A Characterization of Askey-Wilson polynomials

Maurice Kenfack Nangho[1,2] and Kerstin Jordaan[3]

[1] Department of Mathematics and Applied Mathematics, University of Pretoria

[2] Department of Mathematics and Computer Science,
University of Dschang, Cameroon
maurice.kenfack@univ-dschang.org

[3] Department of Decision Sciences,
University of South Africa, PO Box 392, Pretoria, 0003, South Africa
jordakh@unisa.ac.za

March 12, 2019

Abstract

We show that the only monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ that satisfy

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j}P_{n+j}(x),\ x = \cos\theta,\ \ a_{n,n-2} \neq 0,\ n = 2, 3, \dots,$$

where $\pi(x)$ is a polynomial of degree at most 4 and $\mathcal{D}_q$ is the Askey-Wilson operator, are Askey-Wilson polynomials and their special or limiting cases. This completes and proves a conjecture by Ismail concerning a structure relation satisfied by Askey-Wilson polynomials. We use the structure relation to derive upper bounds for the smallest zero and lower bounds for the largest zero of Askey-Wilson polynomials and their special cases.

1 Introduction

A sequence of polynomials $\{p_n\}_{n=0}^{\infty}$, $\deg(p_n) = n$, is orthogonal with respect to a positive measure μ on the real numbers $\mathbb{R}$, if

$$\int_S p_m(x)p_n(x)d\mu(x) = d_n\delta_{m,n},\ m, n \in \mathbb{N},$$

where S is the support of μ, $d_n > 0$ and $\delta_{m,n}$ the Kronecker delta. A sequence $\{P_n\}_{n=0}^{\infty}$ of monic polynomials orthogonal with respect to a positive measure satisfies a three-term recurrence relation

$$P_{n+1} = (x - a_n)P_n - b_nP_{n-1}, \quad n = 0, 1, 2, \dots \tag{1}$$

with initial conditions $P_{-1} \equiv 0$, $P_0 \equiv 1$ (note that with this choice of P_{-1}, the initial value of b_0 is irrelevant) and recurrence coefficients $a_n \in \mathbb{R}$, $n = 0, 1, 2 \dots$, $b_n > 0$, $n = 1, 2, \dots$.

A sequence of monic orthogonal polynomials is classical if the sequence $\{P_n\}_{n=0}^{\infty}$ as well as D^mP_{n+m}, $m \in \mathbb{N}$, where D is the usual derivative $\frac{d}{dx}$ or one of its extensions (difference, q-difference or divided-difference operator) satisfies a three-term recurrence of the form (1). When $D = \frac{d}{dx}$, Hahn [13] showed that a sequence of monic orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying

$$\frac{1}{n+1}\frac{dP_{n+1}}{dx}(x) = (x - a_n')\frac{1}{n}\frac{dP_n}{dx}(x) - \frac{b_n'}{n-1}\frac{dP_{n-1}}{dx}(x),\ \ a_n',\, b_n' \in \mathbb{R}, b_n' \neq 0,$$

satisfies a second order Sturm-Liouville differential equation of the form

$$\phi(x)\frac{d^2}{dx^2}P_n(x) + \psi(x)\frac{d}{dx}P_n(x) + \lambda_n\, P_n = 0. \tag{2}$$

where, ϕ and ψ are polynomials independant of n with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$ while λ_n is a constant dependant on n. Bochner [4] first considered sequences of polynomials satisfying (2) and showed that the orthogonal polynomial solutions of (2) are Jacobi, Laguerre and Hermite polynomials, a result known as Bochner's theorem. Bochner's theorem has been generalized and used to characterize Askey-Wilson polynomials (cf. [15]). See also [12, 23].

A related problem, due to Askey (cf. [1]), is to characterize the orthogonal polynomials whose derivatives satisfy a structural relation of the form

$$\pi(x)\frac{d}{dx}P_n(x) = \sum_{j=-r}^{s} a_{n,n+j}P_{n+j}(x), \quad n = 1, 2, \ldots$$

and this problem was considered by Maroni (cf. [21], [22]) who called such orthogonal polynomial sequences semi-classical.

Al-Salam and Chihara [1] characterized Jacobi, Laguerre and Hermite as the only orthogonal polynomials with a structure relation of form

$$\pi(x)\frac{d}{dx}P_n(x) = \sum_{j=-1}^{1} a_{n,n+j}P_{n+j}(x), \quad n = 1, 2, \ldots \tag{3}$$

where $\pi(x)$ is a polynomial of degree at most two. Replacing the usual derivative in (3) by the forward difference operator

$$\Delta f(s) = f(s+1) - f(s),$$

García, Marcellán and Salto [11] proved that Hahn, Krawtchouk, Meixner and Charlier polynomials are the only orthogonal polynomial sequences satisfying

$$\pi(x)\Delta P_n(x) = \sum_{j=-1}^{1} a_{n,n+j}P_{n+j}(x), \quad n = 1, 2, \ldots$$

with $\pi(x)$ a polynomial of degree two or less. More recently, replacing the derivative in (3) by the Hahn operator (cf. [17, (11.4.1)], [14]), also known as the q-difference operator or Jackson derivative [18],

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

Datta and Griffin [7] characterized the big q-Jacobi polynomial or one of its special or limiting cases (Al-Salam-Carlitz 1, little and big q-Laguerre, little q-Jacobi, and q-Bessel polynomials) as the only orthogonal polynomials that satisfy

$$\pi(x)D_qP_n(x) = \sum_{j=-1}^{1} a_{n,n+j}P_{n+j}, \quad n = 1, 2, \ldots \tag{4}$$

where $\pi(x)$ is a polynomial of degree at most two.

The polynomials mentioned above are all special or limiting cases of the Askey-Wilson polynomials [2, (1.15)], [19, (14.1.1)]

$$\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = {}_4\phi_3\left(\begin{array}{c} q^{-n},\, abcdq^{n-1},\, ae^{-i\theta}, ae^{i\theta} \\ ab,\, ac,\, ad \end{array}; q, q\right),\ x = \cos\theta, \tag{5}$$

with the multiple q-shifted factorials defined by $(a_1, \ldots, a_i; q)_k = \prod_{j=1}^{i}(a_j; q)_k$ where the q-shifted factorials are given by $(a; q)_0 = 1, \quad (a; q)_k = \prod_{j=0}^{k-1}\left(1 - aq^j\right),$
$k = 1, 2, \ldots$ or ∞ and

$${}_{s+1}\phi_s\left(\begin{array}{c} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{array}; q, z\right) = \sum_{k=0}^{\infty}\frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k}\frac{z^k}{(q; q)_k}.$$

Askey-Wilson polynomials do not satisfy either (3) or (4) but they do satisfy the shift relation (cf. [19, (14.1.9)])

$$\mathcal{D}_q p_n(x, a, b, c, d|q) = \frac{2q^{\frac{1-n}{2}}(1-q^n)(1-abcdq^{n-1})}{1-q}p_{n-1}(x; aq^{\frac{1}{2}}, bq^{\frac{1}{2}}cq^{\frac{1}{2}}, dq^{\frac{1}{2}}|q)$$

where $\mathcal{D}_q$ is the Askey-Wilson divided difference operator (cf. [2, p.35], [19, (1.16.4)], [17, (12.1.12)])

$$\mathcal{D}_q f(x) = \frac{\breve{f}(q^{\frac{1}{2}}e^{i\theta}) - \breve{f}(q^{-\frac{1}{2}}e^{i\theta})}{(e^{i\theta} - e^{-i\theta})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})/2}, \quad \breve{f}(z) = f\left(\frac{z+z^{-1}}{2}\right), \quad z = e^{\pm i\theta}. \tag{6}$$

The Askey problem involving the Askey-Wilson operator $\mathcal{D}_q$ is still open but in 2005, Ismail [17] gave an important hint to the solution of this problem with the following conjecture.

Conjecture 1.1. *[17, Conjecture 24.7.9] Let $\{P_n\}$ be orthogonal polynomials and π be a polynomial of degree at most 4. Then $\{P_n(x)\}$ satisfies*

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-r}^{s} a_{n,n+j} P_{n+j}(x)$$

if and only if $\{P_n(x)\}$ are Askey-Wilson polynomials or special cases of them.

The aim of this paper is to complete and prove this conjecture in §3 and to apply the explicit structure relation that characterizes Askey-Wilson polynomials to obtain inequalities satisfied by the extreme zeros of these polynomials in §4.

2 Preliminaries

Before moving to our main result let us recall some basic results. Taking $e^{i\theta} = q^s$, the operator (6) reads

$$\mathcal{D}_q f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad x(s) = \frac{q^{-s}+q^s}{2}.$$

Moreover, $x(s)$ satisfies (cf. [3])

$$\begin{aligned} x(s+n) - x(s) &= \gamma_n \left(x\left(s+\tfrac{1}{2}n+\tfrac{1}{2}\right) - x\left(s+\tfrac{1}{2}n-\tfrac{1}{2}\right)\right), \\ x(s+n) + x(s) &= 2\alpha_n\, x\left(s+\tfrac{1}{2}n\right), \end{aligned} \tag{7}$$

for $n = 0, 1, \ldots$, with the sequences (α_n), (γ_n) given explicitly by

$$2\alpha_n = q^{\frac{n}{2}} + q^{-\frac{n}{2}}, \quad (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\gamma_n = q^{\frac{n}{2}} - q^{-\frac{n}{2}}, \quad \alpha_1 = \alpha \tag{8}$$

The following hold (cf. [17, p.302], [9, p.169])

$$\mathcal{D}_q(fg) = \mathcal{S}_q(f)\mathcal{D}_q(g) + \mathcal{D}_q(f)\mathcal{S}_q(g) \tag{9}$$

$$\mathcal{S}_q(fg) = \mathcal{S}_q(f)\mathcal{S}_q(g) + U_2\mathcal{D}_q(f)\mathcal{D}_q(g) \tag{10}$$

$$\mathcal{D}_q\,\mathcal{S}_q = \alpha\,\mathcal{S}_q\,\mathcal{D}_q + U_1\,\mathcal{D}_q^2 \tag{11}$$

$$\mathcal{S}_q^2 = U_1\,\mathcal{S}_q\,\mathcal{D}_q + \alpha\,U_2\,\mathcal{D}_q^2 + \mathbb{I}, \tag{12}$$

where $U_1(x) = (\alpha^2-1)x$, $U_2(x) = (\alpha^2-1)(x^2-1)$, $\mathbb{I}(f) = f$ and $\mathcal{S}_q$ is the averaging operator [17, (12.1.21)]

$$\mathcal{S}_q\, f(x(s)) = \tfrac{1}{2}\left(f(x(s+\tfrac{1}{2})) + f(x(s-\tfrac{1}{2}))\right).$$

Unless otherwise indicated, $0 < q < 1$.

3 Proving the conjecture due to Ismail

We begin by proving a lemma that generalizes a result proved by Hahn in [13]. We will denote a monic orthogonal polynomial of precise degree n, $n = 1, 2, \ldots$ by $P_n(x)$ which implies that $\frac{1}{\gamma_n}\mathcal{D}_q P_n(x)$ will be monic. To see this, normalise the basis in [17, (20.3.9)], to obtain the monic polynomial base $\{F_k(x)\}$ where $F_k(x) = \frac{q^{-\frac{k^2}{4}}}{(-2)^k}(q^{\frac{1}{4}}q^s, q^{\frac{1}{4}}q^{-s}; q^{\frac{1}{2}})_k = \prod_{j=0}^{k-1}[x - \zeta_j]$, for $k = 0, 1, \ldots$, $x = \cos\theta$ with $\zeta_j = \frac{1}{2}(q^{-\frac{1}{4}-\frac{j}{2}} + q^{\frac{1}{4}+\frac{j}{2}})$. It follows from [16, Thm 2.1] that $P_n(x) = F_n(x) + \ldots$ and, since $\mathcal{D}_q F_k(x) = \gamma_k F_{k-1}(x)$ (cf. [17, 20.3.11]), $\mathcal{D}_q P_n(x) = \gamma_n F_{n-1}(x) + \ldots$.

Lemma 3.1. *Let $\{P_n\}_{n=0}^{\infty}$ a sequence of monic orthogonal polynomials. If there are two sequences (a'_n) and (b'_n) such that*

$$\frac{1}{\gamma_{n+1}}\mathcal{D}_q P_{n+1}(x) = (x - a'_n)\frac{1}{\gamma_n}\mathcal{D}_q P_n(x) - \frac{b'_n}{\gamma_{n-1}}\mathcal{D}_q P_{n-1}(x) + c_n,\ c_n \in \mathbb{R}, \tag{13}$$

then there are two polynomials $\phi(x)$ and $\psi(x)$ of degree at most two and of degree one respectively and a sequence $\{\lambda_n\}_{n=0}^{\infty}$ depending on n such that $P_n(x)$ satisfies the divided difference equation

$$\phi(x)\mathcal{D}_q^2 P_n(x) + \psi(x)\mathcal{S}_q\mathcal{D}_q P_n(x) + \lambda_n P_n(x) = 0,\ n \geq 5. \tag{14}$$

Proof. Since $\{P_n\}_{n=0}^{\infty}$ is monic and orthogonal, there exist sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that the recurrence relation (1) is satisfied. If $f(x) = x - a_n$, it follows from (7) and (8) that

$$\mathcal{S}_q\, f(x) = \alpha x - a_n. \tag{15}$$

Applying the operator $\mathcal{D}_q$ to both sides of (1) and using the product rule (9) together with (15), yields

$$\mathcal{D}_q P_{n+1}(x) = (\alpha x - a_n)\mathcal{D}_q P_n(x) + \mathcal{S}_q P_n(x) - b_n \mathcal{D}_q P_{n-1}(x). \tag{16}$$

If we apply $\mathcal{S}_q$ to both sides of (13) and (16), and use the products (10) and (12), we obtain respectively

$$\begin{aligned}\frac{1}{\gamma_{n+1}}\mathcal{S}_q\mathcal{D}_q P_{n+1}(x) &= (\alpha x - a'_n)\,\frac{1}{\gamma_n}\mathcal{S}_q\mathcal{D}_q P_n(x) + \frac{1}{\gamma_n}U_2(x)\mathcal{D}_q^2 P_n(x)\\ &\quad - \frac{b'_n}{\gamma_{n-1}}\mathcal{S}_q\mathcal{D}_q P_{n-1}(x) + c_n. &(17a)\\ \mathcal{S}_q\mathcal{D}_q P_{n+1}(x) &= \left(\alpha^2 x + U_1(x) - a_n\right)\mathcal{S}_q\mathcal{D}_q P_n(x) + 2\alpha\, U_2(x)\mathcal{D}_q^2 P_n(x)\\ &\quad + P_n(x) - b_n\mathcal{S}_q\mathcal{D}_q P_{n-1}(x) &(17b)\end{aligned}$$

Applying $\mathcal{D}_q$ to both sides of (13) and (16) and then using (9) and (11) we obtain respectively

$$\frac{1}{\gamma_{n+1}}\mathcal{D}_q^2 P_{n+1}(x) = \frac{(\alpha x - a'_n)}{\gamma_n}\mathcal{D}_q^2 P_n(x) + \frac{1}{\gamma_n}\mathcal{S}_q\mathcal{D}_q P_n(x) - \frac{b'_n}{\gamma_{n-1}}\mathcal{D}_q^2 P_{n-1}(x). \tag{18a}$$

$$\mathcal{D}_q^2 P_{n+1}(x) = \left(\alpha^2 x + U_1(x) - a_n\right)\mathcal{D}_q^2 P_n(x) + 2\alpha\mathcal{S}_q\mathcal{D}_q P_n(x) - b_n\mathcal{D}_q^2 P_{n-1}(x), \tag{18b}$$

Eliminating $\mathcal{S}_q\mathcal{D}_q P_{n-1}(x)$ in the system (17), by subtracting b_n times (17a) from $\frac{b'_n}{\gamma_{n-1}}$ times (17b), we have

$$A_n\mathcal{S}_q\mathcal{D}_q P_{n+1}(x) = D_n U_2(x)\mathcal{D}_q^2 P_n(x) + \frac{b'_n}{\gamma_{n-1}}P_n(x) - b_n c_n + B_n(x)\mathcal{S}_q\mathcal{D}_q P_n(x) \tag{19}$$

where $A_n = \dfrac{b'_n}{\gamma_{n-1}} - \dfrac{b_n}{\gamma_{n+1}}$, $B_n(x) = \left(\frac{\alpha^2 b'_n}{\gamma_{n-1}} - \frac{\alpha\, b_n}{\gamma_n}\right)x + \frac{b'_n}{\gamma_{n-1}}U_1(x) + \frac{b_n a'_n}{\gamma_n} - \frac{b'_n a_n}{\gamma_{n-1}}$ and $D_n = \left(\dfrac{2\alpha b'_n}{\gamma_{n-1}} - \dfrac{b_n}{\gamma_n}\right)$.
Eliminating $\mathcal{S}_q\mathcal{D}_q P_{n+1}(x)$ in (17), by subtracting $\frac{1}{\gamma_{n+1}}$ times (17b) from (17a), using the relation $\gamma_{n+1} = \alpha_n + \alpha\gamma_n$ obtained by direct computation from (8) and substituting n by $n+1$, yields

$$\frac{P_{n+1}(x)}{\gamma_{n+2}} = C_n(x)\mathcal{S}_q\mathcal{D}_q P_{n+1}(x) - E_n U_2(x)\mathcal{D}_q^2 P_{n+1}(x) - A_{n+1}\mathcal{S}_q\mathcal{D}_q P_n(x) + c_{n+1} \tag{20}$$

where $C_n(x) = \dfrac{\alpha\,\alpha_{n+1}}{\gamma_{n+1}\gamma_{n+2}}x - \dfrac{U_1(x)}{\gamma_{n+2}} + \dfrac{a_{n+1}}{\gamma_{n+2}} - \dfrac{a'_{n+1}}{\gamma_{n+1}}$ and $E_n = \left(\dfrac{2\alpha}{\gamma_{n+2}} - \dfrac{1}{\gamma_{n+1}}\right)$. Subtracting $\frac{b'_n}{\gamma_{n-1}}$ times (18b) from b_n times (18a) we obtain

$$A_n\mathcal{D}_q^2 P_{n+1}(x) = B_n(x)\mathcal{D}_q^2 P_n(x) + D_n\mathcal{S}_q\mathcal{D}_q P_n(x) \tag{21a}$$

Subtracting $\frac{1}{\gamma_{n+1}}$ times (18b) from (18a), using again the relation $\gamma_{n+1} = \alpha_n + \alpha\gamma_n$ and substituting n by $n+1$, yields

$$E_n\mathcal{S}_q\mathcal{D}_q P_{n+1}(x) = C_n(x)\mathcal{D}_q^2 P_{n+1}(x) - A_{n+1}\mathcal{D}_q^2 P_n(x). \tag{21b}$$

Eliminating $\mathcal{D}_q^2 P_{n+1}(x)$ in (21b), by substituting (21a) into (21b), we obtain

$$A_n E_n \mathcal{S}_q \mathcal{D}_q P_{n+1}(x)(C_n(x)B_n(x) - A_n A_{n+1})\mathcal{D}_q^2 P_n(x) + C_n(x) D_n \mathcal{S}_q \mathcal{D}_q P_n(x). \tag{22}$$

Using (19), we eliminate $\mathcal{S}_q \mathcal{D}_q P_{n+1}(x)$ from (22) to obtain

$$\phi_n(x)\mathcal{D}_q^2 P_n(x) + \psi_n(x)\mathcal{S}_q \mathcal{D}_q P_n(x) - E_n \frac{b_n'}{\gamma_{n-1}} P_n(x) = -E_n b_n c_n, \tag{23}$$

where

$$\begin{aligned}\phi_n(x) &= C_n(x)B_n(x) - A_n A_{n+1} - E_n D_n U_2(x)\\ \psi_n(x) &= C_n(x) D_n - B_n(x) E_n.\end{aligned}$$

Similarly, eliminating $\mathcal{D}_q^2 P_n(x)$ in (21b) by adding $B_n(x)$ times (21b) to A_{n+1} times (21a), and then substituting the resulting relation into (20) to eliminate $\mathcal{S}_q \mathcal{D}_q\, P_n(x)$, yields

$$\phi_n(x)\mathcal{D}_q^2 P_{n+1}(x) + \psi_n(x)\mathcal{S}_q \mathcal{D}_q\, P_{n+1}(x) - \frac{D_n}{\gamma_{n+2}} P_{n+1}(x) = -D_n c_{n+1}, \tag{24}$$

where $\phi_n(x)$ and $\psi_n(x)$ are the polynomial coefficients of (23). Substituting $U_1(x) = (\alpha^2 - 1)x$ into (18b) and subtracting $\frac{1}{\gamma_n}$ times the obtained equation from 2α times (18a) to elliminate $\mathcal{S}_q \mathcal{D}_q P_n(x)$, yields

$$\frac{x}{\gamma_n}\mathcal{D}_q^2 P_n(x) = E_{n-1}\mathcal{D}_q^2 P_{n+1}(x) + \frac{(2\alpha a_n' - a_n)}{\gamma_n}\mathcal{D}_q^2 P_n(x) + D_n \mathcal{D}_q^2 P_{n-1}(x). \tag{25}$$

Substituting $\mathcal{S}_q \mathcal{D}_q P_{n+1}$, $\mathcal{S}_q \mathcal{D}_q P_n$ and $\mathcal{S}_q \mathcal{D}_q P_{n-1}$ obtained from (21b), into (17a) and repeatedly applying (25), we obtain $c_n = \displaystyle\sum_{k=-2}^{2} d_{n,k}\mathcal{D}_q^2 P_{n+k}(x), n \geq 2$. Since $\mathcal{D}_q^2 P_{j+2}$ is of degree j, $\{\mathcal{D}_q^2 P_{j+2}\}_{j=0}^{\infty}$ forms a basis for the space of polynomials and therefore $c_n = 0$ for $n \geq 5$. In the sequel of this proof, we will assume that $n \geq 5$.

Using the relation

$$\mathcal{S}_q f(x(s)) = \mathcal{T}_1 f(x(s)) - \frac{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}{2}\mathcal{D}_q f(x(s)), \quad \mathcal{T}_\nu f(x(s)) = f(x(s+\tfrac{\nu}{2})),$$

that follows from the definitions of $\mathcal{S}_q$ and $\mathcal{D}_q$, in (23) with n replaced by $n+1$ and also in (24), we obtain respectively

$$\sigma_{n+1}(x(s))\mathcal{D}_q^2 P_{n+1}(x(s)) + \psi_{n+1}(x(s))\mathcal{T}_1 \mathcal{D}_q P_{n+1}(x(s)) - \frac{E_{n+1}b_{n+1}'}{\gamma_n} P_{n+1}(x(s)) = 0, \tag{26a}$$

$$\sigma_n(x(s))\mathcal{D}_q^2 P_{n+1}(x) + \psi_n(x(s))\mathcal{T}_1 \mathcal{D}_q P_{n+1}(x(s)) - D_n \frac{1}{\gamma_{n+2}} P_{n+1}(x(s)) = 0, \tag{26b}$$

where

$$\sigma_n(x(s)) = \phi_n(x(s)) - \frac{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}{2}\psi_n(x(s)).$$

Subtracting $\sigma_{n+1}(x(s))$ times (26b) from $\sigma_n(x(s))$ times (26a), yields

$$\begin{aligned}&\left(\phi_n(x(s))\psi_{n+1}(x(s)) - \phi_{n+1}(x(s))\psi_n(x(s))\right)\mathcal{T}_1 \mathcal{D}_q P_{n+1}(x(s)) +\\ &\left(\frac{\sigma_{n+1}(x(s))D_n}{\gamma_{n+2}} - \frac{\sigma_n(x(s))E_{n+1}b_{n+1}'}{\gamma_n}\right) P_{n+1}(x(s)) = 0,\end{aligned} \tag{27}$$

where $\mathcal{T}_1 \mathcal{D}_q P_{n+1}(x(s)) = \frac{P_{n+1}(x(s+1)) - P_{n+1}(x(s))}{x(s+1) - x(s)}$ by definition. Since P_{n+1} is a function of the variable $x = \cos\theta$, its zeros are in the interval $(-1, 1)$. Let $-1 < x(s_1) < x(s_2) < ... < x(s_{n+1}) < 1$ denote the zeros of $P_{n+1}(x(s))$. For $j = 1, 2, ..., n+1$ there is $\theta_j, 0 < \theta_j < \pi$, such that $x(s_j) = \frac{q^{s_j} + q^{-s_j}}{2} = \frac{e^{i\theta_j} + e^{-i\theta_j}}{2}$ and it follows that $x(s_j + 1) = \frac{q e^{i\theta_j} + q^{-1}e^{-i\theta_j}}{2} = \frac{(q^2+1)\cos\theta_j + i(q^2-1)\sin\theta_j}{2q} \notin \mathbb{R}$ for $0 < q < 1$. Therefore $P_{n+1}(x(s_j + 1)) \neq 0$ and hence $\mathcal{T}_1 \mathcal{D}_q P_{n+1}(x(s_j)) \neq 0$ for $j = 1, 2, ..., n+1$. So, by (27), the polynomial $F_n(x(s)) = \phi_n(x(s))\psi_{n+1}(x(s)) - \phi_{n+1}(x(s))\psi_n(x(s))$, which is of degree at most 3, will vanish at n+1 zeros of P_{n+1}, $n \geq 5$. Hence $F_n(x)$ is equal to zero for all x and there exists G_n, $n \in \mathbb{N}$, such that $\phi_{n+1}(x) = G_n \phi_n(x)$ and $\psi_{n+1}(x) = G_n \psi_n(x)$. Iterating these relations, we obtain $\phi_n(x) = H_n \phi_5(x)$ and $\psi_n(x) = H_n \psi_5(x)$, $H_n = \displaystyle\prod_{j=5}^{n-1} G_j$. Finally, dividing both sides of (23) by H_n and keeping in mind that $c_n = 0$ for $n \geq 5$, we obtain the result. $\square$

We now state and prove our main result.

Theorem 3.2. *Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials orthogonal with respect to a positive weight function $w(x)$. The following properties are equivalent.*

(a) There is a polynomial $\pi(x)$ of degree at most 4 and constants $a_{n,n+k}$, $k \in \{-2,-1,0,1,2\}$ with $a_{n,n-2} \neq 0$ such that P_n satisfies the structure relation

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{k=-2}^{2} a_{n,n+k} P_{n+k}(x), \quad n = 2,3,\ldots;$$

(b) There is a polynomial $\pi(x)$ of degree at most four such that $\{\mathcal{D}_q^2 P_j\}_{j=2}^{\infty}$ is orthogonal with respect to $\pi(x)\, w(x)$;

(c) There are two polynomials $\phi(x)$ and $\psi(x)$ of degree at most two and of degree one respectively and a constant λ_n such that

$$\phi(x)\mathcal{D}_q^2 P_n(x) + \psi(x)\mathcal{S}_q\mathcal{D}_q P_n(x) + \lambda_n P_n(x) = 0, \quad n = 5,6,\ldots. \tag{28}$$

Proof of Theorem 3.2. The proof is organized as follows.
Step 1 $(a) \Rightarrow (b) \Rightarrow (a)$ which is equivalent to $(a) \Leftrightarrow (b)$.
Step 2 $(b) \Rightarrow (c) \Rightarrow (a)$ which, taking into account Step 1, is equivalent to $(b) \Leftrightarrow (c)$.

Step 1: Assume that (a) is satisfied. Let $m, n \in \mathbb{N}$, $m, n \geq 2$ and $m \leq n$. From (a), there is a polynomial $\pi(x)$ of degree at most four and there exist constants $a_{n,n+j}$, $j \in \{-2, -1, 0, 1, 2\}$ such that

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x), \text{ with } a_{n,n-2} \neq 0. \tag{29}$$

Since $m \leq n$ we have that $m-2 \leq n-2 \leq n+j \leq n+2$ for $j \in \{-2, -1, 0, 1, 2\}$. Multiplying both sides of (29) by $w(x)\mathcal{D}_q^2 P_m(x)$, integrating on (a,b) and then taking into account the fact that $\{P_j\}_{j=0}^{\infty}$ is orthogonal on the interval (a,b) with respect to the weight function $w(x)$, we obtain

$$\int_a^b \mathcal{D}_q^2 P_m(x)\mathcal{D}_q^2 P_n(x)\pi(x)w(x)dx \begin{cases} = 0 & \text{if} \quad m < n \\ \neq 0 & \text{if} \quad m = n. \end{cases}$$

If $n < m$, interchanging m and n in the above argument yields

$$\int_a^b \mathcal{D}_q^2 P_n(x)\mathcal{D}_q^2 P_m(x)\pi(x)w(x)dx = 0.$$

Now let $n \in \mathbb{N}$, $n \geq 2$ and assume (b). Since $\pi(x)\mathcal{D}_q^2 P_n(x)$ is a polynomial of degree less or equal to $n+2$, it can be expanded in the orthogonal basis $\{P_j\}_{j=0}^{\infty}$ as $\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{k=0}^{n+2} a_{n,k} P_k(x)$, where, for $k \in \{0, ..., n+2\}$, $a_{n,k}$ is given by

$$a_{n,k} \int_a^b (P_k(x))^2 w(x)dx = \int_a^b P_k(x)\mathcal{D}_q^2 P_n(x)\pi(x)w(x)dx.$$

Since $\mathcal{D}_q^2 P_n(x)$ is of degree $n-2$ we deduce from the hypothesis that $a_{n,k} = 0$ for $k \in \{0, ..., n-3\}$ and $a_{n,n-2} \neq 0$.

Step 2: We suppose (b) and we prove (c). Firstly, we prove that polynomials in the sequence $\{P_n\}_{n=0}^{\infty}$ satisfy an equation of type (13). Let $n \in \mathbb{N}$, $n \geq 2$ and denote the leading coefficient of P_n by γ_n, then, since $\frac{x}{\gamma_n}\mathcal{D}_q P_n$ is a monic polynomial of degree n, it can be expanded as

$$x\frac{1}{\gamma_n}\mathcal{D}_q P_n(x) = \frac{1}{\gamma_{n+1}}\mathcal{D}_q P_{n+1}(x) + \sum_{j=1}^{n} \frac{e_{n,j}}{\gamma_j}\mathcal{D}_q P_j(x), \; e_{n,j} \in \mathbb{R}. \tag{30}$$

Applying $\mathcal{D}_q$ to both sides of (30) and using (9), we obtain

$$(\alpha\, x)\frac{1}{\gamma_n}\mathcal{D}_q^2 P_n(x) + \frac{1}{\gamma_n}\mathcal{S}_q\mathcal{D}_q P_n(x) = \frac{1}{\gamma_{n+1}}\mathcal{D}_q^2 P_{n+1}(x) + \sum_{j=2}^{n} \frac{e_{n,j}}{\gamma_j}\mathcal{D}_q^2 P_j(x). \tag{31}$$

Substituting $U_1(x) = (\alpha^2 - 1)x$ into (18b), yields

$$\mathcal{D}_q^2 P_{n+1}(x) = [\left(2\alpha^2 - 1\right) x - a_n]\mathcal{D}_q^2 P_n(x) + 2\alpha \mathcal{S}_q \mathcal{D}_q\, P_n(x) - b_n \mathcal{D}_q^2 P_{n-1}(x). \tag{32}$$

Eliminating $\mathcal{S}_q \mathcal{D}_q\, P_n(x)$ in (31) by subtracting $\frac{1}{\lambda_n}$ times (32) from 2α times (31), we obtain

$$\begin{aligned} &\frac{x + a_n}{\gamma_n} \mathcal{D}_q^2\, P_n(x) + \frac{b_n}{\gamma_n} \mathcal{D}_q^2 P_{n-1}(x) \\ &\qquad = \left(\frac{2\alpha}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right) \mathcal{D}_q^2\, P_{n+1}(x) + \sum_{j=2}^{n} \frac{2\alpha\, e_{n,j}}{\gamma_j} \mathcal{D}_q^2 P_j(x). \end{aligned} \tag{33}$$

Since $\{\frac{\mathcal{D}_q^2\, P_n}{\gamma_n \gamma_{n-1}}\}$ is a family of monic orthogonal polynomials, there are a_n'' and $b_n'' > 0$ such that

$$x \frac{\mathcal{D}_q^2 P_n(x)}{\gamma_n} = \frac{\gamma_{n-1}}{\gamma_{n+1}\gamma_n} \mathcal{D}_q^2 P_{n+1}(x) + a_n'' \mathcal{D}_q^2 P_n(x) + b_n'' \mathcal{D}_q^2 P_{n-1}(x). \tag{34}$$

Substituting (34) into (33) and using the relation $\gamma_{n+1} - 2\alpha\gamma_n + \gamma_{n-1} = 0$, obtained by direct computation from (8), we obtain

$$\left(a_n'' + \frac{a_n}{\gamma_n}\right) \mathcal{D}_q^2 P_n(x) + \left(b_n'' + \frac{b_n}{\gamma_n}\right) \mathcal{D}_q^2 P_{n-1}(x) = \sum_{j=2}^{n} \frac{2\alpha e_{n,j}}{\gamma_j} \mathcal{D}_q^2 P_j(x).$$

Therefore, $e_{n,j} = 0$ for $j \in \{2, 3, ...n-2\}$ and (30) can be written as

$$\frac{x}{\gamma_n} \mathcal{D}_q\, P_n(x) = \frac{1}{\gamma_{n+1}} \mathcal{D}_q\, P_{n+1}(x) + \frac{e_{n,n}}{\gamma_n} \mathcal{D}_q\, P_n(x) + \frac{e_{n,n-1}}{\gamma_{n-1}} \mathcal{D}_q\, P_{n-1}(x) + e_{n,1}.$$

The result follows from Lemma 3.1. Finally, we prove that $(c) \Rightarrow (a)$.
Adding $\psi(x)$ times (17b) to $\phi(x)$ times (18b) and then using the assumption (c), we obtain

$$\begin{aligned} \lambda_{n+1} P_{n+1}(x) =& \lambda_n \left(\alpha^2 x + U_1(x) - a_n\right) P_n(x) - 2\alpha(\phi(x) \mathcal{S}_q \mathcal{D}_q\, P_n(x) \\ &+ U_2(x)\psi(x)\mathcal{D}_q^2 P_n(x)) - \psi(x)\, P_n(x) - b_n \lambda_{n-1} P_{n-1}(x). \end{aligned} \tag{35}$$

Multiplying (35) by $\psi(x)$ and substituting $\psi(x)\mathcal{S}_q \mathcal{D}_q\, P_n(x) = -\phi(x)\mathcal{D}_q^2 P_n(x) - \lambda_n P_n(x)$ obtained from (28) and $U_1(x) = \left(\alpha^2 - 1\right) x$, yields

$$\begin{aligned} 2\alpha \left(\phi^2(x) - U_2(x)\psi^2(x)\right) \mathcal{D}_q^2 P_n(x) &= \lambda_{n+1}\psi(x)\, P_{n+1}(x) + [\psi^2(x) - 2\alpha\lambda_n \phi(x) \\ &\quad - \lambda_n \psi(x) \left((\alpha^2 - 1)x - a_n\right)] P_n(x) + \lambda_{n-1} b_n \psi\, P_{n-1}(x). \end{aligned}$$

Taking $\phi\,(x) = \phi_2 x^2 + \phi_1 x + \phi_0$ and $\psi\,(x) = \psi_1 x + \psi_0$ and using the three-term recurrence relation (1), we transform the above equation into

$$\left(\phi^2(x) - U_2(x)\psi^2(x)\right) \mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x), \tag{36}$$

where $2\alpha a_{n,n-2} = \psi_1 b_{n-1} b_n \left(\psi_1 - \lambda_n(2\alpha\phi_2 + (\alpha^2 - 1) + \lambda_{n-1}\right)$. Clearly $a_{n,n-2} \neq 0$ for $b_n > 0$, since $\psi_1 \neq 0$ and ψ_1 also does not depend on n. This yields the required result. ☐

Corollary 3.3. *A sequence of monic orthogonal polynomials satisfies the relation*

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{k=-2}^{2} a_{n,n+k} P_{n+k}(x), \quad a_{n,n-2} \neq 0, \quad x = \cos\theta, \tag{37}$$

where π is a polynomial of degree at most 4, if and only if $P_n(x)$ is a multiple of the Askey-Wilson polynomial for some parameters a, b, c, d, including limiting cases as one or more of the parameters tend to ∞.

Proof. Let $\{P_n(x)\}_{n=0}^{\infty}$, $x = \cos\theta$, be a sequence of monic orthogonal polynomials and $\pi(x)$ be a polynomial of degree at most 4. It follows from Theorem 3.2 that $\{P_n(x)\}$ satisfies (37) if and only if $P_n(x)$ is polynomial solution of (28). It was proved in [15, Thm. 3.1] that (28) has a polynomial solution of degree n if and only if the solution is up to a multiplactive factor equal to an Askey-Wilson polynomial, a special case or a limiting case of an Askey-Wilson polynomial when one or more of the parameters tend to ∞ and these limiting cases are orthogonal [15, Remark 3.2], which yields the result. ☐

Remark 3.4. *It follows from (36) and Theorem 3.2 that $\{\mathcal{D}_q^2 P_n\}_{n=2}^{\infty}$ is orthogonal with respect to $\left(\phi^2(x) - U_2(x)\psi^2(x)\right) w(x)$. So, there is a positive constant c such that $\pi(x) = c\left(\phi^2(x) - U_2(x)\psi^2(x)\right)$. Without loss of generality, we can take $c = 1$ so that*

$$\pi(x) = \phi^2(x) - U_2(x)\psi^2(x). \tag{38}$$

In the following remark we provide the polynomial coefficients $\phi(x)$ and $\psi(x)$ of (28) as well as the polynomial $\pi(x)$ in (37) for the monic Askey-Wilson polynomials.

Remark 3.5. *Let $a_n := a_n(a,b,c,d)$ and $b_n := b_n(a,b,c,d)$ be the coefficients of (1) for the monic Askey-Wilson polynomials*

$$2^n(abcdq^{n-1};q)_n P_n(x;a,b,c,d|q) = p_n(x;a,b,c,d|q).$$

Since $\mathcal{D}_q P_n(x;a,b,c,d|q) = \gamma_n P_{n-1}(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q)$, the coefficients of (13) can be deduced from those of (1) as follows

$$a_n' = a_{n-1}(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}) \quad \text{and} \quad b_n' = b_{n-1}(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}). \tag{39}$$

It is shown in the proof of Lemma 3.1 that $\phi(x)$ and $\psi(x)$ in (28) *are obtained by letting $n = 5$ in the polynomial coefficients of (23). Hence, taking $n = 5$ in the expressions for $\phi_n(x)$ and $\psi_n(x)$ (cf. (23)) and using* (39) *together with the three-term recurrence relation for monic Askey-Wilson polynomials (cf. [19, 14.1.5]), we obtain, up to a multiplicative factor,*

$$\begin{aligned}\phi(x) =& 2(abcd+1)x^2 - (abc+abd+acd+bcd+a+b+c+d)x \\ &+ ab+ca+ad+bc+bd+cd-dcba-1; \end{aligned}\tag{40}$$

$$\psi(x) = \frac{(abcd-1)4\sqrt{q}x}{q-1} + \frac{(a+b+c+d-abc-abd-acd-bcd)2\sqrt{q}}{q-1}. \tag{41}$$

Substituting the expressions (40) and (41) for $\phi(x)$ and $\psi(x)$ into (38) and taking into account the fact that $U_2(x) = (\alpha^2-1)(x^2-1)$, we obtain after simplification,

$$\pi(x) = 16abcd(x - \tfrac{a^{-1}+a}{2})(x - \tfrac{b^{-1}+b}{2})(x - \tfrac{c^{-1}+c}{2})(x - \tfrac{d^{-1}+d}{2}).$$

Ismail [15, Remark 3.2] points out that solutions to (28) do not necessarily satisfy the orthogonality relation of Askey-Wilson polynomials using the example $\lim\limits_{d\to\infty} p_n(x;a,b,c,d)$ to show that the moment problem is indeterminate for $0<q<1$ and $\max\{ab,ac,ad\}<1$ while, for $q>1$ and $\min\{ab,ac,ad\}>1$, the moment problem is determinate and the polynomials are special Askey-Wilson polynomials. In the next proposition, we explicitly state the various limiting cases for Askey-Wilson polynomials.

Proposition 3.6. *Let $q>0$, $q\neq 1$. Then, for the Askey-Wilson polynomials $p_n(x;a,b,c,d|q)$, we have*

(i) $\lim\limits_{d\to\infty} \dfrac{p_n(x;a,b,c,d|q)}{(ad;q)_n} = (bc)^n q^{n(n-1)} p_n(x;a^{-1},b^{-1},c^{-1}|q^{-1})$, *where $p_n(x;a^{-1},b^{-1},c^{-1}|q^{-1})$ denotes continuous dual q-Hahn polynomials with the orthogonality relation for $q>1$ given by [19, (14.4.2)]).*

(ii) $\lim\limits_{c,d\to\infty} \dfrac{a^n p_n(x;a,b,c,d|q)}{(ac;q)_n(ad;q)_n} = (-b)^n q^{\frac{n(n-1)}{2}} Q_n(x;a^{-1},b^{-1}|q^{-1})$, *where Q_n denotes the Al-Salam-Chihara polynomials with the orthogonality relation for $q>1$ given by [19, (14.8.2)].*

(iii) $\lim\limits_{b,c,d\to\infty} \dfrac{a^n p_n(x;a,b,c,d|q)}{(ab;q)_n(ac;q)_n(ad;q)_n} = a^{-n} H_n(x;a^{-1}|q^{-1})$, *where H_n is the continuous big q-Hermite polynomials with the orthogonality relation for $q>1$ given by [19, (14.8.2)] .*

(iv) $\lim\limits_{a,b,c,d\to\infty} \dfrac{a^{2n} p_n(x;a,b,c,d|q)}{(ab;q)_n(ac;q)_n(ad;q)_n} = H_n(x|q^{-1})$, *where H_n denotes the continuous q-Hermite polynomials [19, (14.26.2)].*

Proof.

$$\begin{aligned}\lim_{d\to\infty} \frac{a^n p_n(x;a,b,c,d|q)}{(ab;q)_n(ac;q)_n(ad;q)_n} &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(bcq^n)^k}{(ab;q)_k(ac;q)_k(q;q)_k} \prod_{j=0}^{k-1}(1-2aq^jx+a^2q^{2j}) \\ &= \frac{(2\,abc)^n\, q^{n(n-1)}}{(ab;q)_n(ac;q)_n} q_n(x;a,b,c|q),\end{aligned}$$

where q_n is a monic polynomial satisfying the three-term recurrence relation

$$q_{n+1}(x;a,b,c|q) = (x-\widetilde{a}_n)q_n(x;a,b,c|q) - \widetilde{b}_n q_{n-1}(x;a,b,c|q), \tag{42}$$

where $\widetilde{a}_n = \frac{abq^n+acq^n+bcq^n+q^nq-q-1}{2ac(q^n)^2b}$ and $\widetilde{b}_n = \frac{(q^n-1)(bcq^n-q)(acq^n-q)(abq^n-q)}{2a^2c^2(q^n)^4b^2}$. From (42) and [19, (14.3.5)] , we obtain $2^nq_n(x;a,b,c|q) = p_n(x;a^{-1},b^{-1},c^{-1}|q^{-1})$ where $p_n(x;a^{-1},b^{-1},c^{-1}|q^{-1})$ denotes continuous dual q-Hahn polynomials [19, (14.3.1)]. Therefore $\lim_{d\to\infty}\frac{p_n(x;a,b,c,d|q)}{(ad;q)_n} = (bc)^nq^{n(n-1)}p_n(x;a^{-1},b^{-1},c^{-1}|q^{-1})$. The other limits are obtained in an analogous manner. □

In [20], Koornwinder obtained another structure relation for Askey -Wilson polynomials in the form $Lp_n = r_np_{n+1} + s_np_{n-1}$, where L is the divided q-difference linear operator defined by [20, (1.8)]. The connection of the structure relation [20, (4.7)] to (37) is provided in the following proposition.

Proposition 3.7. *Let $P_n(x) = P_n(x;a,b,c,d|q) = \frac{p_n(x;a,b,c,d|q)}{2^n(abcdq^{n-1};q)_n}$ denote the monic Askey-Wilson polynomials. Then, for the operator L defined by [20, (1.8)] we have that, for $x=\cos\theta$,*

$$\begin{aligned}\psi(x)(LP_n)(x) &= \frac{1-q^2}{2q}\pi(x)\mathcal{D}_q^2P_n(x) + \frac{q-1}{\sqrt{q}}\\ &\times[\psi(x)^2 + \frac{4\sqrt{q}(q^n-1)(q^{n-1}abcd-1)}{(q-1)^2q^{n-1}}(\frac{1}{\sqrt{q}}\phi(x) + \frac{(q-1)^2}{2q}x\psi(x))]P_n(x),\end{aligned}$$

where $\phi(x)$ and $\psi(x)$ are the polynomial coefficients of (28) given by (40) and (41).

Proof. It follows from [10, Thm 6] that the structure relation [20, (4.7)] can be written as $(LP_n)(x(s)) = \xi\left(2\phi(x(s))\mathcal{D}_q\mathcal{S}_q + 2\psi(x(s))\mathcal{S}_q^2 - \psi(x(s))\right)P_n(x(s))$, where $x(s) = \frac{q^{-s}+q^s}{2}$ ($q^s = e^{i\theta}$) and ξ is a constant. Take $n=1$, to obtain, after simplification, $2q\xi = 1-q^2$. Use (11) and (12) to write LP_n in terms of $\mathcal{D}_q^2$ and $\mathcal{S}_q\mathcal{D}_q$. Now, multiply the relation by ψ and use the fact that Askey-Wilson polynomials satisfy (28) with polynomial coefficients ϕ and ψ and the constant $\lambda_n = -4\frac{\sqrt{q}(q^n-1)(q^nabcd-q)}{(-1+q)^2q^n}$ given in [17, (16.3.19) and (16.3.20)], to obtain the result. □

In the following proposition we consider the conditions under which the nth degree polynomial $P_n(x)$ in a sequence of polynomials orthogonal with respect to a weight $w(x)$ can be written as a linear combination of the polynomials $\mathcal{D}_q^2P_{n+j}(x)$, $j,n\in\mathbb{N}$. A structure relation of this type involving the forward arithmetic mean operator $\frac{1}{2}(f(s+1)+f(s))$ is proved in [5].

Proposition 3.8. *Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials orthogonal with respect to a weight function $w(x)$ defined on (a,b). Suppose $\{\mathcal{D}_q^2P_j\}_{j=2}^{\infty}$ is a sequence of polynomials orthogonal with respect to the weight function $\pi(x)\,w(x)$ on (a,b) where $\pi(x)$ is a polynomial of degree at most 4. Then for each $n\in\mathbb{N}$, $n\geq 4$, there exist constants $b_{n,n+j}$, $j\in\{-2,\,-1,\,0,\,1,\,2\}$ such that*

$$P_n(x) = \sum_{j=-2}^{2} b_{n,n+j}\mathcal{D}_q^2P_{n+j}(x). \tag{43}$$

Proof. Let $n\in\mathbb{N}$, $n\geq 4$. Since $\{\mathcal{D}_q^2P_j\}_{j=2}^{\infty}$ is orthogonal with respect to a weight function $\pi(x)\,w(x)$ on (a,b), P_n can be expanded in terms of the orthogonal basis as $P_n(x) = \sum_{k=2}^{n+2} b_{n,k}\mathcal{D}_q^2P_k(x)$, where, for each fixed k, $k\in\{2,3,...,n+2\}$, $b_{n,k}$ is given by

$$b_{n,k}\int_a^b\left(\mathcal{D}_q^2P_k(x)\right)^2\pi(x)\,w(x)dx = \int_a^b\mathcal{D}_q^2P_k(x)P_n(x)\pi(x)w(x)dx.$$

Since $\pi(x)\,\mathcal{D}_q^2P_k(x)$ is a polynomial of degree at most $k+2$ and $\{P_j\}_{j=0}^{\infty}$ is orthogonal with respect to $w(x)$ on $(a,\,b)$, it follows that $b_{n,k}=0$, for $k\in\{2,..,n-3\}$. □

4 Extreme zeros of Askey-Wilson polynomials and special cases

In this section we obtain the explicit structure relation (37) characterizing Askey-Wilson polynomials and then use the relation to derive bounds for the extreme zeros of the Askey-Wilson polynomials and their special cases.

Lemma 4.1. *The monic Askey-Wilson polynomials $P_n(x;a,b,c,d|q)$ satisfy the following contiguous relations*

$$\begin{aligned}
\left(x-\tfrac{a^{-1}+a}{2}\right)P_n(x;aq,b,c,d|q) &= P_{n+1}(x;a,b,c,d|q)+k_n^{(a,b,c,d)}P_n(x;a,b,c,d|q),\\
\left(x-\tfrac{b^{-1}+b}{2}\right)P_n(x;a,bq,c,d|q) &= P_{n+1}(x;a,b,c,d|q)+k_n^{(b,a,c,d)}P_n(x;a,b,c,d|q),\\
\left(x-\tfrac{c^{-1}+c}{2}\right)P_n(x;a,b,cq,d|q) &= P_{n+1}(x;a,b,c,d|q)+k_n^{(c,b,a,d)}P_n(x;a,b,c,d|q),\\
\left(x-\tfrac{d^{-1}+d}{2}\right)P_n(x;a,b,c,dq|q) &= P_{n+1}(x;a,b,c,d|q)+k_n^{(d,b,c,a)}P_n(x;a,b,c,d|q),
\end{aligned}$$

with $k_n^{(a,b,c,d)}=-\dfrac{(1-abq^n)(1-acq^n)(1-adq^n)\left(1-abcdq^{n-1}\right)}{2a\,(1-abcdq^{2n-1})\,(1-abcdq^{2n})}.$

Proof. Substitute $P_n(x;a,b,c,d|q)$ into [2, (2.15)] to obtain the first relation. For the others, permute a and e, $e\in\{b,c,d\}$ in the first relation and use the fact that $P_n(x;a,b,c,d|q)$ is symmetric with respect to a,b,c,d, (cf. [2, p.6]), to obtain the result. ☐

Proposition 4.2. *The structure relation (37) for monic Askey-Wilson polynomials is*

$$\begin{aligned}
16abcd&\left(x-\tfrac{a^{-1}+a}{2}\right)\left(x-\tfrac{b^{-1}+b}{2}\right)\left(x-\tfrac{c^{-1}+c}{2}\right)\left(x-\tfrac{d^{-1}+d}{2}\right)D_q^2P_n(x;a,b,c,d|q)\\
&=\sum_{j=-2}^{2}a_{n,n+j}P_{n+j}(x;a,b,c,d|q),\ \text{where}
\end{aligned}\tag{44}$$

$$\begin{aligned}
a_{n,n+2}&=16abcd\gamma_n\gamma_{n-1},\\
a_{n,n+1}&=a_{n,n+2}\left(k_{n-2}^{(a,bq,cq,dq)}+k_{n-1}^{(b,a,cq,dq)}+k_n^{(c,b,a,dq)}+k_{n+1}^{(d,b,c,a)}\right),\\
a_{n,n}&=a_{n,n+2}\Big[k_{n-2}^{(a,bq,cq,dq)}k_{n-2}^{(b,a,cq,dq)}+k_{n-1}^{(c,b,a,dq)}\left(k_{n-2}^{(a,bq,cq,dq)}+k_{n-1}^{(b,a,cq,dq)}\right)\\
&\quad+k_n^{(d,b,c,a)}\left(k_{n-2}^{(a,bq,cq,dq)}+k_{n-1}^{(b,a,cq,dq)}+k_n^{(c,b,a,dq)}\right)\Big],\\
a_{n,n-1}&=a_{n,n+2}\Big[k_{n-2}^{(a,bq,cq,dq)}k_{n-2}^{(b,a,cq,dq)}k_{n-2}^{(c,b,a,dq)}+k_{n-1}^{(d,b,c,a)}k_{n-2}^{(a,bq,cq,dq)}k_{n-2}^{(b,a,cq,dq)}\\
&\quad+k_{n-1}^{(d,b,c,a)}k_{n-1}^{(c,b,a,dq)}\left(k_{n-2}^{(a,bq,cq,dq)}+k_{n-1}^{(b,a,cq,dq)}\right)\Big],\\
a_{n,n-2}&=a_{n,n+2}\left(k_{n-2}^{(a,bq,cq,dq)}k_{n-2}^{(b,a,cq,dq)}k_{n-2}^{(c,b,a,dq)}k_{n-2}^{(d,b,c,a)}\right),
\end{aligned}$$

and k_n is given in Lemma 4.1.

Proof. Using the fact that $\mathcal{D}_q^2P_n(x;a,b,c,d|q)=\gamma_n\gamma_{n-1}P_{n-2}(x;aq,bq,cq,dq|q)$ (cf. [19, (14.1.9)]) and taking into account the expression for the polynomial $\pi(x)$, given in Remark 3.5, (37) can be written as

$$\begin{aligned}
&\left(x-\tfrac{a^{-1}+a}{2}\right)\left(x-\tfrac{b^{-1}+b}{2}\right)\left(x-\tfrac{c^{-1}+c}{2}\right)\left(x-\tfrac{d^{-1}+d}{2}\right)P_{n-2}(x;aq,bq,cq,dq|q)\\
&=\sum_{j=-2}^{2}\frac{a_{n,n+j}}{16abcd\gamma_n\gamma_{n-1}}P_{n+j}(x;a,b,c,d|q).
\end{aligned}\tag{45}$$

Replace n by $n-2$, b by bq, c by cq and d by dq in the first equation of Lemma 4.1 to obtain

$$\begin{aligned}
&\left(x-\tfrac{a^{-1}+a}{2}\right)P_{n-2}(x;aq,bq,cq,dq|q)\\
&=P_{n-1}(x;a,bq,cq,dq|q)+k_{n-2}^{(a,bq,cq,dq)}P_{n-2}(x;a,bq,cq,dq|q).
\end{aligned}\tag{46}$$

Multiply (46) by $\left(x-\frac{b^{-1}+b}{2}\right)\left(x-\frac{c^{-1}+c}{2}\right)\left(x-\frac{d^{-1}+d}{2}\right)$ and use the other relations in Lemma 4.1 to transform (46) into (45) where the coefficients $a_{n,n+j}$, $j\in\{-2,...,2\}$ are written in terms of k_{n+j}. ☐

Theorem 4.3. *Let $x_{n,1}$ $(x_{n,n})$ be the smallest (largest) zero of the Askey-Wilson polynomial $P_n(x;a,b,c,d|q)$. Then*

$$x_{1,n} < \frac{2(\,q^{n-1}+1)\left(q^{n-1}\,(aA+C)-a-B\right)\left(aCq^{n-1}-1\right)-\sqrt{I_n}}{8\,(\,aCq^{2\,n-2}-1)\,(aCq^{n-1}-1)} \tag{47}$$

$$x_{n,n} > \frac{2(\,q^{n-1}+1)\left(q^{n-1}\,(aA+C)-a-B\right)\left(aCq^{n-1}-1\right)+\sqrt{I_n}}{8\,(\,aCq^{2\,n-2}-1)\,(aCq^{n-1}-1)} \tag{48}$$

where $A = bc+bd+cd, \quad B = b+c+d, \quad C = bcd$ and

$$\begin{aligned} I_n = {} & -16\left(aCq^{2\,n-2}-1\right)\left(aCq^{n-1}-1\right)\,\left[\left(-q^{3\,n-3}aC-1\right)(aC-aB-A+1)\right.\\ & +((C^2+b^2c^2+b^2d^2+c^2d^2+bcdB-A)a^2+A\,(C-B)\,a+C^2-CB)q^{2n-2}\\ & \left.+((1-A)a^2-(A-1)\,Ba-CB+b^2+A+c^2+d^2+1)q^{n-1}\right]\\ & +4\left(q^{n-1}+1\right)^2\left(q^{n-1}aA+q^{n-1}C-a-B\right)^2\left(aCq^{n-1}-1\right)^2. \end{aligned}$$

Proof. Use the three-term relation [19, (14.1.5)] to transform (44) into

$$\begin{aligned} & \left(x-\tfrac{a^{-1}+a}{2}\right)\left(x-\tfrac{b^{-1}+b}{2}\right)\left(x-\tfrac{c^{-1}+c}{2}\right)\left(x-\tfrac{d^{-1}+d}{2}\right)P_{n-2}(x;aq,bq,cq,dq|q)\\ & = \frac{(1-q)\left(dbca\,(q^n)^2-q\right)}{4a\sqrt{q}\,(-q+q^n)\,(-1+q^n)\,cdb}\psi(x)P_{n+1}(x;a,b,c,d|q)+G_{2,n}(x)P_n(x;a,b,c,d|q) \end{aligned}$$

where ψ is the polynomial coefficient of (28) given in (41) and

$$\begin{aligned} & \frac{4abcd(q^n;q^{-1})_2(abcdq^{2n}-1)}{abcdq^{n-1}-1}G_{2,n}(x) = 4(abcdq^{2n}-1)(abcdq^n-1)x^2\\ & -(2q^n+2)(q^n(abc+abd+acd+qbcd)-a-b-c-d)(abcdq^{n-1}-1)x\\ & -(q^{3n}abcd+1)(dbca-ab-ac-ad-bc-bd-cd+1)+((b^2c^2d^2+b^2c^2+b^2cd\\ & +b^2d^2+bc^2d+bcd^2+c^2d^2-bc-bd-cd)a^2+(bc+bd+cd)(dbc-b-c-d)a\\ & +bdc(dbc-b-c-d))q^{2n}+((1-bc-bd-cd)a^2-(bc+bd+cd-1)(d+c+b)a\\ & -b^2cd-bc^2d-bcd^2+b^2+bc+bd+c^2+cd+d^2+1)q^n. \end{aligned}$$

It follows from [6, Cor. 2.2] that the zeros of the second degree polynomial $G_{2,n-1}$ yield inner bounds for the extreme zeros of $P_n(x;a,b,c,d|q)$ and the result follows. ☐

Bounds for the zeros of Askey-Wilson polynomials obtained in Theorem 4.3 for some special values of the parameters n, a, b, c, d and q are illustrated in Table 1.

Table 1: Zeros of monic Askey-Wilson polynomials for $n = 7, 9, 12$ respectively and $(a,b,c,d,q) = (\frac{6}{7},\frac{5}{7},\frac{4}{7},\frac{3}{7},\frac{1}{9})$

Value of n	7	9	12
Smallest zeros of $P_n(x;a,b,c,d\|q)$	-0.864348856	-0.922505234	-0.95879261
Upper bound (47)	0.33690627	0.336904827	0.336904809
Lower bound (48)	0.948809497	0.948809477	0.948809477
Largest zeros of $P_n(x;a,b,c,d\|q)$	0.981913401	0.986122226	0.990012586

Special cases of Askey-Wilson polynomials arise when one or more of the parameters vanish and bounds for the extreme zeros of these special cases, namely continuous dual q-Hahn, Al-Salam Chihara, continuous big q-Hermite and continuous q-Hermite polynomials, can be deduced from the bounds in Theorem 4.3.

Acknowledgments

The authors thank Mourad Ismail and Tom Koornwinder for helpful discussions and comments and the anonymous referee for constructive comments that resulted in substantial improvements to the paper.

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