and two spin 3/2 multiplets (T_s and S_s).

As in (2.20), we will gauge away the radial dependence on the connection, and have the charges carried only by the the angular component of the connection. This component ends up looking like

$$a_\phi = L_1 - \mathcal{L}L_{-1} - iQ_1 J - Q_2 A_{-1} - iQ_3 W_{-2}, \tag{3.23}$$

with $\mathcal{L}$, Q_1, Q_2 and Q_3 all real and constant. Using the form of this connection, we redefine the charges in terms of the eigenvalues of $a_\phi + iQ_1 J$, which we will label as

$$\text{eigen}(a_\phi + iQ_1 J) = [\lambda_1, -\lambda_1 + \lambda_2, -\lambda_2, \frac{1}{2}\lambda_3, -\frac{1}{2}\lambda_3]. \tag{3.24}$$

To motivate as to the specific choice of these eigenvalues, it is simply chosen in such a way as to ensure a traceless connection (and even supertraceless if necessary). We specifically aim to define each eigenvalue in terms of the previous diagonal, depending on the size of the block under consideration. What we mean by a block is the separation between what we call the upper- and lower-block, in the current case the upper-block being the 3×3 matrix along the diagonal, and similarly the lower being the 2×2 on the bottom right diagonal. We will see the same methodology being applied in the construction of the higher dimensional case in the next section.

This equation has a characteristic polynomial that factorizes into a cubic and quadratic part,

$$\det(\lambda - a_\phi - iQ_1 J) = (\lambda^3 - 4(\mathcal{L} + Q_2)\lambda + 8iQ_3)(\lambda^2 - \mathcal{L} + Q_2) \tag{3.25}$$

$$= (\lambda - \lambda_1)(\lambda + \lambda_1 - \lambda_2)(\lambda + \lambda_2)(\lambda - \frac{1}{2}\lambda_3)(\lambda + \frac{1}{2}\lambda_3). \tag{3.26}$$

These have respective discriminants as

$$\Delta_3 = 64(4(\mathcal{L} + Q_2)^3 + 27Q_3^2) \tag{3.27}$$

$$= (2\lambda_1 - \lambda_2)^2(2\lambda_2 - \lambda_1)^2(\lambda_1 + \lambda_2)^2, \tag{3.28}$$

and

$$\Delta_2 = 4(\mathcal{L} - Q_2) \tag{3.29}$$

$$= \lambda_3^2. \tag{3.30}$$

Due to the fact that $\mathcal{L}$, Q_2 and Q_3 are real, the eigenvalues λ_1, $-\lambda_1 + \lambda_2$ and $-\lambda_2$ are then purely imaginary when $\Delta_3 < 0$. For $\Delta_3 > 0$, two eigenvalues are minus complex conjugates of each other and the third is purely imaginary. For λ_3 it is easy to see that it is imaginary for $\Delta_2 < 0$ and real for $\Delta_2 > 0$. We split these solutions into two different sectors, namely black holes, and smooth conical defects:

Table 3.1: Discriminant conditions for black holes and smooth conical defects in $sl(3|2)$

$sl(3\|2)$ solutions	Δ_3	Δ_2
Black holes	≥ 0	≥ 0
Smooth conical defects	< 0	< 0

It was proposed in [42] that for the case of $\Delta_3 = 0$ or $\Delta_2 = 0$, the solutions are extremal. Using this assumption we have that

$$\mathcal{L} = \frac{1}{8}(\lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2 + \lambda_3^2), \tag{3.31}$$

$$Q_2 = \frac{1}{8}(\lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2 - \lambda_3^2), \tag{3.32}$$

$$Q_3 = -\frac{i}{8}(-\lambda_1 + \lambda_2)\lambda_1\lambda_2. \tag{3.33}$$

These charges are symmetric polynomials in the eigenvalues. We have chosen the normalization in the eigenvalues such that the choice of $\lambda_1 = \lambda_2 = \lambda_3$ results in the $sl(2|1)$ theory

$$\mathcal{L} = \frac{1}{4}\lambda_1^2, \quad Q_2 = 0, \quad Q_3 = 0, \tag{3.34}$$

as well as

$$\Delta_3 = 4\lambda_1^6, \quad \Delta_2 = \lambda_1^2. \tag{3.35}$$

From this point on one can specialise the classification further, by considering specific Jordan classes, looking at the signs of the respective discriminants, and then classify them based on their eigenvalues. This will be looked at in more detail in the following chapter. Following on the method used in this chapter, in the next chapter we will turn to the construction of the $sl(4|3)$ black hole solutions in full.

Chapter 4

The $sl(N|N-1)$ Racah Basis and the Construction of $sl(4|3)$

An incredibly useful tool in the construction of our basis generators comes from the Racah basis [43]. This basis is set in a continuous-parametric family of infinite-dimensional associative and Lie algebras, and we will specifically be studying the supersymmetric versions of this algebra. One possible application of this basis is a possible method of constructing higher spin generalizations of the Virasoro algebra, which involves an infinite tower of higher spin generators. This will be looked at briefly in a following section.

4.1 The $sl(N|N-1)$ Racah Basis

One of the main ingredients that is necessary for the study of higher-spin black holes are the generators of the $sl(N|N-1)$ superalgebra[1]. The general $sl(N|N-1)$ element can be decomposed [43] into

$$sl(N|N-1) = sl(2) \oplus (\bigoplus_{s=2}^{N-1} g^{(s)}) \oplus (\bigoplus_{s=0}^{N-2} g^{(s)}) \oplus 2 \times (\bigoplus_{s=0}^{N-2} g^{(s+\frac{1}{2})}) \tag{4.1}$$

where $g^{(s)}$ is defined as the chosen spin-s multiplet of $sl(2)$. This quantity is one less than the conformal/spacetime spin. The actual spacetime spin of the corresponding fields is one higher, and as such the current setup describes a system with a maximal spin of 4. The total amount of generators found for any N is discussed more thoroughly in the Appendix.

Typically we take the normal Chern-Simons action at level k to be

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{tr}(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma) \tag{4.2}$$

however in our case we have to consider the difference between two super-Chern Simons actions:

$$S_{CS}[\Gamma, \bar{\Gamma}] = \frac{k}{4\pi} \int \mathrm{str}(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma) - \frac{k}{4\pi} \int \mathrm{str}(\bar{\Gamma} \wedge d\bar{\Gamma} + \frac{2}{3}\bar{\Gamma} \wedge \bar{\Gamma} \wedge \bar{\Gamma}) \tag{4.3}$$

where 'str' is the super-trace, Γ is the connection one form's chiral sector and $\bar{\Gamma}$ is the anti-chiral sector. It is necessary to specify how the gravitational $sl(2)$ sector is embedded in the gauge algebra, specifically when we have the higher-spin theory cast as a Chern-Simons theory.

[1] Additional information on the construction of this superalgbera can be found in the Appendix

This can be seen if we look at, for example, the $sl(N)$ case, where the Chern-Simons theory based upon this algebra can realize distinct higher spin theories from inequivalent embeddings of $sl(2)$. If we demand that the gravitational $sl(2)$ is part of the $osp(2|2)$ superalgebra, then it is allowed to use the Racah basis for the generators and as such the commutation relations. The Racah basis is better suited to identify any physical interpretations of the fields.

So, following [43], we start from $gl(N|N-1)$ and quotient out its center. These generators are $\mathbb{Z}_2$-graded by the Grassmann parity function $P(T) = P(U) = 0, P(Q) = P(\bar{Q}) = 1$. We have that the bosonic generators are denoted by T and U, which generate the $gl(N)$ and $gl(N-1)$ in (4.1) and the fermionic generators by Q and $\bar{Q}$, which generate the half-integer multiplets. The following are the supercommutation relations in schematic form (the complete expressions can be found in the Appendix), with the structure constants being multiples of the Wigner $6j$-symbols, however they are omitted here:

$$\begin{aligned} &[T,T] \sim T, \qquad [U,U] \sim U, \qquad \{Q,\bar{Q}\} \sim T+U, \\ &[T,\bar{Q}] \sim \bar{Q}, \qquad [U,\bar{Q}] \sim \bar{Q}, \qquad [T,Q] \sim Q, \qquad [U,Q] \sim Q. \end{aligned} \tag{4.4}$$

In the complete description of the generators, we denote T^s_m to generate each $g^{(s)}$ multiplet, where $-s \leq m \leq s$ and similarly for U, Q and $\bar{Q}$. We have an identity matrix $\mathbf{1} = \sqrt{N}T^0_0 + \sqrt{N-1}U^0_0$ as the center, which we previously stated is modded out, so that $sl(N|N-1) \sim gl(N|N-1)/\mathbf{1}$.

Now it is necessary to re-define the generators such that they can form a basis for the $sl(2)$ in (4.1), where all the other generators will transform under its irreducible representations. Within the Appendix, it is carefully explained how the generators are constructed for the principal embedding of $sl(2)$ within $sl(4|3)$. Considering linear combinations of T^s_m and U^s_m, we have the following for generators of $sl(N|N-1)$ with $N=3$:

$$L_0 = \frac{1}{\sqrt{2}}(2T^1_0 + U^1_0), \tag{4.5}$$

$$L_{\pm 1} = 2T^1_{\pm 1} + U^1_{\pm 1}, \tag{4.6}$$

$$A_0 = \frac{1}{\sqrt{2}}(2T^1_0 - U^1_0), \tag{4.7}$$

$$A_{\pm 1} = 2T^1_{\pm 1} - U^1_{\pm 1}, \tag{4.8}$$

$$W_{\pm 2} = 4T^2_{\pm 2}, \qquad W_{\pm 1} = 2T^2_{\pm 1}, \qquad W_0 = \sqrt{\frac{8}{3}}T^2_0, \tag{4.9}$$

$$U_0 = \frac{1}{\sqrt{3}}T^0_0 + \frac{1}{\sqrt{2}}U^0_0. \tag{4.10}$$

It might prove necessary to explain in detail the form of the generators. Depending on the choice of N, the size of the generators will of course differ. We will look at the choice of $N=3$ for an example. If $N=3$, then we are working in $sl(3|2)$, with generators that are comprised of 5×5 matrices, sorted into specific 'blocks'.

As a very generic example, let us look at the generator L_0 for $sl(3|2)$:

$$L_0 = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{array}\right]. \tag{4.11}$$

The structure essentially describes along the diagonal certain blocks which are the sizes of our N and $N-1$. For our bosonic cases, these generators will only have elements on the diagonal blocks, with the off-diagonal blocks being empty, and vice versa for our fermionic blocks. These blocks can further be refined into what is called 'Jordan blocks', however we will go into more detail concerning those at a later stage. Note that the set of generators $\{(L_i + A_i)/2, W_i\}$ forms the $sl(3)$ algebra.

4.2 The $sl(4|3)$ Generators

The $sl(4|3)$ generators were constructed using a linear combination of the Racah basis generators mentioned in the previous section, however needed to be properly normalized accordingly. The construction of the generators in the $sl(3|2)$ case from [42] was vague, and as such much time had to be spent in understanding and de-constructing the methodology such that we may define any dimensional algebra's generators. This is an on-going attempt, especially in the generalization to spin-N, however the framework laid out in the current study will make the process much simpler in future.

Initially the generators are constructed with the appropriate Racah generators being added while being multiplied by an unknown coefficient, for example $W_1 = aT_1^2 + bU_1^2$. Let us note, the notation for these generators are associated with their individual spin cases, such that in the current example, the W matrices are the generators for spin-2, hence the use of the upper indices. The lower indices indicate the specific range of matrices inherent to each spin class, such that for any generator A_n^m, we have $-m \leq n \leq m$. Following this, the generators are then commuted with the raising and lowering generators L_1 and L_{-1} respectively. We do know the commutation relations for these operators, since they remain coherent for any spin and any dimension. We relate these to the coefficients and then compare the two to find the values of the coefficients in terms of a central coefficient. In our case these were related to the zero elements of our generators' coefficients i.e. W_0 has coefficients e and f, these were related to e_0 and f_0.

From this, we start commuting the generators with each other. This unfortunately is largely a trial and error process, to see which combinations give valid commutation relations, but the same process is followed as with the raising and lowering generator commutations, comparing the coefficients, and solving. However, certain cases contain elements of other dimensional generators, for example $[X_0, X_1]$ contains a combination of L_1 and A_1, but also X_1. These add yet more coefficients which we call x and y, and are then solved in the form a polynomial.

The generators are then finally achieved, and can be found in the Appendix. They satisfy:

$$\begin{gathered} L_i^\dagger = (-1)^i L_{-i}, \qquad A_i^\dagger = (-1)^i A_{-i}, \\ W_m^\dagger = (-1)^m W_{-m}, \qquad Z_m^\dagger = (-1)^m Z_{-m}, \qquad X_n^\dagger = (-1)^n X_{-n} \end{gathered} \tag{4.12}$$

with the following commutation relations:

$$[L_i, L_j] = (i-j)L_{i+j}, \tag{4.13}$$
$$[L_i, A_j] = (i-j)A_{i+j}, \tag{4.14}$$
$$[L_i, W_j] = (2i-j)W_{i+j}, \tag{4.15}$$
$$[L_i, Z_j] = (2i-j)Z_{i+j}, \tag{4.16}$$
$$[L_i, X_j] = (3i-j)X_{i+j} \tag{4.17}$$

and with the non-vanishing commutations as

$$[A_i, A_j] = (i-j)L_{i+j}, \tag{4.18}$$
$$[A_i, W_j] = (2i-j)W_{i+j}, \tag{4.19}$$
$$[A_i, Z_j] = (2i-j)W_{i+j}, \tag{4.20}$$
$$[A_i, X_j] = (3i-j)X_{i+j}, \tag{4.21}$$

$$\begin{aligned}[X_m, X_n] =& \frac{1}{72}(m-n)(3m^4 - 2m^3 - 39m^2 + 4m^2n^2 + 20mn - 2n^3m \\ & + 108 - 39n^2 + 3n^4)L_{m+n} + \frac{1}{6}(m-n)(m^2 - mn - 7 + n^2)X_{m+n},\end{aligned} \tag{4.22}$$

$$\begin{aligned}[W_m, W_n] =& -\frac{1}{40}(m-n)(2m^2 - mn + 2n^2 - 8)(51L_{m+n} + 21A_{m+n}) \\ & + \frac{54}{5}(m-n)X_{m+n}\end{aligned} \tag{4.23}$$

$$\begin{aligned}[Z_m, Z_n] =& -\frac{1}{40}(m-n)(2m^2 - mn + 2n^2 - 8)(51L_{m+n} + 21A_{m+n}) \\ & + \frac{54}{5}(m-n)X_{m+n}\end{aligned} \tag{4.24}$$

As a quick summary, Table 4.1 is simply a reminder of the different spins each generator represents, including the fermionic generators (which were not included above, since they are not strictly necessary in the construction of the connection, but are vital in the description of the Killing spinors. This will be covered in more detail in a following chapter).

Table 4.1: Generators and their corresponding spins

Spin	**Generators**	**Spin**	**Generators**
1	L, A	$\frac{1}{2}$	G, H
2	W, Z	$\frac{3}{2}$	S, T
3	X	$\frac{5}{2}$	P, R

In essence, we tried a principal embedding of $sl(2|1)$ in $sl(4|3)$. By doing this, the even-graded sector of the superalgebra is decomposed into the $sl(2)$ generators L_i, a spin 1 multiplet A_i, two spin 2 multiplets W_m and Z_m, a spin 3 multiplet X_n and a spin 0 element J. The indices of these multiplets range from $-S$ to S, where S is the $sl(2)$ spin. This of course then gives us a total of $1+3+3+5+5+7 = 24$ bosonic generators. The odd-graded elements are made up of two spin $\frac{1}{2}$ multiplets, H_r and G_r, two spin $\frac{3}{2}$ multiplets, T_s and S_s and two spin $\frac{5}{2}$ multiplets R_t and P_t. There are $2+2+4+4+6+6 = 24$ fermionic generators.

The full list of commutation relations between the bosonic generators and fermionic generators can be found in the Appendix.

We now have all the tools necessary to start the construction of our $sl(4|3)$ solutions and discussing the definition of extremality in our framework and what that implies for an N-dimensional case.

Chapter 5

$sl(4|3)$ Solutions

We currently study the case of non-perturbative solutions of $sl(4|3) \oplus sl(4|3)$ Chern-Simons supergravity. We will in future simply refer to these as '$sl(4|3)$ black holes' for the sake of simplicity.

We start by looking at the relevant superalgebra. As explained in the previous section, it is required to embed $sl(2|1)$ into $sl(4|3)$, where the even-graded sector of the superalgebra decomposes into the $sl(2)$ generators (L_i), a spin 0 element (J), a spin 1 multiplet (A_i), two spin 2 multiplets (W_i, Z_i) and a spin 3 multiplet (X_i). These together then span the bosonic sub-algebra $sl(4) \oplus sl(3) \oplus sl(2) \oplus u(1)$. Of course by spin we imply the $sl(2)$ spin S, so that each multiplet has indices that range from $-S$ to S. The complex superalgebra $sl(4|3, \mathbb{C})$ has several real forms, it proves useful to look at $su(3,1|2,1)$, which is linked to the dual $\mathcal{W}_{(4|3)}$ symmetry.

The aim is to characterize a wide class of solutions that is supported by the even-graded sector of the $su(3,1|2,1)$ superalgebra, which will include solutions to black holes and smooth conical defects. We must incorporate the higher spin sources to the non-extremal case. To this end, after gauge fixing the radial dependence of our connection [42], we are left with only the angular component a_ϕ carrying the charges. We are then left with the following connection:

$$a_\phi = L_1 - \mathcal{L}L_{-1} - iQ_1 J - Q_2 A_{-1} - iQ_3 W_{-2} - Q_4 Z_{-2} - iQ_5 X_{-3} \tag{5.1}$$

with $\mathcal{L}$, Q_1, Q_2, Q_3, Q_4 and Q_5 taken to be constant and real. This is required so that a_ϕ lies on the real form $su(3,1|2,1)$. As to whether or not the generators are real, the properties of these generators are discussed in the appendix, and can be referred to in order to know the state of each individual generator. The above is of course the same type of expression we would expect for $\bar{a}_\phi$, the complex conjugate case, however those solutions will be omitted simply to avoid redundancy for the sake of simplicity.

Now it is possible to show that the charges are real relating to the $\mathcal{W}$-symmetry, however we focus on the eigenvalues of these charges. Simply put, we redefine the charges to be in terms of the eigenvalues of $a_\phi + iQ_1 J$, which are explicitly written as

$$\text{eigen}(a_\phi + iQ_1 J) = [\lambda_1, -\lambda_1 + \lambda_2, -\lambda_2 + \lambda_3, -\lambda_3, \lambda_4, -\lambda_4 + \lambda_5, -\lambda_5]. \tag{5.2}$$

Certain structures must be noted here. We have subtracted the $U(1)$ piece from the connection simply because it already commutes with the rest of the generators and it is already diagonal. It also forces the matrix $a_\phi + iQ_1 J$ to be traceless as well as super-traceless. Also the choice of eigenvalues are not unique. Any choice of corresponding eigenvalues that causes the matrix to be traceless will have the desired effect, for example choosing $[\lambda_1 + \lambda_2 + \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3, \lambda_{4+}\lambda_5, -\lambda_4, -\lambda_5]$ would have been sufficient.

Now since this matrix is block-diagonal, it has a characteristic polynomial which can factorise into a quartic and cubic part, such that

$$\begin{aligned} det(\lambda - a_\phi - iQ_1 J) &= (\lambda^4 - 10\lambda^2\mathcal{L} - 10\lambda^2 Q_2 - 12Q_3 i\lambda + 9\mathcal{L}^2 + 18\mathcal{L}Q_2 + 9Q_2^2 \\ &\quad - 12Q_4\lambda - 6iQ_5)(\lambda^3 - 4\lambda\mathcal{L} - 4Q_4\sqrt{5} + 4\lambda Q_2 - 4iQ_3\sqrt{5}) \\ &= (\lambda - \lambda_1)(\lambda + \lambda_1 - \lambda_2)(\lambda + \lambda_2 - \lambda_3)(\lambda + \lambda_3)(\lambda - \lambda_4)(\lambda + \lambda_4 - \lambda_5)(\lambda + \lambda_5). \end{aligned} \tag{5.3}$$

Typically, we do not expect to have a closed-form expression for the eigenvalues. However, since we assign the eigenvalues ourselves, instead of trying to calculate them using the discriminant of a polynomial, we can configure the discriminants accordingly. These discriminants for the quartic and cubic parts respectively, are:

$$\Delta_4 = (\lambda_3 + \lambda_1)^2(2\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_2 - \lambda_3)^2(\lambda_1 - 2\lambda_2 + \lambda_3)^2(\lambda_2 - 2\lambda_3)^2 \tag{5.4}$$

and

$$\Delta_3 = (\lambda_4 + \lambda_5)^2(2\lambda_4 - \lambda_5)^2(\lambda_4 - 2\lambda_5)^2. \tag{5.5}$$

We are now extending the definition of [42], which was reviewed in Chapter 3.2, to the $sl(4|3)$ case. It is possible to classify the existence of black holes or smooth conical defects if they fall within certain sectors. These sectors are partitioned based on the sign of the respective discriminants, and they can be classified as follows:

Table 5.1: Discriminant conditions for black holes and smooth conical defects in $sl(4|3)$

$sl(4\|3)$ solutions	Δ_4	Δ_3
Black holes	≥ 0	≥ 0
Smooth conical defects	< 0	< 0

There are of course other possibilities, such as $\Delta_4 < 0$,$\Delta_3 \geq 0$ and $\Delta_4 \geq 0$,$\Delta_3 < 0$, however they won't be explicitly solved in this review. We aim to find specifically extremal black holes, and according to the definition of extremality in [42], the conditions indicate that such cases are found when either $\Delta_4 = 0$ or $\Delta_3 = 0$. As such, we can naturally classify the connection in terms of the discriminants to generalize the pure gravity solutions as hyperbolic, elliptic and parabolic conjugacy classes in $SL(2)$.

Following up on our discriminant cases, we can match the discriminants of the eigenvalues with the discriminants of our connection in terms of the charges to explicitly solve the charges in terms of the eigenvalues. By doing this, we find that the charges are:

$$\mathcal{L} = -\frac{1}{8}\lambda_5^2 - \frac{1}{20}\lambda_1\lambda_2 + \frac{1}{20}\lambda_2^2 + \frac{1}{20}\lambda_1^2 + \frac{1}{20}\lambda_3^2 - \frac{1}{8}\lambda_4^2 - \frac{1}{20}\lambda_2\lambda_3 + \frac{1}{8}\lambda_4\lambda_5 \tag{5.6}$$

$$Q_2 = \frac{1}{8}\lambda_5^2 - \frac{1}{20}\lambda_1\lambda_2 + \frac{1}{20}\lambda_2^2 + \frac{1}{20}\lambda_1^2 + \frac{1}{20}\lambda_3^2 + \frac{1}{8}\lambda_4^2 - \frac{1}{20}\lambda_2\lambda_3 + \frac{1}{8}\lambda_4\lambda_5 \tag{5.7}$$

$$Q_3 = \frac{i}{144}(-\lambda_2\lambda_3^2 + 18\lambda_4\lambda_5^2 - 18\lambda_4^2\lambda_5 - \lambda_1\lambda_2^2 + \lambda_2^2\lambda_3 + \lambda_1^2\lambda_2) \tag{5.8}$$

$$Q_4 = -\frac{1}{144}(\lambda_2\lambda_3^2 + 18\lambda_4\lambda_5^2 - 18\lambda_4^2\lambda_5 + \lambda_1\lambda_2^2 - \lambda_2^2\lambda_3 - \lambda_1^2\lambda_2) \tag{5.9}$$

$$Q_5 = -\frac{i}{6000}(82\lambda_1^2\lambda_2\lambda_3 + 82\lambda_1\lambda_2\lambda_3^2 - 82\lambda_1\lambda_2^2\lambda_3 - 82\lambda_1^2\lambda_3^2 + 27\lambda_1^2\lambda_2^2$$
$$-18\lambda_1\lambda_2^3 - 18\lambda_1^3\lambda_2 + 27\lambda_2^2\lambda_2^2 - 18\lambda_2^3\lambda_3 - 18\lambda_3^3\lambda_2 + 9\lambda_2^4 + 9\lambda_1^4 + 9\lambda_3^4) \tag{5.10}$$

We have chosen the relative normalization of the eigenvalues in such a way that the $sl(2|1)$ theory corresponds to $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$, which gives

$$\mathcal{L} = -\frac{3}{40}\lambda_1^2, Q_2 = \frac{7}{40}\lambda_1^2, Q_3 = 0, Q_4 = 0, Q_5 = -\frac{3i}{2000}\lambda_1^4 \tag{5.11}$$

and

$$\Delta_4 = 0, \Delta_3 = 4\lambda_1^6. \tag{5.12}$$

We once again refer back to our definition of extremality, particularly the condition that involves the diagonalizability of the angular component of the connection. Looking at our case, it is easily seen that a_ϕ is diagonalizable if and only if $\Delta_4 \neq 0$ and $\Delta_3 \neq 0$. If that is the case, there exists a similarity matrix V that will bring the connection to the form

$$V^{-1}a_\phi V = a_\phi^D \tag{5.13}$$

where a_ϕ^D is constructed of a linear combination of the diagonal generators with the eigenvalues, and lies in the Cartan subalgebra of $sl(4|3)$. However, if a_ϕ is not diagonalizable, and is then according our definition extremal, either $\Delta_4 = 0$ or $\Delta_3 = 0$. This leads to a Jordan decomposition of the form

$$V^{-1}a_\phi V = a_\phi^D + a_\phi^N \tag{5.14}$$

where a_ϕ would be the same diagonal matrix as the one above and a_ϕ^N a nilpotent matrix which commutes with a_ϕ^D. Depending on the class under consideration, the exact form of a_ϕ^N may change, as it depends on the multiplicity of zeroes in the discriminants.

We can then construct a classification system which states that a general $sl(4|3)$ Drinfeld-Sokolov connection can be found in Table 5.2.

We can further refine this classification if we investigate specific signs of the discriminants Δ_4 and Δ_3. A quick look back at some changes that we implemented begs the question as to the importance of the charge Q_1. While the $U(1)$ charge is important in other sections that we consider, it plays no role in the characterization of our connections' Jordan class. This is because it is already diagonal and commutes with all the other generators of $sl(4|3)$. We must state that while our Jordan form of the connection does belong to $sl(4|3;\mathbb{C})$, our similarity matrix does not axiomatically belong in the supergroup $SU(3,1|2,1)$, which is to say that our connection involving the nilpotent and diagonal matrices do not necessarily take values in the

Table 5.2: Classes for extremal and non-extremal solutions solutions of $sl(4|3)$ black hole

Eigenvalues (4 x 4)	Δ_4
$\lambda_1 = \lambda_2 = \lambda_3 = 0$	$= 0$
$\lambda_1 = -\lambda_3 \neq 0, \lambda_2 = 0$	$= 0$
$\lambda_1 = -\lambda_3 \neq 0, \lambda_2 \neq 0$	$= 0$
$\lambda_2 = 2\lambda_1 \neq 0, \lambda_3 = 0$	$= 0$
$\lambda_2 = 2\lambda_1 \neq 0, \lambda_3 \neq 0$	$= 0$
$\lambda_2 = 2\lambda_3 \neq 0, \lambda_1 = 0$	$= 0$
$\lambda_2 = 2\lambda_3 \neq 0, \lambda_1 \neq 0$	$= 0$
$\lambda_3 = \lambda_1 \neq 0, \lambda_2 = 0$	$= 0$
$\lambda_3 = \lambda_2 \neq 0, \lambda_1 = 0$	$= 0$
$\lambda_1 = \lambda_2 \neq 0, \lambda_3 = 0$	$= 0$
$\lambda_1 = 2\lambda_2 \neq 0, \lambda_3 = 0$	$= 0$
$2\lambda_1 \neq \lambda_2 \neq \lambda_1 \neq \lambda_3 \neq 2\lambda_3$	$\neq 0$

$\otimes$

Eigenvalues (3 x 3)	Δ_3
$\lambda_4 = \lambda_5 = 0$	$= 0$
$\lambda_5 = 2\lambda_4 \neq 0$	$= 0$
$\lambda_4 = 2\lambda_5 \neq 0$	$= 0$
$\lambda_4 = -\lambda_5 \neq 0$	$= 0$
$2\lambda_5 \neq \lambda_4 \neq -\lambda_5 \neq -2\lambda_4$	$\neq 0$

real form of $su(3,1|2,1)$ of $sl(4,3;\mathbb{C})$. We will impose the necessary reality properties which will vastly decrease the amount of possible black hole classes.

A quick note to the existence of smooth conical defects. These so-called defects are solutions to the Lorentzian theory which have trivial holonomy along a cycle of the form $\phi \sim \phi + 2\pi$. By trivial we simply imply it belongs to the center of the gauge group. This triviality works much the same as for black holes along the thermal direction, such that it has the form

$$H_\phi \equiv \mathcal{P}e^{\oint a_\phi d\phi} = \Gamma^\pm \tag{5.15}$$

It is assumed that the background topology is related to AdS_3, such that smoothness of the fermionic fields at the origin requires that $H_\phi = \Gamma^-$ since the ϕ cycles are contractible. Similarly we allow the possibility of periodic fermions such that $H_\phi = \Gamma^+$. From [42] however, this implies that we introduce a singularity at the origin, which can only appear in the bulk if the theory is coupled to matter in a UV complete fashion.

The holonomy condition implies that all the eigenvalues of a_ϕ are purely imaginary, such that we are working in the sector $\Delta_4 < 0$ and $\Delta_3 < 0$. Our choices of eigenvalues that will adhere to the holonomy condition can then be generalized with the only restriction that the eigenvalues be non-degenerate.

As was stated earlier, it might prove insightful to review our knowledge on supersymmetry, but now in the context of our solutions.

5.1 Supersymmetry

It is now necessary to consider the supersymmetric Chern-Simons theory. Within this theory, a BPS solution is a solution where a gauge transformation exists that is supported by the odd elements of the gauge group which leaves the connection invariant. Much of the work follows closely that of [44], [45] and [46], specifically where the odd roots of the superalgebra and the eigenvalues of the connection were used. This is because we allow the possibility that the connection can be non-diagonalizable, since that is the main point on our definition of extremality. Following from [47], the boundary conditions used are similar to those used here, however the superalgebra used is $osp(1|4)$, which then displays hypersymmetry instead of supersymmetry.

We now move on to the supersymmetry conditions for Chern-Simons theory. From our construction of the connection, the radial dependence has been eliminated, and as such the residual gauge transformations on the boundary connection are given by

$$a \to a' = e^{-\epsilon} a e^{\epsilon} + d\epsilon \tag{5.16}$$

with ϵ being an arbitrary element of the gauge superalgebra. If $a' = a$ for odd transformation parameters, then the background is supersymmetric as stated above. The BPS conditions can be found after linearising (5.16):

$$\partial_t \epsilon + [a_t, \epsilon] = 0, \qquad \partial_\phi \epsilon + [a_\phi, \epsilon] = 0. \tag{5.17}$$

Depending on the amount of independent solutions to the above conditions, they will determine the amount of supersymmetries preserved by the background. These equations will be referred to as 'Killing equations', with its solutions being called 'Killing spinors'.

Killing spinors exists for fermionic generators and arbitrary connections, and since we focus on constant a_t and a_ϕ in the backgrounds, we then have an integrability condition $[a_t, a_\phi] = 0$ which allows us to write a general solution for the BPS conditions:

$$\epsilon(t, \phi) = e^{-a_t t - a_\phi \phi} \epsilon_0 e^{a_t t + a_\phi \phi}. \tag{5.18}$$

ϵ_0 in this case is a constant odd element of the superalgebra $su(3,1|2,1)$. We are interested in the admissible, globally defined solutions, and these are only possible if they possess the correct periodicity in ϕ, such that the spinors are periodic in the Ramond sector or anti-periodic in the Neveu-Schwarz sector. For a black hole, the ϕ-cycle is non-contractible, so NS and R boundary conditions are allowed. We will focus on the ϕ-dependence of $\epsilon(\phi) \equiv \epsilon(0, \phi)$.

We now bring a_ϕ to its Jordan normal form, remembering that a_ϕ^N is nilpotent and commutes with a_ϕ^D and vanishes for diagonalizable connections. From this decomposition, the general solution becomes:

$$\varepsilon(\phi) = e^{-a_\phi^D \phi} e^{-a_\phi^N \phi} \varepsilon_0 e^{a_\phi^D \phi} e^{a_\phi^N \phi} \tag{5.19}$$

with

$$\varepsilon(\phi) \equiv V^{-1} \epsilon(\phi) V, \qquad \varepsilon_0 \equiv V^{-1} \epsilon_0 V, \tag{5.20}$$

and V a constant matrix which is independent of ϕ as was had previously. An important note to point out is that $a_\phi^D, a_\phi^N, \varepsilon$ and ε_0 need not belong to $su(3,1|2,1)$. It is only necessary to belong there once the transformation that takes the connection to its Jordan form is reverted. We also move over to the E_{IJ} basis of $sl(4|3)$ generators. There are 49 of these 7×7 matrices of the form

$$(e_{IJ})_{KL} = \delta_{IK} \delta_{JL}. \tag{5.21}$$

We can split the index $I = (1,2,3,4,5,6,7)$ into $I = (i, \bar{i})$ with $i = (1,2,3,4)$ and $\bar{i} = (5,6,7)$. Then the even elements of the superalgebra have a basis of the form

$$E_{ij} = e_{ij} - \delta_{ij} \mathbf{1}, \tag{5.22}$$
$$E_{\bar{i}\bar{j}} = e_{\bar{i}\bar{j}} + \delta_{\bar{i}\bar{j}} \mathbf{1}, \tag{5.23}$$

and odd elements spanned by

$$E_{i\bar{j}} = e_{i\bar{j}}, \tag{5.24}$$
$$E_{\bar{i}j} = e_{\bar{i}j}. \tag{5.25}$$

Lastly concerning the construction of this basis, is that the basis is over-complete, since

$$\sum_i E_{ii} = -\sum_{\bar{i}} E_{\bar{i}\bar{i}}. \tag{5.26}$$

Following the discussion of this new basis, a_ϕ^D takes the form

$$\begin{aligned} a_\phi^D =& \lambda_1(E_{11} - E_{22}) + \lambda_2(E_{22} - E_{33}) + \lambda_3(E_{33} - E_{44}) + \lambda_4(E_{55} - E_{66}) + \lambda_5(E_{66} - E_{77}) \\ &+ iQ_1(E_{11} + E_{22} + E_{33} + E_{44}), \end{aligned} \tag{5.27}$$

and a_ϕ^N can be constructed from a linear combination of $E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}, E_{56}, E_{57}$ and E_{67}, depending on the Jordan class that is being inspected. This basis is also useful since it diagonalizes the adjoint action of the Cartan elements. We use this advantage to write

$$[a_\phi^D, E_{IJ}] = \omega_{IJ} E_{IJ}, \tag{5.28}$$

with

$$\omega_{IJ} = (a_\phi^D)_{II} - (a_\phi^D)_{JJ}. \tag{5.29}$$

The frequencies $\omega_{IJ} = -\omega_{JI}$ are determined by the roots of the superalgebra and the connection holonomy [47], so following this, let α_j represent a root of the bulk gauge superalgebra. Also, let a_ϕ^D denote the Jordan normal form's diagonal part of the Drinfeld-Sokolov connection a_ϕ with the appropriate boundary conditions. We know that a_ϕ^D belongs to the Cartan subalgebra $\mathcal{C}$, so we can associate a particular element $\vec{\Lambda}_\phi \in \mathcal{C}^*$ of the root space with it. We may also associate another Cartan element H_j with α_j if the isomorphism between $\mathcal{C}$ and $\mathcal{C}^C$ is used. If we then use the bilinear form on $\mathcal{C}^*$, then the frequencies can be written in a way that is representation-independent:

$$\omega_j = \langle \vec{\Lambda}_\phi, \alpha_j \rangle. \tag{5.30}$$

We are currently interested in fermionic symmetries only, so looking at only the odd symmetries, we have the following:

$$\omega_{15} = \lambda_1 - \lambda_4 + iQ_1, \tag{5.31}$$
$$\omega_{25} = -\lambda_1 + \lambda_2 - \lambda_4 + iQ_1, \tag{5.32}$$
$$\omega_{35} = -\lambda_2 + \lambda_3 - \lambda_4 + iQ_1, \tag{5.33}$$
$$\omega_{45} = -\lambda_3 - \lambda_4 + iQ_1, \tag{5.34}$$
$$\omega_{16} = \lambda_1 + \lambda_4 - \lambda_5 + iQ_1, \tag{5.35}$$
$$\omega_{26} = -\lambda_1 + \lambda_2 + \lambda_4 - \lambda_5 + iQ_1, \tag{5.36}$$
$$\omega_{36} = -\lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 + iQ_1, \tag{5.37}$$
$$\omega_{46} = -\lambda_3 + \lambda_4 - \lambda_5 + iQ_1, \tag{5.38}$$
$$\omega_{17} = \lambda_1 + \lambda_5 + iQ_1, \tag{5.39}$$
$$\omega_{27} = -\lambda_1 + \lambda_2 + \lambda_5 + iQ_1, \tag{5.40}$$
$$\omega_{37} = -\lambda_2 + \lambda_3 + \lambda_5 + iQ_1, \tag{5.41}$$
$$\omega_{47} = -\lambda_3 + \lambda_5 + iQ_1. \tag{5.42}$$

We must also expand the element ε_0 into $U(1)$ eigenstates

$$\varepsilon_0 = \varepsilon_0^- + \varepsilon_0^+, \tag{5.43}$$

with

$$\varepsilon_0^- = \sum_{i,\bar{j}} \varepsilon_{i\bar{j}} E_{i\bar{j}}, \qquad \varepsilon_0^+ = \sum_{\bar{i},j} \varepsilon_{\bar{i}j} E_{\bar{i}j}, \tag{5.44}$$

where $i = (1, 2, 3, 4)$ and $\bar{i} = (5, 6, 7)$. In this case there are 24 complex parameters $\varepsilon_{i\bar{j}}$ and $\varepsilon_{\bar{i}j}$. Only half of these are independent however, due to the reality constraints that are satisfied by the elements within $su(3,1|2,1)$, where the two $U(1)$ sectors are tied by complex conjugation. From this then it can be deemed that a fully supersymmetric solution is one that will permit a total of 24 real parameters, one 1/2-BPS solution that will preserve 12 symmetries, a 1/3-BPS background that will preserve 8, etc. We must remember that the sector $\bar{a}$ is not currently being counted.

From this point the necessary conditions for invariant $sl(4|3)$ solutions under supersymmetric transformations can be studied. Firstly, within the decomposition of our general solution for ε, the expansion of $e^{a_\phi^N \phi}$ will be truncated at some finite order. This is because we specifically chose a_ϕ^N to be nilpotent. We must also require that

$$[a_\phi^N, \varepsilon_0] = 0, \tag{5.45}$$

to avoid any ϕ-dependence within the Killing spinors, since the spinors are neither periodic nor anti-periodic. This also restricts the number of independent coefficients that appear within (5.44). Then the only ϕ-dependence for $\epsilon(\phi)$ is found within $[a_\phi^D, \epsilon_0]$. Then by the Baker-Campbell-Hausdorff formula and (5.28), our general solution becomes

$$\varepsilon(\phi) = \varepsilon^-(\phi) + \varepsilon^+(\phi), \tag{5.46}$$

where

$$\varepsilon^-(\phi) = \sum_{i,\bar{j}} \varepsilon_{i\bar{j}} E_{i\bar{j}} e^{\omega_{i\bar{j}}\phi}, \qquad \varepsilon^+(\phi) = \sum_{\bar{i},j} \varepsilon_{\bar{i}j} E_{\bar{i}j} e^{\omega_{\bar{i}j}\phi}. \tag{5.47}$$

The above then suggests that the frequencies $\omega_{i\bar{j}} = -\omega_{\bar{j}i}$ must be quantized into either integer or half-integer imaginary values so that the solution has the correct periodicity,i.e:

$$\omega_{i\bar{j}} \in \begin{cases} \mathbb{Z} & \text{R sector} \\ \mathbb{Z} + \frac{1}{2} & \text{NS sector} \end{cases} \tag{5.48}$$

The quantization condition imposes additional constraints, and some of these might not be satisfied automatically, which in turn then lowers the amount of preserved charges as well.

After the Killing spinor $\varepsilon(\phi)$ is found explicitly, the similarity transformation is undone and the solution is then expressed in terms of $\epsilon(\phi)$[1]. We then have the general fermionic gauge parameter which still preserves the Drinfeld-Sokolov form of our connection as:

$$\begin{aligned}\epsilon(\phi) =& \epsilon^+_{-\frac{1}{2}}(\phi) H_{\frac{1}{2}} + \epsilon^-_{-\frac{1}{2}}(\phi) G_{\frac{1}{2}} + \epsilon^+_{-\frac{3}{2}}(\phi) T_{\frac{3}{2}} + \epsilon^-_{-\frac{3}{2}}(\phi) S_{\frac{3}{2}} + \epsilon^+_{-\frac{5}{2}}(\phi) R_{\frac{5}{2}} \\ &+ \epsilon^-_{-\frac{5}{2}}(\phi) P_{\frac{5}{2}} + h_{-\frac{1}{2}}(\phi) H_{-\frac{1}{2}} + g_{-\frac{1}{2}}(\phi) G_{-\frac{1}{2}} + t_{-\frac{3}{2}}(\phi) T_{-\frac{3}{2}} + t_{-\frac{1}{2}}(\phi) T_{-\frac{1}{2}} \\ &+ t_{\frac{1}{2}}(\phi) T_{\frac{1}{2}} + s_{-\frac{3}{2}}(\phi) S_{-\frac{3}{2}} + s_{-\frac{1}{2}}(\phi) S_{-\frac{1}{2}} + s_{\frac{1}{2}}(\phi) S_{\frac{1}{2}} + p_{-\frac{5}{2}}(\phi) P_{-\frac{5}{2}} \\ &+ p_{-\frac{3}{2}}(\phi) P_{-\frac{3}{2}} + p_{-\frac{1}{2}}(\phi) P_{-\frac{1}{2}} + p_{\frac{1}{2}}(\phi) P_{\frac{1}{2}} + p_{\frac{3}{2}}(\phi) P_{\frac{3}{2}} + r_{-\frac{5}{2}}(\phi) R_{-\frac{5}{2}} \\ &+ r_{-\frac{3}{2}}(\phi) R_{-\frac{3}{2}} + r_{-\frac{1}{2}}(\phi) R_{-\frac{1}{2}} + r_{\frac{1}{2}}(\phi) R_{\frac{1}{2}} + r_{\frac{3}{2}}(\phi) R_{\frac{3}{2}}.\end{aligned} \tag{5.49}$$

[1]Refer to the appendix on the use of the real form $su(3,1|2,1)$

In the above equation, the higher weight terms are fixed in terms of the lower weight components, and algebraically is simple to solve, however the higher spin cases gets much more complicated and as such much messier to work with with each higher spin, as will be seen. Technically, every coefficient in the above expression has a time dependence, however in none of the calculations is this a necessary dependence, and as such has been omitted. The above was achieved from explicitly calculating the ϕ-dependence of (5.17), since the globally defined solutions are the ones that have the correct periodicity in ϕ. The lower weight components were calculated by first commuting the above expression with the connection, after which many individual commutations had to be considered. One then considers a single generator (in descending order of weights) after which one gathers all like-terms of the generators and sets to zero. The coefficients commute out naturally, after which one simply solves for the coefficient. This process continues, where each following case depends on the preceding case. As one might expect, the final cases are exceptionally cluttered. These lower weight components are:

$$h_{-\frac{1}{2}} = -\partial_\phi \epsilon^+_{-\frac{1}{2}} - iQ_1\epsilon^+_{-\frac{1}{2}} - 5Q_2\epsilon^+_{-\frac{3}{2}} + 60Q_4\epsilon^+_{-\frac{5}{2}} \tag{5.50}$$

$$g_{-\frac{1}{2}} = -\partial_\phi \epsilon^-_{-\frac{1}{2}} + iQ_1\epsilon^-_{-\frac{1}{2}} + 5Q_2\epsilon^-_{-\frac{3}{2}} - 60Q_4\epsilon^-_{-\frac{5}{2}} \tag{5.51}$$

$$t_{\frac{1}{2}} = -\partial_\phi \epsilon^+_{-\frac{3}{2}} - iQ_1\epsilon^+_{-\frac{3}{2}} + \frac{11}{2}Q_2\epsilon^+_{-\frac{5}{2}} \tag{5.52}$$

$$s_{\frac{1}{2}} = -\partial_\phi \epsilon^-_{-\frac{3}{2}} + iQ_1\epsilon^-_{-\frac{3}{2}} + \frac{11}{2}Q_2\epsilon^-_{-\frac{5}{2}} \tag{5.53}$$

$$r_{\frac{3}{2}} = -\partial_\phi \epsilon^+_{-\frac{5}{2}} - iQ_1\epsilon^+_{-\frac{5}{2}} \tag{5.54}$$

$$p_{\frac{3}{2}} = -\partial_\phi \epsilon^-_{-\frac{5}{2}} + iQ_1\epsilon^-_{-\frac{5}{2}} \tag{5.55}$$

$$\begin{aligned} t_{-\frac{1}{2}} =& \frac{1}{2}\partial_\phi^2\epsilon^+_{-\frac{3}{2}} + iQ_1\partial_\phi\epsilon^+_{-\frac{3}{2}} + \frac{4}{5}Q_2\partial_\phi\epsilon^+_{-\frac{5}{2}} + \frac{1}{2}(3\mathcal{L} - \frac{7}{5}Q_2 - Q_1^2)\epsilon^+_{-\frac{3}{2}} \\ &+ \frac{1}{2}(72iQ_3 + 24Q_4 - \frac{47i}{5}Q_1Q_2)\epsilon^+_{-\frac{5}{2}} \end{aligned} \tag{5.56}$$

$$\begin{aligned} s_{-\frac{1}{2}} =& \frac{1}{2}\partial_\phi^2\epsilon^-_{-\frac{3}{2}} - iQ_1\partial_\phi\epsilon^-_{-\frac{3}{2}} - \frac{47}{20}Q_2\partial_\phi\epsilon^-_{-\frac{5}{2}} - \frac{1}{2}(3\mathcal{L} + \frac{7}{5}Q_2 + Q_1^2)\epsilon^-_{-\frac{3}{2}} \\ &+ \frac{1}{2}(72iQ_3 + 24Q_4 + \frac{47i}{5}Q_1Q_2)\epsilon^-_{-\frac{5}{2}} \end{aligned} \tag{5.57}$$

$$r_{\frac{1}{2}} = \frac{1}{2}\partial_\phi^2\epsilon^+_{-\frac{5}{2}} + iQ_1\partial_\phi\epsilon^+_{-\frac{5}{2}} - \frac{1}{2}(5\mathcal{L} - Q_2 + Q_1^2)\epsilon^+_{-\frac{5}{2}} \tag{5.58}$$

$$p_{\frac{1}{2}} = \frac{1}{2}\partial_\phi^2\epsilon^-_{-\frac{5}{2}} - iQ_1\partial_\phi\epsilon^-_{-\frac{5}{2}} - \frac{1}{2}(5\mathcal{L} - Q_2 + Q_1^2)\epsilon^-_{-\frac{5}{2}} \tag{5.59}$$

$$\begin{aligned} r_{-\frac{1}{2}} =& -\frac{1}{6}\partial_\phi^3\epsilon^+_{-\frac{5}{2}} - \frac{i}{2}Q_1\partial_\phi^2\epsilon^+_{-\frac{5}{2}} + (\frac{13}{6}\mathcal{L} + \frac{1}{10}Q_2 + \frac{1}{2}Q_1^2)\partial_\phi\epsilon^+_{-\frac{5}{2}} - \frac{1}{5}Q_2\epsilon^+_{-\frac{3}{2}} \\ &+ \frac{i}{30}(65\mathcal{L}Q_1 + 3Q_1Q_2 + 5Q_1^3 - 120Q_3 + 120iQ_4)\epsilon^+_{-\frac{5}{2}} \end{aligned} \tag{5.60}$$

$$\begin{aligned} p_{-\frac{1}{2}} =& -\frac{1}{6}\partial_\phi^3\epsilon^-_{-\frac{5}{2}} + \frac{i}{2}Q_1\partial_\phi^2\epsilon^-_{-\frac{5}{2}} + (\frac{13}{6}\mathcal{L} + \frac{1}{10}Q_2 + \frac{1}{2}Q_1^2)\partial_\phi\epsilon^-_{-\frac{5}{2}} - \frac{1}{5}Q_2\epsilon^-_{-\frac{3}{2}} \\ &- \frac{i}{30}(65\mathcal{L}Q_1 + 3Q_1Q_2 + 5Q_1^3 + 120Q_3 - 120iQ_4)\epsilon^-_{-\frac{5}{2}} \end{aligned} \tag{5.61}$$

$$
\begin{aligned}
t_{-\frac{3}{2}} =& -\frac{1}{6}\partial_\phi^3\epsilon^+_{-\frac{3}{2}} - \frac{i}{2}Q_1\partial_\phi^2\epsilon^+_{-\frac{3}{2}} + \frac{7}{60}Q_2\partial_\phi^2\epsilon^+_{-\frac{5}{2}} + \frac{1}{6}(\mathcal{L} + \frac{49}{15}Q_2 + 3Q_1^2)\partial_\phi\epsilon^+_{-\frac{3}{2}} \\
&+ \frac{i}{15}(31Q_1Q_2 - 324Q_3 + 108iQ_4)\partial_\phi\epsilon^+_{-\frac{5}{2}} + \frac{4}{9}Q_2\epsilon^+_{-\frac{1}{2}} - \frac{i}{90}(-60\mathcal{L} + 45\mathcal{L}Q_1 - 49Q_1Q_2 \\
&- 15Q_1^3 + 1260Q_3 - 1044iQ_4)\epsilon^+_{-\frac{3}{2}} - \frac{1}{3}(\frac{67}{4}\mathcal{L}Q_2 - \frac{324}{5}Q_1Q_3 + \frac{108i}{5}Q_1Q_4 \\
&+ \frac{117}{20}Q_1^2Q_2 + \frac{239}{60}Q_2^2 + 40iQ_5)\epsilon^+_{-\frac{5}{2}}
\end{aligned} \tag{5.62}
$$

$$
\begin{aligned}
s_{-\frac{3}{2}} =& -\frac{1}{6}\partial_\phi^3\epsilon^-_{-\frac{3}{2}} + \frac{i}{2}Q_1\partial_\phi^2\epsilon^-_{-\frac{3}{2}} + \frac{7}{6}Q_2\partial_\phi^2\epsilon^-_{-\frac{5}{2}} + \frac{1}{6}(7\mathcal{L} + \frac{49}{15}Q_2 - Q_1^2)\partial_\phi\epsilon^-_{-\frac{3}{2}} \\
&- \frac{4i}{5}(2Q_1Q_2 + 81Q_3 - 27iQ_4)\partial_\phi\epsilon^-_{-\frac{5}{2}} - \frac{4}{9}Q_2\epsilon^-_{-\frac{1}{2}} + \frac{i}{90}(60\mathcal{L} + 45\mathcal{L}Q_1 + 7Q_1Q_2 \\
&- 15Q_1^3 - 1260Q_3 - 1044iQ_4)\epsilon^-_{-\frac{3}{2}} + \frac{1}{3}(\frac{67}{4}\mathcal{L}Q_2 + \frac{36}{5}Q_1Q_3 - \frac{12i}{5}Q_1Q_4 \\
&- \frac{71}{20}Q_1^2Q_2 + \frac{239}{60}Q_2^2 - 40iQ_5)\epsilon^-_{-\frac{5}{2}}
\end{aligned} \tag{5.63}
$$

$$
\begin{aligned}
r_{-\frac{3}{2}} =& \frac{1}{24}\partial_\phi^4\epsilon^+_{-\frac{5}{2}} + \frac{i}{6}Q_1\partial_\phi^3\epsilon^+_{-\frac{5}{2}} - \frac{1}{4}(\frac{11}{3}\mathcal{L} + \frac{2}{5}Q_2 + Q_1^2)\partial_\phi^2\epsilon^+_{-\frac{5}{2}} + \frac{1}{5}Q_2\partial_\phi\epsilon^+_{-\frac{3}{2}} + \frac{i}{120}(65\mathcal{L}Q_1 \\
&+ 21Q_1Q_2 + 5Q_1^3 - 336Q_3 + 336iQ_4 + 30)\partial_\phi\epsilon^+_{-\frac{5}{2}} + \frac{i}{40}(8Q_1Q_2 - 27Q_3 + 18iQ_4)\epsilon^+_{-\frac{3}{2}} \\
&+ \frac{1}{4}(\frac{15}{2}\mathcal{L}^2 - \frac{3}{2}\mathcal{L}Q_2 + \frac{11}{3}\mathcal{L}Q_1^2 + \frac{1}{10}Q_1^2Q_2 + \frac{1}{6}Q_1^4 - \frac{56}{5}Q_1Q_3 + 4iQ_4 + \frac{3}{2}\mathcal{L}Q_1 - \frac{3}{10}Q_1Q_2 \\
&+ \frac{3}{10}Q_1^3 - \frac{33}{10}Q_2^2 + \frac{36i}{5}Q_1Q_4 - 10iQ_5)\epsilon^+_{-\frac{5}{2}}
\end{aligned} \tag{5.64}
$$

$$
\begin{aligned}
p_{-\frac{3}{2}} =& \frac{1}{24}\partial_\phi^4\epsilon^-_{-\frac{5}{2}} - \frac{i}{6}Q_1\partial_\phi^3\epsilon^-_{-\frac{5}{2}} - \frac{1}{4}(\frac{11}{3}\mathcal{L} + \frac{2}{5}Q_2)\partial_\phi^2\epsilon^-_{-\frac{5}{2}} + \frac{4}{5}Q_2\partial_\phi\epsilon^-_{-\frac{3}{2}} + \frac{i}{120}(65\mathcal{L}Q_1 \\
&- 15Q_1Q_2 + 5Q_1^3 + 336Q_3 - 336iQ_4 + 30)\partial_\phi\epsilon^-_{-\frac{5}{2}} + \frac{i}{40}(4Q_1Q_2 - 27Q_3 + 18iQ_4)\epsilon^-_{-\frac{3}{2}} \\
&+ \frac{1}{4}(\frac{15}{2}\mathcal{L}^2 - \frac{3}{2}\mathcal{L}Q_2 - \frac{2}{3}\mathcal{L}Q_1^2 - \frac{1}{10}Q_1^2Q_2 - \frac{1}{6}Q_1^4 - \frac{16}{5}Q_1Q_3 - 4iQ_4 + \frac{3}{2}\mathcal{L}Q_1 - \frac{3}{10}Q_1Q_2 \\
&+ \frac{3}{10}Q_1^3 - \frac{33}{10}Q_2^2 + \frac{36i}{5}Q_1Q_4 - 10iQ_5)\epsilon^-_{-\frac{5}{2}}
\end{aligned} \tag{5.65}
$$

$$
\begin{aligned}
r_{-\frac{5}{2}} =& -\frac{1}{120}\partial_\phi^5\epsilon^+_{-\frac{5}{2}} - \frac{i}{24}Q_1\partial_\phi^4\epsilon^+_{-\frac{5}{2}} + \frac{1}{10}(\frac{5}{2}\mathcal{L} + \frac{1}{3}Q_2 + \frac{5}{6}Q_1^2)\partial_\phi^3\epsilon^+_{-\frac{5}{2}} - \frac{1}{10}Q_2\partial_\phi^2\epsilon^+_{-\frac{3}{2}} + \frac{i}{3000}(825\mathcal{L}Q_1 \\
&+ 285Q_1Q_2 + 125Q_1^3 - 2760Q_3 + 2760iQ_4 - 150 + 288iQ_2^2)\partial_\phi^2\epsilon^+_{-\frac{5}{2}} - \frac{i}{200}(40Q_1Q_2 \\
&- 99Q_3 + 66iQ_4)\partial_\phi\epsilon^+_{-\frac{3}{2}} + \frac{1}{5}(\frac{8}{5}Q_1Q_3 + \frac{61}{200}Q_2^2 - \frac{13i}{5}Q_1Q_4 - \frac{83}{120}\mathcal{L}Q_2 + iQ_4 - \frac{3}{40}Q_1^3 \\
&+ \frac{3}{40}Q_1Q_2 - \frac{3}{8}\mathcal{L}Q_1 + \frac{1}{4}Q_1 - \frac{7}{24}\mathcal{L}Q_1^2 - \frac{97}{12}\mathcal{L}^2 + \frac{1}{12}Q_1^4 + \frac{13i}{2}Q_5)\partial_\phi\epsilon^+_{-\frac{5}{2}} - \frac{6}{25}Q_4\epsilon^+_{-\frac{1}{2}} - \frac{1}{10}(\mathcal{L}Q_2 \\
&- Q_1^2Q_2 - \frac{99}{20}Q_1Q_3 + \frac{33i}{10}Q_1Q_4 + Q_2^2 + \frac{3i}{2}Q_5)\epsilon^+_{-\frac{3}{2}} - \frac{i}{3000}(-1800iQ_1Q_4 - 8400iQ_2Q_4 \\
&- 1500iQ_1Q_5 + 2160iQ_2^2Q_4 + 19020Q_2Q_3 - 2163Q_1Q_2^2 + 3725\mathcal{L}^2Q_1 + 750\mathcal{L}Q_1^3 - 10200\mathcal{L}Q_3 \\
&+ 55Q_1^3Q_2 - 2760Q_1^2Q_3 + 225\mathcal{L}Q_1^2 - 45Q_1^2Q_2 + 25Q_1^4 + 415\mathcal{L}Q_1Q_2 + 10200i\mathcal{L}Q_4)\epsilon^+_{-\frac{5}{2}}
\end{aligned} \tag{5.66}
$$

$$
\begin{aligned}
p_{-\frac{5}{2}} =& -\frac{1}{120}\partial_\phi^5\epsilon^-_{-\frac{5}{2}} - \frac{i}{120}Q_1\partial_\phi^4\epsilon^-_{-\frac{5}{2}} + \frac{1}{10}(\frac{5}{2}\mathcal{L} + \frac{1}{3}Q_2 + \frac{1}{2}Q_1^2)\partial_\phi^3\epsilon^-_{-\frac{5}{2}} - \frac{11}{50}Q_2\partial_\phi^2\epsilon^-_{-\frac{3}{2}} \\
& - \frac{i}{3000}(1475\mathcal{L}Q_1 + 105Q_1Q_2 + 25Q_1^3 + 2760Q_3 - 2760iQ_4 + 150 + 846iQ_2^2)\partial_\phi^2\epsilon^-_{-\frac{5}{2}} \\
& - \frac{i}{200}(-52Q_1Q_2 - 99Q_3 + 66iQ_4)\partial_\phi\epsilon^-_{-\frac{3}{2}} - \frac{1}{5}(\frac{1}{20}Q_1^2Q_2 + \frac{28}{5}Q_1Q_3 - \frac{157}{200}Q_2^2 - \frac{23i}{5}Q_1Q_4 \\
& + \frac{83}{120}\mathcal{L}Q_2 - iQ_4 + \frac{3}{40}Q_1Q_2 + \frac{3}{8}\mathcal{L}Q_1 + \frac{1}{4}Q_1 + \frac{11}{8}\mathcal{L}Q_1^2 + \frac{149}{24}\mathcal{L}^2 - \frac{13i}{2}Q_5)\partial_\phi\epsilon^-_{-\frac{5}{2}} - \frac{6}{25}Q_4\epsilon^-_{-\frac{1}{2}} \\
& - \frac{1}{50}(13\mathcal{L}Q_2 + 2Q_1^2Q_2 + \frac{171}{4}Q_1Q_3 - \frac{33i}{2}Q_1Q_4 + 5Q_2^2 - 30iQ_5)\epsilon^-_{-\frac{3}{2}} + \frac{i}{3000}(-600iQ_1Q_4 \\
& + 8400iQ_2Q_4 - 3900iQ_1Q_5 - 495iQ_1^2Q_4 - 12000Q_2Q_3 - 1668Q_1Q_2^2 + 3725\mathcal{L}^2Q_1 + 100\mathcal{L}Q_1^3 \\
& + 6200\mathcal{L}Q_3 + 25Q_1^3Q_2 + 600Q_1^2Q_3 + 225\mathcal{L}Q_1^2 - 45Q_1^2Q_2 - 25Q_1^5 + 45Q_1^4 + 415\mathcal{L}Q_1Q_2 \\
& - 10200i\mathcal{L}Q_4 - 1080Q_1Q_3)\epsilon^-_{-\frac{5}{2}}
\end{aligned}
\tag{5.67}
$$

A final note on the use of supersymmetry. In principle, it would be possible to find the Killing spinors by solving the differential equations for $(\epsilon^+_{-\frac{5}{2}}, \epsilon^+_{-\frac{3}{2}}, \epsilon^+_{-\frac{1}{2}})$ without the need to use the Jordan form of a_ϕ, however that makes it extremely difficult to differentiate between the cases that are extremal and those that are not, due to the use of the Drinfeld-Sokolov form which is written in terms of the charges instead of its eigenvalues.

Chapter 6

Black Hole Class Analysis

We now have the necessary information to start with the analysis of the supersymmetry conditions for the Jordan class of the black hole. We have the tools available to consider all the classes, however we will only look at some sample classes. It is important to remember that the diagonal section of the connection for black holes is automatically part of $su(3,1|2,1)$, due to the reality properties of our eigenvalues. We must also make sure that the nilpotent part in the Jordan decomposition also has values in the superalgebra, and that any similarity transformations of a_ϕ which puts it in its Jordan form is part of the corresponding supergroup. This will ensure that the Killing spinors that are found are in the correct real form, regardless of the basis used. We will be generalizing [42] from $sl(3|2)$ to $sl(4|3)$, but will be following the same procedure as the source material. According to [42], it is then only necessary to perform the analysis on ε_0^- since the parameters are then tied to $\varepsilon_0^{\pm\dagger} = -K\varepsilon_0^{\mp}K$ such that $\varepsilon_0 = \varepsilon_0^- + \varepsilon_0^+ \in su(3,1|2,1)$.

Class I: $\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = \lambda_5 = 0$

In this example we will look at the construction of Killing spinors with a specific choice of eigenvalues which are identified with the supersymmetric sector of the $sl(2|1)$ truncation which happens to map to $\mathcal{N} = 2$ supergravity. These can be called "BPS charged BTZ black holes."

Our choice of eigenvalues gives us our charges as

$$\mathcal{L} = 0, \qquad Q_2 = 0, \qquad Q_3 = 0, \qquad Q_4 = 0, \qquad Q_5 = 0, \tag{6.1}$$

which leads to a Jordan decomposition of the form

$$a_\phi^D = iQ_1(E_{11} + E_{22} + E_{33} + E_{44}), \qquad a_\phi^N = -(E_{12} + E_{23} + E_{34} + E_{56} + E_{67}) \tag{6.2}$$

Now, this choice of a_ϕ^N is by no means unique, any choice of matrix which is related to this one by a similarity transformation will yield similar results. We then use the expansions of ε_0^+ and ε_0^- to see that

$$\begin{aligned}\varepsilon_0^- &= \sum_{i,\bar{j}} \varepsilon_{i\bar{j}} E_{i\bar{j}} \\ &= \varepsilon_{15}E_{15} + \varepsilon_{16}E_{16} + \varepsilon_{17}E_{17} + \varepsilon_{25}E_{25} + \varepsilon_{26}E_{26} + \varepsilon_{27}E_{27} + \varepsilon_{35}E_{35} \\ &\quad + \varepsilon_{36}E_{36} + \varepsilon_{37}E_{37} + \varepsilon_{45}E_{45} + \varepsilon_{46}E_{46} + \varepsilon_{47}E_{47}\end{aligned} \tag{6.3}$$

such that

$$\begin{aligned}[a_\phi^N, \varepsilon_0^-] = &-\varepsilon_{25}E_{15} + (\varepsilon_{15} - \varepsilon_{26})E_{16} + (\varepsilon_{16} - \varepsilon_{27})E_{17} - \varepsilon_{35}E_{25} + (\varepsilon_{25} - \varepsilon_{36})E_{26} \\ &+ (\varepsilon_{26} - \varepsilon_{37})E_{27} - \varepsilon_{45}E_{35} + (\varepsilon_{35} - \varepsilon_{46})E_{36} + (\varepsilon_{36} - \varepsilon_{47})E_{37} + \varepsilon_{45}E_{46} + \varepsilon_{46}E_{47}.\end{aligned} \tag{6.4}$$

We have argued that the commutator above needs to vanish, and as such, it sets the following:

$$\varepsilon_{25} = \varepsilon_{35} = \varepsilon_{45} = \varepsilon_{46} = \varepsilon_{36} = \varepsilon_{47} = 0, \tag{6.5}$$

$$\varepsilon_{16} = \varepsilon_{27}, \tag{6.6}$$

$$\varepsilon_{15} = \varepsilon_{26} = \varepsilon_{37}. \tag{6.7}$$

This leaves us then with three independent complex coefficients, namely $\varepsilon_{17}, \varepsilon_{27}$ and ε_{37}. From this then, we can see for Class **I** solutions, we can preserve at most 6 real supercharges.

We must also ensure that the Killing spinors have the required periodicity. From (5.31) and (5.1), we have

$$\varepsilon^{-}(\phi) = (\varepsilon_{17}E_{17} + \varepsilon_{27}(E_{27} + E_{16}) + \varepsilon_{37}(E_{37} + E_{26} + E_{15}))e^{-iQ_1\phi}. \tag{6.8}$$

This implies that

$$-Q_1 = \eta + \frac{1}{2} \in \begin{cases} \mathbb{Z} & \text{R sector} \\ \mathbb{Z} + \frac{1}{2} & \text{NS sector} \end{cases} \tag{6.9}$$

Such that the $U(1)$ charge must be appropriately quantized [1].

We then undo the similarity transformation which places a_ϕ into its Jordan form, and casting the generators into the general Killing spinor, we then have the following for the transformation parameters

$$\begin{aligned} \epsilon^{+}_{-\frac{1}{2}} =& \epsilon^{+}_{-\frac{1}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}, \\ \epsilon^{+}_{-\frac{3}{2}} =& \epsilon^{+}_{-\frac{3}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}, \\ \epsilon^{+}_{-\frac{5}{2}} =& \epsilon^{+}_{-\frac{5}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}, \end{aligned} \tag{6.10}$$

where we have exchanged the coefficients $\varepsilon_{17}, \varepsilon_{27}$, and ε_{37} for $\epsilon^{+}_{-\frac{1}{2}}(0), \epsilon^{+}_{-\frac{3}{2}}(0)$ and $\epsilon^{+}_{-\frac{5}{2}}(0)$. Naturally, all the other components for the Killing spinor vanishes due to our choice of eigenvalues and as such we are left with the Killing spinor

$$\epsilon(\phi) = \epsilon^{+}_{-\frac{1}{2}}(\phi)H_{\frac{1}{2}} + i\bar{\epsilon}^{+}_{-\frac{1}{2}}(\phi)G_{\frac{1}{2}} + \epsilon^{+}_{-\frac{3}{2}}(\phi)T_{\frac{3}{2}} - i\bar{\epsilon}^{+}_{-\frac{3}{2}}(\phi)S_{\frac{3}{2}} + \epsilon^{+}_{-\frac{5}{2}}(\phi)R_{\frac{5}{2}} + i\bar{\epsilon}^{+}_{-\frac{5}{2}}(\phi)P_{\frac{5}{2}} \tag{6.11}$$

Class II: $\lambda_2 = 2\lambda_1 \neq 0, \lambda_3 \neq 0, \qquad \lambda_4 = \lambda_5 = 0$

For these solutions we see that $\lambda_3 \neq 0$, so for convenience we will choose $\lambda_1 = -\lambda_3$. From this, the charges are carried by

$$\mathcal{L} = \frac{3}{10}\lambda_1^2, \quad Q_2 = \frac{3}{10}\lambda_1^2, \quad Q_3 = -\frac{i}{18}\lambda_1^3, \quad Q_4 = \frac{i}{18}\lambda_1^3, \quad Q_5 = -\frac{13i}{125}\lambda_1^4. \tag{6.12}$$

The Jordan form of the connection due to our choice of eigenvalues then reads

$$a^D_\phi = \lambda_1(E_{11} + E_{22} - 3E_{33} + E_{44}) + \frac{1}{6}iQ_1(E_{11} + E_{22} + E_{33} + E_{44}) \tag{6.13}$$

$$a^N_\phi = -(E_{12} + E_{24} + E_{56} + E_{67}). \tag{6.14}$$

[1] As a small note, the use of the η is mostly just for convention. We could just as simply have said that the value of the charge must be either an integer value, thus making it periodic (in the R sector) or half-integer valued, thus making it anti-periodic (in the NS sector).

As a reminder, it is important to check that the nilpotent and diagonal connections commute. Now, the condition $[a_\phi^N, \varepsilon_0^-] = 0$ sets the following conditions

$$\varepsilon_{25} = \varepsilon_{35} = \varepsilon_{36} = \varepsilon_{45} = \varepsilon_{46} = 0 \tag{6.15}$$

$$\varepsilon_{16} = \varepsilon_{27}, \quad \varepsilon_{15} = \varepsilon_{26} = \varepsilon_{47} \tag{6.16}$$

We work with the parameter ε_{i7}, which we use to describe the Killing spinors with the aid of the fermionic frequencies

$$\begin{aligned}\varepsilon^-(\phi) =& (\varepsilon_{17}E_{17} + \varepsilon_{27}(E_{16} + E_{27}) + \varepsilon_{47}(E_{15} + E_{26} + E_{47}))e^{-i(-i\lambda_1 + \frac{1}{6}Q_1)\phi} \\ &+ \varepsilon_{37}(E_{26} + E_{37})e^{-i(3i\lambda_1 + \frac{1}{6}Q_1)\phi}.\end{aligned} \tag{6.17}$$

There are two exponentials in the above expression, each one requiring its own periodicity. We can impose different quantization conditions, which will preserve different supersymmetries. This then creates two separate subclasses. The first subclass we consider is the vanishing of the second term in (6.17)

$$i\lambda_1 - \frac{1}{6}Q_1 = \eta + \frac{1}{2} \in \begin{cases} \mathbb{Z} & \text{R sector} \\ \mathbb{Z} + \frac{1}{2} & \text{NS sector} \end{cases} \tag{6.18}$$

which preserves 6 supercharges. We can define the transformation parameters in this quantization by

$$\epsilon^+_{-\frac{1}{2}}(\phi) = \epsilon^+_{-\frac{1}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}, \quad \epsilon^+_{-\frac{3}{2}}(\phi) = \epsilon^+_{-\frac{3}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}, \quad \epsilon^+_{-\frac{5}{2}}(\phi) = \epsilon^+_{-\frac{5}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}. \tag{6.19}$$

If we then instead study subclass two, then

$$-3i\lambda_1 - \frac{1}{6}Q_1 = \eta + \frac{1}{2} \in \begin{cases} \mathbb{Z} & \text{R sector} \\ \mathbb{Z} + \frac{1}{2} & \text{NS sector} \end{cases} \tag{6.20}$$

then this preserves only two supersymmetries which in terms of the Drinfeld-Sokolov form of the connection is read as

$$\epsilon^+_{-\frac{5}{2}}(\phi) = \epsilon^+_{-\frac{5}{2}}(0)e^{i(\eta+\frac{1}{2})\phi}, \quad \epsilon^+_{-\frac{3}{2}}(\phi) = k_1\epsilon^+_{-\frac{5}{2}}(\phi), \quad \epsilon^+_{-\frac{1}{2}}(\phi) = k_2\epsilon^+_{-\frac{5}{2}}(\phi) \tag{6.21}$$

where

$$k_1 = -\frac{3\sqrt{2}}{18\sqrt{2}\lambda_1 + 90\sqrt{6} + 5iQ_1\sqrt{2}} \tag{6.22}$$

and

$$k_2 = -\frac{72i\sqrt{2}\lambda_1}{-125i\sqrt{2}Q_1^2\lambda_1 - 900\lambda_1^2\sqrt{2}Q_1 + 1512i\sqrt{2}\lambda_1^3 - 3600\lambda_1 Q_1\sqrt{6} + 12960i\sqrt{6}\lambda_1^2 - 1800i\sqrt{6}}. \tag{6.23}$$

It is possible to satisfy both quantization conditions simultaneously, which is possible when

$$\lambda_1 = \frac{i}{4}(\eta_2 - \eta_1) \quad \text{and} \quad Q_1 = \frac{1}{4}(\eta_2 - 3\eta_1) - \frac{1}{2}. \tag{6.24}$$

This case preserves eight supersymmetries. It is also an interesting case to study, since this shows that a charged BTZ black hole is not the most supersymmetric configuration for higher spin theory. All the solutions for Class **II** or Class **I** have entropy $S = 0$ + other sector, due to the fact that the entropy of an $sl(4|3)$ black hole is [42]

$$S = 2\pi k_{CS}(\lambda_1 + \lambda_2 - 2\lambda_3 + \lambda_4 - \lambda_5) + \text{other sector}. \tag{6.25}$$

Now, the most symmetric configuration minimizes the entropy, however between the two current classes, this argument cannot distinguish between them. This is quite unorthodox, since one would naïvely expect that these classes be distinguishable at the entropy level.

Class III: $\lambda_1 = \lambda_2 = \lambda_3 = 0, \qquad \lambda_4 = -\lambda_5 \neq 0$

This class has the following for the charges

$$\mathcal{L} = -\frac{1}{8}\lambda_5^2 - \frac{1}{8}\lambda_4^2 + \frac{1}{8}\lambda_4\lambda_5, \quad Q_2 = \frac{1}{8}\lambda_5^2 + \frac{1}{8}\lambda_4^2 - \frac{1}{8}\lambda_4\lambda_5, \quad Q_3 = \frac{i}{144}(18\lambda_4\lambda_5^2 - 18\lambda_4^2\lambda_5),$$
$$Q_4 = \frac{1}{8}\lambda_4\lambda_5^2 - \frac{1}{8}\lambda_4^2\lambda_5, \quad Q_5 = 0. \tag{6.26}$$

The Jordan form of the connection is

$$a_\phi^D = \lambda_4(E_{55} - E_{66}) - \lambda_5(E_{66} - E_{77}) + iQ_1(E_{11} + E_{22} + E_{33} + E_{44}), \quad a_\phi^N = -(E_{12} + E_{23} + E_{34}). \tag{6.27}$$

However, this class does not contain any supersymmetric solutions, due to the fact that the exponential dependence of the Killing spinor contains all non-zero real parts. This is an interesting case due to the fact that this is a extremal solution (at least according to our definition of extremality), but is not supersymmetric.

Class IV: $\lambda_1 \neq \lambda_2 \neq \lambda_3, \qquad \lambda_4 \neq -\lambda_5$

Within this class all the charges are generically independent, such that the connections are diagonalizable, with the Jordan form:

$$a_\phi^D = \lambda_1(E_{11} - E_{22}) + \lambda_2(E_{22} - E_{33}) + \lambda_3(E_{33} + E_{44}) + \lambda_4(E_{55} - E_{66}) + \lambda_5(E_{66} - E_{77}) + iQ_1(E_{11} + E_{22} + E_{33} + E_{44}), \tag{6.28}$$

$$a_\phi^N = 0. \tag{6.29}$$

According to our definition of extremality, these solutions are therefore not extremal. We also have a trivial a_ϕ^N, which automatically satisfies (5.45).

We must ensure that the $\varepsilon^-(\phi)$ are single- or double-valued along the ϕ-cycle, which we can achieve by manipulation the fermionic frequencies. Now, if we take into account our eigenvalue parameters $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\lambda_4 \neq -\lambda_5$ as well as some reality conditions that follow from [42] ($\lambda_1 = \lambda_2^*, \lambda_2 = \lambda_3^*, \lambda_1 = \lambda_3^*, \lambda_4 = \lambda_5^*$) then we can see that ω_{16}, ω_{25}, ω_{26}, ω_{27}, ω_{35}, ω_{36}, ω_{37} and ω_{46} all cannot be purely imaginary in the current class. Using the frequencies we have left, we may set

$$\lambda_4 = \lambda_5 = \frac{1}{2}(\lambda_1 + \lambda_3) \quad \text{and} \quad \frac{i}{2}(\lambda_1 - \lambda_3) - Q_1 = \eta + \frac{1}{2} \in \begin{cases} \mathbb{Z} & \text{R sector} \\ \mathbb{Z} + \frac{1}{2} & \text{NS sector} \end{cases} \tag{6.30}$$

then we have ω_{15} and ω_{47} properly quantized. This has a corresponding global Killing spinor

$$\varepsilon^-(\phi) = (\varepsilon_{15}E_{15} + \varepsilon_{47}E_{47})e^{i(\eta + \frac{1}{2})\phi}, \tag{6.31}$$

which conserves 4 real and independent supersymmetries. We may also impose as an alternative

$$\lambda_4 = \lambda_5 = -\frac{1}{2}(\lambda_1 + \lambda_3) \quad \text{and} \quad \frac{i}{2}(\lambda_1 - \lambda_3) - Q_1 = \eta + \frac{1}{2} \in \begin{cases} \mathbb{Z} & \text{R sector} \\ \mathbb{Z} + \frac{1}{2} & \text{NS sector} \end{cases} \tag{6.32}$$

which properly quantizes ω_{17} and ω_{45} with a Killing spinor of

$$\varepsilon^-(\phi) = (\varepsilon_{17} E_{17} + \varepsilon_{45} E_{45}) e^{i(\eta + \frac{1}{2})\phi}. \tag{6.33}$$

This solution also preserves 4 real supercharges, and in both cases the following is true

$$\epsilon^+_{-\frac{1}{2}}(\phi) = \epsilon^+_{-\frac{1}{2}}(0) e^{i(\eta+\frac{1}{2})\phi}, \quad \epsilon^+_{-\frac{3}{2}}(\phi) = \epsilon^+_{-\frac{3}{2}}(0) e^{i(\eta+\frac{1}{2})\phi}, \quad \epsilon^+_{-\frac{5}{2}}(\phi) = \epsilon^+_{-\frac{5}{2}}(0) e^{i(\eta+\frac{1}{2})\phi}. \tag{6.34}$$

The charges in (5.6) are all independent of the choice of sign of λ_4 and λ_5 (there would normally be a sign dependence for Q_3 and Q_4, but due to our choice of λ_4 and λ_5, those specific terms cancel in the (5.6)). As such, we have a corresponding set of bosonic charges that will always read

$$\mathcal{L} = \frac{1}{4}\left(\frac{3}{40}\lambda_1^2 - \frac{1}{4}\lambda_1\lambda_3 + \frac{3}{40}\lambda_3^2 - \frac{1}{5}\lambda_1\lambda_2 + \frac{1}{5}\lambda_2^2 - \frac{1}{5}\lambda_2\lambda_3\right), \tag{6.35}$$

$$Q_2 = \frac{1}{4}\left(\frac{13}{40}\lambda_1^2 + \frac{1}{5}\lambda_2^2 + \frac{13}{40}\lambda_3^2 - \frac{1}{5}\lambda_1\lambda_2 + \frac{1}{4}\lambda_1\lambda_3 - \frac{1}{5}\lambda_2\lambda_3\right), \tag{6.36}$$

$$Q_3 = \frac{i}{144}(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 - \lambda_3)\lambda_2, \tag{6.37}$$

$$Q_4 = -\frac{i}{144}(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 - \lambda_3)\lambda_2, \tag{6.38}$$

$$\begin{aligned} Q_5 = &-\frac{i}{6000}(82\lambda_1^2\lambda_2\lambda_3 + 82\lambda_1\lambda_2\lambda_3^2 - 82\lambda_1\lambda_2^2\lambda_3 - 82\lambda_1^2\lambda_3^2 + 27\lambda_1^2\lambda_2^2 \\ &- 18\lambda_1\lambda_2^3 - 18\lambda_1^3\lambda_2 + 27\lambda_2^2\lambda_3^2 - 18\lambda_2^3\lambda_3 - 18\lambda_3^3\lambda_2 + 9\lambda_2^4 + 9\lambda_1^4 + 9\lambda_3^4). \end{aligned} \tag{6.39}$$

It is important to note however that the chemical potentials (as well as the entropy) are sensitive to the choice of signs for λ_4 and λ_5. This is the point that is slightly unsettling. We see now that if we choose our eigenvalues in a specific way we can conserve supersymmetry within our class of diagonalizable connections, however we have deduced that according to our definition of extremality these solutions are not extremal. This seems to go against the convention regarding supersymmetric theories, where extremality is usually a condition for supersymmetry. These solutions correspond to the solutions found for the $sl(3|2)$ [42], albeit in a higher spin dimension. It should be possible to determine the stability of each solution, be it extremal and not supersymmetric, or vice versa, by identifying the sign of each eigenvalue corresponding to each class, then classifying the solutions to describe thermodynamical stability, for example. This classification is beyond the scope of the thesis, but could be interesting to study for future work. We can rather look at the particular non-extremal supersymmetric solution in more detail to perhaps attempt to explain the unorthodox phenomenon. As a side note, we stated that there are many different classes to consider, and we have only investigated four of these. However, we intentionally examine the classes that we could expect beforehand would have certain characteristics, such that the main differences between the classes may become apparent. We can explore some of the features we have uncovered:

1. In our current configuration we have only focussed on the ϕ-dependence of the Killing spinors. We have been purposefully aloof concerning the thermal cycles. We know that the spinor $\epsilon(t, \phi)$ is anti-periodic, which is required for a smooth fermionic field. For our Class **IV** solutions, the topology of the solutions is a solid torus, which is not true for the other solutions due to vanishing temperature [42].

2. The Class **IV** solutions have BPS conditions that do not allow solutions in the $sl(2|1)$ truncation. As such, supersymmetric BTZ black holes in higher spin are not affected by our findings.

3. Regarding the supercharges conserved by Class **IV** solutions, they are a non-trivial mixture of spin-3 and spin-4 multiplets constructed from fermionic generators. The corresponding variation of the charges contain non-linear bosonic and fermionic field relations, which corresponds to the asymptotic symmetry group. The belief is that the non-linearities allow the solutions to balance the periodic bosons with the anti-periodic fermions on a topology with a contractible cycle, which is not a feature that exists in standard supergravity, due to the BPS conditions always involving a linear fermionic field relation.

4. It should be possible to reproduce the BPS bounds from calculating the Kac determinant in a $\mathcal{W}_{(3|2)}$ symmetry CFT. That is not the focus of this thesis, but is theoretically possible due to similar calculations done in [42].

This concludes the main section of the thesis, containing a partial classification of higher spin $sl(4|3)$ black holes. The classification was made with the aim that some insight could be achieved on how to generalize the construction of solution for any N dimensions, i.e. $sl(N|N-1)$ solutions. Following from this we would like to lightly touch on the concept of hs$[\lambda]$ theory, which is a different method used to describe higher spin black holes. This topic has much depth to it and is not specifically the aim of this thesis, hence why we cover it only in minor detail. However we would like to possibly use the information achieved from the study to gather insights into a duality between the construction of higher spin black hole solutions in both representations, which in turn could again be used to describe an $sl(N|N-1)$ theory.

Chapter 7

hs[λ] theory

Throughout this study, we have focused on defining smooth extremal black holes in $sl(3,R)$ and $sl(4,R)$, with the hopes to somehow generalize the process to construct extremal black holes within $sl(N,R)$. One such attempt is to study the Vasiliev theory rather than moving to progressively higher spins. The aim is then to define a Vasiliev theory that is dual to this, since the Vasiliev theory is said to be more useful in providing a link between higher spin gravity theories and string theory.

The three dimensional Vasiliev theory contains an infinite tower of higher spins $s \geq 2$. The method used to construct higher spin black holes using this theory is that we require the higher spin sector to be cast as two Chern-Simons theory copies with the connections being valued in the infinite-dimensional hs[λ] Lie algebra, hence the name. To accomplish this, the matter fields in the Vasiliev theory are set to zero in the black hole background [17]. We now focus on the unbarred Chern-Simons connection, with the barred being implied. By this we mean we mostly work with the following simplified connection: $a = a_z dz + a_{\bar{z}} d\bar{z}$, which has no ρ-dependence. We will now quickly review the necessary background to be able to define the non-supersymmetric case of $sl(N,R)$.

7.1 hs[λ] higher spin gravity

The hs[λ] Lie algebra is spanned by generators that have two labels, each integers, that consist of a spin and a mode index. This can be represented as

$$V^s_m, \quad s \geq 2, \quad |m| < 2. \tag{7.1}$$

Following [48], these have the following commutation relation

$$[V^s_m, V^t_n] = \sum_{u=2,4,6,\ldots}^{s+t-|s-t|-1} g^{st}_u(m,n;\lambda) V^{s+t-u}_{m+n}, \tag{7.2}$$

where the structure constants can be found in the appendix. We use the parameter λ to signify inequivalent algebras; it appears via generalized hypergeometric functions but simplify to polynomials in λ^2 when evaluated for integer (s,m). The appearance of the λ^2 polynomial is due to the construction of hs[λ] as a subspace of a quotient of the universal $sl(2,R)$ by its quadratic Casimir. The Casimir C_2 for $sl(2,R)$ is fixed as

$$C_2 = (L_0)^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) = \frac{1}{4}(\lambda^2 - 1). \tag{7.3}$$

One then constructs the hs[λ] generators using the $sl(2,R)$ generators

$$V^s_m = (-1)^{m+s-1}\frac{(m+s-1)!}{(2s-2)!}\underbrace{[L_{-1},...[L_{-1},[L_{-1},}_{s\text{-}1\text{-}m} L_1^{s-1}]]]. \tag{7.4}$$

These generate hs[λ] when modding out the Casimir ideal, and dropping V^1_0, which is the identity element.

The generators with $s = 2$ form an $sl(2,R)$ subalgebra, where the remaining generators transform under the adjoint $sl(2,R)$ action as

$$[V^2_m, V^t_n] = (m(t-1)-n)V^t_{m+n}. \tag{7.5}$$

These will be required for constructing the BTZ solution. We can also move back from the a-connection to the ϕ-dependent connection A by simply making $b = e^{\rho V_0^2}$ and giving generators with mode index m a factor of e^{mp}.

The reason as to why this algebra might prove more useful is that unlike the $sl(N,R)$ algebras, hs[λ] has only a single $sl(2,R)$ subalgebra, without any other nontrivial subalgebras. Any commutator of two generators with spin $s > 2$ produces a generator with spin $t > s$. This changes at $\lambda = N$, where an ideal forms, which consists of all generators with $s > N$, and by modding out this ideal, one recovers $sl(N,R)$ [48]. This causes the viewpoint that $sl(N,R)$ are limiting cases of hs[λ] gravity. There is however the problem that it is not yet fully understood how to couple matter to $sl(N,R)$ gauge fields, such that it is possible to truncate the full Vasiliev theory while still including non-trivial fields other than the gauge fields.

The general λ commutator can be realized by using a star commutator

$$[V^s_m, V^t_n] = V^s_m \star V^t_n - V^t_n \star V^s_m \tag{7.6}$$

where the associative product is defined as

$$V^s_m \star V^t_n = \frac{1}{2}\sum_{u=1,2,3,...}^{s+t-|s-t|-1} g^{st}_u(m,n;\lambda)V^{s+t-u}_{m+n}. \tag{7.7}$$

This is known as the "lone star product", and from here on out, all multiplication of hs[λ] generators will be done by using this star product. The hs[λ] structure constants are given by

$$g^{st}_u(m,n;\lambda) = \frac{q^{u-2}}{2(u-1)!}\phi^{st}_u(\lambda)N^{st}_u(m,n) \tag{7.8}$$

with

$$N^{st}_u(m,n) = \sum_{k=0}^{u-1}(-1)^k\binom{u-1}{k}[s-1+m]_{u-1-k}[s-1-m]_k[t-1+n]_k[t-1-n]_{u-1-k} \tag{7.9}$$

$$\phi^{st}_u(\lambda) = {}_4F_3\left[\begin{matrix}\frac{1}{2}+\lambda, & \frac{1}{2}-\lambda, & \frac{2-u}{2}, & \frac{1-u}{2} \\ \frac{3}{2}-s, & \frac{3}{2}-t, & \frac{1}{2}+s+t-u & \end{matrix}\middle|\, 1\right] \tag{7.10}$$

This made use of the descending Pochhammer symbol

$$[a]_n = a(a-1)...(a-n+1). \tag{7.11}$$

Also q is a normalization constant which can be scaled away by simply taking $V^s_m \to q^{s-2}V^s_m$. We set it to be $q = 1/4$.

Useful structure constant properties:

$$\phi_u^{st}\left(\frac{1}{2}\right) = \phi_2^{st}(\lambda) = 1 \tag{7.12}$$

$$N_u^{st}(m,n) = (-1)^{u+1} N_u^{ts}(n,m) \tag{7.13}$$

$$N_u^{st}(0,0) = 0, \quad u \quad \text{even} \tag{7.14}$$

$$N_u^{st}(n,-n) = N_u^{ts}(n,-n). \tag{7.15}$$

The first three of these imply a few things, chief among them the isomorphism hs[$\frac{1}{2}$]≅hs(1,1); that the lone star product can be used to define the Lie algebra of hs[λ] and that the zero modes commute.

To factor out the ideal as we stated previously, we construct the bilinear trace, by picking out the V_0^1 element of the lone star product, and up to some normalization this is found to be

$$Tr(V_m^s V_n^t) \propto g_{s+t-1}^{st}(m,n;\lambda)\delta^{st}\delta_{m,-n}. \tag{7.16}$$

The structure constants can be written as

$$g_{2s-1}^{ss}(m,-m;\lambda) = (-1)^m \frac{2^{3-2s}\Gamma(s+m)\Gamma(s-m)}{(2s-1)!!\,(2s-3)!!} \prod_{\sigma=1}^{s-1}(\lambda^2-\sigma^2). \tag{7.17}$$

To remain consistent with [32], we choose to normalize the trace as

$$Tr(V_m^s V_n^t) = \frac{12}{(\lambda^2-1)} g_{2s-1}^{ss}(m,-m;\lambda). \tag{7.18}$$

The asymptotic symmetry group of hs[λ] gravity with generalized AdS boundary conditions was found to be $W_\infty[\lambda]$, the infinite-dimensional W-algebra [32]. We can construct the unbarred sector of the connection in this algebra by using the highest-weight gauge in the hs[λ]-valued connection,

$$a_z = V_1^2 - \frac{2\pi}{k}\mathcal{L}(z)V_{-1}^2 + \sum_{s=3}^{\infty} J^{(s)}(z)V_{1-s}^s \tag{7.19}$$

with $J^{(s)}(z)$ as spin-s currents. This algebra does not contain hs[λ] as a proper subalgebra, but in the infinite charge limit where the nonlinear terms drop off, hence we can consider hs[λ] as the "wedge subalgebra" of $W_\infty[\lambda]$.

Perhaps this is then a good place to ask how we can define black holes in this new language. This is a good question, and the link between how the hs[λ] theory solutions can be matched to the AdS solutions is still under investigation.

7.2 Constructing the hs[λ] black hole

We aim to construct higher spin black holes in a generic theory of higher spin gravity with a Lie algebra that contains at least one $sl(2,R)$ subalgebra. The following follows essentially identical to [48]. Before starting, it is required to write out the solution for the BTZ black hole using $sl(2,R)$ generators and to compute its Euclidean time circle holonomy eigenvalues. Following this is the simple steps to construct the black hole [48]:

1. Write the higher spin black hole connection with some non-zero higher spin chemical potentials.

2. Make the black hole smooth.

What is meant by smoothness of the black hole is in terms of the holonomy. As a brief overview, one compactifies the Euclidean time direction to form the topology of a solid torus in the $3d$ manifold. It then has boundary conditions $(w, \bar{w})$ which are subject to identifications $w \simeq w + 2\pi \simeq w + 2\pi\tau$, where τ is the modular parameter of the boundary torus. The smoothness condition for higher spin black holes was given by 3.22. Recall that the $sl(3, R)$ connection has Ward identities that indicates that the leading term in $a_{\bar{z}}$ represents a nonzero spin-3 chemical potential, which we will generalize to our Lie algebra. The unbarred connection for a higher spin black hole with nonzero spin-s chemical potential μ_s in the wormhole gauge is generalized to

$$a_z = a_z^{BTZ} + \text{(higher spin charges)} \tag{7.20}$$
$$a_{\bar{z}}\ \mu_s[(a_z)^{s-1} - \text{trace}] \tag{7.21}$$

and the ρ-dependence can again be restored by putting $b = e^{\rho L_0}$. Now it must be made smooth. This can be accomplished by fixing all the charges in terms of the potentials (μ_s, τ) consistent with an integrability condition. To complete this, return to the BTZ holonomy condition, to compute the Euclidean time circle holonomy matrix ω,

$$\omega = 2\pi(\tau a_z + \bar{\tau} a_{\bar{z}}) \tag{7.22}$$

and demand that the eigenvalues equal the BTZ solution.

$$Tr(\omega^n) = Tr(\omega^n_{BTZ}), \quad n = 2, 3, ..., rk(\omega) \tag{7.23}$$

will then be known as the holonomy condition with $rk(\omega)$ being the rank of ω.

The smooth Euclidean horizon from the perspective of the tower of higher spin fields produces the holonomy equations exactly.

For an hs[λ] black hole with spin-3 chemical potential, the smooth BTZ solution is

$$a_z = V_1^2 + \frac{1}{4\tau^2} V_{-1}^2 \tag{7.24}$$
$$a_{\bar{z}} = 0 \tag{7.25}$$

and the BTZ holonomy matrix is

$$\omega_{BTZ} = 2\pi\tau(V_1^2 + \frac{1}{4\tau^2} V_{-1}^2). \tag{7.26}$$

All of the odd-n traces vanish, and as such some low even-n traces are found to be

$$Tr(\omega^2_{BTZ}) = -8\pi^2 \tag{7.27}$$
$$Tr(\omega^4_{BTZ}) = \frac{8\pi^4}{5}(3\lambda^2 - 7) \tag{7.28}$$
$$Tr(\omega^6_{BTZ}) = -\frac{8\pi^6}{7}(3\lambda^4 - 18\lambda^2 + 31). \tag{7.29}$$

An ansatz is then constructed for the simplest case of a higher spin black hole with spin-3 chemical potential [48]

$$a_z = V_1^2 - \frac{2\pi\mathcal{L}}{k} V_{-1}^2 - N(\lambda)\frac{\pi\mathcal{W}}{2k} V_{-2}^3 + J \tag{7.30}$$
$$a_{\bar{z}} = -\mu N(\lambda)(a_z \star a_z - \frac{2\pi\mathcal{L}}{3k}(\lambda^2 - 1)) \tag{7.31}$$

where

$$J = J^{(4)}V^4_{-3} + J^{(5)}V^5_{-4} + ... \tag{7.32}$$

allows an infinite series of higher spin charges. $N(\lambda)$ is a normalization factor

$$N(\lambda) = \sqrt{\frac{20}{(\lambda^2 - 4)}} \tag{7.33}$$

chosen in such a way that it simplifies the comparison to the $sl(3, R)$ result.

It proves to be instructive to compare the hs[λ] black hole with the $sl(3, R)$ black hole, where we have a few interesting infinities:

- The amount of holonomy equations, due to the infinite dimensionality of hs[λ]. We fix smoothness at the horizon for the metric and an infinite tower of higher spin fields.
- The UV behaviour of the tower of metric-like higher spin fields. Generically, spin-s fields will involve traces over s vielbeins.
- The number of non-zero higher spin $J^{(s)}$ charges, which is a generic solution to the infinite set of holonomy equations.

Note that in our attempt to write out solutions for our $sl(4|3)$ black holes, we required supersymmetry with certain solutions, and as such it would make sense to include supersymmetry to the hs[λ] theory to best represent any possible link between the two theories. As such we will now incredibly briefly look at the so-called super higher spin λ theory, or simply shs[λ] theory.

7.3 shs(λ) theory

The previous section on hs[λ] is useful in describing higher spin black hole solutions and has been fairly well researched, however the continuation to include extremal solutions are lacking. As we have seen in our analysis, the number of classes increases dramatically as N increases, however if it is possible to describe a theory much the same as hs[λ], where a chosen spin is just a specific case concerning the infinite tower of spins, then the same should hold for our $sl(N|N-1)$. It is possible to extend the hs(λ) theory to a supersymmetric one which admits a super-higher-spin algebra, which should hold for all $sl(N|N-1)$ theories. The aim is to better understand how to define extremality in this framework such that defining extremality in $sl(N|N-1)$ is simpler.

Now, the super-higher-spin algebra is generated by the bosonic generators $L^{(s)\pm}_m$ as well as the newly-added fermionic generators $G^{(s)\pm}_r$. It is achievable from the wedge subalgebra of the super-$W_\infty[\lambda]$ algebra. The wedge condition can be seen as $|m| \leq s-1,\ \ |r| \leq s-\frac{3}{2}$. The wedge condition restrict the spin s generators to be in the finite-dimensional irreducable representations of the bosonic section of the $sl(2, R)$ algebra. The super-$W_\infty[\lambda]$ algebra constucted as well as the operator realizations as $\mathcal{N} = 1$ differential operators can be found in [28].

The shs[λ] algebra can be defined in the following way [49]. We consider the universal enveloping algebra of $Osp(1, 2)$. Since $U(Osp(1, 2))$ is an associative algebra, we denote the associative multiplication between any two elements in $U(Osp(1, 2))$ as $\mathcal{J} \star \mathcal{L}$. The commutator is then defined by

$$[\mathcal{J}, \mathcal{L}] = \mathcal{J} \star \mathcal{L} - \mathcal{L} \star \mathcal{J}. \tag{7.34}$$

We also define a bilinear trace of the product of two elements in $SB[\lambda]$:

$$Tr(\mathcal{K}, \mathcal{L}) = \left.\frac{\mathcal{K} \star \mathcal{L}}{(2\lambda^2 - \lambda)}\right|_{\mathcal{J}=0}, \quad \forall \mathcal{J} \neq \mathbf{1}. \tag{7.35}$$

The algebraic structure of the shs$[\lambda]$ algebra can be seen from the following commutation relations

$$[L_m^{(s)}, L_n^{(t)}] = \sum_{u=1}^{s+t-1} g_u^{st}(m, n, \lambda) L_{m+n}^{(s+t-u)} \tag{7.36}$$

$$\{G_p^{(s)}, G_q^{(t)}\} = \sum_{u=1}^{s+t-1} \tilde{g}_u^{st}(p, q, \lambda) L_{p+q}^{(s+t-u)} \tag{7.37}$$

$$[L_m^{(s)}, G_q^{(t)}] = \sum_{u=1}^{s+t-1} h_u^{st}(m, q, \lambda) G_{p+q}^{(s+t-u)} \tag{7.38}$$

$$[G_p^{(s)}, L_n^{(t)}] = \sum_{u=1}^{s+t-1} \tilde{h}_u^{st}(p, n, \lambda) G_{p+n}^{(s+t-u)} \tag{7.39}$$

$$\tag{7.40}$$

with $h_u^{st}(m, q, \lambda) = -\tilde{h}_u^{ts}(q, m, \lambda)$. The rest of the structure constants can all be found by the associative product found above. These commutation relations show that for any $N > 2$, the generators $L_m^{(s)}, G_n^{(s)}$ with an $s > N$ will generate a subalgebra at $\lambda = \frac{1-N}{2}$. Also, the bilinear trace we defined earlier will degenerate

$$Tr(L_m^{(s)} L_n^{(t)}) = 0, \quad Tr(G_p^{(s} G_q^{(t)}) = 0, \quad \text{for } s > N. \tag{7.41}$$

What this all implies is that we can set all the generators $L_m^{(s)}, G_n^{(s)}$ with $s > N$ to be zero and obtain a finite Lie superalgebra $sl(N, N-1)$. Knowing this, an aim we might have for the future is to incorporate some of the information regarding shs$[\lambda]$ in defining extremality.

Chapter 8

Discussion

At this point we are left with some interesting concepts that was discovered throughout the study. We have seen that it is possible to find solutions which are both supersymmetric as well as extremal, as was usually the norm for these types of solutions. However we have also found solutions which are extremal but not supersymmetric, which is odd but not against reason. The oddity is however that we have found supersymmetric solutions which are not extremal as well.

One is then right to ask, why are the solutions required to be supersymmetric then? Is the definition of extremality incomplete? Or are there other reasons as to why these solutions appear to be supersymmetric and non-extremal? Some of these questions have simple answers while others are interesting topics to perhaps investigate in a future study.

Our main aim was to replicate results found in [42] as well as construct our own solutions following from the previous work to gain possible insight into constructing solutions for spin-N.

As such, we started by defining a general definition of extremal black hole solutions, which is expressed in terms of the diagonalizability of the angular component of the connection. Then that extremality is compatible with the notion of zero Hawking temperature using the non-trivial Jordan classes in the connection. We also identify our real forms of the algebra corresponding to Lorentzian theories, which was $su(3,1|2,1)$. It was notable that unlike zero-temperature BTZ solutions, our extremal black holes in higher spin gravity carried residual entropy in the extremal sector. Following from this, we define the discriminants of the characteristic polynomial of a_ϕ, which initially we thought was impossible to achieve due to the nature of discriminant solutions with more than five eigenvalues. We were able to circumvent this issue by assuming that the discriminants can always be expressed in terms of these eigenvalues, even if the specific values of these are not explicitly known. These discriminants were vital in describing the extremal conditions of the solutions, particularly when the discriminant would equal zero.

We then began with the actual classification of the black holes, which was time consuming, especially deciding on which classes to use. It can very easily be seen that the amount of classes that appear in the higher spin dwarf the amount in the lower. As such we can say with relative certainty that the amount of classes increases dramatically as we increase the spin. Luckily to illustrate the core concepts of extremality and supersymmetry only a few classes had to be examined in detail. From this study we found that the solutions need not be extremal to be supersymmetric, which is contrary to the convention previously assumed.

As was alluded to in the final chapter, is the existence of the hs[λ] theory, or more relevant to our description, the shs[λ] theory. The difficulty of the shs[λ] theory is that in the bulk it proves to be difficult to impose holonomy conditions. However, the advantage of this framework is that one can study the tensionless limit of string theories and the dual CFT. Ideally we would have liked to study this connection in much more detail, and perhaps try and describe the $sl(4|3)$ solutions in terms of a simple hs$[\lambda]$ representation, but due to time constraints this was simply not possible. On shs$[\lambda]$, the framework has the versatility to describe many sections on the $sl(N|N-1)$ solutions simply, but we would perhaps like to investigate the analogues statements to the discriminant conditions which describes the extremality of our solutions. Finding the equivalent in that framework should simplify the generalization and improve a general definition of extremality in shs$[\lambda]$.

To summarise the extent of the work done throughout this study, specifically in writing down solutions for the black holes we are working with, one has to go through many steps:

1. Firstly, one must define the generators that one is working with in the Racah basis, according to the spin under consideration. These of course must be normalised appropriately, as well as determining the correct commutation relations, since these are vital later on.

2. One then constructs the appropriate ϕ-dependent connection, which will be used in the full connection that satisfies the Chern-Simons action. This is done by coupling charges to the lowest-weight components.

3. The eigenvalues of this connection must then be found, with corresponding discriminants, since these are essential in determining extremality. The charges can then also be found in terms of the eigenvalues.

4. Following the previous steps, it is finally possible to start classifying each class based on individual eigenvalues, with the amount of classes growing rapidly with each increase in spin. Note, in this study we have merely looked at four classes, however we have laid the foundation to be able to classify any of the available classes, should the need arise. We merely investigate the classes which clearly showcase each class' difference.

We can also make some comments as to how we might suspect the construction of solutions might entail in even higher spins. As was seen throughout the thesis, specifically in the classification of the solutions, the supersymmetries that were preserved according to our choice of eigenvalues increased at a linear rate with respect to the lower spin case. However, the cases themselves grew at an exponential rate, with subclasses emerging from them according to specific choices of the eigenvalues. This was not a feature that we expected when simply looking at the $sl(3|2)$ case, and we must expect that the amount of classes in higher spins to be more complicated and numerous still. It was encouraging to note that the same principle stands in both spin cases: That we found evidence of extremal black hole solutions that were not supersymmetric, and vice versa as well. As such, it is reasonable to assume that the property holds for higher spin cases as well, however it is merely speculation at this point. It might come to pass that above a certain spin threshold the property is not valid, however as it stands we have no reason to think this.

As to the use and addition of the hs$[\lambda]$ theory and its supersymmetric continuation, one might see the $sl(N|N-1)$ theory to be a specific case of the hs$[\lambda]$ theory, with certain restrictions and conditions imposed to lower the infinite tower of spins to an integer number of spins. Of course, to revert the $sl(N|N-1)$ theory back into the shs$[\lambda]$ counterpart will be by no means a simple procedure, and such studies might prove fruitful in the future.

As a matter of interest, it would be intriguing to study the higher dimensional cases of lower spin gravity, or perhaps non-AdS theories. These are all of course not very relevant to this thesis, but might prove insightful for future study.

Appendix A

$sl(4|3)$ Generators, Clebsch-Gordan Symbols and hs$[\lambda]$

A.1 The $sl(4|3)$ Racah Generators

The superalgebra $sl(m|n;\mathbb{C})$ consists of all complex $(m+n)\times(m+n)$ supermatrices of the form

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{A.1}$$

The matrices required to construct the generators for the $sl(4|3)$ connections were derived from the Racah basis of generators and then adapted to fit our needs. The final product of these generators are as follows:

$$T^3_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T^3_{-3} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.2}$$

$$T^3_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T^3_{-2} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.3}$$

$$T^3_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{5}}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T^3_{-1} = \begin{bmatrix} 0 & \frac{\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{5}}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.4}$$

$$T_0^3 = \begin{bmatrix} \frac{\sqrt{5}}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3\sqrt{5}}{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\sqrt{5}}{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{5}}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.5}$$

$$T_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T_{-2}^2 = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.6}$$

$$T_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T_{-1}^2 = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.7}$$

$$T_0^2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.8}$$

$$T_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{30}}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{10}}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{30}}{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T_{-1}^1 = \begin{bmatrix} 0 & \frac{\sqrt{30}}{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{10}}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{30}}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.9}$$

$$T_0^1 = \begin{bmatrix} \frac{3\sqrt{5}}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{5}}{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3\sqrt{5}}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.10}$$

$$T_0^0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.11}$$

$$U_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad U_{-2}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.12}$$

$$U_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \qquad U_{-1}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.13}$$

$$U_0^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{6} \end{bmatrix} \tag{A.14}$$

$$U_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \qquad U_{-1}^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.15}$$

$$U_0^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{A.16}$$

$$U_0^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \tag{A.17}$$

$$Q^{\frac{1}{2}}_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 \end{bmatrix} \quad Q^{\frac{1}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \end{bmatrix} \tag{A.18}$$

$$Q^{\frac{3}{2}}_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{10}}{5} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad Q^{\frac{3}{2}}_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{10}}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{15}}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\sqrt{30}}{15} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.19}$$

$$Q^{\frac{3}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\sqrt{30}}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{15}}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{10}}{5} & 0 & 0 & 0 \end{bmatrix} \quad Q^{\frac{3}{2}}_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{10}}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{15}}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.20}$$

$$Q^{\frac{5}{2}}_{-\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad Q^{\frac{5}{2}}_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{10}}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.21}$$

$$Q^{\frac{5}{2}}_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{10}}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{30}}{10} & 0 & 0 & 0 & 0 \end{bmatrix} \quad Q^{\frac{5}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{10}}{10} & 0 & 0 & 0 \end{bmatrix} \tag{A.22}$$

$$Q^{\frac{5}{2}}_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{15}}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{10}}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q^{\frac{5}{2}}_{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.23)}$$

$$\overline{Q}^{\frac{1}{2}}_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \overline{Q}^{\frac{1}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.24)}$$

$$\overline{Q}^{\frac{3}{2}}_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{10}}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \overline{Q}^{\frac{3}{2}}_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\sqrt{30}}{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{10}}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.25)}$$

$$\overline{Q}^{\frac{3}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{10}}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{15}}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\sqrt{30}}{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \overline{Q}^{\frac{3}{2}}_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{15}}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{10}}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.26)}$$

$$\overline{Q}^{\frac{5}{2}}_{-\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \overline{Q}^{\frac{5}{2}}_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{10}}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.27)}$$

$$\overline{Q}^{\frac{5}{2}}_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{30}}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{15}}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{30}}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \overline{Q}^{\frac{5}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{10}}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{30}}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.28)}$$

$$\overline{Q}^{\frac{5}{2}}_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{10}}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \overline{Q}^{\frac{5}{2}}_{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.29}$$

with commutation relations

$$[T^s_m, T^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''|N+1) C^{s,s',s''}_{m,m',m''} T^{s''}_{m''} \tag{A.30}$$

$$[U^s_m, U^{s'}_{m'}] = \sum_{s'',m''} f(s,s',s''|N) C^{s,s',s''}_{m,m',m''} U^{s''}_{m''} \tag{A.31}$$

$$\left\{Q^s_m, \bar{Q}^{s'}_{m'}\right\} = \sum_{s'',m''} C^{s,s',s''}_{m,m',m''} (g_1(s,s',s''|N) T^{s''}_{m''} + g_2(s,s',s''|N) U^{s''}_{m''}) \tag{A.32}$$

$$[T^s_m, \bar{Q}^{s'}_{m'}] = \sum_{s'',m''} h_1(s,s',s''|N) C^{s,s',s''}_{m,m',m''} \bar{Q}^{s''}_{m''} \tag{A.33}$$

$$[U^s_m, \bar{Q}^{s'}_{m'}] = \sum_{s'',m''} h_2(s,s',s''|N) C^{s,s',s''}_{m,m',m''} \bar{Q}^{s''}_{m''} \tag{A.34}$$

$$[T^s_m, Q^{s'}_{m'}] = \sum_{s'',m''} (-1)^{s+s'-s''-1} h_1(s,s',s''|N) C^{s,s',s''}_{m,m',m''} Q^{s''}_{m''} \tag{A.35}$$

$$[U^s_m, Q^{s'}_{m'}] = \sum_{s'',m''} (-1)^{s+s'-s''-1} h_2(s,s',s''|N) C^{s,s',s''}_{m,m',m''} Q^{s''}_{m''} \tag{A.36}$$

where

$$f(s,s',s''|N) = (1-(-1)^{s+s'-s''})(-1)^{s''+N-1}\sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ j & j & j \end{Bmatrix}, j = \frac{N-1}{2} \tag{A.37}$$

$$g_1(s,s',s''|N) = (-1)^{s+s'+N-1}\sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ \frac{N}{2} & \frac{N}{2} & \frac{N-1}{2} \end{Bmatrix} \tag{A.38}$$

$$g_2(s,s',s''|N) = (-1)^{s''+N}\sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N}{2} \end{Bmatrix} \tag{A.39}$$

$$h_1(s,s',s''|N) = (-1)^{s''+N-\frac{1}{2}}\sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N}{2} & \frac{N}{2} \end{Bmatrix} \tag{A.40}$$

$$h_2(s,s',s''|N) = (-1)^{s+s'+N+\frac{1}{2}}\sqrt{(2s+1)(2s'+1)} \begin{Bmatrix} s & s' & s'' \\ \frac{N}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \tag{A.41}$$

$$\tag{A.42}$$

A.2 The $sl(4|3)$ Generators

The generators for $sl(4|3)$:

$$J = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix} \tag{A.43}$$

$$L_0 = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{A.44}$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \end{bmatrix}, \quad L_{-1} = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.45}$$

$$A_0 = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.46}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix} \quad A_{-1} = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.47}$$

$$W_0 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.48}$$

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{2}} & 0 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \tag{A.49}$$

$$W_{-1} = \begin{bmatrix} 0 & 3\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad W_{-2} = \begin{bmatrix} 0 & 0 & 6\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.50}$$

$$Z_0 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{A.51}$$

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & 0 \end{bmatrix} \quad Z_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \tag{A.52}$$

$$Z_{-1} = \begin{bmatrix} 0 & 3\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad Z_{-2} = \begin{bmatrix} 0 & 0 & 6\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.53}$$

$$X_0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.54}$$

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad X_{-1} = \begin{bmatrix} 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.55}$$

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad X_{-2} = \begin{bmatrix} 0 & 0 & \frac{5}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.56}$$

$$X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad X_{-3} = \begin{bmatrix} 0 & 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.57}$$

$$L_i^\dagger = (-1)^i L_{-i}, \qquad A_i^\dagger = (-1)^i A_{-i}, \tag{A.58}$$

$$W_m^\dagger = (-1)^m W_{-m}, \qquad Z_m^\dagger = (-1)^m Z_{-m}, \qquad X_n^\dagger = (-1)^n X_{-n} \tag{A.59}$$

with the following commutation relations:

$$[L_i, L_j] = (i-j)L_{i+j}, \tag{A.60}$$

$$[L_i, A_j] = (i-j)A_{i+j}, \tag{A.61}$$

$$[L_i, W_m] = (2i-m)W_{i+m}, \tag{A.62}$$

$$[L_i, Z_m] = (2i-m)Z_{i+m}, \tag{A.63}$$

$$[L_i, X_n] = (3i-n)X_{i+n}, \tag{A.64}$$

$$[J, L_i] = 0 \tag{A.65}$$

and with the non-vanishing commutations as

$$[A_i, A_j] = (i-j)L_{i+j}, \tag{A.66}$$
$$[A_i, W_m] = (2i-j)W_{i+m}, \tag{A.67}$$
$$[A_i, Z_m] = (2i-j)W_{i+m}, \tag{A.68}$$
$$[A_i, X_n] = (3i-j)X_{i+n}, \tag{A.69}$$
$$[J, A_i] = 0 \tag{A.70}$$

$$\begin{aligned}[X_m, X_n] =& \frac{1}{72}(m-n)(3m^4 - 2m^3 - 39m^2 + 4m^2n^2 + 20mn - 2n^3m + 108 \\ & - 39n^2 + 3n^4)L_{m+n} + \frac{1}{6}(m-n)(m^2 - mn - 7 + n^2)X_{m+n},\end{aligned} \tag{A.71}$$

$$\begin{aligned}[W_m, W_n] =& -\frac{1}{40}(m-n)(2m^2 - mn + 2n^2 - 8)(51L_{m+n} + 21A_{m+n}) \\ & + \frac{54}{5}(m-n)X_{m+n}\end{aligned} \tag{A.72}$$

$$\begin{aligned}[Z_m, Z_n] =& -\frac{1}{40}(m-n)(2m^2 - mn + 2n^2 - 8)(51L_{m+n} + 21A_{m+n}) \\ & + \frac{54}{5}(m-n)X_{m+n}\end{aligned} \tag{A.73}$$

$$[J, W_n] = 0, \tag{A.74}$$
$$[J, Z_n] = 0, \tag{A.75}$$
$$[J, X_n] = 0. \tag{A.76}$$

$$G_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix} \quad G_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}\sqrt{3} & 0 & 0 & 0 \end{bmatrix} \tag{A.77}$$

$$H_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\sqrt{2}\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad H_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.78}$$

$$S_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3^{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3\sqrt{2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad S_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2^{\frac{3}{2}} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.79}$$

$$S_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2}\sqrt{3} & 0 & 0 & 0 \end{bmatrix} \quad S_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3^{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.80}$$

$$T_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3^{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad T_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2^{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.81}$$

$$T_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2^{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad T_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3^{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.82}$$

$$P_{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2^{\frac{3}{2}}5\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad P_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^{\frac{3}{2}}3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.83}$$

$$P_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix} \quad P_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}\sqrt{3} & 0 & 0 & 0 \end{bmatrix} \tag{A.84}$$

$$P_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{\frac{3}{2}}3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad P_{-\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{\frac{3}{2}}5\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.85}$$

$$R_{\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2^{\frac{3}{2}}5\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^{\frac{3}{2}}3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.86}$$

$$R_{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_{-\frac{1}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.87}$$

$$R_{-\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 4\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2^{\frac{3}{2}}3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_{-\frac{5}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2^{\frac{3}{2}}5\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.88}$$

$$\{G_m, H_n\} = 2(m-n)J + 2L_{m+n} \tag{A.89}$$

$$\{G_m, T_n\} = \frac{1}{4}(3m-n)(7L_{m+n} - 3A_{m+n}) + 4m^2 W_{m+n} \tag{A.90}$$

$$\{G_m, R_n\} = \frac{1}{6}(5m-n)(7W_{m+n} - 5Z_{m+n}) + 24m^2 X_{m+n} \tag{A.91}$$

$$\{H_m, S_n\} = -\frac{1}{4}(3m-n)(7L_{m+n} - 3A_{m+n}) + 4m^2 W_{m+n} \tag{A.92}$$

$$\{H_m, P_n\} = -\frac{1}{6}(5m-n)(7W_{m+n} - 5Z_{m+n}) + 24m^2 X_{m+n} \tag{A.93}$$

$$\begin{aligned}\{S_m, T_n\} =& -\frac{5}{8}(2m^2 + 2n^2 - 5)(m-n)J - \frac{1}{10}(\frac{3}{4}m^2 + \frac{3}{4}n^2 - mn - \frac{9}{8})(L_{m+n} + 21A_{m+n}) \\ &+ \frac{1}{8}(m-n)(7W_{m+n} - 9Z_{m+n}) + \frac{27}{10}X_{m+n}\end{aligned} \tag{A.94}$$

$$\begin{aligned}\{S_m, R_n\} =& \frac{3}{4}(\frac{17}{8}m - \frac{31}{40}n - m^3 + \frac{3}{5}m^2 n - \frac{3}{10}n^2 m + \frac{1}{10}n^3)(7L_{m+n} - 3A_{m+n}) \\ &+ \frac{1}{24}(40m^2 + 12n^2 - 32mn - 45)(W_{m+n} - 2Z_{m+n}) + (\frac{3}{5}n - m)6X_{m+n}\end{aligned} \tag{A.95}$$

$$\begin{aligned}\{P_m, R_n\} =& \frac{5}{96}(16m^4 - 140m^2 + 16m^2n^2 - 140n^2 + 16n^4 + 259)(m-n)J + (\frac{31}{8}m^2 - \frac{135}{32} - \frac{26}{5}mn \\ &+ \frac{31}{8}n^2 - \frac{9}{10}m^2n^2 - \frac{1}{2}m^4 - \frac{1}{2}n^4 + \frac{4}{5}m^3 n + \frac{4}{5}mn^3)(L_{m+n} - 4A_{m+n}) + \frac{1}{24}(10m^2 \\ &- 8mn - 55 + 10n^2)(m-n)(3W_{m+n} + Z_{m+n}) + 3(m^2 + \frac{5}{2} - \frac{8}{5}mn + n^2)X_{m+n}\end{aligned} \tag{A.96}$$

$$[J,G_s]=\frac{1}{2}G_s,\quad [J,H_s]=-\frac{1}{2}H_s,\quad [J,S_s]=\frac{1}{2}S_s,\quad [J,T_s]=-\frac{1}{2}T_s,$$
$$[J,P_s]=\frac{1}{2}P_s,\quad [J,R_s]=-\frac{1}{2}R_s \tag{A.97}$$

$$[L_r,G_s]=(\frac{r}{2}-s)G_{r+s},\quad [L_r,H_s]=(\frac{r}{2}-s)H_{r+s} \tag{A.98}$$

$$[L_r,S_s]=(\frac{3r}{2}-s)S_{r+s},\quad [L_r,T_s]=(\frac{3r}{2}-s)T_{r+s} \tag{A.99}$$

$$[L_r,P_s]=(\frac{5r}{2}-s)P_{r+s},\quad [L_r,R_s]=(\frac{5r}{2}-s)R_{r+s} \tag{A.100}$$

$$[A_r,G_s]=\frac{7}{6}(r-2s)G_{r+s}-\frac{4}{3}S_{r+s} \tag{A.101}$$

$$[A_r,H_s]=\frac{7}{6}(r-2s)H_{r+s}+\frac{4}{3}T_{r+s} \tag{A.102}$$

$$[A_r,S_s]=\frac{5}{24}(12r^2-8rs+4s^2-9)G_{r+s}+\frac{7}{30}(3r-2s)S_{r+s}-\frac{3}{5}P_{r+s} \tag{A.103}$$

$$[A_r,T_s]=-\frac{5}{24}(12r^2-8rs+4s^2-9)H_{r+s}+\frac{7}{30}(3r-2s)T_{r+s}-\frac{3}{5}R_{r+s} \tag{A.104}$$

$$[A_r,P_s]=\frac{1}{10}(40r^2-16rs-25)S_{r+s}+\frac{5r-2s}{10}P_{r+s} \tag{A.105}$$

$$[A_r,R_s]=\frac{1}{10}(40r^2-16rs-25)T_{r+s}+\frac{5r-2s}{10}R_{r+s} \tag{A.106}$$

$$[W_r,G_s]=-(4s-r)S_{r+s} \tag{A.107}$$

$$[W_r,H_s]=-(4s-r)T_{r+s} \tag{A.108}$$

$$\begin{aligned}[W_r,S_s]=&\frac{5}{32}(16s^3-12rs^2+8r^2s-36s-4r^3+19r)G_{r+s}\\&-\frac{7}{4}(4s^2-4rs+2r^2-5)S_{r+s}-\frac{9}{20}(4s-3r)P_{r+s}\end{aligned} \tag{A.109}$$

$$\begin{aligned}[W_r,T_s]=&\frac{5}{32}(16s^3-12rs^2+8r^2s-36s-4r^3+19r)H_{r+s}\\&-\frac{7}{4}(4s^2-4rs+2r^2-5)T_{r+s}-\frac{9}{20}(4s-3r)R_{r+s}\end{aligned} \tag{A.110}$$

$$[W_r,P_s]=\frac{3}{40}(16s^3-36rs^2+48r^2s-100s-40r^3+145r)S_{r+s}-\frac{1}{20}(12s^2-24rs+20r^2-35)P_{r+s} \tag{A.111}$$

$$[W_r,R_s]=\frac{3}{40}(16s^3-36rs^2+48r^2s-100s-40r^3+145r)T_{r+s}-\frac{1}{20}(12s^2-24rs+20r^2-35)R_{r+s} \tag{A.112}$$

$$[Z_r,G_s]=-\frac{7}{5}(4s-r)S_{r+s}-\frac{6}{5}P_{r+s} \tag{A.113}$$

$$[Z_r,H_s]=-\frac{7}{5}(4s-r)T_{r+s}+\frac{6}{5}R_{r+s} \tag{A.114}$$

$$\begin{aligned}[Z_r,S_s]=&\frac{7}{32}(16s^3-12rs^2+8r^2s-36s-4r^3+19r)G_{r+s}\\&+\frac{29}{20}(4s^2-4rs+2r^2-5)S_{r+s}-\frac{3}{20}(4s-3r)P_{r+s}\end{aligned} \tag{A.115}$$

$$[Z_r, T_s] = \frac{7}{32}(16s^3 - 12rs^2 + 8r^2s - 36s - 4r^3 + 19r)H_{r+s} - \frac{29}{20}(4s^2 - 4rs + 2r^2 - 5)T_{r+s} - \frac{3}{20}(4s - 3r)R_{r+s} \tag{A.116}$$

$$[Z_r, P_s] = -\frac{1}{32}(16s^4 - 32rs^3 + 48r^2s^2 - 136s^2 - 64r^3s + 264rs + 80r^4 - 380r^2 + 225)G_{r+s} + \frac{1}{40}(16s^3 - 36rs^2 + 48r^2s - 100s - 40r^3 + 145r)S_{r+s} - \frac{1}{20}(12s^2 - 24rs + 20r^2 - 35)P_{r+s} \tag{A.117}$$

$$[Z_r, R_s] = \frac{1}{32}(16s^4 - 32rs^3 + 48r^2s^2 - 136s^2 - 64r^3s + 264rs + 80r^4 - 380r^2 + 225)H_{r+s} + \frac{1}{40}(16s^3 - 36rs^2 + 48r^2s - 100s - 40r^3 + 145r)T_{r+s} - \frac{1}{20}(12s^2 - 24rs + 20r^2 - 35)R_{r+s} \tag{A.118}$$

$$[X_r, G_s] = -\frac{6s - r}{6}P_{r+s} \tag{A.119}$$

$$[X_r, H_s] = -\frac{6s - r}{6}R_{r+s} \tag{A.120}$$

$$[X_r, S_s] = \frac{1}{40}(40s^3 - 20rs^2 + 8r^2s - 82s - 2r^3 + 23r)S_{r+s} - \frac{1}{120}(60s^2 - 40rs + 12r^2 - 63)P_{r+s} \tag{A.121}$$

$$[X_r, T_s] = \frac{1}{40}(40s^3 - 20rs^2 + 8r^2s - 82s - 2r^3 + 23r)T_{r+s} + \frac{1}{120}(60s^2 - 40rs + 12r^2 - 63)R_{r+s} \tag{A.122}$$

$$[X_r, P_s] = -\frac{5}{1152}(96s^5 - 80rs^4 + 54r^2s^3 - 816s^3 - 48r^3s^2 + 632rs^2 + 32r^4s + 1350s - 16r^5 + 220r^3 - 729r)G_{r+s} + \frac{1}{720}(240s^4 - 320rs^3 + 288r^2s^2 - 1752s^2 - 192r^3s + 1808rs + 80r^4 - 920r^2 + 1575)S_{r+s} - \frac{1}{240}(40s^3 - 60rs^2 + 48r^2s - 202s - 20r^3 + 155r)P_{r+s} \tag{A.123}$$

$$[X_r, R_s] = -\frac{5}{1152}(96s^5 - 80rs^4 + 54r^2s^3 - 816s^3 - 48r^3s^2 + 632rs^2 + 32r^4s + 1350s - 16r^5 + 220r^3 - 729r)H_{r+s} - \frac{1}{720}(240s^4 - 320rs^3 + 288r^2s^2 - 1752s^2 - 192r^3s + 1808rs + 80r^4 - 920r^2 + 1575)T_{r+s} - \frac{1}{240}(40s^3 - 60rs^2 + 48r^2s - 202s - 20r^3 + 155r)R_{r+s} \tag{A.124}$$

As noted with the definition of the Killing spinors, we work with the real form of $su(3,1|2,1)$. We work with the real forms associated to $sl(4|3,\mathbb{C})$, specifically

$$su(p, 3-p|q, 2-q) \supset su(p, 3-p) \oplus su(q, 2-q) \oplus i\mathbb{R} \tag{A.125}$$

with $p = 3$ and $q = 2$. We use this bulk superalgebra due to the fact that it naturally makes contact with the boundary $W_{(3|2)}$ theory and by extension with the $W_{(4|3)}$ theory, which we are using. We define the superalgebra $su(3,1|2,1) \supset su(3,1) \oplus su(2,1) \oplus i\mathbb{R}$ by the set of supertraceless 7×7 supermatrices M which satisfy

$$M^\dagger K + KM = 0 \tag{A.126}$$

with K a non-degenerate Hermitian form of signature $(3,1|2,1)$. From our representation of $sl(4,3)$ the generators

$$L_i, \quad A_i, \quad iW_m, \quad iJ, \quad iZ_m, \quad X_m, \tag{A.127}$$

as well as

$$\begin{aligned} &e^{i\pi/4}(H_r + G_r), \quad e^{i\pi/4}(H_r - G_r), \\ &e^{3i\pi/4}(T_r + S_r), \quad e^{3i\pi/4}(T_r - S_r), \\ &e^{5i\pi/4}(P_r + R_r), \quad e^{5i\pi/4}(P_r - R_r), \end{aligned} \tag{A.128}$$

do satisfy the property with

$$K = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \end{pmatrix}. \tag{A.129}$$

We also decompose the fermionic parameter as

$$\epsilon = \epsilon^- + \epsilon^+ \tag{A.130}$$

where the $\epsilon^\pm$ are the $U(1)$ eigenstates. If we demand that this belongs to $su(3,1|2,1)$ then

$$\epsilon^\dagger K + K\epsilon = 0 \quad \Leftrightarrow \quad \epsilon^{\pm\dagger} = -K\epsilon^\mp K. \tag{A.131}$$

A.3 The Non-zero Clebsch-Gordan Symbols

Clebsch-Gordan Symbol	Numerical Value	Clebsch-Gordan Symbol	Numerical Value
$< \frac{3}{2}\ 3\ ;\ \frac{-3}{2}\ 3 \mid \frac{3}{2}\ \frac{3}{2} >$	$-2\sqrt{\frac{1}{7}}$	$< \frac{3}{2}\ 3\ ;\ \frac{-1}{2}\ 2 \mid \frac{3}{2}\ \frac{3}{2} >$	$\sqrt{\frac{2}{7}}$
$< \frac{3}{2}\ 3\ ;\ \frac{-3}{2}\ 2 \mid \frac{3}{2}\ \frac{1}{2} >$	$-\sqrt{\frac{2}{7}}$	$< \frac{3}{2}\ 3\ ;\ \frac{1}{2}\ 1 \mid \frac{3}{2}\ \frac{3}{2} >$	$-2\sqrt{\frac{1}{35}}$
$< \frac{3}{2}\ 3\ ;\ \frac{-1}{2}\ 1 \mid \frac{3}{2}\ \frac{1}{2} >$	$2\sqrt{\frac{3}{35}}$	$< \frac{3}{2}\ 3\ ;\ \frac{-3}{2}\ 1 \mid \frac{3}{2}\ \frac{-1}{2} >$	$-2\sqrt{\frac{1}{35}}$
$< \frac{3}{2}\ 3\ ;\ \frac{3}{2}\ 0 \mid \frac{3}{2}\ \frac{3}{2} >$	$\sqrt{\frac{1}{35}}$	$< \frac{3}{2}\ 3\ ;\ \frac{1}{2}\ 0 \mid \frac{3}{2}\ \frac{1}{2} >$	$-3\sqrt{\frac{1}{35}}$
$< \frac{3}{2}\ 3\ ;\ \frac{-1}{2}\ 0 \mid \frac{3}{2}\ \frac{-1}{2} >$	$3\sqrt{\frac{1}{35}}$	$< \frac{3}{2}\ 3\ ;\ \frac{-3}{2}\ 0 \mid \frac{3}{2}\ \frac{-3}{2} >$	$-\sqrt{\frac{1}{35}}$
$< \frac{3}{2}\ 2\ ;\ \frac{-1}{2}\ 2 \mid \frac{3}{2}\ \frac{3}{2} >$	$\sqrt{\frac{2}{5}}$	$< \frac{3}{2}\ 2\ ;\ \frac{-3}{2}\ 2 \mid \frac{3}{2}\ \frac{1}{2} >$	$\sqrt{\frac{2}{5}}$
$< \frac{3}{2}\ 2\ ;\ \frac{1}{2}\ 1 \mid \frac{3}{2}\ \frac{3}{2} >$	$-\sqrt{\frac{2}{5}}$	$< \frac{3}{2}\ 2\ ;\ \frac{-3}{2}\ 1 \mid \frac{3}{2}\ \frac{-1}{2} >$	$\sqrt{\frac{2}{5}}$
$< \frac{3}{2}\ 2\ ;\ \frac{3}{2}\ 0 \mid \frac{3}{2}\ \frac{3}{2} >$	$\sqrt{\frac{1}{5}}$	$< \frac{3}{2}\ 2\ ;\ \frac{1}{2}\ 0 \mid \frac{3}{2}\ \frac{1}{2} >$	$-\sqrt{\frac{1}{5}}$
$< \frac{3}{2}\ 2\ ;\ \frac{-1}{2}\ 0 \mid \frac{3}{2}\ \frac{-1}{2} >$	$-\sqrt{\frac{1}{5}}$	$< \frac{3}{2}\ 2\ ;\ \frac{-3}{2}\ 0 \mid \frac{3}{2}\ \frac{-3}{2} >$	$\sqrt{\frac{1}{5}}$
$< \frac{3}{2}\ 1\ ;\ \frac{1}{2}\ 1 \mid \frac{3}{2}\ \frac{3}{2} >$	$-\sqrt{\frac{2}{5}}$	$< \frac{3}{2}\ 1\ ;\ \frac{-1}{2}\ 1 \mid \frac{3}{2}\ \frac{1}{2} >$	$-2\sqrt{\frac{2}{15}}$
$< \frac{3}{2}\ 1\ ;\ \frac{-3}{2}\ 1 \mid \frac{3}{2}\ \frac{-1}{2} >$	$-\sqrt{\frac{2}{5}}$	$< \frac{3}{2}\ 1\ ;\ \frac{3}{2}\ 0 \mid \frac{3}{2}\ \frac{3}{2} >$	$\sqrt{\frac{3}{5}}$
$< \frac{3}{2}\ 1\ ;\ \frac{1}{2}\ 0 \mid \frac{3}{2}\ \frac{1}{2} >$	$\sqrt{\frac{1}{15}}$	$< \frac{3}{2}\ 1\ ;\ \frac{-1}{2}\ 0 \mid \frac{3}{2}\ \frac{-1}{2} >$	$-\sqrt{\frac{1}{15}}$
$< \frac{3}{2}\ 1\ ;\ \frac{-3}{2}\ 0 \mid \frac{3}{2}\ \frac{-3}{2} >$	$-\sqrt{\frac{3}{5}}$	$< \frac{3}{2}\ 0\ ;\ \frac{3}{2}\ 0 \mid \frac{3}{2}\ \frac{3}{2} >$	1
$< \frac{3}{2}\ 0\ ;\ \frac{1}{2}\ 0 \mid \frac{3}{2}\ \frac{1}{2} >$	1	$< 1\ 2\ ;\ -1\ 2 \mid 1\ 1 >$	$\sqrt{\frac{3}{5}}$
$< \frac{3}{2}\ 0\ ;\ \frac{-3}{2}\ 0 \mid \frac{3}{2}\ \frac{-3}{2} >$	1	$< 1\ 2\ ;\ -1\ 1 \mid 1\ 0 >$	$\frac{1}{2}\sqrt{\frac{6}{5}}$
$< 1\ 2\ ;\ 0\ 1 \mid 1\ 1 >$	$-\frac{1}{2}\sqrt{\frac{6}{5}}$	$< \frac{3}{2}\ 0\ ;\ \frac{-1}{2}\ 0 \mid \frac{3}{2}\ \frac{-1}{2} >$	1
$< 1\ 2\ ;\ -1\ 1 \mid 1\ 0 >$	$\frac{1}{2}\sqrt{\frac{2}{5}}$	$< 1\ 2\ ;\ 0\ 0 \mid 1\ 0 >$	$-\sqrt{\frac{2}{5}}$

Clebsch-Gordan Symbol	Numerical Value	Clebsch-Gordan Symbol	Numerical Value
$< 1\ 2\ ;\ -1\ 0 \mid 1\ -1 >$	$\frac{1}{2}\sqrt{\frac{2}{5}}$	$< 1\ 1\ ;\ 0\ 1 \mid 1\ 1 >$	$-\sqrt{\frac{1}{2}}$
$< 1\ 1\ ;\ -1\ 1 \mid 1\ 0 >$	$-\sqrt{\frac{1}{2}}$	$< 1\ 0\ ;\ 1\ 0 \mid 1\ 1 >$	1
$< 1\ 0\ ;\ 0\ 0 \mid 1\ 0 >$	1	$< 1\ 1\ ;\ 1\ 0 \mid 1\ 0 >$	$\sqrt{\frac{1}{2}}$
$< 1\ 0\ ;\ -1\ 0 \mid 1\ -1 >$	1	$< 1\ \ ;\ -1\ 0 \mid 1\ -1 >$	$-\sqrt{\frac{1}{2}}$
$< \frac{3}{2}\ \frac{1}{2}\ ;\ \frac{3}{2}\ \frac{-1}{2} \mid 1\ 1 >$	$\frac{1}{2}\sqrt{3}$	$< \frac{3}{2}\ \frac{1}{2}\ ;\ \frac{1}{2}\ \frac{-1}{2} \mid 1\ 0 >$	$\frac{1}{2}\sqrt{2}$
$< \frac{3}{2}\ \frac{1}{2}\ ;\ \frac{-1}{2}\ \frac{-1}{2} \mid 1\ -1 >$	$\frac{1}{2}$	$< \frac{3}{2}\ \frac{1}{2}\ ;\ \frac{1}{2}\ \frac{1}{2} \mid 1\ 1 >$	$\frac{-1}{2}$
$< \frac{3}{2}\ \frac{1}{2}\ ;\ \frac{-1}{2}\ \frac{1}{2} \mid 1\ 0 >$	$-\frac{1}{2}\sqrt{2}$	$< \frac{3}{2}\ \frac{1}{2}\ ;\ \frac{-3}{2}\ \frac{1}{2} \mid 1\ -1 >$	$-\frac{1}{2}\sqrt{3}$
$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{3}{2}\ \frac{-3}{2} \mid 1\ 0 >$	$\frac{3}{10}\sqrt{5}$	$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{1}{2}\ \frac{-3}{2} \mid 1\ -1 >$	$\frac{1}{10}\sqrt{30}$
$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{3}{2}\ \frac{-1}{2} \mid 1\ 1 >$	$\frac{1}{10}\sqrt{30}$	$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{1}{2}\ \frac{-1}{2} \mid 1\ 0 >$	$-\frac{1}{10}\sqrt{5}$
$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{-1}{2}\ \frac{-1}{2} \mid 1\ -1 >$	$-\frac{1}{5}\sqrt{10}$	$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{1}{2}\ \frac{1}{2} \mid 1\ 1 >$	$-\frac{1}{5}\sqrt{10}$
$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{-1}{2}\ \frac{1}{2} \mid 1\ 0 >$	$-\frac{1}{10}\sqrt{5}$	$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{-3}{2}\ \frac{1}{2} \mid 1\ -1 >$	$\frac{1}{10}\sqrt{30}$
$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{-1}{2}\ \frac{3}{2} \mid 1\ 1 >$	$\frac{1}{10}\sqrt{30}$	$< \frac{3}{2}\ \frac{3}{2}\ ;\ \frac{-3}{2}\ \frac{3}{2} \mid 1\ 0 >$	$\frac{3}{10}\sqrt{5}$
$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{3}{2}\ \frac{-5}{2} \mid 1\ -1 >$	$\frac{1}{2}\sqrt{2}$	$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{3}{2}\ \frac{-3}{2} \mid 1\ 0 >$	$\frac{1}{5}\sqrt{5}$
$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{1}{2}\ \frac{-3}{2} \mid 1\ -1 >$	$-\frac{1}{10}\sqrt{30}$	$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{3}{2}\ \frac{-1}{2} \mid 1\ 1 >$	$\frac{1}{10}\sqrt{5}$
$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{1}{2}\ \frac{-1}{2} \mid 1\ 0 >$	$-\frac{1}{10}\sqrt{30}$	$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{-1}{2}\ \frac{-1}{2} \mid 1\ -1 >$	$\frac{1}{10}\sqrt{15}$
$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{-1}{2}\ \frac{1}{2} \mid 1\ 0 >$	$\frac{1}{10}\sqrt{30}$	$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{-3}{2}\ \frac{1}{2} \mid 1\ -1 >$	$-\frac{1}{10}\sqrt{5}$
$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{-1}{2}\ \frac{3}{2} \mid 1\ 1 >$	$\frac{1}{10}\sqrt{30}$	$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{-3}{2}\ \frac{3}{2} \mid 1\ 0 >$	$-\frac{1}{5}\sqrt{5}$
$< \frac{3}{2}\ \frac{5}{2}\ ;\ \frac{-3}{2}\ \frac{5}{2} \mid 1\ 1 >$	$-\frac{1}{2}\sqrt{2}$	$< 1\ \frac{1}{2}\ ;\ -1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$\frac{1}{3}\sqrt{3}$
$< 1\ \frac{1}{2}\ ;\ 0\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$\frac{1}{3}\sqrt{6}$	$< 1\ \frac{1}{2}\ ;\ -1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{-3}{2} >$	1
$< 1\ \frac{1}{2}\ ;\ 1\ \frac{1}{2} \mid \frac{3}{2}\ \frac{3}{2} >$	1	$< 1\ \frac{1}{2}\ ;\ 0\ \frac{1}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$\frac{1}{3}\sqrt{6}$
$< 1\ \frac{1}{2}\ ;\ -1\ \frac{1}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$\frac{1}{3}\sqrt{3}$	$< 1\ \frac{3}{2}\ ;\ 1\ \frac{-3}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$\frac{1}{5}\sqrt{10}$
$< 1\ \frac{3}{2}\ ;\ 0\ \frac{-3}{2} \mid \frac{3}{2}\ \frac{-3}{2} >$	$\frac{1}{5}\sqrt{15}$	$< 1\ \frac{3}{2}\ ;\ 1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$\frac{2}{15}\sqrt{30}$

Clebsch-Gordan Symbol	Numerical Value	Clebsch-Gordan Symbol	Numerical Value
$< 1\ \frac{3}{2}\ ;\ 0\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$\frac{1}{15}\sqrt{15}$	$< 1\ \frac{3}{2}\ ;\ -1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{-3}{2} >$	$-\frac{1}{5}\sqrt{10}$
$< 1\ \frac{3}{2}\ ;\ 1\ \frac{1}{2} \mid \frac{3}{2}\ \frac{3}{2} >$	$\frac{1}{5}\sqrt{10}$	$< 1\ \frac{3}{2}\ ;\ 0\ \frac{1}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$-\frac{1}{15}\sqrt{15}$
$< 1\ \frac{3}{2}\ ;\ -1\ \frac{1}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$-\frac{2}{15}\sqrt{30}$	$< 1\ \frac{3}{2}\ ;\ 0\ \frac{3}{2} \mid \frac{3}{2}\ \frac{3}{2} >$	$-\frac{1}{5}\sqrt{15}$
$< 1\ \frac{3}{2}\ ;\ -1\ \frac{3}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$-\frac{1}{5}\sqrt{10}$	$< 1\ \frac{5}{2}\ ;\ 1\ \frac{-5}{2} \mid \frac{3}{2}\ \frac{-3}{2} >$	$\frac{1}{3}\sqrt{6}$
$< 1\ \frac{5}{2}\ ;\ 1\ \frac{-3}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$\frac{1}{5}\sqrt{10}$	$< 1\ \frac{5}{2}\ ;\ 0\ \frac{-3}{2} \mid \frac{3}{2}\ \frac{-3}{2} >$	$\frac{2}{15}\sqrt{15}$
$< 1\ \frac{5}{2}\ ;\ 1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$\frac{1}{5}\sqrt{5}$	$< 1\ \frac{5}{2}\ ;\ 1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$-\frac{1}{5}\sqrt{10}$
$< 1\ \frac{5}{2}\ ;\ -1\ \frac{-1}{2} \mid \frac{3}{2}\ \frac{-3}{2} >$	$\frac{1}{15}\sqrt{15}$	$< 1\ \frac{5}{2}\ ;\ 1\ \frac{1}{2} \mid \frac{3}{2}\ \frac{3}{2} >$	$\frac{1}{15}\sqrt{15}$
$< 1\ \frac{5}{2}\ ;\ 0\ \frac{1}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$\frac{1}{5}\sqrt{10}$	$< 1\ \frac{5}{2}\ ;\ -1\ \frac{1}{2} \mid \frac{3}{2}\ \frac{-1}{2} >$	$\frac{1}{5}\sqrt{5}$
$< 1\ \frac{5}{2}\ ;\ 0\ \frac{3}{2} \mid \frac{3}{2}\ \frac{3}{2} >$	$-\frac{2}{15}\sqrt{15}$	$< 1\ \frac{5}{2}\ ;\ -1\ \frac{3}{2} \mid \frac{3}{2}\ \frac{1}{2} >$	$\frac{1}{5}\sqrt{10}$
$< 1\ \frac{5}{2}\ ;\ -1\ \frac{5}{2} \mid \frac{3}{2}\ \frac{3}{2} >$	$\frac{1}{3}\sqrt{6}$		

A.4 Maple Code

It was found throughout the study that most of the mathematics calculated could not be found by working out the solutions explicitly by hand, and as such, we have turned to using Maple for the calculations. Specifically, the version number that was used was Maple 15, which as of this writing was not the most up-to-date version. This could impact some of the procedures that could be used in the calculations to perhaps shorten some of the processes, however they were unavailable at the time. Also, certain features might be updated in future releases which could cause the process to not compile correctly, but no such issues exist in the version used.

Here follows a rough explanaition of the process create in describing the classification of solutions in Maple. Of course, this is only for Class I, but it can easily be adapted for any class under consideration.

We start off by first defining all the necessary generators, which is not shown here to conserve space. We can the define the connection we will be using:

$$> a_{\mathrm{phi}} := simplify\left(L_1 - \mathcal{L} \cdot L_{-1} - I \cdot Q_1 \cdot J_0 - Q_2 \cdot A_{-1} - I \cdot Q_3 \cdot W_{-2} - Q_4 \cdot Z_{-2} - I \cdot Q_5 \cdot X_{-3}\right)$$

Then we define the charges as was explicitly calculated in Chapter 5:

$$> \mathcal{L} := \frac{1}{8}\lambda_5^2 - \frac{1}{20}\lambda_1\lambda_2 + \frac{1}{20}\lambda_2^2 + \frac{1}{20}\lambda_1^2 + \frac{1}{20}\lambda_3^2 + \frac{1}{8}\lambda_4^2 - \frac{1}{20}\lambda_2\lambda_3 - \frac{1}{8}\lambda_4\lambda_5;$$

$$Q_2 := -\frac{1}{20}\lambda_1\lambda_2 + \frac{1}{20}\lambda_2^2 + \frac{1}{20}\lambda_1^2 + \frac{1}{20}\lambda_3^2 - \frac{1}{8}\lambda_4^2 - \frac{1}{20}\lambda_2\lambda_3 - \frac{1}{8}\lambda_5^2 + \frac{1}{8}\lambda_4\lambda_5;$$

$$Q_3 := \frac{1}{144} \mathrm{I}\left(-\lambda_2\lambda_3^2 - 18\lambda_4\lambda_5^2 + 18\lambda_4^2\lambda_5 + \lambda_2^2\lambda_3 + \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2\right);$$

$$Q_4 := \frac{1}{144}\lambda_2\lambda_3^2 - \frac{1}{8}\lambda_4\lambda_5^2 + \frac{1}{8}\lambda_4^2\lambda_5 - \frac{1}{144}\lambda_2^2\lambda_3 - \frac{1}{144}\lambda_1^2\lambda_2 + \frac{1}{144}\lambda_1\lambda_2^2;$$

$$Q_5 := -\frac{1}{6000} \mathrm{I}\left(-82\lambda_1^2\lambda_3^2 + 27\lambda_1^2\lambda_2^2 - 18\lambda_1\lambda_2^3 - 18\lambda_1^3\lambda_2 + 27\lambda_2^2\lambda_3^2 - 18\lambda_2^3\lambda_3 - 18\lambda_3^3\lambda_2 + 82\lambda_1^2\lambda_2\lambda_3 + 82\lambda_1\lambda_2\lambda_3^2 - 82\lambda_1\lambda_2^2\lambda_3 + 9\lambda_2^4 + 9\lambda_1^4 + 9\lambda_3^4\right):$$

We can then start with the specific class we are considering. We must first define the general spinor which we will be using:

$$> \varepsilon_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & \varepsilon_{15} & \varepsilon_{16} & \varepsilon_{17} \\ 0 & 0 & 0 & 0 & \varepsilon_{25} & \varepsilon_{26} & \varepsilon_{27} \\ 0 & 0 & 0 & 0 & \varepsilon_{35} & \varepsilon_{36} & \varepsilon_{37} \\ 0 & 0 & 0 & 0 & \varepsilon_{45} & \varepsilon_{46} & \varepsilon_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And then we may substitute the correct eigenvalues of this case into the charges

$$> \mathcal{L} := subs\left(\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \mathcal{L}\right):$$

$$> Q_2 := subs\left(\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, Q_2\right):$$

$$> Q_3 := subs\left(\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, Q_3\right):$$

$$> Q_4 := subs\left(\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, Q_4\right):$$

$$> Q_5 := subs\left(\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, Q_5\right):$$

The solution of the charges is then substituted into the connection, which we will rename to Snew

$$> Snew := simplify\left(subs\left(\mathcal{L} = 0, Q_2 = 0, \mathrm{Q}_3 = 0, \mathrm{Q}_4 = 0, \mathrm{Q}_5 = 0, \mathrm{a}_{\mathrm{phi}}\right)\right)$$

However, the following step where we must calculate the Jordan form of this specific connection is calculated by Maple in a strange way. It automatically permutes the eigenvalues according to its own algorithms and the final solution does not correspond to what we expect. As such, we must construct a permutation matrix, using Maple to calculate the transformational matrix which it would use to calculate the Jordan form, and then manually calculate the Jordan form:

$$> P := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}:$$

$> V := JordanForm(Snew, output='Q').P$

$> MatrixInverse(V).Snew.V$

The output of the above will then give the correct Jordan form of the connection. It is then split into a diagonal section (D) as well as a nilpotent section (n), which we will now use. We calculate the commutation of the spinor ε_0 with the nilpotent matrix which will impose conditions on the elements of the spinor:

$> n.\varepsilon_0 - \varepsilon_0.n$

We then substitute the corresponding elements which should equate zero into our case specific spinor:

$> \varepsilon_0 := subs(\varepsilon_{25} = 0, \varepsilon_{35} = 0, \varepsilon_{45} = 0, \varepsilon_{46} = 0, \varepsilon_{36} = 0, \varepsilon_{47} = 0, \varepsilon_{15} = \varepsilon_{37}, \varepsilon_{26} = \varepsilon_{37}, \varepsilon_{16} = \varepsilon_{27}, \varepsilon_0)$

Once again, Maple has it's limitations in that it struggles to simplify certain expressions, and as such we manually define the exponentiated diagonal into

$$> E_D := \begin{bmatrix} \exp(I \cdot Q_1 \cdot \mathrm{phi}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \exp(I \cdot Q_1 \cdot \mathrm{phi}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \exp(I \cdot Q_1 \cdot \mathrm{phi}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \exp(I \cdot Q_1 \cdot \mathrm{phi}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We may then find the Killing spinor in this class if we first find

$> MatrixInverse(E_D).\varepsilon_0.E_D$

(1)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\varepsilon_{37}}{e^{I Q_1 \phi}} & \frac{\varepsilon_{27}}{e^{I Q_1 \phi}} & \frac{\varepsilon_{17}}{e^{I Q_1 \phi}} \\ 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon_{37}}{e^{I Q_1 \phi}} & \frac{\varepsilon_{27}}{e^{I Q_1 \phi}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon_{37}}{e^{I Q_1 \phi}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1)}$$

and following

> $V.\%MatrixInverse(V)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}\,\varepsilon_{37}}{e^{I Q_1 \phi}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\sqrt{3}\,\varepsilon_{27}}{e^{I Q_1 \phi}} & -\frac{\sqrt{3}\,\varepsilon_{37}\sqrt{2}}{e^{I Q_1 \phi}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{6\,\varepsilon_{17}}{e^{I Q_1 \phi}} & \frac{3\,\varepsilon_{27}\sqrt{2}}{e^{I Q_1 \phi}} & -\frac{3\,\varepsilon_{37}}{e^{I Q_1 \phi}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(2)}$$

we may find the expressions of the Killing spinor. This following expression is attained by computing specific entries of the above matrix and equating them to the corresponding elements and coefficients of the appropriate fermionic generators which make up the Killing spinor in (5.49) and using a differential solve function

> $dsolve\left(\left[\left[-3\,\mathrm{sqrt}(2)\cdot r_{\frac{1}{2}} + -2^{\frac{3}{2}}\cdot t_{\frac{1}{2}} - \mathrm{sqrt}(2)\cdot h_{\frac{1}{2}} = -\frac{\sqrt{3}\,\varepsilon_{37}}{e^{I Q_1 \phi}}, 3\,\mathrm{sqrt}(2)\cdot t_{\frac{3}{2}} + 3\cdot 2^{\frac{3}{2}}\cdot r_{\frac{3}{2}} = \frac{2\sqrt{3}\,\varepsilon_{27}}{e^{I Q_1 \phi}}, -5\cdot 2^{\frac{3}{2}}\cdot \mathrm{sqrt}(3)\cdot r_{\frac{5}{2}} = -\frac{6\,\varepsilon_{17}}{e^{I Q_1 \phi}}\right]\right]\right)$

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