A digital-friendly mathematical notation for series.

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Abstract

Mathematics in the South African Secondary School context is a subject under a lot of scrutiny and discussion. Enrollment figures for Mathematics in Secondary Schools are declining and qualified teachers that produce learners with good results are hard to find. In this climate, the use of on-line tools and portals could be a solution for teacher self-education and a useful and cheap supplementary source of material to learners of mathematics.

We conducted a survey to establish the usage of on-line portals by Secondary School Mathematics teachers in the province of Gauteng in South Africa. The survey showed that the use of on-line portals is very low for a variety of reasons. One of the reasons, after the more obvious ones such as training, access and infrastructure, was the difficulty that non IT-literate people have in using the tools, and in particular entering mathematical expressions electronically. " In a parallel activity to present a possible improvement to the problem stated, we proposed a new notation for the well-known sigma notation for series. The sigma notation is one of the first complex and multidimensional notations that learners encounter in the final secondary school mathematics curriculum. Research showed that the use of a well-designed and uniform notation is imperative to improve communication and understanding in the field of mathematics. Currently, a fairly consistent notation is used to notate the elements of sets, but the notations for sequences, series and other equations often vary from one author to another. Furthermore, the currently used sigma notation for series is ambiguous. The notation proposed by us involves a modification of the traditional set notation because the most widely used notation for sets has limitations and cannot easily be generalized to sequences and series. The proposed modification addresses these limitations and provides an elegant extension of the set notation to denote sequences and series more uniformly. The proposed notation has the additional benefit not using any special character or symbol. It is also a linear notation and therefore simpler to use electronically.

We tested the comprehension of the proposed notation in a live school study for improved understanding. The outcome of the study did not show that

the new notation brings any improvement in the learners ability to grasp the underlying concept, but it highlighted issues with the writing of the traditional notation that might benefit from a shift towards a more explicit notation using familiar symbols, such as the one proposed.

The survey results and the school experiment indicated that notations that use familiar and easy-to-write symbols, and are simple to input electronically using a standard keyboard, could lower the barrier to entry towards wider use of on-line portals for mathematics education. For wide-scale and general benefit, all mathematical notations will need to be adapted to adhere to the principal of one-dimensional, standard keyboard symbols. This is an ambitious goal but we hope that this dissertation can instigate thinking and experimenting in this area to slowly migrate mathematical notations to digital on-line friendly formats.

Contents

List of Figures

List of Tables

Declaration

I declare that this dissertation is my own work. It is being submitted to the degree of Master of Science Computer Science at the University of Pretoria.

It has not been submitted before for any degree to any university.

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(Electronic signature of candidate)

Date: 1 March 2018

Dedication and Acknowledgement

I dedicate this dissertation to my late brother Peet du Plessis, who always believed the best of me.

Further, I would like to acknowledge and thank Dr. Vreda Pieterse, my sponsor and advisor, for her encouragement, advice and patience. It has been a long journey but a lot of fun.

Part I

Introduction and Background

Chapter 1

Introduction

Mathematics is an important cornerstone for most forms of tertiary education and the quality of mathematical education is an important determinant of nations economic and technological progress [52]. South Africa's achievement in high school mathematics has improved in the last 20 years, but is still dismal compared to the rest of the world, as was shown by the TIMMS (Trends in International Mathematics and Science Study) report [35]. According to this report South Africa is still one of the lower performing participating countries. The achievement in mathematics is also still highly unequal between school types (independent, public fee-paying and public no-fee schools).

The CDE (Centre for Development and Enterprise) report [25] showed that mathematical education in South Africa is still a big problem although teachers in South Africa show a higher level of confidence in their own skills than many of the top countries. This is at odds with the TIMMS outcome mentioned above that puts South Africa at the bottom of the worldwide rankings for mathematics.

South Africa's dismal worldwide position for mathematics on school level is a subject that receives much attention and solutions for our lack of performance are popular subjects for research and speculation. The quote below from the South African Center of Development and Enterprise confirms the problem that South Africa has and it also shows that the problem starts in the lower grades:

> South Africa's public schooling system, particularly Mathematics education, is in crisis. In the 2012 Annual National Assessment the average result for Grade 9 Maths was 13 per cent, with only 2,2 per cent of pupils scoring 50 per cent or above [44].

The problem with mathematical education is not unique to South Africa. The report of Vorderman [49] highlights huge problems with mathematical education in the United Kingdom. She finds that only 15% of learners in the British schooling system take Mathematics further than GCSE (Grade 10) level and that more than half fails Mathematics on GCSE level. The Leitch report [20] emphasizes the necessity of a high level of mathematics education for sustainable competitive advantage of any nation.

Makopa [22] finds that the availability of study material and general infrastructure like seating has an effect on the outcome of primary school learners performance in reading and calculation.

An interesting outcome of the study by Makopa [22] is that Zimbabwe fared better than South Africa in the SACMEQ[1] III project. Zimbabwe's economic decline in the past 15 years explains their decrease in performance between SACMEQ 1 and SACMEQ III but the fact that Zimbabwe managed to perform better than South Africa is worrying.

The use of on-line resources to improve the level of mathematics education in South Africa could be a cost convenient and efficient way to get quality material to teachers and learners. Not much data exists on the question of cost of on-line vs traditional methods in the context of high-school education but college educational data shows that on-line teaching can be cheaper than traditional classroom teaching. A report by the Boston Consulting Group and the Arizona State University, Bailey *et al.* [3] found that institutions can lower their costs by offering well designed on-line courses. These savings can be passed onto students in the forms of lower credit hour fees.

To estimate costs differences for Matric learners, the cost between a Matric Mathematics tutor and on-line resources should be compared. The average private tutor costs around R200 per hour, a Mastermaths subscription about R120 per hour, while an on-line subscription to Siyavula cost, at the time of writing, R59 per month for unlimited access to all Mathematical content and exercises.

There is a lot of focus on improving mathematical education in South Africa and a large part of it concentrates on alternative methods and structure for mathematical education. The number of private companies offering tutorship such as Mastermaths(www.mastermaths.co.za), KipMcgrath(www.kipmcgrath.co.za) and Kumon (www.kumon.co.za) are growing. On-line resource offerings are also expanding and are increasingly used by educators.

To explore the state of on-line Mathematics education in Gauteng and the

[1]The Southern and Eastern Africa Consortium for Monitoring Educational Quality

level of usage of the on-line tools, an on-line survey was conducted. In the survey, the extent of current usage of on-line tools in secondary schools was investigated as a starting point. The results gave an indication of the current usage of the Web as an educational tool as well as providing some information on the reasons for the usage levels. The method and results of this survey are discussed in Chapter 5.

In a concurrent activity aimed at improving mathematical education, the deficiency of some mathematical notations when used in on-line teaching and learning has been observed. To address this, a new notation is proposed for series (sum of sequences) that is a derivative of a notation proposed by Dijkstra [12]. The proposed notation is more explicit with less chance of confusion. It is also linear which should make it better adapted to electronic use.

Mathematics and its notation should not be viewed as one and the same thing. Mathematical notations are representations of mathematical concepts that uses many components such as numbers, letters and other symbols. Different notations and symbols can be used to describe the same concept. Biletch et.al [5] analyzed different mathematicians opinions on what constitutes a good mathematical notation and concludes that attributes of a good notation are:

1. Conciseness
2. Unambiguity
3. Consistency
4. Keep focus on underlying concept of study
5. Easy to produce typographically
6. Compatible with existing notations

Mathematical notation has evolved over the years to what is generally accepted as standard notation today. A good example of such an evolution is described by Mazur[24] concerning notation for negative numbers that underwent a few changes (inverted letters, half moons in different directions) before the mathematics fraternity settled on the notation that is mostly used today (minus sign in front of the number). The time might be right for the introduction of mathematical notation that is more adapted to the tools of the modern era.

The notation that is proposed as an alternative for the sigma notation for series, has the general form:

$$\Sigma\ (\ x \mid P(x) \mid f(x)\ )$$

In this notation the meaning of the elements in the equation are:

- Σ means sum of the elements specified in the expression that defines the series.
- x specifies the dummy variable used in the functions that describe the series.
- $P(x)$ is a predicate that specifies the scope of the series.
- $f(x)$ is an expression that describes the elements of the series.

Using the proposed notation, a traditional expression for a series,

$$\sum_{m=3}^{7}(m^2+1) \qquad \text{is written as} \qquad \Sigma(\ m \mid 3 \leq m \leq 7 \mid m^2+1\ ).$$

The new notation is proposed with the expectation that it could serve 2 purposes:

1. improves ease of electronic use
2. improved understanding of the underlying concepts

With the Internet and the Web becoming ubiquitous in educational environments, email and other social media are increasingly used for communication, replacing paper and traditional post and fax, as ways to record and send information to one or many recipients. In this digital era, the problem of coding mathematics to facilitate communication has become increasingly important. Input of information into electronic media happens largely via keyboards with standard layouts and characters. The spatial significance of symbols is difficult to render on a standard keyboard and mathematical equations often assigns meaning to a specific position. A very simple example of the difference that symbol positioning makes in the value of an equation is: a^b versus ab. The new proposed notation is one-dimensional in format and spatial positioning of any symbol has no particular meaning. The effect of the notation on learners' understanding of the concepts was examined in a study conducted in a traditional teaching environment at a secondary school in Gauteng. The results of this study are described in Chapter 7.

Given the scenarios described above, a seemingly developed country with dismal achievement in mathematics, the high-level motivation for this research is to contribute in some way to the improvement of mathematical achievement in South Africa by enabling easier use of on-line tools for education in secondary schools

> *The essence of mathematics is not to make simple things complicated, but to make complicated things simple.* – Stan Gudder [2]

[2] http://platonicrealms.com/quotes/Stan-Gudder

Chapter 2

Mathematical notation and electronic processing

As discussed in the introduction, mathematical notation is a system of symbolic representations of mathematical objects and ideas. Mathematicians assign symbols to represent complex ideas for easier digestion[1]. The field of mathematical information retrieval is getting attention as can be seen in Schubotz [40] that defines a new component for MathJax (a LaTex and MathML rendering engine), called the mathoid server, that augments mathematical expressions for easier searching. Schubotz [40] developed the concept of Mathematical Language Processing (MLP) that uses methods from NLP(Neuro Linguistic Processing) to process mathematical expressions.

The interest in mathematical notation in this dissertation is driven by the ambition to modify the currently used notations to be more suitable to electronic processing rather than devising ever more complex methods for processing current notations. In the rest of this chapter methods for on-line mathematical expression processing is explored.

2.1 Markup languages

Markup languages are the standard for Web communication. Examples are Latex, HTML, XML (of which MathML is an extension). A markup language typically separates the presentation from semantics. Markup languages are a form of programming and novices typically find it hard to learn or interpret. Markup languages use plain-text and the concept of “tags“ to specify format. It requires an understanding of the concept of processing or “programming”

[1] *https : //en.wikipedia.org/wiki/Mathematical_notation*

a document. This level of understanding is not normally mastered by the typical secondary school teacher or learner.

Users of WYSIWYG word processors will find the concept difficult to process, so entering equations in this format may not be a viable option.

2.2 Equation editors in word processors

Most word processors have equation editors that offers templates where the spatial positioning of the elements of an equation is pre-determined. The use of these equation editors is not complicated for advanced word processor users but high school learners are generally not aware of the function and attempt to construct equations by manually drawing them and by other means. An example of this can be seen in Figure 7.8 in Chapter 7. The number of clicks used to enter an equation using these editors is also a deterrent for frequent use. The different steps needed to enter an expression using the the sigma notation compared with the steps needed to enter the same expression using the proposed new notation described in Chapter 3 electronically in a few different editing tools were investigated. The comparison is shown in terms of the number of mouse-clicks needed to enter the different expressions. The results of this comparison can be seen in Table 2.1.

Tool	Format	Mouseclicks
MS-Word Equation editor sigma notation	$\sum_{m=3}^{7}(m^2+1)$	7
No equation editor - proposed notation	$+(\ m \mid 3 <= m <= 7 \mid m^2+1\ )$	0/1
MS-Word equation editor - proposed notation	$\Sigma(\ m \mid 3 \leq m \leq 7 \mid m^2+1\ )$	3

Table 2.1: Different notations for equivalent concepts

The second row of Table 2.1 shows a different notation entered in Word without using an equation editor. This alternative notation is discussed in more detail in Chapter 3. The result is not as visually appealing as the third row (proposed equation entered with equation editor), but it can be done with 0 or 1 click making this approach appealing when the work is not published, such as school homework. The third row shows the same equation done with the equation editor in Word. It has more compact presentation of the less than/more than or equals sign.

2.3 Images and online communications

Electronic processing of mathematical equations today is largely done through image processing. Images typically take up much more bandwidth in electronic format than plain text so using images to communicate in low-bandwidth environment is not advisable.

MathML is a markup language (as discussed in Section 2.1) that is frequently used to encode mathematical expression in an online environment. MathML has two major components [2]:

1. Presentation MathML: information about the visual structure of a mathematical expression.
2. Content MathML: information about the logical meaning of a mathematical expression

An example of an equation expressed in MathML as a text file can be seen in Figure 2.1.

If the equation expressed in the MathML format in Figure 2.1, is presented in JPG format as an image, it appears as in Figure 2.2. In the example in Figure 2.1 and Figure 2.2, the size of the MathML text file is 1 KB and the size of the image is 8KB. This constitutes a factor of 8 which soon escalates when the volumes of information in the image increases.

A more complex sequence of equations such as shown in Figure 2.4 has the text equivalent in MathML of size 15KB (Figure 2.3) and the JPG format of size 258KB.

[2] https://reference.wolfram.com/language/XML/tutorial/MathML.html

```
<math xmlns='http://www.w3.org/1998/Math/MathML'
display='block'>
  <munderover>
    <mo>&sum;</mo>
    <mrow>
        <mi>i</mi>
        <mo>=</mo>
        <mn>0</mn>
    </mrow>
    <mi>&infin;</mi>
  </munderover>
  <mi>sin</mi>
  <mfenced separators=' '>
<mi>i</mi>
<mi>x</mi>
  </mfenced>

</math>
```

Figure 2.1: Equation in MathML format

$$\sum_{i=0}^{\infty} \sin(ix)$$

Figure 2.2: Equation in JPG format - 8KB

```
<math xmlns='http://www.w3.org/1998/Math/MathML'
display='block'>
  <munderover>
    <mo>&sum;</mo>
    <mrow> <mi>i</mi>   <mo>=</mo> <mn>0</mn>  </mrow>
    <mi>&infin;</mi>
  </munderover>
  <mi>sin</mi>
  <mfenced separators=' '><mi>i</mi><mi>x</mi>  </mfenced>
</math>
<math xmlns='http://www.w3.org/1998/Math/MathML'>
 <mrow>
  <mi>sin</mi>
  <mo>&#8290;</mo>
  <msup><mi>x</mi> <mn>4</mn> </msup>
 </mrow>
 <math xmlns='http://www.w3.org/1998/Math/MathML'>
 <mrow>
  <mi>sin</mi>
  <mo>&#8289;</mo>
  <mo>(</mo>
  <msup>
   <mi>x</mi>
   <mn>4</mn>
  </msup>
  <mo>)</mo>
 </mrow>
</math>
<math xmlns='http://www.w3.org/1998/Math/MathML'>
  <mrow>
     <mrow>
     <mi>a</mi>
     <mn>2</mn>
     <mo>+</mo>
      <mi>b</mi>
     <mn>2</mn>
     <mo>=</mo>
      <mi>c</mi>
     <mn>2</mn>
  </mrow>
</math>
```

Figure 2.3: More complex equations in MathML- Size = 15KB

$$\sum_{i=0}^{\infty} \sin(ix)$$

$$\sin x^4$$

$$a^2 + b^2 = c^2$$

Figure 2.4: Larger sequence of equations in JPG format. Size = 258KB

2.4 Electronic equation entering approaches

Pérez-Navarro and Sancho-Vinuesa [30] found in a study at a University in Spain that students of mathematical subjects, when communicating with each others on mathematical topics via email, use one of the following methods:

1. Write Greek letters by name when only a symbol is needed
2. Use a pseudo-language to describe formula
3. Cite formulas by number
4. Attach OpenOffice or Word documents

The most popular of the above methods for the communication of formulas were found to be number 2 and 3, pseudo-language and citing formulas by number. It has to be noted that these were engineering students who are familiar with programming. The concept of a pseudo-language was not unfamiliar to them.

The variety of methods used indicates an unsolved problem in the field of on-line mathematics. Perez-Navarro and team developed a special formula editor for use in online communications on the campus that was widely used and appreciated by the students.

The use of mathematics on-line and electronically has become more important in the last 15 years or since the proliferation of the internet. Despite this, the preferred way for students to do mathematics problems remains pen and paper because of the complexity of entering formulas in editors. An experiment has been done to measure the effectiveness of reading electronically (tablets) by Sackstein *et al.* [37]. They found that students read just as fast, or even faster using e-books or tablets when studying and that the comprehension of material is also on par with that on paper. This research did not focus on mathematical learning and also did not cover information input at all but it does confirm that electronic devices is likely become more entrenched in educational environments.

Roux [36] found that although the introduction of technologically enriched study environments brought some benefits to prospective mathematics teachers, it did not effect dramatic changes in attitude and conceptual understanding of the topics. Her study was limited and no general conclusions can be drawn from it but she made the interesting observation based on interviews of the participants in her study, that the prospective mathematics teachers have problems with formal mathematical notation.

She used Geometra's Sketchpad as software for her studies. Geometra is educational software that can be used to create visual representations of mathematical functions.

The field of electronic recognition of handwritten mathematical texts has received a lot of attention and has been the subject of many studies. Mathematical expression recognition consists of two different areas of study, namely:

- symbol recognition
- structure analysis of two dimensional patterns.

In 1999 Chan and Yeung [8] did a survey of the work that existed at that time related to this problem. According to them, at the time, the areas listed below had not received sufficient attention or proper research:

- error detection
- use of contextual information
- ambiguities
- performance

The main parameters for evaluating the efficiency of algorithms and software attempting such recognition are error-rate and work rate. The CROHME competition[3] for on-line recognition was established in 2011 and ran its it's fifth competition in 2016 . The main task in this competition is formula recognition from handwritten strokes. The results of a study by Mouchère *et al.* [28] shows that the progress in this field is slow. Electronic recognition of handwritten mathematical expressions is still a difficult and error prone task. The highest recognition rates that were obtained for the main task in the 2016 competition was 67.65%. Simistira *et al.* [41] proposes an elaborate algorithm to improve the spatial recognition on mathematical expressions. They achieved an improvement on the CROHME 2013 results but their work shows that a lot of effort needs to be put in to affect an improvement in the electronic recognition of mathematical expressions.

The sigma notation introduced by Euler in 1755 [15] as discussed in the next chapter, apart from some ambiguity problems, poses a problem for electronic recognition due to the dependence on specific spatial positioning of the parameters in order to extract the correct expression.

[3] http://www.nlpr.ia.ac.cn/icfhr2016/competitions.htm

2.5 Mathematical notation

Mathematical notation is a language in its own right — unique and constantly evolving. Mathematicians, speaking a range of indigenous languages, use it to express mathematical concepts in a way which everyone can understand. The continued invention of mathematical notation has similarities with the development of natural languages.

The use of mathematical language has evolved over many centuries as users of this language share a mutual understanding of the meaning of its symbols, words and sentences. Good notation enhances the precision of expression and at the same time simplifies communication. Whitehead, a famous English mathematician and philosopher, states:

> *By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and, in effect, increases the mental power of the race.* [51]

Cajori [6, 7] describes the origin and development of a wide range of the notations commonly used today. His descriptions, however, include various obsolete notations that have been abandoned in favour of more versatile and/or expressive notations. There is a need for more recent work of this range and rigor. Abadir and Magnus [1] attempted to fill this gap in the field of econometrics. Dijkstra and Van Gasteren [13] refer to Cajori's work as "a graveyard of notations".

Mathematical notations are introduced to provide concise and accurate ways of communicating complex yet well-understood concepts. It is well known that brevity is the leading characteristic of mathematical elegance, but this is not the only requirement. Although some may think that symbolic notation is merely introduced to save space [50], most mathematicians agree on the value of clever notations. Dijkstra and Van Gasteren [13] emphasize that the use of appropriate notation can make a difference in mathematical work. Lipton [21] gives an example of the European mathematicians who used Leibniz's $\frac{dx}{dt}$ differential notation, which enabled them to progress faster than their British counterparts who used Newton's $\dot{x}$ to express the same concept. Fekete [16] states that Von Ettinghausen's notation $\binom{n}{m}$ for binomial coefficients has been called beautiful. He similarly praises Knuth's $\left[{n \atop m}\right]$ and $\left\{{n \atop m}\right\}$ notations for the Stirling numbers. To support mathematical thinking, notation should not only be designed to enhance the brevity of the text but should also control the number of rules governing the manipulation of expressions [13].

Mathematical notation is constantly evolving as mathematicians find ingenious ways to express concepts. It is not without dialects, as there is no guarantee that the community will adopt a proposed notation. Variations in notation for similar concepts are often a reflection of personal preferences. For this reason, it is common practice in mathematical writing to expect the author to explain the notation used in a given publication.

2.6 Notation in education

The influence of notation on the understanding, or lack of it, in learners at high school level has been studied by Chirume [9]. He concluded that a clear notation definitely plays a role at the initial stage of learning a new concept. He found that students fail to grasp mathematical concepts because the students consider the symbol as the object of the mathematics rather than the concept that the symbol represents. His students also indicated a difference in preference for certain symbols depending on whether they have to write it or understand it. In an example from his study, students indicated that y' and y'' are easier to write, but $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, are easier to understand.

This would point to the fact that more explicit notations does aid understanding, an observation that was also made by Van Gasteren [47].

Another finding from Chirume's study [9] involved discussions with the teachers. He interviewed 7 teachers that took part in this experiment and 4 of them indicated that positioning of symbols plays a very important part in maths learning. They said they had difficulties in explaining, for example, that $a^{(b+c)}$, $ab + c$ and $a^b + c$ mean different things although they are the same symbols in left to right order.

In theory it is argued that notation should remain static to avoid confusion. In practice this can not so easily be achieved as can easily be illustrated by looking at the difference in notations for the same concept currently applied across different subjects in South African schools.

Concept	Notation Options
Multiplication	$2 \times n$, $2.n$,$2n$, $2 * n$
Division	$18 \div n$,$18/n$, $\frac{18}{n}$
Functions	$f(x)$, f
Derivative	$f'(x)$,$\frac{d}{dx}$, $\frac{d^2y}{dx^2}$

Table 2.2: Different notations for equivalent concepts

From the examples in Table 2.2, it can be seen that there are by no means only one accepted notation for many mathematical expressions. Another criteria besides the traditional ones for selecting a specific notation as listed on page 4, could be the degree of ease of on-line usage.

2.7 Summary

The current state of affairs in the field of electronic recognition of mathematical expressions has promoted the thought that an alternative approach to solving the issue of electronic recognition of handwritten expression could be to modify the notation used for such expressions. Expressions which have a more linear nature, such as Dijkstra and Wolfram's notations shows promise in this regard. Using well-thought out notations may also aid learners in understanding the concept that the notation describes.

An alternative for the sigma notation for series is proposed in this study. The development of this notation is discussed in Chapter 3. The understanding of this notation is explored in an experiment in a school using traditional teaching methods. Simultaneously, an online survey was conducted to determine the level of usage of on-line tools in mathematical education. This was done to verify the hypothesis that the level of usage is lower than what it could be considering the availability of electronic devices to teachers.

Chapter 3

Development of a new notation

The use of a good mathematical notation enhances understanding and effective communication. Notation is used to describe complex problems or situation in a precise and compact way. Brevity is seen as the first goal or measurement of a good notation but there are more factors to be considered.

As mentioned in Chapter 2, a new notation is developed as an alternative to the sigma notation for series. The sigma notation is cumbersome to enter electronically and could also be ambiguous. The history of the sigma notation and the process of developing the new notation is discussed in this chapter. The discussion starts by introducing the reader to the *de facto* standard methods for describing sets in Sections 3.1 and 3.2. Thereafter Dijkstra's critique of this notation and his proposal to address the shortcomings are discussed, followed by a critique of Dijkstra's proposed notation culminating in the new notation for sets proposed in this dissertation. In Section 3.3 this new notation is subjected to a natural adaptation to produce a notation for the intentional specification of sequences. Finally Section 3.4 is a critique of existing notations for series followed by a proposal for a new notation for series.

3.1 Extensional specification of sets

A set is a collection of objects. One way of describing or specifying the members of a set is by extension, i.e. by listing each member of the set. When specifying a set by extension, according to generally accepted convention, the members are enclosed in the curly bracket pair $\{$ and $\}$.

For example, the set containing the first five even numbers can be written as:

$$\{2, 4, 6, 8, 10\}$$

The set of *non-negative integers* = $\{0,\ 1,\ 2,\ \ldots\}$ is known as the natural numbers. This set is denoted by $\mathbb{N}$. For $n \in \mathbb{N}$, $\mathbb{N}_n$ denotes the set containing the first n natural numbers. Thus $\mathbb{N}_0 = \varnothing$, $\mathbb{N}_1 = \{0\}$ and $\mathbb{N}_n = \{0, 1, 2, \ldots, n-1\}$.

3.2 Intentional specification of sets

When the members of a set follow a pattern, the set can be specified by intentional definition, i.e. by using a rule or semantic description as specified in the international standard ISO 80000-2:2009 [18]. The general format for the notation is:

$$\{x \mid P(x)\}$$

The symbol | means *such that*. Therefore, the above means the set of all elements x *such that* $P(x)$ is a true statement. The property P can be expressed in either words or symbols. The variable on the left of the | specifies a dummy variable whereas the expression on the right delineates the scope. One may use formulas to specify the dummy variable. For example, the set of odd natural numbers can be expressed as:

$$\{2t + 1 \mid t \in \mathbb{N}\}$$

Dijkstra [12] criticizes this notation. He gives the following example which uses this notation with an ambiguous meaning. It reveals an inherent flaw in this notation:

$$\{i^n \mid i < n\}$$

It can be interpreted as $\{1^n, 2^n, 3^n, \ldots, n^n\}$ or as $\{i^{i+1}, i^{i+2}, i^{i+3}, \ldots\}$. This ambiguity arises because the specification of the dummy variable is not separated from the description of the elements. He proposes a notation to remedy this deficiency. This notation requires the clear separation of three aspects, namely (i) the dummy variable elements, (ii) the scope description, and (iii) the description of the elements of the set in terms of the dummy variable elements. Dijkstra's notation for the intentional definition of a set specifies these aspects separated by the : character and enclosed in angle brackets. This notation not only removes ambiguities, it is also a more versatile expression of the conditions for membership of a set. The following is the general format for Dijkstra's notation:

$$\langle\ x : P(x) : f(x)\ \rangle$$

Here x is the dummy variable, $P(x)$ is a predicate that specifies the scope and $f(x)$ is an expression that describes the elements of the set. More than one dummy variable, as well as more than one predicate to specify the scope, may be used. When doing so, they should be separated by commas. This allows a distinction between the following:

$$\langle\ i : i < n : i^n\ \rangle, \langle\ n : i < n : i^n\ \rangle \text{ and even } \langle\ i, n : i < n : i^n\ \rangle$$

Dijkstra states that he has no logical objection to declaring the type of the dummy variable when identifying the dummy variable. For example, the following are equivalent specifications of the set of even numbers less than 100 using Dijkstra's notation:

$$\langle\ i \in \mathbb{N} : i < 100 : 2 \times i\ \rangle$$
$$\langle\ i : i \in \mathbb{N}, i < 100 : 2 \times i\ \rangle$$

This notation poses an element of interpretation complexity by allowing multiple specifications in each section and also by having a low-key character which separates the different sections.

In this dissertation a notation similar to Dijkstra's notation but closer to the ISO standard is introduced. The following is the general format for the proposed notation:

$$\{\ x \mid P(x) \mid f(x)\ \}$$

An obvious difference between this notation and Dijkstra's is the use of the punctuation prescribed in the ISO standard, i.e. $\{ \mid \mid \}$ instead of $\langle\ :\ \ :\ \rangle$. To avoid further ambiguities the following restrictions are also proposed:

- Type specifications of dummy variables are not allowed.
- Multiple scope predicates are not allowed.

Usually the types of dummy variables are clear in the context. If needed, the type of a dummy variable may be specified by including it in the scope predicate. Limiting the specification of the scope to a single predicate is not a real restriction, because if both P_1 and P_2 should hold, it can just as well be specified with the single predicate $P_1 \wedge P_2$ where $\wedge$ is the symbol for the Boolean AND operation.

It could be argued that the use of the comma to indicate multiple predicates, as suggested by Dijkstra [12], might be more readable, but linking the predicates by the AND symbol is more explicit and groups the specification in neatly. Given that the notation is targeted to eventual easier electronic interpretation, it is felt that the introduction of another symbol to signify the grouping of predicates would make for easier parsing. This is a matter that can be debated in further studies and the $\wedge$ symbol will be used in the rest of this dissertation. The following is therefore the only legitimate specification of the set of even numbers less than 100, using the proposed notation:

$$\{ \ i \mid (i \in \mathbb{N}) \wedge (i < 100) \mid 2 \times i \ \}$$

3.3 Intentional specification of sequences

A sequence is an ordered list of objects. A sequence differs from a set because the order of the objects matters. In addition, exactly the same elements can appear multiple times at different positions in the sequence. A sequence M with n entries is called an *n-tuple*. The following shows how a sequence is usually specified by extension:

$$(m_0, m_1, m_2, \ldots, m_{n-1})$$

The values of the terms in a sequence may be random, with no relationship between i and the value of the i^{th} term in the sequence. If this is the case, such a sequence can only be described by extension.

No standard has been specified to notate an intentional definition of sequences as far as can be determined from literature. In this dissertation a notation for cases where there is a relationship between i and the value of the i^{th} term in the sequence is proposed. This is an adaptation of the proposed notation for the intentional definition of sets in Section 3.2. Similar to the notation for sets, the three aspects — namely the dummy variables, the range predicate and the expression describing the entries — are specified and separated by the | character. This is a variation of the notation proposed by Pieterse [32]. Here the punctuation prescribed in the ISO standard for series is used. To indicate that it is a series and not a set, parentheses are used instead of curly braces, for example, the following specifies a quintuple of natural numbers:

$$(\ i \mid i \in \mathbb{N}_5 \wedge m_i \in \mathbb{N} \mid m_i \ )$$

Often the value of a term in a sequence is related to its index. Consider the sextuple described by extension as $(3, 5, 7, 9, 11, 13)$. The formula for the i^{th} element in this sextuple is $2 \times i + 3$. It can therefore be described by using the following intentional definition:

$$(\ i \mid i \in \mathbb{N}_6 \mid 2 \times i + 3 \ )$$

3.4 Series

A series is the sum of the terms of a sequence. The well-known sigma notation for the summation of sequences was introduced by Leonard Euler in 1755 [15]. Euler's notation and modern refinements of this notation are well established in the mathematical community. When using Euler's notation, $3^2 + 4^2 + 5^2 + 6^2 + 7^2$ is written as:

$$\sum_{m=3}^{7} m^2$$

Wees [50] contends that students find Euler's sigma notation difficult to understand at first. He attributes this to the complexity of the expression, which is a function that takes as many as four parameters, all of which have to be understood at once. He proposes a programming-like notation as a substitute for Euler's sigma notation. His notation requires descriptive names for the parameters. Though his proposal contributes significantly to the clarity of the expression, it loses two essential attributes of viable notational systems, namely conciseness and independence from natural language.

Van Gasteren [47, p.141] propose that explicit declaration of dummies paves the way to teachability.

Gries and Schneider [17] introduce a similar notation to that of Dijkstra. Their notation takes the form:

$$(+i|0 \leq i \leq n : i^2)$$

They give the following reasons for their proposed notation:

1. the scope of the dummy variable is explicit
2. it is easier to write more general ranges for i
3. it is easier to add more than one dummy variable

Van de Snepscheut [46, p.33] also uses a similar notation to that of Dijkstra for proving of iterative statements. His notation takes the form:

$$\langle \sum i : 1 \leq i \leq n : a[i] \rangle$$

A study conducted by Strand and Larsen [43] highlights the difficulties that students have with the sigma notation. As illustration, they described the mental constructions involved in resolving the task below:

Express the sum of the first five odd integers using Summation Notation

According to Strand and Larsen [43], the mental constructions needed to solve the problem above need to be:

1. Construct the long-hand sum, e.g.: 1 + 3 + 5 + 7 + 9, mentally or physically.
2. Construct an indexing process a) Identify an appropriate starting value of the index. b) Identify successive integer values of the index with each term in the sum. c) Identify the appropriate terminating value of the index.
3. Construct a function that takes index values as its input and outputs the appropriate term of the sum. For this example, with index k, the function ($2k$ - 1) would generate the appropriate addends for integer values $1 \leq k \leq 5$.
4. Arrange the elements of the notation to indicate the desired sum.

In conclusion, their investigation found that students have difficulty in steps 2 and 3 offering solutions such as below for the given task:

$$\sum_{i=2}^{4}(i-1) = (2-1) + (4-1) + (6-1) + (8-1)$$

This example clearly showing some confusion as to the functioning of the index.

Dijkstra [12] points out a flaw in Euler's notation which is beyond the problem observed by Wees, namely an inherent semantic ambiguity resulting from uncertainty related to the extent of the description of the elements in the series, for example,

$$\sum_{m=3}^{7} m^2 + 1 \text{ can be interpreted as } \left(\sum_{m=3}^{7} m^2\right) + 1 \text{ or as } \sum_{m=3}^{7} (m^2 + 1)$$

This ambiguity is usually addressed by requiring the use of brackets. Dijkstra proposes that his set notation should be adapted and that the adaptation should replace Euler's notation. Dijkstra follows Euler's idea of using the Σ symbol to indicate summation. In Dijkstra's proposal, this symbol is specified along with the dummy variable.

The notation for series proposed was published in the paper by Du Plessis and Pieterse [14]. It has the advantage of clearly distinguishing the description of the terms in the sequence from the surrounding text. Since addition is commutative, the order of the elements in a series is not relevant. For this reason it would make sense to extend the set notation as specified in Section 3.1 rather than the notation for sequences as specified in Section 3.3 for series in this manner. There is, however, sometimes a need to specify a series in which terms may be repeated and the order of the terms are significant. For this reason the notation for series proposed here is an extension of the notation for sequences as specified in Section 3.3.

The following is the general format of the notation for series proposed in the paper by Du Plessis and Pieterse [14] cited above and further explored in this dissertation:

$$+\ (\ x \mid P(x) \mid f(x)\ )$$

The meaning of the parameters is the same as in the notation for sequences: x is the dummy variable, $P(x)$ is a predicate that specifies the scope and $f(x)$ is an expression that describes the terms in the series. This proposed notation for series can further be generalized to quantifications other than $+$. Further generalization is, however, beyond the scope of this dissertation.

This notation is similar to Dijkstra's notation, yet it has the following differences:

- It uses the $+$ symbol instead of the Σ symbol as the operation symbol to denote summation. Note that at the time of the school study the Σ symbol was used. Observations of the written work received from the learners during the study, prompted the replacement of the Σ symbol with the $+$ sign.

- The operation symbol is written as a prefix to the sequence instead of placing it along with the specification of the dummy variable.
- It uses parentheses where Dijkstra's notation uses angle brackets. This notation therefore applies explicitly to series (not sets) and uses the punctuation prescribed in the ISO standard.

Using the proposed notation, the equation:

$$\sum_{m=3}^{7}(m^2+1), \qquad \text{is written as:} \qquad +(\ m \mid 3 \leq m \leq 7 \mid m^2+1\ ).$$

A more complex quantification such as:

$$\sum_{m=3}^{5}\sum_{n=1}^{9}(mn+1), \qquad \text{is written as:}$$

$$+(\ m,n \mid (3 \leq m \leq 5) \wedge (1 \leq n \leq 9) \mid mn+1\ ).$$

The use of parenthesis bearing different meanings in the above expression may cause some confusion. As part of the proposed notation the outer parenthesis indicates that the elements are part of a series while the parenthesis used inside the expression is needed to ensure the correct order of interpretation. It is likely that Dijkstra [12] chose to use angle brackets rather than parenthesis or curly brackets to avoid this potential confusion. The decision to use parenthesis in the proposed notation is maintained to adhere to the decided meaning of parenthesis to indicate the important fact that the elements in the series are ordered.

The proposed notation is more versatile than other notations. It does not introduce a new symbol and can therefore be used without adaptation for quantifications involving operations other than $+$. Other notations require the introduction of additional symbols when used for quantifications involving other operations. For example, when using Euler's notation, the symbol Π is introduced to indicate multiplication over a series, but the proposed notation would use $\times$.

3.5 Summary

This chapter describes the development of an alternative notational system for sets, sequences, and series. It begins by defining these constructs and specifying the symbols used to explain the proposed notation. Each section justifies the proposed notation for the discussed construct and uses examples to illustrate it.

The value of using the proposed new notation to promote an understanding of the concept of series is investigated by means of a study conducted at a secondary-school with learners who had not previously been introduced to series or the sigma notation. The school study is described in detail in Part III of this dissertation.

During the school study it was observed that the structure of the new proposed notation could be useful when having to enter expressions describing sets and series electronically. Easier electronic use might promote the use of available on-line tools. An online survey was done to gather data on the extent of use of on-line mathematical tools in secondary schools in Gauteng and to see if there is a correlation between the usage level and the difficulty of entering mathematical equations electronically. This survey is discussed in detail in Part II of this dissertation.

Part II

Online Survey

Chapter 4

On-line Survey Research Design

Technology usage in schools is a topic that receives a lot of attention in government and academia alike. The common approach in the quest to modernize education systems, is to hand out devices such as tablets or laptops to students[1] or to equip schools with computer laboratories that can be used in a shared fashion by all the students. Young [53] conducted a study that involved teachers and students in schools on military bases in Germany. The students were American children from military families. She found training on technology is an important factor for success. 50% of the teachers indicated that they need more training in order to successfully apply the technology in their classes.

Once the issue of availability of technology to teachers and learners are resolved, the next step is to investigate the usage of the technology. The primary use of technology for educational purposes is to access subject matter material and courses. There is currently a vast amount of material available on the Internet, not all customized for the South African syllabus, but nevertheless useful to someone that has a clear idea of the topic(s) for which they need help.

A further use of technology in education is automated assessment tools to streamline and speed up feedback to learners. Such timely feedback has been shown to improve learner progress and interest. In a study on teacher classroom practices in 2017, Arends *et al.* [2] found that feedback with remediation can have a positive influence on learners' mathematics grades. In the crowded classrooms of South African public schools, timely and informative feedback is difficult to achieve. Automated tools that can

[1] http://ewn.co.za/2015/07/20/Over-300-Gauteng-public-schools-to-get-tablets

provide qualitative feedback and distinguish between a wrong answer and lack of understanding of the subject matter in a timely fashion, could improve learners' capabilities and performance.

This topic was studied and discussed by Pieterse [31] in the field of computer science and more particular, MOOCs for programming courses. She found several problems with the implementation of a tool that can grade student programs in a consistent way. A large percentage of the submissions received from students failed due to formatting or alignment errors. This shows a similar problem as with mathematical expressions where the spatial positioning of components of an expression can make it difficult to recognize or interpret correctly.

Pieterse [31] concluded nevertheless that there is educational benefit for students in providing a tool that can be used in a repetitive way to facilitate deliberate practice. Immediate feedback is regarded as beneficial in any deliberate practice. Well designed on-line assessment tools can provide this function. In the context of MOOCs, issues such as plagiarism and summative assessment is not important. In schools it might be that on-line tools will remain in the context of voluntary practice and preparation to circumvent the potential plagiarism problem.

4.1 Tool selection

This section justifies the tool that was selected to conduct the survey reported in this chapter.

On-line surveys are popular for gathering information and doing opinion polls in a cost-efficient and time saving way. There are many commercial offerings on the web that can be used to conduct surveys and analyze the feedback. Free options exist that limits the number of respondents and general flexibility of the survey.

Google Forms was used since it offers a simple but adequate interface for constructing the survey. The number of questions are limited but there are no limit on the number of respondents.

4.2 Minimum expected response rate

This section predict the expected response rate for the survey reported in this chapter based on reports regrading response rates for on-line surveys.

Response rate to surveys is not an exact science since there are many factors that will contribute to the response rate of a particular survey. Porter and Whitcomb [34] received a response rate of 14.8% on their survey with 2 follow-ups after the initial request. Interestingly, they found that personalization of the request has little effect on the response rate.

To verify what could be considered as a realistic response rate, expected response rates reported by a few on-line commercial survey providers that published expected response rates on their websites were investigated. The data gathered in this investigation is shown in Table 4.1 and the webpages from which the data was obtain shown in the list below. These are commercial sites and not researched data, so the webpages may change or disappear over time.

- Surveygizmo: https://www.surveygizmo.com/resources/blog/survey-response-rates/
- Peoplepulse: https://www.peoplepulse.com/resources/useful-articles/survey-response-rates/
- Benchmarkemail: https://www.benchmarkemail.com/help-FAQ/answer/what-is-a-typical-survey-response-rate

Survey company	Typical response	Parameters
Surveygizmo	10 - 15%	External survey *
Peoplepulse	<10%	General survey without incentive
Benchmarkemail	<10%	General survey with open rate of 15-20%

Table 4.1: Survey Response rates.(*Could be as low as 2% for less targeted and low incentive surveys.)

Given the fact that the survey reported in this chapter was without any incentive or obligation and completely unsolicited, a very high response rate was not expected. A response rate of 10% was targeted. In other words, if a response rate of 10% was not achieved with the survey, the study would be abandoned and a different request method explored.

4.3 Survey goals

The main goal of the survey is to determine the extent of usage of electronic tools for mathematical education in schools and possible reasons or motivations for lack of use of such tools.

A secondary objective was to determine if there is a correlation between the difficulty in using modern day word processors for mathematics expression entering, and the actual usage of on-line or electronic methods for the teaching and marking of mathematics in schools.

4.4 Hypotheses

The hypotheses that will be tested in the survey is described as below based on the literature study in part I. Despite projects to provide schools, teachers and learners with equipment, small benefit is derived from these efforts and in many cases, large amounts of money is lost.

- Part 1: On-line tools are not used extensively for mathematics in the high school environment in South Africa.
- Part 2: The reason for this is not the lack of access to equipment. It is rather the lack of knowledge. Difficulty of doing mathematical expressions electronically may also inhibit the use of on-line tools.

The survey is structured broadly in three high-level topics:

1. respondents and availability of infrastructure (Questions 1,2,3)
2. use of on-line portals in teaching and self-education (Questions 4,5)
3. use of electronic portals to receive and mark assignments and tests (Questions 6,7)
4. improvement ideas (Questions 8,9,10)

4.5 Survey coverage and structure

The detail regarding the invitations that were sent is shown in Table 5.1. The request to complete the survey explicitly asked that whoever receives the request would forward it to the Mathematics department of the school. The text for the invitation can be seen in Appendix A on page 87. The request for participation was automated and sent out 3 times at intervals of about a month.

The survey was conducted using Google Forms. The generic analysis that Google does is shown interleaved with the survey questions. The results of questions as independent sources of data did not yield enough insights, therefore analysis of the data was done to try and uncover hidden correlations. The findings of this analysis can be seen in Section 5.3.

The survey was completely anonymous and no email or other contact information was collected. The invitation (see Appendix A on page 87) included a commitment of anonymity and implicit informed consent. This removed the opportunity to follow-up on some of the suggestions and opinions voiced, but was necessary to give people greater freedom to be honest in their responses. It also removed any privacy or ethics issues that could have arisen if addresses were collected.

The time needed to complete the response was kept under 5 minutes and this requirement limited the questions to 10. De Ruyter and Oosterveld [11] found that short surveys have a better response rate than long ones. They also found that incentives and follow-up had a positive effect on response rate. Two follow-up mail requests after the initial one on 17 October 2017 was done and this had a positive effect on the response rate (see Table 5.1).

None of the questions were compulsory in order to prevent survey abandonment. Although a 5 minute limit was targeted to complete the survey, the invitation stated that it should not take more than 10 minutes to complete.

The survey used a non probabilistic approach (voluntary sample). The possibility to apply sophisticated statistical analysis on the data obtained using this method is limited and the focus was on high-level observations to guide us to possible conclusions and directions for further studies.

The survey covered the Gauteng province of South Africa. The targets were all the Secondary (Grade 8 to Grade 12) and Combined (both primary and secondary grades) schools, both private and public in Gauteng. A publicly available list was obtained that held a list of email addresses that were published as the contact mail address for each of the schools. The majority of the addresses were the general school contact mail address (eg. info@schoolname.co.za). Over 1000 addresses was obtained in the list that was used and 146 responses(14%) were received over the period of 3 months starting mid October 2017 and ending December 2017. The survey request was personalized in that the name of the school that the address was linked to, was visible, but not a person's name (where known) associated with the address.

Chapter 5

On-line Survey Execution and Results

In this chapter the results that were obtained from the on-line survey will be discussed. The process and results of soliciting the responses will be shown in the next section and in the following section each of the questions will be discussed separately. Finally, conclusions from combinations of results are drawn.

5.1 Survey invitations and responses

In this section the responses received compared to the invitations sent will be discussed.

	17 Oct 2017	10 Nov 2017	1 Dec 2017
Nr. of requests sent	1148	1063	1036
Bounced	83	55	16
Addresses Changed	68	55	0
Resp. ack.	7	3	2
Resp. Opt out	2	0	0
Number of responses received	64	44	38

Table 5.1: Survey requests

From the three requests sent with the number of targets as in the row "Number of requests sent" of Table 5.1, 146 responses were received. If a request bounced with "address unknown" the mistake was corrected if the mistake is obvious. For example, an address such as `name@gamil.com` was corrected to `name@gmail.co`. Similar typos were corrected. If a respondent acknowledged that they received and completed the survey, the address was not included in the next round. Only 2 targets opted out after the first round and they were excluded from the next rounds.

The response percentage was calculated using this formula:

Final number of target addresses = Total number of addresses(1148) - addresses that bounced without correction(16) - persons that opted-out (2).

This gave us a final number of addresses of 1130. The received responses from the 3 invitations on the dates as in Table 5.1 were 146 giving us a response rate of 12.9%.

This was greater than the target of 10% as described in section 4.2 and therefore this percentage was regarded as sufficient to continue with the rest of the steps.

5.2 Description of survey questions and basic results

In this section the survey questions will be discussed one by one and the summary results for each question provided. Percentages were calculated for questions where respondents could only select one answer. For the other questions the number of mentions that the options received are shown and in some cases percentages calculated where it provided extra insights. The results for the survey was one-hot encoded. One-hot encoding encoding is a process by which categorical variables (label encoding) are converted to a binary form. A column is created for each categorical variable and the fact whether it was selected or not is indicated using 0 for not selected and 1 for selected. This was done to simplify analysis and cross-question correlations such as those shown in Section 5.3 (Deduced Results).

Below an extract of the resultant matrix (first two questions and first 8 respondents):

Resp nr	Math Lit jnr	Math Lit snr	Math jnr	Math snr	Laptop	Desktop	Tablet	Smart phone	None
1	0	0	0	1	1	0	1	1	0
2	0	0	1	1	1	0	1	1	0
3	0	0	0	1	1	0	1	1	0
4	0	0	0	1	1	1	1	1	0
5	0	0	0	1	1	1	1	1	0
6	0	0	0	1	1	0	0	1	0
7	0	0	0	1	1	0	0	1	0
8	0	0	1	1	1	0	0	1	0

Table 5.2: Extract of results in one-hot encoded format

5.2.1 Question 1: "Please indicate your position?" (Question Type = Checkboxes)

In South Africa learners have a choice between Mathematical Literacy or Mathematics as compulsory subjects for Matric. Mathematical Literacy is a more practical subject that focuses on the application of mathematical concepts rather than the theoretical basis for the concepts as described by Clark [10] in "Maths vs. Maths Literacy: the continuing debate". The curriculum was introduced to SA in 2008. Due to its short lifetime, one would not expect a lot on-line resources that caters for this subject. Out of all the respondents, only 6 were exclusively Mathematical Literacy teachers. The rest of the respondents teach both Mathematical Literacy and Mathematics with the majority of the respondents teaching Mathematics at Senior Secondary level.

The summary of the respondents that indicated that they teach more than one category is shown in Table 5.4.

The survey had an error in that Math Lit is not a subject presented at grades lower than Grade 10, therefore, the first option "Teacher of Math Lit junior secondary" is not a valid option. It is assumed that the single response received for that option was a misread and the respondent meant to enter teacher of mathematics junior secondary. Nevertheless, the response was discarded in the further analysis of this survey.

Position	Number of mentions	% of mentions
Teacher of Math Lit. Junior Secondary	1	0.5%
Teacher of Math Lit. Senior Secondary	17	9.6%
Teacher of Mathematics Junior Secondary	31	17.6%
Teacher of Mathematics Senior Secondary	108	61.4%
Other	19	10.8%
Total mentions	176	100.0%

Table 5.3: Respondent's function in the school

Position	Number of respondents
Junior Mathematics and Senior Secondary Mathematics	17
Junior Mathematics and Senior Math Lit	5
Senior Math Lit and Senior Mathematics	7
Junior Maths, Senior Math Lit and Senior Mathematics	2
Other and Senior Mathematics	7
Other only (no other option selected)	8

Table 5.4: Respondent's function in the school by combinations

5.2.2 Question 2: "Which of the following devices do you own? (tick all that apply)"

The results of this question are as shown in Table 5.5.

This question determined if any of the teachers are without any electronic device and as can be seen in Table 5.5, none of the respondents indicated that they do not own an electronic device at all.

An analysis of the individual responses showed that 15 teachers have no smart-phone, but 14 of those do have a laptop. This was not expected but it could be that the term "smart-phone" was not familiar to the respondents and that there phones are likely to be smart-phones.

The fact that 134 out of the 146 of the respondents have a laptop indicates a high level of equipment availability. A large number of respondents also more than 1 device listed.

Device	Number
Laptop	134
PC	52
Smartphone	31
Tablet	81
None of the above	0

Table 5.5: Devices owned by respondent.

5.2.3 Question 3: "What percentage of your learners own tablets? or laptops?"

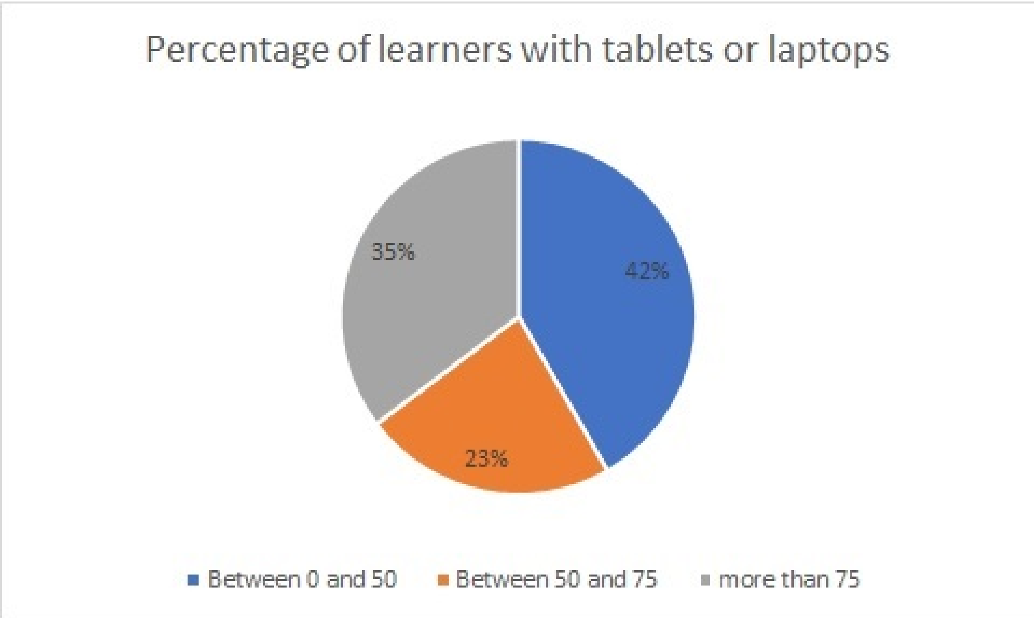

Figure 5.1: Question 3 Results summary

7 respondents did not answer this question. The reason might be that the options provided were difficult to choose from, especially if a teacher have many classes spanning different age groups.

The question sought to establish whether the lack of electronic tools available to learners could be an indicator for usage or non-usage of tools.

42% of respondents (58) indicated that only between 0% and 50% of their learners have access to electronic devices.

After label encoding of the results we could determine that 65% of respondents have environments where less than 75% of learners have access to electronic devices.

A more detailed look at student access and use of online portals is done in Section 5.3.

5.2.4 Question 4: "Do you use any of the on-line course platforms below as part of your teaching? (tick all that apply)"

The online course platform options listed in Table 5.6 were given by the survey to respondents and the number of respondents selecting the tools can be seen in this table. Respondents were given the option to provide additional online course platforms for Question 4. these are listed in Table 5.7.

Tool or website	Number of mentions
ITunesU	4
Kahn Academy	46
Siyavula	57
ITSI	18
Mathsbuddy	12
ALEKS	0
Udemy	1
Mathplanet	2
Mymathlab	4
MathXL	12
Mumie	0
Sagemath	1
Moodle	13
Blackboard	13

Table 5.6: Usage of online tools and platforms for teaching

This question explored which of the on-line platforms most commonly available, are used by those teachers that are venturing into the on-line world for additional material for their students. The selection of tools presented was drawn up by on-line searches by the author of this dissertation as well as previous knowledge and use of these platforms. The list was not exhaustive and an option was added where other sources could be listed. These platforms can be see in Table 5.7.

Alternative Platforms — Count —	
Mathletics	3
Limu	1
Youtube	1
MOOCs	1
Mathletics	3
Google Classroom	1
Maths is Fun	1
Geogebra	3
Socrative	1
Kahoot	1
IXL	1
Investec	1
VAW	4

Table 5.7: Other teaching tools mentioned by respondents

Khan Academy (46 mentions) and Siyavula (57 mentions) were the most popular tools. Khan Academy is a free US based portal with general topics. It does not follow any particular curriculum. Khan academy offers excellent video material. Siyavula is a South African site with some free material but access to premium content entails a nominal fee of R59 per month. Both learner and teacher textbooks in Afrikaans and English can be downloaded for free. Siyavula is built on the South Africa NSC (National Senior Certificate) curriculum so is the most relevant. It is basic as far as material is concerned. It presents only text explanations with some line drawings. It does not have videos or animations to explain concepts. Siyavula, however, has a section that allows users to do exercises that are evaluated on submission. This is coupled with a reward program.

Siyavula and Khan Academy offer a relatively simple user interface and cover all topics. If you sign up for them there is little need to go elsewhere. Very little technical knowledge is required to use them, but they do not provide the interactive classroom type facility of the portals discussed below.

26 Respondents used both Khan Academy and Siyavula, and 20 of those are teachers of Senior Secondary Mathematics.

The next most used portals are ITSI, Mathsbuddy[1], MathXL, Moodle and Blackboard. These portals differ from the first two in that the primary mode of engagement is via the school. The school would subscribe and customize

[1] Mathsbuddy also provides a private subscription facility at R87 per month.

the portal to their requirements and provide educators and students access codes. Participation to the portal is then mandated and sponsored by the school. Using these portals as a school, would require some technical expertise beyond just subjects in the school itself and more training of teachers in order to fully utilize the functions on offer by the portals.

Only 116 respondents completed this question and 5 of them indicated that they do not use any tool for teaching in the "Other" option. This constitutes 24.1% that never use online tools for self education.

$$\text{Percentage non-tool use for teaching (24.1\%)}$$

$$= \frac{\text{(total of question respondents (116) } - \text{ those that answered NONE (5))}}{\text{Total survey respondents(145)}}$$

5.2.5 Question 5: "Which of the online platforms below do you use as a tool for self-education? (tick all that apply)"

This question explored which of the online platforms most commonly available are used by teachers to improve their own knowledge of the subject.

Similar to Question 4, the selection of tools presented was compiled using on-line searches and previous knowledge. On this question an "Other" option was added for respondents to supply tools that they use that were not listed. The list of tools supplied by respondents can be seen in Table 5.9.

Note that Moodle as an option was erroneously omitted from this question but 3 respondents used the "Other" option to indicate that they use Moodle. This is included in this table for consistency with Question 4. It could be that others did not bother to do that although they use Moodle, so the actual number of users could be more. Here again Khan Academy and Siyavula came out on top with 40 and 47 mentions respectively, but there is a bit more variety in the tools used compared to the usage for teaching purposes.

In this question 35 respondents chose "None". The reason for this is hard to determine but could be due to the unexpected finding in the SACMEQ (The Southern and Eastern Africa Consortium for Monitoring Educational Quality) report [25]: mathematics teachers in South Africa show a higher level of confidence in their own skills than many of the top countries. It could be that the confidence in one's skill level is maybe not conducive to further study or that the respondents are using paper-based methods to keep up with the syllabus.

125 respondents completed this question and 33 of them indicated that they

Tool or website	Number of mentions
ITunesU	5
Kahn Academy	47
Siyavula	40
ITSI	9
Mathsbuddy	16
ALEKS	0
Udemy	1
Mathplanet	3
Mymathlab	3
MathXL	15
Mumie	0
Sagemath	2
Moodle	3
Blackboard	6
None of the above	33

Table 5.8: Usage of online tools and platforms for respondent self-education

do not use any tool for teaching in the "None" option with 2 writing None in the "Other" option. This constitutes 38% that never use online tools for self education.

$$\text{Percentage non-tool use for self-education (38\%)}$$

$$= \frac{\text{(total of question respondents (125)} - \text{those that answered NONE (35))}}{\text{Total survey respondents(145)}}$$

Other tools provided by respondents	Count
VAW	3
aqa and edexcel and examsolutions	1
Moocs	1
Youtube	3
Google Classroom	1
Maths is fun	1
Video teaching	1
TED	1
Linkedin articles	1
Vodacom e-learning	1
WOLKSKOOL	1
own website	1

Table 5.9: Other tools for self-education mentioned by respondents

5.2.6 Question 6: "How often do you use electronic tools to mark homework or tests?"

This question explored the use of electronic resources for entering and marking mathematical assignments and tests using a Likert scale of 5 ranging from 1 = NEVER and 5 = ALWAYS. This is different from merely using an on-line tools for teaching or self-education. It moves into the area of electronically processing mathematical exercises.

This question allowed for only one answer per respondent. 2 Respondents did not answer this question. One of them only responded to the last 2 questions and the other one responded to all other question except 6 and 7. They were left out of the calculations for this question. As can be seen in Table 5.10,

Frequency	Number	%
1 (NEVER)	108	75%
2	13	9%
3	14	9.7%
4	3	2.1%
5 (ALWAYS)	6	4.2%
TOTAL	144	100%

Table 5.10: Usage of electronic tools to mark homework or tests?

the majority of respondents indicated that they never use on-line tools to mark homework or tests - 75% - versus 25% that indicated they use on-line tools with some frequency. Only 6 (4.2%) indicated intensive use of on-line tools for marking.

Although quite a high percentage (75%) never use tools for processing of tests, it is encouraging that a quarter of the respondents at least use it sometimes. If the online educational tool usage for self-education (Question 5 page 41), is considered, where 62% (100% - 38%) respondents indicated that they use online tools for self-education and Question 4, page 38 where 76% (100% - 24%) respondents indicated that they use tools for teaching, it might be fairly easy to increase the tool usage for electronic marking given that a high number of teachers already use it for teaching and self-education. Whether there is a correlation between tool usage for teaching and more technical use for marking, needs further investigation.

5.2.7 Question 7: "How often do you receive homework or tests electronically from learners?"

This question is the follow-on of Question 6. If you never receive homework or assignments electronically, you would also not employ automated tools to attempt the marking of such homework or assignments and vice versa. The responses show that the distinction between using a tool for marking and just collecting homework or assignments electronically might have been a bit subtle to most of the respondents. It would have been better to explain the distinction between the 2 questions more explicitly to the participants.

Frequency	Number	%
1 (NEVER)	105	72.9%
2	20	13.9%
3	12	8.3%
4	5	3.5%
5 (ALWAYS)	2	1.4%
TOTAL	144	100%

Table 5.11: Usage of electronic tools to collect home work or tests.

Similar to Question 6, the majority of respondents (72.9%) indicated that they never receive homework or tests electronically. 72.9% versus 27.1%, that indicated they use on-line tools with some frequency. Only 2 (1.4%) indicated intensive use of on-line tools for the receipt of homework or tests.

The responses to Questions 6 and 7 confirms the first hypotheses that on-line tools are not used extensively in high school mathematics education in the South Africa province of Gauteng. Despite the fact that the respondents reported that many online platforms are used as evident in their responses to Question 4, the responses to Questions 6 and 7 suggests that the use of the platforms are only considered to be optional for learners and that the tools offered on the platforms are not extensively used for assessment or homework assignments.

5.2.8 Question 8: "What will improve your use of use electronic devices and e-learning platforms in your teaching of mathematics? (tick all that apply)"

This question presented the participants with a number of possible answers to the question plus an option to provide other possibilities. The question was a check-box format question with any number of choices allowed. It also had a free-form input box for additional suggestions from respondents. The options provided by respondents can be seen in Table 5.13. This question was devised to address the second hypothesis in trying to determine reasons for the low usage of on-line tools for secondary school mathematics education. The response options can be grouped in the following topics:

1. Training
2. Free internet access
3. Infrastructure and device
4. Software tools

The mentions of each of the provided options plus the totals per group can be seen in Table 5.12. Table 5.12 also shows the importance of the options according to the respondents in the column labeled "Priority".

Method	Number of mentions	Priority	Group	**Group total**
Training for teachers	112	1	Training	
Training for learners	74	3	Training	**186**
Free access to high speed internet and wifi	93	2	Free internet access	**93**
Free laptops and tablets for teachers	58	6	Infrastructure and device	
Free laptops and tablets to learners	55	7	Infrastructure and device	**113**
Better editing tools for mathematics	70	4	Software tools	
Easy to use grading tools for mathematical problems	65	5	Software tools	**135**

Table 5.12: Opinions on methods for improving the use of electronic tools

Option provided	Count
Mathematics Olympiads	1
Don't want to use online tools	1
Cheaper cost of joining online platforms	1
more time with learners	1

Table 5.13: Options for improvement provided by respondents

As shown in Table 5.12, the priority value of editing and grading tools for mathematics is higher than the priority value of equipment availability on the list of reasons for low on-line tool usage. The most important issues selected by respondents are access to Internet and training with the highest priority values.

This result partly confirms the second hypotheses: The reason for low usage of on-line tools in mathematical education this is not primarily the lack of access to equipment. Lack of training on the use of online tools and the perceived inconvenience of using such on-line tools may be reasons for low usage.

5.2.9 Question 9: “Which of the following do you think may be problematic when students have to submit mathematics homework or tests electronically? (tick all that apply)”

This was the last question with formatted answers and it focused on automated marking of mathematics homework or tests. Again, the distinction between this question and Question 8 was perhaps a little subtle and more specific results might have been obtained by adding more detail or description.

Question 9 solicited advice for improving the use of on-line tools for education by offering a few options and providing an opportunity to receive other ideas from the respondents. The options provided were general and covered all the aspects addressed before, namely teaching, self-education, receipt of homework and marking. Multiple options could be selected by respondents making the number of mentions more than the number of respondents to the question.

The most popular option selected by respondents was related to the perception of possibilities with electronic processing (86 mentions) as can be seen in Table 5.14. “Multiple choice tests do not teach learners to write out problems”.

The second most frequently selected option (57 mentions) was one of competency “Learners cannot use editors properly”. This is consistent with the findings in the school experiment after the request to submit a question electronically. The details of these findings will be discussed in Chapter 7.

Problem	Number
Multiple choice tests do not teach learners to write out problems	86
Difficult to grade freeform problems electronically	52
Teachers struggle with the tools	41
Learners cannot use editors properly	57
Have no problem	23

Table 5.14: Problems with electronic submission of homework and tests.

5.2.10 Question 10: "If you have a view on any aspect of on-line math teaching not covered in the previous questions or advice for improving the survey, please enter below."

Question 10 was a free format question that is a catch all and gave the opportunity to suggest any other factor that the respondent felt was relevant. 18 inputs to this question was received out of the 146 respondents. In other words, 12% of respondents felt strongly enough about the subject that they supplied free-form comments.

All responses to this request can be seen in Appendix B on 88. In this section the comments will be categorised in two categories based on whether the comment is pro the use of on-line tools for mathematics education or against it. A grouping is also made with regard to technical and pedagogical suggestions.

Seven respondents were *opposed* to the use of on-line tools in mathematical education. Their respective reasons are given below exactly as given by the respondents.

> - Define the roles of a teacher when using online teaching tool....[sic]
> - Electronics is a great aid, but it cannot replace pen and paper for math.
> - The small screen smart-boards not ideal for teaching Mathematics - microsoft math and vodacom eschool
> - Learners need to write. Typing answers and ticking boxes does not help with the cognitive process when doing maths. I am old school. I will project past papers for them to do, or show one or two video clips on specific topics, but I prefer paper. Learners don't need tablets, internet, or smartphones to pass maths. They need confidence, a strong work ethic and dedication. It should be an extra source, but not the main source of teaching.
> - I still think that the method of writing out problems is the best way for kids to learn.
> - I don't believe online teaching is generally suitable for use in Mathematics.
> - I feel maths needs the personal touch as so many battle with the subject.

The rest of the comments were positive towards the concept of on-line tools for mathematical education but highlighted problems such as training, access to resources and funding, and learner discipline.

Two respondents to this question partly confirmed the second hypotheses as described on page 31:

“Part 2: The reason for this is not the lack of access to equipment. It is rather the lack of knowledge. Difficulty of doing mathematical expressions electronically may also inhibit the use of on-line tools.”

with the comments:

> - I think *I would find an assessment tool beneficial*, especially for immediate feedback and more for junior grades than seniors.
> - We need to integrate different systems. Using Itsi with Geogebra and *making the use of equation editors more user friendly* so learners will be able to type answer that does not appear on a normal keyboard.

If the comments are evaluated with regard to the category of suggestion, pedagogical vs technical and financial, they can be grouped as below:

Pedagogical

1. There is too much stereotyping in South Africa. I looked at Siyavula and Khan’s Academy - not relevant to South African situation.

2. Learner skills and Educators to use the electronic skills successfully.

3. Online teaching maths is good, problem our learners are not disciplined.

4. Learners need to write. Typing answers and ticking boxes does not help with the cognitive process when doing maths. I am old school. I will project past papers for them to do, or show one or two video clips on specific topics, but I prefer paper. Learners don’t need tablets, internet, or smartphones to pass maths. They need confidence, a strong work ethic and dedication.

5. It should be an extra source, but not the main source of teaching.

6. I still think that the method of writing out problems is the best way for kids to learn.

7. I don’t believe online teaching is generally suitable for use in Mathematics.

8. Khan's videos are excellent. I use it extensively. I also love i-Pathways tutorials and tests especially for wordsums
9. I feel maths needs the personal touch as so many battle with the subject
10. ... it is a problem to have learners express their maths knowledge in a systematic manner. Learners are not yet mature enough to manage the responsible use of tablets in their learning. The devices are often not treated carefully and are subsequently damaged, which impacts on the e-learning process.

Technical and financial

1. I do love using online math teaching but sometimes its very difficult to keep up with all the changes that takes which schools cannot afford to pay for for renewals or on going support from the people that offer the services.
2. Definitely not enough online South African resources available and I therefore use many overseas resources. To the detriment of my budget as many are not always freely available.
3. In my institution we use Fathom, geogebra, autograph, Sketchpad Why and when should you use online math teaching? Can be asked. Differentiate maybe between teaching tool and assessment tool, because I think I would find an assessment tool beneficial, especially for immediate feedback and more for junior grades than seniors. Define the roles of a teacher when using online teaching tool....
4. Electronics is a great aid, but it cannot replace pen and paper for math
5. The small screen smart-boards not ideal for teaching Mathematics
6. microsoft math and vodacom eschool
7. Online education is often not seen as a useful tool by school administrators or they do not have the funding available to ensure that e-learning is facilitated
8. The main problem is one of access. Many of our learners rely on free wifi, which is available at school. Once they get home it becomes an issue as many of them do not have access to the internet at home.
9. Our school has learners with a very big variety of socio-economic backgrounds. To level the playing fields, will be financially tough.

5.3 Deduced results

In this section the result of responses to questions in combination with other questions will be provided. The one-hot encoded data shown in figure 5.2 was used to select subsets of the responses that satisfy certain criteria.

A calculation of the number of respondents with student access between 0% and 50% that do use some on-line portal was done (Question 3 combined with questions 4 and 5). 55 of the 58 respondents with student access between 0% and 50% indicated that they never use on-line portals for assignment marking and 53 indicated that they never use it for assignment collection. 12 (22.64%) of the 58 responded that they never use on-line media for self-education or teaching.

77.36% (32) of respondents with less than 50% of learners with access to an electronic device use on-line portals for self education or teaching.

Of the 32 that indicated that only between 50 and 75% of their learners have access to electronic devices, only 2 (6%) never use on-line tools for self-education or teaching.

Of the 49 respondents with 75% or more of their students with access to electronic devices, only 6 (13%) never use on-line tools for self-education or teaching.

This is an encouraging result in that it appears that lack of student access to equipment does not impede teachers to explore on-line possibilities and where a high percentage of the students have access to electronic devices, the teachers are inclined to use online tools for teaching and self-education.

The first 6 questions showed that the use of on-line tools in mathematical education is at a fairly low level and some attention was given to the opinions of teachers on the factors that in their opinion would improve their usage of such tools.

Table 5.15 shows the difference between the responses to Q6 (How often do you use electronic tools to mark homework or tests?) and to Q7 (How often do you receive homework or tests electronically from learners?). It was expected that Q7 would get more positive responses since receiving homework or assignments in electronic format is a prerequisite for marking such homework using some electronic tool.

The responses are consistent with the expectation that more people will use electronic tools to receive homework and tests as opposed to using such tools to *mark* such homework or tests. 27 respondents indicated that they use electronic tools at some frequency to mark mathematics homework,

Frequency	How often do you use electronic tools to mark homework or tests?	How often do you receive homework or tests electronically from learners?
1(never)	117	105
2	10	20
3	10	12
4	2	5
5(Always)	5	2

Table 5.15: Comparison between Q6 and Q7

Never uses any on-line tool	Total	No problem	Learners cannot use editors properly	Teachers struggle with the tools	Difficult to grade free-form problems electronically	Multiple choice tests not adequate
Number of respondents	105	15	41	32	38	68

Table 5.16: Reason for not using online tools of those that *never* use them.

versus 39 respondents indicating that they receive mathematical homework in electronic format at some frequency.

The option "Multiple choice tests do not teach learners to write problems" received 35% of the mentions of those that never use tools vs 28% of mentions of those that use tools at any frequency. See tables 5.16 and 5.17. The limitations of multiple choice question in a subject such as mathematics seems to be a barrier for the usage of electronic tools. Improvements in the marking capabilities of electronic tools to the point that they can handle the ambiguities of extended answers might improve the use of such tools.

Use on-line tool(s) at some frequency	Total	No problem	Learners cannot use editors properly	Teachers struggle with the tools	Difficult to grade freeform problems electronically	Multiple choice tests not adequate
Nr. of respondents	41	8	16	9	14	18

Table 5.17: Reason for not using online tools of those that use them at *some frequency.*

	Total	Q7 Never(1)	Q7 Sometime(2)	Q7 Average(3)	Q7 Frequently(4)	Q7 Always(5)
Maths Snr. Sec.	112	80 (72%)	16 (14%)	9(8%)	5(5%)	1(1%)
Rest (Math. Lit. and Junior)	33	25 (78%)	4(13%)	2(6%)	0(0%)	1 (3%)

Table 5.18: Comparison of usage levels between senior secondary teachers and the rest.

From Table 5.18 it is clear that the senior secondary teachers are more active users of on-line tools. 38% percent of math senior teachers indicated that they use tools at some frequency compared to 7% of the rest. The reason for this is not explained by this survey but it could be due to the lack of material in online portals for lower grades. This will have to be investigated before a conclusion can be drawn.

5.4 Limitations of survey

The survey could be subject to non-response bias. It was not a formal survey conducted by the Gauteng Department of Education and as such participation could not be made compulsory.

It is possible that only people that had strong feelings or prior thoughts on the topics, either pro or con, would respond. Teachers not using on-line tools at all might also not have responded. This would skew the results in favor of a higher reported percentage of using on-line tools.

Only people with access to internet could have responded. This would also have skewed the results in that the respondents needed to have some level of technical ability and access in order to respond.

It is thus likely that the usage of on-line tools for mathematics in Gauteng could even be lower than what is reported here.

5.5 Summary

The survey produced some interesting results but not all unexpected. No definite conclusions can be drawn on the overall availability of equipment since it is likely that only teachers with some equipment and internet access would have responded to the survey. Nevertheless, all respondents have access to equipment and 91% of respondents have laptops which indicates that the main impediment for the use of online tools in secondary schools is not lack of equipment.

On-line tools are being used by about 77% of respondents in some format. The tools are mainly being used for education and teaching but not widely for communications to learners in activities such as assignment collection and marking.

The results of the survey were valuable but there could be merit in a follow-up survey after the conclusion of this research activity, applying some form of incentive to increase participation.

Part III

School Study to investigate the impact of the use of Notation

Chapter 6

Planning and Design

As explained in Section 3.5, a study was designed to investigate the comprehension aspect of the proposed notation. The design, motivation for the design and planning of the study is discussed in this chapter.

6.1 Research design and method

The study of the comprehension benefits of a new notation, can best be defined as *exploratory research* as characterized by Van Wyk [48].

> This is the most useful (and appropriate) research design for those projects that are addressing a subject about which there are high levels of uncertainty and ignorance and when the problem is not very well understood (i.e. very little existing research on the subject matter). Such research is usually characterized by a high degree of flexibility and lacks a formal structure. The main aim of exploratory research is to identify the boundaries of the environment in which the problems, opportunities or situations of interest are likely to reside, and to identify the salient factors or variables that might be found there and be of relevance to the research.

In this case there is an intuitive feeling that there must be some benefit with regards to comprehension in using a notation with a more linear style and less chance for confusion, but no study on these specific aspects could be found.

The initial approach to the research can be characterized as a combination of deductive and inductive research. Inductive research is characterized as being more exploratory and open in that it does not try and put all the focus on a specific hypothesis.

Deductive research is the more generally accepted method for pure scientific research since it focuses on a hypothesis and gathers data to prove or disprove the hypothesis.

Both approaches are well described by Soiferman [42] and Table 6.1 has been compiled using Soiferman's article as a basis.

	Quantitative/deductive	Qualitative/Inductive
Intent of research	Test theories to prove or disprove	Gather info to formulate themes.
Literature usage	Major and extensive	Brief and more limited
Focus	Pointed using close-ended questions	Adaptive/versatile using open-ended questions
Data collection	Many participants in rigid fashion	Data or images from few participants
Data analysis	Numerical statistical analysis-descriptive	Look for patterns or themes-Iterative
Role of researcher	Objective approach	Open to adjusting view according to data and observations
Data validation	Standard Validation procedures	Participants, peers

Table 6.1: Characteristics of research approaches

Tuckman [45] describes the problems often encountered in education research and labels it as 'dealing with reality'. He describes two principles of research, namely:

1. Internal validity: The study outcome is only a function of the approach being tested. In this case this would translate to the certainty that improved results of the two groups are only due to the changed notation and not any other factor.

2. External validity: Results are applicable in the real world to other similar programs. In this case this would mean that changing the notation as proposed would bring improved results to any group and even wider. Modifying other notations to a more linear and explicit form, would bring similar results.

In this case it was anticipated that it will be difficult to obtain complete internal validity in the study due to the many factors that play a role in student's understanding of a subject. These factors will be discussed in Section 6.1.1.

Schanzenbach [39] describes difficulties with educational research in the context of policy making, but the aspects she outlines are applicable in this case as well. Setting up a randomized controlled trial in the educational field and especially in the primary and high school phases, is almost impossible due to the many variables. Basic controls such as determining the outcome in a different location, at a different time, using a different teacher or students, are extremely hard to arrange. In the case of policy making, the questions might be irrelevant by the time it is completed. In particular she states that the biggest obstacle in conducting education research of reasonable complexity, is external validity. Nevertheless she advises that the obstacles should not deter one from conducting educational research but that one should be cognisant of the limitations of such research.

6.1.1 Factors that influence learning

A study done by Mji and Makgato [26] on the factors influencing mathematics and science marks, found that the following direct factors play a very important role:

1. teaching methods
2. teacher's knowledge and understanding of the subject matter
3. motivation and interest of the teacher
4. use and availability of laboratories (science)
5. completion of syllabus

Mji and Makgato [26] also found that indirect factors such as language and parents (domestic circumstances) play a big role in educational outcomes.

In a different study, Saritas and Akdemir [38] found that curriculum, teaching methods, school context and facilities all play a big role in the outcome of teaching.

Considering all factors elaborated above, it will be difficult to construct a short term, scientifically objective study to measure the capacity of learners to assimilate and use any new mathematical concept. In accordance with this conclusion, a study was designed that had the research question as described in Section 6.1.2 but approached the study with an open mind and endeavored to observe any other benefit or drawback of the proposed notation.

6.1.2 Notation School Study Research question

Is there any difference in the understanding of mathematical concepts, and in particular the concepts underlying sequences and series, when using different notation schemes? In other words, does the notation used for mathematical expression, play a measurable role in understanding?

Given the mixed (qualitative/quantitative) nature of the research, all data and material gathered will be analyzed for any other conclusions or insights.

6.1.3 Notation school study research method

It was decided to use a mixed method that includes quantitative as well as qualitative studies as used by Pieterse and Sonnekus [33] in their research on the lack of good computer educators and to identify qualities of good candidates for this role. The validity of this approach is also explained by Niglas [29]. He argued that a mixed methodology is acceptable in research and that a combination of methods can complement one another and may give different perspectives on the same data.

The primary study focus was the question of whether an alternative notation has an effect on the comprehension of sequences and series. This investigation needed learners that have never been exposed to the sigma notation before. Enough students had to participate so that they could be divided into 2 groups with each group receiving primary instruction in one of the 2 notations. To avoid learner confusion between the 2 notations and ensure that no-one would be disadvantaged by participation in the study, the study was designed to ensure exposure of both groups to both notations (within-subject design). This was also a requirement from the school and teachers that participated in the study.

6.2 Notation school study structure and planning

The study was conducted in 2016 to investigate the research question as described in Subsection 6.1.2 .

To conduct the research with learners who complied with the requirements as discussed in 6.1.3 , an experiment was designed that contained the following high level steps :

- Step1: Find a school and recruit participants
- Step2: Find and prepare lecture material
- Step3: Present lectures and gather data
- Step4: Process the data
- Step5: Draw conclusions and expand the study as necessary

The rest of this chapter will describe the steps that dealt with the preparation of the study and Chapter 7 will discuss steps 3 to 5.

6.2.1 Step 1: School selection and participant recruiting

A public secondary school was found in an affluent area of Johannesburg for the experiment, mainly because the researchers had links with the school and were able to obtain all the necessary permissions within a reasonable time. The language of instruction in this school is Afrikaans, which meant that Afrikaans lesson material had to be sourced and the lessons and tests were conducted in Afrikaans. Obtaining permission to proceed took some time and effort. The school in question is a “model C” school which in South Africa means that although it is a public school, the school board, consisting of parents, has a big influence in the running of the school and any new activity needed permission from the school board as well as permission from the principal. On top of that, an experiment involving mathematics, one of the most sensitive subjects in the school since it is crucial to get good marks for mathematics in order to be accepted for most university degrees, got some initial resistance from the school authorities as well as the teachers. A meeting was held with the principal of the school in which the main assurance that had to be given was that learners will not be confused by the new notation. This assurance was also important for the mathematics teachers that are in principle not in favor of any outside lessons in topics such as mathematics and science. Their main argument being that learners are taught different methods by outsiders than those prescribed by the department of education causing confusion to the learners. This argument was the driving force behind the decision to implement a within-subject study, exposing all participants to both notations.

The timing of the study was such that it fell at the end of the school year for the Grade 11 learners. This strategy ascertained that all participants would have covered the general topic of sequences and number patterns but they have not yet encountered series and sigma notation. The mathematics teachers were co-opted to distribute invitations to a workshop to all the Grade 11 learners with mathematics as subject, with a strong recommendation to all to attend. The fact that Sequences and Series is the first topic covered in the matric year, gave the learners an added incentive to attend since it is well known that the first few weeks in the matric year are chaotic for the learners. During the these weeks lots of time are taken up with other activities such as Grade 8 camps, concerts and other school cultural and sports events. The advantage of having covered the material of the first chapter already once when the matric year begins, was emphasized in the invitation to attend and reinforced by the teachers. See Figure 10.2 in Appendix C on page 90 for the invitation text.

From the invitation, a few months before the end of the Grade 11 year, about 50 learners out of a total mathematics learner group of about 120 accepted the invitation. This number was deemed sufficient to proceed with the study.

The logistical issues around organizing a study of this kind are daunting. Permissions are needed from governing bodies and teachers, and the parents also have to give their consent. The permissions alone could take several months. Since the school where the research was conducted, is semi-private and dependent on parental funding and goodwill, it is understandable that they were reluctant to introduce anything that could be seen as remotely controversial.

Once all the stakeholders had been persuaded and requisite permissions obtained, a suitable time slot had to be found in the school's extremely packed agenda. For the experiment, this date ended up being after the final exams just before the summer holidays. In the South African school system, learners who have completed their end-of-year examinations, do not wait for the official closing date of the schools to go on holiday. This meant that the pool of available learners was small and that the participating learners' motivation to sit through lessons and tests was very low. This also accounts for the high dropout rate of participating learners in the study.

To realize the scope, the learners were divided into two groups of equal capability. To this end, the result of a pre-test was used (See Appendix D on page 92) given to all participants before the start of the lessons. The pre-test consisted of ten questions, eight of which were mathematics related and two were brainteasers. The purpose of the brainteasers was to judge the participants' basic problem-solving and logic skills, independent of previously acquired mathematical knowledge. Due to the restricted time available to conduct this whole activity, mathematical questions that required more than a few minutes to answer could not be included.

6.2.2 Step 2: Find and prepare lesson material

For completeness sake, it was decided to cover all the topics in the chapter on Sequences and series of the syllabus. This meant the following topics:

- Arithmetic sequences
- Geometric sequences
- Series
- Finite arithmetic series
- Finite geometric series
- Infinite series

The same material was used for both groups with the one group's material modified to use the Dijkstra notation. The same teacher taught both groups to avoid differences in presentation. The Afrikaans version of the chapter on sequences and series of the Grade 12 mathematics textbook published by Siyavula[1] was used.

An experienced mathematics teacher was selected to present the lectures and followed a traditional lesson schema for the lessons:

1. Recap prior knowledge
2. State the learning objective
3. Explain by means of an example
4. Allow learners to solve problems
5. Evaluation and feedback

No attempt was made to be innovative in lesson structure and approaches since the learning environment needed to stay as close as possible to a traditional school setup.

[1]https://www.siyavula.com/maths/grade-12

At the end of the three days, an opinion poll was conducted, involving the participants who had remained in the experiment to the end.

Four questions were asked in this opinion poll:

1. Which notation is easier to write?
2. Which notation is easier to read?
3. Which notation is easier to understand?
4. Which notation do you prefer?

Consistent with the research methodology, the data was analyzed in an iterative manner to identify aspects that might point to other aspects and factors that can be studied with the end goal the improvement of mathematics understanding in high school learners.

Apart from the test results and opinion polls obtained from the first research implementation, the written artifacts produced by the learners during the 3 days were also analysed. This material was used to uncover other potential issues with either notation that could be used in further studies.

Chapter 7

School Study Execution and Results

This chapter will describe the grouping, lecture process and results of the school study according to the steps outlined on page 59. In particular this chapter will address steps 3 to 5.

7.1 Step 3: Present lectures and gather data

The school study took place 5, 6 and 7 December 2016 from 9h00 to 12h00 each day at Linden Secondary School in Johannesburg. As described in Chapter 6, grade 11 mathematics learners of one selected secondary school in Johannesburg were targeted. Of the 50 learners who had initially accepted to participate in the study, 33 arrived for the first day and only 26 finished the three-day workshop. The 33 learners that arrived for the first day were divided into two groups using the results of a pre-test.

The purpose of the pre-test was solely to have some criteria with which to divide the participants in more or less equal groups. The number of participants were too few to attempt a completely randomized selection method. The results of the pre-test were not used for anything else apart from serving as a crude selection tool.

The results of the pre-test are shown in Table 7.1. The results reflect only the 26 learners who completed all three days. The variation in the results of the first eight questions was too small to use as a decider. We decided to split the learners who managed to answer Question 9, evenly across the two groups and to allocate the rest of the learners randomly.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
% Students with a correct answer	85	96	77	92	92	96	88	77	15	0

Table 7.1: Performance of students in the pre-test

The experiment was conducted over a period of three days with a work period of approximately three hours every day. The same teacher presented all the lessons to remove any variability associated with different teaching styles which different teachers could have employed. The teacher moved from one classroom to the next, presenting the same topic but using a different notation. While the first group was receiving a lesson, the other group was doing exercises under supervision of another researcher.

At the end of the instruction phase, both groups wrote the same tests. Examples of the questions and memoranda can be seen in Appendix E on 96.

The groups were then inverted, giving the first group a view of the Dijkstra notation and the second group a view of the traditional notation, and again gave the same level of test to both groups, requesting that they used the notation of their choice to give the answers. The second test was intended to ensure that the learners were able to use both notations.

The detailed lesson schedule that was used can be seen in Table 7.2.

The lectures given to the two classes were tightly scheduled with little time for relaxation or reflection by the teacher. This was necessary to get through the planned material in time and maintain the within subject design by having two groups. Reflections on this approach can be found in Chapter 9.

The study proceeded as planned in the research design although time constraints necessitated the shortening of some of the components.

The erosion of the participants from 33 to 26 as mentioned on page 63 could have been due to a few factors that are described below:

- Study participation was free so there was little incentive to attend.
- The study was conducted during the last 3 days of the school year. Many of the intended participants left early on holiday and did not bother to attend. Some excused themselves for this reason halfway through the study.
- The study demanded some work and attention which was difficult to

Day	Time	Dijkstra Group(D)	Sigma Group(T)	Subject	Chapter in studymaterial
Day1	09:00	Pretest to all learners. Divide into groups.			
	09:30	Lesson 1	Lesson 1	Arithmetic Sequences	1,1
	10:00	Practise	Practise		
	10:30	Lesson 2	Lesson 2	Quadratic Sequences	1,1
	11:00	Practise	Practise		
	11:30	Lesson 3	Lesson 3	Geometric Sequences	1,2
	12:00	Practise	Practise		
Day2	09:00		Lesson 4	Series (T)	1,3
	09:30	Lesson 4	Practise	Series (D)	1,3
	10:00	Practise	Lesson 5	Finite Arithmetic series (T)	1,4
	10:30	Lesson 5	Practise	Finite Arithmetic series (D)	1,4
	11:00	Practise	Lesson 6	Finite Geometric Series (T)	1,5
	11:30	Lesson 6	Practise	Finite Geometric Series (D)	1,5
	12:00	Practise	Test 1		
	12:30	Test 1			
Day3	09:00	Lesson 7		Sigma Notation (T)	1,3
	09:30	Practise	Lesson 7	Sigma Notation (D)	1,3
	10:00	Practise	Practise		
	10:30	Lesson 8	Lesson 8	Infinite Series	1,6
	11:00	Practise	Practise		
	11:30	Lesson 9	Lesson 9	Summary (T +D)	1,7
	12:00	Practise	Practise		
	12:30	Test 2	Test 2		
	13:00	Opinion Poll	Opinion Poll		

Table 7.2: Lesson Schedule

do for some of the learners that were clearly already in a holiday mood.

The small number of learners who completed the study had a negative effect on the data and subsequent results. An attempt was nevertheless made to reach some conclusions from the available data. These conclusions are discussed in Section 7.2.

7.2 Step 4a: Analysis of learners' performance in the tests

In this section the results of the sequence of tests that were administered during the course of the study will be discussed.

At the end of the lessons, both groups wrote the same tests that covered the same material but using the different notations. This test is called "Test 1".

The groups were then inverted, giving the first group a view of the material using the Dijkstra notation and the second group, using the traditional notation. The same level of test were then given to both groups, requesting that they used the notation of their choice to give the answers. The second test was intended to ensure that the learners were able to use both notations. This test is called "Test 2".

The results for both these test for the group using the sigma notation can be seen in Figure 7.1. Similarly, the results for both these tests for the group using the Dijkstra notation can be seen in Figure 7.2.

Table 7.3 shows the descriptive statistics of the marks that the learners achieved in the tests, while the individual performance of each participant is shown in detail in Figs. 7.1 and 7.2.

	Test 1			Test 2		
	Average	Median	Std Dev	Average	Median	Std Dev
Sigma group	37.00%	37.70%	0.09	20.71%	19.19%	0.09
Dijkstra group	17.21%	13.93%	0.17	13.30%	7.07%	0.11

Table 7.3: Descriptive statistics

The results of the final test were lower than the first, which is unexpected, since one would imagine that learners would perform better after having had more exposure to the underlying concepts. The lack of performance could have been due to the learners having had little incentive to complete the tests, so the learners did not put much effort into the final test. It is also likely that the introduction of another notation could have had a negative effect on the confidence of the learners and as a consequence they performed worse. This is more obvious for the sigma group (Figure 7.1) than for the Dijkstra group (Figure 7.2).

Statistical analysis is not feasible in this case, due to the small number of participants and the large deviation in the results of the 2 groups that cannot be explained by the different notation only. The one group was clearly

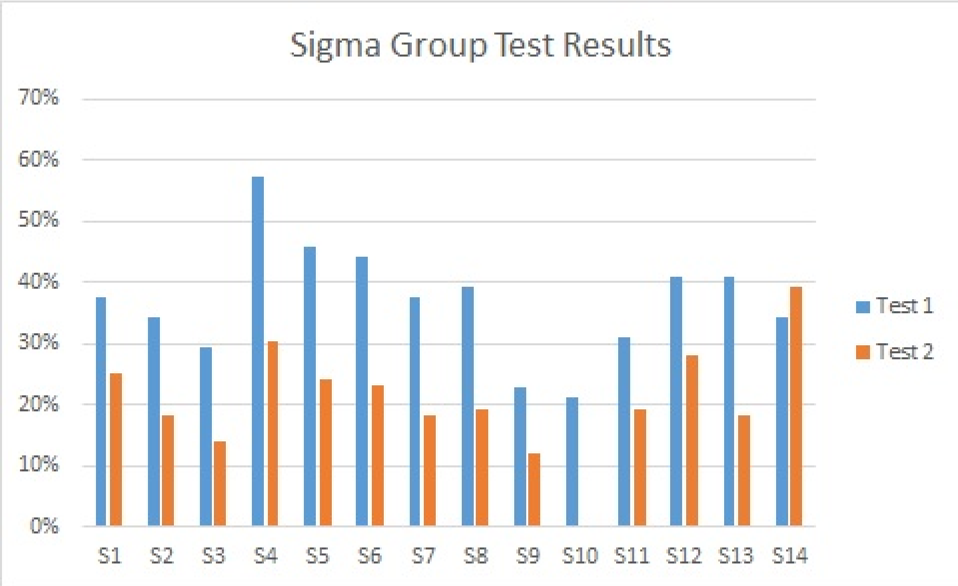

Figure 7.1: Sigma Group Test Results

stronger than the other. The initial selection method (refer to Table 7.1 Pre-Test Results) was not adequate as it did not render comparable groups.

A factor which might have played a role is that the two groups were not equal in gender distribution. To make matters worse, the dropout rate over the three days skewed the gender distribution even more (Table 7.4). The group with a majority of girls performed much better on average than the group with predominantly boys. The difference in the performance of the groups is so obvious that it is unlikely to be coincidental. The reason for this difference is, however, unclear. It could be gender differences rather than the way in which the material was presented in this experiment. This opinion is a topic for another investigation, which is beyond the scope of the present research.

	Start			Finish		
	Girls	Boys	Total	Girls	Boys	Total
Sigma group	11	5	16	10	4	14
Dijkstra group	5	12	17	4	8	12
Total	16	17	33	14	12	26

Table 7.4: Start and Finish Number of Participants

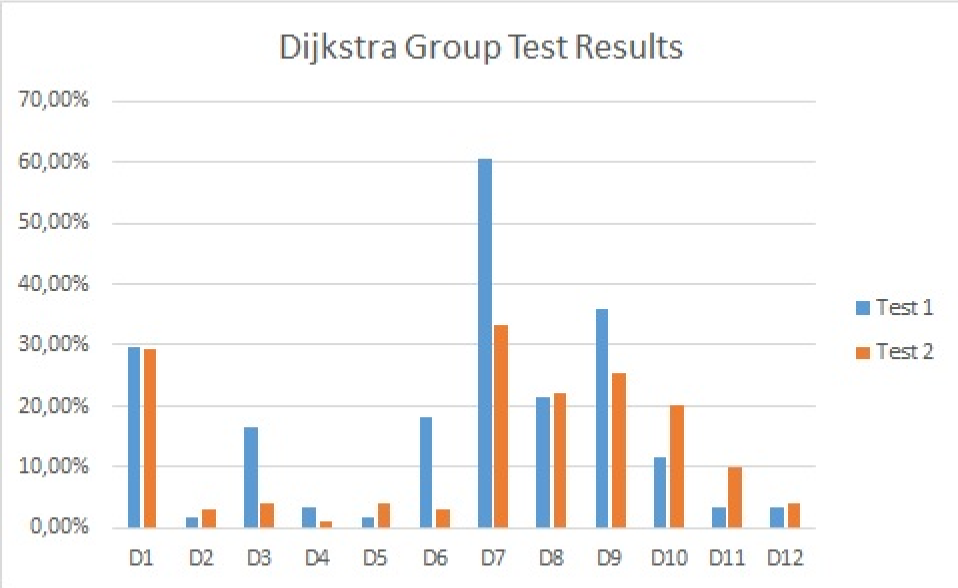

Figure 7.2: Dijkstra Group Test Results

7.2.1 Step 4b: Opinion poll results

The text of the opinion poll can be seen in Appendix F on page 103. The results of the opinion poll were inconclusive but quite interesting, since the stronger academic group had a higher number of learners who preferred the new notation, although they had been primarily instructed in using the sigma notation. This indicates that there might be merit in conducting a full-scale experiment to obtain more reliable results.

Table 7.5 shows the notational preferences of the 26 participants who completed the experiment. The majority of the learners favoured the sigma notation. Regardless of which notation they were taught at first, 18 preferred the sigma notation and 8 the Dijkstra notation.

The following can be observed:

- 35% of the group receiving the first instruction using the sigma notation felt that the Dijkstra notation was easier to write.
- 66% of the group receiving the first instruction using the Dijkstra notation preferred the sigma notation.

All the tests and exercises were done in a hand-written format. It is clear that the participating learners preferred the sigma notation when they had to use pen and paper. This could be due to the easier visual separation of

	Sigma notation	Dijkstra notation	No preference	Sigma notation	Dijkstra notation	No preference
	Easier to write			Easier to read		
Sigma group	8	5	1	10	4	0
Dijkstra group	10	1	1	7	4	1
Total	18	6	2	17	8	1
	Easier to understand			Preferred		
Sigma group	9	4	1	10	4	0
Dijkstra group	8	3	1	8	4	0
Total	17	7	2	18	8	0

Table 7.5: Opinion Poll Results

the elements in this notation but more research should be conducted before conclusions can be drawn about this preference.

Consistent with the selected methodology of mixed qualitative and quantitative research, other aspects of the gathered data were analyzed from a few different perspectives. The observations are discussed below.

7.2.2 Step 4c: Written material analysis

The available material was studied for patterns or examples of the aspects below:

1. Evidence of confusion using both notation in tests and exercises
2. Visual reading preference for authors

Analyzing the accumulated written material the same conclusion as Strand and Larsen [43] was reached, which is that students have difficulty in grasping all the cognitive processes that are involved when dealing with sigma notation and the sum of series. Evidence of that is clear in the examples shown in figures 7.3, 7.4 and 7.5.

The lack of handwriting skills of learners as can be seen in Figure 7.6 also indicated some area where improvement can be made, possibly by introducing symbols and the proper writing thereof, earlier in a learner's school career.

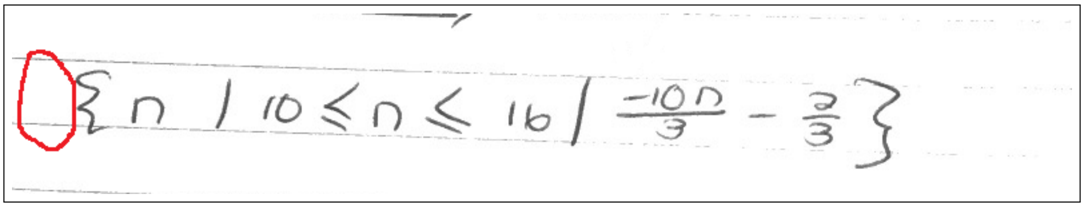

Figure 7.3: Illustration of written confusion

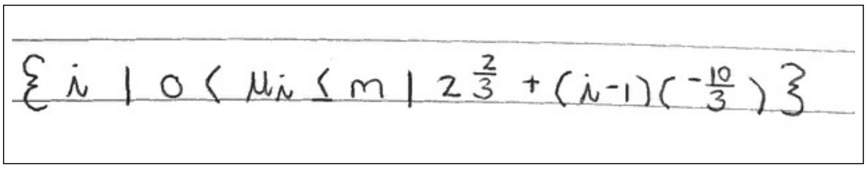

Figure 7.4: Further illustration of written confusion

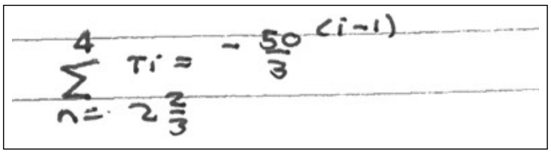

Figure 7.5: Illustration of concept confusion

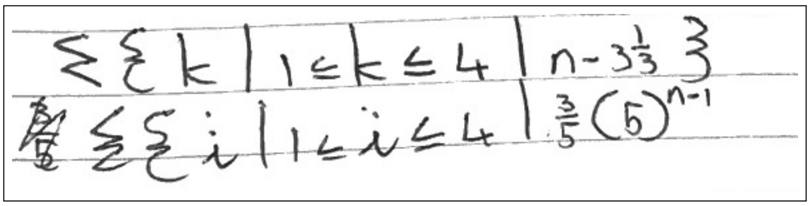

Figure 7.6: Symbol confusion

7.2.3 Visual preference

From a reading point of view, the sigma notation gives a clearer and easier visual image to process. The Dijkstra notation, while more consistent, in some cases resulted in rather strange visual representations such as can be seen in Figure 7.7. the distinction between the sigma sign and the parenthesis is not obvious.

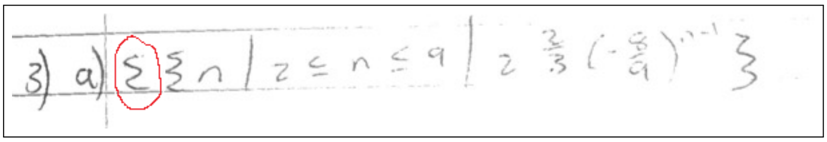

Figure 7.7: Illustration of written ambiguity

It would be an option to replace the curly brackets in the general format with square brackets to counter the type of visual confusion seen in the example in Figure 7.7. This would result in a visual representation as below, resembling the Wolfram notation, which might be easier to read.

$$\sum[n|1 \leq n \leq 7|n^3]$$

This will however have the same objection as the original choice by Dijkstra to have "<>" brackets which do not conform to the ISO standard for sets.

Another approach could be to replace the sigma sign simply with a plus sign. With reference to Figure 7.7, it could be that the learners have not learned to write these symbols as they did the letters of the alphabet and numbers. It is not part of any curriculum and as a result, when they first encounter the symbol, their attempts at writing it can cause confusion. The notation which is finally proposed in this dissertation replaces the sigma sign with a plus sign and use round brackets instead of curly brackets as shown below. The use of the plus sign is seen as a major advantage as it eliminates the introduction of a new symbol. The decision to use round brackets is two-fold. Firstly it is easier to write and secondly it is a more accurate representation of the concept of a series as the summation of a sequence (and not of a set). These changes applied to the Dijkstra notation, could resolve the confusions that have been encountered.

$$+(\ n \mid 3 \leq n \leq 7 \mid n^2 + 1\ )$$

7.2.4 Electronic processing

The process of marking the handwritten work by the learners and capturing the results raised another important factor. All current word processors have special functions that can be used to enter mathematical expressions. For a mature or expert user, these functions are second nature but for high school pupils with little or no knowledge of word processing packages, using sophisticated features such as these remains a problem that will not be solved until schools provide basic computer literacy a compulsory subject.

Figure 7.8 shows the answer to a question when the constraint was that it should be submitted in electronic format. Rather than use the equation editor of a word processor, the learner used the graphics feature to manually construct a sigma sign. This shows a weakness in general computing skills that should be addressed in the school curriculum.

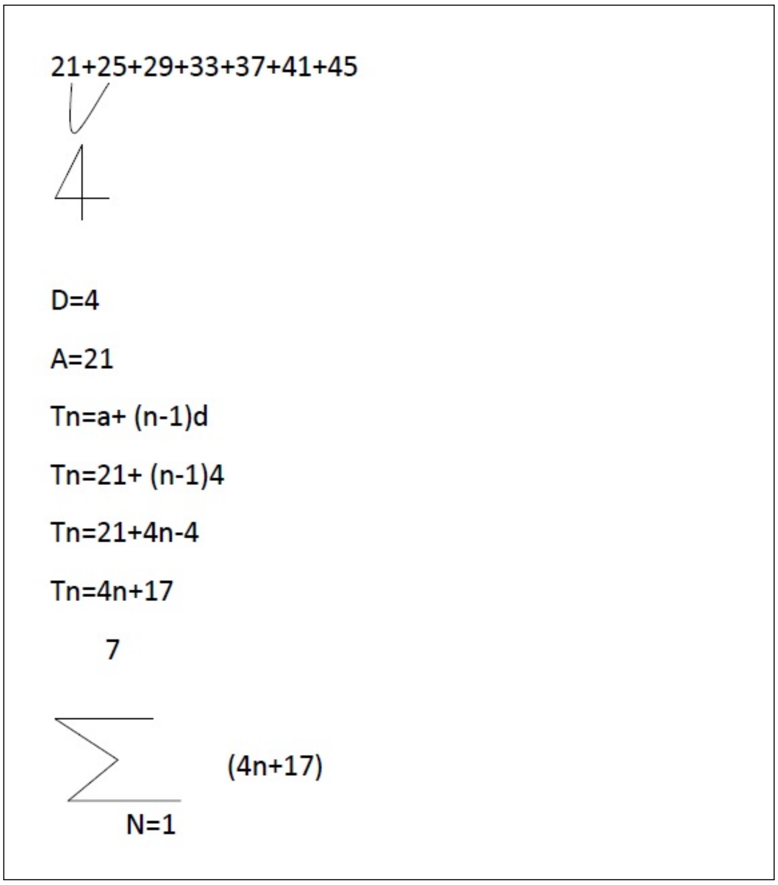

Figure 7.8: Electronic submission

7.3 Step 5: Conclusions

The mathematical tests conducted during the school study did not provide sufficient results for meaningful conclusions. Both groups performed poorly in both tests although the sigma group did a little better. Taking all the external factors that could have influenced the outcome into account, it was felt that a conclusion cannot be drawn about the effect of the notation on comprehension of the material.

More learners preferred using the sigma notation in the second test where they were given the option to use the notation that they prefer. This could point to a slight preference for that variation. The opinion poll conducted after the study also showed a preference for the sigma notation when having to write the equation by hand in the tests.

Part IV

Summary and Conclusion

Chapter 8

Conclusion Online Survey

On-line surveying is a popular and economic method to gather information from a large number of respondents. The survey conducted with the mathematics teachers in Secondary and combined schools in Gauteng had a satisfactory response rate and provided some interesting insights into the level of sophistication in the usage of on-line tools and the most popular tools that are used by educators.

8.1 Major findings

Availability of equipment is not a large obstacle. All teachers have access to some equipment and the Internet. Access to equipment for learners, whilst not at the level of teachers, is sufficient to warrant some focus and effort on enabling the use of such technology more productively.

The most popular sites for self-education and for teaching are Siyavula and Kahn academy. Siyavula is based on the South African curriculum and is easier to use, but not free. Kahn Academy is free but not shaped towards a particular curriculum.

Although availability of equipment is not a problem, only 77% of the respondents use on-line tools for self-education. The largest obstacles to the widespread use of on-line tools in secondary school Mathematics education, as indicated by respondents, are: improved(free) Internet access and training for teachers and students. Some of the more experienced respondents indicated that tools that are better adapted to electronic input of mathematical equations would be beneficial for increased use of on-line tools.

The survey concurs with the study done by Mashile [23] in 2016. Her study on the use of technology in schools concluded that teacher education will be a major factor in the future adoption of technology in schools. She also found that availability of technical equipment is not really an issue. In her study, similar to this one, 100% of the respondents had access to a laptop or a desktop. Mashile [23] also found that attitude plays a large role in the adoption of on-line tools in the educational environment which is similar to the findings of this survey. Some teachers are opposed to technological aids in the classroom and to convince them to explore such aids will take more than just improved tools.

The on-line survey indicated that the teachers with experience with on-line tools would prefer tools that can provide mathematics assessment beyond multiple choice only. Tools that will enable basic computer users, such as secondary school learners, to submit answers in an easy and efficient manner and offer accurate electronic assessment of the solution method, could encourage use of on-line tools by learners and teachers. The proposed notation could simplify the development of such tools due to its explicit and linear structure.

8.2 Related findings

Not all teachers agreed that it is necessary to initiate or improve the use of on-line tools as they preferred teaching using traditional methods. This can be seen in some of the free form responses in the survey as reported in Section 5.2.10. To convince these teachers to incorporate on-line resources as part of their teaching or encourage students to use such resources, will take a significant change management effort as well as substantial training.

Chapter 9

Conclusion Notation School Study

Inventing a useful notation that is economical and aesthetic is an art. A notation should be equally convenient to write by hand and to typeset with the use of contemporary tools. More important, it has to be clear and easy to read and understand. If a proposed notation lacked any of these properties, it would be less likely to be accepted. Having all the required features, however, does not guarantee acceptance. If it is not promoted in the right place at the right time, it may remain unnoticed and unused. Abadir and Magnus [1] concedes that it is likely that authors will not adhere to proposed notational standards even if the notation complies with all the mentioned requirements. He uses the example of Bernoulli [4], who did not adopt the = sign for equality 150 years after it had been proposed, even though many other mathematicians used it.

9.1 Development of notation

A new notation for the currently used sigma notation for series is proposed in this dissertation. The effect of its use on comprehension in secondary school learners was tested. The new notation is a derivative of a notation proposed by Dijkstra [12].

The first iteration of the new proposed notation was:

$$\Sigma \ \{ \ x \mid P(x) \mid f(x) \ \}$$

After analysis of the results of the school study, the proposed notation was modified to an even more keyboard and writing friendly form, as shown below:

$$+ \; (\; x \mid P(x) \mid f(x) \;)$$

9.2 Major findings

The study in the secondary school environment did not reach any conclusions on the hypothesis that the new notation might promote a better understanding of the underlying concept. Both the groups performed below expectation in the use of both the sigma notation and the proposed updated Dijkstra notation. The understanding of the concepts was not sufficient in both study groups to draw any conclusion on the benefit or drawback of the new proposed notation.

9.3 Related findings

The study unveiled some issues with the sigma notation such as difficulty to write the notation in a clear and legible way as well as some general problems with understanding the use of symbols.

Chapter 10

Summary

10.1 Future directions

With regard to the new proposed notation and the school study, it is acknowledged that the introduction of a new notation would not in itself solve the problems with mathematical education in South Africa, but it could be a first step in changing scientific notations to be friendlier to electronic processing.

For follow-up studies we want to explore the practical advantages of this new notation in the area of e-learning and mathematics using electronic devices. This will require development of an editing tool using the new notation and data comparing the ease of use and computational efficiency of the new notation vs the traditional sigma notation. Simultaneously, we would also like to expand the concept of more electronic processing friendly notations to other mathematical topics and use the tools developed to investigate the possible benefits of such new notations.

A second issue of the on-line survey in a revised format to address the shortcomings of this one, could be done from which the data can be used to estimate progress in the Gauteng province in the lapsed time.

10.2 Reflection

To solve South Africa's educational problem, specifically in the scientific subject on secondary school level, we will have to make a jump similar to what was in the communications industry where South Africa adopted cellphones and cellular technology without the majority of users ever passing the fixed-line phase. One of the major problems in South Africa in the scientific subjects, is the availability of qualified teachers. Solving this problem organically or in an evolutionary way by just training more teachers and pushing them into the system, is going to take too long. We will have to be more creative about it. On-line or electronic tools, once working, are easily deployable and scalable. A possible approach could be to position the use of such tools as the primary source of education, with teachers being coached and trained to provide guidance and assistance on the usage of the tools, not necessarily the subject matter. Whether this approach will render positive results is a subject for further research.

The IRR(Institute of Race Relations) report of John Kane-Berman [19] highlighted the inequality that exists in South African schools by comparing Public (free and low fee paying) vs independent schools. He found that it is possible for public no-fee schools to achieve the same results as independent fee paying counterparts but that it is the exception, not the rule. He also found that good and motivated teachers and principals are essential to good performance. The quest to provide good education to all South African school children is a very difficult one that is hampered by lack of teachers. A quality alternative to classic classroom teaching will have to be devised to alleviate the shortage.

In their book "Practical design patterns for teaching and learning with technology", Mor *et al.* [27] describe the concepts of "Hint on Demand", and "Try once, refine once" and the benefits which meaningful feedback could have on student learning. In order to alleviate South Africa's educational problems, we might need to "activate learners as owners of their own learning" [27]. This will require more than just providing learning material on-line but will also require intelligent systems that can guide students during their education journey. This aspect of learning tools borders on artificial intelligence, in itself a topic receiving a lot of focus. Combining disciplines (Computer Science, specifically Artificial Intelligence, and Education) in creative ways might result in South Africa transforming itself into a model for turn-around mathematical education.

In conclusion, the insights gained by investigating the current state of affairs in South Africa with regard to mathematics education, combined with the experience in the school study where the difficulties children have in grasping mathematical concepts was highlighted by the results of the study, is a motivation to contribute in some way to the improvement of the situation.It is acknowledged that providing an alternative for the currently predominant sigma notation will not be revolutionary but perhaps it can start a natural evolution to adapt to the era of e-learning and on-line activities and be expanded to cover more notations in the future.

Further research in methods and tools that would expand the mathematical teaching capability in South Africa, combined with tools that give learners the ability to progress without dedicated teachers that are masters of the subjects, seems a worthwhile cause.

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Appendices

A. Online Survey

Dear Sir/Madam,

RE: Mathematics Online at VUWANI SECONDARY SCHOOL

I am supervising Ms SH du Plessis doing research on the use of online tools in mathematical education towards a Master's degree in the Computer Science Department of the University of Pretoria. To this end, we would appreciate it very much if you could complete the short survey in the link below as part of a pilot study to determine the effectiveness of our research methods. It should not take more than 10 minutes.

You may have received this request previously (or receive more in the future) – if this is the case, you need to complete the form only once.

http://tinyurl.com/yc38nljp

If you are not personally involved with Mathematical Education it would be very helpful if you could forward this email to any of your colleagues that would be able to complete the survey. The survey may be answered by different mathematics teachers at VUWANI SECONDARY SCHOOL individually.

By participating in the survey, you give implicit consent that the data gathered can be used for the research as described above. The survey is conducted anonymously and no data is kept that can trace any response back to a person or institution.

Thank you very much in advance for your time.

Sincerely

Dr Vreda Pieterse
Department of Computer Science
UNIVERSITY OF PRETORIA

Figure 10.1: Original Invitation text for participation in Survey

B. Free-form responses to Question 10

"If you have a view on any aspect of on-line math teaching not covered in the previous questions or advice for improving the survey, please enter below."

1. I do love using online math teaching but sometimes its very difficult to keep up with all the changes that takes which schools cannot afford to pay for for renewals or on going support from the people that offer the services.
2. Definitely not enough online South African resources available and I therefore use many overseas resources. To the detriment of my budget as many are not always freely available.
3. In my institution we use Fathom, geogebra, autograph, Sketchpad Why and when should you use online math teaching? Can be asked. Differentiate maybe between teaching tool and assessment tool, because I think I would find an assessment tool beneficial, especially for immediate feedback and more for junior grades than seniors. Define the roles of a teacher when using online teaching tool....
4. Electronics is a great aid, but it cannot replace pen and paper for math
5. There is too much stereotyping in South Africa. I looked at Siyavula and Khan's Academy - not relevant to South African situation.
6. WE need to integrate different systems. Using Itsi with Geogebra and making the use of equation editors more user friendly so learners will be able to type answer that does not appear on a normal keyboard.
7. Learner skills and Educators to use the electronic skills successfully.
8. Online teaching maths is good, problem our learners are not disciplined.
9. The small screen smart-boards not ideal for teaching Mathematics
10. microsoft math and vodacom eschool
11. Learners need to write. Typing answers and ticking boxes does not help with the cognitive process when doing maths. I am old school. I will project past papers for them to do, or show one or two video clips on specific topics, but I prefer paper. Learners don't need tablets, internet, or smartphones to pass maths. They need confidence, a strong work ethic and dedication.
12. It should be an extra source, but not the main source of teaching.

13. I still think that the method of writing out problems is the best way for kids to learn.

14. I don't believe online teaching is generally suitable for use in Mathematics.

15. Khan's videos are excellent. I use it extensively. I also love i-Pathways tutorials and tests especially for wordsums

16. No view

17. Online education is often not seen as a useful tool by school administrators or they do not have the funding available to ensure that e-learning is facilitated

18. The main problem is one of access. Many of our learners rely on free wifi, which is available at school. Once they get home it becomes an issue as many of them do not have access to the internet at home.

19. i feel maths needs the personal touch as so many battle with the subject

20. See the above. Added to this, it is a problem to have learners express their maths knowledge in a systematic manner. Learners are not yet mature enough to manage the responsible use of tablets in their learning. The devices are often not treated carefully and are subsequently damaged, which impacts on the e-learning process.

21. Our school has learners with a very big variety of socio-economic backgrounds. To level the playing fields, will be financially tough.

C. School Study Marketing

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

WISKUNDE NOTASIE ONDERSOEK

1 September 2016

Geagte Ouer en Graad 11 leerder,

Die eerste paar weke van enige nuwe jaar is geweldig besig, veral vir die matrieks, en die eerste paar onderwerpe van alle vakke is gewoonlik onder druk vir voldoende aandag. Hier is 'n geleentheid om die jaar met 'n voorsprong te begin, ten minste sover dit Wiskunde aangaan!

Die Universiteit van Pretoria beplan 'n ondersoek om die effektiwiteit van 'n nuwe notasie in die gebied van Rye en Reekse vas te stel. Die ondersoek is in die vorm van 'n Graad 12 Wiskunde voorskou vir Graad 11 leerlinge. Die voorskou behels lesings en oefeninge wat die eerste hoofstuk in die matriek leerplan dek. Beide die nuwe notasie en die voorgeskrewe notasie sal aangeleer word en sorg sal gedra word dat leerders aan die einde van die ondersoek die voorgeskrewe notasie korrek kan gebruik.

Die ondersoek vind plaas **5de, 6de, en 7de Desember 2016 van 9h00 to 12h00 by die Hoërskool Linden**.

Enige graad 11 leerling wat Wiskunde as vak neem kwalifiseer vir deelname. Deelnemers sal blootstelling kry aan die onderwerp 'Rye en Reekse', wat heel eerste in die Matriek jaar behandel word. Deelname aan hierdie program behoort dus aan leerders 'n goeie voorsprong te verskaf. Deelname aan die ondersoek is gratis maar ons versoek dat almal wat inskryf, seker maak dat hulle al drie dae kan bywoon.

Me. Vreda Pieterse, dosent aan die Universiteit van Pretoria en ervare Wiskunde onderwyseres, sal die lesings aanbied. 'n Baie kort evaluasie toets sal voor die aanvang van die ondersoek aan deelnemers verskaf word om 'n basislyn vir die ondersoek vas te stel.

Indien jy wil deelneem aan die ondersoek, vul asseblief die onderstaande toestemmingsvorm in en besorg dit terug aan Me. Van Schaik voor 1 Oktober 2016. (Alternatief epos aan santjie@gmail.com)

By voorbaat dank vir die ondersteuning,

Santjie du Plessis **Vreda Pieterse**

Ek, ______________________________, ouer van ____________________________ (Graad 11) gee hiermee toestemming dat hy/sy mag deelneem aan die ondersoek vanaf 5 tot 7 Desember 2016 by Hoërskool Linden en onderneem dat hy/sy teenwoordig sal wees vir al drie dae en deelneem aan al die lesse, oefeninge en toetse.

Naam van leerling: ______________________________

Telefoon nr. leerling: ______________________________

Telefoon nr. ouer/voog: ______________________________

Kontak Epos adres: ______________________________

Handtekening ouer/voog: ______________________________

Figure 10.2: Invitation to participate - original

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

MATHEMATICAL NOTATION STUDY

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1 September 2016

Dear parent and grade 11 learner,

The first few weeks of any schoolyear are normally extremely busy, especially for the Grade 12s, and the first few topics of all subjects normally suffers from lack of attention and time. Here is an opportunity to start your matric year with an advantage, at least as far as mathematics is concerned.

The University of Pretoria plans a study to investigate the effectiveness of a new notation on the topic of Sequences and Series. This study is in the form of a Grade 12 Mathematics preview for learners currently in Grade 11. This preview will contain lectures and exercises covering the first chapter of the grade 12 curriculum. Both the new and the prescribed notations will be covered to ensure that learners are able to use both at the end of the preview.

The study is scheduled to take place on the 5**th, 6th, and 7th December 2016 from 9h00 till 12h00 at the Hoërskool Linden**.

Any grade 11 learner with Mathematics as subject can participate. Learners will be exposed to the subject 'Sequences and Series' that is the first topic covered in the Grade 12 syllabus. Participation in the study should provide learners with a good start. Participation is free, but we request that anyone enrolling be sure that they will be available for all three days.

Me. Vreda Pieterse, lecturer at the University of Pretoria and an experienced Mathematics teacher will give the lectures. A short evaluation test will be given to learners before the start of the study.

If you want to participate, please complete the form below and return to Me. Van Schaik before 1 October 2016. (Alternatively send an email to santjie@gmail.com)

Thanks for your support,

Santjie du Plessis **Vreda Pieterse**

I, ______________________________________, parent of _______________________________ (Grade 11) ive permission the he/she can participate in the study from to 7 Desember 2016 at Hoërskool Linden and commit that he/she will be present for al three dates and participate in all the lectures, exercises and tests.

Name of learner: ____

Telephone nr. learner: ____

Telephone nr. parent: ____

Contact Email address: ____

Signature parent: ____

Figure 10.3: Invitation text for participation in Survey - English

D. Pretest

Notasie Ondersoek Pre-toets

Naam: ________________________________

Omkring die korrekte antwoord.

1. Jannie ontwerp 'n reghoekige tuin met afmetings $5x$ en $5x-4$. Vind die oppervlakte van die tuin in terme van x?
   a. $25x^2 - 20x$
   b. $25x^2 - 4$
   c. $25x^2 + 5x - 4$
   d. $25x - 20$

Die standaard vorm van 'n kwadratiese funksie is?

   a. $y = mx + b$
   b. $y = x^3$
   c. $y = ax^2 + bx + c$
   d. $y = x^2$

2. Wat is die produk van $(x+5)(x^3 - 2x - 3)$?
   a. $x^4 + 5x^3 - 2x^2 - 7x - 15$
   b. $x^4 + 5x^3 - 2x^2 - 13x - 15$
   c. $x^4 + 5x^3 - 2x^2 - 10x - 15$
   d. $x^4 + 5x^3 - 2x^2 - 3x - 15$

3. Wat is die som van die eerste 5 terme in die reeks 1,2,4,8,16,32 ...
   a. 16
   b. 31
   c. 32
   d. 63

4. Vind die a,b en c van die die kwadratiese vergelyking $-6x^2 - 12 = 0$?
   a. $a = -6, b = 0, c = -12$
   b. $a = 6, b = 0, c = 12$
   c. $a = 0, b = 12, c = -6$
   d. $a = -12, b = -6, c = 0$

5. Faktoriseer die kwadratiese trinominaal $x^2 - 29x + 100$.
   a. $(x+25)(x-4)$
   b. $(x+25)(x+4)$
   c. $(x-25)(x-4)$
   d. $(x-25)(x+4)$

Page **1** of **2**

Figure 10.4: Pre-test - original page 1

6. $\frac{-4\pi}{3}$ is dieselfde as?

  a. $\frac{-2\pi}{6}$

  b. $\frac{4\pi}{-3}$

  c. $\frac{\pi}{3}$

  d. $\frac{2\pi}{3}$

7. Watter tipe vergelyking is $-2g - 3g^2$?

  a. Kubies

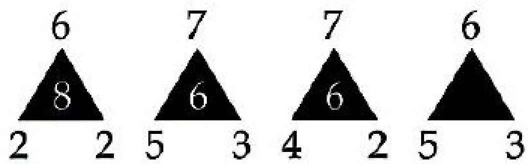
  b. Liniêr

  c. Kwadraties

  d. Nie een van die bogenoemde nie.

Verskaf die gepaste antwoord:

9. Watter getal moet in die laaste driehoek inkom? ....



10. Korrigeer die vergelyking 101 - 102 = 1 deur net een syfer te skuif?

....

Figure 10.5: Pre-test - original page 2

Notation Study Pre-test

Name: ____

Circle the correct response:

1. Jannie designs a rectangular garden with dimensions: $5x$ and $5x - 4$. Find the surface of the garden in terms of x?
   a. $25x^2 - 20x$
   b. $25x^2 - 4$
   c. $25x^2 + 5x - 4$
   d. $25x - 20$

2. The standard form of a quadratic function is?
   a. $y = mx + b$
   b. $y = x^3$
   c. $y = ax^2 + bx + c$
   d. $y = x^2$

3. What is the product of $(x + 5)(x^3 - 2x - 3)$?
   a. $x^4 + 5x^3 - 2x^2 - 7x - 15$
   b. $x^4 + 5x^3 - 2x^2 - 13x - 15$
   c. $x^4 + 5x^3 - 2x^2 - 10x - 15$
   d. $x^4 + 5x^3 - 2x^2 - 3x - 15$

4. What is the sum of the first 5 terms in the sequence 1,2,4,8,16,32 ...
   a. 16
   b. 31
   c. 32
   d. 63

5. Find the values of a, b and c in the quadratic expression $-6x^2 - 12 = 0$?
   a. $a = -6, b = 0, c = -12$
   b. $a = 6, b = 0, c = 12$
   c. $a = 0, b = 12, c = -6$
   d. $a = -12, b = -6, c = 0$

6. Factorize the quadratic trinomial $x^2 - 29x + 100$.
   a. $(x + 25)(x - 4)$
   b. $(x + 25)(x + 4)$
   c. $(x - 25)(x - 4)$
   d. $(x - 25)(x + 4)$

Page **1** of **2**

Figure 10.6: Pre-test - English p1

7. $\frac{-4\pi}{3}$ is equal to?

a. $\frac{-2\pi}{6}$

b. $\frac{4\pi}{-3}$

c. $\frac{\pi}{3}$

d. $\frac{2\pi}{3}$

8. What type of expression is $-2g - 3g^2$?

a. Cubic

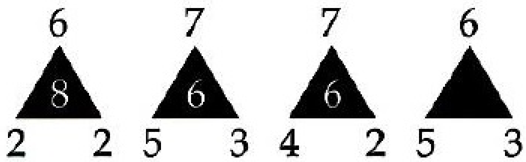
b. Linear

c. Quadratics

d. None of the above.

Provide the correct answer:

9. Which number must complete the last triangle? ....



10. Correct the equation 101 - 102 = 1 , by only moving one number?

....

Page 2 of 2

Figure 10.7: Pre-test - English p2

E. Tests and memorandum

The first test given to the 2 groups was the same for both groups.

Toets 1

Wys al jou berekeninge wanneer jy die volgende vrae be-antwoord. Indien die vraag nie ge-antwoord kan word nie moet jy sê waarom nie. Byvoorbeeld as 'n versameling getalle nie 'n ry vorm nie, kan 'n mens nie die algemene term daarvoor spesifiseer nie.

1. Bepaal vir elkeen van die volgende versamelings of die getalle in die versameling 'n ry vorm. Indien die getalle wel 'n ry form, spesifiseer watter soort ry dit is:
   (a) $\{26;\ -26;\ 26;\ -26\ \ldots\}$
   (b) $\{8;\ 13;\ 32;\ 132;\ 632;\ \ldots\}$
   (c) $\{7;\ 8\frac{1}{2};\ 10;\ 11\frac{1}{2};\ 13;\ \ldots\}$
2. Skryf die algemene term vir elkeen van die rye wat jy in Vraag 1 geïdentifiseer het.
3. Gebruik sigma notasie om 'n reeks van terme, soos hieronder gespesifiseer is, te skryf met die getalle soos gegee in die ooreenstemmende nommer in Vraag 1:
   (a) Die eerste 40 terme
   (b) Die eerste 2 terme
   (c) Begin by die derde term en hou op met die term waarvan die waarde 37 is.
4. Bereken die waarde van elk van die reekse wat jy in Vraag 3 geskryf het.

Figure 10.8: First Test: Original

Test 1

Show all your calculations when answering the questions. If the question has no answer you have to specify the reason. For example, if a set of numbers does not form a sequence, a general term for the set cannot be specified.

1. For each of the sets of numbers below, determine whether they form a sequence. If yes, specify the type of sequence.

   (a) $\{26;\ -26;\ 26;\ -26\ \ldots\}$ (9)

   Solution:
   $T_2 - T_1 = -26 - 26 = -52$ ✓
   $T_3 - T_2 = 26 - (-26) = 26 + 26 = 52$ ✓
   $\therefore$ the difference between the terms is not constant so it is not an arithmetic sequence. ✓✓

   $T_2 \div T_1 = -26 \div 26 = -1$ ✓
   $T_3 \div T_2 = 26 \div (-26) = 26 + 26 = -1$✓
   $T_4 \div T_3 = -26 \div 26 = -1$ ✓
   $\therefore$ The ratio between successive terms is constant. It is a geometric sequence with common ratio $= -1$. ✓✓

   (b) $\{8;\ 13;\ 32;\ 132;\ 632;\ \ldots\}$ (13)

   Solution:
   $T_2 - T_1 = 13 - 8 = 5$ ✓
   $T_3 - T_2 = 32 - 13 = 19$ ✓
   $\therefore$ the difference between the terms is not constant so it is not an arithmetic sequence. ✓✓

   $T_2 \div T_1 = 13 \div 8 = \frac{13}{8}$ ✓
   $T_3 \div T_2 = 32 \div 13 = \frac{32}{13}$ ✓
   $\therefore$ The ratio between successive terms is not constant. It is not a geometric sequence. ✓✓

   $(T_3 - T_2) - (T_2 - T_1) = 19 - 5 = 14$ ✓
   $T_4 - T_3 = 132 - 32 = 100$ ✓
   $(T_4 - T_3) - (T_3 - T_2) = 100 - 19 = 81$ ✓
   $\therefore$The second difference between successive terms is not constant. It is not a quadratic sequence. ✓✓

   (c) $\{7;\ 8\frac{1}{2};\ 10;\ 11\frac{1}{2};\ 13;\ \ldots\}$ (6)

1

Figure 10.9: First test with memorandum - English - page 1

Solution:
$T_2 - T_1 = 8\frac{1}{2} - 7 = 1\frac{1}{2}$ ✓
$T_3 - T_2 = 10 - 8\frac{1}{2} = 1\frac{1}{2}$ ✓
$T_4 - T_3 = 11\frac{1}{2} - 10 = 1\frac{1}{2}$ ✓
$T_5 - T_4 = 13 - 11\frac{1}{2} = 1\frac{1}{2}$ ✓
$\therefore$ The difference between successive terms is constant. It is an arithmetic sequence with constant difference $1\frac{1}{2} = \frac{3}{2}$. ✓✓

2. Give the general term for each of the sequences that your identified in question 1. (5)

Solution:
a) $T_n = ar^{n-1} = 26(-1)^{n-1}$ ✓✓
b) It is not a sequence therefor no general term exists. ✓
c) $T_n = a + (n-1)d = 7 + (n-1)\frac{3}{2} = 7 - \frac{3}{2} + \frac{3n}{2} = 5\frac{1}{2} + \frac{3n}{2}$ ✓✓

3. Use the sigma notation to express a series as described below. Use the same terms as in question 1 wees:

(a) The first 40 terms (3)

Solution:

$$\sum_{n=1✓}^{40✓} 26(-1)^{n-1}✓$$

(b) The first 2 terms (2)

Solution: There is no general term to describe the terms in the sequence so the sum cannot be expressed using the sigma notation✓✓

(c) Start at the third term and stop at the term with value 37. (8)

Solution:
$T_n = 37$ ✓
$\therefore 5\frac{1}{2} + \frac{3n}{2} = 37$
$\therefore \frac{3n}{2} = 37 - 5\frac{1}{2}$
$\therefore \frac{3n}{2} = 31\frac{1}{2}$
$\therefore \frac{3n}{2} = \frac{63}{2}$
$\therefore 3n = 63$
$\therefore n = 21$ ✓✓✓

$$\sum_{n=3✓}^{21✓} (5\frac{1}{2} + \frac{3n}{2})✓✓$$

2

Figure 10.10: First test with memorandum - English - page 2

Parentheses are important

4. Calculate the value of each of the series that you defined in question 3. (15)

Solution:
a = 26 ✓
r = -1 ✓
n = 40 ✓

$$S_{40} = \frac{a(1-r^n)}{1-r} \checkmark = \frac{26(1-(-1)^{40})}{1-(-1))} \checkmark = \frac{26(1-1)}{1+1} = \frac{26 \times 0}{2} = 0 \checkmark$$

Solution: b) No answer. (There was no series) ✓

Solution: c)
$a = 7$ ✓
$l = 37$ ✓
$S_n = \frac{n}{2}(a + \ell)$ ✓

$$\sum_{n=3}^{21} (5\frac{1}{2} + \frac{3n}{2})$$

Solution: $= S_{21} - S_2$ ✓
$= \frac{21}{2}(7 + 37) - \frac{2}{2}(7 + 8\frac{1}{2})$ ✓
$= \frac{21}{2}(44) - 15\frac{1}{2}$
$= 21 \times 22 - 15\frac{1}{2}$
$= 462 - 15\frac{1}{2}$
$= 446\frac{1}{2}$ ✓✓✓

3

Figure 10.11: First test with memorandum - English - page 3

Toets 3

Wys al jou berekeninge wanneer jy die volgende vrae be-antwoord. Indien die vraag nie ge-antwoord kan word nie moet jy sê waarom nie.

1. Bepaal vir elkeen van die volgende versamelings watter soort ry dit vorm:
   (a) $\{2\frac{2}{3};\ -\frac{2}{3};\ -4;\ -7\frac{1}{3};\ \ldots\}$
   (b) $\{\frac{3}{5};\ 3;\ 15;\ 75;\ \ldots\}$
   (c) $\{1;\ 4;\ 9;\ 16;\ 25;\ \ldots\}$
   (d) $\{512;\ 256;\ 128;\ 64;\ \ldots\}$
2. Gebruik Dijkstra se notasie vir versamelings om elkeen van die versamelings van Vraag 1 te spesifiseer.
3. Skryf 'n versameling-reeks van terme, soos hieronder gespesifiseer, met die getalle soos gegee in die versameling met ooreenstemmende nommer in Vraag 1:
   (a) Sewe terme waarvan die middelste term se waarde -44 is.
   (b) Die eerste n terme sodat die reeks se waarde $2343\frac{3}{4}$ is.
   (c) Al die terme.
   (d) Al die terme.
4. Gebruik sigma-notasie om elk van die versameling-reekse wat jy in Vraag 3 geskryf het, te spesifiseer.
5. Bereken die waarde van elk van die versameling-reekse wat jy in Vraag 3 geskryf het.

Figure 10.12: Test 3: Original

Test 3

Show all your calculations when answering the question. If the question does not have answer, explain why.

1. Specify for each of the following collections, which type of sequence it is:
   (a) (5 points) $\{2\frac{2}{3};\ -\frac{2}{3};\ -4;\ -7\frac{1}{3};\ \ldots\}$
   (b) (9 points) $\{\frac{3}{5};\ 3;\ 15;\ 75;\ \ldots\}$
   (c) (15 points) $\{1;\ 4;\ 9;\ 16;\ 25;\ \ldots\}$
   (d) (9 points) $\{512;\ 256;\ 128;\ 64;\ \ldots\}$
2. (23 points) Use Dijkstra's notation to express each of the sequences in question 1.
3. Write an expression for the number of terms below for each of the sequences of question 1:
   (a) (10 points) Seven terms of which the middle term has the value -44.
   (b) (10 points) The first n terms so that the series will have a value of $2343\frac{3}{4}$.
   (c) (1 point) All the terms.
   (d) (1 point) All the terms.
4. (4 points) Use sigma notation to express each of the series in question 3.
5. (12 points) Calculate the value of each of the series in question 3.

Figure 10.13: Test 3 - English

Test 3

Show all your calculations when answering the question. If the question does not have answer, explain why.

1. Specify for each of the following collections, which type of sequence it is:

   (a) (5 points) $\{2\frac{2}{3};\ -\frac{2}{3};\ -4;\ -7\frac{1}{3};\ \ldots\}$

   Solution:
   $T_2 - T_1 = -\frac{2}{3} - 2\frac{2}{3} = -3\frac{1}{3}$ ✓
   $T_3 - T_2 = -4 - (-\frac{2}{3}) = -4 + \frac{2}{3} = -3\frac{1}{3}$ ✓
   $T_4 - T_3 = -7\frac{1}{3} - (-4) = -7\frac{1}{3} + 4 = -3\frac{1}{3}$ ✓
   $\therefore$ The difference between the successive terms is constant. It is an arithmetic sequence with common difference $-3\frac{1}{3}$. ✓✓

   (b) (9 points) $\{\frac{3}{5};\ 3;\ 15;\ 75;\ \ldots\}$

   Solution:
   $T_2 - T_1 = 3 - \frac{3}{5} = 2\frac{2}{5}$ ✓
   $T_3 - T_2 = 15 - 3 = 12$ ✓
   $\therefore$ The difference between the successive terms is not constant. It is not an arithmetic sequence. ✓✓

   $T_2 \div T_1 = 3 \div \frac{3}{5} = 3 \times \frac{5}{3} = 5$ ✓
   $T_3 \div T_2 = 15 \div 3 = 5$ ✓
   $T_4 \div T_3 = 75 \div 15 = \frac{3\times5\times5}{3\times5} = 5$ ✓
   $\therefore$The ratio between successive terms is constant. It is a geometric sequence with constant ratio $= 5$. ✓✓

   (c) (15 points) $\{1;\ 4;\ 9;\ 16;\ 25;\ \ldots\}$

   Solution:
   $T_2 - T_1 = 4 - 1 = 3$ ✓
   $T_3 - T_2 = 9 - 4 = 5$ ✓
   $\therefore$ The difference between the successive terms is not constant. It is not an arithmetic sequence.. ✓✓

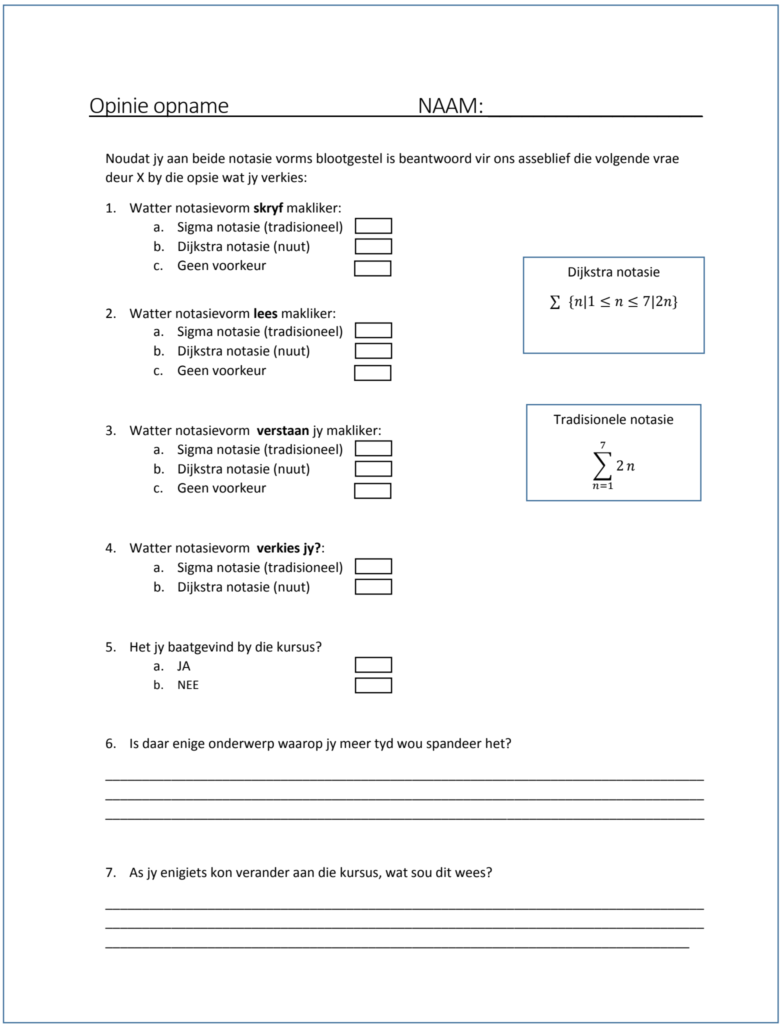
   $T_2 \div T_1 = 4 \div 1 = 4$ ✓
   $T_3 \div T_2 = 9 \div 4 = 2\frac{1}{4}$ ✓
   $\therefore$ The ratio between successive terms is not constant. It is not a geometric sequence. ✓✓

   $(T_3 - T_2) - (T_2 - T_1) = 5 - 3 = 2$ ✓
   $T_4 - T_3 = 16 - 9 = 7$ ✓

1

Figure 10.14: Test 3 with memorandum -extract page 1 - English

F. Notation Opinion poll



Opinie opname NAAM: ____________________

Noudat jy aan beide notasie vorms blootgestel is beantwoord vir ons asseblief die volgende vrae deur X by die opsie wat jy verkies:

1. Watter notasievorm **skryf** makliker:
   a. Sigma notasie (tradisioneel) ☐
   b. Dijkstra notasie (nuut) ☐
   c. Geen voorkeur ☐

2. Watter notasievorm **lees** makliker:
   a. Sigma notasie (tradisioneel) ☐
   b. Dijkstra notasie (nuut) ☐
   c. Geen voorkeur ☐

3. Watter notasievorm **verstaan** jy makliker:
   a. Sigma notasie (tradisioneel) ☐
   b. Dijkstra notasie (nuut) ☐
   c. Geen voorkeur ☐

4. Watter notasievorm **verkies jy?**:
   a. Sigma notasie (tradisioneel) ☐
   b. Dijkstra notasie (nuut) ☐

5. Het jy baatgevind by die kursus?
   a. JA ☐
   b. NEE ☐

6. Is daar enige onderwerp waarop jy meer tyd wou spandeer het?

____
____
____

7. As jy enigiets kon verander aan die kursus, wat sou dit wees?

____
____
____

Dijkstra notasie

$\sum \{n | 1 \leq n \leq 7 | 2n\}$

Tradisionele notasie

$\sum_{n=1}^{7} 2n$

Figure 10.15: Opinion poll - original

Opinion Poll NAME: ____________________

After you have been exposed to both notations. Please answer the question below by selecting the option you prefer by an X:

1. Which notation writes the easiest?
   a. Sigma notation (traditional) ☐
   b. Dijkstra notation (new) ☐
   c. No preference ☐

2. Which notation reads the easiest?
   a. Sigma notation (traditional) ☐
   b. Dijkstra notation (new) ☐
   c. No preference ☐

3. Which notation is the easiest to understand?
   a. Sigma notation (traditional) ☐
   b. Dijkstra notation (new) ☐
   c. No preference ☐

4. Which notation do you prefer?:
   a. Sigma notation (traditional) ☐
   b. Dijkstra notation (new) ☐

5. Do you feel the course was valuable?
   a. YES ☐
   b. NO ☐

6. Would you have liked to spend more time on any topic?

____
____
____

7. If you could change anything in the course, what would it be?

____
____
____

Dijkstra notation

$\sum \ \{n | 1 \le n \le 7 | 2n\}$

Traditional notation

$$\sum_{n=1}^{7} 2\,n$$

Figure 10.16: Opinion poll - English