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FINANCIAL CONTAGION IN LARGE, INHOMOGENEOUS STOCHASTIC INTERBANK NETWORKS

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We consider the fraction of nodes that default in large, stochastic, inhomogeneous financial networks following an initial shock to the system. Results for deterministic sequences of networks are generalized to stochastic networks to account for interbank lending relationships that change frequently. A general class of inhomogeneous stochastic networks is proposed for use in systemic risk research, and we illustrate how results that hold for Erdős-Rényi networks can be generalized to the proposed network class. The network structure of a system is determined by interbank lending behavior which may vary according to the relative sizes of the banks. We then use the results of the paper to illustrate how network structure influences the systemic risk inherent in large banking systems.

Keywords: Systemic risk; banking networks; large networks; core-periphery structures; inhomogeneous networks.

1. Introduction

The research considers the fraction of a financial system that defaults following a shock to a small proportion of institutions. When dealing with financial systems, network theory can be a useful tool for explicitly modeling contractual relationships between financial entities [2, 35]. This allows a clear distinction between individual

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firms and the network [11]. The early works of [20, 12] support this, as it was shown that financial systems can naturally be represented as a network of interconnected financial institutions such as insurance companies, banks and shadow banking institutions. The focus of this research is on banks, although the concepts intuitively carry over to other types of financial institution.

The banks are represented by nodes in the network, and the exposures between them form the edges. As the direction of lending and exposure amounts are important, the resulting graphs are weighted and directed. Network models of systemic risk are typically balance sheet driven, where capital levels are used to monitor the financial health of institutions. Once an institution's capital is depleted, it is deemed unable to fulfill its loan obligations and its counterparties consequently suffer losses. This is a commonly used mechanism through which losses are assumed spread within the system.

It is important to note that this is not the only mechanism by which losses can be transmitted throughout the system. Contagion is usually amplified by liquidity losses through fire-sale haircuts, market-to-market losses and increased funding costs resulting from a deterioration of confidence within the financial system. Contagion within the system would necessarily be underestimated without including all channels of contagion [17], and may lead to low probabilities of default within the system [28]. For the purpose of this study, other contagion mechanisms are not included as the focus here is on demonstrating the application of a general class of networks that can be used to generate a wide range of different interbank network structures. While other contagion mechanisms should not be disregarded in practice and interbank exposures alone have not led to defaults in the past, this may be because government intervention has prevented such failures [42, 31]. The fear of direct contagion may cause losses through other channels of contagion and it is therefore important to understand the direct mechanism for losses between banks before exploring indirect losses [31]. However, it is also important to emphasize that losses due to direct exposures underestimate contagion and hence is important to bear in mind when interpreting the results.

The topology of banking networks is an important consideration when investigating how losses spread following initial bank failures in the system [25, 26, 33]. The authors in [32] and [22], among others, have found that hierarchical or core-peripheral structures are prevalent in real-world interbank networks [31, 29]. Additionally, [19] formally define a tiered network structure and show that the German banking system follow this structure. Hierarchical structures been considered by [36, 5, 6, 41, 30] for example. The core of the network consists of a few, highly interconnected banks and the peripheral part of smaller, less interconnected banks. However, the distinction between the two types of bank is not always clear in practice [29]. For this reason it may be beneficial to use more than one group of interacting banks, which is attainable in our setting.

Empirical investigations have further shown that many banking systems exhibit a power law distribution of degrees [12, 23, 18]. However, it is also of interest to look at structures that imply degree distributions other than a power law distribution. This is because, as [29] points out, there is not a definite consensus on whether real life systems necessarily exhibit this behavior. Instead it is agreed that degree distributions are highly skewed and that the distributions of in- and out-degrees are not the same. This study therefore proposes a broad class of stochastic networks that can naturally take account of an arbitrary number of bank groups as well as differences between their in- and out-degree distributions.

We further recognize that financial systems can become very large, consisting of hundreds or thousands of entities [18, 43], and therefore focus on analytical methods as opposed to simulation methods. The paper shows how the asymptotic results derived in [3] can be applied to the above mentioned class of networks that can be used to extend results pertaining to Erdős-Rényi graphs. The results concern the fraction of the total network that defaults after an initial shock to the system. These results are then used to compare three different network structures that form part of the family of networks that are formally defined in section 3.2.

For the purpose of this study the ‘structure’ of the network not only refers to the degree distribution of nodes, but the way in which the degree distribution is influenced by the relative asset values of banks. Previous studies considering the structural effect of networks on systemic risk interpret the structure/topology as the level of interconnectedness in the system, or the degree distribution as seen independently of the relative asset values between individual banks (see for example [25, 26, 37, 1]). While the way in which financial institutions are connected to one another plays an important role in the propagation of shocks [38], we argue that it must not be investigated in isolation, but in conjunction with lending preferences[a] and other network characteristics such as capital levels, average interconnectedness etc.

Empirical evidence of the role of lending preferences in network structure differ. For example, [34] find that asset size is not always clearly associated with lending preferences. On the other hand, [19] find that the German core-peripheral network’s highly connected core consists of money centre banks which act as intermediaries between other banks and are identified by their size, specialization and balance sheet ratios. Similarly, [15] find evidence that factors such as bank size, sector and type are indicative of a bank’s in- and out-degrees.

The class of stochastic networks defined in this study is based on the concept of multiple interacting networks [13]. The difference between this study and studies that make use of such networks (for example [8]) is that we do not consider different *types* of network connections (such as different types of loan), but rather consider banks being grouped together according to some criteria, such as their asset size. We therefore assume the existence of multiple groups of banks (that can each be

[a]The work in [33] takes account of lending preferences between banks, but in a different way than in this study.

seen as a network on its own) that interact with one another via interbank links. Banks within any one group are assumed to have similar characteristics and exhibit similar lending behavior towards one another and towards banks belonging to other groups. Since banks within a group exhibit homogeneous behavior and heterogeneous lending preferences between groups are allowed, graphs belonging to this class are called *semi-heterogeneous Erdős-Rényi graphs* for the purpose of this study.

The class of networks proposed by this study accounts for the empirical evidence noted above as follows:

- Banks can explicitly be grouped into core and non-core banks.
- In cases where the distinction between the core and the periphery banks is not clear, more than two groups of banks can be considered.
- An explicit distinction can be made between money centre banks (which are both lenders and borrowers) and banks that only act as either lenders or borrower.
- Banks can be grouped according to any relevant characteristics that are indicative of lending preferences. In the illustrative application considered in section 4, banks within any one group are assumed to be similar in size, although a combination of other characteristics can be used as well.

The class of graphs considered here bears similarities to [4], who consider groups of graphs. That study also extends the results of [3] to a more general setting. However, the randomness of that model originates from the probability of a contagious link existing between any two nodes, whereas the randomness considered in this study originates from the probability of one bank lending to another which leads to random degree sequences. This is motivated by empirical evidence from [21] who show that interbank relationships are formed randomly on a daily basis, based on the true underlying structure [40].

To this end, the research makes the following contributions:

(i) A class of networks are formally defined and we illustrate how this can be used to generalize results for Erdős-Rényi (i.e. homogeneous) networks to inhomogeneous networks. This creates the opportunity for existing research on Erdős-Rényi graphs to be applied to a much richer collection of networks. This is illustrated in section 3.2, where the results from section 3.1 (which apply to Erdős-Rényi networks) are shown to also hold for semi-heterogeneous Erdős-Rényi graphs with any finite number of interacting groups.

(ii) We build upon the results by [3] to show that it can be applicable to sequences of networks with random degree distributions and not only to deterministic degree distributions. This is of practical interest as the interbank connections between banks change continuously over time.

(iii) We illustrate how semi-heterogeneous Erdős-Rényi networks can be used to model and compare different types of core-peripheral financial networks. These networks can explicitly take account of the lending preferences of banks based on asset sizes or any other network characteristic.

The structure of this paper is as follows: Section 2 describes the interbank network model, defines the relevant notation and introduces the required concepts. The preliminary results on which the main theorem is based are shown to hold for Erdős-Rényi graphs in section 3.1. This is then extended to the semi-heterogeneous Erdős-Rényi case in section 3.2. Section 4 illustrates the main theorem of this paper and uses it to compare the systemic risk inherent in different network structures. Thereafter section 5 concludes the study.

2. Notation and Banking Network Model

2.1. *Network description*

Suppose there are n banks in the system, and that the links between the banks are represented by their interbank exposures. In other words, whenever a bank i has lent money to a bank j, there exists a directed edge from node i to node j. Note that the results presented here require the use of results from [3] and a number of notational conventions and definitions introduced in this section are based on that paper.

For this study we will first consider an Erdős-Rényi graph of size n, where the average number of connections is given by $\lambda \in (0, \infty)$ and the resulting connection probability is then $q_n = \frac{\lambda}{n-1}$. Note that the Erdős-Rényi graph definition adopted by this study does not assume a fixed number of edges in the network. For each n, the connection probability $q_n = \frac{\lambda}{n-1}$ is fixed but the number of edges in the systems is a random variable. Such an Erdős-Rényi graph will be denoted by $K_{\lambda,n}$. Accordingly we let $\kappa_{\lambda,n}$ denote a realization of the random network $K_{\lambda,n}$, chosen uniformly over all the possible networks that $K_{\lambda,n}$ can result in. The initial results of the study are applicable to standard Erdős-Rényi graphs. These are then extended to an arbitrary number of connected groups of Erdős-Rényi networks.

A node's out-degree is the number of outgoing edges originating from it (in other words the number of banks that it has lent money to). The out-degree of a node i in a network consisting of n nodes is the random variable given by $D_n^+(i)$. Similarly, the in-degree of a node i in a network of size n is the number of incoming edges connected to it and represents the number of banks that it has borrowed money from. It is denoted by the random variable $D_n^-(i)$. For notational convenience we will say that node i has degree (j, k) if $D_n^+(i) = j$ and $D_n^-(i) = k$, where $j, k \in \mathbb{N}_0$.

Now let h_n^+ and h_n^- be the probability mass functions of $D_n^+(i)$ and $D_n^-(i)$ respectively. Note that D_n^+ and D_n^- are independent of i, but may depend on n. This is because the edges of nodes in an Erdős-Rényi graph all have the same probability of being present, and this probability is a function of n. Now if h_n is the two-dimensional probability mass function whose marginals are h_n^+ and h_n^-, then $h_n(j, k)$ is the probability that a node i has degree (j, k) (i.e. that $D_n^+(i) = j$ and

$D_n^-(i) = k$.). For the random network $K_{\lambda,n}$ we will have that

$$h_n(j,k) = \binom{n-1}{j} q_n^j \left(1-q_n\right)^{n-1-j} \binom{n-1}{k} q_n^k \left(1-q_n\right)^{n-1-k}, \tag{1}$$

since each node has $n-1$ other nodes to which it can be connected via incoming and/or outgoing edges.

Since $q_n = \frac{\lambda}{n-1}$ for $\lambda \in (0,\infty)$ and $\lim_{n\to\infty} nq_n = \lambda$, the Poisson limit theorem implies that for each $j,k \in \mathbb{N}_0$,

$$h_n(j,k) \to e^{-\lambda}\frac{\lambda^j}{j!}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-2\lambda}\frac{\lambda^{j+k}}{j!k!} := h(j,k)\,. \tag{2}$$

Furthermore for each $j,k \in \mathbb{N}_0$, let $\tilde{\mu}(j,k)$ represent the proportion of nodes that have degree (j,k). This will be called the degree distribution, keeping in mind that for any fixed j,k, this proportion will be a random variable.

Each bank in the network has a simplified balance sheet structure which determines the bank's initial financial position (i.e. before any defaults have occurred). An illustration of this balance sheet is given in table 2.1. For the purpose of this study, only the capital and the interbank assets are of interest. It is assumed that a bank's capital is used to absorb losses and that a bank defaults whenever its capital is depleted. Due to the seniority of interbank liabilities banks will generally not default on their obligations to one another. However, if one bank defaults, funding requirements are imposed on them. This funding requirement is approximated by the exposure that a bank had to the failing bank.

Table 1. Illustration of a simplified bank balance sheet.

Assets	**Liabilities**
Interbank Assets	Capital/Net worth
	Interbank liabilities
Other assets	
	Other liabilities

For a network with given edges, let $e_n(i,j)$ be the amount that bank i has lent to bank j in a network of size n (i.e. bank i's exposure to j). Bank i's total interbank assets is therefore given by $\sum_{k=1}^{n} e_n(i,k)$, where $e_n(i,j) = 0$ when there does not exist a directed edge from node i to j.

All n banks' exposures can be represented by a matrix $\mathbf{e}_n$ where the $(i,j)^{\text{th}}$ entry is the exposure of i to j. This matrix contains all the information about the links between the banks, as well as the weights of those links. This is because the non-zero entries in $\mathbf{e}_n$ indicate the exposures between banks in the system, whereas the zero entries in the matrix indicate which banks are not exposed each other. The system can therefore be represented by a weighted directed graph with n vertices, whose edges are given by $\mathbf{e}_n$. This matrix indicates the presence, direction and weight of all edges in the network. Note that for any exposure matrix, the diagonal entries $e_n(i,i)$ must be zero, whereas the rest of the entries must be non-negative.

Let $\gamma_n(i)$ be bank i's ratio of capital to interbank assets, so that bank i's total capital is given by $\gamma_n(i) \sum_{j=1}^{n} e_n(i,j)$. The vector containing the capital ratios of the banks is given by $\vec{\gamma}_n$. Even though capital ratios normally refer to the ratio of capital to total assets, it refers here to the ratio of capital to interbank assets. This avoids the need to specify the amount of external assets of banks. A network of banks with known edges is then characterized by its exposure matrix $\mathbf{e}_n$ and its vector of capital ratios $\vec{\gamma}_n$. In other words, two networks are considered to be identical if and only if they have the same exposure matrix and capital ratio vector. We can therefore define a financial network to be the pair $(\mathbf{e}_n, \vec{\gamma}_n)$ [3].

Once the network $\kappa_{\lambda,n}$ has been determined, the out- and in-degrees of the nodes are known and are no longer treated as random variables. In this case we denote the out-degree of a node i by $d_n^+(i)$ and the in-degree of node i by $d_n^-(i)$. We define $\vec{d}_n^+ = (d_n^+(1), d_n^+(2), \dots, d_n^+(n))$ and $\vec{d}_n^- = (d_n^-(1), d_n^-(2), \dots, d_n^-(n))$ to be the vectors containing the out-degrees and in-degrees of the nodes in the system respectively.

Note that a financial network's exposure matrix $\mathbf{e}_n$ can only be known once the edges in the network are known and the exposure amounts decided on. The fixed network $\kappa_{\lambda,n}$ will determine the edges, and thereafter exposure amounts can be assigned to each existing edge. Restrictions regarding determining the exposure amounts will be dealt with in section 3.

Definition 1 below is essentially from [3], and emphasizes that the degree vectors are obtained from $\mathbf{e}_n$ and remain fixed for a random financial network.

Definition 1. (Random financial network) *Suppose* $(\mathbf{e}_n, \vec{\gamma}_n)$ *is a financial network of size* n *with fixed degree vectors* $\vec{d}_n^+$ *and* $\vec{d}_n^-$ *as determined by* $\mathbf{e}_n$. *The set of exposures for a fixed bank* i *in this financial network is given by* $\{e_n(i,j) \mid e_n(i,j) > 0, j \in \{1, \dots, n\}\}$.

Let $\mathcal{G}(\mathbf{e}_n)$ *be the set of all possible exposure matrices of size* n *that*

(i) have the same associated degree vectors $\vec{d}_n^+$ *and* $\vec{d}_n^-$ *as* $\mathbf{e}_n$ *and*
(ii) for each node i, *have the same set of exposure amounts as* $\mathbf{e}_n$ *(each of which need not be assigned to the same counterparty as in the original financial network, as long as the node* i *which lent the money remains the same as in the original financial network).*

Define $\mathbf{E}_n \colon \Omega(\mathbf{e}_n, \vec{\gamma}_n) \to \mathcal{G}(\mathbf{e}_n)$ *to be a random exposure matrix, uniformly distributed on* $\mathcal{G}(\mathbf{e}_n)$. *The nodes of* $\mathbf{E}_n$ *are endowed with the capital ratios* $\vec{\gamma}_n$, *and the resulting financial network* $(\mathbf{E}_n, \vec{\gamma}_n)$ *is called a* random financial network.

Every node in a random financial network retains its original number of debtors and creditors, as well as the monetary amounts of the interbank loans that it granted. The element of randomness in a network should reflect the daily change in interbank relationship because of the high frequency of this change. The definition of a random financial network does not completely capture this daily change in interbank relationships since degree vectors and the exposure amounts are assumed to remain the same.

It is for this reason that we do not assume a fixed exposure matrix $\mathbf{e}_n$ to begin with, but rather let $\mathbf{e}_n$ be determined via a random Erdős-Rényi graph $K_{\lambda,n}$. A second reason for using a random graph is that the results for the default Erdős-Rényi case can be extended to the case where there are multiple Erdős-Rényi networks that interact with one another. This allows us to apply the results to a very flexible network and thus examine a range of different network structures in section 4 based on bank lending behavior.

2.2. *Shock propagation and the final fraction of defaults*

Let the fixed network $\kappa_{\lambda,n}$, the corresponding exposure matrix $\mathbf{e}_n$ and the capital ratio vector $\vec{\gamma}_n$ be given. The set $\mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n)$ is defined to be the set of initial defaults in the financial system $(\mathbf{e}_n, \vec{\gamma}_n)$. As mentioned in section 2.1, an institution defaults whenever its capital is depleted. In order to assess the effect of a shock to the financial system, there needs to be one or more initial defaults. Banks whose capital ratios are zero therefore constitute the set of initial defaults so that

$$\mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n) = \{i \in \{1, \ldots, n\} \mid \vec{\gamma}(i) = 0\}\,. \tag{3}$$

These are then the institutions that may cause a default cascade in the system. Once the initial defaults have occurred, losses are spread through the defaulted nodes' incoming edges. This is because the direction of edges imply the direction of lending in the system. A bank $i \in \mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n)$ will cause a loss of $(1 - R(i))\, e_n(j, i)$ to each of lenders j, where $R(i)$ is the recovery rate associated with the defaulted node i.

The losses caused by the nodes in $\mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n)$ might lead to additional defaults in the system. The set of nodes that have defaulted up to this point is given by

$$\mathbb{D}_1(\mathbf{e}_n, \vec{\gamma}_n) = \left\{ i \in \{1, \ldots, n\} \mid \gamma_n(i) \sum_{j=1}^{n} e_n(i,j) \le \sum_{j \in \mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n)} (1 - R(j))\, e_n(i,j) \right\}. \tag{4}$$

These losses form the first 'round' of defaults, as these are the first nodes to have defaulted as a result of the initial set of defaults. For subsequent rounds of default we have that

$$\mathbb{D}_k(\mathbf{e}_n, \vec{\gamma}_n) = \left\{ i \in \{1, \ldots, n\} \mid \gamma_n(i) \sum_{j=1}^{n} e_n(i,j) \le \sum_{j \in \mathbb{D}_{k-1}(\mathbf{e}_n, \vec{\gamma}_n)} (1 - R(j))\, e_n(i,j) \right\} \tag{5}$$

for $k \ge 1$.

The sequence $\mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n) \subseteq \mathbb{D}_1(\mathbf{e}_n, \vec{\gamma}_n) \subseteq \cdots \subseteq \mathbb{D}_{n-1}(\mathbf{e}_n, \vec{\gamma}_n)$ is nested. Here $\mathbb{D}_{n-1}(\mathbf{e}_n, \vec{\gamma}_n) \subseteq \{1, \ldots, n\}$, since the set $\mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n)$ must be nonempty and hence there can be at most $n-1$ rounds of default. If the default cascade stops when there have only been $k_0 < n-1$ rounds of default, then $\mathbb{D}_k(\mathbf{e}_n, \vec{\gamma}_n) = \mathbb{D}_{k+1}(\mathbf{e}_n, \vec{\gamma}_n)$ for all $k = k_0, \ldots, n-2$.

From the defaults $\mathbb{D}_{n-1}(\mathbf{e}_n, \vec{\gamma}_n)$ caused by nodes in the set $\mathbb{D}_0(\mathbf{e}_n, \vec{\gamma}_n)$, we can find the final/total fraction $\alpha_n(\mathbf{e}_n, \vec{\gamma}_n)$ of defaults in a financial network $(\mathbf{e}_n, \vec{\gamma}_n)$ given by

$$\alpha_n(\mathbf{e}_n, \vec{\gamma}_n) := \frac{|\mathbb{D}_{n-1}(\mathbf{e}_n, \vec{\gamma}_n)|}{n}. \tag{6}$$

3. Results for Stochastic Networks

In this section we consider assumptions that are required for the results in [3]. In that work, the assumptions apply to a sequence of networks with deterministic numbers of degrees, while the goal of this section is to show that similar assumptions hold for certain sequences of networks with random numbers of degrees. Section 3.1 deals with the Erdős-Rényi case, while section 3.2 generalizes the results of section 3.1 by showing similar results for a more general, inhomogeneous class of networks

3.1. *The Erdős-Rényi case*

Suppose $\lambda \in (0, \infty)$ is fixed and consider the Erdős-Rényi graph $K_{\lambda,n}$. The following proposition follows from the fact that the in- and out-degrees of any two nodes are asymptotically uncorrelated:

Proposition 1. *Let $K_{\lambda,n}$ be an Erdős-Rényi network with n nodes, each with average degree $\lambda \in (0, \infty)$, and fix the integers $j, k \in \mathbb{N}_0$. If $\tilde{\mu}_n(j,k)$ is the sample proportion of nodes with degree (j,k) after a realization of the network and $h(j,k) = e^{-2\lambda} \frac{\lambda^{j+k}}{j!k!}$, then for any $\epsilon > 0$*

$$P\left(|\tilde{\mu}_n(j,k) - h(j,k)| > \epsilon\right) \underset{\infty}{\overset{n}{\to}} 0. \tag{7}$$

Proof. Let $\epsilon > 0$ be given and fix the integers $j, k \in \mathbb{N}_0$. Further let $h_n(j,k)$ be defined as in Eq. (1). From (2) we know that $|h_n(j,k) - h(j,k)| \underset{\infty}{\overset{n}{\to}} 0$. Hence we can choose $N \in \mathbb{N}$ large enough so that $|h_n(j,k) - h(j,k)| < \frac{\epsilon}{2}$ for all $n \geq N$. By using the fact that

$$|\tilde{\mu}_n(j,k) - h(j,k)| \leq |\tilde{\mu}_n(j,k) - h_n(j,k)| + |h_n(j,k) - h(j,k)|, \tag{8}$$

we have that for all $n \geq N$

$$\begin{aligned} &P(|\tilde{\mu}_n(j,k) - h(j,k)| > \epsilon) \\ \leq &P\Big(|\tilde{\mu}_n(j,k) - h_n(j,k)| + \frac{\epsilon}{2} > \epsilon\Big) \\ = &P\Big(|\tilde{\mu}_n(j,k) - h_n(j,k)| > \frac{\epsilon}{2}\Big), \end{aligned} \tag{9}$$

since $|h_n(j,k) - h(j,k)| \underset{\infty}{\overset{n}{\to}} 0$.

We know that $\tilde{\mu}_n(j,k)$ is the sample proportion of nodes with degree (j,k), where there are n nodes in total. Therefore it can be seen as the average of n Bernoulli trial outcomes. The probability of success is equal to $h_n(j,k)$ since this is the probability that a node will be of degree (j,k) when there are n nodes in the system. It then also follows that $E[\tilde{\mu}_n(j,k)] = h_n(j,k)$.

For fixed $j,k \in \mathbb{N}_0$, define the random variables $X_i^{(n,j,k)}$, $i = 1,2,\ldots,n$ to indicate whether a node i is of type (j,k) or not. Then $X_i^{(n,j,k)} \sim \text{Bern}(h_n(j,k))$ for each $n \in \mathbb{N}$ and each $i \in \{1,2,\ldots,n\}$.

The $X_i^{(n,j,k)}$'s are not independent of each other, since the degrees of one node affect the degrees of the nodes connected to it. They are however, asymptotically mutually uncorrelated. This is because if we let $b(n,q,j) = \binom{n}{j} q^j (1-q)^{n-j}$ denote the binomial probability mass function, then for $i \neq l$ we have that

$$\begin{aligned} &E\left[X_i^{(n,j,k)} X_l^{(n,j,k)}\right] \\ =&q_n^2 \left[b(n-2,q_n,j-1)\, b(n-2,q_n,k-1)\right]^2 \\ &+ 2q_n(1-q_n)\, b(n-2,q_n,j-1)\, b(n-2,q_n,k)\, b(n-2,q_n,j)\, b(n-2,q_n,k-1) \\ &+ (1-q_n)^2 \left[b(n-2,q_n,j)\, b(n-2,q_n,k)\right]^2 \\ \underset{\infty}{\overset{n}{\rightarrow}}& \left[h(j,k)\right]^2, \end{aligned} \tag{10}$$

and therefore $cov\left(X_i^{(n,j,k)}, X_l^{(n,j,k)}\right) \underset{\infty}{\overset{n}{\rightarrow}} 0$. Now since

$$var[\tilde{\mu}_n(j,k)] = \frac{1}{n}\left(h_n(j,k) - (h_n(j,k))^2 + (n-1)\, cov\left(X_i^{(n,j,k)}, X_l^{(n,j,k)}\right)\right), \tag{11}$$

Chebyshev's inequality implies that

$$\begin{aligned} &P\Big(|\tilde{\mu}_n(j,k) - h_n(j,k)| > \frac{\epsilon}{2}\Big) \\ \leq&\frac{4}{\epsilon^2 n}\left(h_n(j,k) - (h_n(j,k))^2\right) + \frac{(n-1)}{n}\frac{4\, cov\left(X_i^{(n,j,k)}, X_l^{(n,j,k)}\right)}{\epsilon^2} \\ \underset{\infty}{\overset{n}{\rightarrow}}&0. \end{aligned} \tag{12}$$

Since $\epsilon > 0$ was arbitrary, this concludes the proof. □

This serves as a building block for similar results for a variant of the Erdős-Rényi graph where there are groups of connected Erdős-Rényi graphs. This will allow us to apply the results considered in this paper to a very versatile type of network. This can be used to compare the risk implied by different types of network structure. The type of network that this study is focused on is discussed below.

3.2. *Semi-heterogeneous Erdős-Rényi graphs*

Suppose that there are d groups of Erdős-Rényi networks that all interact with one another to form a new network of size n. Each group α comprises n_α nodes, so that $\sum_{\alpha=1}^d n_\alpha = n$. The probabilities of edges existing between any two nodes are predetermined based on the groups to which the nodes belong. Suppose a node i is in group α and a node j is in group β. The probability that the edge from i to j exists is denoted by $q^{(n)}_{\alpha\beta}$. The n in the superscript is included to indicate that the value of this probability will be dependent on the size of the network.

For each group α, let $w_\alpha = \frac{n_\alpha}{n}$ and assume that these remain constant when $n \to \infty$. Let $\lambda_{\alpha\beta}$ be the expected number of edges from any node in group α to nodes in group β. If $\alpha = \beta$ then a node in group α can connect to $n_\alpha - 1$ other nodes in the same group, this means that $q^{(n)}_{\alpha\alpha} = \frac{\lambda_{\alpha\alpha}}{n_\alpha - 1}$. If $\alpha \neq \beta$ then a node in group α can connect to n_β nodes in group β and therefore $q^{(n)}_{\alpha\beta} = \frac{\lambda_{\alpha\beta}}{n_\beta}$.

For the purpose of this research we will call this a semi-heterogeneous Erdős-Rényi graph. To formalize this, we have the following definitions:

Definition 2. (Average connection matrix) *For a group of $d \geq 1$ Erdős-Rényi graphs $K_{\lambda_1,n_1}, K_{\lambda_2,n_2}, \ldots, K_{\lambda_d,n_d}$ where there exists edges linking nodes between different graphs, the average connection matrix λ is defined to be the $d\times d$ matrix whose elements $\lambda_{\alpha\beta}$, $\alpha, \beta = 1, 2, \ldots, d$ represent the expected number of edges from any node in graph α to nodes in graph β. The diagonal entries are given by $\lambda_{\alpha\alpha} = \lambda_\alpha$, $\alpha = 1, 2, \ldots, d$. An average connection matrix is said to be positive if $\lambda_{\alpha\beta} > 0$ for all α, β.*

Definition 3. (Semi-heterogeneous Erdős-Rényi graph) *Let $d \in \mathbb{N}$ and consider the set $\{K_{\lambda_{11},n_1}, \ldots, K_{\lambda_{dd},n_d}\}$ of d Erdős-Rényi graphs with positive average connection matrix λ. If $\vec{n} = (n_1, \ldots, n_d)$ and the nodes from these graphs may be connected to one another such that $\lambda_{\alpha\beta}$ is the expected number of edges from any node in group α to nodes in group β, then we call the resulting graph a semi-heterogeneous Erdős-Rényi graph. This graph will be denoted by $K^d_{\lambda,\vec{n}}$.*

Recall that $D^+_n(i)$ and $D^-_n(i)$ are the random variables representing the number of out- and in-degrees of a randomly chosen node i in the network. Now let $D^{+,\alpha}_n(i)$ and $D^{-,\alpha}_n(i)$ be the random variables representing the number of out- and in-degrees of any node i in group α. Similar to h^+_n and h^-_n which denote the respective probability mass functions of $D^+_n(i)$ and $D^-_n(i)$, we let $h^{+,\alpha}_n$ and $h^{-,\alpha}_n$ denote the probability mass functions of $D^{+,\alpha}_n(i)$ and $D^{-,\alpha}_n(i)$ respectively.

Now let h^α_n be the joint probability mass function of $h^{+,\alpha}_n$ and $h^{-,\alpha}_n$. Then $h^\alpha_n(j,k)$ is the probability that a randomly chosen node in group α (where the total network size is n) has j and k outgoing and incoming edges connected to it respectively. Then $h_n(j,k)$, the probability that any node i has degree (j,k), is given by

$$\begin{aligned} h_n(j,k) &= \sum_{\alpha=1}^{d} \frac{n_\alpha}{n} h^\alpha_n(j,k) \\ &= \sum_{\alpha=1}^{d} \frac{n_\alpha}{n} h^{+,\alpha}_n(j)\, h^{-,\alpha}_n(k)\,. \qquad (13) \end{aligned}$$

First consider a semi-heterogeneous Erdős-Rényi graph $K^d_{\lambda,\vec{n}}$ where $d = 2$. For a preliminary discussion we will consider the expressions for $h_n(j,k)$ and its limit. When $d = 2$ we have four connection probabilities, namely $q_{11}^{(n)}$, $q_{12}^{(n)}$, $q_{21}^{(n)}$ and $q_{22}^{(n)}$. For notational convenience, let the binomial probability mass function be denoted by

$$b(n,p,l) = \binom{n}{l} p^l (1-p)^{n-l} . \tag{14}$$

Then for $d = 2$ groups, we have that

$$\begin{aligned} h_n(j,k) =& \frac{n_1}{n} h_n^{+,1}(j)\, h_n^{-,1}(k) + \frac{n_2}{n} h_n^{+,2}(j)\, h_n^{-,2}(k) \\ =& \frac{n_1}{n} \left(\sum_{l=0}^{j} b\Big(n_1 - 1, q_{11}^{(n)}, l\Big)\, b\Big(n_2, q_{12}^{(n)}, j-1\Big) \right) \\ & \cdot \left(\sum_{l=0}^{k} b\Big(n_1 - 1, q_{11}^{(n)}, l\Big)\, b\Big(n_2, q_{21}^{(n)}, k-l\Big) \right) \\ +& \frac{n_2}{n} \left(\sum_{l=0}^{j} b\Big(n_2 - 1, q_{22}^{(n)}, l\Big)\, b\Big(n_1, q_{21}^{(n)}, j-l\Big) \right) \\ & \cdot \left(\sum_{l=0}^{k} b\Big(n_2 - 1, q_{22}^{(n)}, l\Big)\, b\Big(n_1, q_{12}^{(n)}, k-l\Big) \right) . \end{aligned} \tag{15}$$

For the expressions to make sense it is assumed, without loss of generality, that $j,k < n_1$ and $j,k < n_2$.

Note that since w_1 and w_2 remain constant as $n \to \infty$, then n_1 and n_2 tend to infinity at the same rate as n. This means that all of the $q_{\alpha\beta}^{(n)}$ probabilities converge to 0 as $n \to \infty$ and hence all of the binomial factors above converge to Poisson probability mass functions. Therefore the limit of $h_n(j,k)$ is given by

$$\begin{aligned} & \lim_{n\to\infty} h_n(j,k) \\ =& h(j,k) \\ =& w_1 \left(e^{-\lambda_{11}-\lambda_{12}} \sum_{l_1=0}^{j} \frac{\lambda_{11}^{l_1} \lambda_{12}^{j-l_1}}{l_1!\,(j-l_1)!} \right) \left(e^{-\lambda_{11}-\frac{w_2}{w_1}\lambda_{21}} \sum_{l_2=0}^{k} \frac{\lambda_{11}^{l_2} \left(\frac{w_2}{w_1}\lambda_{21}\right)^{k-l_2}}{l_2!\,(k-l_2)!} \right) \\ & + w_2 \left(e^{-\lambda_{22}-\lambda_{21}} \sum_{l_3=0}^{j} \frac{\lambda_{22}^{l_3} \lambda_{21}^{j-l_3}}{l_3!\,(j-l_3)!} \right) \left(e^{-\lambda_{22}-\frac{w_1}{w_2}\lambda_{12}} \sum_{l_4=0}^{k} \frac{\lambda_{22}^{l_4} \left(\frac{w_1}{w_2}\lambda_{12}\right)^{k-l_4}}{l_4!\,(k-l_4)!} \right) \\ =& w_1 e^{-2\lambda_{11}-\lambda_{12}-\frac{w_2}{w_1}\lambda_{21}} \left(\sum_{l_1=0}^{j} \frac{\lambda_{11}^{l_1} \lambda_{12}^{j-l_1}}{l_1!\,(j-l_1)!} \right) \left(\sum_{l_2=0}^{k} \frac{\lambda_{11}^{l_2} \left(\frac{w_2}{w_1}\lambda_{21}\right)^{k-l_2}}{l_2!\,(k-l_2)!} \right) \end{aligned}$$

$$+ w_2 e^{-2\lambda_{22}-\lambda_{21}-\frac{w_1}{w_2}\lambda_{12}} \left(\sum_{l_3=0}^{j} \frac{\lambda_{22}^{l_3}\lambda_{21}^{j-l_3}}{l_3!\,(j-l_3)!} \right) \left(\sum_{l_4=0}^{k} \frac{\lambda_{22}^{l_4}\left(\frac{w_1}{w_2}\lambda_{12}\right)^{k-l_4}}{l_4!\,(k-l_4)!} \right). \quad (16)$$

This probability mass function has a finite mean where the average number of outgoing edges connected to a randomly chosen node in the system is given by $w_1(\lambda_{11}+\lambda_{12}) + w_2(\lambda_{21}+\lambda_{22})$.

We can now consider the expression for $h_n(j,k)$ when $d > 2$. In this case we have d^2 connection probabilities $q_{\alpha\beta}^{(n)}$, where $\alpha, \beta = 1, 2, \ldots, d$. Recall that for d groups, $h_n(j,k)$ can be expressed as follows:

$$h_n(j,k) = \sum_{\alpha=1}^{d} w_\alpha h_n^{+,\alpha}(j)\, h_n^{-,\alpha}(k)\,. \quad (17)$$

In order to find general expressions for $h_n^{+,\alpha}(j)$ and $h_n^{-,\alpha}(k)$, we look at the different ways in which a node in group α can have j out-degrees and k in-degrees. Without loss of generality we assume that $n_\beta > j,k$ for $\beta = 1, 2, \ldots, d$, since for $\beta = 1, 2, \ldots, d$ we have that $n_\beta \to \infty$ at the same rate as n. Then we have that for $d > 2$ and $\alpha < d$

$$\begin{aligned} &h_n^{+,\alpha}(j) \\ &= \sum_{m_1=0}^{j} \sum_{m_2=0}^{j-m_1} \cdots \sum_{m_\alpha=0}^{j-m_1-\cdots-m_{\alpha-1}} \cdots \sum_{m_{d-1}=0}^{j-m_1-\cdots-m_{d-1}} b\Big(n_1, q_{\alpha 1}^{(n)}, m_1\Big)\, b\Big(n_2, q_{\alpha 2}^{(n)}, m_2\Big) \\ &\quad \cdots b\Big(n_{\alpha-1}, q_{\alpha,\alpha-1}^{(n)}, m_{\alpha-1}\Big)\, b\Big(n_\alpha - 1, q_{\alpha\alpha}^{(n)}, m_\alpha\Big)\, b\Big(n_{\alpha+1}, q_{\alpha,\alpha+1}^{(n)}, m_{\alpha+1}\Big) \\ &\quad \cdots b\Big(n_{d-1}, q_{\alpha,d-1}^{(n)}, m_{d-1}\Big)\, b\Big(n_d, q_{\alpha d}^{(n)}, j - m_1 - \cdots - m_{d-1}\Big), \end{aligned} \quad (18)$$

with a similar expression for $h_n^{-,\alpha}(k)$ and for the case $\alpha = d$.

Each of the factors above is a binomial probability mass function and will converge to a Poisson probability mass function as $n \to \infty$. Therefore $h_n^{+,\alpha}(j)$ and $h_n^{-,\alpha}(k)$ will converge for all groups α. Hence there exists an h such that $h_n(j,k) \xrightarrow[\infty]{n} h(j,k)$ for all $j,k \in \mathbb{N}$.

Similar to Proposition 1 we have the following result:

Proposition 2. *Let $K_{\lambda,\vec{n}}^{d}$ be a semi-heterogeneous Erdős-Rényi graph. Let $n = \|\vec{n}\|_1$ and for $\alpha, \beta = 1, 2, \ldots, d$, let $q_{\alpha\beta}^{(n)}$, w_α, h_n^α, h^α and h be defined as before. If $\tilde{\mu}(j,k)$ is the sample proportion of nodes with degree (j,k), then for every $j,k \in \mathbb{N}_0$ and any $\epsilon > 0$*

$$P\left(|\tilde{\mu}_n(j,k) - h(j,k)| < \epsilon\right) \xrightarrow[\infty]{n} 0 \quad . \quad (19)$$

Proof. Let j, k and d be given. Recall that $h_n^\alpha(j,k)$ is the probability that a node in group α is of type (j,k), where the system is of size n. Analogous to this we let $h_n^\alpha(j,k,\beta)$ be the probability that a node in group α is of type (j,k), where one node in group β is disregarded, and the system is treated as if it has $n-1$ nodes. For any given $j,k \in \mathbb{N}_0$, let $X_i^{(n,j,k)}$, $i=1,2,\ldots,n$ be the indicator random variable which is equal to one when node i is of type (j,k). Then for any two nodes $i \neq l$ we have that

$$\begin{aligned}
&E\left[X_i^{(n,j,k)} X_l^{(n,j,k)}\right] \\
&= \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} w_\alpha w_\beta \\
&\quad \cdot P\Big(X_i^{(n,j,k)} = 1, X_l^{(n,j,k)} = 1 \mid \{\text{node } i \text{ is in group } \alpha\} \cap \{\text{node } l \text{ is in group } \beta\}\Big) \\
&= \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} w_\alpha w_\beta \Big[q_{\alpha\beta}^{(n)} q_{\beta\alpha}^{(n)} h_n^\alpha(j-1,k-1,\beta)\, h_n^\beta(j-1,k-1,\alpha) \\
&\quad + q_{\alpha\beta}^{(n)} \left(1-q_{\beta\alpha}^{(n)}\right) h_n^\alpha(j-1,k,\beta)\, h_n^\beta(j,k-1,\alpha) \\
&\quad + \left(1-q_{\alpha\beta}^{(n)}\right) q_{\beta\alpha}^{(n)} h_n^\alpha(j,k-1,\beta)\, h_n^\beta(j-1,k,\alpha) \\
&\quad + \left(1-q_{\alpha\beta}^{(n)}\right)\left(1-q_{\beta\alpha}^{(n)}\right) h_n^\alpha(j,k,\beta)\, h_n^\beta(j,k,\alpha)\Big]. \qquad (20)
\end{aligned}$$

If $n \to \infty$ then for all $\alpha,\beta = 1,2,\ldots,d$ we have that $q_{\alpha\beta}^{(n)} \to 0$ and that w_α remains constant. Therefore

$$\begin{aligned}
E\left[X_i^{(n,j,k)} X_l^{(n,j,k)}\right] &\to \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} w_\alpha w_\beta h^\alpha(j,k)\, h^\beta(j,k) \\
&= [h(j,k)]^2 \\
&= E\left[X_i^{(n,j,k)}\right] E\left[X_l^{(n,j,k)}\right], \qquad (21)
\end{aligned}$$

and hence $cov\left(X_i^{(n,j,k)}, X_l^{(n,j,k)}\right) \to 0$ for $i \neq l$ when $n \to \infty$.

In order to show that $\tilde{\mu}_n(j,k) \overset{n}{\underset{\infty}{\to}} h(j,k)$ for all $j,k \in \mathbb{N}$, the same steps as for Proposition 1 can be followed. □

Assume now that after the links for any graph $\kappa_{\lambda,\vec{n}_i}^d$ have been determined, the exposure amounts of any node in group α are i.i.d. random variables with distribution function F_α and that the non-zero exposure amounts of any two nodes are independent. Suppose that the fraction of initial defaults is π_0 (chosen uniformly over all the nodes), that these nodes have capital ratios equal to zero and that all other nodes have capital ratios equal to $c>0$.

Let $p(j,k,\theta)$ be the expected fraction of nodes of degree (j,k) that default after θ of its counterparties have defaulted. Similarly for $\alpha = 1,2,\ldots,d$, let $p^\alpha(j,k,\theta)$ denote the expected fraction of nodes in group α with degree (j,k) that default after θ counterparties have defaulted. Note that $p(j,k,0) = p^\alpha(j,k,0) = \pi_0$. Suppose therefore that $\theta > 0$. The fact that the order of default has not yet been determined can be ignored, as the exposures of nodes are i.i.d. within each group.

For a fixed $j \in \mathbb{N}$, let $X_1^\alpha, X_2^\alpha, \ldots, X_j^\alpha$, $\alpha = 1, 2, \ldots, d$ be d sequences of i.i.d. random variables, where F_α is the distribution function of the α^{th} sequence's random variables. Let $LGD = 1 - R$ denote the loss given default for any counterparty. Note that a node i can only have a default threshold greater than zero if $\gamma(i) = c < LGD$.

Therefore since the capital ratios are independent of the exposures, and the capital ratios satisfy

$$\gamma(i) = \gamma = \begin{cases} c \text{ with probability } 1 - \pi_0 \\ 0 \text{ with probability } \pi_0, \end{cases}$$

then

$$\begin{aligned} p_n^\alpha(j,k,\theta) =& P\left(LGD\, X_\theta^\alpha > \gamma \sum_{l=1}^{j} X_l^\alpha - LGD \sum_{m=1}^{\theta-1} X_m^\alpha > 0 \right) \\ =& (1-\pi_0)\, P\left(LGD\, X_\theta^\alpha > c \sum_{l=1}^{j} X_l^\alpha - LGD \sum_{m=1}^{\theta-1} X_m^\alpha > 0 \right), \end{aligned} \tag{22}$$

with the appropriate adjustments whenever $\theta = 1$ and/or $j = 1$. Eq. (22) depends on j and through the joint distribution of $X_1^\alpha, X_2^\alpha, \ldots, X_j^\alpha$, but does not depend on n and therefore $p^\alpha(j,k,\theta) = p_n^\alpha(j,k,\theta)$. By using Bayes' theorem we then have that

$$p(j,k,\theta) = \sum_{\alpha=1}^{d} p^\alpha(j,k,\theta)\, \frac{h^\alpha(j,k)\, w_\alpha}{h(j,k)}. \tag{23}$$

For a network of size n and fixed j, k, let the random variable $\tilde{\mu}_n(j,k)$ be defined on the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ and assume that the capital ratios $\vec{\gamma}_n$ are given for each n. Define the mapping $\mathcal{H}_n : \Omega_n \to M_n$, where M_n is the set of all exposure matrices of size n and $\mathcal{H}_n(\omega_n) = \mathbf{e}_n$.

Let π^* be the smallest fixed point of the function $I\colon\ [0,1] \to [0,1]$, where

$$I(\pi) = \sum_{j,k} \frac{h(j,k)\, k}{\bar{\lambda}} \sum_{\theta=0}^{j} p(j,k,\theta)\, \bar{B}(j,\pi,\theta), \tag{24}$$

and where $\bar{B}(j,\pi,\theta) = P(X \geq \theta) = \sum_{l \geq \theta}^{j} \binom{j}{l} \pi^l (1-\pi)^{j-l}$ denotes the survival function of a binomial random variable. Since I is non-decreasing, Kleene's fixed point theorem (see [7]) can be used to show that $\pi^* = \lim_{k\to\infty} I^k(0)$, where $I(0)$ is the fraction of initially defaulted nodes.

Theorem 1 below, which constructs a measure on the product space and is a special case of a theorem in [39], is used together with Theorem 3.8 in [3] in order to prove Theorem 2. Theorem 2 makes the results in [3] (which are based on deterministic in- and out-degree sequences) applicable to semi-heterogeneous Erdős-Rényi graphs where the in- and out-degree sequences are random. It shows that it is possible to find a subsequence of semi-heterogeneous Erdős-Rényi graphs for which the results in [3] hold almost surely.

Theorem 1. (Special case of Theorem 2.4.4 in [39]) *For each $n \in \mathbf{N}$, let $(X_n, \mathcal{A}_n, P_n)$ be a probability space where X_n is a locally compact, σ-compact metric space with Borel σ-algebra $\mathcal{A}_n$. Then there exists a unique probability measure $P = \prod_{i=1}^{\infty} P_n$ on $(X, \mathcal{A}) := (\prod_{n=1}^{\infty} X_n, \prod_{n=1}^{\infty} \mathcal{A}_n)$ with the property that*

$$P\left(\prod_{n=1}^{\infty} U_n\right) = \prod_{n=1}^{\infty} P_n(U_n) \tag{25}$$

whenever $U_n \in \mathcal{A}_n$ for each $n \in \mathbf{N}$ and one has $U_n = X_n$ for all but finitely many of the n.

Theorem 2. *For each $n \in \mathbb{N}$ and $j, k \in \mathbb{N}_0$, let $\tilde{\mu}_n(j,k)$ denote the fraction of nodes with degree (j,k) in the semi-heterogeneous Erdős-Rényi graph $K^d_{\lambda,\vec{n}}$, where $\lambda = (\lambda_{il})$, $\bar{\lambda} = \sum_{i,l} w_i \lambda_{il}$ and $n = \sum_{i=1}^{d} n_i$. Then there exists a sequence $(n_m)_{m\geq 1}$ in $\mathbb{N}$ such that for any $\omega = (\omega_{n_m})_{m\geq 1}$ in the product space $\prod_{m=1}^{\infty} \Omega_{n_m}$, the corresponding sequence of exposure matrices $(\mathcal{H}_{n_m}(\omega_{n_m}))_{m\geq 1} = (\mathbf{e}_{n_m})_{m\geq 1}$ will satisfy the following with probability one:*

(1) If $\pi^* = 1$, i.e. if $I(\pi) > \pi$ for all $\pi \in [0,1)$, then

$$\alpha_{n_m}(\mathbf{E}_{n_m}, \vec{\gamma}_{n_m}) \to 1 \tag{26}$$

weakly as $m \to \infty$. In other words, almost all nodes default as the network size goes to infinity.

(2) If $\pi^* < 1$, and π^* is a stable fixed point of I (i.e. $I'(\pi^*) < 1$), then

$$\alpha_{n_m}(\mathbf{E}_{n_m}, \vec{\gamma}_{n_m}) \to \sum_{j,k} h(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\, \bar{B}(j,\pi^*,\theta) \tag{27}$$

weakly as $m \to \infty$. This is then the asymptotic fraction of defaults as the network size tends to infinity.

Proof. Fix $j, k \in \mathbb{N}$ and let $(\Omega, \mathcal{F}) = (\prod_{i=1}^{\infty} \Omega_i, \prod_{i=1}^{\infty} \mathcal{A}_i)$. Then for each $i \in \mathbb{N}$, define the projection $\Pi_i : \Omega \to \Omega_i$ by $(x_1, x_2, \dots) \mapsto x_i$. From Theorem 1, there exists a unique probability measure $\mathbb{P}$ on Ω such that if $\tilde{\mu}(j,k) = (\tilde{\mu_1}(j,k), \tilde{\mu_2}(j,k), \dots)$ is a random variable on $(\Omega, \mathcal{F})$, then for all $\epsilon > 0$

$$\mathbb{P}(|\Pi_n \tilde{\mu}(j,k) - h(j,k)| > \epsilon) = \mathbb{P}_n(|\tilde{\mu}_n(j,k) - h(j,k)| > \epsilon) \to 0. \tag{28}$$

The left-hand side of Eq. (28) follows from Theorem 1, and the convergence from Proposition 2. This shows that $\Pi_n \tilde{\mu}(j,k) \to h(j,k)$ in probability. Therefore there exists a subsequence $n_1, n_2, \dots$ such that $\Pi_{n_k} \tilde{\mu}(j,k) \to h(j,k)$ almost surely.

Now let χ_C denote the indicator function of the set C. In a system of size n, the number of nodes with degree (j,k) can be expressed as

$$\frac{\sum_{i=1}^{n} \left[(D_n^+(i))^2 + (D_n^-(i))^2 \right] \chi_{\{D_n^+(i)=j\}} \chi_{\{D_n^-(i)=k\}}}{j^2 + k^2}, \tag{29}$$

and hence

$$\tilde{\mu}_n(j,k) = \frac{1}{n}\frac{\sum_{i=1}^{n}\left[\left(D_n^+(i)\right)^2 + \left(D_n^-(i)\right)^2\right]\chi_{\left\{D_n^+(i)=j\right\}}\chi_{\left\{D_n^-(i)=k\right\}}}{j^2+k^2}. \tag{30}$$

Therefore

$$\begin{aligned}\frac{1}{n_m}\sum_{i=1}^{n_m}\left[\left(D_{n_m}^+(i)\right)^2 + \left(D_{n_m}^-(i)\right)^2\right] &= \sum_{j,k}\tilde{\mu}_{n_m}(j,k)\left(j^2+k^2\right)\\ &\to \sum_{j,k} h(j,k)\left(j^2+k^2\right) < \infty \end{aligned}\tag{31}$$

almost surely, since h is the joint probability mass function of two random variables with finite second moments. Now we have that

$$\sum_{i=1}^{n_m}\left[\left(D_{n_m}^+(i)\right)^2 + \left(D_{n_m}^-(i)\right)^2\right] = O(n_m) \tag{32}$$

almost surely. By Theorem 3.8 in [3] we now have that for any $(\omega_{n_1}, \omega_{n_2}, \dots) \in \prod_{m=1}^{\infty}\Omega_{n_m}$ the corresponding sequence of exposure matrices $(\mathcal{H}_{n_m}(\omega_{n_m}))_{m\geq 1} = (\mathbf{e}_{n_m})_{m\geq 1}$ satisfies Eq. (26) and (27). □

Section 4 now deals with illustrating Theorem 2 and shows how semi-heterogeneous Erdős-Rényi graphs can be used to compare different types of network structures

4. Application to Stochastic Financial Networks

4.1. *Illustration of theoretical results*

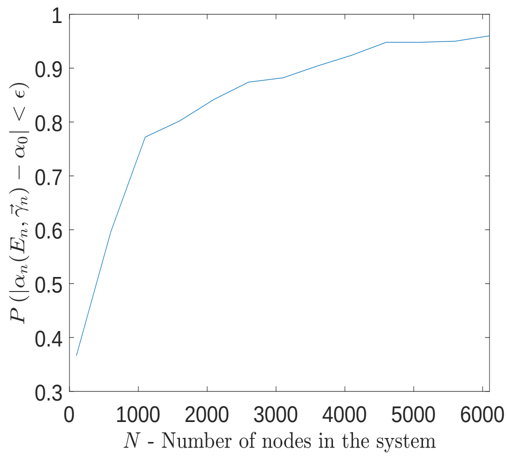
A simple Erdős-Rényi structure is used for this section as the computational inefficiencies of evaluating large networks are exasperated when dealing with multiple Erdős-Rényi networks that interact with one another. Two cases are considered regarding the non-zero exposures of each bank. The first case is where all exposures are assumed to be equal, and the second is where the positive exposures are assumed to be exponentially distributed with parameter η. This keeps the function $p(j,k,\theta)$ mathematically tractable while ensuring that counterparty exposures remain positive.

Theorem 2 is now illustrated by means of the following:

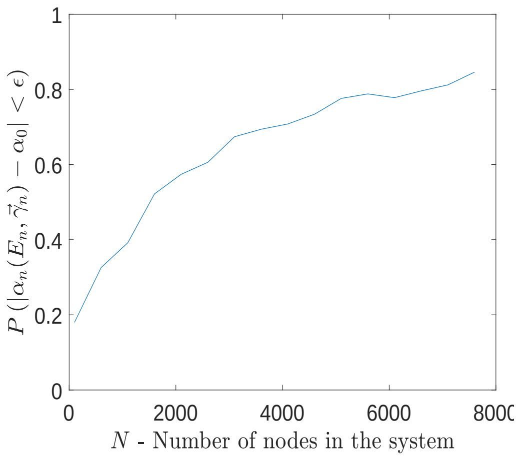
$$\mathbb{P}\left(|\alpha_n(\mathbf{E}_n, \vec{\gamma}_n) - \alpha_0| < \epsilon\right) \tag{33}$$

where $\epsilon > 0$ and $\alpha_0 = \sum_{j,k} h(j,k)\sum_{\theta=0}^{j} p(j,k,\theta)\,\bar{B}(j,\pi^*,\theta)$. The value of $\alpha_n(\mathbf{E}_n, \vec{\gamma}_n)$ is determined via simulation for increasing values of n, and α_0 is determined analytically. Table 4.1 contains the parameters used for the purpose of this illustration.

Fig. 1 shows how Eq. (33) moves closer to one for increasing values of N, which supports the conclusion of Theorem 2. For equal exposures (Fig. 1a), convergence is achieved much faster than for random exposure amounts (Fig. 1b). This is expected, since there is less variation between nodes in the network. Similarly, convergence is expected to be slower when groups of interacting Erdős-Rényi graphs are considered.



(a) Equal exposures



(b) Exponentially distributed i.i.d. exposures

Fig. 1. Illustrating the convergence as given by Theorem 2.

Note that the fraction of defaults based on simulation results only start to converge to the theoretical quantity α_0 for very large values of n, which may not be attained in a practical setting. For example, the German banking system had approximately 1,800 banks as at 2014 [9]. In this case Fig. 1 implies that the theoretical results would be close to the observed fraction of defaults approximately 80% of the time for equal exposures, and approximately 40% of the time for exponential i.i.d. exposures. However, computationally it is still much more efficient than simulation methods to assess sensitivities resulting from changes to combinations of network and bank characteristics. Furthermore, analytical results are often useful tools for understanding complex systems as these can assist in understanding the underlying components before carrying out simulations. Results for large financial networks are also of interest in cases where multiple countries and/or multiple types of financial institutions are considered. In these cases the number of nodes in the network can increase significantly, making an asymptotic approach appropriate.

4.2. *Applying the results to different network structures*

For this section we consider a semi-heterogeneous Erdős-Rényi graph $K^d_{\lambda,\vec{n}}$ with $d = 2$ groups of connected Erdős-Rényi graphs, even though the theory presented in this paper can deal with any finite number of interacting graphs. This will be used to compare different network structures based on the matrix λ. In the setting discussed in section 3.2, the groups can be determined in any way as long as the exposure amounts satisfy requirements lined out in [3]. In our case we assume that banks are grouped according to size in order to relate this section to banking systems commonly found in practice. Hence we let group one consist of a small fraction w_1 of large banks and group two of a larger fraction w_2 of small banks.

We will assume that the non-zero exposure amounts follow an exponential distribution with means η_1 and η_2 for groups one and two respectively. The exponential distribution is used because of its analytical tractability. However in practice it would make sense to use a truncated distribution for the exposures since a bounded support is more realistic for balance sheet figures.

Recall now that the total asset value of each bank does not feature in any of the results that this study considers. Therefore in this stylized setting we will assume that banks that generally have large counterparty exposures have high asset values and vice versa for banks with lower exposure amounts. This is equivalent to assuming that loans granted by large banks are generally larger than any loans granted by small banks, which is a reasonable assumption to make. Therefore we must have that $\eta_1 > \eta_2$ and these parameters will be used to differentiate between banks of different size.

The matrix λ will in turn be used to differentiate between different network structures by varying the level of interconnectedness between the different groups and within each group. In order to make the structures comparable it is assumed that the average out-degree (or equivalently the in-degree) of a randomly chosen node in each type of network is a fixed quantity $\bar{\lambda}$. Three network structures will be compared to one another. These structures together with their connection probabilities in the case of a finite network of size n are as follows:

(i) Standard Erdős-Rényi graph, with $q_{11}^{(n)} = q_{12}^{(n)} = q_{21}^{(n)} = q_{22}^{(n)} = \frac{\bar{\lambda}}{n-1}$.
(ii) Tiered type I - Large banks are the most likely to be exposed to one another and small banks less likely to be exposed to one another. The probability of a small bank and a large bank being exposed to one another is in between the former two probabilities. The probabilities are given by $q_{ij}^{(n)} = \frac{\eta_i+\eta_j}{2\eta_1} L_2^{(n)}$.
(iii) Tiered type II - Large banks have a relatively high probability of lending to any other bank, small banks have a smaller probability of lending to large banks and the probability of small banks lending to one another is the least. Here we have that $q_{ij}^{(n)} = \frac{\eta_i+\eta_j+\max\{\eta_i-\eta_j,0\}}{3\eta_1} L_3^{(n)}$.

The $L_m^{(n)}$ quantities in the formulae above are adjustment factors that ensure that the structures exhibit the required average out-degree $\bar{\lambda}$ and that the connection probabilities are functions of n that tend to zero as $n \to \infty$. It is noted that the structures and formulae chosen are used for illustrative purposes and for investigating how network structure may affect systemic risk. Hence they are not necessarily the most realistic structures for banking systems. However, the second and third structures are representative of core-peripheral networks which explicitly place larger banks in the tightly connected core, and therefore contain elements of structures found in practice.

It now remains to determine the matrix $\lambda = (\lambda_{ij})$ for each network structure based on the above probabilities so that Theorem 2 can be applied. Note that in the case of a finite network of size n, the average out-degree (or in-degree) of a node in the network would be given by

$$\bar{\lambda} = w_1 (n_1 - 1) q_{11}^{(n)} + w_1 w_2 n q_{12}^{(n)} + w_1 w_2 n q_{21}^{(n)} + w_2 (n_2 - 1) q_{22}^{(n)}. \tag{34}$$

This equation will be used to determine the functional form of $L_m^{(n)}$, $m = 1, 2, 3$, so that we can find $\lambda_{ii} = \lim_{n\to\infty} (n_i - 1) q_{ii}^{(n)}$, $i = 1, 2$ and $\lambda_{ij} = \lim_{n\to\infty} w_j n q_{ij}^{(n)}$, $i \neq j$.

1. Erdős-Rényi This structure is straightforward, since $\lambda_{ij} = \bar{\lambda} w_j$ for $i, j = 1, 2$.

2. Tiered Type I For this structure we have that $q_{11}^{(n)} = L_2^{(n)}$, $q_{12}^{(n)} = q_{21}^{(n)} = \frac{\eta_1+\eta_2}{2\eta_1} L_2^{(n)}$ and $q_{22}^{(n)} = \frac{\eta_2}{\eta_1} L_2^{(n)}$. Using Eq. (34) it can be seen that

$$\begin{aligned} \bar{\lambda} &= w_1 (n_1 - 1) q_{11}^{(n)} + 2 w_1 w_2 n q_{12}^{(n)} + w_2 (n_2 - 1) q_{22}^{(n)} \\ &= L_2^{(n)} \left[w_1 (n_1 - 1) + w_1 w_2 n \frac{\eta_1 + \eta_2}{\eta_1} + w_2 (n_2 - 1) \frac{\eta_2}{\eta_1} \right] \end{aligned} \tag{35}$$

so that

$$L_2^{(n)} = \bar{\lambda} \frac{\eta_1}{\eta_1 w_1 (w_1 n - 1) + w_1 w_2 n (\eta_1 + \eta_2) + w_2 \eta_2 (w_2 n - 1)}. \tag{36}$$

Hence

$$\begin{aligned} \lambda_{11} &= \lim_{n\to\infty} (n_1 - 1) L_2^{(n)} \\ &= \bar{\lambda} \frac{\eta_1 w_1}{\eta_1 w_1^2 + w_1 w_2 (\eta_1 + \eta_2) + \eta_2 w_2^2}, \end{aligned} \tag{37}$$

and in general $\lambda_{ij} = \bar{\lambda} \frac{(\eta_i+\eta_j) w_j}{2\eta_1 w_1^2 + 2 w_1 w_2 (\eta_1+\eta_2) + 2\eta_2 w_2^2}$.

3. Tiered Type II In this case $q_{11}^{(n)} = q_{12}^{(n)} = \frac{2}{3} L_3^{(n)}$, $q_{21}^{(n)} = \frac{\eta_1+\eta_2}{3\eta_1} L_3^{(n)}$ and $q_{22}^{(n)} = \frac{2\eta_2}{3\eta_1} L_3^{(n)}$. Based on Eq. (34) we now have

$$\begin{aligned} \bar{\lambda} &= w_1 (n - 1) q_{11}^{(n)} + w_1 w_2 n q_{21}^{(n)} + w_2 (n_2 - 1) q_{22}^{(n)} \\ &= \frac{L_3^{(n)}}{3} \left(2 w_1 (n - 1) + w_1 w_2 n \frac{\eta_1 + \eta_2}{\eta_1} + w_2 (n_2 - 1) \frac{2\eta_2}{\eta_1} \right), \end{aligned} \tag{38}$$

$$L_3^{(n)} = \bar{\lambda} \frac{3\eta_1}{2\eta_1 w_1 (n - 1) + w_1 w_2 n (\eta_1 + \eta_2) + 2\eta_2 w_2 (w_2 n - 1)} \tag{39}$$

and

$$\lambda_{11} = \lim_{n\to\infty} \bar{\lambda} \frac{2\eta_1 w_1 (w_1 n - 1)}{2\eta_1 w_1 (n-1) + w_1 w_2 n (\eta_1 + \eta_2) + 2\eta_2 w_2 (w_2 n - 1)}$$
$$= \bar{\lambda} \frac{2\eta_1 w_1}{2\eta_1 w_1 + w_1 w_2 (\eta_1 + \eta_2) + 2\eta_2 w_2^2}. \tag{40}$$

Furthermore $\lambda_{ij} = \frac{(\eta_k + \eta_i) w_j}{2\eta_1 w_1 + w_1 w_2 (\eta_1 + \eta_2) + 2\eta_2 w_2^2}$, where $k = \min\{i, j\}$.

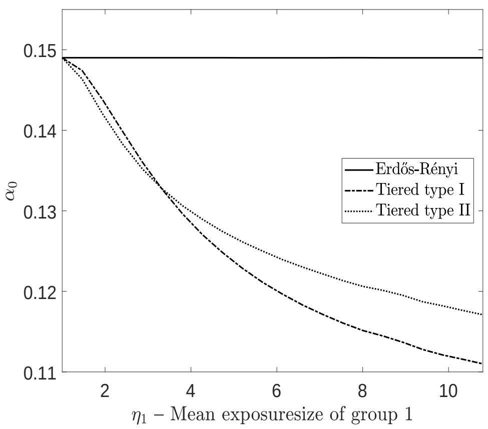
For all of the structures above, the expressions for λ_{ij} then satisfies the identity $\bar{\lambda} = w_1 (\lambda_{11} + \lambda_{12}) + w_2 (\lambda_{21} + \lambda_{22})$. Table 3 now shows the parameter values that were chosen for this analysis. Parameters either have the default value as indicated by the table or are varied within the range given in the final column.

Consider first the variation in the final fraction of defaults as the relative sizes of the two groups are changed. The mean exposure amount of group one is varied from one to 11, whereas the mean exposure amount of group two is kept fixed at one. Therefore when $\eta_1 = 1$, we have a completely homogeneous network where all banks are of the same size. The three structures should therefore yield precisely the same fraction of final defaults in this case, since the discriminatory factor is eliminated when $\eta_1 = \eta_2$. This is illustrated in Fig. 2a, where the graph starts out with all three lines on top of one another. As the heterogeneity between the banks is increased along with η_1, the structures begin to discriminate between banks of different size.

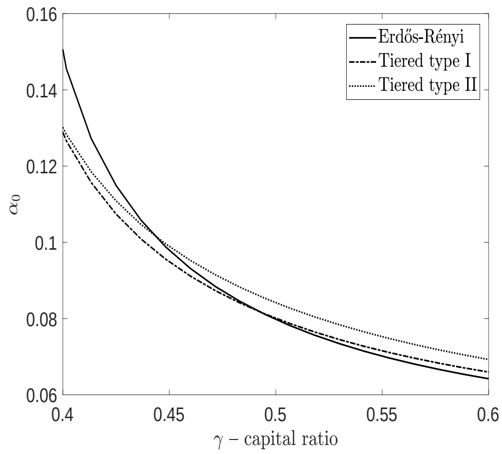
It is interesting to see that both Tiered structures immediately start to deviate from the standard Erdős-Rényi case. These two structures exhibit decreasing risk for increasing heterogeneity between the groups of banks. This suggests that systems may benefit from having a core-peripheral structure, with lending preferences that depend thereon.

Consider now the effect of varying the capital ratio from 0.4 to 0.6. Since this is a ratio of capital to interbank assets, then if interbank exposures consist of roughly 20% [3] of total capital, it corresponds to a range of 0.08 to 0.12 of capital to total assets. The results are given by Fig. 2b. As expected, the final fraction of defaults declines for all structures, though the final fraction of default declines more steeply for the Erdős-Rényi structure compared to the Tiered type I and Tiered Type II structures in Fig. 2b.

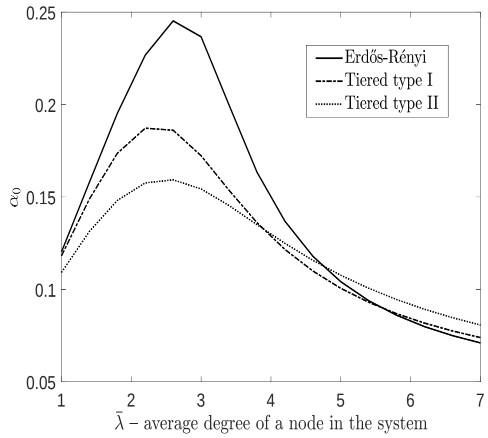
The average degree of the system is considered in Fig. 2c. This parameter is varied from one (an extremely sparse network) to seven. It can be seen that for a very sparse network, the different structures do not result in significantly different levels of default fractions. When the average out-degree is increased, the additional links in the system facilitate the spread of contagion for all structures. When the average out-degrees is just over 2.5, the default fractions start do decline when the additional links in the system serve as a safety mechanism. The peak at $\bar{\lambda} \approx 2.5$ and subsequent decline is more pronounced for the Erdős-Rényi case than for the other structures. This indicates a higher sensitivity to the level of interconnectedness in the system for our base parameters. It shows that conclusions regarding the optimal



(a) Varying the mean asset value of group 1



(b) Varying the capital ratio



(c) Varying the average degree per node

Fig. 2. Illustrating parameter sensitivities of different semi-heterogeneous Erdős-Rényi graphs, where $\alpha_0 = \sum_{j,k} h(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\, \bar{B}(j,\pi^*,\theta)$.

network structure can be highly dependent on network characteristics.

5. Conclusion

This study firstly shows how the fraction of defaults in a financial system can be approximated for large, random networks. A sequence of financial networks of increasing size with random in- and out- degree sequences that satisfies certain limiting conditions is considered. It is shown that there will be a subsequence for which the fraction of defaults following an initial shock can be determined.

A class of inhomogeneous graphs is defined and it is illustrated how results that apply to Erdős-Rényi networks may be generalized to apply to this class. This is done by considering the theoretical results developed in this paper, which can be applied to Erdős-Rényi networks. It is then showed how these results hold for multiple networks that are connected to one another, leading to results that hold for inhomogeneous networks. This brings existing theoretical results closer to the complexities found in real-world financial network structures. Potential uses of such a class of networks include modeling the interaction between different types of financial entities (e.g. between banks, investment companies and insurance companies) or between the financial systems of different countries.

As a simple illustration of the versatility of the proposed class of networks, three different structures that comply with the definition are compared to one another. The first is the standard Erdős-Rényi graph, where lending behavior is independent of relative asset sizes. The remaining structures assume different kinds of lending behavior for banks based on their relative asset sizes. In other words, banks' preferred creditors and debtors are determined by their asset sizes, although the framework allows for other characteristics to infer such preferences instead. While the illustration may not be based on entirely realistic structures, the second and third structures do account for a hierarchical formation of edges based on bank size.

The illustration considered here suggests that for large systems, the sensitivity of systemic risk to network characteristics is dependent on the network structure. For example, where one structure may show a significant change in systemic risk when the interconnectedness is varied, another structure may only show a modest change. It further suggests that while systemic risk can be lower for tiered structures compared to non-tiered structures (this is supported by e.g. [41]), network characteristics such as heterogeneity between banks, capital ratios and interconnectedness influence whether this is indeed the case. Therefore, whether or not the level of tiering in a network serves to strengthen or weaken the system potentially depends on a combination of network characteristics. It is noted that the realism of these observations are influenced by the shocks being transmitted via direct exposures, and that indirect mechanisms such as liquidity risk may well serve to increase the importance of network structure even further [28].

One possible course of action for managing the contagion risk surrounding too-big-to-fail institutions is to not discourage core-peripheral banking structures (as opposed to 'breaking up the big banks'). Instead, focus can be placed on incentivizing lending preferences (i.e. banks' preferred creditors and/or debtors based on characteristics such as bank size, sector, type etc.) that lead to lowered contagion risk. Comparing this possibility with other options where the structure of the system is taken into account is an important direction for future research. While there are many other aspects to take into consideration when managing contagion risk [16], we illustrate why the structure of the network as influenced by lending preferences should play an important role.

Further important considerations for future research is the inclusion of a central bank (e.g. as in [27]), the investigation of targeted shocks [14], different resolution outcomes for banks in distress [24] and different measures of systemic risk (e.g. as proposed in [18] and [10]). The inclusion of liquidity/market confidence effects are also important aspects to include when considering contagion risk. The interplay between these aspects and the structure of the system would then provide valuable insights for practitioners.

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Table 2. The parameter values used for illustrating the convergence given by Theorem 2.

Parameter	Description	Parameter value
γ	Ratio of interbank assets to capital	0.4
π_0	Initial fraction of defaults	0.05
$\bar{\lambda}$	Average out-degree/in-degree of the system	4
η	Mean exposure amount	1
ϵ	Error term used for evaluating Eq. (33)	0.025

Parameter	Description	Default value	Range
γ	Ratio of interbank assets to capital	0.4	$[0.4, 0.6]$
π_0	Initial fraction of defaults	0.05	
$\bar{\lambda}$	Average out-degree as given by eq. (34)	4	$[1, 7]$
η_1, η_2	Mean exposure amounts for groups one and two	$4, 1$	$[1, 11]$
w_1	Weight for group one	0.15	

Table 3. The default parameter values and their respective ranges used for comparing network structures.