UNIVERSITY OF PRETORIA

DOCTORAL THESIS

Combinatorial and Analytic properties of partition functions in AdS/LCFT

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Declaration of Authorship

I, Yannick MVONDO-SHE, declare that this thesis titled, "Combinatorial and Analytic properties of partition functions in AdS/LCFT" and the work presented in it are my own. I confirm that:

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"That I may know Him, and the power of His resurrection, and the fellowship of His sufferings, being made conformable unto His death;"

Philippians 3:10

UNIVERSITY OF PRETORIA

Abstract

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by Yannick MVONDO-SHE

Quantum field theory has over the years shown to be an enthralling subject of study in both theoretical physics and mathematics, by reason of its description of nature and of its beauty. A leitmotiv in this study is the role played by symmetry, which pervades many areas of theoretical physics, be it gauge theories, relativity or string theories to name a few.

Two dimensional conformal field theories are quantum field theories which possess infinite dimensional symmetry algebras, a property which renders them exactly solvable. The mathematical structure of these theories is well under control, especially for the so called rational conformal field theories. Besides their applications in statistical physics, they play an important role in string theory, via the AdS/CFT correspondence. For about a quarter century, a generalization by the name of *logarithmic* conformal field theory has been intensively studied by mathematicians and physicists. On the physics side, a new type of holographic duality, the $AdS_3/LCFT_2$, has been the object of ardent investigation.

The primary goal of this thesis is the study of the 1-loop partition function of the critical topologically massive gravity, a theory conjectured to be dual to a logarithmic conformal field theory through the $AdS_3/LCFT_2$ correspondence. In particular, a better understanding of the combinatorics of the multi-log sector has been desired, in order to give the partition function a more concrete interpretation from an LCFT perspective.

This text begins with two chapters intended to present the theoretical foundations. The third chapter deals with the combinatorial recasting of the partition function of topologically massive gravity using the so called Bell polynomials. While studying combinatorial aspects of the partition function, it was found that Bell polynomials are connected to the Plethystic Exponential. In our case, the relationship is made explicit in chapter 4. Furthermore, as a mathematical excursion, we show in chapter 5 that some algebraic properties arise in the partition function once differential operators are defined. Finally, in chapter 6, we draw a conclusion on the work done, and project ourselves towards future work in this exciting area of the $AdS_3/LCFT_2$ correspondence.

Acknowledgements

Thus saith the Lord, which maketh a way in the sea, and a path in the mighty waters;

Which bringeth forth the chariot and horse, the army and the power; they shall lie down together, they shall not rise: they are extinct, they are quenched as tow.

Remember ye not the former things, neither consider the things of old.

Behold, I will do a new thing; now it shall spring forth; shall ye not know it? I will even make a way in the wilderness, and rivers in the desert.

The beast of the field shall honour me, the dragons and the owls: because I give waters in the wilderness, and rivers in the desert, to give drink to my people, my chosen.

This people have I formed for myself; they shall shew forth my praise.

Isaiah 43:16-21

Thank you Lord for the Path of Redemption.

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And finally, last but by no means least, to my "Mpingi" Mimi, and my little baby girl Ivy, thank you so so much for your unconditional love, and for your spiritual, moral and emotional support in this life of hardship. You are a wonderful, precious gift from the Lord. You both are my raison d' être.

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Dedicated to Mireille and Ivannah

Introduction

The advent of conformal field theory is established in the seminal paper of Belavin, Polyakov and Zamolodichikov [1]. In those days quantum field theories (QFTs) were almost exclusively studied in a perturbative fashion. Yet, owing to the infinite dimensional symmetry algebra of conformal field theories in two dimensions, Belavin *et al* were able to introduce a suitable non-perturbative description of scale invariant (one-dimensional quantum or two dimensional classical) systems. The impact of this new approach was tremendous in both physics and mathematics, and 2D (two dimensional) CFT found many applications.

In physics, CFTs were first used in statistical mechanics and condensed matter physics, as models for phase transitions of the second order at the critical point, characterized by divergent correlation lengths of some physical quantities such as magnetisation. A field theory describing such a model with fluctuations of typical configurations on various length scales is said to be scale invariant or again *conformally* invariant, hence conformal field theory. Furthermore, at the critical point, it was noticed that many seemingly disparate physical models could be described by the same CFT. This principle called *universality* became a motivation to classify CFTs into so called *universality classes*. Despite many ongoing efforts, this ambitious program has only been achieved for rational CFTs, *i.e* theories characterized by a parameter c called *central charge* for which in this case $c < 1$ [2, 3, 4].

Another field in which CFT has been of important use is string theory, arguably the best candidate for a theory unifying all fundamental forces, and in particular for a consistent theory of quantum gravity. The proposal brought by string theory is to consider elementary particles as vibrational modes on strings that can interact exclusively by splitting apart or joining together. As the strings evolve in time, they map out a Riemann surface called the *world-sheet*, that is, a two-dimensional manifold with a conformal structure. The relationship between CFT and string theory can thus be identified by the fact that CFT lives on the world-sheet traced by the strings in time evolution.

CFT and string theory have also greatly impacted the field of mathematics, and it is not just by a fortunate stroke of serendipity that five of the twelve Field medalists of the 1990s (Drinfel'd, Jones, Witten; Borcherds, Kontsevich) received the prestigeous award based on works related to CFT. For instance, the mathematical study of CFT has shed light on a mysterious connection between string theory, algebra and number theory (or again between Lie theory, finite groups and automorphic forms) through monstrous moonshine [5, 6, 7]. Furthermore, CFT is also connected to topological field theory on three manifolds [8].

Just three years into the discovery of CFT, V. Knizhnik— a first-class soviet theoretical physicist who unfortunately passed away at the young age of 25— noted the appearance of logarithmic singularities in ghost systems, in contrast with the usual occurrence of poles in ordinary CFT [9]. This phenomenon was observed in subsequent works [10, 11, 12], but it was really twenty five years ago, with the ground-breaking work of Victor Gurarie [13], that what is known today as *logarithmic conformal field theory* was established. Just like for ordinary CFT, LCFTs became an interesting object of study in mathematics and physics, and efforts to understand these theories from a mathematical point of view as well as to relate them to other fields of physics are eagerly pursued to this day. Applications in condensed matter theory and statistical physics include the description of phase transitions in disordered non-interacting electronic systems such as the transition between plateaus in the integer quantum Hall effect [14, 15], critical geometrical models like polymers [11, 16, 17, 18] and percolation [19, 20, 21, 22, 23, 24, 25, 26], or generally speaking, critical systems with quenched disorder [27, 28, 29].

LCFTs also found interesting applications in string theory. Indeed, soon after the discovery of the celebrated AdS/CFT correspondence [30], suggestions about an AdS/LCFT duality arose [31, 32, 33, 34]. But just about ten years ago, a new type of correspondence was proposed after observing that Jordan cells, the hallmark of LCFTs appear on the gravity side [35] for a certain critical tuning of the coupling constants [36] in topologically massive gravity (TMG) [37, 38]. Within this $AdS_3/LCFT_2$ correspondence proposal, the 1-loop partition function of critical cosmological topologically massive gravity (CCTMG) was calculated in [39]. But up to now, the partition function begs for a better understanding, in particular of the multi-particle sector. Progress in studying the structure of the single- and multi-particle sectors of the partition function would enable to make more concrete statements about the correspondence, as well as its possible applications. This work is an effort to find answers in that direction.

The structure of this manuscript is as follows. In the first chapter we give a presentation of the basics of CFT and LCFT. Then we proceed in the second chapter with a review of gravity in three dimensions and holography. The third chapter has the main result of our work, *i.e* a recasting of the partition function of CCTMG found in [39] in terms of combinatorial Bell polynomials. This allows for a systematic treatment of the single- and multi-particle sectors. Along the way, while studying the combinatorial properties of the partition function in terms of Bell polynomials, a connection between the latter and the celebrated Plethystic Exponential was noted. In the present case, a specific identification is derived in chapter 4. Next to that, chapter 5 shows how upon an appropriate choice of differential operators acting on the Bell polynomials, an sl(2) action appears within successive constituents of the partition function. Lastly, in chapter 6, a conclusion is drawn and prospective lines of research are given.

Chapter 1

(Logarithmic) conformal field theory

This chapter presents (logarithmic) conformal field theories. Before commencing, we would like to plead the indulgence of the more technically inclined reader towards this chapter as well as the next chapter, both primarily written to give an overview of theories necessary for the following chapters. As such, chapter 1 and 2 are written at the expense of mathematical rigor, and almost all proofs are omitted. In turn, we hope that the reader will find some solace in the scores of introductory bodies of literature mentioned throughout the text. These references should compensate the interested reader for this choice and hopefully close all persisting gaps.

1.1 Conformal field theory

This section gives a brief introduction of conformal field theory. We refer to the excellent "*big yellow book*" [40] as well as [41], that are among the earliest textbooks dedicated to the study of conformal field theory. A very readable and more recent textbook on CFT with applications to string theory is [42]. For the reader interested in a more mathematical approach to CFT, a good place to start is [43], and an introductory textbook from an algebraic geometry perspective is [44]. There are also loads of lecture notes in the literature, with among the first ones, the very nice notes by Ginsparg [45].

With the ever increasing use of sophisticated mathematics over the past decades, theoretical physicists have been trained in the knowledge that groups and symmetries go hand in hand. Indeed, whenever one encounters a symmetry, a group of transformations is expected to play a role in the background. A revolution in the past thirty years has been attributable to the presence of *conformal symmetry* in a number of physical systems. In particular, a conformal transformation allows not only for the rescaling of space, but also for the space to be twisted in such a way that angles are preserved. In that setting, the invariance under scale transformation is generalized under the name of *conformal invariance*. That being given, CFT really begins with *geometry*, since it turns to account that a system is invariant under some types of geometric transformations, the angle preserving ones.

In the next subsection we will introduce CFT in a fairly general way, *i.e* on curved space in arbitrary dimension $\mathbb{d}$. Thereafter, we will consider the theory in two dimensions, for which the space is conformally flat.

1.1.1 Conformal theory in arbitrary dimensions

Conformal transformation and conformal group

Let the pair $(\mathbb{M}, g)$ consisting of a smooth manifold $\mathbb{M}$ of dimension $\mathbb{d}$ and a metric tensor g be a pseudo-Riemannian manifold, where g assigns to each point $m \in \mathbb{M}$ a nondegenerate and symmetric bilinear form on the tangent space $T_m\mathbb{M}$

$$g_m : T_m\mathbb{M} \times T_m\mathbb{M} \to \mathbb{R}. \tag{1.1}$$

In local coordinates $x^1, x^2, \ldots, x^{\mathbb{d}}$ of the manifold $\mathbb{M}$, the bilinear form g_m on $T_m\mathbb{M}$ can be written

$$g_m(X, Y) = g_{ab}(m) X^a Y^b, \tag{1.2}$$

where X and Y are vectors fields in the tangent space $T_m\mathbb{M}$ denoted by

$$X = X^a \partial_a, \qquad Y = Y^b \partial_b, \tag{1.3}$$

and described with respect to the basis

$$\partial_a = \frac{\partial}{\partial x_a}, \qquad a = 1, \ldots, \mathbb{d}. \tag{1.4}$$

The line element ds, or loosely speaking the distance associated with an infinitesimal scale (the distance squared between two points with coordinates x^a and $x^a + dx^a$), can then be expressed as

$$ds^2 = g_{ab}(x) dx^a dx^b, \tag{1.5}$$

Let now h be another metric on $\mathbb{M}$. Then g and h are said to be *conformally equivalent* if there exists a smooth scalar function called the *conformal factor* $\Omega : \mathbb{M} \to \mathbb{R}_+$ such that $g(\mathbf{x}) = \Omega^2(\mathbf{x})\, h(\mathbf{x})$, where $\mathbf{x}$ is the set of local coordinates. This is defined as an equivalence relation on $\mathbb{M}$. The corresponding classes of equivalence are called *conformal structures* on $\mathbb{M}$, and the pair $(\mathbb{M}, g)$ is a *conformal manifold*.

The diffeomorphisms $f : \mathbb{M} \to \mathbb{M}$ in the category of conformal manifolds are smooth maps that preserve the conformal structure of the manifolds. More specifically, they form the group Diff($\mathbb{M}$) such that any smooth function $f \in$ Diff($\mathbb{M}$) fulfills

$$f^* g_{f(m)} = \Omega^2 g_m, \qquad \forall m \in \mathbb{M}, \tag{1.6}$$

where $f^* g_{f(m)}(X, Y) = g_{f(m)}(f_* X, f_* Y)$ is the pullback of $g_{f(m)}$ via f. The map f is called a *conformal transformation*, and the set of conformal transformations on $\mathbb{M}$ forms a group called the *conformal group*, and denoted Conf($\mathbb{M}$). In the special case $\Omega = 1$, Eq. (1.6) defines an *isometry*, and the group of isometries Iso($\mathbb{M}$) is said to be a subgroup of Conf($\mathbb{M}$).

On a Riemannian manifold, the angle θ between two vectors X and Y of the tangent space $T_m\mathbb{M}$ can be expressed by

$$\theta = \frac{g_m(X,Y)}{\sqrt{g_m(X,X)g_m(Y,Y)}}. \tag{1.7}$$

Then, from Eq. (1.6), it can be shown that the angle is invariant under conformal transformations. Therefore, as much as conformal transformations can locally change the scale, they cannot change the shape of a manifold.

Things can be made more concrete by using components. Consider the change of coordinate $x \mapsto x'(x)$, such that

$$dx^a = \frac{\partial x^a}{\partial x'^c} dx'^c. \tag{1.8}$$

Such a coordinate tranformation keeps distance between two nearby points ds invariant. As a result one can write

$$ds^2 = g_{ab}(x)\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d} dx'^c dx'^d = g'_{cd}(x')dx'^c dx'^d. \tag{1.9}$$

Therefore, under a change of coordinates, the metric transforms according to

$$g'_{cd}(x') = g_{ab}(x)\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}. \tag{1.10}$$

As an example, let us consider the scale transformation $x^a \mapsto x'^a = \lambda x^a$, for a real number λ). Then the metric transforms like

$$g'_{cd}(x') = g'_{cd}(\lambda x) = \Omega^2(x) g_{ab}(x), \tag{1.11}$$

such that

$$\Omega(x) := e^{\frac{w}{2}}, \tag{1.12}$$

with $w = -2\ln\lambda$, and for which we can write $g'_{cd}(x') = \frac{1}{\lambda^2} g_{ab}(x)$. More generally, the coordinate transformation $g'_m = \Omega^2 g_m$ is referred to as *Weyl transformation*.

Conformal Killing vector fields

Here, we look at conformal transformations infinitesimally, *i.e* we consider vector fields whose infinitesimal displacement generates a conformal transformation.

Let us consider a vector field X that induces an infinitesimal displacement ϵX on points $m \in \mathbb{M}$ with coordinates x^a. In the case where $\Omega(x) = 1$, *i.e* the infinitesimal displacement of the vector field generates an isometry, plugging the change of coordinates given by the infinitesimal coordinate transformation $x'^a = x^a + \epsilon X^a(x)$ into Eq. (1.11) yields

$$\mathcal{L}_X g_{ab} := X^e \partial_e g_{ab} + g_{eb}\partial_a X^e + g_{ae}\partial_b X^e = 0, \tag{1.13}$$

where the operator $\mathcal{L}_X$ is called the *Lie derivative*. Eq. (1.13) is defined as the *Killing equation*, and any vector X satisfying it is a *Killing vector field*.

In the more general case $\Omega(x) \neq 0$, introducing the *Conformal Killing factor* $\varkappa$, $\Omega^2(x')$ can be expanded as $\Omega^2(x') \simeq 1 + \epsilon\varkappa(x') = 1 + \epsilon\varkappa(x) + \mathcal{O}\left(\epsilon^2\right)$. The previous equation becomes

$$\mathcal{L}_X g_{ab} := X^e \partial_e g_{ab} + g_{eb}\partial_a X^e + g_{ae}\partial_b X^e = \varkappa g_{ab}. \tag{1.14}$$

Eq. (1.14) is called the *conformal Killing equation*, and any vector X satisfying it is a *conformal Killing vector field*.

A simplified and concrete picture of the above can be given by considering the flat space $\mathbb{R}^{p,q}$ with $p + q = \mathbb{d} > 2$ endowed with the flat metric tensor η_{ab}. An infinitesimal transformation $x \mapsto x' = x + \epsilon(x) + \mathcal{O}\left(\epsilon^2\right)$ yields

$$\eta_{cd}\left(\delta_a^c + \partial_a \epsilon^c\right)\left(\delta_b^d + \partial_b \epsilon^d\right) + \mathcal{O}(\epsilon^2) = \Omega^2(x)\eta_{ab}. \tag{1.15}$$

Working the left-hand side gives

$$\eta_{ab} + \eta_{ab}\left(\partial_a \epsilon_b + \partial_b \epsilon_a\right) + \mathcal{O}\left(\epsilon^2\right) = \Omega^2(x)\eta_{ab} = \eta_{ab} + \kappa(x)\eta_{ab}. \tag{1.16}$$

It is then easy to see that

$$\left(\partial_a \epsilon_b + \partial_b \epsilon_a\right) = \varkappa(x)\eta_{ab}. \tag{1.17}$$

Tracing with η^{ab} leads to

$$2\partial^a \epsilon_a = \varkappa(x)\mathbb{d}, \tag{1.18}$$

or again

$$\varkappa(x) = \frac{2}{\mathbb{d}}\left(\partial \cdot \epsilon\right). \tag{1.19}$$

The final result is the *conformal Killing equation*

$$\left(\partial_a \epsilon_b + \partial_b \epsilon_a\right) = \frac{2}{\mathbb{d}}\left(\partial \cdot \epsilon\right)\eta_{ab}. \tag{1.20}$$

In a nutshell, a conformal Killing equation simply gives the condition for conformal invariance. In the case $\mathbb{d} = 2$, as we will see later, it leads to a system of two differential equations that can be solved exactly. Next we discuss the conformal algebra and its generators in dimension $\mathbb{d} \geq 3$.

Conformal algebra

For $\mathbb{d} \geq 3$, by contracting the conformal Killing equation further, one gets

TABLE 1.1: Generators of the conformal group in $\mathbb{d} > 2$ dimensions

Parameter	Transformation	Generator
α^a	$x'^a = x^a + \alpha^a$	$P_a = i\partial_a$
n	$x'^a = (1+n)x^a$	$D = -ix^a\partial_a$
m_{ab}	$x'^a = x^a + m^a_b x^b$	$L_{ab} = i\left(x_a\partial_b - x_b\partial_a\right)$
β_a	$x'^a = x^a + 2(x\cdot\beta)x^a - (x\cdot x)\beta^a$	$K_a = -i\left(2x_a x^b\partial_b - (x\cdot x)\partial_a\right)$

$$\left[-\eta_{ab}\Box + (\mathbb{d}-2)\partial_a\partial_b\right](\partial\cdot\epsilon) = 0, \tag{1.21}$$

or again

$$2\partial_a\partial_b\epsilon_c = \frac{2}{\mathbb{d}}\left(\eta_{ab}\partial_c + \eta_{ca}\partial_b + \eta_{bc}\partial_a\right)(\partial\cdot\epsilon)\,. \tag{1.22}$$

From Eq. (1.21), the following ansatz can be made

$$(\partial\cdot\epsilon) = A + B_a x^a \tag{1.23}$$

leading to

$$\epsilon_a = \alpha_a + \beta_{ab}x^b + \gamma_{abc}x^a x^b, \tag{1.24}$$

with $\alpha_a, \beta_{ab}, \gamma_{abc}$ as constants and $\gamma_{abc} = \gamma_{acb}$. Decomposing β_{ab} in a symmetric part that is required by Eq. (1.21) to be proportional to the metric and an antisymmetric part yields $\beta_{ab} = n\cdot\eta_{ab} + m_{ab}$, with m_{ab} as the antisymmetric part. Then, using Eq. (1.22) and $\beta_a = \mathbb{d}^{-1}\cdot\gamma^c_{ca}$, γ_{abc} can be recast such that the infinitesimal conformal transformations take the general expression

$$x'^a = x^a + \underbrace{\alpha^a}_{\text{Translation}} + \underbrace{n\cdot x^a}_{\text{Dilatation}} + \underbrace{m^a_b x^b}_{\text{Rotation}} + \underbrace{2(x\cdot\beta)x^a - (x\cdot x)\beta^a}_{\text{Special Conformal Transformation}}\,. \tag{1.25}$$

As indicated in the above equation, each parameter corresponds to a specific transformation, whose generators are given in Table 1.1.

The commutation relations of the generators of conformal transformations can easily be found, forming the *conformal algebra*.

Energy-momentum tensor and conserved currents

We close the review on conformal symmetry in dimension $\mathbb{d}$ with a discussion about the role of the *energy-momentum tensor*, and its relation to conserved currents.

Generally, a classical field theory of arbitrary dimension is defined by the action S of a Lagrangian $\mathcal{L}$, that encodes properties of the theory. In particular, the energy-momentum tensor can be defined by varying the action with respect to the metric tensor as

$$T^{ab} := \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g_{ab}}, \tag{1.26}$$

thus keeping track of the behaviour of the theory under infinitesimal transformations $g_{ab} \mapsto g_{ab} + \delta g_{ab}$. Noether's theorem states that if there is a continuous symmetry in a theory, then there is a current j_a which is conserved. In other words, if the Lagrangian of the theory is invariant under transformation $x^a \mapsto x^a + \delta x^a$, then $\partial^a j_a = 0$ is conserved.

In the case of a CFT in any dimension for which a Lagrangian may not be well defined, the energy-momentum tensor can be described as a linear map from infinitesimal conformal transformations $x \mapsto x + \epsilon(x)$ to a *conformal current* j, such that

$$j^a(\epsilon) = T^{ab}\epsilon_b. \tag{1.27}$$

From Noether's theorem, the conformal current associated to the infinitesimal conformal transformation ϵ is conserved for $\epsilon(x)$ constant. It can be shown in that case that $T^a_{\ a} = 0$, *i.e* that the energy momentum is traceless.

1.1.2 Conformal field theory in two dimensions

The geometric considerations of the previous subsection were on a pseudo-Riemannian manifold $(\mathbb{M}, g)$ in arbitrary dimension $\mathbb{d}$. As we shall see now, the dimension $\mathbb{d}$=2 is quite special in many ways.

Conformal invariance in 2 dimensions

On two dimensional manifolds $\mathbb{R}^{1,1}$ or $\mathbb{R}^{2,0}$, a Weyl transformation of the flat metric tensor in the form $\Omega(x)\eta_{ab}$ is always possible using a general coordinate system. Thanks to the *Wick rotation*, one can go from the Minkowski metric to the Euclidean one.

Here, we consider the Euclidean plane with coordinate (z^0, z^1). For reasons of symmetry, the condition (1.20) for invariance under infinitesimal conformal transformations in two dimensions leads to the following two equations

$$\partial_0\epsilon_0 + \partial_0\epsilon_0 = \partial_0\epsilon_0 + \partial_1\epsilon_1 \tag{1.28a}$$
$$\partial_0\epsilon_1 + \partial_1\epsilon_0 = 0, \tag{1.28b}$$

for which the solution gives

$$\partial_0\epsilon_0 = +\partial_1\epsilon_1, \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0, \tag{1.29}$$

These are the well known Cauchy-Riemann differential equations of complex analysis. We can therefore exploit the power of complex analysis and introduce the following complex variables

$$z = z^0 + iz^1, \quad \epsilon = \epsilon^0 + i\epsilon^1, \quad \partial_z = \frac{1}{2}\left(\partial_0 - i\partial_1\right), \tag{1.30a}$$

$$\bar{z} = z^0 - iz^1, \quad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_0 + i\partial_1\right). \tag{1.30b}$$

A complex function that satisfies Eq. (1.29) is holomorphic (in an open set). Since $\epsilon(z)$ is holomorphic, we can actually define a function $f(z) = z + \epsilon(z)$ such that the metric tensor would transform under $z \mapsto f(z)$ as

$$ds^2 = dzd\bar{z} \mapsto \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}dzd\bar{z}. \tag{1.31}$$

From complex analysis, it is known that any holomorphic transformation on the complex plane is conformal. The two dimensional conformal group is therefore the set of all analytic maps whose composition of maps is the multiplication law of the group.

From complex analysis, it is also known that any function $z \mapsto f(z)$ that is holomorphic on an open set can be expanded as a Laurent series $f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$. This implies that an infinite number of parameters is required to specify any element of the group. In that sense, the local group in two dimensions is infinite dimensional.

Generators of conformal transformations, conformal group and conformal algebra

As we have just seen, every holomorphic or antiholomorphic function gives rise to a conformal transformation on a given open set. This can also be applied to the meromorphic functions, and in such case, a Laurent series expansion of meromorphic functions can be performed around say $z = 0$ giving

$$f(z) = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n \left(-z^{n+1}\right), \tag{1.32}$$

$$f(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \left(-\bar{z}^{n+1}\right), \tag{1.33}$$

with the infinitesimal parameters ϵ_n and $\bar{\epsilon}_n$ taken as constants. This corresponds to the expansion of the transformation in a basis of infinitesimal generators

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{1.34}$$

Computing the commutators of the above generators yields two independent copies of the *classical conformal algebra*, also known as the *Witt algebra* expressed as follows

$$[l_m, l_n] = (m-n)l_{m+n}, \tag{1.35a}$$

$$\left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n}, \tag{1.35b}$$

$$\left[l_m, \bar{l}_n\right] = 0. \tag{1.35c}$$

We deduce that the algebra of infinitesimal conformal transformations in a two dimensional Euclidean space is infinite dimensional.

The fact that the conformal algebra is infinite dimensional is only a local structure of the conformal transformations, and does not guarantee a group structure. The reason is that for the complex plane that is extended by the point at infinity forming its stereographic projection, *i.e* the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$, not every local transformation is globally well-defined. To get the (global) conformal group, one has to perform a conformal compactification (of $\mathbb{C}$ to $\mathbb{C} \cup \{\infty\}$), and find the set of conformal transformations that are non-singular on that space. It turns out that only the generators $\{l_{-1}, l_0, l_1\}$ (and $\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$) are globally well defined. The global conformal transformations induced by these generators can be classified as in the follwoing table

Transformation	**Generator**
Translation	$l_{-1}, \bar{l}_{-1}$
Dilatation	$l_0 + \bar{l}_0$
Rotation	$\mathrm{i}\,(l_0 - \bar{l}_0)$
SCT	$l_1, \bar{l}_1$

TABLE 1.2: Generators of global conformal transformations in two dimensions

The set of global conformal transformations above is known as *projective conformal transformations* or again, the *Möbius transformations*. They are of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{1.36}$$

To be more specific, a, b, c, d are four complex variables, *i.e* 8 real variables. The renormalization condition being one complex constraint, six variables remain, which coincides precisely with the number of globally well-defined generators mentioned above. Furthermore, the above transformations are invariant under $(a, b, c, d) \mapsto (-a, -b, -c, -d)$. As result of all the above restrictions, the global conformal group of the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ is the *Möbius group* $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$.

The energy-momentum tensor

Applying the results found in the previous subsection to the two dimensional scenario, the tracelessness of the energy-momentum tensor implies $T_{z\bar{z}}(z, \bar{z}) = T_{\bar{z}z}(z, \bar{z}) = 0$, and in addition $\partial_z T_{\bar{z}\bar{z}}(z, \bar{z}) = \partial_{\bar{z}} T_{zz}(z, \bar{z}) = 0$. The remaining components are the purely holomorphic and the purely anti-holomorphic energy-momentum tensors

$$T_{zz}(z, \bar{z}) = T(z), \qquad T_{\bar{z}\bar{z}}(z, \bar{z}) = \bar{T}(\bar{z}) \tag{1.37}$$

In this instance, the conformal current reads

$$j_z = T_{zz}\epsilon(z), \qquad \bar{j}_{\bar{z}} = T_{\bar{z}\bar{z}}\bar{\epsilon}(\bar{z}). \tag{1.38}$$

Primary, quasi-primary and secondary fields

Next we give some important definitions concerning the fields $\phi(z,\bar{z})$ of the theory.

A field $\phi(z,\bar{z})$ is said to be *chiral* if $\partial_{\bar{z}}\phi(z,\bar{z}) = 0$. Analogously, it is *anti-chiral* if $\partial_z\phi(z,\bar{z}) = 0$.

A field $\phi(z,\bar{z})$ is said to have *conformal dimension* $h, \bar{h}$ if under the scaling transformation $z \mapsto \lambda z$ ($\lambda \in \mathbb{C}$) it transforms as

$$\phi(z,\bar{z}) \mapsto \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}). \tag{1.39}$$

A field $\phi(z,\bar{z})$ is called *primary field* of conformal dimension $(h, \bar{h})$ if under conformal transformations $z \mapsto f(z)$ it transforms as

$$\phi(z,\bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi\left(f(z), \bar{f}(\bar{z})\right). \tag{1.40}$$

A field $\phi(z,\bar{z})$ is called *quasi-primary field* of conformal dimension $(h, \bar{h})$ if Eq. (1.40) holds only for global conformal transformations.

A field $\phi(z,\bar{z})$ is called *secondary field* if it does not transform as Eq. (1.40).

The conformal weights h and $\bar{h}$ are real quantities. The sum $\Delta = h + \bar{h}$ is also called *conformal dimension* or again *scaling dimension*. The difference $s = h - \bar{h}$ is usually referred to as *conformal spin*.

Radial quantization

The quantization of a field $\phi(z,\bar{z})$ of conformal dimension $(h, \bar{h})$ requires one to expand it first as

$$\phi(z,\bar{z}) = \sum_{n,\bar{n}\in\mathbb{Z}} \phi_{n,\bar{n}} z^{-n-h} \bar{z}^{-\bar{n}-\bar{h}}. \tag{1.41}$$

Then the modes $\phi(z,\bar{z})$ with scaling dimension $(n, \bar{n})$ are promoted to operators. This approach can be motivated by considering the theory on a cylinder. Following this consideration, the first step is to compactify the space coordinate by the identification

$$w \sim w + 2\pi i, \qquad \bar{w} \sim \bar{w} + 2\pi i, \tag{1.42a}$$

$$w = x^0 + ix^1, \qquad \bar{w} = \bar{x}^0 + i\bar{x}^1. \tag{1.42b}$$

As a result of imposing these periodic boundary conditions, the momenta are quantized, and the compactification prescribed prevents the appearance of infrared divergences in the theory.

In order to continue exploiting the power of complex analysis, a conformal mapping from the cylinder back to the complex plane is possible using the trick

$$w \mapsto z = e^w = e^{x^0} e^{ix^1}, \tag{1.43}$$

which describes the map from an infinite cylinder with coordinates x^0 and x^1 to the complex plane with coordinates z. The infinite past in the Euclidean time coordinate $x^0 = -\infty$ is mapped to $z = 0$, and the infinite future $x^0 = +\infty$ is mapped to the infinite circle $|z| = \infty$ as illustrated in Fig. 1.1. Surfaces of equal time are mapped to circles of constant radius.

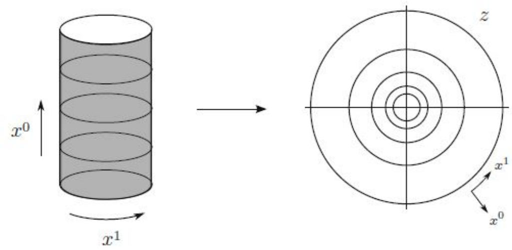


FIGURE 1.1: Conformal map from the cylinder to the complex plane [42]

Mapping the cylinder into the complex plane changes the quantum field theory *time ordered product* to a *radially ordered product*, defined for two operators as follows

$$\mathcal{R}\left(A(z)B(w)\right) = \begin{cases} A(z)B(w), \text{ if } |z| > |w| \\ B(w)A(z), \text{ if } |z| < |w| \end{cases} \tag{1.44}$$

Operator Product Expansion

As mentioned earlier, Noether's theorem states that to every symmetry in a given theory, there is a corresponding conserved current $j_a(\epsilon) = T_{ab}\epsilon^b$ with $\partial_a j^a = 0$. In its most general form, the conserved charge is given as

$$Q = \int d^{\mathbb{d}-1}x j^0(x,t), \tag{1.45}$$

where $\mathbb{d} - 1$ denotes the space dimensions over which the integral is taken. In the two-dimensional theory under consideration, this simply gives

$$Q = \int dx^1 j_0, \qquad \text{with } x^0\text{=constant.} \tag{1.46}$$

From Field theory, the conserved charge is the generator of symmetry transformations for an operator A as

$$\delta A = [Q, A]\,. \tag{1.47}$$

The change of coordinates (1.43) implies that x^0=constant corresponds to $|z|$=constant. Therefore, the conserved charge is calculated by considering the integral over space $\int dx^1$ as a contour integral

$$Q = \frac{1}{2\pi i}\oint [T(z)\epsilon(z)dz + \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})d\bar{z}] \tag{1.48}$$

The integration is by convention performed anti-clockwise, with the centre of the circle as the origin of the complex plane. In turn, Eq. (1.48) allows the evaluation of the infinitesimal conformal transformation of $\phi(z,\bar{z})$ generated by a conserved charge Q. Combining Eq. (1.48) and $\delta\phi = [Q,\phi]$ yields

$$\delta_{\epsilon,\bar{\epsilon}}\phi\left(w,\bar{w}\right) = \frac{1}{2\pi i}\oint\left[T(z)\epsilon(z),\phi(w,\bar{w})\right]dz + \frac{1}{2\pi i}\oint\left[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}),\phi(w,\bar{w})\right]d\bar{z}. \quad (1.49)$$

Using Cauchy's theorem together with the radially ordered product, the commutator can be defined as the difference between contour integrals of the radially ordered product around 0, as illustrated in Fig. 1.2.

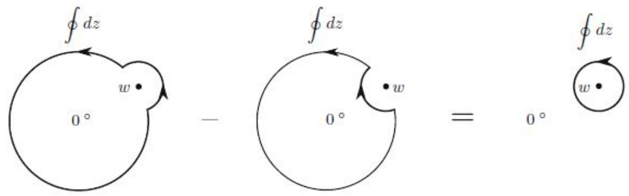


FIGURE 1.2: Contour integral difference for radially ordered product [42]

This gives

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) &= \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\left(\epsilon(z)\mathcal{R}\left(T(z)\phi(w,\bar{w})\right)dz\right) + \text{anti-chiral}\\ &= \frac{1}{2\pi i}\oint_{w}\left(\epsilon(z)\mathcal{R}\left(T(z)\phi(w,\bar{w})\right)dz\right) + \text{anti-chiral}. \quad (1.50)\end{aligned}$$

As a fundamental result

$$\mathcal{R}\left(T(z)\phi(w,\bar{w})\right) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \ldots, \quad (1.51)$$

where the ellipsis represents the remainder of the expansion in terms of non-singular terms. Equation (1.51) defines an *operator product expansion* (OPE).

The central charge and the Virasoro algebra

Similarly to Eq. (1.51), it is possible to determine the OPE of the energy-momentum tensor with itself by successively applying two conformal transformations. The result reads

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w), \quad (1.52a)$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial\bar{T}(\bar{w}). \quad (1.52b)$$

The constants c and $\bar{c}$ are called (holomorphic/antiholomorphic) *central charges* or *conformal anomaly*. They are proportional to the *Casimir energy*, *i.e* to the change in the vacuum energy density resulting from the periodic boundary condition on the cylinder.

A Laurent expansion series of the energy-momentum tensor can be performed through the expressions

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n, \tag{1.53}$$

where the symbols L_n and $\bar{L}_n$ can in turn be expressed as

$$L_n = \frac{1}{2\pi i} \oint z^{n+1} T(z) dz, \qquad \bar{L}_n = \frac{1}{2\pi i} \oint \bar{z}^{n+1} \bar{T}(\bar{z}) d\bar{z}. \tag{1.54}$$

An important property of these modes is that their commutators satisfy an important algebra called the *Virasoro algebra*, expressed as follows

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n+1)(n-1)\delta_{n+m,0} \quad \forall n, m \in \mathbb{Z}, \tag{1.55a}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12} n(n+1)(n-1)\delta_{n+m,0} \quad \forall n, m \in \mathbb{Z}, \tag{1.55b}$$

$$[L_n, \bar{L}_m] = 0. \tag{1.55c}$$

Correlation functions of primary fields

Vacuum expectation values of quasi-primaries are constrained by conformal symmetry. By exploiting the corresponding symmetry of correlation functions as well as the transformation properties of quasi -primary fields, it is possible to determine the structure of 2- and 3-point functions up to a constant.

Let us first take a look at the 2-point function which we will denote as $G^{(2)}(z_i, \bar{z}_i)$, and restrict ourselves to the holomorphic sector.

- The invariance under L_{-1} (translations) implies that $G^{(2)}(z_1, z_2) = G^{(2)}(z_1 - z_2)$,
- the invariance under L_0 (dilatations of the form $f(z) = \lambda z$) implies that $G^{(2)}(z_1, z_2) = \lambda^{h_1+h_2} G^{(2)}(\lambda(z_1 - z_2))$,
- The invariance under L_1 (special conformal transformations with $z \mapsto -z^{-1}$) implies that $G^{(2)}(z_1, z_2) = \frac{G^{(2)}\left(-z_1^{-1} + z_2^{-1}\right)}{z_1^{2h_1} z_2^{2h_2}}$ which leads to $h_1 = h_2$.

Treating the anti-holomorphic sector analogously, the $SL(2, \mathbb{C})$ subgroup imposes the v.e.v of the product of two primary fields to take the form

$$G^{(2)}(z_i, \bar{z}_i) = \langle \phi(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}} \text{ for } \left\{ \begin{array}{l} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{array} \right. , \tag{1.56}$$

where C_{12} is a constant.

Similarly, taking $z_{ij} = z_i - z_j$, the 3-point function yields

$$G^{(3)}(z_i, \bar{z}_i) = \frac{C_{123}}{z_{12}^{h_{123}} z_{23}^{h_{231}} z_{13}^{h_{312}} \bar{z}_{12}^{\bar{h}_{123}} \bar{z}_{23}^{\bar{h}_{231}} \bar{z}_{13}^{\bar{h}_{312}}} \text{ for } \left\{ \begin{array}{l} h_{ijk} = h_i + h_j - h_k \\ \bar{h}_{ijk} = \bar{h}_i + \bar{h}_j - \bar{h}_k \end{array} \right. . \tag{1.57}$$

Just like C_{12}, C_{123} is a constant. They are called *structure constants* of the theory.

The constraints imposed on n-point correlation functions can be understood as follows. Assume one wants to construct scalar conformal invariants from distinct points z_i. So long as the Poincaré invariance only is considered, *i.e* translations, rotations and boosts, the invariants are the distance $|z_{ij}| = |z_i - z_j|$. Adding scale invariance requires more constraints on the structure of the above invariants, as distances are not scale invariant. This is translated by rather considering a ratio of two distances, which clearly satisfies scale invariance. Then adding inversions means adding more structure to the previously improved invariants. The simplest objects that would satisfy conformal invariance are therefore

$$u_{ijkl} = \frac{z_{ij}z_{kl}}{z_{ik}z_{jl}}, \qquad \bar{u}_{ijkl} = \frac{\bar{z}_{ij}\bar{z}_{kl}}{\bar{z}_{ik}\bar{z}_{jl}}, \tag{1.58a}$$

and

$$v_{ijkl} = \frac{z_{ij}z_{kl}}{z_{il}z_{jk}}, \qquad \bar{v}_{ijkl} = \frac{\bar{z}_{ij}\bar{z}_{kl}}{\bar{z}_{il}\bar{z}_{jk}}, \tag{1.58b}$$

also known as *conformal ratios*. In order to have well-defined non-zero conformal ratios, one needs four distinct points. By deduction, the 2- and 3-point correlation functions must be determined up to a constant. As for 4- and higher point functions, they are determined up to an arbitrary function of conformal ratios. In the case of the 4-point functions, we have

$$G^{(4)} = f\left(u, \bar{u}, v, \bar{v}\right) \prod_{1\leq i\leq j\leq 4} z_{ij}^{\frac{h}{3}-h_i-h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j} \quad \text{with} \begin{cases} h = \sum_{i=1}^{4} h_i \\ \bar{h} = \sum_{i=1}^{4} \bar{h}_i \end{cases} \tag{1.59}$$

In general, there are $\frac{n(n-3)}{2}$ independent conformal ratios for the n-point function. In the particular case of the 4-point functions, it amounts to two independent conformal ratios.

Since the global symmetries employed are also present in higher dimensions, identical results are valid in CFTs of dimension $\mathbb{d} > 2$.

Conformal Ward identities

Ward identities are quantum manifestations of classical laws of conservation. Equations of classical symmetries such as $\partial_a j^a = 0$ do not hold at the quantum level in general. This is because in classical theory, $\partial_a j^a = 0$ only holds on shell. In the quantum case, the laws of conservation are formulated in terms of Ward identities, which express n-point functions by way of inserting a divergence of a conserved current in terms of $(n-1)$-point correlation functions.

An expression of the primary fields $\phi_1, \ldots, \phi_n$ can be obtained using the OPE defined in Eq. (1.51). The result reads

$$\langle T(w)\phi(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i}\right)\langle\phi(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle\,. \tag{1.60}$$

Equation (1.60) is the conformal Ward identity for primary fields.

Representation of the Virasoro algebra

Earlier on, it was mentioned that the holomorphic sector of the Virasoro algebra decouples from the anti-holomorphic one. We therefore consider only the holomorphic Virasoro representation, bearing in mind that an anti-holomorphic counterpart with identical formalism exists, and that the full representation can be recovered by tensoring the two sectors.

The spectrum of the theory, *i.e* the space of states, can be decomposed into the direct sum of an irreducible representation of the Virasoro algebra. A starting point in looking at the structure of the representation, is the Virasoro mode L_0. From the identification of the radial coordinate as the time in the radial quantization formalism, and in analogy with the classical Witt algebra generator l_0, the quantum counterpart L_0 can be considered as the chiral energy operator.

There exists a vector $|h\rangle$ that is an eigenstate of L_0, with smallest eigenvalue h, the conformal dimension of the vector. Then $L_n |h\rangle$ is also a L_0 eigenvector as seen in the following equation

$$L_0 L_n |h\rangle = L_n L_0 |h\rangle + [L_0, L_n] |h\rangle = (h - n) L_n |h\rangle . \tag{1.61}$$

Since h is the smallest eigenvalue, $n > 0$ implies that $L_n |h\rangle = 0$ (otherwise, we would have $h - n$ as the smallest eigenvalue). Hence, another definition of primary states, *any state that satisfies* $L_n |h\rangle = 0$ *is called a primary state*.

From Eq. (1.61), we also see that states created by the action of Virasoro modes $L_n |h\rangle$ with $n < 0$ can appear in the representation. They form the basis

$$\left\{ \prod_{i=1}^{k} L_{-n_i} |h\rangle \right\}_{0 < n_1 \leq \cdots \leq n_k} , \tag{1.62}$$

of a representation called a *Verma module*. Any state in that space is called a *descendant state*, and has level $N = \sum_{i=1}^{k} n_i$. The diagram below shows a basis of primary and descendent states up to the level 3.

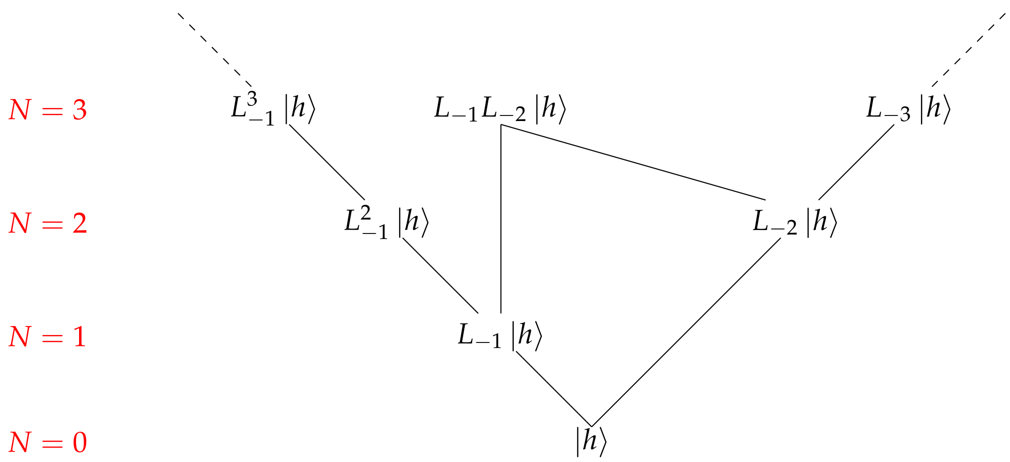


FIGURE 1.3: Graphical representation of a Verma module

Note that the state $L_{-2}L_{-1}$ is not included, as $L_{-2}L_{-1} = L_{-1}L_{-2} - L_{-3}$.

It is also possible to find descendant states that are also primary. Such states are called *null vectors* or *singular vectors*. To describe these vectors, we introduce the notion of inner product on the space of states. Consider two vector states $|\phi_1\rangle$ and $|\phi_2\rangle$ denoted as

$$|\phi_1\rangle = L_{-i_1} \cdots L_{-i_m} |h\rangle \tag{1.63a}$$
$$|\phi_2\rangle = L_{-j_1} \cdots L_{-j_n} |h\rangle . \tag{1.63b}$$

Then, their inner product is defined as

$$\langle\phi_1|\phi_2\rangle = \langle h| L_{i_m} \cdots L_{i_1} L_{-j_1} \cdots L_{-j_n} |h\rangle . \tag{1.64}$$

For specific values of the central charge c and the conformal weight h, it is possible to have

$$\langle\phi_1|\phi\rangle = 0 \qquad \forall\, |\phi\rangle \in \mathcal{V}_{c,h}, \tag{1.65}$$

$\mathcal{V}_{c,h}$ denoting the Verma module in which the vectors live. States that decouple from all other states of the Verma module are called *null vectors* or *singular vectors*. A Verma module containing null vectors is called *reducible* Verma module. Such a module can be rendered irreducible by removing all null vectors. Only then an irreducible representation of the Virasoro algebra is obtained.

Let us look for null vectors at the first two levels. Starting at level $N = 1$, a good question to ask is whether $L_{-1}|h\rangle$ is a null vector. It is easy to understand that for any $n \geq 2$, the states $L_n(L_{-1}|h\rangle)$ vanish as they have negative levels. Now, for $n = 1$

$$L_1 (L_{-1}|h\rangle) = [L_1, L_{-1}] |h\rangle = 2L_0 |h\rangle = 2h |h\rangle , \tag{1.66}$$

and we see that there is a null vector only if the conformal dimension of the primary state $|h\rangle$ is $h = 0$.

At level $N = 2$, the most general form of the descendant of a primary state is

$$|\phi\rangle = \left(aL_{-1}^2 + bL_{-2}\right)|h\rangle . \tag{1.67}$$

In analogy with the discussion for the case $N = 1$, we have $L_{n\geq 3}|\phi\rangle = 0$. Then, computing $L_1|\phi\rangle$ and $L_2|\phi\rangle$ yields

$$L_1|\phi\rangle = [(4h+2)a + 3b]\, L_{-1}|h\rangle \tag{1.68a}$$
$$L_2|\phi\rangle = \left[6ha + (4h + \frac{1}{2}c)b\right]|h\rangle , \tag{1.68b}$$

the constant c representing the central charge. Then, requiring that $L_1|\phi\rangle$ and $L_2|\phi\rangle$ vanish leads to a system of two linear equations for the two unknown (a, b) that reads

$$\begin{cases} (4h+2)a + 3b = 0 \\ 6ha + (4h + \frac{c}{2})b = 0 \end{cases} , \tag{1.69}$$

and whose determinant is

$$D_2(h) = \begin{vmatrix} 4h+2 & 3 \\ 6h & 4h+\frac{c}{2} \end{vmatrix} = 4(2h+1)^2 + (c-13)(2h+1) + 9. \tag{1.70}$$

Singular vectors at level $N = 2$ exist if $D_2(h) = 0$, or again if

$$h = \frac{1}{16}\left[5 - c \pm \sqrt{(c-1)(c-25)}\right]. \tag{1.71}$$

An important concept associated to null vectors is the one of *unitarity*. The inner product mentioned above naturally leads to the notion of *norm* of a state in a Verma module. The norm of a state $|\phi\rangle$ is defined as

$$\| \, |\phi\rangle \, \|^2 = \langle h| \, L_{i_k} \cdots L_{i_1} L_{-i_1} \cdots L_{-i_k} \, |h\rangle \, . \tag{1.72}$$

From Eq. (1.72), it is clear that null vectors have zero norm. However, besides the presence of zero-norm states in a Verma module, states of negative norm can also exist. A highest weight representation of the Virasoro algebra that contains no states of negative norm is called a *unitary representation*. The Virasoro algebra can be used to show that

$$\|L_{-n} \, |h\rangle \, \|^2 = \langle h| \, L_n L_{-n} \, |h\rangle = \left[2nh + \frac{1}{12}cn(n^2-1)\right] \langle h|h\rangle \, . \tag{1.73}$$

From there, it can be inferred that all representations with central charge $c < 0$ are non-unitary. The non-unitarity also holds for $n = 1$ and $h < 0$. A necessary condition for unitarity of representations is $\{c \geq 0, h \geq 0\}$.

Modular invariance and partition function

A generic upshot that applies to a large category of CFTs is that the crossing symmetry featured by the correlation functions together with the modular invariance of the partition function on the torus suffices to render those theories well defined on arbitrary Riemann surfaces. The consistent theory on the Riemann surface is then invariant under the *Fuchsian group* (a discret subgroup of $PSL(2, \mathbb{R})$) corresponding to the *genus g* of the Riemann surface.

The simplest Riemann surface is $\mathbb{C}$, or open subsets of $\mathbb{C}$. Previously, we considered theories on the simplest possible worldsheet, the cylinder, mapped by conformal transformation to the complex plane. This was part of the process of conformal compactification in order to have a consistent (radial) quantization of the theory.

There are two important examples of compactification of a Riemann surface. The first one is the Riemann sphere $S^2 \subset \mathbb{R}^3$, that can be conformally mapped to the extended complex plane $\mathbb{C} \cup \{\infty\}$. The second example of a compact manifold where all conformal transformations are defined as smooth non-singular maps enjoying a group structure is the torus. In this case, the theory defined on a torus is *de facto* invariant under the *modular group* $\Gamma = PSL(2, \mathbb{Z})$, a subgroup of the more general Fuchsian group, due to the periodic boundary conditions imposed.

A torus can be defined as the set

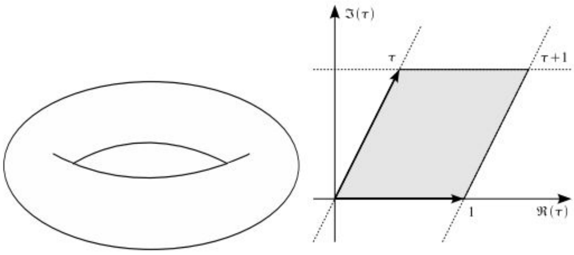
$$\mathbb{T} = \mathbb{C}/L = \{z|z \simeq z + n\lambda_1 + m\lambda_2\}, \tag{1.74}$$

with L as the torus lattice and λ_1 and λ_2 as two linear independent lattice vectors on the complex plane represented by two complex numbers, called the *periods* of the lattice. A torus is therefore a complex plane modulo a lattice. It can be constructed by identification of opposite sides of parallelograms such as the one formed by the points $0, \lambda_1, \lambda_1 + \lambda_2, \lambda_2$ in the complex plane.

Defining the ratio of lattice vectors on the upper half-plane $\mathbb{H}$ as

$$\tau = \frac{\lambda_1}{\lambda_2} \in \mathbb{H} \subset \mathbb{C}, \tag{1.75}$$

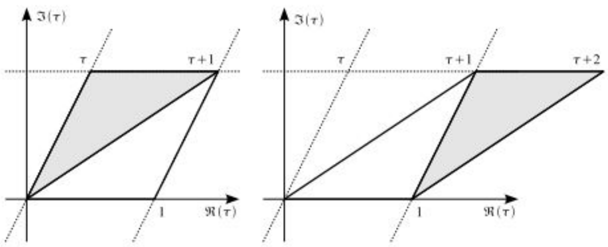
it turns out from the scale invariance of the theory that τ is the coordinate needed to distinguish between inequivalent tori. With this notation λ_1 and λ_2 simply become 1 and τ on the complex plane and the torus' identification on the plane is in terms of the parallelogram with vertices $0, 1, \tau + 1, \tau$, as illustrated in Fig 1.4.



(a) a torus $\mathbb{T}$ (b) the torus lattice L

FIGURE 1.4: The torus and its lattice [46]

A torus is invariant under two transformations of the lattice vectors. The first one is the map $\tau \mapsto \tau + 1$. Indeed, as illustrated in Fig. 1.5, the lattice vector 1 and $\tau + 1$ (Fig. 1.5 (b))is identical to the choice of lattice 1 and τ (Fig. 1.5 (a)).

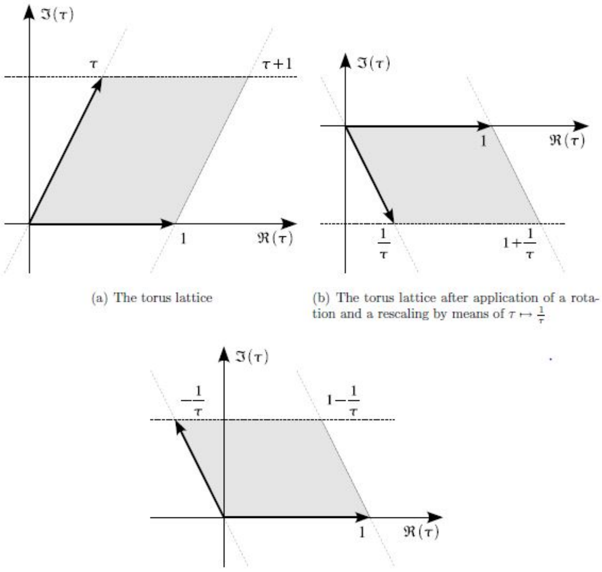


(a) The torus lattice (b) The torus lattice after application of $\tau \mapsto \tau + 1$

FIGURE 1.5: Modular transformation $\mathcal{T} : \tau \mapsto \tau + 1$ [46]

The other discrete symmetry is a bit more involved. It basically consists of first rotating vector τ into the real axis and doing a rescaling by the transformation $\tau \mapsto \frac{1}{\tau}$ (Fig. 1.6 (b)), and thereafter bringing the new τ vector in the upper half-plane by taking -$\frac{1}{\tau}$ (Fig. 1.6 (c)), as illustrated in Fig. 1.6.

The two symmetries generating the modular group Γ may be represented by 2×2 matrices acting on the column vector $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ as



(a) The torus lattice

(b) The torus lattice after application of a rotation and a rescaling by means of $\tau \mapsto \frac{1}{\tau}$

(c) Choosing a new lattice vector $\tau' = -\frac{1}{\tau}$ in the upper half-plane

FIGURE 1.6: Modular transformation $\mathcal{S} : \tau \mapsto -\frac{1}{\tau}$ [46]

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{1.76}$$

and

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.77}$$

which respectively describe a translation and a reflection. In turn, the general modular transformation can be expressed as

$$\mathcal{M} : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \tag{1.78}$$

or again in matrix form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.79}$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ (which guarantees that M has an integer inverse, *i.e* preserves area), and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim - \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.80}$$

as an overall sign in the numerator and denominator cancels. Hence, $M \in PSL(2, \mathbb{Z})$.

Any change on a Riemann surface due to the action of the modular group leaves the data coming from correlation functions invariant up to some multipliers that can be computed. This property of the n-point correlation functions is called *modular covariance*. The zero-point functions, which are the characters of the theory are also the building blocks of the partition function of the theory. They form representation spaces of the modular group. When the representation is finite-dimensional, the theory is called *rational*. In that case, the modular transformation properties of the zero-point function are expressed as

$$\chi_h\left(-\frac{1}{\tau}\right) = \sum_{h'} S_h^{h'} \chi_{h'}(\tau), \tag{1.81}$$

and

$$\chi_h\,(\,\tau+1) = \sum_{h'} T_h^{h'} \chi_{h'}(\tau), \tag{1.82}$$

which are sums over finitely countable values of weights h'.

An advantage of choosing the torus as our Riemann surface for the theory is that, the partition function is independent of the choice of lattice vector λ_1 and λ_2, *i.e* it is *modular invariant*. The partition function can generally be defined as

$$Z = Tre^{-\beta H}, \tag{1.83}$$

where H stands for the Hamiltonian operator, represented by the Virasoro zero-modes $L_0 + \bar{L}_0$ in a CFT. It is important to mention that the torus is actually twisted before being glued together, since in general the real part $\mathcal{R}(\tau) \neq 0$. Also, space and time directions are usually represented on the real and imaginary axes, respectively. The translation operator of the model along the lattice vector λ_2 over a distance d is then

$$e^{-\frac{d}{|\lambda_2|}H\mathcal{I}(\lambda_2) - iP\mathcal{R}(\lambda_2)}. \tag{1.84}$$

Considering a cylinder of circumference D, we can write the generators along time and space respectively

$$H = \frac{2\pi}{D}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \tag{1.85}$$

and

$$P = \frac{2\pi}{D}\left(L_0 - \bar{L}_0\right). \tag{1.86}$$

Identifying λ_1 to D, and rescaling the lattice by a factor of 2π, the partition function becomes

$$\begin{aligned} Z &= Tre^{\pi i\left[(\tau-\bar{\tau})\left(L_0+\bar{L}_0-\frac{c}{12}\right)+(\tau+\bar{\tau})(L_0-\bar{L}_0)\right]} \\ &= Tre^{2\pi i\left[\tau\left(L_0-\frac{c}{24}\right)-\bar{\tau}\left(L_0-\frac{c}{24}\right)\right]}. \end{aligned} \tag{1.87}$$

Then, using the expressions $q = e^{2\pi i\tau}$ and $\bar{q} = e^{2\pi i\bar{\tau}}$, the partition function takes the more conventional form

$$Z(\tau) = Tr\left(q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{c}{24}}\right). \tag{1.88}$$

Finally, due to conformal invariance the torus partition function can be expressed in terms of bilinear combinations of characters of the representation of Virasoro algebras as

$$Z(\tau) = \sum_{h,\bar{h}} \chi_h(q) \mathcal{N}_{h,\bar{h}} \bar{\chi}_{\bar{h}}(\bar{q}) \tag{1.89}$$

with respective holomorphic and antiholomorphic highest weight h and $\bar{h}$, and $\mathcal{N}_{h,\bar{h}}$ as integer coefficients.

Minimal models

We succinctly discuss a special subclass of rational CFTs, the so-called *minimal models*. Introduced by Belavin *et al.* in [1], these are models with central charge parametrized by two coprime integers $p > q$ as

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq}. \tag{1.90}$$

An interesting feature of minimal models is that only a finite number of primary fields (with infinitely many descendants) appears in the theory. Furthermore, for the given central charge, the allowed values that the conformal weights of the primary fields can take are given by

$$h_{r,s} = \frac{(rq - sp)^2 - (p-q)^2}{4pq}, \tag{1.91}$$

where $1 \le r \le p-1$ and $1 \le s \le q-1$. They are determined by the *Kac determinant*

$$detM^l = \alpha_l \prod_{1 \le r,s \le l} (h - h_{r,s}(c))^{p(l-rs)} = 0, \tag{1.92}$$

where $p(l-rs)$ is the number of partitions of the positive integer $l - rs$, and α_l is a positive constant depending on h. Organizing the conformal weights $h_{r,s}$ on a grid, we get what is known as the *Kac table*, which is symmetric in $r \mapsto q - r$ and $s \mapsto p - s$. The special case $p = q + 1$ describes a subset of the minimal model which is unitary, *i.e* which do not contain any negative norm. In that case

$$c_q = 1 - \frac{6}{q(q+1)}, \qquad q = 3, 4, 5, \ldots \tag{1.93}$$

such models are also called *unitary discrete models*.

Minimal models are very important because they describe statistical models at their critical points. A famous example is the one for which $q = 3$, and the corresponding central charge is $c_{4,3} = \frac{1}{2}$. This is the so-called *Ising model*, the non-trivial minimal model with smallest Kac table

$\mathbb{1}_{h=0}$	$\sigma_{h=\frac{1}{16}}$	$\epsilon_{h=\frac{1}{2}}$
$\epsilon_{h=\frac{1}{2}}$	$\sigma_{h=\frac{1}{16}}$	$\mathbb{1}_{h=0}$

where σ represents the lattice spin, and ϵ the interaction between two nearest neighboring spins.

1.2 Logarithmic Conformal field theory

The presence of logarithmic divergences in the correlation functions of some systems in two dimensions was noticed a few years after the discovery of conformal field theories [9]. Subsequently, several aspects of logarithmic conformal field theory were observed in the literature [10, 11, 12], but it is really about a quarter century ago that logarithmic conformal field theory established itself, through the work of Gurarie [13].

In the next subsections, we first discuss the main features of LCFTs, and then give two specific examples: the celebrated $c = -2$ model, and the $c = 0$ model which is the model of interest for us.

1.2.1 Indecomposable structure and Jordan cells

The concept of LCFT introduced in [13] was based on the indecomposable representations that are present in the fusion of primary operators. In ordinary CFTs, the primary operators forming irreducible representations of the Virasoro algebra and their descendants make a complete set of operators, such that all others are expressible in terms of linear combinations of them. However, in the case of LCFTs the computation of correlation functions reveals that in order to a have complete set of operators, a new family of operators must be added. This induces the "logarithmic" properties of LCFTs, featuring the appearance of a non-diagonal action and a Jordan cell structure.

Considering two operators $A(z)$ and $B(z)$ of same conformal weight h, a non-diagonal action of the Hamiltonian L_0 can occur, generating a Jordan cell as follows

$$L_0 \left| A \right\rangle = h \left| A \right\rangle \tag{1.94a}$$
$$L_0 \left| B \right\rangle = h \left| B \right\rangle + \left| A \right\rangle \tag{1.94b}$$

or equivalently

$$L_0 \begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix} \begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} . \tag{1.95}$$

Eq. (1.95) is an example of a non-diagonalizable Jordan cell. Such a structure can arise in any arbitrary size, but for the sake of simplicity, we restrict ourselves to the two by two cell as in Eq. (1.95). An n by n cell is also said to be of *rank* $n-1$.

As we can see from the first line of Eqs. (1.94), $|A\rangle$ resembles a conventional primary state. However, from the second line, the action of L_0 on $|B\rangle$ is slightly different as it includes $|A\rangle$. For reasons that will be made clear below, $|B\rangle$ was coined the *logarithmic partner* of $|A\rangle$. As a result of Eqs. (1.94), the Hamiltonian is not hermitian since $L_0^\dagger \neq L_0$. LCFTs are therefore non-unitary.

1.2.2 Logarithmic correlators

The two-point correlation functions of operators $A(z)$ and $B(z)$ can be calculated using conformal invariance. Under an infinitesimal conformal transformation given by $z \mapsto z + \epsilon(z)$, the primary operator $|A(z)\rangle$ transforms as

$$\delta A(z) = \epsilon(z)\frac{\partial A(z)}{\partial z} + h\frac{\partial \epsilon(z)}{\partial z}A(z). \tag{1.96}$$

The operator $|B(z)\rangle$ transforms as

$$\delta B(z) = \epsilon(z)\frac{\partial B(z)}{\partial z} + \frac{\partial \epsilon(z)}{\partial z}\left(hB(z) + A(z)\right). \tag{1.97}$$

We recall the general result of ordinary CFT which is that the constraints imposed on the two point correlation function of a primary operator $\mathcal{O}(z)$ by translation invariance (with $\epsilon(z) = \epsilon =$ constant), dilatation (with with $\epsilon(z) = \epsilon z$) and SCT (with with $\epsilon(z) = \epsilon z^2$) fix the structure of the correlation function up to a constant K as

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle = \frac{K}{(z_1 - z_2)^{2h}}. \tag{1.98}$$

In turn, when applied to the primary operator $A(z)$, the differential equations satisfying the above correlation function can be expressed as [47]

$$\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)\langle A(z_1)A(z_2)\rangle = 0 \qquad \text{(translation)}, \tag{1.99a}$$

$$\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + 2h\right)\langle A(z_1)A(z_2)\rangle = 0 \qquad \text{(dilatation)}, \tag{1.99b}$$

$$\left(z_1^2\frac{\partial}{\partial z_1} + z_2^2\frac{\partial}{\partial z_2} + 2h\left(z_1 + z_2\right)\right)\langle A(z_1)A(z_2)\rangle = 0 \qquad \text{(SCT)}. \tag{1.99c}$$

When it comes to the operator $B(z)$, the differential equations satisfying the 2-point correlation function are more involved, as we see now translation:

$$\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\langle A(z_1)B(z_2)\rangle = 0, \tag{1.100a}$$

$$\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\langle B(z_1)A(z_2)\rangle = 0, \tag{1.100b}$$

$$\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\langle B(z_1)B(z_2)\rangle = 0, \tag{1.100c}$$

dilatation :

$$\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+2h\right)\langle A(z_1)B(z_2)\rangle + \langle A(z_1)A(z_2)\rangle = 0, \tag{1.100d}$$

$$\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+2h\right)\langle B(z_1)A(z_2)\rangle + \langle A(z_1)A(z_2)\rangle = 0, \tag{1.100e}$$

$$\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+2h\right)\langle B(z_1)B(z_2)\rangle + \langle A(z_1)B(z_2)\rangle + \langle B(z_1)A(z_2)\rangle = 0, \tag{1.100f}$$

SCT:

$$\left(z_1^2\frac{\partial}{\partial z_1}+z_2^2\frac{\partial}{\partial z_2}+2h\left(z_1+z_2\right)\right)\langle A(z_1)B(z_2)\rangle + 2z_2\langle A(z_1)A(z_2)\rangle = 0, \tag{1.100g}$$

$$\left(z_1^2\frac{\partial}{\partial z_1}+z_2^2\frac{\partial}{\partial z_2}+2h\left(z_1+z_2\right)\right)\langle B(z_1)A(z_2)\rangle + 2z_1\langle A(z_1)A(z_2)\rangle = 0, \tag{1.100h}$$

$$\left(z_1^2\frac{\partial}{\partial z_1}+z_2^2\frac{\partial}{\partial z_2}+2h\left(z_1+z_2\right)\right)\langle B(z_1)B(z_2)\rangle + 2z_1\langle A(z_1)B(z_2)\rangle + 2z_2\langle B(z_1)A(z_2)\rangle = 0. \tag{1.100i}$$

These differential equations are uniquely solved for

$$\langle A(z_1)A(z_2)\rangle = 0, \tag{1.101a}$$

$$\langle A(z_1)B(z_2)\rangle = \langle B(z_1)A(z_2)\rangle = \frac{K}{\left(z_1-z_2\right)^{2h}}, \tag{1.101b}$$

$$\langle B(z_1)B(z_2)\rangle = -2K\frac{\ln\left(z_1-z_2\right)}{\left(z_1-z_2\right)^{2h}}. \tag{1.101c}$$

Eqs. (1.101) justifies the appellation "logarithmic partner" for the operator $B(z)$.

Another way from which the logarithmic feature of the theory can be seen is through consideration of the four-point function of a primary operator $\mathcal{O}(z)$. From ordinary CFT, we recall that such a primary operator with conformal dimension h has a four-point function that looks like

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\rangle = \frac{1}{\left(z_1-z_3\right)^{2h}\left(z_2-z_4\right)^{2h}}F(x), \tag{1.102}$$

with x as

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \tag{1.103}$$

Then, if $\mathcal{O}(z)$ is in the Kac table, the function $F(x)$ is known to satisfy a differential equation whose solutions are singular for $x = \{0, 1, \infty\}$, and at those points, $F(x)$ behaves as a power law. In the case $x \to 0$ for instance, $F(x) \sim x^\alpha$ and it can be shown that the OPE of $\mathcal{O}(z)$ with itself looks like

$$\mathcal{O}(z)\mathcal{O}(0) \sim z^\alpha C(0) + \cdots , \tag{1.104}$$

with $C(0)$ as a primary operator with dimension $\delta = \alpha + 2h$.

However, in some cases, the differential equations may exhibit logarithmic singularities rather than the power laws at the specific points. Taking again the case $x \to 0$, this leads to $F(x) \sim x^\alpha \ln x + \cdots$, which means

$$\mathcal{O}(z)\mathcal{O}(0) \sim z^\alpha \left[A(0) \ln z + B(0)\right] + \cdots . \tag{1.105}$$

Just as before, $B(z)$ is the logarithmic partner of $A(z)$.

Next we turn our attention to the logarithmic generalization of minimal models.

1.2.3 Logarithmic minimal models

In this subsection, we give a brief review of logarithmic extensions to minimal models previously discussed.

Logarithmic minimal models originate from the work of Kausch [48] who noticed that it was possible to extend the Virasoro algebra by a multiplet of fields at certain values of the central charge.These models can roughly be classified in two categories. The first category consists of models in which the Virasoro algebra is extended only by a series of singlet solutions. Under the notation $\mathcal{LM}_{p',p}$, these models have been studied from a lattice perspective in [49, 17, 23, 50, 51, 52]. Interesting works can also be found in [53, 24, 54, 55]. The second category, denoted as $\mathcal{WLM}_{p',p}$ has been studied more extensively than the former. In particular, $\mathcal{WLM}_{p',p}$ models are rational not with respect to the Virasoro algebra, but with respect to an enlarged symmetry algebra called the W-algebra. As lattice integrable models, they have been studied in [56, 25, 57, 58, 59, 60, 61, 62, 63]. From a mathematical physics perspective, these models were studied in [64, 65, 66, 67].

Virasoro representations

Logarithmic minimal models are defined for sets of coprime integers p, p' such that the central charge of these theories is

$$c = 1 - 6\frac{(p' - p)^2}{pp'}. \tag{1.106}$$

Depending on (groups of) authors, either $p < p'$ or $p' < p$. Here we use the convention $p < p'$.

These models involve an infinite number of Virasoro representations which close under fusion, with an infinitely extended Kac table. In that setting, the conformal weights are given by

$$h_{r,s} = \frac{(rp' - sp)^2 - (p' - p)}{4p'p}, \qquad r,s \in \mathbb{N}. \tag{1.107}$$

The Kac table admits a $\mathbb{Z}_2$ symmetry such that $h_{r,s} = h_{p-r,p'-s}$.

Although logarithmic minimal models admit an infinite number of representations in the Virasoro algebra, these representations can be rearranged into a finite number of representations of a larger algebra, the *W algebra*.

W- irreducible representations

Under the extended $W(p, p')$ symmetry, the infinite number of Virasoro representations are reorganized into a finite number of W indecomposable representations that close under fusion.

The $W_{p,p'}$ algebra is generated by the stress tensor $T(z)$ and two Virasoro primaries $W^+(z)$ and $W^-(z)$ of conformal dimension $(2p-1)(2p'-1)$. This yields $2pp' + \frac{1}{2}(p-1)(p'-1)$ W-irreducible representations, where the $\frac{1}{2}(p-1)(p'-1)$ contribution corresponds to the representations of rational minimal models.

Because of the fixed number of representations generated in the W-logarithmic minimal models ($\mathcal{WLM}$), they are called *rational* LCFTs, in contrast with the logarithmic minimal models called *irrational* LCFTs. Next we proceed with the famous example of the $c = -2$ model.

An example: the $c = -2$ model

The $c = -2$ model is the simplest of the series of logarithmic minimal models, with $(p', p) = (1, 2)$. We follow the paper by Gurarie, as it is a key example of where an operator was used to show that logarithmic terms cannot be avoided in the system.

Gurarie computed the four-point function of an operator denoted by μ, of conformal dimension $h = -\frac{1}{8}$, which takes the form

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1 - z_3)^{\frac{1}{4}}(z_2 - z_4)^{\frac{1}{4}}\left[x(1-x)\right]^{\frac{1}{4}} F(x). \tag{1.108}$$

$F(x)$ is a holomorphic function of the anharmonic ratio x, that is determined using the null vector condition

$$\left(L_{-2} - 2L_{-1}^2\right)|\mu\rangle = 0, \tag{1.109}$$

which in conjunction the definition of mode expansion and normal ordering implies that $F(x)$ satisfies a second order differential equation given as

$$x(1-x)\frac{d^2F(x)}{dx^2} + (1-2x)\frac{dF(x)}{dx} - \frac{1}{4}F(x) = 0. \tag{1.110}$$

The differential equation (1.110) has two independent solutions

$$F(x) = k_1 J(x) + k_2 J(1-x), \tag{1.111}$$

where k_1 and k_2 are constants, $J(x)$ is the hypergeometric function $F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right)$ and $J(1-x)$ is the hypergeometric function $F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-x\right)$. An interesting feature of these hypergeometric functions is that while $J(x)|_{x=0} = 1$ and it can be expanded about $x = 0$ as a standard Taylor series, $J(1-x)$ behaves as $\ln x$ as $x \to 0$ [68]. The logarithmic singularity sitting in the four-point function can therefore be made more visible by rewriting the second solution as

$$J(1-x) = J(x)\ln x + H(x), \tag{1.112}$$

where, $H(x)$ is a regular function in the vicinity of $x = 0$ [68].

1.2.4 $c = 0$ LCFTs

Over the past decades, the interest on theories with vanishing central charge has considerably grown due to the fact that they play a prominent role in the understanding of statistical mechanics and condensed matter physics models. Applications of such models include the statistical properties of critical geometrical models like self-avoiding walk (polymers) or percolation [11], and the critical properties of non-interacting $2+1$ dimensional (quenched) disordered electronic systems [28, 69]. Other applications are in the area of high energy physics with for instance the description of super-symmetric sigma models beyond the topological sector [70], or with the AdS/CFT correspondence which will occupy us from the next chapter.

LCFTs with vanishing central charges were first encountered in the work of Rozansky and Saleur [12], who studied a particular type of $c = 0$ theory with $U(1,1)$ supergroup symmetry. In their study, they came across the same form of logarithmic dependent four-point function as the ones discussed above. Their results and conclusions were confirmed by Gurarie [13], who later on with Ludwig in [27] and [29] formalized the $c = 0$ theory, through a fundamental requirement that the theory must possess a field other than the energy-momentum tensor $T(z)$, whose holomorphic part denoted by $t(z)$ has conformal weight 2. Furthermore, the requirement is such that the action of the Virasoro mode L_2 on this operator should give a non vanishing constant. The argument given by Gurarie to explain the statement is that a primary operator with non-zero two point function in any $c \neq 0$ CFT should have an OPE with itself that looks like

$$\mathcal{O}(z)\mathcal{O}(0) = \frac{1}{z^{2h}}\left(1 + \frac{2h}{c} z^2 T(0) + \cdots\right). \tag{1.113}$$

Then, taking the limit $c \to 0$ in Eq. (1.113) would lead to a divergence of the term $\frac{2h}{c}$ also called the *c=0 catastrophe*.

The origin of the term $\frac{2h}{c}$ can first be recalled from the fact that applying L_2 on both sides of Eq. (1.113) gives on the right side

$$\frac{2h}{c} z^{2-2h} L_2 T(0) = h z^{2-2h}, \tag{1.114}$$

since $L_2 T(0) = \frac{c}{2}$, while using the standard CFT result

$$[L_n, \mathcal{O}(z)] = z^{n+1}\frac{\partial \mathcal{O}(z)}{\partial z} + h(n+1)z^n \mathcal{O}(z), \tag{1.115}$$

and applying it with L_2 on the left side of Eq. (1.113) gives

$$\begin{aligned} L_2 \mathcal{O}(z)\mathcal{O}(0) &= [L_2, \mathcal{O}(z)]\, \mathcal{O}(0) && \text{(1.116a)} \\ &= \left[z^3 \frac{\partial}{\partial z} + 3z^2 h\right] \mathcal{O}(z)\mathcal{O}(0) && \text{(1.116b)} \\ &\simeq \left[z^3 \frac{\partial}{\partial z} + 3z^2 h\right] \frac{1}{z^{2h}} = hz^{2-2h}. && \text{(1.116c)} \end{aligned}$$

Hence the only way for both sides to agree in the same answer is for the specific choice of the coefficient $\frac{2h}{c}$. Then, to resolve the divergence conundrum, Gurarie and Ludwig observed that the only way to make Eqs. (1.114) and (1.116) compatible at $c = 0$ is to assume the existence of another dimension 2 operator $t(z)$ such that $L_2 t(0) = b$. The non-zero coeficient b now featuring in the following new OPE expression

$$\mathcal{O}(z)\mathcal{O}(0) \simeq \frac{1}{z^{2h}}\left(1 + \frac{h}{b} z^2 t(0) + \cdots\right), \tag{1.117}$$

plays an important physical role. The operator $t(z)$ also satisfies

$$L_n t(0) = 0, \qquad \forall n > 2, n = 1. \tag{1.118}$$

As an example, one could consider the direct product of two CFTs with central charges $c_1 = c$ and $c_2 = -c$, so that the total central charge is zero. Having the operator $t(z)$ as a quasiprimary field, *i.e* $L_2 t(0) = b$ and $L_0 t(0) = 2t(0)$, one could argue that $T = T_1 + T_2$ and $t = T_1 - T_2$ and construct the OPE of the energy-momentum tensor $T(z)$ with the operator t. This gives

$$T(z)t(w) = \frac{b}{(z-w)^4} + \frac{2t(w)}{(z-w)^2} + \frac{t'(w)}{z-w} + \cdots, \tag{1.119}$$

with $b = c$.

Alternatively, $t(z)$ could be thought of as the logarithmic partner of the energy-momentum tensor $T(z)$. This eventuality could arise by considering the fact that for $c = 0$, $T(z)$ has a vanishing norm, *i.e* $\langle T(z)T(w)\rangle = 0$. This is consistent with the set of equations (1.101), and leads to the more general OPE expression

$$T(z)t(w) = \frac{b}{(z-w)^4} + \frac{2t(w) + T(w)}{(z-w)^2} + \frac{t'(w)}{z-w} + \cdots \tag{1.120}$$

As a result, applying Eq. (1.101) gives

$$\langle T(z)T(w)\rangle = 0, \tag{1.121a}$$

$$\langle T(z)t(w)\rangle = \langle t(z)T(w)\rangle = \frac{b}{(z-w)^4}, \tag{1.121b}$$

$$\langle t(z)t(w)\rangle = -2b\frac{\ln(z-w)}{(z-w)^4}, \tag{1.121c}$$

and shows how the logarithmic singularity arises in the two-point correlation function.

As we have seen in this chapter, in contrast with its ordinary counterpart, logarithmic conformal field theory at $c = 0$ turns out to be nontrivial due to the appearance of t, the logarithmic partner of the energy momentum tensor T. Many issues related to these logarithmic operators still need to be addressed in order to have a better understanding of vanishing central charge LCFTs, and their various applications in condensed matter physics and string theory.

After reviewing the general features of CFT and LCFT, we will now take a look at certain three dimensional gravity theories that have been proposed as possible holographic duals of $c = 0$ logarithmic conformal field theories.

Chapter 2

Gravity in three dimensions and holography

The search for a consistent theory of gravity had been going on for a long time now. Such a theory presents formidable challenges, technically and conceptually. On one hand, at the technical level for instance, the non renormalizability of general relativity as a perturbative quantum field theory has for a long time been a hindrance for progress in the field. On the other hand, the gauge invariance of observables in a theory that is diffeomorphism invariant like general relativity has conceptually been a difficult problem to solve.

The easiest way out when studying a particularly complicated system is to turn to a toy model with similar conceptual features. Gravity in three dimensions is one such toy model that has enabled theoretical physicists to investigate many aspects of gravity that would otherwise be very difficult to fathom. For that reason, the study of three dimensional gravity is originally motivated by the desire to understand its four dimensional analog. Indeed, while the dynamics of $3\mathbb{d}$ gravity is relatively simpler than the $4\mathbb{d}$ one in the sense that it has no propagating degrees of freedom and the gauge constraints can be solved exactly, it still retains many properties of the higher dimensional theories that to this day remain poorly understood.

An example of a not very well understood phenomenon in $\mathbb{d} = 4$ gravity is the black hole thermodynamics. Interestingly enough, $3\mathbb{d}$ gravity admits black hole solutions [71] with properties similar to the ones of $4\mathbb{d}$ gravity, illustrated for instance by the entropy that obeys the Bekenstein-Hawking area law. In particular, the black hole solutions that $3\mathbb{d}$ gravity admits requires the theory to have a negative cosmological constant [72]. This renders $3\mathbb{d}$ gravity even more interesting in the context of string theory, as it allows to make use of the AdS/CFT correspondence [30].

Many questions about quantum gravity can be addressed in terms of their well understood $2\mathbb{d}$ conformal field theory duals through the celebrated $\mathrm{AdS}_{\mathbb{d}+\mathbb{1}}/\mathrm{CFT}_{\mathbb{d}}$ correspondence. However, in the specific dimensional case of $\mathrm{AdS}_3/\mathrm{CFT}_2$, it is acknowledged to have been discovered long before the work of of Maldacena in [30]. Indeed, Brown and Henneaux showed in [73] that the symmetry algebra of asymptotically AdS_3 spaces is generated by two copies of the Virasoro algebra with non-zero central charges c_L, c_R, which as previously discussed is known as the algebra of local conformal transformations in two dimensions.

In this chapter, after a quick introduction to gravity in $2 + 1$ dimensions, we give a brief review of the precursor of the AdS/CFT correspondence in the work of Brown and Henneaux. Subsequently, we go through the basics of $\mathrm{AdS}_3/\mathrm{CFT}_2$ before discussing the logarithmic $\mathrm{AdS}_3/\mathrm{LCFT}_2$ correspondence which motivates the work done in this thesis.

2.1 Gravity in $2+1$ dimensions

Pure gravity in $2+1$ dimensions is defined by the Einstein-Hilbert action [74]

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}(R-2\Lambda)+C, \tag{2.1}$$

with G as the 3$\mathbb{d}$ Newton constant, $g \equiv \det g_{ab}(a,b=0,1,2)$ with a metric g_{ab} of signature $(-,+,+)$, $R \equiv R_{ab}g^{ab}$ as the curvature scalar, R_{ab} as the Ricci tensor, and the natural units setting $\hbar = c = 1$. $\mathcal{M}$ is a manifold in 3 dimensions, and Λ is the cosmological constant that depending on whether its value is positive, negative or null, describes locally de Sitter (dS), Anti-de-Sitter (AdS) or flat spacetimes. In the case where the spacetime is Anti-de-Sitter, the cosmological constant is expressed in terms of the AdS radius l as $\Lambda = -1/l^2$. C represents a boundary term that enables the action to have a well behaved action principle.

The (Einstein) equations of motion for the action (2.1) are

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 0, \tag{2.2}$$

which are diffeomorphism covariant.

A particular aspect of the solutions of the vacuum ($T_{ab}=0$) Einstein equations in 3$\mathbb{d}$ is that they are locally de Sitter (dS), Anti-de-Sitter (AdS) or flat depending on the sign of Λ as specified above. This can be checked by realizing that the Riemann curvature tensor is totally determined by the Ricci tensor. Indeed, the number of independent component of Ricci and Riemann tensors in $\mathbb{d}$ dimensions (for $\mathbb{d}>2$) are

$$\frac{\mathbb{d}(\mathbb{d}+1)}{2}, \tag{2.3}$$

and

$$\frac{\mathbb{d}(\mathbb{d}-1)}{4}\left(\frac{\mathbb{d}(\mathbb{d}-1)}{2}+1\right), \tag{2.4}$$

respectively. In $2+1$ dimensions, this means that both tensors have 6 independent components. Hence, using the symmetries of the Riemann tensor, the full curvature tensor can be expressed in terms of Ricci tensor given by

$$R_{abcd} = g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc} - \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{2.5}$$

As a result, any solution of equations of motion (2.2) has constant curvature. Physically this means that $2+1$ dimensional Einstein spacetime does not have local propagating degrees of freedom, and that there are no gravitational waves in the theory.

At this stage, a fundamental question is how can a toy model without degrees of freedom be used to study gravity theories in higher dimension? In the case of $\Lambda < 0$ it turns out that $2+1$ dimensional gravity is not "empty", but admits black hole solutions as shown by Bañados, Teitelboim and Zanelli in [71]. The BTZ black hole of mass M and angular momentum J is described, in Schwarzschild coordinates, by the metric

$$ds^2 = -\left(N(r)\right)^2 dt^2 + \left(N(r)\right)^{-2} dr^2 + r^2 \left(d\phi + N^{\phi}(r)dt\right)^2, \tag{2.6}$$

where the lapse and shift functions are expressed as

$$N(r) \equiv \sqrt{-8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}}, \qquad N^{\phi}(r) = -\frac{4GJ}{r^2}, \tag{2.7}$$

with $-\infty < t < +\infty$, $0 < r < +\infty$ and $0 < \phi < 2\pi$.

The BTZ solution (2.6) ensures that every point of the black hole has a neighborhood isometric to AdS$_3$.

2.2 Asymptotic symmetries in AdS$_3$ spacetime

In this section we discuss asymptotically AdS spacetimes in the spirit of Brown/Henneaux. For that, one needs to consider a set of metrics that tend to the AdS$_3$ metric. This amounts to a prescription of fall-off conditions on the metric component at large distances, that is equivalent to imposing *boundary conditions*.

First consider a 3d manifold $\mathcal{M}$ with the topology of a cylinder, parametrized by the time coordinate $x^0 \equiv \tau$ and the 2d space manifold parametrized by the coordinates r and $x^1 \equiv \varphi$ such that $\varphi \sim \varphi + 2\pi$. Then, introducing light cone coordinates $x^{\pm} \equiv \tau \pm \varphi$ with $\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\varphi)$, the boundary of the spacetime at (space) infinity ($r = \infty$) is the cylinder of coordinates τ, φ as illustrated in Fig. 2.1.

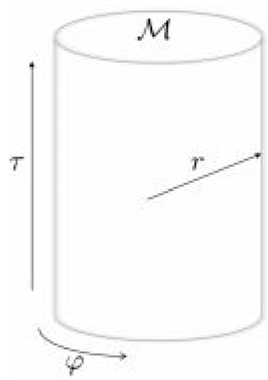


FIGURE 2.1: Manifold $\mathcal{M}$ with the topology of a cylinder [75]

Now that the boundary conditions of the spacetime have been specified, one can look at the boundary conditions.

2.2.1 Boundary conditions

Using Fefferman-Graham coordinate system [76] with the following metric

$$ds^2 = \frac{l^2}{r^2}dr^2 + \gamma_{ij}(r, x^k)dx^i dx^j, \tag{2.8}$$

where $i = 0, 1$ and

$$\gamma_{ij} = r^2 g^{(0)}_{ij}(x^k) + \mathcal{O}(1) \qquad \text{for} \quad r \to \infty, \tag{2.9}$$

asymptotically AdS_3 can be defined in terms of metrics such as Eq. (2.8) for which the boundary metric $g^{(0)}_{ij}$ is

$$g^{(0)}_{ij} = -dx^+ dx^-. \tag{2.10}$$

These are called the Brown-Henneaux boundary conditions.

As was shown in [77], the most general solution of the Einstein equations with $\Lambda = -1/l^2$ and boundary conditions (2.8) and (2.10) is

$$ds^2 = \frac{l^2}{r^2} dr^2 - \left(r dx^+ - \frac{l^2}{r^2} L(x^-) dx^- \right) \left(r dx^- - \frac{l^2}{r^2} \bar{L}(x^+) dx^+ \right), \tag{2.11}$$

with $L(x^-)$ and $\bar{L}(x^+)$ as two single-valued functions of the light-cone coordinates.

An important property of the 3d gravity action (2.1) is that it can be expressed in terms of ordinary gauge fields [78, 79]. This property holds for any sign of the cosmological constant and simplifies considerably the structure of the action and of the equations of motion. More specifically, three-dimensional gravity is equivalent to a Chern-Simons gauge theory. This being noted, it is possible to translate the Brown-Henneaux boundary conditions in terms of the Chern-Simons formalism.

We start by a discussion on *vielbeins* and *spin connections*. Within the *first-order*, or Palatini framework of general relativity, the usual metric $g_{\mu\nu}$ can be expressed in terms of an object called vielbein and denoted by e^a_μ as

$$g_{\mu\nu}(x) = e^a_\mu(x) \eta_{ab} e^b_\nu(x), \tag{2.12}$$

where a can take the values $\{0, 1, 2\}$, and η_{ab} is the metric of flat 3D Minkowski spacetime. Eq. (2.12) can be interpreted as a tensor transformation under change of coordinates described by a non-singular matrix e^a_μ, with inverse $e^\mu_a(x)$ such that

$$e^a_\mu e^\mu_b = \delta^a_b, \qquad \text{and} \qquad e^\mu_a e^a_\nu = \delta^\mu_\nu. \tag{2.13}$$

The vielbein can be used to define a basis in the space of differential forms. From the one-form $e^a \equiv e^a_\mu dx^\mu$ and the Levi-Civita components ϵ_{abc}, one obtains

$$\epsilon_{\mu\nu\rho} \equiv e^{-1} \epsilon_{abc} \epsilon^a_\mu \epsilon^b_\nu \epsilon^c_\rho, \tag{2.14a}$$
$$\epsilon^{\mu\nu\rho} \equiv e \epsilon^{abc} \epsilon^\mu_a \epsilon^\nu_b \epsilon^\rho_c, \tag{2.14b}$$

where $e = det(e^a_\mu)$. A covariant derivative is often expressed as $D = \partial + \Gamma$, where the first term on the right-hand side stands for a normal derivative, and the second term on the right-hand side is the affine connection. In the tetrad formalism, the one-forms $\omega^{ab} = \omega^{ab}_\mu dx^\mu$, with $\omega^{ab} = -\omega^{ba}$, play the role of the connections.

They are useful in constructing objects that transform as vectors under local Lorentz transformation. One such object is the *torsion* 2-form of the connection expressed as

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b, \tag{2.15}$$

that transforms as

$$T^a \to \Lambda^{-1a}{}_b T^b, \tag{2.16}$$

with $\Lambda \in SO(2,1)$ if the term $\omega^a{}_b$ whose components ω^{ab}_μ are the spin connections, transforms as

$$\omega^a{}_b \to \Lambda^{-1a}{}_c d\Lambda^c{}_b + \Lambda^{-1a}{}_c \omega^c{}_d \Lambda^d{}_b. \tag{2.17}$$

Eq. (2.15) is called the *first Cartan structure equation*. Now specifying to the metric in Eq. (2.11), a choice of dreibein e^a that satisfies $ds^2 = \eta_{ab}\epsilon^a\epsilon^b$ is the following

$$e^0 = -\frac{r}{\sqrt{2}}dx^- + \frac{l^2}{\sqrt{2}r}\bar{L}(x^+)dx^+, \tag{2.18a}$$
$$e^1 = -\frac{r}{\sqrt{2}}dx^+ + \frac{l^2}{\sqrt{2}r}\bar{L}(x^-)dx^-, \tag{2.18b}$$
$$e^2 = \frac{l}{r}dr. \tag{2.18c}$$

For such a choice, one recovers $ds^2 = 2\epsilon^0\epsilon^1 + (\epsilon^2)^2$, and the first Cartan structure equation (2.15) fixes the associated spin connections

$$\omega^0 = -\frac{r}{\sqrt{2}}dx^- + \frac{l}{\sqrt{2}r}\bar{L}(x^+)dx^+, \tag{2.19a}$$
$$\omega^1 = -\frac{r}{\sqrt{2}}dx^+ + \frac{l}{\sqrt{2}r}\bar{L}(x^-)dx^-, \tag{2.19b}$$
$$\omega^2 = 0. \tag{2.19c}$$

The corresponding Chern-Simons flat connections

$$A = \left(\omega^a + \frac{e^a}{l}\right) j_a, \qquad \bar{A} = \left(\omega^a - \frac{e^a}{l}\right) j_a \tag{2.20}$$

with j_a as the generators can then be expressed as follows

$$A = \begin{pmatrix} \frac{dr}{2r} & \frac{l}{r}\bar{L}(x^+)dx^+ \\ \frac{r}{l}dx^+ & -\frac{dr}{2r} \end{pmatrix}, \qquad \bar{A} = \begin{pmatrix} -\frac{dr}{2r} & \frac{r}{l}dx^- \\ \frac{l}{r}L(x^-)dx^- & \frac{dr}{2r}. \end{pmatrix} \tag{2.21}$$

One can factorize out the r-dependence of the gauge fields. This yields the reduced connections

$$a = \begin{pmatrix} 0 & l\bar{L}(x^+)dx^+ \\ \frac{dx^+}{l} & 0 \end{pmatrix}, \qquad \bar{a} = \begin{pmatrix} 0 & \frac{dx^-}{l} \\ lL(x^-)dx^- & 0. \end{pmatrix} \tag{2.22}$$

Finally, from the off-shell reduced gauge connections expressions $a = a^a_\mu j_a dx^\mu$ and $\bar{a} = \bar{a}^a_\mu j_a dx^\mu$, the following boundary conditions can be derived

$$\begin{cases} a_- = \bar{a}_+ = 0, \\ a_+ = \frac{\sqrt{2}}{l} j_1 + 0 j_2 + \sqrt{2} l L(x^+) j_0, \quad \bar{a}_- = \sqrt{2} l \bar{L}(x^-) j_1 + 0 j_2 + \frac{\sqrt{2}}{l} j_0 \end{cases} . \tag{2.23}$$

Here, $\{j_0, j_1, j_2\}$ is the set of $sl(2,\mathbb{R})$ generators identified by

$$j_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad j_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.24}$$

2.2.2 Asymptotic symmetries

The asymptotic symmetries can be identified to the following set of gauge transformations

$$\delta a = d\lambda + [a, \lambda], \qquad \delta \bar{a} = d\bar{\lambda} + [\bar{a}, \bar{\lambda}] \tag{2.25}$$

that preserve the asymptotic behaviour of the off-shell reduced gauge connections, *i.e* Eq. (2.23). Given that $\lambda = \lambda^a j^a$ and $\bar{\lambda} = \bar{\lambda}^a \bar{j}_a$, these gauge parameters can be rewritten as

$$\lambda = l^2 \left(L\lambda^1 - \frac{1}{2}\partial_+^2 \lambda^1 \right) j_0 + \lambda^1 j_1 - \frac{l}{\sqrt{2}} \partial_+ \lambda^1 j_2, \tag{2.26a}$$

$$\bar{\lambda} = \bar{\lambda}^0 j_0 + l^2 \left(\bar{L}\bar{\lambda}^0 - \frac{1}{2}\partial_-^2 \bar{\lambda}^0 \right) j_1 + \frac{l}{\sqrt{2}} \partial_- \lambda^0 j_2, \tag{2.26b}$$

with $\lambda^1, \bar{\lambda}^0$ as functions of x^+ and x^- respectively. Then, writing $Y \equiv l\lambda^1/\sqrt{2}$ and $\bar{Y} \equiv l\bar{\lambda}^0/\sqrt{2}$, it is found that

$$\delta L = Y\partial_+ L + 2L\partial_+ Y - \frac{1}{2}\partial_+^3 Y, \tag{2.27a}$$

$$\delta \bar{L} = \bar{Y}\partial_- \bar{L} + 2\bar{L}\partial_- \bar{Y} - \frac{1}{2}\partial_-^3 \bar{Y}. \tag{2.27b}$$

Within the Chern-Simons formalism, the variation of the canonical generators associated to the asymptotic symmetries spanned by λ gives [80, 81]

$$\delta Q[\lambda] = -\frac{k}{2\pi} \int_0^{2\pi} (\lambda, \delta a_+)\, d\varphi, \qquad \delta \bar{Q}[\bar{\lambda}] = -\frac{k}{2\pi} \int_0^{2\pi} (\bar{\lambda}, \delta a_-)\, d\varphi. \tag{2.28}$$

These equations can be directly integrated to give

$$Q_Y = -\frac{k}{2\pi}\int_0^{2\pi} YLd\varphi, \qquad \bar{Q}_{\bar{Y}} = -\frac{k}{2\pi}\int_0^{2\pi} \bar{Y}\bar{L}d\varphi. \tag{2.29}$$

Then, using the Poisson brackets $\delta_{Y_1}Q_{Y_2} = \{Q_{Y_2}, Q_{Y_1}\}$, the algebra of canonical generators can be worked out from Eqs. (2.27). Defining the generators

$$L_m \equiv \frac{k}{2\pi}\int_0^{2\pi} e^{im\phi}Ld\varphi, \qquad \bar{L}_m \equiv \frac{k}{2\pi}\int_0^{2\pi} e^{im\phi}\bar{L}d\varphi, \tag{2.30}$$

one arrives at

$$i\{L_m, L_n\} = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}, \tag{2.31a}$$
$$i\{L_m, \bar{L}_n\} = 0, \tag{2.31b}$$
$$i\{\bar{L}_m, \bar{L}_n\} = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m^3\delta_{m+n,0}, \tag{2.31c}$$

with central charges given by

$$c = \bar{c} = 6k = \frac{3l}{2G}. \tag{2.32}$$

The algebra in Eqs. (2.31) from purely classical considerations first appeared in the seminal paper of Brown and Henneaux [73], for that reason is considered as a precursor of the AdS$_3$/CFT$_2$ correspondence.

2.3 AdS$_3$/CFT$_2$ in a nutshell

The Brown-Henneaux results of the previous subsection have been generalized to other 3d gravity theories that are asymptotically AdS$_3$. Here, we just review how such constructions work in a general way. For simplicity, we set the AdS radius l to unity ($l = 1$).

After identifying the bulk theory, one needs to impose suitable boundary conditions for all fields, *i.e* locally asymptotically AdS boundary conditions. To start with, one can consider a metric in global AdS with expression

$$ds^2 = dr^2 - (\cosh r dt)^2 + (\sinh r d\varphi)^2\,, \tag{2.33}$$

where r is the radial coordinate on the AdS cylinder, so that the asymptotic boundary of the cylinder is reached in the limit $r \to \infty$. The asymptotic expansion of Eq. (2.33) can be written as

$$ds^2 = dr^2 + \gamma_{ij}dx^i dx^j, \qquad \text{with} \quad \gamma_{ij} = \gamma^{(0)}_{ij}e^{2r} + \mathcal{O}\left(e^{2r}\right)_{ij}, \tag{2.34}$$

with x^i as boundary coordinates (such as the light cone ones $x^{\pm} = t \pm \varphi$), $\gamma^{(0)}_{ij}$ as the boundary metric and $\mathcal{O}\left(e^{2r}\right)$ as terms diverging less rapidly than the leading terms, finite terms and vanishing terms at $r \to \infty$. In the case of Einstein gravity, the subleading terms reduce to the Feffermann-Graham expansion, corresponding to the Brown-Henneaux boundary conditions. Replacing the Brown-Henneaux conditions by subleading terms that grow polynomially in r leads to the appearance of log-partners of gravitons in critical cosmological topologically massive gravity.

Next, in analogy with the Brown-Henneaux formalism, canonical generators of gauge transformation must be determined. The variation of the canonical generators associated to their corresponding asymptotic symmetries takes the form

$$\delta Q\left[\lambda\right] = \oint dx \sqrt{|\sigma|} \lambda \mathcal{L}(g, \pi, \delta g, \delta \pi), \tag{2.35}$$

with σ as the induced volume-element at the asymptotic boundary circle, such that the canonical boundary charge Q are integrable (finite and conserved in time).

After that, one can finally proceed to derive the (classical) asymptotic symmetry algebra and its central charges. For that, one first observes that from Eq. (2.35), the fields can be integrated. Then, by exploiting the Poisson brackets $\delta_{\lambda_2} Q\left[\lambda_1\right] = \left\{Q\left[\lambda_1\right], Q\left[\lambda_2\right]\right\}$, the algebra of the canonical generators can be computed, using the standard result $\delta_\lambda \mathcal{L} = 2\lambda\mathcal{L}' + \lambda'\mathcal{L} + \frac{k}{\pi}\lambda'''$ and the Fourier decompositions $\mathcal{L} = \sum_n L_n e^{-inx^+}, \bar{\mathcal{L}} = \sum_n \bar{L}_n e^{-inx^-}$. The result is the Virasoro algebra with $c = \bar{c} = 6k = \frac{3l}{2G}$. Once all the above steps have been executed, the dual CFT can be constrained, and the AdS_3/CFT_2 correspondence can be verified.

Two routes can be taken to check the consistency of the the AdS_3/CFT_2 correspondence. The first one is by computing the two- and three- point correlation functions of the stress tensor on the gravity side and to relate them to the CFT counterparts using the standard AdS/CFT dictionary [82]. For instance,

$$\left\langle \psi^{L/R} \psi^{L/R} \right\rangle_{\text{CFT}} \sim \frac{\delta^2 \Gamma_{\text{grav}}}{\delta \psi^{L/R}_{NN} \delta \psi^{L/R}_{NN}}, \tag{2.36}$$

with on the left-hand side of Eq. (2.36), the two point correlation function between two (anti-) holomorphic flux components of the stress-energy tensor, and on the right-hand side the second variation of the holographically renormalized on-shell action Γ_{grav} with respect to non-normalizable left-(right-)moving solutions $\psi^{L/R}_{NN}$ of the linearized EOM on the AdS background (2.33).

The other way is by comparing the partition functions. Indeed, for the correspondence to hold, the Euclidean CFT partition function on the torus must be on par with the Euclidean quantum gravity partition function on the filled AdS torus. Taking $q \equiv e^{i\tau}$ as the modular parameter with $\tau = \theta + i\beta$ (θ being the the angular momentum and β the inverse temperature or also the periodicity in Euclidean time), on the gravity side, the partition function reads

$$Z_{\text{grav}}(q, \bar{q}) = e^{-h\Gamma^{(0)} + \Gamma^{(1)} + \frac{1}{k}\Gamma^{(2)} + \cdots}. \tag{2.37}$$

k is the inverse Newton constant, $\Gamma^{(0)}$ the classical on-shell action, and $\Gamma^{(1)}, \Gamma^{(2)}, \cdots$ the one-, two-, and higher loop contributions that can be calculated using heat-kernel techniques [83]. Up to the one-loop contribution, the quantum gravity partition function yields

$$Z_{\text{grav}}(q,\bar{q}) = |q|^{-\frac{k}{2}} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}. \tag{2.38}$$

This expression is to be compared with the CFT partition function that counts the Virasoro descendants of the vacuum, which for the central charge c of the CFT takes the form

$$Z_{\text{grav}}(q,\bar{q}) = \text{Tr}\left\{q^{L_0}\bar{q}^{\bar{L}_0}\right\} = |q|^{-\frac{c}{12}} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}, \tag{2.39}$$

given that $h = \bar{h} = -\frac{c}{24} = -\frac{k}{4}$ (since from the Brown-Henneaux analysis $c = 6k$).

2.4 AdS$_3$/LCFT$_2$ correspondence

We now turn to the logarithmic extension of the AdS$_3$/CFT$_2$ correspondence. A few years after the discovery of LCFT, an AdS$_3$/LCFT$_2$ correspondence was proposed, suggesting the identification of a higher derivative action for a scalar field on an AdS background to a singleton dipole [32] (see also [33, 34]). Those types of correspondence are different from the one recently proposed by Grumiller *et al* [35, 84], in which the energy-momentum tensor can only acquire a log partner if there are at least two spin-2 modes with degenerate weights.

The action of TMG is expressed as [38, 37]

$$\Gamma_{\text{TMG}} = \frac{1}{16\pi G_N}\int_M d^3x \left[\sqrt{-g}\left(R+\frac{2}{l^2}\right)+\frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\sigma}{}_{\lambda\rho}\left(\partial_\mu\Gamma^{\rho}{}_{\nu\sigma}+\frac{2}{3}\Gamma^{\rho}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\sigma}\right)\right] + \frac{1}{8\pi G_N}\int_{\partial M} d^2x\sqrt{-\gamma}\left[K-\frac{1}{l}+\frac{1}{4\mu}\left(K^{\alpha\beta}K_{\alpha\beta}-\frac{1}{2}K^2\right)\right], \tag{2.40}$$

with two dimensionless combinations of coupling constants, $\frac{l}{G_N}$ and μl, where μ is the Chern-Simons coupling constant and K the extrinsic curvature. TMG admits a local physical degree of freedom, the massive graviton, for any non-singular value of the coupling constant. Applying the previously outlined Brown-Henneaux analysis on TMG leads to two copies of the Virasoro algebra with central charges

$$c = \frac{3l}{2G_N}\left(1-\frac{1}{\mu l}\right), \qquad \bar{c} = \frac{3l}{2G_N}\left(1+\frac{1}{\mu l}\right). \tag{2.41}$$

At the critical tuning $\mu l = 1$, the left central charge vanishes and TMG is called critical topologically massive gravity.

Following Witten's proposal in 2007 to find a CFT dual to Einstein gravity [85], the Einstein graviton 1-loop partition function was calculated in [86]. However, discrepancies were found in the results. In particular, the left- and right-moving contributions did not factorise, therefore clashing with the proposal of [85]. Soon after, Li, Song and Strominger [36] showed that the situation can be improved if one replaces Einstein gravity by *chiral gravity*, which can be viewed as the special case of topologically massive gravity at the critical tuning $\mu l = 1$, and which is asymptotically defined with Brown-Henneaux boundary conditions. A particular feature of the theory was that one of the two central charges vanishes. This gave an indication that the partition function could factorise. However, a controversial feature of the chiral gravity conjecture was the absence of massive graviton excitation. Carlip, Deser, Waldron and Wise found in [87] that local excitations exist even at the critical point. These contradictory results stirred up an intense debate [35, 88, 89, 90, 91, 92, 84, 93]. This was eventually resolved and we will say a few words about that in a moment.

In transverse gauge, the linearized EOM

$$\nabla_\mu \left(\psi^{\mu\nu} - g^{\mu\nu} \psi^\lambda{}_\lambda \right) = 0, \tag{2.42}$$

for the graviton excitations $\psi_{\mu\nu}$ around the AdS background takes the form [36]

$$\left(\mathcal{D}^L \mathcal{D}^R \mathcal{D}^{(\mu)} \psi \right)_{\mu\nu} = 0, \tag{2.43}$$

where appear the mutually commuting first order differential operators

$$\mathcal{D}^{(L/R)} = \mathcal{D}^{(\mu)} \Big|_{\mu = \pm \frac{1}{l}} \tag{2.44}$$

and

$$(\mathcal{D})_\alpha^{\ \beta} = \epsilon_\alpha{}^{\gamma\beta} \nabla_\gamma + \mu \delta_\alpha^\beta. \tag{2.45}$$

Eq. (2.43) implies that all linearized solutions in transverse gauge are traceless. Modes $\psi^M_{\alpha\beta}$ that are annihilated by operators like $\mathcal{D}^{(\mu)}$ can possess different properties such as regularity or normalizability. In light cone coordinates, this means that

$$\psi^M_{\alpha\beta} = e^{-ihx^+ - i\bar{h}x^-} F_{\alpha\beta}(r), \tag{2.46}$$

with h and $\bar{h}$ $sl(2)$ weights. More precisely, the six Killing vectors are the Virasoro modes $\{L_{-1}, L_0, L_1\}$ and $\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$, such that for any value of μ, the Virasoro zero mode generators defined as $L_0 = i\partial_+, \bar{L}_0 = i\partial_-$ act on the fields as [36]

$$L_0 \psi^M = h \psi^M, \qquad \bar{L}_0 \psi^M = \bar{h} \psi^M, \tag{2.47}$$

and the $\{L_1, \bar{L}_1\}$ modes act on the fields as

$$L_1 \psi^M = \bar{L}_1 \psi^M = 0. \tag{2.48}$$

Note that Eq. (2.48) is only true for gravitational modes corresponding to primaries, and not generically. Depending on the sign of μ, the $sl(2)$ weights of the primaries take the values

$$(h,\bar{h}) = \begin{cases} \left(\frac{3}{2}+\frac{\mu l}{2}, -\frac{1}{2}+\frac{\mu l}{2}\right) & \text{for} \quad \mu > 0, \\ \left(-\frac{1}{2}-\frac{\mu l}{2}, \frac{3}{2}-\frac{\mu l}{2}\right) & \text{for} \quad \mu < 0. \end{cases} \tag{2.49}$$

In each case, the difference of weight is $|h-\bar{h}| = 2$, as is the case for a graviton excitation. This means that the primaries ψ^L and ψ^R respectively annihilated by $\mathcal{D}^L$ and $\mathcal{D}^R$ have weights $(2,0)$ and $(0,2)$. They correspond to the L_{-2} and $\bar{L}_{-2}$ descendants of the vacuum on the CFT side. At the critical tuning, the operators $\mathcal{D}^L$ and $\mathcal{D}^{(\mu)}$ degenerate with each other, just as the weights of the primaries ψ^L and ψ^M.

In particular, the fact that the left moving boundary graviton ψ^L and the massive graviton degenerate implies that there is a logarithmic mode such that

$$\left(\mathcal{D}^L\mathcal{D}^L\psi^{log}\right)_{\mu\nu} = 0, \qquad \left(\mathcal{D}^L\psi^{log}\right)_{\mu\nu} \propto \psi^L_{\mu\nu}. \tag{2.50}$$

The degeneracy of the two vectors suggests the emergence of a generalized eigenvector, and the appearance of a Jordan cell in critical topological massive gravity. Such a structure is the hallmark of logarithmic conformal field theories, and its occurence in TMG at the critical point is at the origin of this recently proposed $AdS_3/LCFT_2$ correspondence. The logarithmic partner of the massive graviton can be expressed as

$$\psi^{log}_{\alpha\beta} = -2(it + \ln\cosh r)\psi^L_{\alpha\beta}. \tag{2.51}$$

For the Hamiltonian $H = L_0 + \bar{L}_0 = i\partial_t$, the result is the Jordan structure

$$H\begin{pmatrix}\psi^{log}\\ \psi^L\end{pmatrix} = \begin{pmatrix}2 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}\psi^{log}\\ \psi^L\end{pmatrix}, \tag{2.52}$$

while the angular momentum operator $J = L_0 - \bar{L}_0 = i\partial_\varphi$ yields a diagonal matrix as

$$J\begin{pmatrix}\psi^{log}\\ \psi^L\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\psi^{log}\\ \psi^L\end{pmatrix}. \tag{2.53}$$

Among the conclusive tests in favor of the correspondence is the computation of the conformal Ward identities on the gravity side, with a precise match between the two-point correlation functions calculated by Skenderis, Taylor and van Rees in [94] and previously known results from LCFT. The match was confirmed in [95] where in addition, three-point correlation functions were calculated again with precise agreements with long known results from LCFT.

Another interesting check was performed by calculating the one-loop partition function of TMG at the critical point in [39]. The result was given as

$$Z_{\text{TMG}}(q,\bar{q}) = \prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=2}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}}. \tag{2.54}$$

It was rewritten in the following form

$$Z_{\text{TMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q|^2}\left(1+\frac{q^2}{|1-q|^2}\right) + \sum_{h,\bar{h}} N_{h,\bar{h}} q^h \bar{q}^{\bar{h}} \prod_{n=1}^{\infty} \frac{1}{|1-q|^2}, \tag{2.55}$$

in order to be compared to the LCFT partition function

$$Z_{\text{LCFT}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q|^2}\left(1+\frac{q^2}{|1-q|^2}\right) + \cdots \tag{2.56}$$

Following [39], we briefly explain the motivation to rewrite Eq. (2.54) into Eq. (2.55). We start recalling that in critical TMG, the holomorphic central charge vanishes ($c_L = 0$), while the antiholomorphic central charge is $\bar{c} = c_R = 3l/G_N$. The holomorphic energy-momentum tensor $T(z)$ possesses a logarithmic partner $t(z)$, such that

$$L_0 t = 2t + T, \qquad L_0 T = 2T, \qquad L_1 t = L_1 T = 0. \tag{2.57}$$

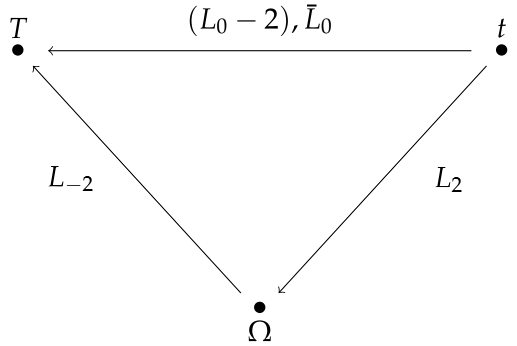
Then, from the fact that the two-point correlation function $\langle T(z)t(w)\rangle = b_L/(z-w)^4$ is non-zero, and the relations $T = L_{-2}\Omega$ and $L_2 T = 0$, Ω being the ground state vacuum of the LCFT, one gets

$$L_2 t = b_L \Omega, \qquad b_L = -c_R = -\frac{3l}{G_N}. \tag{2.58}$$

Besides, T and t are annihilated by all positive $\bar{L}_n$ modes, as well as by all modes $L_n \forall n \geq 3$. Lastly, for the LCFT to be locally consistent, $L_0 - \bar{L}_0$ must be diagonalizable, and therefore

$$\bar{L}_0 t = T. \tag{2.59}$$

The above properties can be summarized in the following drawing



The contribution of the above states to the partition function is evaluated by first considering the descendants of the vacuum. From the above diagram, clearly they are not affected by the presence of t. Therefore one obtains

$$Z_\Omega = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}. \tag{2.60}$$

The states that remain are descendants of t that are not already descendants of Ω. They can be expressed as [39]

$$Z_t = q^2 \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2}, \tag{2.61}$$

where the term q^2 accounts from the fact that t has eigenvalue $(2,0)$ under the diagonal part of $(L_0, \bar{L}_0)$. The result is the partition function of the Virasoro descendants given by

$$Z^0_{\text{LCFT}} = Z_\Omega + Z_t = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left(1 + \frac{q^2}{|1-q|^2}\right). \tag{2.62}$$

The term in brackets in the above equation can be written as a sum

$$\left(1 + \frac{q^2}{|1-q|^2}\right) = 1 + \sum_{m=2}^{\infty} \sum_{\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}}, \tag{2.63}$$

and compared to the double product of Eq. (2.54). The result is that the double product is a sum generating single- and multi-particle representations of the logarithmic t-excitations as

$$\prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}} = 1 + \sum_{m=2}^{\infty} \sum_{\bar{m}=0}^{\infty} q^m \bar{q}^{\bar{m}} + \text{multiparticle}. \tag{2.64}$$

Eventually, the fact that Z_{TMG} was able to be rewritten as Eq. (2.55), confirmed on one hand that there was a match between Z_{TMG} and Z_{LCFT} up to single-particle, and on the other hand that a consistent description of the counting of states in the multi-particle sector was found, as the coefficients $N_{h,\bar{h}}$ gave non-negative integers. However, despite the perfect agreements found on the two sides of the correspondence, a better understanding of the LCFT partition function is still lacking.

This work initiates a program to learn more about the dual CFT of critical TMG from the partition function. As such, we first undertake the task to show that the partition function can be rewritten in a way that systematically accounts for single- and multi-log excitations. Along the way, we also find that the partition function can also be expressed in terms of the so called Plethsytic Exponential. Given the extensive work done on the Plethystic Exponential, this interesting relationship allows one to envision a further study of the theory through the properties of the Plethystic Exponential. As a by-product, using an appropriate set of differential ladder operators, we also derive new ladder and sl(2) actions between the multi-particle components of the partition function, showing from the Bell polynomial expansions how it applies to the Plethystic Exponential, in the case under consideration.

Chapter 3

Combinatorial properties of the partition function of critical TMG

In this chapter, we proceed with the main result of the thesis, the expression of the partition function of critical TMG in terms of Bell polynomials.

3.1 Multipartite generating functions

Following the theory developed in [96], we show how multipartite generating functions can be written in terms of Bell polynomials, also known as Faà di Bruno formula as we will see below.

For any *multipartite* (or m-partite) numbers $\vec{k} = (k_1, k_2, \ldots, k_m)$, i.e any ordered m-tuple of non negative integers not all zeros, let $N^{(z;m)}(\vec{k}) = N^m(z; k_1, k_2, \ldots, k_m)$ be the number of partitions of $\vec{k}$, i.e the number of distinct representations of $(k_1, k_2, \ldots, k_m)$ as a sums of multipartite numbers. The generating functions of $N^{(z;m)}(\vec{k})$ can be defined as [96]

$$G(z; X) = \prod_{\vec{k}\geq 0} \frac{1}{1 - z x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}} = \sum_{\vec{k}\geq 0} N^{(z;m)}(\vec{k}) x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}. \tag{3.1}$$

It follows that

$$\log G(z; X) = -\sum_{\vec{k}\geq 0} \log\left(1 - z x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}\right) \tag{3.2a}$$

$$= \sum_{\vec{k}\geq 0} \sum_{n=1}^{\infty} \frac{z^n}{n} x_1^{nk_1} x_2^{nk_2} \cdots x_m^{nk_m} \tag{3.2b}$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{1 - x_1^n} \frac{1}{1 - x_2^n}, \cdots \frac{1}{1 - x_m^n} \tag{3.2c}$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{n} \prod_{j=1}^{m} \frac{1}{1 - x_j^n} \tag{3.2d}$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{n!} (n-1)! \prod_{j=1}^{m} \frac{1}{1 - x_j^n}. \tag{3.2e}$$

Introducing the quantity $g_m(n)$ such that

$$g_m(n) = (n-1)! \prod_{j=1}^{m} \frac{1}{1-x_j^n} = (n-1)! \sum_{(k_1,\ldots,k_m)\geq 0} \left(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \right)^n , \tag{3.3}$$

we can write

$$\log G(z;X) = \sum_{n=1}^{\infty} \frac{z^n}{n!} g_m(n), \tag{3.4}$$

and finally

$$\sum_{\vec{k}\geq 0} N^{(z;m)}(\vec{k}) x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n!} g_m(n) \right). \tag{3.5}$$

As we will see later (Eq. (3.21)) in our specialization, $g_m(n)$ will become $g_2(n)$ (because of the two variables m and $\bar{m}$), and we will just denote the quantity by g_n.

3.1.1 Faà di Bruno formula: a combinatorial argument

Bell polynomials are intimately connected to the Faà di Bruno formula. The former were defined in 1934 by E.T. Bell in [97], but their name is due to Riordan [98] who studied the Faà di Bruno formula ([99], [100]) that suggests expressing the n-th derivative of a composite function $f \circ g$ in terms of the derivatives of f and g [101].

Consider two functions f and g such that $f, g : \mathbb{R} \to \mathbb{R}$ that admits n derivatives. The Faà di Bruno formula enumerates the terms in the expansion of the n-th derivative

$$\frac{d^n}{dx^n} f\left(g(x)\right) = \left(f \circ g\right)^{(n)}(x). \tag{3.6}$$

The computation of the derivative can be done using on one hand the *chain rule* for derivation of a composition of functions, and on the other hand the *rule for products* for deriving the product of functions, where the result is a sum of monomials.

The calculation of the n-th derivative applies both rules many times, and up to third order, can be summarized in the beautiful *tree of derivatives* in Fig. 3.1.

In summary, the derivative is always the sum of several monomials of shape

$$a f^{(k)}\left(g(x)\right) g'(x)^{b_1} \cdots g^{(j)}(x)^{b_j}, \tag{3.7}$$

with the proper integer coefficients $a, k, j, b_1, \cdots, b_j$. Because the Faà di Bruno formula enumerates such monomials, it can be considered from a combinatorial perspective. For that, we first introduce the Bell polynomials, which are very useful in the study of set partitions. The partition {1} can be associated to the monomial x_1. The set {1} has only one partition, so one can define the term $B_{1,1}(x_1) = x_1$. The set {1,2} has two partitions {1,2} and {1},{2}, the former with one block and the latter with two blocks. In this case, we can associate to these sets the monomials x_2 and x_1^2 respectively, such that $B_{2,1}(x_1, x_2) = x_2$ and $B_{2,2}(x_1) = x_1^2$.

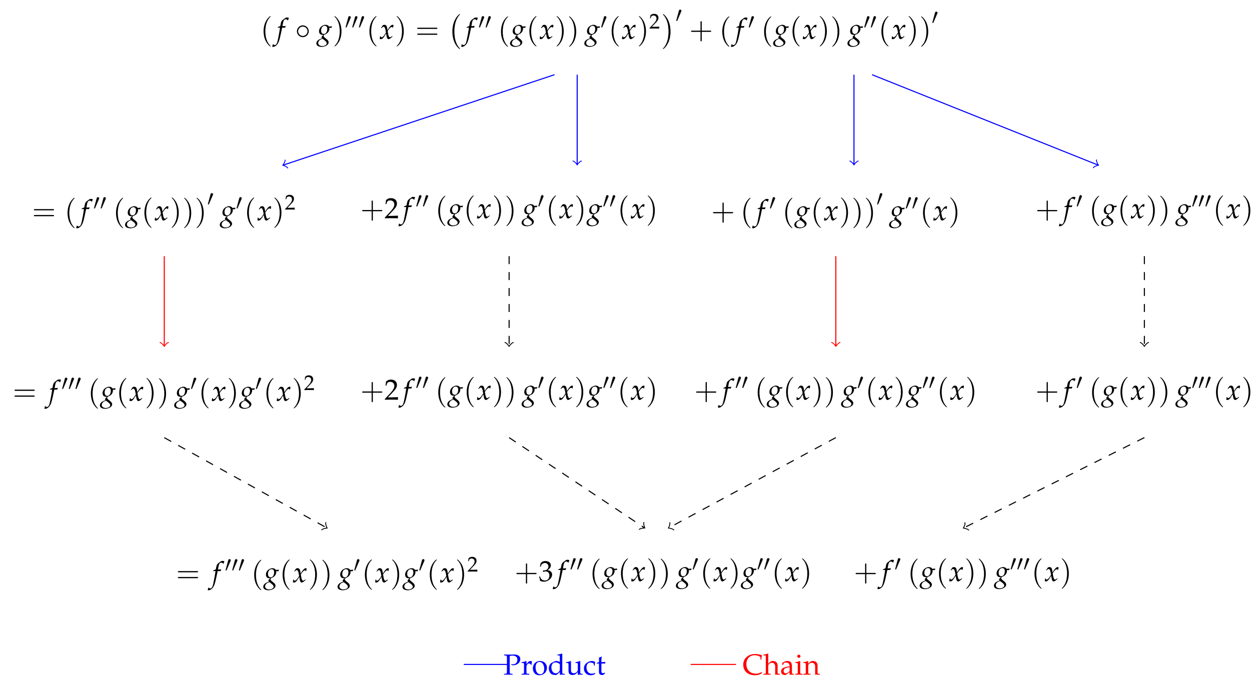


FIGURE 3.1: Tree of derivatives

The set {1,2,3} has five partitions. Three partitions with two blocks, namely {1,2},{3}, {1,3},{2}, and {1,2},{3} are associated with the monomial $x_1 x_2$ such that $B_{3,2}(x_1, x_2) = 3x_1 x_2$. One partition with three blocks, *i.e* {1},{2},{3} is associated with x_1 such that $B_{3,3}(x_1) = x_1^3$. The other partition is {1,2,3} associated to x_3, for which $B_{3,1}(x_1, x_2, x_3) = x_3$. The general rule is given by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \frac{1}{k!} \sum_{j_1+\cdots+j_k=n, j_i \geq 1} \binom{n}{j_1, \ldots, j_k} x_{j_1} \cdots x_{j_k}, \tag{3.8}$$

with

$$\binom{n}{j_1, \ldots, j_k} = \frac{n!}{j_1! j_2! \cdots j_k!}, \tag{3.9}$$

and $B_{0,0}(x_1) = 1$. What this means is that the sum is over set partitions of $\{1, 2, \ldots, n\}$ with block sizes $j_1, \ldots, j_k$ with the factor $1/k!$ making the correction for the multiple counting in the sum. The number of variables necessary is $n - k + 1$ as no block can take more than $n - k + 1$ elements. The terms $B_{n,k}$ in Eq. (3.8) are called *partial Bell polynomials*. The polynomial considered by E.T. Bell [97] are

$$Y_n = Y_n(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}). \tag{3.10}$$

Eq. (3.10) is known as the *complete Bell polynomial*. At level three for instance,

$$Y_3 = Y_3(x_1, x_2, x_3) = B_{3,1} + B_{3,2} + B_{3,3} = x_3 + 3x_1x_2 + x_1^3. \tag{3.11}$$

In a similar way, one can associate set partitions to derivatives of composite functions. {1} will be associated to $f'(g(x)) \cdot g'(x)$ for instance. At the next level, the partitions of {1,2} and {1},{2} correspond to $f'(g(x))\, g''(x)$ and $f''(g(x))\, g'(x)^2$ respectively, and so on. So, we see that to each partition of $\{1, 2, \ldots, n\}$ with k blocks corresponds a term $d^n f(g(x))/dx^n$, with the factor $f^{(k)}(g(x))$, where the block sizes determine its other factors (the derivatives of g).

The result of all this is the set partition version of the Faà di Bruno formula given by

$$\frac{d^n}{dx^n} f(g(x)) = \sum f^{(k)}(g(x)) \left(g'(x)\right)^{b_1} \left(g''(x)\right)^{b_2} \cdots \left(g^n(x)\right)^{b_n}, \tag{3.12}$$

where the sum is over all partitions of $\{1, 2, \ldots, n\}$, and for each partition, k is its number of blocks and b_i is the number of block with exactly i elements. An immediate corollary of this is the Bell polynomial version of the Faà di Bruno formula that takes the form

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=0}^{n} f^{(k)}(g(x))\, B_{n,k}\left(g'(x), g''(x), \cdots, g^{n-k+1}(x)\right). \tag{3.13}$$

It turns out that the study of n-th derivative of a composite function simply reduces to the study of Bell polynomials. These polynomials find applications in combinatorics, number theory, analysis, probability, algebra, *etc* .... We will limit ourselves to their application in multipartite partition problems. [96].

3.1.2 Bell polynomials

Useful recurrence relations for the Bell polynomials $Y_n(g_1, g_2, \ldots, g_n)$ and their generating function $\mathcal{G}(z)$ have the form [96]

$$Y_{n+1}(g_1, g_2, \ldots, g_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(g_1, g_2, \ldots, g_{n-k}) g_{k+1}, \tag{3.14}$$

and

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} \frac{Y_n z^n}{n!} \quad \Rightarrow \quad \log \mathcal{G}(z) = \sum_{n=0}^{\infty} \frac{g_n z^n}{n!}. \tag{3.15}$$

The term g_n in Eq. (3.15) expresses the monomials $g_1, g_2, \ldots, g_n$ that constitute the complete Bell polynomials $Y_n(g_1, g_2, \ldots, g_n)$. Those were labelled $x_1, x_2, \ldots, x_n$ in the general discussion leading to Eq. (3.10).

From Eq. (3.15), one obtains the following explicit expression for the Bell polynomials

$$Y_n(g_1, g_2, \dots, g_n) = \sum_{\vec{k} \vdash n} \frac{n!}{k_1! \cdots k_n!} \prod_{j=1}^{n} \left(\frac{g_j}{j!} \right)^{k_j}. \tag{3.16}$$

We refer the reader to Appendix A for a discussion on the partition notation and the meaning of $\vec{k} \vdash n$.

3.1.3 Specialization

We apply the results of the previous subsections to show that Z_{TMG} can be rewritten as (exponential) generating function of Bell polynomials. We start by rewriting eq. (2.55) as

$$Z_{\mathrm{TMG}}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1 - q^n|^2} \prod_{m=0}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1 - q^2 q^m \bar{q}^{\bar{m}}}, \tag{3.17}$$

and we see that we have just specialized z to be q^2. Then, if we write

$$Z_{\mathrm{TMG}} = \mathcal{A}(q, \bar{q}) \mathcal{B}(q, \bar{q}), \tag{3.18}$$

with

$$\mathcal{A}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1 - q^n|^2} \quad ; \quad \mathcal{B}(q, \bar{q}) = \prod_{m=0}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1 - q^2 q^m \bar{q}^{\bar{m}}}, \tag{3.19}$$

and we focus on $\mathcal{B}(q, \bar{q})$, it follows that

$$\log \mathcal{B}(q, \bar{q}) = - \sum_{m \geq 0, \bar{m} \geq 0} \log(1 - q^2 q^m q^{\bar{m}}) \tag{3.20a}$$

$$= - \sum_{m \geq 0, \bar{m} \geq 0} \left(- \sum_{n=1}^{\infty} \frac{(q^2)^n}{n} q^{nm} \bar{q}^{n\bar{m}} \right) \tag{3.20b}$$

$$= \sum_{m \geq 0, \bar{m} \geq 0} \sum_{n=1}^{\infty} \frac{q^{2n}}{n} q^{nm} \bar{q}^{n\bar{m}} \tag{3.20c}$$

$$= \sum_{n=1}^{\infty} \frac{q^{2n}}{n} \left(\sum_{m \geq 0, \bar{m} \geq 0} q^{nm} \bar{q}^{n\bar{m}} \right) \tag{3.20d}$$

$$= \sum_{n=1}^{\infty} \frac{q^{2n}}{n!} \left[(n-1)! \sum_{m \geq 0, \bar{m} \geq 0} q^{nm} \bar{q}^{n\bar{m}} \right]. \tag{3.20e}$$

Now, if we write

$$g_n = (n-1)! \sum_{m \geq 0, \bar{m} \geq 0} q^{nm} \bar{q}^{n\bar{m}}, \tag{3.21}$$

it is easy to see that $\log \mathcal{B}(q, \bar{q})$ becomes

$$\log \mathcal{B}(q,\bar{q}) = \sum_{n=1}^{\infty} \frac{q^{2n}}{n!} g_n. \tag{3.22}$$

Hence, in the present case, Eq. (3.21) gives the monomials of the complete Bell polynomials Y_n. According to Eq. (3.15), for the expression of g_n given in Eq. (3.21), we have the corresponding expression in terms of the complete Bell polynomials Y_n

$$\mathcal{B}(q,\bar{q}) = \sum_{n=0}^{\infty} \frac{Y_n}{n!} q^{2n}. \tag{3.23}$$

Finally, we have

$$Z_{\text{TMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \sum_{n=0}^{\infty} \left(\sum_{\vec{k}\vdash n} \frac{n!}{k_1! \cdots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \cdots \left(\frac{g_n}{n!}\right)^{k_n} \right) q^{2n}. \tag{3.24}$$

Let us see how this works. If we focus on $\mathcal{B}(q,\bar{q})$, and referring to Appendix A, we can write

$$\begin{aligned}
\mathcal{B}(q,\bar{q}) &= \frac{1}{0!} Y_0 \left(q^2\right)^0 + \frac{1}{1!} Y_1 \left(q^2\right)^1 + \frac{1}{2!} Y_2 \left(q^2\right)^2 + \frac{1}{3!} Y_3 \left(q^2\right)^3 + \ldots && (3.25a)\\
&= 1 + Y_1 \left(q^2\right) + \frac{1}{2!} Y_2 \left(q^2\right)^2 + \frac{1}{3!} Y_3 \left(q^2\right)^3 + \ldots, && (3.25b)
\end{aligned}$$

with [98]

$$\begin{aligned}
Y_1 &= g_1 = \sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^m \bar{q}^{\bar{m}}, && (3.26a)\\
Y_2 &= g_1^2 + g_2 && (3.26b)\\
&= \left(\sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^m \bar{q}^{\bar{m}} \right)^2 + \sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^{2m} \bar{q}^{2\bar{m}} && (3.26c)\\
&= \sum_{m\geq 0} \sum_{\bar{m}\geq 0} (m+1)(\bar{m}+1) q^m \bar{q}^{\bar{m}} + \sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^{2m} \bar{q}^{2\bar{m}}, && (3.26d)\\
Y_3 &= g_1^3 + 3g_1 g_2 + g_3 && (3.26e)\\
&= \left(\sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^m \bar{q}^{\bar{m}} \right)^3 + 3 \left(\sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^m \bar{q}^{\bar{m}} \right) \left(\sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^{2m} \bar{q}^{2\bar{m}} \right) && (3.26f)\\
&\quad + 2 \sum_{m\geq 0} \sum_{\bar{m}\geq 0} q^{3m} \bar{q}^{3\bar{m}},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Then, we get the following expression for Z_{TMG}

$$Z_{\text{TMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left[1 + \frac{g_1}{1!} q^2 + \frac{g_1^2 + g_2}{2!} \left(q^2\right)^2 + \frac{g_1^3 + 3g_1 g_2 + g_3}{3!} \left(q^2\right)^3 + \cdots \right]. \tag{3.27}$$

A straightforward calculation of the above equation allows us to verify that the coefficients of the different representations match the ones in [39].

3.1.4 Bell polynomial expansion as a multi-particle generating function

In this section we would like to give an interpretation of the combinatorial results obtained. In particular, we would like to suggest that in the expansion $\mathcal{B}(q,\bar{q})$, while the terms $(q^2)^n$ represent single particle and multiparticle highest weights (of states t and $t \otimes_n t$ respectively), the terms Y_n for $n \geq 1$ are in fact character representations of descendants of single particles t when $n = 1$, and of multiparticles $t \otimes_n t$ for $n \geq 2$.

We start by recalling the well known identity for geometric sums

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}. \tag{3.28}$$

From the above equation, it is easy to see that

$$\begin{aligned}
g_n &= (n-1)! \sum_{m\geq 0, \bar{m}\geq 0} q^{nm}\bar{q}^{n\bar{m}} && \text{(3.29a)}\\
&= (n-1)! \left(\sum_{m=0}^{\infty} (q^n)^m\right)\left(\sum_{\bar{m}=0}^{\infty} (\bar{q}^n)^{\bar{m}}\right) && \text{(3.29b)}\\
&= (n-1)! \left(\frac{1}{1-q^n}\right)\left(\frac{1}{1-\bar{q}^n}\right) && \text{(3.29c)}\\
&= (n-1)! \frac{1}{|1-q^n|^2} && \text{(3.29d)}
\end{aligned}$$

This allows us to rewrite Equation (3.27) up to third order in the expansion of $\left(q^2\right)$ as

$$\begin{aligned}
Z_{\text{LCFT}}(q,\bar{q}) &= \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left\{ 1 + \frac{\left(q^2\right)^1}{|1-q|^2} + \frac{1}{2!}\left[\left(\frac{1}{|1-q|^2}\right)^2 + \frac{1}{|1-q^2|^2}\right]\left(q^2\right)^2 \right.\\
&+ \left. \frac{1}{3!}\left[\left(\frac{1}{|1-q|^2}\right)^3 + 3\frac{1}{|1-q|^2}\frac{1}{|1-q^2|^2} + \frac{2}{|1-q^3|^2}\right]\left(q^2\right)^3 + \cdots \right\}
\end{aligned} \tag{3.30}$$

One can immediately see that the first two terms in the above expansion are identical to the ones in [39]. Besides, it is known that [102]

$$\prod_{i=1}^{j} \frac{1}{1-q^i} \tag{3.31}$$

is also the partition function of j bosonic one-dimensional harmonic oscillators. Hence, one can think of $\mathcal{B}(q,\bar{q})$ as a character generating function of single particles when $n = 1$ and multiparticles when $n \geq 2$.

3.2 Counting of the single particle states

We start by rewritting Z_{TMG} as

$$\begin{aligned} Z_{\text{TMG}}(q,\bar{q}) &= \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \qquad (3.32a)\\ &+ \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left[\frac{g_1}{1!} q^2 + \frac{g_1^2 + g_2}{2!} \left(q^2\right)^2 + \frac{g_1^3 + 3g_1 g_2 + g_3}{3!} \left(q^2\right)^3 + \cdots \right] \\ &= \mathcal{A}(q,\bar{q}) + \mathcal{A}(q,\bar{q}) \cdot \mathcal{C}(q,\bar{q}), \qquad (3.32b) \end{aligned}$$

where

$$\mathcal{C}(q,\bar{q}) = \frac{g_1}{1!} q^2 + \frac{g_1^2 + g_2}{2!} \left(q^2\right)^2 + \frac{g_1^3 + 3g_1 g_2 + g_3}{3!} \left(q^2\right)^3 + \cdots \qquad (3.33)$$

From the expansion $\mathcal{C}(q,\bar{q})$, it is possible to write down an explicit counting of the single- and multi-particle states. For now, we restrict ourselves to the single particle sector of the expansion of $\mathcal{C}(q,\bar{q})$, leaving the multi-particle sector for the next section. The single particle sector can be expressed in orders of conformal dimension $(h+\bar{h})$ as

$$\begin{aligned} Y_1 q^2 = g_1 q^2 = q^2 & \qquad (3.34a)\\ + \left(q^3 + q^2\bar{q}^1\right) & \qquad (3.34b)\\ + \left(q^4 + q^3\bar{q}^1 + q^2\bar{q}^2\right) & \qquad (3.34c)\\ + \left(q^5 + q^4\bar{q}^1 + q^3\bar{q}^2 + q^2\bar{q}^2\right) & \qquad (3.34d)\\ + \left(q^6 + q^5\bar{q}^1 + q^4\bar{q}^2 + q^3\bar{q}^3 + q^2\bar{q}^4\right) & \qquad (3.34e)\\ + \cdots\cdots\cdots\cdots\cdots\cdots & \qquad (3.34f) \end{aligned}$$

The above expansion can be identified as the following tower of states, again organized in levels of conformal dimension

$$\begin{array}{ccccccc} & & & |t\rangle & & & \\ & & L_{-1}|t\rangle & & \bar{L}_{-1}|t\rangle & & \\ & L_{-1}^2|t\rangle & & L_{-1}\bar{L}_{-1}|t\rangle & & \bar{L}_{-1}^2|t\rangle & \\ L_{-1}^3|t\rangle & & L_{-1}^2\bar{L}_{-1}|t\rangle & & L_{-1}\bar{L}_{-1}^2|t\rangle & & \bar{L}_{-1}^3|t\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

This is in perfect agreement with the results that already are in the literature, as it was found in [35] that descendants of the logarithmic partner are obtained by acting with $L_{-1}^n \bar{L}_{-1}^m$, $n,m \geq 0$.

3.3 Construction of the multi-particle states

Here, we would like to show how the multi-particle states are constructed from the coefficients obtained by the Bell polynomial generating function. Before taking care of the specific 2-particle case, we first briefly introduce the notion of *Hopf algebras*.

3.3.1 Hopf algebras

Let $\mathcal{A}$ be an associative complex algebra with unit element $\mathbf{1}$. Then, for all $a, b \in \mathcal{A}$ and all complex numbers z, the following maps can be defined

(i) Multiplication map: $f(a \otimes b) = ab$,

(ii) Unitality map: $g(z) = z\mathbf{1}$,

(iii) Identical map: $\mathrm{id}(a) = a$.

Using linear extensions, the following algebra morphisms can be written

$$f : \mathcal{A} \otimes \mathcal{A} \mapsto \mathcal{A}, \qquad g : \mathbb{C} \mapsto \mathcal{A}, \qquad \mathrm{id}{:}\mathcal{A} \to \mathcal{A}. \tag{3.35}$$

The above canonical morphisms of the algebra $\mathcal{A}$ have the following properties

(A) Associativity: $f(f \otimes \mathrm{id}) = f(\mathrm{id} \otimes f)$,
(U) Unitality: $f(g \otimes \mathrm{id}) = f(\mathrm{id} \otimes g) = \mathrm{id}$.

This can be proved quite easily. Relation (A) follows from the associative law $a(bc) = (ab)c$ for all $a, b, c \in \mathcal{A}$. Actually

$$f(\mathrm{id} \otimes f)(a \otimes b \otimes c) = f(a \otimes f(b \otimes c)) = f(a \otimes bc) = a(bc). \tag{3.36}$$

Similarly $f(f \otimes \mathrm{id})(a \otimes b \otimes c) = (ab)c$, which proves (A). Relation (U) follows from

$$f(g \otimes \mathrm{id})(1 \otimes a) = f(g(1) \otimes a) = f(\mathbf{1} \otimes a) = \mathbf{1}a = a \tag{3.37}$$

Similarly, $f(\mathrm{id} \otimes g)(a \otimes 1) = a\mathbf{1} = a$. Relation (U) is then proved if we identify $a \otimes 1$ and $1 \otimes a$ with a. This corresponds to the isomorphisms $\mathcal{A} \otimes \mathbb{C} = \mathcal{A} = \mathbb{C} \otimes \mathcal{A}$.

The relations (A) and (U) can be dualized by replacing the symbols f and g by Δ and ε respectively, and by commuting the factors. The resulting dual relations read

(CA) Coassociativity: $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$,
(CU) Counitality: $(\mathrm{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$.

Let us consider an associative unital complex algebra $\mathcal{A}$. This algebra is called complex *bialgebra* iff there exist two algebra morphisms

(i) $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (coproduct),

(ii) $\varepsilon : \mathcal{A} \to \mathbb{C}$ (counit),

satisfying the relations (CA) and (CU). The complex bialgebra is called a Hopf algebra iff there exists a linear map $S : \mathcal{A} \to \mathcal{A}$ such that

$$f(S \otimes \mathrm{id})\Delta = f(\mathrm{id} \otimes S)\Delta = g\varepsilon. \tag{3.38}$$

It is a standard procedure to make use of the so-called *Sweedler notation* when it comes to carrying out concrete computations. For all $a \in \mathcal{A}$, the coproduct Δa is contained in the tensor product $\mathcal{A} \otimes \mathcal{A}$. Therefore, there are elements $a_1, \ldots, a_n, b_1, \ldots b_n$ such that

$$\Delta a = \sum_{k=1}^{n} a_k \otimes b_k. \tag{3.39}$$

Using Sweedler notation, one can write Eq. (3.39) as

$$\Delta a = \sum_{a} a_{(1)} \otimes a_{(2)}. \tag{3.40}$$

For all $a, b \in \mathcal{A}$, one can further write

$$\Delta a \Delta b = \Delta(ab), \qquad \Delta 1 = 1 \otimes 1. \tag{3.41}$$

We close this subsection by making the distinction between two coproducts. When $\mathcal{A}$ is a Lie algebra type, the coproduct of the Hopf algebra reads

$$\Delta a = a \otimes 1 + 1 \otimes a, \tag{3.42}$$

and is often called a *primitive* coproduct. When $\mathcal{A}$ is a group algebra, the coproduct of the Hopf algebra reads

$$\Delta a = a \otimes a, \tag{3.43}$$

and is usually called a group-like coproduct.

3.3.2 Constrution of 2-particles states

We first consider the 2-particle highest weight state $|t\rangle \otimes |t\rangle$, and its descendants. The counting of these states is in $\mathcal{C}(q, \bar{q})$ (Eq. 3.33) as

$$\frac{g_1^2 + g_2}{2!} \left(q^2\right)^2. \tag{3.44}$$

This gives the following counting organization in levels of conformal dimension

$$\frac{1}{2!} Y_2 \left(q^2\right)^2 = \frac{g_1^2 + g_2}{2!} \left(q^2\right)^2 = q^4 \tag{3.45a}$$

$$+ \left(q^5 + q^4 \bar{q}^1\right) \tag{3.45b}$$

$$+ \left(2q^6 + 2q^5 \bar{q}^1 + 2q^4 \bar{q}^2\right) \tag{3.45c}$$

$$+ \left(2q^7 + 3q^6 \bar{q}^1 + 3q^5 \bar{q}^2 + 2q^4 \bar{q}^3\right) \tag{3.45d}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots \tag{3.45e}$$

We now want to show how the corresponding states built upon $|t\rangle \otimes |t\rangle$ are constructed. In what follows, we will see that the standard use of primitive coproducts expected to construct the multi-particle states does not work. In order to obtain the correct construction of the states, we have to use a special type of coproducts called *half-coproducts*. Such objects are typical of Hopf algebroids and quantum groupoids [103, 104], which are generalizations of Hopf algebras. They are defined as

$$\Delta^L(x) = x \otimes 1; \quad \Delta^R(x) = 1 \otimes x, \tag{3.46}$$

where $\Delta^L(x)$ is the left co-product, and $\Delta^R(x)$ is the right co-product. Starting from the lowest descendant levels, we first consider the only state counted by q^5. It could be constructed using the left half coproduct as

$$\Delta^L(L_{-1})(|t\rangle \otimes |t\rangle) = (L_{-1} \otimes 1)(|t\rangle \otimes |t\rangle) = L_{-1}|t\rangle \otimes |t\rangle\,, \tag{3.47}$$

or using the right half coproduct as

$$\Delta^R(L_{-1})(|t\rangle \otimes |t\rangle) = (1 \otimes L_{-1})(|t\rangle \otimes |t\rangle) = |t\rangle \otimes L_{-1}|t\rangle\,, \tag{3.48}$$

since the tensor product is between indistinguishable particles. In order to preserve the symmetrization of identical particles, we write the state as

$$\left(\frac{\Delta^L(L_{-1}) + \Delta^R(L_{-1})}{2}\right)(|t\rangle \otimes |t\rangle) = \frac{L_{-1}|t\rangle \otimes |t\rangle + |t\rangle \otimes L_{-1}|t\rangle}{2}. \tag{3.49}$$

In the same way, the state counted by $q^4\bar{q}^1$ looks like

$$\left(\frac{\Delta^L(\bar{L}_{-1}) + \Delta^R(\bar{L}_{-1})}{2}\right)(|t\rangle \otimes |t\rangle) = \frac{\bar{L}_{-1}|t\rangle \otimes |t\rangle + |t\rangle \otimes \bar{L}_{-1}|t\rangle}{2}. \tag{3.50}$$

We now move to the two states counted by $q^5\bar{q}^1$. The multiplication of coproducts for $\Delta(L_{-1}\bar{L}_{-1})$ gives many options

$$\Delta^L(L_{-1})\Delta^L(\bar{L}_{-1}) = (L_{-1} \otimes 1)(\bar{L}_{-1} \otimes 1) = L_{-1}\bar{L}_{-1} \otimes 1 \tag{3.51a}$$
$$\Delta^R(L_{-1})\Delta^R(\bar{L}_{-1}) = (1 \otimes L_{-1})(1 \otimes \bar{L}_{-1}) = 1 \otimes L_{-1}\bar{L}_{-1} \tag{3.51b}$$
$$\Delta^L(L_{-1})\Delta^R(\bar{L}_{-1}) = (L_{-1} \otimes 1)(1 \otimes \bar{L}_{-1}) = L_{-1} \otimes \bar{L}_{-1} \tag{3.51c}$$
$$\Delta^L(\bar{L}_{-1})\Delta^R(L_{-1}) = (\bar{L}_{-1} \otimes 1)(1 \otimes L_{-1}) = \bar{L}_{-1} \otimes L_{-1}. \tag{3.51d}$$

Clearly from the possible combinations, with respect to symmetrization of identical particles, the two states are

$$\left(\frac{\Delta^L(L_{-1}\bar{L}_{-1}) + \Delta^R(L_{-1}\bar{L}_{-1})}{2}\right)(|t\rangle \otimes |t\rangle) = \frac{L_{-1}\bar{L}_{-1}|t\rangle \otimes |t\rangle + |t\rangle \otimes L_{-1}\bar{L}_{-1}|t\rangle}{2}, \tag{3.52}$$

and

$$\left(\frac{\Delta^L(L_{-1})\Delta^R(\bar{L}_{-1})+\Delta^L(\bar{L}_{-1})\Delta^R(L_{-1})}{2}\right)(|t\rangle\otimes|t\rangle)=$$
$$\frac{L_{-1}\,|t\rangle\otimes\bar{L}_{-1}\,|t\rangle+\bar{L}_{-1}\,|t\rangle\otimes L_{-1}\,|t\rangle}{2}. \tag{3.53}$$

Looking at the last example, we can see that the classical structure of a Lie (primitive) coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$ would spoil the counting for $x = L_1$ or $x = L_{-1}$, as the expression of $\Delta(L_{-1}\bar{L}_{-1})$ is different from the one obtained above (basically the four coproducts in Eqs. (3.51)) would give us one state instead of the two obtained). Another example is the two states counted by $q^6\bar{q}^0$. In analogy with the above exposition, one of the states is

$$\left(\frac{\Delta^L(L_{-1}^2)+\Delta^R(L_{-1}^2)}{2}\right)(|t\rangle\otimes|t\rangle)=\frac{L_{-1}^2\,|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}^2\,|t\rangle}{2}. \tag{3.54}$$

The other state is $L_{-1}\,|t\rangle\otimes L_{-1}\,|t\rangle$, which requires the need of the grouplike coproduct. A list of states constructed in the same way up to counting multiplicity for the 2- and 3-particle states of the LCFT dual to CCTMG, and for the 2-particle states of the LCFT dual to CCNMG appears in Appendices B, C and D respectively.

3.4 Full log-spectrum in holographic LCFT

In this section, we consider the full spectrum of logarithmic states, *i.e* the states created from combinations involving modes L_{-n} with $n \geq 1$. At the level of the partition function, this means multiplying the prefactor

$$\prod_{n=2}^{\infty}\frac{q^2}{|1-q^n|^2}, \tag{3.55}$$

into the expansion $\mathcal{C}(q,\bar{q})$.

3.4.1 Single particle sector

As far as the single particle sector is concerned, from the sub-partition function given as

$$\prod_{n=1}^{\infty}\frac{q^2}{|1-q|^2}, \tag{3.56}$$

all holomorphic, anti-holomorphic and mixed descendant states appear trivially.

3.4.2 Multi particle sector

When it comes to multi particle states, one could ask whether the formalism exposed in section 3.3 holds. This turns not to be the case, and such modes can only work in a classical Lie algebra scenario with a coproduct of the type $\Delta(x) = x \otimes 1 + 1 \otimes x$, according to the counting of the partition function. In the case of 2-particles

$$\prod_{n=2}^{\infty} \frac{q^2}{|1-q^n|^2} \left[\frac{1}{2!} Y_2 \left(q^2\right)^2\right] = \prod_{n=2}^{\infty} \frac{q^2}{|1-q^n|^2} \left[\frac{g_1^2 + g_2}{2!} \left(q^2\right)^2\right] \tag{3.57a}$$
$$= q^4 \tag{3.57b}$$
$$+ (q^5 + q^4 \bar{q}^1) \tag{3.57c}$$
$$+ (3q^6 + 2q^5 \bar{q}^1 + 3q^4 \bar{q}^2) \tag{3.57d}$$
$$+ (4q^7 + 4q^6 \bar{q}^1 + 4q^5 \bar{q}^2 + 4q^4 \bar{q}^3) \tag{3.57e}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots \tag{3.57f}$$

If we consider the four 2-particle states counted by $q^7 \bar{q}^0$, the first three states are

$$\left(\frac{\Delta^L(L_{-1}^3) + \Delta^R(L_{-1}^3)}{2}\right)(|t\rangle \otimes |t\rangle) = \frac{L_{-1}^3 |t\rangle \otimes |t\rangle + |t\rangle \otimes L_{-1}^3 |t\rangle}{2}, \tag{3.58}$$

$$\left(\frac{\Delta^L(L_{-1}^2)\Delta^R(L_{-1}) + \Delta^L(L_{-1})\Delta^R(L_{-1}^2)}{2}\right)(|t\rangle \otimes |t\rangle) = \frac{L_{-1}^2 |t\rangle \otimes L_{-1} |t\rangle + L_{-1} |t\rangle \otimes L_{-1}^2 |t\rangle}{2}, \tag{3.59}$$

and using the classical primitive like element

$$\Delta L_{-3}(|t\rangle \otimes |t\rangle) = \frac{1}{2}\left(L_{-3} \otimes 1 + 1 \otimes L_{-3}\right)(|t\rangle \otimes |t\rangle) \tag{3.60a}$$
$$= \frac{1}{2}(L_{-3} |t\rangle \otimes |t\rangle + |t\rangle \otimes L_{-3} |t\rangle). \tag{3.60b}$$

Then, the fourth state being a combination of L_{-1} and L_{-2} modes, the result is

$$(\Delta L_{-2})(\Delta L_{-1})(|t\rangle \otimes |t\rangle) = \frac{1}{2}(L_{-2} |t\rangle \otimes |t\rangle + |t\rangle \otimes L_{-2} |t\rangle) \times \frac{1}{2}(L_{-1} |t\rangle \otimes |t\rangle + |t\rangle \otimes L_{-1} |t\rangle) = \frac{L_{-1}L_{-2} |t\rangle \otimes |t\rangle + L_{-1} |t\rangle \otimes L_{-2} |t\rangle + L_{-2} |t\rangle \otimes L_{-1} |t\rangle + |t\rangle \otimes L_{-1}L_{-2} |t\rangle}{4}. \tag{3.61}$$

Similarly, when considering the space of states outside $\mathcal{C}(q, \bar{q})$, there is one more state counted by $q^4 \bar{q}^2$ than the two states constructed in section 5, which is

$$\frac{L_{-1}\bar{L}_{-2} |t\rangle \otimes |t\rangle + L_{-1} |t\rangle \otimes \bar{L}_{-2} |t\rangle + \bar{L}_{-2} |t\rangle \otimes L_{-1} |t\rangle + |t\rangle \otimes L_{-1}\bar{L}_{-2} |t\rangle}{4}. \tag{3.62}$$

The same prescription can be applied to n-particles.

3.5 Generalization to new massive and higher spin topological critical massive gravities

In this section, we generalize the above results to New Massive Gravity and Topologically Massive Spin-3 Gravity at the critical point.

3.5.1 Partition function of Critical New Massive Gravity

New Massive Gravity (NMG) is a recently discovered three-dimensional theory of gravity that propagates massive positive-energy spin 2 modes of helicities ± 2 in a Minkowski vacuum [105, 106]. Thanks to properties such as super-renormalizability [107], it has been regarded as a promising candidate for a fully consistent three-dimensional theory of quantum gravity with massive gravitons.

At the critical point, while TMG has a logarithmic behaviour on the left-hand sector, in NMG, both left-hand and right-hand sectors are logarithmic. Furthermore, in the case of NMG, $c_L = c_R = 0$ and $b_L = b_R = \frac{-12l}{G_N}$.

The partition function of NMG at the critical point was obtained in [39], and given the form

$$Z_{\text{NMG}}(q,\bar{q}) \quad = \quad \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}} \prod_{l=0}^{\infty} \prod_{\bar{l}=2}^{\infty} \frac{1}{1-q^l \bar{q}^{\bar{l}}} \tag{3.63}$$

It was then compared to the partition function of the dual LCFT, with the following single-particle match

$$Z^{(0)\text{NMG}_{\text{LCFT}}}(q,\bar{q}) = Z_{\Omega} + Z_t \tag{3.64a}$$

$$= \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left(1 + \frac{q^2+\bar{q}^2}{|1-q|^2}\right). \tag{3.64b}$$

Following the derivation obtained for Z_{TMG} in the previous section, in the case of Z_{NMG}, we have

$$Z_{\text{NMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=0}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^2 q^m \bar{q}^{\bar{m}}} \prod_{l=0}^{\infty} \prod_{\bar{l}=0}^{\infty} \frac{1}{1-\bar{q}^2 q^l \bar{q}^{\bar{l}}} \tag{3.65a}$$

$$= \mathcal{A}(q,\bar{q}) \cdot \mathcal{B}(q,\bar{q}) \cdot \bar{\mathcal{B}}(q,\bar{q}). \tag{3.65b}$$

Using the Bell polynomial prescription, Eq. (3.65) can be expressed as

$$Z_{\text{NMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left(\sum_{j=0}^{\infty} \frac{Y_j}{j!} \left(q^2\right)^j\right) \left(\sum_{k=0}^{\infty} \frac{Y_k}{k!} \left(\bar{q}^2\right)^k\right). \tag{3.66}$$

The equation above can be rewritten in a binomial-type form as

$$Z_{\text{NMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left[\sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{Y_j}{j!}\right) \left(\frac{Y_{k-j}}{(k-j)!}\right) (q^2)^j (\bar{q}^2)^{k-j}\right]. \tag{3.67}$$

3.5.2 Partition function of Critical Topologically Massive Spin-3 Gravity

In [108], topologically massive gravity was generalized to higher spins, with a special attention given to spin-3, and in [109], the 1-loop partition function for topologically massive higher spin gravity (TMHSG) for arbitrary spin was calculated, and given the closed form

$$Z^{(s)}_{TMHSG} = \prod_{s=2}^{N}\left[\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=0}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}}\right]\times\left[\prod_{s=3}^{N}\prod_{t=2}^{s-1}\prod_{p=r(s,t)}^{\infty}\prod_{\bar{p}=k(s,t)}^{\infty}\frac{1}{1-q^p\bar{q}^{\bar{p}}}\right], \tag{3.68}$$

with

$$k(s,m) = \frac{s(s-1)-(s-m+1)(s-m-1)}{2(s-m)}, \qquad r(s,m) = k(s,m)+s-m. \tag{3.69}$$

In deriving the above expression, the contribution to the full partition function from an arbitrary spin-s field at the chiral point was found to be

$$Z^{(s)} = \prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=0}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}}\prod_{t=2}^{s-1}\prod_{p=r(s,t)}^{\infty}\prod_{\bar{p}=k(s,t)}^{\infty}\frac{1}{1-q^p\bar{q}^{\bar{p}}}. \tag{3.70}$$

In the case of a spin-3 field added, the partition function then takes the form

$$\begin{aligned} Z^{(3)}_{\text{TMHSG}}(q,\bar{q}) &= \prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=2}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}} \\ &\times\left[\prod_{n=3}^{\infty}\frac{1}{|1-q^n|^2}\prod_{l=3}^{\infty}\prod_{\bar{l}=0}^{\infty}\frac{1}{1-q^l\bar{q}^{\bar{l}}}\prod_{k=4}^{\infty}\prod_{\bar{k}=3}^{\infty}\frac{1}{1-q^k\bar{q}^{\bar{k}}}\right]. \end{aligned} \tag{3.71}$$

Before writing the expression of $Z^{(3)}_{\text{TMHSG}}$ in terms of Bell polynomials, we first note that quite interestingly, starting from an expression coming from gravity on the left-hand side, one has an expression that features W-algebra characters on the right-hand side. Indeed

$$\begin{aligned} Z^{(3)}_{\text{TMHSG}}(q,\bar{q}) = &\left\{\prod_{n=2}^{\infty}\frac{1}{|1-q^n|^2}\prod_{n=3}^{\infty}\frac{1}{|1-q^n|^2}\right\}\prod_{m=2}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}} \\ &\times\left[\prod_{l=3}^{\infty}\prod_{\bar{l}=0}^{\infty}\frac{1}{1-q^l\bar{q}^{\bar{l}}}\prod_{k=4}^{\infty}\prod_{\bar{k}=3}^{\infty}\frac{1}{1-q^k\bar{q}^{\bar{k}}}\right] \end{aligned} \tag{3.72a}$$

$$\begin{aligned} &= \chi_0(\mathcal{W}_3)\times\bar{\chi}_0(\mathcal{W}_3)\prod_{m=2}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{1-q^m\bar{q}^{\bar{m}}} \\ &\times\left[\prod_{l=3}^{\infty}\prod_{\bar{l}=0}^{\infty}\frac{1}{1-q^l\bar{q}^{\bar{l}}}\prod_{k=4}^{\infty}\prod_{\bar{k}=3}^{\infty}\frac{1}{1-q^k\bar{q}^{\bar{k}}}\right], \end{aligned} \tag{3.72b}$$

where $\chi_0(\mathcal{W}_3)$ and $\bar{\chi}_0(\mathcal{W}_3)$ are the holomorphic and antiholomorphic vacuum characters of the W_3-algebra.
In terms of Bell polynomial expansions, the above partition function then reads

$$\begin{aligned} Z^{(3)}_{\mathrm{TMHSG}}(q,\bar{q}) &= \chi_0(\mathcal{W}_3) \times \bar{\chi}_0(\mathcal{W}_3) \left(\sum_{j=0}^{\infty} \frac{Y_j}{j!} \left(q^2\right)^j \right) \\ &\times \left(\sum_{k=0}^{\infty} \frac{Y_k}{k!} \left(q^3\right)^k \right) \left(\sum_{l=0}^{\infty} \frac{Y_l}{l!} \left(q^4 \bar{q}^3\right)^l \right). \end{aligned} \tag{3.73}$$

The interpretation and counting of the states leading to the first and second square brackets in the above expression is very similar to that discussed for TMG: the n-th term in the first bracket corresponds to states built upon $t \otimes_n t$, the n-th term in the second bracket corresponds to n-particle states of w, the logarithmic partner of $\mathcal{W}$, while the mixed terms will be counted similarly to the NMG case above. The counting that leads to the third bracket is less clear, and as mentioned in [109] this term arises from the trace part of the spin-3 field.

In conclusion to this chapter, we have shown that the Bell polynomial recasting of the partition function is applicable not only to TMG, but to many other critical gravities. In the next chapter, we show that the Bell polynomials expansion can be rewritten as the Plethystic Exponential.

Chapter 4

Connections to the Plethystic Exponential

In this chapter, We would like to draw a parallel between our work and results coming from the *Plethystic Program* initiated in [102] and [110]. Before we make the connection explicit, we first briefly discuss what *plethysm* means, and how it relates to the Bell polynomials.

4.1 Plethysm of exponential functions

The term plethysm originates from the work of Littlewood in 1936 [111] who introduced it as an operation. Also called *substitution* or *composition*, plethysm has been widely used in combinatorics, group theory or invariant theory, and has played a fundamental role in physics when one applies the theory of group representation [112, 113, 114]. Considering the operation of composition for symmetric functions, if one takes f and g to be symmetric functions, the plethysm of f and g is denoted $f[g]$ or $f \circ g$.

As an application of plethysm, it can be shown that $\mathcal{B}(q, \bar{q})$ is a composition of functions. Indeed, let us consider a function g such that

$$g(q) = \sum_{n=1}^{\infty} \frac{g_n}{n!} q^n, \tag{4.1}$$

and the exponential function

$$h(q) = \exp\{q\}. \tag{4.2}$$

Then the composition $f = h \circ g$ yields

$$f(q) = \exp\left\{ \sum_{n=1}^{\infty} \frac{g_n}{n!} q^n \right\}. \tag{4.3}$$

Up to order 3, the exponential function can be expanded in the following way

$$f(q) = e^{\left(g_1 q + \frac{1}{2} g_2 q^2 + \frac{1}{6} g_3 q^3 + \cdots\right)} \tag{4.4a}$$

$$= e^{(g_1 q)} \cdot e^{\left(\frac{1}{2} g_2 q^2\right)} \cdot e^{\left(\frac{1}{6} g_3 q^3\right)} \ldots \tag{4.4b}$$

$$= \left(1 + g_1 q + \frac{1}{2} g_1^2 q^2 + \frac{1}{6} g_1^3 q^3 + \cdots\right)\left(1 + \frac{1}{2} g_2 q^2 + \cdots\right)\left(1 + \frac{1}{6} g_3 q^3 + \cdots\right)\cdots$$

$$= 1 + g_1 q + \frac{1}{2}\left(g_2 + g_1^2\right) q^2 + \frac{1}{6}\left(g_3 + 3 g_1 g_2 + g_1^3\right) q^3 + \cdots \tag{4.4c}$$

If now we take $g_n = (n-1)! \frac{1}{|1-q|^2}$, it is easy to see that $f(q, \bar{q}) = \mathcal{B}(q, \bar{q})$. This shows the relationship between the plethysm of exponential functions and Bell polynomials.

4.2 Plethystic exponential and Bell polynomials

A method of counting operators from generating functions was proposed in [102]. An essential ingredient in that program was the so-called *Plethystic Exponential*, used to get the generating function of multi-trace operators from the generating function of single-trace operators at large N. Under the name of *Plethystic Program*, this method of counting found many applications (see for instance [115] [116] [117] [118] [119] [120] [121] [122] and references therein).

In particular, we recall the *bosonic Plethystic function* [123]

$$\prod_{n=0}^{\infty} \frac{1}{(1-\nu q^n)^{a_n}} = PE^{\mathcal{B}}[\mathcal{G}_1(q)] \equiv \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k}{k} \mathcal{G}_1\left(q^k\right)\right) = \sum_{N=0}^{\infty} \nu^N \mathcal{G}_N(q), \tag{4.5}$$

with

$$\mathcal{G}_1(q) = \sum_{n=0}^{\infty} a_n q^n, \tag{4.6}$$

where the integer a_n indicates the number of operators with dimension n. The relation between Eqs. (4.5) and (4.6) can be explained following [102] as follows. Let

$$\mathcal{G}_1(q) = \sum_{n=0}^{\infty} a_n q^n \tag{4.7}$$

be a Taylor expansion, where a_n is the number of independent variables q of degree n. By definition [102]

$$PE^{\mathcal{B}}[\mathcal{G}_1(q)] := \exp\left(\sum_{k=1}^{\infty} \frac{\mathcal{G}_1\left(q^k\right) - \mathcal{G}_1(0)}{k}\right). \tag{4.8}$$

By series-expansion, it yields

$$PE^{\mathcal{B}}[\mathcal{G}_1(q)] = \exp\left(\sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \frac{q^{nk}}{k} - a_0 \sum_{k=0}^{\infty} \frac{1}{k}\right) \tag{4.9a}$$

$$= \exp\left(-\sum_{n=0}^{\infty} a_n \log\left(1-q^n\right) - a_0 \sum_{k=0}^{\infty} \frac{1}{k}\right) \tag{4.9b}$$

$$= \exp\left(-\sum_{n=1}^{\infty} a_n \log\left(1-q^n\right)\right) \tag{4.9c}$$

$$= \frac{1}{\prod_{n=1}^{\infty}\left(1-q^n\right)^{a_n}}. \tag{4.9d}$$

In addition, introducing one more variable ν to be inserted into the summand such that upon expansion, the power of ν counts the number of single-trace functions, one recovers Eq. (4.5).

As explained in [123], the above function takes a certain function $\mathcal{G}_1(q)$ and generates new partition functions $\mathcal{G}_N(q)$ counting all possible N times symmetric products of the constitutents of $g_1(q)$, implementing in this way the bosonic statistics. Depending on what is being counted, the variable ν is sometimes referred to as root coordinates, fugacities or monomials in weight [120].

In [102], it is mentioned that the Plethystic Exponential in one variable q can be generalized by considering a set of variables q_i, such that

$$\prod_{p_1,\ldots,p_m}^{\infty} \frac{1}{\left(1-\nu q_1^{p_1}\cdots q_m^{p_m}\right)^{a_{p_1,\ldots,p_m}}} = PE^{\mathcal{B}}\left[\mathcal{G}_1\left(q_1,\ldots,q_m\right)\right] \tag{4.10a}$$

$$\equiv \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k}{k}\mathcal{G}_1\left(q_1^k,\ldots,q_m^k\right)\right) \tag{4.10b}$$

$$= \sum_{N=0}^{\infty} \nu^N \mathcal{G}_N(q_1,\ldots,q_m), \tag{4.10c}$$

with:

$$\mathcal{G}_1(q_1,\ldots,q_m) = \sum_{p_1,\ldots,p_m=0}^{\infty} a_{p_1,\ldots,p_m} q_1^{p_1}\cdots q_m^{p_m}. \tag{4.11}$$

We want to show that in the case of two variables, our results would be the same using the Plethystic Exponential prescription. Starting with the following specialization

$$a_{p_1,p_2} = 1 \tag{4.12a}$$

$$q_1 = q \tag{4.12b}$$

$$q_2 = \bar{q} \tag{4.12c}$$

$$p_1 = m \tag{4.12d}$$

$$p_2 = \bar{m} \tag{4.12e}$$

$$\nu = q^2, \tag{4.12f}$$

we immediately recover the double product $\mathcal{B}\left(q,\bar{q}\right)$. In our case, the term ν is specialized to be a monomial in weight. Then, taking:

$$\mathcal{G}_1(q,\bar{q}) = \sum_{m\geq 0,\bar{m}\geq 0}^{\infty} q^m \bar{q}^{\bar{m}}, \tag{4.13}$$

The plethystic exponential of $\mathcal{G}_1(q,\bar{q})$ is then

$$PE^{\mathcal{B}}\left[\mathcal{G}_1(q,\bar{q})\right] = \exp\left(\sum_{k=1}^{\infty} \frac{\left(q^2\right)^k}{k} \mathcal{G}_1\left(q^k,\bar{q}^k\right)\right) \tag{4.14}$$

Expanding the exponential in the RHS gives a series in powers of q^2:

$$\mathcal{B}\left(q,\bar{q}\right) = PE^{\mathcal{B}}\left[\mathcal{G}_1(q,\bar{q})\right] \tag{4.15a}$$

$$= 1 + \mathcal{G}_1(q,\bar{q})\left(q^2\right) + \frac{\mathcal{G}_1^2(q,\bar{q}) + \mathcal{G}_1\left(q^2,\bar{q}^2\right)}{2}\left(q^2\right)^2 \tag{4.15b}$$

$$+ \frac{\mathcal{G}_1^3(q,\bar{q}) + 3\mathcal{G}_1(q,\bar{q})\mathcal{G}_1\left(q^2,\bar{q}^2\right) + 2\mathcal{G}_1\left(q^3,\bar{q}^3\right)}{6}\left(q^2\right)^3 \tag{4.15c}$$

$$+ \frac{\mathcal{G}_1^4(q,\bar{q}) + 6\mathcal{G}_1^2(q,\bar{q})\mathcal{G}_1\left(q^2,\bar{q}^2\right) + 3\mathcal{G}_1^2\left(q^2,\bar{q}^2\right) + 8\mathcal{G}_1(q,\bar{q})\mathcal{G}_1\left(q^3,\bar{q}^3\right) + 6\mathcal{G}_1\left(q^4,\bar{q}^4\right)}{24}\left(q^2\right)^4$$

$$+ \cdots \tag{4.15d}$$

Finally, it is easy to see that the coefficients of $\left(q^2\right)^N$ are equal to the Bell polynomials at each order, by using the identification:

$$(k-1)!\mathcal{G}_1\left(q^k,\bar{q}^k\right) = g_k\left(q,\bar{q}\right). \tag{4.16}$$

From the above derivation, the Plethystic Program ascertains us that we have reorganized Z_{LCFT} in a way that clearly shows the single particle and multi particle Hilbert spaces of t.

We close this chapter by mentioning that the similarity between our method and the one of the Plethystic Program is quite normal since as we have seen, the Bell polynomials expansion is a result of plethysm associated with multipartite exponential generating functions. In the next section, we will see that using Bell polynomials allows to uncover hidden symmetry actions.

Chapter 5

Ladder operators and sl(2) action in the partition function of critical TMG

In this chapter, we wish to uncover some hidden structure in the sub partition function $\mathcal{B}(q,\bar{q})$ of Z_{TMG}, in terms of algebraic (ladder) operators acting on the Bell polynomials and on the Plethystic exponential. As we shall see, these ladder operators are the building blocks of of an sl_2 algebra that acts on characters of $\mathcal{B}(q,\bar{q})$.

Before starting, we would like to give a motivation for the construction of the ladder operators. For that, we recall the *Fock space*, which comes from particle physics as the state space for a system of variable number of elementary particles.

Fock spaces were designed as a framework to construct many-particle states. They typically represent the state space of an indefinite number of identical particles (an electron gas, photons, etc...). These particles can be classified in two types, bosons and fermions, and their Fock spaces look quite different. Fermionic Fock spaces are naturally representations of a Clifford algebra, where the generators correspond to adding or removing a particle in a given energy state. In a similar way, bosonic Fock space is naturally a representation of a Weyl algebra. We will be interested on the bosonic Fock space.

In generally, a Fock space is considered on a Hilbert space, but in the simplest case and for the purpose of our discussion, the bosonic vector space is obtained by considering a complex vector space $\mathbb{C}$. Then, the bosonic Fock space as a vector space is essentially a space of polynomials of infinitely many variables. A typical basis can be constructed using Schur symmetric functions. In our case, we consider a space of Bell polynomials Y_n of infinitely many variables $\{g_1, g_2, \cdots, g_n\}$.

The reason why Fock space is of interest to many people is that several important algebras act naturally on it. In the present case of the bosonic Fock space, we construct a combinatorial model of ladder operators that play the role of annihilation and creation operators identical to the model of creation and annihilation of particles in a field. We first show that these operators generate a Heisenberg-Weyl algebra that acts on the Bell polynomials, and that they are building blocks for generators of an $sl(2)$ action on the polynomials. Through the correspondence established between Bell polynomials and the plethystic exponential we show that the same applies to the latter.

5.1 Creation-annihilation operators acting on Bell polynomials

While studying Z_{TMG}, it was found that some ladder operators acting on Bell polynomials can be constructed. From these operators, generators of an $sl(2)$ algebra can be built, and act on characters of $\mathcal{B}(q,\bar{q})$. In what follows,we will show how ladder operators are constructed and how the $sl(2)$ action appears. For that, we will introduce the *monomiality principle*, which is a useful tool for studying properties of special polynomials, such as the Bell polynomial.

5.1.1 Monomiality principle

The idea of monomiality is rooted in the early 1940s, when J.F. Steffensen, in a paper [124] that only recently received attention, suggested the concept of *poweroid*. A resurgence of the theory arose in the work of G. Dattoli *et al*, who systematically made use of the principle [125] [126]. In essence, all polynomial families, in particular special polynomials, are identical as it suffices to transform a basic set of monomials using suitable (*derivative* and *multiplication*) operators to obtain the polynomials. This result, theoretically proved in [127] and [128], is closely related to the theory of *Umbral Calculus* [129] (coined by Sylvester), since the exponent, for instance in the monomial x^n, transforms into its "shadow" in the polynomial $p_n(x)$.

Definition and general properties

Since the advent of Quantum Mechanics, the nilpotent algebra with generators $\hat{P}$ and $\hat{M}$ satisfying the commutation relations

$$[\hat{P},\hat{M}] = 1, \quad [\hat{P},1] = [\hat{M},1] = 0, \tag{5.1}$$

has been widely used to deal with problems associated with canonical quantization. Since then, this algebra called *Heisenberg-Weyl algebra* has enjoyed applications in many areas ranging from quantum optics [126] to string theory for some discrete models in two-dimensional theories [130], as well as (applied with the monomiality principle), in combinatorial physics [131] [132]. As we will see, the generators of the Heisenberg-Weyl algebra can be realized in various ways.

The monomiality principle is based on the fact that a given family of polynomials of order n denoted $p_n(x)$, can be viewed as *quasi-monomial* under the action of two operators $\hat{P}$ and $\hat{M}$, called "derivative" and "multiplicative" operators respectively, if it satisfies the recurrence relations

$$\hat{M}p_n(x) = p_{n+1}(x) \tag{5.2a}$$

$$\hat{P}p_n(x) = np_{n-1}(x) \tag{5.2b}$$

$$p_n(0) = 1. \tag{5.2c}$$

These operators can immediately be seen as raising and lowering operators acting on $p_n(x)$. As a by-product of Eqs. (5.2), the eigenproperty of operator $\hat{M}\hat{P}$ appears as

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{5.3}$$

It is interesting to note that the operators $\hat{P}$ and $\hat{M}$ satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = 1, \tag{5.4}$$

hence displaying a Weyl algebra structure.

5.1.2 Multivariate Bell polynomials: a special case

We now define an operator $\hat{X}$ [1] as

$$\hat{X} = g_1 + \sum_{i=1}^{\infty} g_{i+1} \frac{\partial}{\partial g_i}. \tag{5.5}$$

This operator acts as a multiplication operator on the Bell polynomials in n variables denoted in the previous section as $Y(g_1, g_2, \ldots, g_n)$. For $Y_n = Y(g_1, g_2, \ldots, g_n)$, we therefore have

$$\hat{X}Y_n = Y_{n+1}. \tag{5.6}$$

We then define a second operator $\hat{D}$ as

$$\hat{D} = \frac{\partial}{\partial g_1}, \tag{5.7}$$

that acts as derivative operator on Y_n

$$\hat{D}Y_n = nY_{n-1}. \tag{5.8}$$

Finally, the operator $\hat{X}\hat{D}$ acts on Y_n as

$$\hat{X}\hat{D}Y_n = nY_n. \tag{5.9}$$

It is straightforward to verify that the operators $\hat{X}$ and $\hat{D}$ are generators of the Heisenberg-Weyl algebra. We will next show that these operators generate a $sl(2)$ algebra.

5.2 sl_2 action on the Bell polynomials

Following [134], we write the following definition

Definition 5.2.1 *Denote the basis for a standard sl_2 algebra, $\mathbb{K}$, by {f,e,h}, satisfying*

$$[f, e] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{5.10}$$

[1] This operator appears in [133]

We write $\mathbb{K}$ = *{f,e,h}.*

The following Lemma is well-known [134]

Lemma 5.2.2 *Given a Heisenberg-Weyl algebra, setting*

$$f = \frac{1}{2}\hat{X}^2, \quad h = \hat{X}\hat{D} + \frac{1}{2}, \quad e = \frac{1}{2}\hat{D}^2 \tag{5.11}$$

yields a standard sl_2 *algebra.*

From there, we write the following proposition.

Proposition 5.2.3 *Let* $\mathcal{Y}$ *be a representation of* sl_2*, with generators*

$$f = \frac{1}{2}\hat{X}^2, \quad h = \hat{X}\hat{D} + \frac{1}{2}, \quad e = \frac{1}{2}\hat{D}^2,$$

and Y_n *the multivariate Bell polynomials. If* $Y_n \in \mathcal{Y}$*, then*

$$eY_n = \frac{1}{2}n(n-1)Y_{n-2}, \tag{5.12a}$$

$$fY_n = \frac{1}{2}Y_{n+2}, \tag{5.12b}$$

$$hY_n = \left(n + \frac{1}{2}\right)Y_n. \tag{5.12c}$$

One can easily verify that the sl_2 generators $\{f, e, h\}$ act on the Bell polynomials

$$[e, f]Y_n = (ef - fe)Y_n \tag{5.13a}$$

$$= e\left(\frac{1}{2}Y_{n+2}\right) - f\left(\frac{1}{2}n(n-1)\right)Y_{n-2} \tag{5.13b}$$

$$= \frac{1}{4}\left[(n+2)(n+1) - n(n-1)\right]Y_n \tag{5.13c}$$

$$= \left(n + \frac{1}{2}\right)Y_n \tag{5.13d}$$

$$= hY_n. \tag{5.13e}$$

Similarly

$$[h, f]Y_n = (hf - fh)Y_n \tag{5.14a}$$

$$= h\left(\frac{1}{2}Y_{n+2}\right) - f\left(n + \frac{1}{2}\right)Y_n \tag{5.14b}$$

$$= \frac{1}{2}\left(n + \frac{5}{2}\right)Y_{n+2} - \frac{1}{2}\left(n + \frac{1}{2}\right)Y_{n+2} \tag{5.14c}$$

$$= 2\left(\frac{1}{2}Y_{n+2}\right) \tag{5.14d}$$

$$= 2fY_n, \tag{5.14e}$$

and

$$\begin{aligned}
[e,h]Y_n &= (eh - he)Y_n \\
&= e\left[\left(n+\frac{1}{2}\right)Y_n\right] - h\left[\frac{1}{2}n(n-1)Y_{n-2}\right] && (5.15a)\\
&= \left(n+\frac{1}{2}\right)n(n-1)\frac{1}{2}Y_{n-2} - \frac{1}{2}n(n-1)\left(n-2+\frac{1}{2}\right) && (5.15b)\\
&= 2\left(\frac{1}{2}n(n-1)Y_{n-2}\right) && (5.15c)\\
&= 2eY_n. && (5.15d)
\end{aligned}$$

Pictorially, we have the following two figures

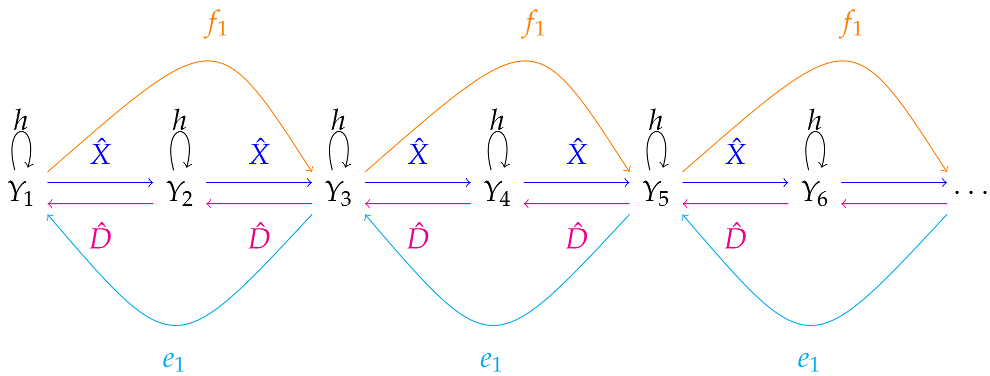


FIGURE 5.1: Ladder operators acting on $\mathcal{Y}$ (n odd)

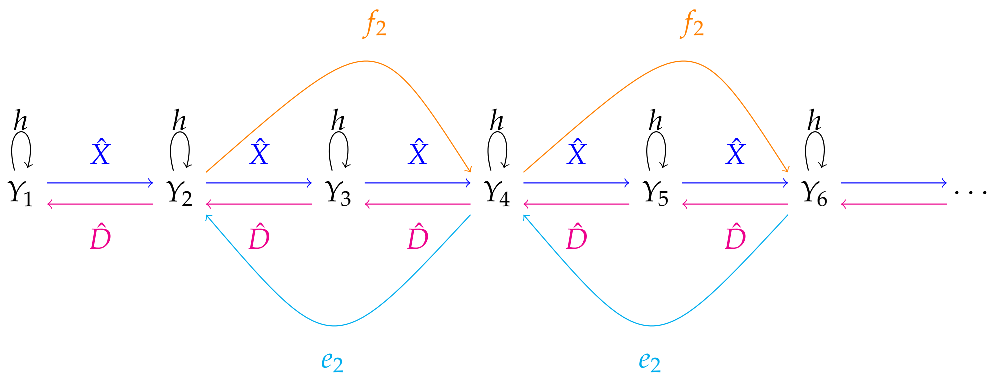


FIGURE 5.2: Ladder operators acting on $\mathcal{Y}$ (n even)

5.3 $sl(2)$ action in the Plethystic Exponential

In this section, we show how from the construction of new operators satisfying the Heisenberg-Weyl algebra, a hidden $sl(2)$ algebra can be obtained in successive terms of the Plethystic Exponential expansion of $PE^{\mathcal{B}}\left[\mathcal{G}_1(q,\bar{q})\right]$.

We start by establishing some notation conventions, writing the Plethystic Exponential expansion as

$$\begin{aligned} PE^{\mathcal{B}}\left[\mathcal{G}_1(q,\bar{q})\right] &= PE_{(1)}\left(q^2\right)^1 + \frac{1}{2!}PE_{(2)}\left(q^2\right)^2 \\ &+ \frac{1}{3!}PE_{(3)}\left(q^2\right)^3 + \dots, \end{aligned} \tag{5.16}$$

with

$$PE_{(1)} = \mathcal{G}_1(q,\bar{q}), \tag{5.17a}$$

$$PE_{(2)} = \mathcal{G}_1^2(q,\bar{q}) + \mathcal{G}_1\left(q^2,\bar{q}^2\right), \tag{5.17b}$$

$$PE_{(3)} = \mathcal{G}_1^3(q,\bar{q}) + 3\mathcal{G}_1(q,\bar{q})\mathcal{G}_1\left(q^2,\bar{q}^2\right) + 2\mathcal{G}_1\left(q^3,\bar{q}^3\right), \tag{5.17c}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{5.17d}$$

For the sake of clarity, we also write $\mathcal{G}_1\left(q^k,\bar{q}^k\right) = \mathcal{G}_{1,k}$.
Now, introducing the operator $\hat{\mathcal{X}}$ as

$$\hat{\mathcal{X}} = \mathcal{G}_{1,1} + \sum_{j=1}^{\infty} j\mathcal{G}_{1,j+1}\frac{\partial}{\partial \mathcal{G}_{1,j}}, \tag{5.18}$$

it is easy to see that it acts as a multiplication operator on $PE_{(k)}$ such that

$$\hat{\mathcal{X}}_k PE_{(k)} = PE_{(k+1)}. \tag{5.19}$$

We then define the operator $\hat{\mathcal{D}}$ as

$$\hat{\mathcal{D}} = \frac{\partial}{\partial \mathcal{G}_{1,1}}, \tag{5.20}$$

that acts as derivative operator on $PE_{(k)}$

$$\hat{\mathcal{D}}PE_{(k)} = kPE_{(k-1)}, \tag{5.21}$$

and finally, the operator $\hat{\mathcal{X}}\hat{\mathcal{D}}$ that acts on $PE_{(k)}$ as

$$\hat{\mathcal{X}}\hat{\mathcal{D}}PE_{(k)} = kPE_{(k)}. \tag{5.22}$$

From there, the following set of generators

$$f = \frac{1}{2}\hat{\mathcal{X}}^2, \quad h = \hat{\mathcal{X}}\hat{\mathcal{D}} + \frac{1}{2}, \quad e = \frac{1}{2}\hat{\mathcal{D}}^2 \tag{5.23}$$

satisfy a sl_2 algebra on $\mathcal{G}_{1,k}$.
The actions on $PE_{(k)}$ can be represented in the quiver diagrams below.

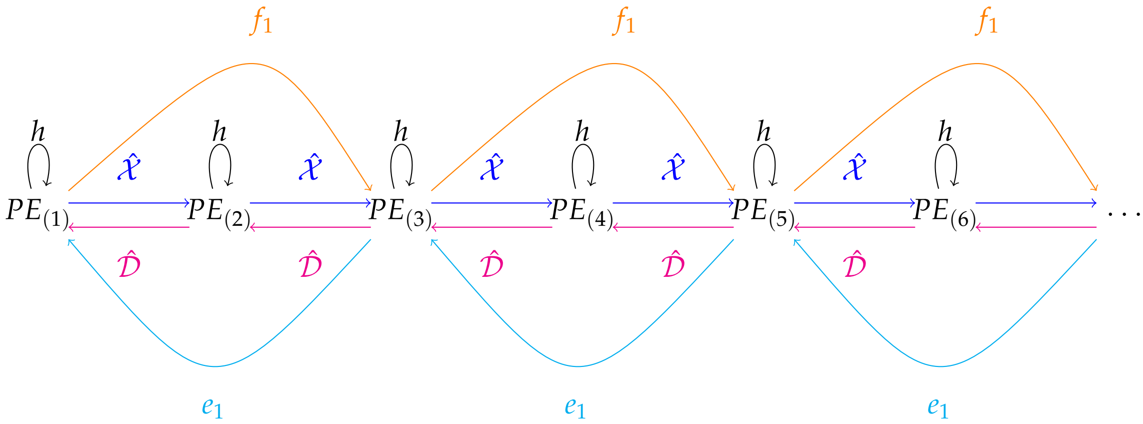


FIGURE 5.3: Ladder operators acting on $PE_{(k)}$ (k odd)

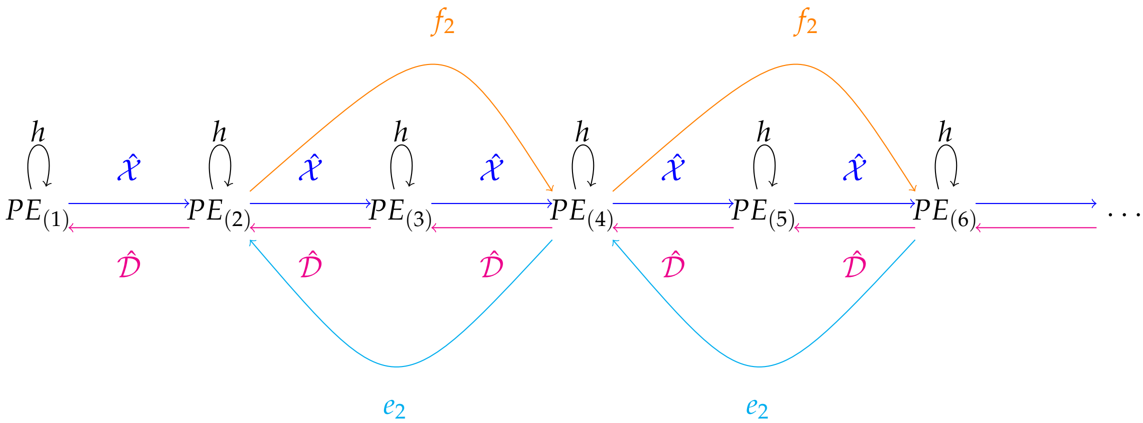


FIGURE 5.4: Ladder operators acting on $PE_{(k)}$ (n even)

Interpretation

As mentioned above, and confirmed through the Plethystic Program, the space of multi-trace representations in $\mathcal{B}(q, \bar{q})$ obtained using Bell polynomials is simply the Fock space of a symmetrized tensor product of single-trace representations [123]. We therefore interpret the above actions as coming from operators acting between Fock space characters. The characters of single and multi particles are all related via the action of ladder (multiplication and derivative) operators.

Chapter 6

Conclusion

The $AdS_3/LCFT_2$ correspondence is an interesting duality that deserves more attention. Indeed, many checks are still lacking in order to have a more concrete picture of the conjecture. In this thesis, we decided to investigate it from the point of view of the dual partition functions. Indeed, since the seminal results of [39] for critical TMG and NMG, the higher-spin generalization of [109] as well as the subsequent works [135] and [136], little progress has been made in relating partition functions of critical massive gravities to the one of their LCFT duals. This is in great part due to the lack of understanding of the multiparticle sector.

Through this work, we make a step further in that understanding by explicitly showing how Z_{TMG} can be beautifully recast in a compact form using Bell polynomials. These polynomials allow us to express at once character representations of single and multi particles. As such, the check formulated in [137] in terms of the combinatorics of the multi particle sector finds a positive answer.

In addition to the Bell polynomial reformulation of the partition functions of critical massive gravities, a precise relationship between the generating function of those polynomials and the Plethystic Exponential with appropriate specialization was derived. This fact is interesting given the amount of work available in the literature when it comes to the Plethystic Exponential. This will allow interesting studies of the logarithmic sector of the theory from algebraic geometry and group theory perspectives, such as the study of the moduli space of the logarithmic states and their associated orbits.

Another result derived as a mathematical excursion is the construction of differential operators that act as ladder operators on the n-th components of the partition function. As a result, further operators were constructed, displaying a hidden sl(2) action within the n-th components of the partition function.

An intriguing result is that the counting needs to be performed in a two-step process, with the combinatorics of the L_{-1} and $\bar{L}_{-1}$ action on the multiparticle sector being non-standard in that it doesn't arise from the usual Lie-algebraic coproduct $\Delta X = X \otimes 1 + 1 \otimes X$. The implications of these combinatorial arguments on the nature of the logarithmic sector merit even more scrutiny. Indeed, in retrospect, looking at the output of this thesis and the unilateral direction taken for many years, it is the opinion of the author that a lot of time was spent in forcefully trying to make a direct connection between the results obtained and results on logarithmic conformal field theory already existing in the literature. A different approach was taken toward the end of the author's doctoral years. Indeed, the choice was made to look at an interpretation of the combinatorial results discussed in this thesis without partiality towards LCFTs. It turns out from this angle that many things can be said, and a good outlook can be given.

Further work is ongoing to show that the partition function of critical TMG reveals the appearance of an orbifold sector in the theory. Indeed, it is well known that the cycle index of the symmetric group can be expressed in terms of Bell polynomials. The work done in this thesis immediately shows that what is being counted on the logarithmic sector is states invariant under action of the symmetric group. In fact, from the work of [138], there is a strong evidence that the sub-partition function of the logarithmic sector is the partition function of the untwisted part of a symmetric orbifold model.

Symmetric orbifold models have already been discussed in the physics literature as groupoids, with a certain Hopf algebra structure [139]. The half-coproducts that give the correct counting of the logarithmic sector are basic *quantum groupoid* objects. Quantum groupoids and Hopf algebroids are generalizations of Hopf algebras. In particular, a class of quantum groupoid called *Weak C^* Hopf Algebras* were introduced by Gabriella Böhm and K. Szlachányi in [103]. In that paper the authors mention that the simplest example for such a weak Hopf algebra can be found in studying the Quasiquantum groups related to orbifold models constructed by Dijkgraaf, Pasquier and Roche in [140]. This is further evidence towards the presence of an orbifold structure in critical TMG, and its dual LCFT.

Another direction of great interest related to the previous one, is to study the modular properties of the partition function. From the results expected above, one would expect a mock modular form that would need completion in order to recover "nice(r)" modular properties.

Lastly, it would be interesting to borrow the ideas in [141, 142, 143, 144], and extend them to recent findings. In fact, the realization that the partition functions of critical massive gravities could be recast in terms of Bell polynomials comes directly from [141, 142]. In these papers, it is shown that multipartite generating functions can be expressed in terms of Bell polynomials, which in turn appear in certain partition functions. These partition functions recast into infinite products eventually lead to the construction of quantum group invariants, as well as knot and link invariants. It would therefore be interesting to see whether our Bell polynomial log-partition function finds a use in the construction of topological invariants.

Appendix A

Bell Polynomials

This appendix is intended to give a succinct explanation of the Bell polynomials formula, with a focus on the partition notation $\vec{k} \vdash n$.
In the expression

$$Y_n(g_1, g_2, \ldots, g_n) = \sum_{\vec{k} \vdash n} \frac{n!}{k_1! \cdots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_n}{n!}\right)^{k_n}, \tag{A.1}$$

we define $\vec{k} \vdash n$ as

$$\vec{k} \vdash n = \{\vec{k} = (k_1, k_2, \ldots, k_n) \mid k_1 + 2k_2 + 3k_3 + \cdots + nk_n = n\}. \tag{A.2}$$

Let us see how this works up to order 3 in n. We actually start at order 2, since order 0 by convention gives $Y_0 = 1$, and order 1 trivially gives $Y_1 = g_1$.
At order 2, we have two options: $\{k_1, k_2\} = \{2, 0\}$ or $\{k_1, k_2\} = \{0, 1\}$. Clearly, in the first case we have $2 = 2 + 2(0)$, and in the second case, $2 = 0 + 2(1)$. This gives:

$$Y_2(g_1, g_2) = \frac{2!}{2!0!} \left(\frac{g_1}{1!}\right)^2 \left(\frac{g_2}{2!}\right)^0 + \frac{2!}{0!1!} \left(\frac{g_1}{1!}\right)^0 \left(\frac{g_2}{2!}\right)^1 \tag{A.3a}$$

$$= g_1^2 + g_2. \tag{A.3b}$$

At order 3, we have three options: $\{k_1, k_2, k_3\} = \{3, 0, 0\}$, $\{k_1, k_2, k_3\} = \{1, 1, 0\}$ and $\{k_1, k_2, k_3\} = \{0, 0, 1\}$.This gives

$$Y_3(g_1, g_2, g_3) = \frac{3!}{3!0!0!} \left(\frac{g_1}{1!}\right)^3 \left(\frac{g_2}{2!}\right)^0 \left(\frac{g_3}{3!}\right)^0 + \frac{3!}{1!1!0!} \left(\frac{g_1}{1!}\right)^1 \left(\frac{g_2}{2!}\right)^1 \left(\frac{g_3}{3!}\right)^0 + \frac{3!}{0!0!1!} \left(\frac{g_1}{1!}\right)^0 \left(\frac{g_2}{2!}\right)^0 \left(\frac{g_3}{3!}\right)^1 \tag{A.4a}$$

$$= g_1^3 + 3g_1 g_2 + g_3. \tag{A.4b}$$

A list of Bell polynomials up to order 10 can be found in [133] for instance.

Appendix B

Low-lying 2-particle states of LCFT dual to CCTMG

In this appendix, we would like to list more low-lying descendant states of the log-excitations constructed in the case of 2-particle states of LCFT dual to CCTMG. For that, we consider the highest weight state $|t\rangle \otimes |t\rangle$, and its descendants counted in Eq. (3.45).

$$
\begin{array}{rc}
q^4: & |t\rangle \otimes |t\rangle \\[2ex]
q^5: & \frac{L_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}|t\rangle}{2} \\
q^4\bar{q}^1: & \frac{\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-1}|t\rangle}{2} \\[2ex]
q^6: & \frac{L^2_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes L^2_{-1}|t\rangle}{2} \\
 & L_{-1}|t\rangle \otimes L_{-1}|t\rangle \\
 & \frac{L_{-2}|t\rangle\otimes L_{-2}|t\rangle}{2} \\
q^5\bar{q}^1: & \frac{L_{-1}\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}\bar{L}_{-1}|t\rangle}{2} \\
 & \frac{L_{-1}|t\rangle\otimes\bar{L}_{-1}|t\rangle+\bar{L}_{-1}|t\rangle\otimes L_{-1}|t\rangle}{2} \\[2ex]
q^4\bar{q}^2: & \frac{\bar{L}_{-1}\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-1}\bar{L}_{-1}|t\rangle}{2} \\
 & \bar{L}_{-1}|t\rangle \otimes \bar{L}_{-1}|t\rangle \\
 & \frac{\bar{L}_{-2}|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-2}|t\rangle}{2} \\[2ex]
q^7: & \frac{L^3_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes L^3_{-1}|t\rangle}{2} \\
 & \frac{L^2_{-1}|t\rangle\otimes L_{-1}|t\rangle+L_{-1}|t\rangle\otimes L^2_{-1}|t\rangle}{2} \\
 & \frac{L_{-3}|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-3}|t\rangle}{2} \\
 & \frac{L_{-1}L_{-2}|t\rangle\otimes|t\rangle+L_{-1}|t\rangle\otimes L_{-2}|t\rangle+L_{-2}|t\rangle\otimes L_{-1}|t\rangle+|t\rangle\otimes L_{-1}L_{-2}|t\rangle}{4}
\end{array}
$$

$$
\begin{array}{ll}
q^6\bar{q}^1: & \frac{L_{-1}^2\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}^2\bar{L}_{-1}|t\rangle}{2} \\
& \frac{L_{-1}^2|t\rangle\otimes\bar{L}_{-1}|t\rangle+\bar{L}_{-1}|t\rangle\otimes L_{-1}^2|t\rangle}{2} \\
& \frac{L_{-1}\bar{L}_{-1}|t\rangle\otimes L_{-1}|t\rangle+L_{-1}|t\rangle\otimes L_{-1}\bar{L}_{-1}|t\rangle}{2} \\
& \frac{L_{-2}\bar{L}_{-1}|t\rangle\otimes|t\rangle+L_{-2}|t\rangle\otimes\bar{L}_{-1}|t\rangle+\bar{L}_{-1}|t\rangle\otimes L_{-2}|t\rangle+|t\rangle\otimes L_{-2}\bar{L}_{-1}|t\rangle}{4} \\
q^5\bar{q}^2: & \frac{L_{-1}\bar{L}_{-1}^2|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}\bar{L}_{-1}^2|t\rangle}{2} \\
& \frac{L_{-1}\bar{L}_{-1}|t\rangle\otimes\bar{L}_{-1}|t\rangle+\bar{L}_{-1}|t\rangle\otimes L_{-1}\bar{L}_{-1}|t\rangle}{2} \\
& \frac{\bar{L}_{-1}^2|t\rangle\otimes L_{-1}|t\rangle+L_{-1}|t\rangle\otimes\bar{L}_{-1}^2|t\rangle}{2} \\
& \frac{L_{-1}\bar{L}_{-2}|t\rangle\otimes|t\rangle+L_{-1}|t\rangle\otimes\bar{L}_{-2}|t\rangle+\bar{L}_{-2}|t\rangle\otimes L_{-1}|t\rangle+|t\rangle\otimes L_{-1}\bar{L}_{-2}|t\rangle}{4} \\
q^4\bar{q}^3: & \frac{\bar{L}_{-1}^3|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-1}^3|t\rangle}{2} \\
& \frac{\bar{L}_{-1}^2|t\rangle\otimes\bar{L}_{-1}|t\rangle+\bar{L}_{-1}|t\rangle\otimes\bar{L}_{-1}^2|t\rangle}{2} \\
& \frac{\bar{L}_{-3}|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-3}|t\rangle}{2} \\
& \frac{\bar{L}_{-2}\bar{L}_{-1}|t\rangle\otimes|t\rangle+\bar{L}_{-2}|t\rangle\otimes\bar{L}_{-1}|t\rangle+\bar{L}_{-1}|t\rangle\otimes\bar{L}_{-2}|t\rangle+|t\rangle\otimes\bar{L}_{-1}\bar{L}_{-2}|t\rangle}{4} \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Appendix C

Low-lying 3-particle states of LCFT dual to CCTMG

In the same way, we consider the 3-particle highest weight state $|t\rangle \otimes |t\rangle \otimes |t\rangle$, and its descendants whose counting is given in the expansion $\mathcal{C}(q, \bar{q})$ of Eq. (3.33) as

$$\frac{g_1^3 + 3g_1g_2 + g_3}{3!}\left(q^2\right)^3 . \tag{C.1}$$

The counting organization in terms of conformal dimension looks like

$$\begin{aligned}
\prod_{n=2}^{\infty}\frac{q^2}{|1-q^n|^2}\left[\frac{1}{3!}Y_3\left(q^2\right)^3\right] &= \prod_{n=2}^{\infty}\frac{q^2}{|1-q^n|^2}\left[\frac{g_1^3+3g_1g_2+g_3}{3!}\left(q^2\right)^3\right] && \text{(C.2a)}\\
&= q^6 && \text{(C.2b)}\\
&+ (q^7 + q^6\bar{q}^1) && \text{(C.2c)}\\
&+ (3q^8 + 2q^7\bar{q}^1 + 3q^6\bar{q}^2) && \text{(C.2d)}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , && \text{(C.2e)}
\end{aligned}$$

and the corresponding states built upon $|t\rangle \otimes |t\rangle \otimes |t\rangle$ are listed as

$$\begin{aligned}
q^6: &\quad |t\rangle \otimes |t\rangle \otimes |t\rangle\\
q^7: &\quad \frac{L_{-1}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes L_{-1}|t\rangle}{3}\\
q^6\bar{q}^1: &\quad \frac{\bar{L}_{-1}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes\bar{L}_{-1}|t\rangle}{3}\\
q^8: &\quad \frac{L_{-1}L_{-1}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}L_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes L_{-1}L_{-1}|t\rangle}{3}\\
&\quad \frac{L_{-1}|t\rangle\otimes L_{-1}|t\rangle\otimes|t\rangle+L_{-1}|t\rangle\otimes|t\rangle\otimes L_{-1}|t\rangle+|t\rangle\otimes L_{-1}|t\rangle\otimes L_{-1}|t\rangle}{3}\\
&\quad \frac{L_{-2}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-2}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes L_{-2}|t\rangle}{3}\\
q^7\bar{q}^1: &\quad \frac{L_{-1}\bar{L}_{-1}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes L_{-1}\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes L_{-1}\bar{L}_{-1}|t\rangle}{3}\\
&\quad \frac{L_{-1}|t\rangle\otimes\bar{L}_{-1}|t\rangle\otimes|t\rangle+L_{-1}|t\rangle\otimes|t\rangle\otimes\bar{L}_{-1}|t\rangle+L_{-1}|t\rangle\otimes\bar{L}_{-1}|t\rangle\otimes|t\rangle}{3}\\
q^6\bar{q}^2: &\quad \frac{\bar{L}_{-1}\bar{L}_{-1}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-1}\bar{L}_{-1}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes\bar{L}_{-1}\bar{L}_{-1}|t\rangle}{3}\\
&\quad \frac{\bar{L}_{-1}|t\rangle\otimes L\bar{L}_{-1}|t\rangle\otimes|t\rangle+\bar{L}_{-1}|t\rangle\otimes|t\rangle\otimes\bar{L}_{-1}|t\rangle+|t\rangle\otimes\bar{L}_{-1}|t\rangle\otimes\bar{L}_{-1}|t\rangle}{3}\\
&\quad \frac{\bar{L}_{-2}|t\rangle\otimes|t\rangle\otimes|t\rangle+|t\rangle\otimes\bar{L}_{-2}|t\rangle\otimes|t\rangle+|t\rangle\otimes|t\rangle\otimes\bar{L}_{-2}|t\rangle}{3}\\
\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Appendix D

Low-lying 2-particle states of LCFT dual to CCNMG

In this appendix, we would like to show how the descendant states of the log-excitations are formed in the case of CCNMG (critical cosmological new massive gravity). The focus is on the lowest level of the multi particle sector.

The partition function for CCNMG was found in terms of Bell polynomials as

$$Z_{\rm NMG}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left[\sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{Y_j}{j!}\right) \left(\frac{Y_{k-j}}{(k-j)!}\right) (q^2)^j (\bar{q}^2)^{k-j} \right] \tag{D.1a}$$

$$= \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \mathcal{D}(q,\bar{q}), \tag{D.1b}$$

where the terms in the square bracket can be expanded in the following way

$$\mathcal{D}(q,\bar{q}) = 1 \tag{D.2a}$$

$$+ \frac{Y_1}{1!}\left(q^2\right) + \frac{Y_1}{1!}\left(\bar{q}^2\right) \tag{D.2b}$$

$$+ \frac{Y_2}{2!}\left(q^2\right)^2 + \frac{Y_1}{1!}\frac{Y_1}{1!}\left(q^2\right)\left(\bar{q}^2\right) + \frac{Y_2}{2!}\left(\bar{q}^2\right)^2 \tag{D.2c}$$

$$+ \frac{Y_3}{3!}\left(q^2\right)^3 + \frac{Y_2}{2!}\frac{Y_1}{1!}\left(q^2\right)^2\left(\bar{q}^2\right) + \frac{Y_1}{1!}\frac{Y_2}{2!}\left(q^2\right)\left(\bar{q}^2\right)^2 + \frac{Y_3}{3!}\left(\bar{q}^2\right)^3 \tag{D.2d}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{D.2e}$$

The single particle sector is composed of the holomorphic part $\frac{Y_1}{1!}\left(q^2\right)$, and the anti-holomorphic sector $\frac{Y_1}{1!}\left(\bar{q}^2\right)$. The states formed in the holomorphic sector were constructed above in the case of CCTMG. The states of the anti-holomorphic sector can be constructed analogously. The multiparticle starts with the lowest level of order 2. The states in $\frac{Y_2}{2!}\left(q^2\right)^2$ were constructed above, and here again, the states counted by $\frac{Y_2}{2!}\left(\bar{q}^2\right)^2$ can be constructed identically. We now show how one can construct the tower of states counted by $\frac{Y_1}{1!}\frac{Y_1}{1!}\left(q^2\right)\left(\bar{q}^2\right)$.

$$
\begin{aligned}
\frac{Y_1}{1!}\frac{Y_1}{1!}\left(q^2\right)\left(\bar{q}^2\right) &= q^2\bar{q}^2 \\
&+ 2q^3\bar{q}^2 + 2q^2\bar{q}^3 \\
&+ 3q^4\bar{q}^2 + 4q^3\bar{q}^3 + 3q^2\bar{q}^4 \\
&+ 4q^5\bar{q}^2 + 6q^4\bar{q}^3 + 6q^3\bar{q}^4 + 4q^2\bar{q}^5 \\
&+ 5q^6\bar{q}^2 + 8q^5\bar{q}^3 + 9q^4\bar{q}^4 + 8q^3\bar{q}^5 + 5q^2\bar{q}^6 \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Some of the corresponding lower level states are

$$
\begin{array}{ll}
q^2\bar{q}^2: & |t\rangle \otimes |\bar{t}\rangle \\
 & \\
q^3\bar{q}^2: & L_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L_{-1}|\bar{t}\rangle \\
q^2\bar{q}^3: & \bar{L}_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \\
 & \\
q^4\bar{q}^2: & L_{-1}L_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad L_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \quad ; \quad |t\rangle \otimes L_{-1}L_{-1}|\bar{t}\rangle \\
q^3\bar{q}^3: & L_{-1}\bar{L}_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L_{-1}\bar{L}_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \\
q^2\bar{q}^4: & \bar{L}_{-1}\bar{L}_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad |t\rangle \otimes \bar{L}_{-1}\bar{L}_{-1}|\bar{t}\rangle \\
 & \\
q^5\bar{q}^2: & L^3_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L^3_{-1}|\bar{t}\rangle\ ; \\
 & L^2_{-1}L_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \quad ; \quad L_{-1}|t\rangle \otimes L^2_{-1}|\bar{t}\rangle \\
q^4\bar{q}^3: & L^2_{-1}\bar{L}_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L^2_{-1}\bar{L}_{-1}|\bar{t}\rangle \\
 & L^2_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes L^2_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}\bar{L}_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \quad ; \quad L_{-1}|t\rangle \otimes L_{-1}\bar{L}_{-1}|\bar{t}\rangle\ ; \\
q^3\bar{q}^4: & L_{-1}\bar{L}^2_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L_{-1}\bar{L}^2_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}|t\rangle \otimes \bar{L}^2_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}^2_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}\bar{L}_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes L_{-1}\bar{L}_{-1}|\bar{t}\rangle \\
q^2\bar{q}^5: & \bar{L}^3_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes \bar{L}^3_{-1}|\bar{t}\rangle\ ; \\
 & \bar{L}^2_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes \bar{L}^2_{-1}|\bar{t}\rangle \\
 & \\
q^6\bar{q}^2: & L^4_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L^4_{-1}|\bar{t}\rangle\ ; \\
 & L^3_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \quad ; \quad L_{-1}|t\rangle \otimes L^3_{-1}|\bar{t}\rangle \quad ; \quad L^2_{-1}|t\rangle \otimes L^2_{-1}|\bar{t}\rangle \\
q^5\bar{q}^3: & L^3_{-1}\bar{L}_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L^3_{-1}\bar{L}_{-1}|\bar{t}\rangle\ ; \\
 & L^3_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes L^3_{-1}|\bar{t}\rangle\ ; \\
 & L^2_{-1}\bar{L}_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \quad ; \quad L_{-1}|t\rangle \otimes L^2_{-1}\bar{L}_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}\bar{L}_{-1}|t\rangle \otimes L^2_{-1}|\bar{t}\rangle \quad ; \quad L^2_{-1}|t\rangle \otimes L_{-1}\bar{L}_{-1}|\bar{t}\rangle \\
q^4\bar{q}^4: & L^2_{-1}\bar{L}^2_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L^2_{-1}\bar{L}^2_{-1}|\bar{t}\rangle\ ; \\
 & L^2_{-1}|t\rangle \otimes \bar{L}^2_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}^2_{-1}|t\rangle \otimes L^2_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}\bar{L}^2_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle \quad ; \quad L_{-1}|t\rangle \otimes L_{-1}\bar{L}^2_{-1}|\bar{t}\rangle\ ; \\
 & L^2_{-1}\bar{L}_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes L^2_{-1}\bar{L}_{-1}|\bar{t}\rangle \quad ; \quad L_{-1}\bar{L}_{-1}|t\rangle \otimes L_{-1}\bar{L}_{-1}|\bar{t}\rangle \\
q^3\bar{q}^5: & L_{-1}\bar{L}^3_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes L_{-1}\bar{L}^3_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}|t\rangle \otimes \bar{L}^3_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}^3_{-1}|t\rangle \otimes L_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}\bar{L}^2_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes L_{-1}\bar{L}^2_{-1}|\bar{t}\rangle\ ; \\
 & L_{-1}\bar{L}_{-1}|t\rangle\,\bar{L}^2_{-1} \otimes |\bar{t}\rangle \quad ; \quad \bar{L}^2_{-1}|t\rangle \otimes L_{-1}\bar{L}_{-1}|\bar{t}\rangle \\
q^2\bar{q}^6: & \bar{L}^4_{-1}|t\rangle \otimes |\bar{t}\rangle \quad ; \quad |t\rangle \otimes \bar{L}^4_{-1}|\bar{t}\rangle\ ; \\
 & \bar{L}^3_{-1}|t\rangle \otimes \bar{L}_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}_{-1}|t\rangle \otimes \bar{L}^3_{-1}|\bar{t}\rangle \quad ; \quad \bar{L}^2_{-1}|t\rangle \otimes \bar{L}^2_{-1}|\bar{t}\rangle\ ; \\
\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

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