Hamilton Cycles in Sparse Locally Connected Graphs

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Abstract

A graph G is *locally connected* if for every $v \in V(G)$ the open neighbourhood $N(v)$ of v is nonempty and induces a connected graph in G. We characterize locally connected graphs of order n with less than $2n$ edges and show that for any natural number k the Hamilton Cycle Problem for locally connected graphs of order n with m edges is polynomially solvable if $m \leq 2n + k \log_2 n$, but NP-complete if $m = 2n + \lfloor n^{1/k} \rfloor$.

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1. Introduction and Background

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the *order* and *size* of G and denoted by $n(G)$ and $m(G)$, respectively. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. When the context is clear we omit the G and simply write n, m, Δ etc. For undefined concepts and notation we refer the reader to [1].

A graph G is *locally connected* (LC) if for every $v \in V(G)$ the open neighbourhood $N(v)$ of v is nonempty and induces a connected graph. The concept of a locally connected graph was introduced by Chartrand and Pippert [2]. One of the first results proved on the topic is the following.

Theorem 1.1 ([2]). *If G is a connected LC graph of order at least 3 with $\Delta \leq 4$, then G is either hamiltonian or isomorphic to the complete tripartite graph $K_{1,1,3}$.*

Global cycle properties of locally connected graphs with $\Delta = 5$ were investigated by Kikust [8], Clark [3], Hendry [6] and Gordon, Orlovich, Pots and Strusevich [4]. Hendry [5] defined a graph G to be *fully cycle extendable* if every vertex of G lies on a 3-cycle and for every nonhamiltonian cycle C in G there is a cycle C^* in G that contains all the vertices of C plus a single new vertex. The combined results of the mentioned authors yield the following.

Theorem 1.2 ([3, 4, 6, 8]). *If G is a connected LC graph with $\Delta = 5$ and $\delta \geq 3$, then G is fully cycle extendable. However, there exist infinitely many nonhamiltonian connected LC graphs with $\Delta = 5$ and $\delta = 2$.*

It is well-known that the Hamilton Cycle Problem (the problem of deciding whether a given graph is hamiltonian) is NP-complete.

Henceforth we abbreviate Hamilton Cycle Problem to HCP. Theorem 1.1 shows that the HCP for LC graphs with $\Delta \leq 4$ is fully solved. Gordon et al. [4] proved that it is NP-complete for LC graphs with $\Delta = 7$. Irzhavski [7] improved on this result by proving the following:

Theorem 1.3 ([7]). *The HCP for LC graphs with minimum degree 2 and maximum degree 5 is NP-complete. Moreover, for any two integers d and D such that $D \geq 6$ and $2 \leq d \leq D$, the HCP for locally connected graphs with minimum degree d and maximum degree D is NP-complete.*

We would like to thank the referee who made us aware of Irzhavski's theorem.

If $\mathcal{R}$ is a finite set of natural numbers, we say a graph G is $\mathcal{R}$-*regular* if the degrees of the vertices in G are all elements of $\mathcal{R}$. In Section 2 we state a corollary of Irzhavski's theorem in terms of $\mathcal{R}$-regular graphs. We also use his technique to prove one of our main results in Section 4. Since Irzhavski's paper [7] has only been published in Russian, we include the main steps of the proof for easy access.

Theorem 1.1 prompted us to investigate the HCP for LC graphs with average degree at most 4, i.e., graphs with $m \leq 2n$. In Section 3 we characterize LC graphs with $m < 2n$. In Section 4 we show that for any natural number k the HCP is NP-complete for LC graphs with $m = 2n + \lfloor n^{1/k} \rfloor$, but polynomially solvable for LC graphs with $m \leq 2n + k \log n$.

2. Hamilton Cycles in R-regular LC graphs

Picouleau [9] proved the following result regarding the complexity of the HCP for r-regular graphs.

Theorem 2.1 ([9]). *For any fixed integer $r \geq 3$ the HCP for r-regular graphs is NP-complete.*

The problem of the complexity of the HCP for $\mathcal{R}$-regular LC graphs is settled by Theorem 1.2 and the following result, which is a corollary of Theorem 1.3, proved by Irzhavski [7].

Theorem 2.2 ([7]). *The HCP for $\mathcal{R}$-regular LC graphs is NP-complete if $\mathcal{R} = \{2,5\}$ or $\mathcal{R}$ is any set of natural numbers with $\max \mathcal{R} \geq 6$.*

Proof. It is sufficient to prove the cases $\mathcal{R} = \{2,5\}$ and $\mathcal{R} = \{r\}$, $r \geq 6$.

Consider any cubic graph G. By Theorem 2.1 it is sufficient to show that G can be transformed in polynomial time to an $\mathcal{R}$-regular LC graph G^* such that G^* is hamiltonian if and only if G is hamiltonian. We briefly explain Irzhavski's transformation for the cases $\mathcal{R} = \{2,5\}$ and $\mathcal{R} = \{6\}$.

For the case $\mathcal{R} = \{2,5\}$, the graph G^* is constructed by replacing each vertex in G with a triangle (henceforth called a *node* of G^*) and each edge in G with a *border* isomorphic to the graph B shown in Figure 1, so that neighbouring nodes of G^* are joined by borders as shown in Figure 2. It is easily seen that G^* is LC and $\{2,5\}$-regular.
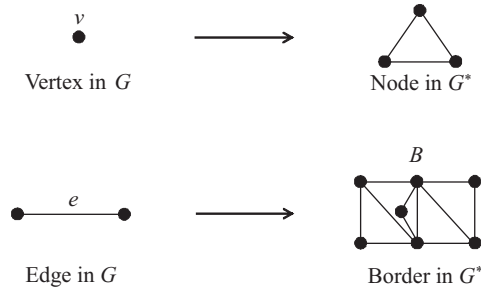
Figure 1: The translation of vertices and edges in G to nodes and borders in G^*.

For a vertex v_1 in G with $N(v_1) = \{u_1, u_2, u_3\}$, let V_1 and U_i, $i = 1, 2, 3$ be the nodes in G^* corresponding to v_1 and u_i, $i = 1, 2, 3$, respectively and let B_i be the borders in G^* corresponding to the edges $v_1 u_i$, $i = 1, 2, 3$. Let the vertices a, b, c, d, x be as shown in Figure 3.
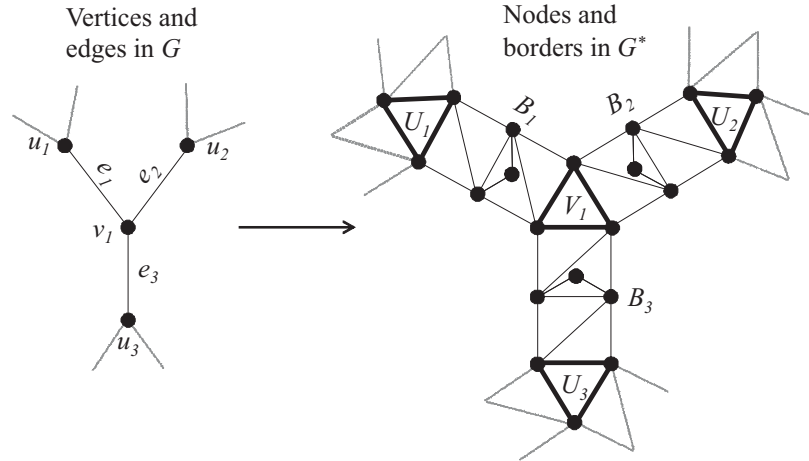

Figure 2: A node in a $\{2, 5\}$-regular LC graph G^* connected with borders to three other nodes in G^*.

Without loss of generality, suppose G has a Hamilton cycle C that contains the path $u_1 v_1 u_2$. Then C also contains the path $v_2 u_3 v_3$, where $N(u_3) = \{v_1, v_2, v_3\}$. Those two paths correspond, for example, to the two paths in G^* indicated by thick lines in Figure 3. Hence, for any Hamilton cycle in G, there is a corresponding Hamilton cycle in G^*.

Now suppose G^* has a Hamilton cycle C^*. If C^* contains the pair ad, bc of "parallel edges", then C^* obviously contains the path $adxcb$. It follows
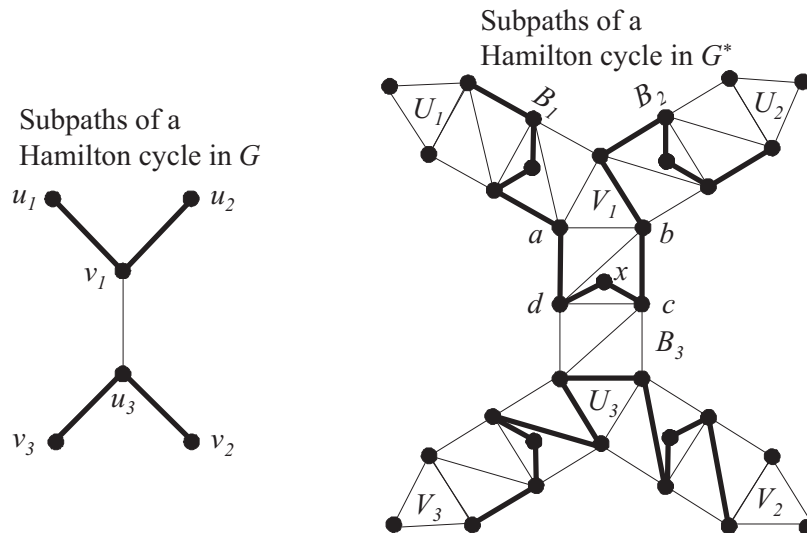
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Figure 3: Illustrating how a Hamilton cycle in G translates to a Hamilton cycle in G^*.

that C^* contains at most one pair of parallel edges incident with any given node of G^*.

Now suppose C^* contains no pair of parallel edges incident with V_1. Since cxd is a subpath of C^*, at most one of the edges $\{ad, bd, bc\}$ of B_3 lies on C^*. The same is true for the corresponding edges of the borders B_1 and B_2. The vertices of V_1 therefore lie on a P_3 in C^*, and in one of the borders, say B_1, the cycle C^* has no edge of the form ad, bc or db incident with V_1. But then C^* contains the pair of parallel edges of B_1 incident with U_1. Thus C^* translates to a Hamilton cycle in G. This proves the result for the case $\mathcal{R} = \{2, 5\}$.

Figure 4 illustrates how two nodes are joined by a border to effect the transformation in the case $\mathcal{R} = \{6\}$. (The sketch that appears in [7] has been redrawn so as to show clearly how the construction for the $\{2, 5\}$-case has been modified to obtain a 6-regular LC graph.)

For $r \geq 7$ it is not difficult to obtain an r-regular LC graph with the desired properties by expanding the nodes and borders in the $\{2, 5\}-$contruction.
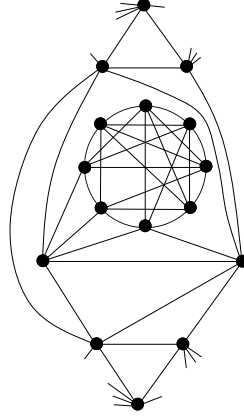
Figure 4: A 6-regular LC construction.

□

3. Characterization of the connected LC graphs with $m < 2n$

We first prove the following.

Proposition 3.1. *If G is a connected LC graph of order n and size m, then $m \geq 2n - 3$.*

Proof. The proof is by induction on n. The result is obviously true if $n \leq 3$. Now let G be a connected LC graph of order $n \geq 4$ and size m. Let x be a vertex of minimum degree in G. We may assume $d(x) = 2$ or 3. Let $H = G - x$.

If $d(x) = 2$, then H is obviously a connected LC graph of order $n-1$ and size $m-2$. Hence, by our induction hypothesis, $m-2 \geq 2(n-1)-3 = 2n-5$, so then $m \geq 2n - 3$.

If $d(x) = 3$, there are two cases to consider.

Case 1. $G[N(x)] \cong K_3$: In this case H is a connected LC graph of size $m-3$ and hence, by our induction hypothesis, $m-3 \geq 2n-5$, so $m \geq 2n-2$.

Case 2. $G[N(x)]$ is a path $v_1v_2v_3$: In this case the graph $H + v_1v_3$ is connected and LC and has size $m - 2$. Hence, by our induction hypothesis, $m - 2 \geq 2n - 5$, so $m \geq 2n - 3$. □

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Suppose that for $i = 1, 2$, the graph G_i has a proper induced subgraph H_i that is isomorphic to a given graph H. If we identify the subgraphs H_1 and H_2 while retaining all other edges present in G_1 and G_2, the resulting graph G of order $n(G) = n(G_1) + n(G_2) - n(H)$ is said to be obtained from G_1 and G_2 by *pasting G_1 and G_2 together on a copy of H*. We observe the following.

Observation 3.2. *Let H be any graph with $\delta(H) \geq 1$ and suppose G is a graph consisting of two LC graphs pasted together on a copy of H. Then G is also LC.*

If an LC graph G consists of two LC graphs of order at least 3 each, pasted together on an edge (i.e., on a copy of K_2), we say that G is a *reducible* LC graph. If G is an LC graph that consists of more than one component, we also consider G to be reducible, since each of its components is a connected LC graph. We define an *irreducible* LC graph G as a graph that is not reducible.

A set of vertices S in a connected graph G is a *vertex cut* if $G - S$ is disconnected. If $\{x, y\}$ is a vertex cut in a connected LC graph G, then it is easily seen that $xy \in E(G)$. This implies the following.

Observation 3.3. *If G is an irreducible LC graph of order n, then $G \in \{K_2, K_3\}$ if $n \leq 3$, and G is 3-connected if $n \geq 4$.*

In order to characterize the connected LC graphs with $m < 2n$ it suffices to characterize the irreducible ones, as the next result shows.

Proposition 3.4. *If G is a connected reducible LC graph with $m < 2n$, then G is obtained by pasting irreducible LC graphs of size less than twice the order together along edges.*

Proof. The proof is by induction on n. By assumption, G is the union of two LC graphs G_1, G_2 having two vertices x, y and the edge xy in common. For $i = 1, 2$, let n_i, m_i be the number of vertices and edges, respectively, in G_i. Proposition 3.1 implies that $m_i \geq 2n_i - 3$ for $i = 1, 2$. As $n = n_1 + n_2 - 2$ and $m = m_1 + m_2 - 1$, it follows that $m_i < 2n_i$ for $i = 1, 2$, and now the result follows by induction. $\square$

A vertex u in a graph G is called a *universal vertex* if u is adjacent to every vertex in $G - u$. A graph is called a *wheel* if it consists of a cycle (called the *rim* of the wheel) plus a universal vertex (called the *centre* of the wheel). Figure 6 depicts a wheel of order 9.
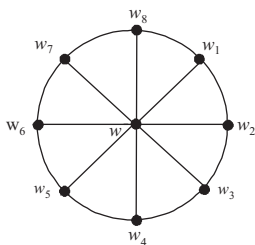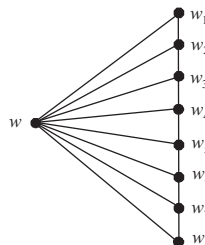
Figure 5: A wheel of order 9.

Figure 6: A fan of order 9.

Let W be a wheel with centre w and rim $w_1 w_2 \ldots w_t$. The edges ww_i, $i = 1, \ldots t$ are called the *spokes* of W. A wheel of order n may be *extended* to a wheel of order $n+1$ by *inserting a spoke*, i.e., by subdividing an edge on the rim of W and joining the inserted vertex to the centre of W. The reverse of this action is called *eliminating a spoke*.

If an edge on the rim of a wheel is deleted, the resulting graph is a *fan*. Thus a fan is a path plus a universal vertex. Let F be a fan with central vertex v and rim $v_1 \ldots v_p$. The edges vv_1 and vv_p are the *outer spokes* of F, and $vv_2, \ldots, vv_{p-1}$ the *inner spokes*. A fan may be *extended* by inserting a spoke between two of its spokes, or by adding a new vertex and joining it to to the central vertex and an end vertex of the rim. A fan may be *reduced* by eliminating an internal spoke or deleting an end vertex of the rim (together with its two incident edges).

A *θ-graph* is the union of three internally disjoint paths with shared end-vertices. An *extended θ-graph* consists of a θ-graph plus a universal vertex.

An extended θ-graph may be viewed as two wheels W and Y pasted together on a fan F such that the centre of Y is pasted to the centre of W. If F is a triangle (a fan with just two spokes) the result is a wheel plus an edge. An example of an extended θ-graph is depicted in Figure 7.

Note that a wheel of order 4 is a K_4, and any vertex in a K_4 may be regarded as its centre. Thus pasting a K_4 and any other wheel together on a triangle necessarily results in an extended θ-graph.

If two wheels W and Y of order at least 5 each are pasted together on a triangle such that the centre of Y is pasted to a vertex on the rim of W, the result is called a *triangle bicycle*.

If two wheels W and Y of order at least 5 each are pasted along a diamond (that is, a K_4 minus an edge) such that the centre of Y is pasted to a vertex on the rim of W, the result is called a *diamond bicycle*.
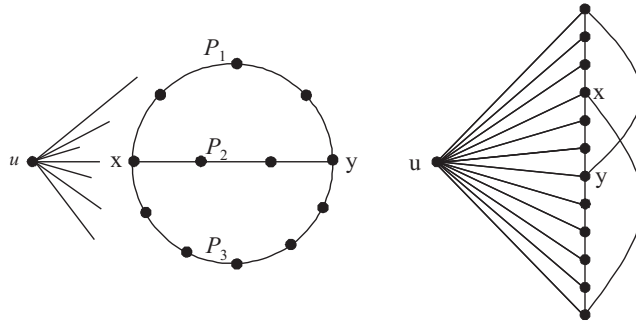
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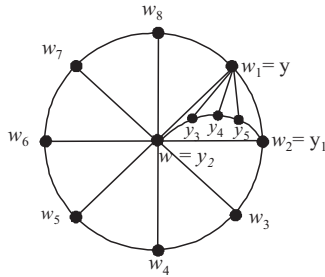Figure 7: Two depictions of an extended θ-graph.
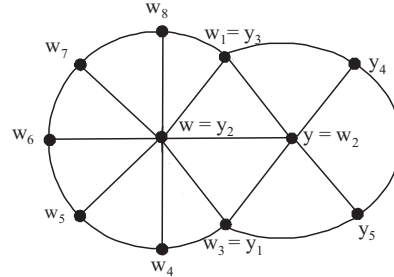

Figure 8: A triangle bicycle

Figure 9: A diamond bicycle.

For example, let W be a wheel with centre w and rim $w_1 w_2 \ldots w_8 w_1$ and let Y be a wheel with centre y and rim $y_1 y_2 \ldots y_5 y_1$. Figure 8 depicts a triangle bicycle, obtained from W and Y by pasting the triangle $Y[\{y, y_1, y_2\}]$ to the triangle $W[\{w_1, w_2, w\}]$. Figure 9 depicts a diamond bicycle, obtained from W and Y by pasting the diamond $Y[\{y, y_2, y_1, y_3\}]$ to the diamond $W[\{w_2, w, w_3, w_1\}]$.

Theorem 3.5. *G is an irreducible LC graph with $m(G) \leq 2n(G) - 1$ if and only if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where*

- *$\mathcal{T}_3 = \{K_2, K_3\}$.*

- *$\mathcal{T}_2$ consists of all the wheels.*

- *$\mathcal{T}_1$ consists of the extended θ-graphs, the diamond bicycles and the triangle bicycles.*

Proof. Adding a universal vertex to any connected graph obviously results in an LC graph. This, together with Observation 3.2, implies that every

graph in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ is LC. It is easily seen that every graph in $\mathcal{T}_1 \cup \mathcal{T}_2$ is 3-connected and that every graph in $\mathcal{T}_i$ satisfies $m = 2n - i$, $i = 1, 2, 3$. Thus, if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then G is an irreducible LC graph with $m(G) \leq 2n(G) - 1$.

To prove the converse, suppose G is an irreducible LC graph with $m(G) \leq 2n(G) - 1$. We prove by induction on the order that $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.

If $n(G) \leq 3$, then $G \in \mathcal{T}_3$, so the basis step is proved.

Now suppose $n(G) \geq 4$. Then G is 3-connected by Observation 3.3. Since $m(G) < 2n(G)$, it therefore follows that G has a vertex, say x, of degree 3. Let $H = G - x$. Since G is locally connected, $G[N(x)]$ is isomorphic to either K_3 or P_3.

We now consider three cases, depending on the relative size of G. In each case we use the fact that if $G[N(x)]$ is a K_3, then H is an irreducible LC graph, and if $G[N(x)]$ is a path $v_1 v_2 v_3$, then $H + v_1 v_3$ is an irreducible LC graph. Note that in the latter case, G can be retrieved from H by subdividing the edge $v_1 v_3$ with a vertex x and then adding the edge xv_2.

Case 1. $m(G) = 2n(G) - 3$:
If $G[N(x)]$ is a K_3, then $m(H) = m(G) - 3 = 2n(H) - 4$, contradicting Proposition 3.1. Hence we may assume that $G[N(x)]$ is a path $v_1 v_2 v_3$. Then $m(H + v_1 v_3) = 2n(H) - 3$. It therefore follows from our induction hypothesis that $H + v_1 v_3 \in \mathcal{T}_3$, so $H + v_1 v_3$ is a K_3. But then G is a diamond, contradicting that G is irreducible. Hence K_2 and K_3 are the only irreducible LC graphs with $m = 2n - 3$.

Case 2. $m(G) = 2n(G) - 2$:
If $G[N(x)]$ is a K_3, then $m(H) = 2n(H) - 3$, so then it follows from our induction hypothesis that $H \in \mathcal{T}_3$ and hence $H \cong K_3$. Then $G \cong K_4$, so $G \in \mathcal{T}_2$.

Now suppose $G[N(x)]$ is a path $v_1 v_2 v_3$. Then $m(H + v_1 v_3) = 2n(H + v_1 v_3) - 2$. Hence, by our induction hypothesis, $H + v_1 v_3$ is a wheel W. Since the set $\{v_1, v_2, v_3\}$ induces a triangle in W, the centre w of W is either v_1, v_2 or v_3.

If $w = v_2$, then v_1 and v_3 are consecutive vertices on the rim of W. Then G is the wheel obtained from W by inserting the spoke wx. Hence $G \in \mathcal{T}_2$.

Now suppose $w = v_1$. Then we may suppose that v_3 is the successor of v_2 on the rim of W. Let v_3^+ be the successor of v_3 on the rim of W. If $n(W) \geq 5$, then since $v_1 v_3 \notin E(G)$, it follows that $G[N(v_3^+)]$ is not connected, contradicting that G is LC. Hence $n(W) = 4$, so $W \cong K_4$. This implies that G is

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a wheel of order 5, centred at v_2. Hence $G \in \mathcal{T}_2$.

Case 3. $m(G) = 2n(G) - 1$:

Suppose $G[N(x)]$ is a K_3 with vertex set v_1, v_2, v_3. Then $m(H) = 2n(H) - 2$. Hence, by our induction hypothesis, H is a wheel. Then $G[\{x, v_1, v_2, v_3\}]$ is a K_4, and G is obtained by pasting this K_4 and the wheel H together on the triangle $G[\{v_1, v_2, v_3\}]$. Thus G is an extended θ-graph, so $G \in \mathcal{T}_1$.

Now suppose $G[N(x)]$ is a path $v_1 v_2 v_3$. Then $m(H + v_1 v_3) = 2n(H + v_1 v_3) - 1$, so by our induction hypothesis $H + v_1 v_3 \in \mathcal{T}_1$. There are three cases to consider.

(a) $H + v_1 v_3$ *is an extended θ-graph*: Let u be the universal vertex of $H + v_1 v_3$. Then $H + v_1 v_3 - u$ consists of two vertices a and z and three internally disjoint $a - z$ paths P_1, P_2, P_3.

If $u = v_2$, then v_1 and v_3 are two consecutive vertices on one of the paths P_1, P_2, P_3, and G is obtained from $H + v_1 v_3$ by inserting x on the edge $v_1 v_3$ and joining x to u. So in this case G is also an extended θ-graph.

If $u = v_1$, then since $G[N(v_3)]$ is connected, we may assume that $P_1 = az$ and P_2 and P_3 have length 2 each. Let y and w be the internal vertices of P_2 and P_3, respectively. First, suppose $v_2 = a$. Then v_2 is a universal vertex of G. If $v_3 = z$, then $G - v_2$ consists of the three internally disjoint $v_1 - v_3$ paths $v_1 x v_3$, $v_1 y v_3$ and $v_1 w v_3$. If $v_3 = y$, then $G - v_2$ consists of the the the three internally disjoint $v_1 - z$ paths $v_1 z$, $v_1 x v_3 z$ and $v_1 w z$. In either case, G is an extended θ-graph. By symmetry, this proves all the cases when $v_2 \in \{a, z\}$. If $v_2 \notin \{a, z\}$, it is sufficient to consider the case $v_2 = w$, $v_3 = z$. Then G consists of the wheels $G[\{v_2, a, v_1, x, v_3\}]$ and $G[\{a, v_1, v_2, v_3, y\}]$, pasted together on the diamond $G[\{a, v_2, v_1, v_3\}]$, so then G is a diamond bicycle.

Now suppose $u \notin \{v_1, v_2, v_3\}$. Then we may assume that $P_1 = az$ and $P_2 = ayz$, where $\{a, y, z\} = \{v_1, v_2, v_3\}$. If $y = v_2$, then G consists of the wheel $G[\{y, u, a, x, z\}]$ and the wheel $G[\{u, y\} \cup V(P_3)]$, pasted together on the diamond $G[\{y, u, a, z\}]$. So in this case G is a diamond bicycle. If $y = v_1$, then G consists of the wheel $G[\{z, u, y, x, a\}]$ and the wheel $G[\{u\} \cup V(P_3)]$, pasted together on the triangle $G[\{u, z, a\}]$, so then G is a triangle bicycle. In either case $G \in \mathcal{T}_1$.

(b) $H + v_1 v_3$ *is a triangle bicycle*: Then $H + v_1 v_3$ consists of two wheels W and Y, each of order at least 5, pasted together on a triangle T such that w and y, the respective centres of W and Y, are distinct vertices in T. We

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may assume that $G[v_1, v_2, v_3]$ is a triangle in W. First, suppose v_1 or v_3 is the centre of W, say $v_1 = w$. Then we may assume that v_3 is the successor of v_2 on the rim of W. Let v_3^+ be the successor of v_3 on the rim of W. Now $v_1 \notin N_G(v_3)$ and since $n(W) \geq 5$, there is no path from v_2 to v_3^+ in $G[N(v_3)]$. Hence $G[N(v_3]$ is not connected, contradicting that G is LC. This proves that $\{v_1, v_3\} \cap \{w, y\} = \emptyset$ and hence $G[\{v_1, v_2, v_3\}] \neq T$.

We may therefore assume that $v_2 = w$ and $v_1 v_3$ is an edge on the rim of W. Since $y \notin \{v_1, v_2, v_3\}$, it follows that $v_1 v_3 \notin E(T)$. Now let W^* be the wheel obtained from W by subdividing the edge $v_1 v_3$ with the vertex x and adding the edge xw. Then G is the triangle bicycle consisting of the wheels W^* and Y, pasted together on the triangle T. Hence $G \in \mathcal{T}_1$.

(c) $H + v_1 v_3$ *is a diamond bicycle*: Then $H + v_1 v_3$ consists of two wheels W and Y, each of order at least 5, that share a diamond D such that w and y, the respective centres of W and Y, are distinct vertices in D.

Suppose $v_1 = w$. Then we may assume that v_3 is the successor of v_2 on W. Let v_3^+ be the successor of v_3 on W. If $v_3 = y$, then $G[N(v_3^+)]$ is not connected, since there is no path from v_1 to v_3 in $N_G(v_3^+)$. If $v_3 \neq y$, then there is no path from v_3^+ to v_2, so then $G[N(v_3)]$ is not connected. This proves that $\{v_1, v_3\} \cap \{w, y\} = \emptyset$ and that $G[\{v_1, v_2, v_3\}]$ is not a triangle in D.

We may therefore assume that $v_2 = w$ and $v_1 v_3$ is an edge on the rim of W such that $v_1 v_3$ is not an edge of the diamond D. Hence G is a diamond bicycle, consisting of two wheels pasted together on D.

$\square$

Corollary 3.6. *Every irreducible LC graph with $n \geq 3$ and $m \leq 2n - 1$ is hamiltonian.*

4. The Hamilton Cycle Problem for sparse LC graphs

A modification of the construction that produced the $\{2, 5\}$-regular graph G^* in the proof of Theorem 1.3 (see Figure 3), yields the following result.

Theorem 4.1. *For every natural number $k > 1$ the Hamilton Cycle Problem for locally connected graphs with $m = 2n + \lfloor n^{1/k} \rfloor$ is NP-complete.*

Proof. Consider any cubic graph G of order n. Let G^* be the $\{2, 5\}$-regular graph obtained from G as in Theorem 1.3. If $k > 1$, we extend the border B

by successively attaching diamonds as shown in Figure 10, until the resulting graph G_N has $N = (3n/2)^k$ vertices. (Note that the order of G^* is $3n+(9n/2)$, which has the same parity as $3n/2$.)
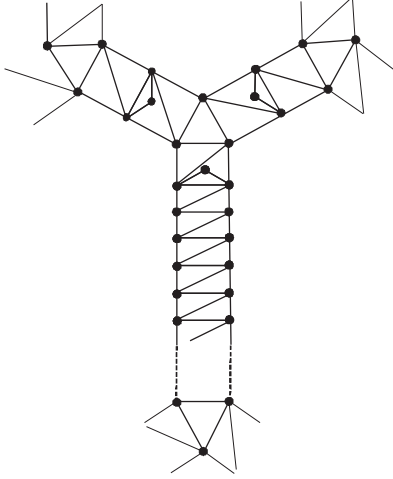

Figure 10: Extending a border in G^* to construct G_N.

This can be done in polynomial time. Each added vertex has degree 4 and the neighbours of vertices of degree 2 are of degree 5 in G_N. Hence the average degree of the vertices that are not in the node-set of G_N is 4. Note that G_N has n nodes and each node has three vertices, each of which has degree 5. Hence

$$\sum_{v \in V(G_N)} d(v) = 5(3n) + 4(N - 3n) = 4N + 3n,$$

so

$$|E(G_N)| = 2N + 3n/2 = 2N + N^{1/k}.$$

G_N is clearly LC, and it follows easily from the proof of Theorem 1.3 that G_N is hamiltonian if and only if G is. □

Our next theorem shows that if the value of m in Theorem 4.1 is slightly decreased, the complexity of the problem is significantly reduced.

Theorem 4.2. *For every natural number k the Hamilton Cycle Problem for LC graphs is in $\mathbb{P}$ for graphs with $2n + k \log_2 n$ edges or fewer.*

The proof of Theorem 4.2 relies on the following reduction procedure, which decreases the order of an LC graph, while retaining local connectivity.

Reduction procedure. Let G be an LC graph of order n and size $2n + p$, where p is a positive integer. We choose a set of edges in G, which we refer to as *special edges* of G. We describe three types of reductions that may be performed on G.

Reduction 1: Suppose $v \in V(G)$ has degree 2, and let $N(v) = \{x, y\}$. Now delete the vertex v. The resulting graph G' is LC and $|E(G')| = 2|V(G')| + p$. We observe that G does not have a Hamilton cycle containing the edge xy. We also note that G is hamiltonian if and only if G' has a Hamilton cycle containing the edge xy. We therefore add xy to the set of special edges of the reduced graph.

Reduction 2: Suppose G has a vertex v of degree 3 such that $\langle N(v) \rangle$ is a path $x_1 x_2 x_3$. Then we delete v and add the edge $x_1 x_3$. The resulting graph G' is LC and $|E(G')| = 2|V(G')| + p$. We call v a *nice vertex* of degree 3. We note that G' is hamiltonian if G is, but the converse need not be true.

Reduction 3: Suppose G has a vertex v of degree 3 such that $\langle N(v) \rangle$ is a triangle. Then we delete v. The resulting graph graph G' is LC and $|E(G')| = 2|V(G')| + p - 1$. We call v a *problematic vertex* of degree 3. It follows from Proposition 3.1 that we meet at most $p + 3$ problematic vertices of degree 3 when we successively delete vertices of degree 3. We note that G' is hamiltonian if G is, but the converse need not be true.

Next, we present some observations, lemmas and definitions that will be used to prove Theorem 4.2

Observation 4.3. *If we reduce a graph G by removing a vertex v by means of a Type 1, 2 or 3 reduction, then the neighbours of v are mutually adjacent in the resulting graph (even if two of the neighbours were not adjacent in G).*

If v is a vertex in a connected graph G such that the neighbours of v are mutually adjacent, then clearly $G - v$ is connected, so by Observation 4.3, Reductions 1, 2 and 3 do not create new vertex cuts. We shall apply this observation in the special version described in Lemma 4.4 below.

Lemma 4.4. *Let G be a connected LC graph and let F be a graph obtained from G by means of a number of consecutive reductions of Types 1, 2 or 3. Let z be the last vertex of G that was removed to obtain F and suppose z had exactly two neighbours x, y in F. Then $\{x, y\}$ is a vertex cut of G.*

The *square of an n-cycle*, denoted by C_n^2, is the 4-regular graph obtained from the n-cycle C_n by adding an edge between each pair of vertices at distance 2 from one another. The graph C_{12}^2 is depicted in Figure 11.
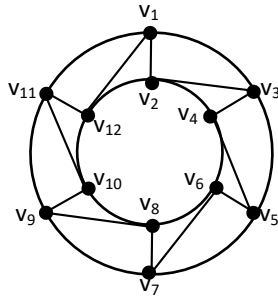

Figure 11: The square of a 12-cycle.

Gordon et al. [4] proved the following.

Lemma 4.5. [4] *Every connected 4-regular LC graph is the square of a cycle.*

Let $v_1 v_2 \ldots v_{2n} v_1$ denote the underlying $2n$-cycle of the graph C_{2n}^2. If the three edges between the sets $\{v_1, v_2\}$ and $\{v_{2n-1}, v_{2n}\}$ are deleted from C_{2n}^2, the result is a a *zig-zag strip* of order $2n$. If the vertex v_{2n} is also deleted, the result is a zig-zag strip of order $2n - 1$. Thus a zig-zag strip consists of two "parallel" paths whose lengths differ by at most one, with edges between the paths in a zig-zag pattern. Figure 12 depicts zig-zag strips of order 11 and 12.

Each triangle in a zig-zag strip may be viewed as a fan with two spokes, having its apex on one of the two parallel paths and its rim on the other. We call a graph obtained from a zig-zag strip by inserting spokes into such fans (as described in Section 3) a *nice generalized strip*. Note that any nice generalized strip is locally connected. Figure 13 depicts an example of a nice generalized strip.

A fan in a graph G is a *nice fan* if every internal vertex on the rim of the fan has degree 3 in G, i.e., only the end vertices of the rim and the centre may
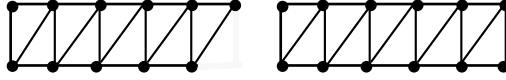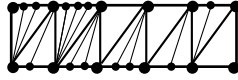
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Figure 12: Zig-zag strips of order 11 and 12.

Figure 13: An example of a nice generalized strip.

have neighbours outside the fan. We note that every fan in a nice generalized strip is a nice fan, provided its rim is on one of the two parallel paths.

A *path cover* of a graph is a set of vertex disjoint paths that contain all the vertices of the graph. An *r-path* cover is a path cover consisting of r vertex disjoint paths.

Lemma 4.6. *Let S be a nice generailzed strip and let a_1, a_2 and b_1, b_2 be the end vertices of the two parallel paths of S. Suppose Q is a path cover of S such that the end vertices of all the paths in Q are in $\{a_1, a_2, b_1, b_2\}$. Then there are at most 26 possibilities for the end vertices of the paths in Q.*

Proof. If Q is a 1-path cover (i.e., a Hamilton path), then the two end vertices can be chosen from the set $\{a_1, a_2, b_1, b_2\}$ in $\binom{4}{2} = 6$ ways.

Now suppose Q is a 2-path cover $\{Q_1, Q_2\}$. If neither Q_1 nor Q_2 is a singleton, then there are only 2 possibilities for their end vertices, since S is planar. If Q_1 is a single vertex v, then there are 4 choices for v and for each such choice there are 3 possibilties for the end-vertices of Q_2. Thus the total number of possibilities in the case of a 2-path cover is 14.

If Q is a 3-path cover, then two of the three paths in Q are singletons, so in this case there are only 6 possibilities for the end vertices.

Thus the total number of possibilities is $6 + 14 + 6 = 26$. $\qquad\square$

If G is a graph with a chosen set of special edges, we call a Hamilton cycle of G that contains all the special edges of G a *special Hamilton cycle* of G.

For simplicity, we only prove Theorem 4.2 for $k = 1$, since it easily generalizes to an arbitrary natural number k. We actually prove the following slightly more general result for $k = 1$.

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Theorem 4.7. *The problem of deciding whether an LC graph of order n and size at most $2n + log_2 n$ has a Hamilton cycle containing a set of prescribed edges is in $\mathbb{P}$.*

We precede the proof with a brief outline, so that the reader can keep track of the main steps while going through the details of the proof.

Outline of proof. Let G be an LC graph of order n and size at most $2n + \log_2 n$, with a prescribed set of special edges. We use our reduction procedure to iteratively remove vertices of degree 2 and 3, until we obtain a graph H that has exactly $\lfloor \log_2 n \rfloor$ vertices or has minimum degree 4, whichever occurs first. We choose the special edges of H to consist of the special edges inherited from G, together with every edge that became special due to a Type 1 reduction. Then any special Hamilton cycle of G reduces to a special Hamilton cycle of H, but the converse is not necessarily true.

We now reverse the reductions by first returning vertices of degree 3 as far as possible without returning vertices of degree 2, to obtain the graph G^*. When reversing a reduction, any edges that were added during the reduction are deleted, so G^* is an induced subgraph of G.

An analysis of the structure, order and size of G^* allows us to shrink G^* to a graph H' of sufficiently small size so that it can be checked in $O(n^c)$ time (for some constant c) whether H' has a special Hamilton cycle, and which is such that H' has a special Hamilton cycle if and only if G^* has one.

If $G = G^*$, we are done. If not, there is at least one vertex z in $G - V(G^*)$ that has two neighbours x, y in G^* and xy is an edge that became special due to a Type 1 reduction. By Lemma 4.4, $\{x, y\}$ is a vertex cut of G, and since G is locally connected, $G - \{x, y\}$ has exactly two components. Let Z be the component of $G - \{x, y\}$ that contains z. Let $G_{xy} = G[V(Z) \cup \{x, y\}]$ and let $G_1 = G[V(G^*) \cup V(G_{xy})]$. Then G_1 has a special Hamilton cycle if and only if G^* has a special Hamilton cycle and G_{xy} has a special Hamilton cycle containing the edge xy. To check whether G_{xy} has such a Hamilton cycle, we repeat our algorithm, starting with the 3-cycle $zxyz$ instead of the graph H. Again this is an $O(n^c)$ algorithm.

Since the number of Type 1 reductions that we need to reverse is less than n, repeated applications of the procedure result in an $O(n^{c+1})$ algorithm for determining whether G has a special Hamilton cycle.

Now we present the detailed proof.

Proof. Let G be an LC graph of order n and size at most $2n + \log_2 n$, with a given set of special edges. We may assume G is connected, since otherwise the problem is trivial. We perform reductions 1, 2 and 3 in any order until the resulting graph H has precisely $\lfloor \log_2 n \rfloor$ vertices or has minimum degree at least 4. In either case it follows from the discussions of the reductions above that $|E(H)| \leq 2|V(H)| + \log_2 n$.

<u>Case 1</u>. *H has precisely $\lfloor \log_2 n \rfloor$ vertices*:

In this case $|E(H)| \leq 2\log_2 n + \log_2 n = 3\log_2 n$. We consider each subset of $E(H)$ and check if that subset induces a special Hamilton cycle of H. This algorithm has time complexity $O(2^{|E(H)|} \log_2 n) = O(2^{3\log_2 n} \log_2 n) = O(n^3 \log_2 n)$.

We now focus on the maximal nice fans. We create the graph G^* by reversing all the reductions of Types 2 and 3 that can be reversed without reversing any reductions of Type 1. We note that $|E(G^*)| \leq 2|V(G^*)| + \log_2 n$.

If a nice vertex whose neighbours form a triangle in a nice fan of order greater than 3 is returned by reversing a Type 2 reduction, then the edge of the triangle that is on the rim of the fan is deleted. (For, if a spoke were deleted we would not get an LC graph.) Thus, reversing a Type 2 reduction can extend a maximal nice fan but does not create additional maximal nice fans. Hence, if no Type 3 reductions were reversed to create G^* from H, then the number of maximal nice fans in G^* would be at most the number of edges of H, i.e., at most $3\log_2 n$.

However, we may need to return problematic vertices of degree 3 to create G^*. Returning a problematic vertex of degree 3 whose neighbours are a triangle in a maximal nice fan divides that fan into two edge disjoint maximal nice fans - see Figure 14.
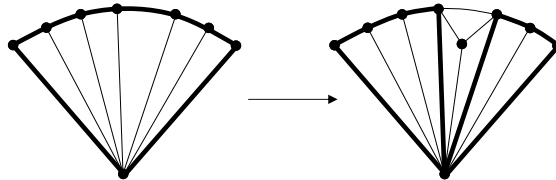

Figure 14: Reversing a reduction of Type 3 divides a maximal nice fan into 2 maximal nice fans.

Since the number of problematic vertices that were deleted from G to create H is at most $\log_2 n + 3$ (as explained in the description of Reduction 3), the addition of problematic vertices to H creates at most $\log_2 n + 3$ additional maximal nice fans. Thus the number of maximal nice fans in G^* is at most $|E(H)| + \log_2 n + 3 = 4\log_2 n + 3$.

We now reduce any maximal nice fan in G^* that has more than 4 spokes to a nice fan with exactly four spokes, by eliminating internal spokes (as described in Section 3). Call the resulting graph H'. Since eliminating a spoke amounts to reducing the number of vertices by one and the number of edges by two, it follows that $|E(H')| \leq 2|V(H')| + \log_2 n$.

We may assume that the set of special edges in G^* does not contain two non-consecutive internal spokes of any nice fan (since otherwise G^*, and hence G, will be nonhamiltonian). It is therefore possible to choose edges to be special in H' so that H' has a special Hamilton cycle if and only if G^* has a special Hamilton cycle.

We note that if F is a nice fan in G^* that was obtained from a nice fan in H by adding only nice vertices of degree 3, the two outer spokes of F are both in H. Thus, if F is reduced to a fan with four spokes in H', it contains at most 2 vertices that were not in H.

If a problematic vertex of degree 3 is inserted into a triangle in a nice fan while we are building G^*, then two of the three the neighbours of that vertex lie on the outer spokes of two new maximal nice fans (see Figure 14). Thus returning a problematic vertex results in at most 3 more vertices in H' that were not in H. Thus

$$|V(H')| \leq |V(H)| + 2(4\log_2 n + 3) + 3(\log_2 n + 3) = 12\log_2 n + 15$$

and

$$|E(H')| \leq 2|V(H')| + \log_2 n \leq 25\log_2 n + 30,$$

so all the Hamilton cycles in H' can be found by means of an $O(2^{25log_2n+30}\log_2 n) = O(n^{25}\log_2 n)$ algorithm.

Thus, in this case there is an $O(n^{26})$ algorithm for determining whether G^* has a special Hamilton cycle.

If $G = G^*$, we are done. If not, we reverse a Type 1 reduction and carry on as explained in the last paragraph of the outline of the proof. The result is an $O(n^{27})$ algorithm for determining whether G has a special Hamilton cycle.

<u>Case 2</u>. *H has more than* $\log_2 n$ *vertices* (and $\delta(H) \geq 4$):

Recall that also in this case, $|E(H)| \leq 2|V(H)| + \log_2 n$. Let

$$X = \{v \in V(H) : d_H(v) > 4\}, \ N = N_{H-X}(X), \ M = G[X \cup N].$$

We assume that X is nonempty. (Otherwise, the argument below is simpler.) Note that

$$\sum_{v \in V(H)} d_H(v) \leq 4|V(H)| + 2\log_2 n.$$

Hence, since $\delta(H) \geq 4$, it follows that $\sum_{v \in X}(d_H(v) - 4) \leq 2\log_2 n$, and hence $|X| \leq 2\log_2 n$. Now

$$|N| \leq \sum_{v \in X} d_H(v) = \sum_{v \in V(H)} d_H(v) - \sum_{v \in V(H) - X} d_H(v).$$

But every vertex in $H - X$ has degree exactly 4 in H, so

$$
\begin{aligned}
|N| &\leq 4|V(H)| + 2\log_2 n - 4(|V(H)| - |X|) \\
&= 4|X| + 2\log_2 n \\
&\leq 10\log_2 n
\end{aligned}
$$

and

$$|V(M)| \leq 12\log_2 n.$$

Now

$$
\begin{aligned}
\sum_{v \in V(M)} d_H(v) &= 4|V(M)| + \sum_{v \in X}(d_H(v) - 4) \\
&\leq 4|V(M)| + 2\log_2 n.
\end{aligned}
$$

Hence

$$|E(M)| \leq 2|V(M)| + \log_2 n \leq 25\log_2 n.$$

Since every vertex in $H - X$ has exactly 4 neighbours in H, it can easily be proved using the same reasoning as in Lemma 4.5 that every component in $H - V(M)$ together with its neighbors in N form a *zig-zag strip* that has its two end edges in N and all the other vertices in $H - V(M)$ - see Figure 15.

Since every vertex in N has degree 4 in H and has at least one neighbour in X, no vertex in N lies in more than one zig-zag strip of H, and each zig-zag strip in H contains 4 vertices of N. Since $|N| \leq 10\log_2 n$, it follows that H has at most $5/2\log_2 n$ zig-zag strips.
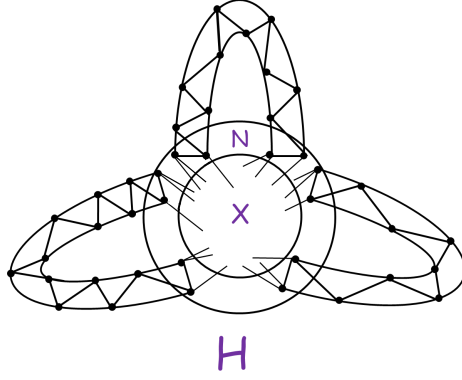
Figure 15: Structure of H.

We now create the graph G^* by reversing all the reductions of Types 2 and 3 that can be reversed without reversing a reduction of Type 1. We first return vertices that have three neighbours in a zig-zag strip. If only nice vertices are added to a zig-zag strip, that strip becomes a *nice generalized strip*. If a problematic vertex is added in a nice generalized strip, then the nice generalized strip becomes two nice generalized strips and we then add the problematic vertex and its neighbours to M, as illustrated in Figure 16. Thus each addition of a problematic vertex to a nice generalized strip creates one more nice generalized strip and increases the number of vertices in M by 4 and the number of edges by 6.

Since there are at most $\log_2 n + 3$ problematic vertices, $|V(M)|$ increases to at most

$$12 \log_2 n + 4(\log_2 n + 3) = 16 \log_2 n + 12$$

and $|E(M)|$ increases to at most

$$25 \log_2 n + 6(\log_2 n + 3) = 31 \log_2 n + 18,$$

while the number of nice generalized strips increases to at most

$$5/2 \log_2 n + \log_2 n + 3 = 7/2 \log_2 n + 3.$$

The intersection of a Hamilton path of G^* with each nice generalized strip is a path cover of which the end vertices of all the paths are in N. By Lemma 4.6 there are 26 possibilities for the end vertices of such path covers. For each of those, there is an $O(n)$ algorithm for deciding whether there exists a corresponding path cover containing all the special edges in the
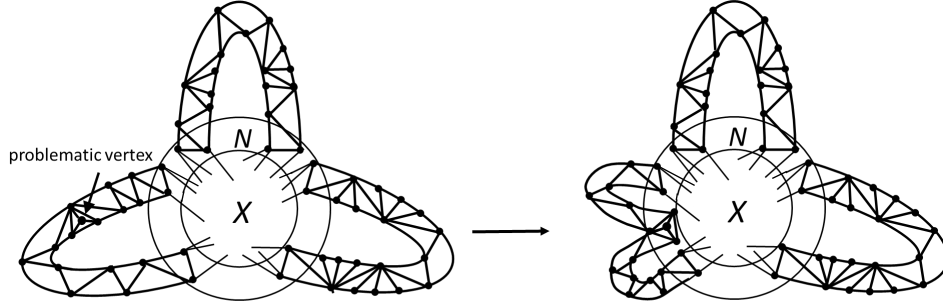
Figure 16: Reversing reductions of Type 2 and 3 to create G^*

strip. Therefore, since there are at most $7/2 \log_2 n + 3$ nice generalized strips, there is an $O(n) \cdot 26^{7/2 \log_2 n + 3} < O(n) \cdot (2^5)^{7/2 \log_2 n + 3} < O(n^{19})$ algorithm for deciding whether there exist corresponding path covers containing all the special edges in all the strips.

So far we have returned vertices that have three neighbours in a zig-zag strip. We now return vertices whose neighbours induce triangles or paths with 3 vertices in M. Denote the resulting extension of M by M^*. Using the method of Case 1, we find that M^* contains at most $|E(M)| + \log_2 n + 3 = 32 \log_2 n + 21$ maximal nice fans.

Now we reduce M^* to the graph M' by reducing all the maximal nice fans in M^* to fans with at most 4 spokes and choose appropriate special edges for M' as in Case 1. Then

$$|V(M')| \leq |V(M)| + 2(32 \log_2 n + 21) + 3(\log_2 n + 3)$$

$$= 16 \log_2 n + 12 + 67 \log_2 n + 51 = 83 \log_2 n + 63,$$

and

$$|E(M')| \leq 2|V(M')| + \log_2 n \leq 166 \log_2 n + 126.$$

Let H' denote the graph consisting of M' together with all the nice generalized strips. Then H' has a special Hamilton cycle if and only if G^* has a special Hamilton cycle. For each subset of $E(M')$ and each choice of path covers of the strips we check, in $O(n)$ time, if the subsets of $E(M')$ and the path covers of the strips form a special Hamilton cycle of H'. The complexity of this algorithm is at most $O(n^{19}) \cdot 2^{166 log_2 n + 126} \cdot O(n) = O(n^{176})$.

Thus, in this case there is an $O(n^{176})$ algorithm for determining whether G^* has a special Hamilton cycle.

If $G = G^*$, we are done. If not, we reverse a Type 1 reduction and carry on as explained in the last paragraph of the outline of the proof. The result is an $O(n^{177})$ algorithm for determining whether G has a special Hamilton cycle.

$\square$

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