A Double Generally Weighted Moving Average Exceedance Control Chart

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Abstract

Since the inception of control charts by W. A. Shewhart in the 1920s they have been increasingly applied in various fields. The recent literature witnessed the development of a number of nonparametric (distribution-free) charts as they provide a robust and efficient alternative when there is a lack of knowledge about the underlying process distribution. In order to monitor the process location, information regarding the in-control process median is typically required. However, in practice this information might not be available due to various reasons. To this end, a generalized type of nonparametric time-weighted control chart labelled as the Double Generally Weighted Moving Average (DGWMA) based on the exceedance statistic (EX) is proposed. The DGWMA-EX chart includes many of the well-known existing time-weighted control charts as special or limiting cases for detecting a shift in the unknown location parameter of a continuous distribution. The DGWMA-EX chart combines the better shift detection properties of a DGWMA chart with the robust in-control performance of a nonparametric chart, by using all the information from the start until the most recent sample to decide if a process is in-control (IC) or out-of-control (OOC). An extensive simulation study reveals that the proposed DGWMA-EX chart, in many cases, outperforms its counterparts.

Keywords: Average run length; Control chart; DGWMA; Exceedance statistic; Nonparametric.

1. Introduction

Statistical process control (SPC) refers to the collection of statistical procedures and problem solving tools used to control and monitor the quality of the output of some production process

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(Balakrishnan et al.[1]). It is often of interest to detect any changes in location and/or dispersion as early as possible and SPC possesses some of the extensively used tools to detect the presence of causes of variation and to maintain stability. One of these tools is the control chart and designed to detect changes in a process from an in-control to an out-of-control state. Control charts are widely used to analyse and understand process variables, monitor effects of the variables on the difference between target and actual performance, and determine if a process is under statistical control. If a charting statistic plots within the upper and lower control limits , it is considered to be in-control (IC) and if a charting statistic plots on or outside either of the limits, it is declared to be out-of-control (OOC). Control charts usually assume a known (normal) distribution for the process, however in many applications, the underlying process distribution is unknown and/or not normal and hence the statistical properties of commonly used charts, designed to perform best under the normal distribution assumption, could be highly affected. Nonparametric control charts provide a robust alternative when there is a lack of knowledge about the underlying process distribution. A chart is called distribution-free or nonparametric if its IC run length distribution remains invariant for all continuous process distributions. However, in some cases, symmetry of the underlying distribution is required for the chart to be nonparametric. The number of plotting statistics to be plotted until the first out-of-control signal occurs, is a discrete random variable and is called the run length.

Walter A. Shewhart (1891-1967) proposed Shewhart-type charts, laying the foundation of SPC. The interested reader is referred to Shewhart[2,3]. Shewhart-type control charts are the most widely known charts in practice because of their global performance. The charting statistic for the Shewhart-type charts is typically the value of the corresponding sample statistic. As an example, assume that the observations from the process being monitored are mutually independent and from a normal distribution with known mean μ and known variance σ^2. Then the symmetrically placed control limits for a Shewhart $\bar{X}$ chart are given by $UCL = \mu + L\frac{\sigma}{\sqrt{n}}$ and $LCL = \mu - L\frac{\sigma}{\sqrt{n}}$, where n denotes the sample size, UCL and LCL are the upper and lower control limits, respectively, and $L > 0$ is the distance of the control limits from the centerline. Because Shewhart-type charts only use the most recent sample to decide if the process is IC or OOC, they are inefficient in detecting small and minor shifts in the process. To overcome the difficulties of Shewhart-type charts in detecting process shifts, it is recommended to use time-weighted or memory-type charts such as the Cumulative Sum (CUSUM) proposed by Page[4], the Exponentially Weighted Moving Average (EWMA) proposed by Roberts[5], the Double Exponentially Weighted Moving Average (DEWMA) proposed by Shamma and Shamma[6] and the Generally Weighted Moving Average (GWMA) proposed by Sheu and Lin[7]; these charts sequentially accumulate information over time to determine the state of statistical control. The interested reader is referred to Montgomery[8] for more details. Sheu and Hsieh[9] proposed a Double Generally Weighted Moving Average (DGWMA) chart for the normal distribution (denoted by DGWMA-$\bar{X}$) by combining the DEWMA- $\bar{X}$ chart proposed by Zhang and Chen[10] and the GWMA-$\bar{X}$ chart proposed by Sheu and Lin[7]. They have shown that the DGWMA-$\bar{X}$ chart is more sensitive in detecting minor shifts in the process. The interested reader is referred to the works by Tai et al.[11] and Huang et al.[12]. In typical applications Shewhart-type and time-weighted charts are based on the fact that the observations of the underlying process are assumed to follow a normal or specified probability distribution. However, in many situations, the assumption of normality may not be justified or valid when the observations are from a non-normal or unknown distribution. The CUSUM signed-rank charts were developed by Bakir and Reynolds[13] and the Shewhart-type signed-rank chart by Bakir[14] . For more details the interested reader is referred to Amin et al.[15], Chakraborti et al.[16] and Bakir[17]. More recently, Lu[18] and Chakraborty et al.[19] proposed nonparametric GWMA charts based on the sign statistic (denoted by GWMA-SN) and Wilcoxon signed-rank statistic (denoted by GWMA-SR), respectively, for the case when the true process median is known; this is referred to as Case K. The parametric DGWMA scheme has been shown to improve the detection ability of the GWMA chart. To this end, Lu[20] proposed a nonparametric DGWMA chart (denoted by DGWMA-SN) when the true process proportion is known. However, the true process median may not be known (referred to as Case U) which limits the applicability of the distribution-free DGWMA charts based on well-known nonparametric statistics, e.g. the sign and Wilcoxon signed-rank statistics. Precedence or exceedance tests, based on precedence or exceedance statistics, are well known nonparametric two-sample tests which do not suffer from the limits of the aforementioned. Precedence statistics are defined as the number of observations from one of the samples that exceeds a specified (r^{th}) order statistic of the other sample. A class of nonparametric Shewhart-type charts, referred to as Shewhart-type precedence charts were studied by Graham et al.[21]. For more information in terms of nonparametric control charts please refer to Chakraborti et al.[22]. More recently, Chakraborty et al.[23] proposed a nonparametric GWMA exceedance chart, referred to as the GWMA-EX chart, which outperforms the EWMA-EX chart. Relatively little work has been done on nonparametric schemes in the context of a DGWMA chart. Motivated by these findings, we construct a distribution-free DGWMA chart based on an exceedance statistics for monitoring the unknown median of a process. This chart is referred to as the DGWMA exceedance (or DGWMA-EX) chart and integrates the virtues of both the GWMA and DEWMA charts to achieve improved detection ability, when compared with the nonparametric GWMA-EX chart. The proposed chart can be viewed as a generalized nonparametric time-weighted control chart which includes other nonparametric time-weighted charts such as the GWMA-EX, EWMA-EX and Shewhart-EX charts as limiting cases. Furthermore, the nonparametric DEWMA chart based on exceedance statistics, labelled as the DEWMA-EX chart, which is a special case of the DGWMA-EX chart, will be proposed and discussed as well. To the best of our knowledge there is no research published on the DEWMA-EX chart, a special case of the proposed DGWMA-EX chart, in the SPC literature, hence this paper also introduces this chart and discusses some of its properties. The structure of the rest of the paper is as follows: Section 2 provides the necessary theoretical framework for the DGWMA-EX chart. In Section 3, the run length distribution and design of the proposed chart are studied. An illustrative example is provided in Section 4, while some conclusions are provided in Section 5.

2. Preliminaries and Statistical Framework of the DGWMA exceedance chart

Let $X_1, X_2, \ldots, X_m \sim$iid $F_X(x)$ denote a Phase I reference sample from an in-control (IC) process with an unknown continuous cumulative distribution function (c.d.f.) $F_X(x)$ where $-\infty < \theta < \infty$ denotes the unknown location parameter. Let $Y_{i1}, Y_{i2}, \ldots, Y_{in}$, $i = 1,2, \ldots$, denote the i^{th} test sample in Phase II of size $n \geq 1$, with an unknown continuous c.d.f. $G_Y(x) = F_X(x - \theta)$. The main intention is to design a control chart for monitoring the unknown process location. The unknown/true value of the location parameter is denoted by θ_0 and the shifted location parameter is denoted by $\theta_1 = \theta_0 + \delta$; where $-\infty < \delta < \infty$ is the location shift. The process is declared to be IC when the unknown continuous c.d.f.'s F and G are equal (i.e. $G = F$ or $\delta = 0$) and out-of-control (OOC) when $G \neq F$ or $\delta \neq 0$.

Let U_{ir} denote the number of Y observations in the i^{th} Phase II sample that exceeds $X_{(r)}$, $r = 1,2,\ldots, m$, i.e. the r^{th} order statistic from the Phase I sample of size $m \geq 1$. The statistic U_{ir} is called the exceedance statistic and the probability $p_r = P\left[Y \geq X_{(r)} \middle| X_{(r)}\right]$, is the exceedance probability. For inference purposes, the exceedance and precedence tests are equivalent in the sense that the two statistics are linearly related. Hereafter, U_i will be used to denote the exceedance statistic for the i^{th} sample in Phase II.

2.1. Charting statistic

The DGWMA-EX chart is an extension of the GWMA-EX chart by invoking the DEWMA technique i.e. performing "smoothing" twice. The GWMA-EX chart proposed by Chakraborty et al.[23] is constructed by taking a weighted average of a sequence of the exceedance statistic U_i's. Let M_1 and M_2 be two discrete random variables denoting the number of samples until the next occurrence of an event since its last occurrence. Then, by summing over all values of M_j, we can write:

$$\sum_{i=1}^{\infty} P[M_j = i] = \sum_{i=1}^{t} P[M_j = i] + P[M_j > t] = 1 \quad \text{for } j = 1,2 \text{ and } t = 1,2,3,... \tag{1}$$

A GWMA is a weighted moving average of a sequence of U_i's, where the probability $P[M_1 = i]$ is known as the weight for the i^{th} most recent statistic U_{t-i+1} among the last t of the U_i's. The probability $P[M_1 > t]$ is considered the weight for the starting value, denoted by Z_0^1, and is typically taken as the unconditional in-control (IC) expected value of the exceedance statistic under consideration, i.e., $Z_0^1 = E(U_i|IC) = n\left(1 - \frac{r}{m+1}\right)$. Hence, the charting statistic for the GWMA-EX chart is as follows:

$$Z_t^1 = \sum_{i=1}^{t} P(M_1 = i)U_{t-i+1} + P(M_1 > t)\,Z_0^1 \quad \text{for} \quad t = 1,2,3,... \tag{2}$$

The distribution of M_1 can be written as (Sheu and Hsieh[9]):

$$P(M_1 = i) = q_1^{(i-1)^{\alpha_1}} - q_1^{i^{\alpha_1}} \tag{3}$$

where $0 < q_1 < 1$ and $\alpha_1 > 0$ are the parameters and $i = 1, 2, ...$. Equation (3) is the probability mass function (p.m.f.) of the two-parameter discrete Weibull distribution introduced by Nakagawa and Osaki[24]. By substituting the p.m.f. of the two-parameter discrete Weibull distribution in Equation (2), the charting statistic for the GWMA-EX is:

$$Z_t^1 = \sum_{i=1}^{t} (q_1^{(i-1)^{\alpha_1}} - q_1^{i^{\alpha_1}})X_{t-i+1} + q_1^{t^{\alpha_1}} Z_0^1 \quad \text{for} \quad t = 1,2,3,... \tag{4}$$

where $Z_0^1 = n\left(1 - \frac{r}{m+1}\right)$.

Now, to propose the DGWMA-EX chart as an extension of the GWMA-EX chart, the DGWMA-EX charting statistic is defined as:

$$Z_t^2 = \sum_{i=1}^{t} P(M_2 = i)Z_{t-i+1}^1 + P(M_2 > t)\,Z_0^2 \tag{5}$$

where $Z_0^2 = Z_0^1 = E(U_i|IC) = n\left(1 - \frac{r}{m+1}\right)$ is the starting value, and

$$P(M_2 = i) = q_2^{(i-1)^{\alpha_2}} - q_2^{i^{\alpha_2}} \tag{6}$$

where $0 < q_2 < 1$ and $\alpha_2 > 0$ are the parameters and $i = 1, 2, ...$, similar to Equation (3).

Note that the superscripts that are used to denote the charting statistics for the GWMA-EX and the DGWMA-EX charts (i.e. Z_t^1 and Z_t^2, respectively) also denote the order in which we apply the first and second "smoothing" of the U_i's; these superscripts should not be confused with the mathematical concept of raising a number or variable to an arbitrary power.

As in Sheu and Hsieh[9], the charting statistic in Equation (5) can be rewritten as:

$$Z_t^2 = P(M_2 = 1)Z_t^1 + P(M_2 = 2)Z_{t-1}^1 + \cdots + P(M_2 = t)Z_1^1 + P(M_2 > t)Z_0^1$$
$$= w_1 U_t + w_2 U_{t-1} + \cdots + w_t U_1 + \left(1 - \sum_{i=1}^{t} w_i\right) Z_0^2 \quad (7)$$

where the weight at time t is defined as:

$$w_t = \sum_{j=1}^{t} P(M_1 = j)P(M_2 = t - j + 1) \quad (8)$$

By substituting the p.m.f. for the discrete Weibull distribution (Equations (3) and (6)) into Equation (8), the weights can be written as:

$$w_t = \sum_{j=1}^{t} \left(q_1^{(j-1)^{\alpha_1}} - q_1^{j^{\alpha_1}}\right)\left(q_2^{(t-j)^{\alpha_2}} - q_2^{(t-j+1)^{\alpha_2}}\right) \quad \text{for} \quad t = 1,2,3,\ldots \quad (9)$$

Finally, the DGWMA-EX charting statistic is defined as:

$$Z_t^2 = \sum_{i=1}^{t} w_i U_{t-i+1} + \left(1 - \sum_{i=1}^{t} w_i\right) Z_0^2 \quad (10)$$

where, $t = 1, 2, \ldots$ and Z_0^2 is considered as the starting value.

The DGWMA-EX statistic Z_t^2 is denoted by *DGWMA-EX* $(q_1, \alpha_1; q_2, \alpha_2)$. In addition, the values of the weights i.e. w_t, using $0 < q_1, q_2 < 1, \alpha_1, \alpha_2 > 0, P(M_1 > t) = q_1^{t^{\alpha_1}}$ and $P(M_2 > t) = q_2^{t^{\alpha_2}}$ are equal to those derived using $P(M_1 > t) = q_2^{t^{\alpha_2}}$ and $P(M_2 > t) = q_1^{t^{\alpha_1}}$. Hence, the DGWMA-EX chart with parameters $(q_1, \alpha_1; q_2, \alpha_2)$ is equivalent (i.e. has the same run length distribution) as the DGWMA-EX chart with parameters $(q_2, \alpha_2; q_1, \alpha_1)$. For the sake of brevity, we write *DGWMA-EX* $(q_1, \alpha_1; q_2, \alpha_2) =$ *DGWMA-EX* $(q_2, \alpha_2; q_1, \alpha_1)$.

Since there is no or little information available with respect to the process distribution, the control limits for the DGWMA-EX chart are determined using the unconditional IC expectation and variance of the charting statistics in Equation (10).

2.2. Control limits

Let $Z_0^2 = n\left(1 - \frac{r}{m+1}\right)$, then the unconditional IC expectation of Z_t^2 can be derived as (see Chakraborty et al.[23]):

$$E(Z_t^2) = n\left(1 - \frac{r}{m+1}\right). \quad (11)$$

The unconditional IC variance of Z_t^2 is:

$$Var(Z_t^2) = \frac{n\left(\frac{r}{m+1}\right)\left(1 - \frac{r}{m+1}\right)}{m+2} \sum_{i=1}^{t} w_i^2 \,(n + m + 1). \quad (12)$$

The exact time-varying, symmetrically placed, control limits (denoted by UCL_e & LCL_e) of the two-sided DGWMA-EX chart are given by:

$$UCL_e/LCL_e = n(1 - \frac{r}{m+1}) \pm L\sqrt{\frac{n\left(\frac{r}{m+1}\right)\left(1 - \frac{r}{m+1}\right)}{m+2}\sum_{i=1}^{t} w_i^2\,(n+m+1)} \tag{13}$$

where $L > 0$ is the distance of the control limits from the centerline and the subscript "e" denotes the exact control limits.

The steady-state control limits, which are based on the asymptotic unconditional variance of the charting statistic, are given by:

$$UCL_s/LCL_s = n(1 - \frac{r}{m+1}) \pm L\sqrt{\frac{n\left(\frac{r}{m+1}\right)\left(1-\frac{r}{m+1}\right)}{m+2}Q'(n+m+1)} \tag{14}$$

with centerline $CL = n\left(1 - \frac{r}{m+1}\right)$, where the subscript "$s$" denotes the steady-state control limits and $Q' = \lim_{t\to\infty} \sum_{i=1}^{t} w_i^2$.

The following points are worth mentioning here:

i. The main focus of this study is to construct a DGWMA-EX chart with control limits equidistance from the centerline. One can also design a one-sided chart depending on the purpose or application;

ii. Steady-state control limits are used in order to simplify the application and implementation of the DGWMA-EX chart. Hence, hereafter we use LCL and UCL to denote the steady-state control limits in Equation (14);

iii. If any plotting statistic Z_t^2, plots on or outside either of the control limits (steady-state) given in Equation (14), the process is declared out-of-control (OOC) and a search for assignable causes is started. Otherwise, the process is considered to be in-control (IC), which implies no location shift has occurred;

iv. For more information in terms of precedence or exceedance type tests and their distributional properties, please refer to Balakrishnan and Ng[25];

v. Since the DGWMA-EX chart has four parameters, the computational aspects can become complex and time consuming. However, choosing specific values for some of the parameters reduces the number of unknown DGWMA-EX parameters and simplifies the implementation of the proposed chart. In this article, we will consider the DGWMA-EX chart with $q_1 = q_2 = q$ and $\alpha_1 = \alpha_2 = \alpha$. For brevity we denote this chart by *DGWMA-EX* $(q; \alpha)$;

vi. Sheu and Hsieh[9] mentioned that the DGWMA chart with four parameters does not perform better than the DGWMA with two parameters. However, it was discovered that there exist DGWMA-EX charts with four parameters that outperforms the DGWMA-EX chart with two parameters. We first computed the in-control *ARL* (ARL_0) for these two charts to ensure both of them are at an equal footing. The ARL_0 of the *DGWMA-EX* ($q_1, \alpha_1; q_2, \alpha_2$) is 370.47 and for the *DGWMA-EX* ($q; \alpha$) is 371.34. We computed the out-of-control *ARL* (ARL_1) for some combinations of q_1, q_2, α_1, α_2 and as a result for some combinations of the aforementioned parameters the DGWMA with four parameters outperforms the DGWMA with two parameters; this is due to the flexibility that is gained by using additional parameters. For example, for $q_1 = 0.8$, $q_2 = 0.7$, $\alpha_1 = 0.9$, $\alpha_2 = 0.7$ and $L=$ 1.984, the OOC *ARL* is equal to $ARL_1 = 348.78$ and $ARL_1 = 107.09$, for shift sizes (δ) 0.05 and 0.25, respectively. For $q_1 = q_2 = q = 0.8$, $\alpha_1 = \alpha_2 = \alpha = 0.9$ and $L=$ 1.925, the OOC *ARL* is equal to $ARL_1 = 364.05$ and $ARL_1 = 111.66$, for shift sizes (δ) 0.05 and 0.25, respectively. The first set of design parameters are referring to the DGWMA with four parameters while the latter one are referring to the DGWMA with two parameters;

vii. The GWMA-EX and EWMA-EX charts are limiting cases of the proposed DGWMA-EX chart. For example, in the DGWMA-EX chart, if we set $q_2 = 0$ and $\alpha_2 = 1$, then the chart simplifies to a GWMA-EX chart with the parameters q_1 and α_1 denoted as *GWMA-EX* (q_1, α_1). In the GWMA-EX chart, if one sets $\alpha_1 = 1$, then it simplifies to the EWMA-EX chart denoted as *EWMA-EX* (q_1). The same result can be obtained, if we set $\alpha_1 = \alpha_2 = 1$ and $q_2 = 0$, then the charting statistic of the DGWMA-EX reduces to the charting statistic of the EWMA-EX chart denoted as *EWMA-EX* (q_1). Hence, as a conclusion, the EWMA-EX chart can be regarded as limiting case for the DGWMA-EX chart, and as a special case of the GWMA-EX chart;

viii. Shamma and Shamma[6] designed a DEWMA chart for the mean. Zhang and Chen [10] have shown that the proposed chart performs better than the EWMA chart for the mean when the process shifts are small. For larger shifts, the DEWMA chart and the EWMA chart perform similarly. We also introduce the nonparametric DEWMA chart (Case U) labeled as DEWMA-EX control chart in this paper which is a special case of the DGWMA-EX chart. Note that, as mentioned by Zhang and Chen[10], there are two cases for the DEWMA chart based on the equality and/or inequality of the smoothing parameters ($\lambda_1 = 1 - q_2$, $\lambda_2 = 1 - q_1$). These two cases of the DEWMA-EX chart are denoted as *DEWMA-EX* ($\lambda =$

$1 - q$) and *DEWMA-EX* ($\lambda_1 = 1 - q_2$, $\lambda_2 = 1 - q_1$), respectively. In the DGWMA-EX chart, if one sets $\alpha_1 = \alpha_2 = 1$, the outcome will be the DEWMA-EX with parameters q_1 and q_2, denoted as *DEWMA-EX* $(1 - q_2, 1 - q_1)$. Zhang and Chen[10] concluded that the DEWMA chart with equal smoothing parameters performs similarly comparing to the DEWMA chart with different smoothing parameters.

3. Implementation and performance

The average run length (*ARL*) is the most important and widely used metric to evaluate the performance of control charts. The performance of a control chart can be evaluated in terms of two *ARL* values:

- ARL_0: the average number of charting statistics until an OOC signal is detected by a control chart when the process is in-control;
- ARL_1: the average number of charting statistics until an OOC signal is detected by a control chart when the process has shifted to an OOC value.

The design of the DGWMA-EX chart typically involves the calculation of the chart parameters so as to obtain a pre-specified in-control ARL (denoted by ARL_0^*) i.e. one wants to solve for the values α_1, α_2 , q_1, q_2 and L such that $ARL_0 \approx ARL_0^*$. In order to make the computational aspects easier, some of the design parameters are set equal to each other, hence $q_1 = q_2 = q$ and $\alpha_1 = \alpha_2 = \alpha$. However, since $X_{(r)}$ is a random variable, computation of the run length distribution for the DGWMA-EX chart is not straightforward. The three standard methods that are often used to evaluate or calculate the *ARL* and that will be investigated in this article are: (i) the exact approach; (ii) the Markov chain approach and, (iii) Monte Carlo simulation.

Exact approach

One can denote K as the run length random variable for the DGWMA-EX chart. Suppose that the signaling event at the i^{th} sample is denoted by S_i. For $\forall i \geq 1$, one can re-write the event $S_i^C = [LCL < Z_i^2 < UCL]$, as $S_i^C = [LC_i < U_i < UC_i]$, where for $i = 2,3,\ldots$,

$$\begin{cases} UC_i = \dfrac{UCL - \sum_{j=2}^{i} w_j U_{i-j+1} - \left(1 - \sum_{j=2}^{i} w_j\right) n(1 - \frac{r}{m+1})}{(1-q_1)(1-q_2)} \\ LC_i = \dfrac{LCL - \sum_{j=2}^{i} w_j U_{i-j+1} - \left(1 - \sum_{j=2}^{i} w_j\right) n(1 - \frac{r}{m+1})}{(1-q_1)(1-q_2)} \end{cases} \quad (15)$$

where $UC_1 = \frac{UCL-(1-w_1)n(1-\frac{r}{m+1})}{(1-q_1)(1-q_2)}$, $LC_1 = \frac{LCL-(1-w_1)n(1-\frac{r}{m+1})}{(1-q_1)(1-q_2)}$ and UCL and LCL are the steady-state control limits defined in Equation (14).

The conditional *ARL* can be written as:

$$ARL|X_{(r)} = \sum_{k=1}^{\infty} I_k \tag{16}$$

where $I_k = \sum_{LC_1}^{UC_1} \sum_{LC_2}^{UC_2} \ldots \sum_{LC_k}^{UC_k} (\prod_{i=1}^{k} P[U_i = u_i | X_{(r)}])$ for $k = 1, 2, 3, \ldots$, $I_0 = 1$ (see Appendix A3) and $P[U_i = u_i|X_{(r)}]$ is given in Appendix A1.

The unconditional *ARL* is, therefore:

$$ARL = E_{X_{(r)}}\left(ARL\middle|X_{(r)}\right) = 1 + \sum_{k=1}^{\infty} E_{X_{(r)}}(I_k). \tag{17}$$

By obtaining $E_{X_{(r)}}(I_k)$ (see Appendix A2), the closed form expression of the unconditional *ARL* is:

$$ARL = 1 + \sum_{k=1}^{\infty} \sum_{LC_1}^{UC_1} \sum_{LC_2}^{UC_2} \ldots \sum_{LC_k}^{UC_k} \left(\prod_{i=1}^{k} \left(\frac{\binom{u_i+m-k}{u_i}\binom{n-u_i+k-1}{n-u_i}}{\binom{m+n}{n}} \right) \right) \tag{18}$$

The following points need to be taken into account when evaluating Equation (18):

i. The closed form expression consists of multiple series and as k inceases, the number of series increase which makes the expression cumbersome to evaluate computationally;
ii. The "IC robustness" property is referred to a control chart based on the exceedance (precedence) statistic which is distribution-free when the process is declared IC. Hence, evaluation of Equation (18) does not require any prior knowledge regarding the distribution of the underlying process when the process is IC.

Markov chain approach

The Markov chain approach is another method that is widely applied in the context of control charts to evaluate the run length distribution and its characteristics. However, due to the complexities of implementing the Markov chain approach raised by Chakraborty et al.[23], calculating the run length distribution and enumerating the states spaces utilizing the Markov chain approach is difficult.

Monte Carlo simulation approach

A numerical Monte Carlo simulation has been implemented in this study to estimate the unconditional run length distribution and its characteristics for the DGWMA-EX chart. Furthermore, to make the calculation easier and less time consuming, as mentioned earlier, we set $q_1 = q_2 = q$ and $\alpha_1 = \alpha_2 = \alpha$. The simulation algorithm includes the following steps:

i. Select a combination of the design parameters, i.e., (q, α, L), the shift to be detected denoted by δ, the reference and test sample sizes $m \geq 1$ and $n \geq 1$, the IC distribution parameter θ_0 and identify a process distribution $F_X(x)$; the latter is only used to investigate the out-of-control run length distribution;

ii. Obtain the r^{th} order statistic $X_{(r)}$ by generating a reference sample of size m from the identified process distribution $F_X(x)$;

iii. A test sample of size $n \geq 1$ is generated to calculate the exceedance statistic U_i by counting the number of observations $Y's$ in the i^{th} sample that met the constraint $Y \geq X_{(r)}$. The test sample is drawn from $F_X(x - \theta_1)$. One needs to note that when an IC run length distribution is desired then $\theta_1 = \theta_0$, whereas $\theta_1 = \theta_0 + \delta$ referred to as an OOC run length;

iv. Calculate the steady-state control limits defined in Equation (14) by using the design parameters values (q, α, L) obtained from step i;

v. Calculate the DGWMA-EX charting statistic Z_t^2 according to Equation (10) with the starting value taken as $\mathrm{Z}_0^2 = n\left(1 - \frac{r}{m+1}\right)$ and compare each plotting statistic with the steady-state control limits obtained from step iv;

vi. After running 10,000 iterations of the steps (i) to (v), the number of samples until the first plotting statistic falls on or outside the steady-state control limits, known as the run length, is calculated for each of the interactions. These 10,000 empirical run length values are then used to calculate the average run length and other characteristics for the run length.

3.1. The in-control (IC) design

The in-control design of the proposed DGWMA-EX chart consists of obtaining the values for the charting constant, i.e. $L > 0$ for chosen values of m (known as the reference sample size) and n (known as the test sample size) and a certain range of values for each (q, α) combination, so that the attained IC ARL is close to the desirable value ARL^* which is typically 370 or 500. Sheu and Hsieh[9], Tai et al.[11] and Huang et al.[12] noted that (q, α) combinations in the intervals $0.5 \leq q \leq 0.9$ and $0.5 \leq \alpha \leq 1.0$ enhanced the sensitivity of the DGWMA-$\bar{X}$ chart and outperformed the GWMA-$\bar{X}$, DEWMA-$\bar{X}$ and EWMA-$\bar{X}$ charts for small shifts. Chakraborty et al.[23] considered $m = 49$ and 99 and $n = 5$ and 10 as the values for the reference sample and test sample sizes, respectively and the following range of values for the GWMA-EX parameters: $q = 0.8, 0.9, 0.95$ and $\alpha = 0.7, 0.8, 0.9, 1.0, 1.3$. In order to make the comparison procedure fair and reliable, in our study, we also considered the same aforementioned values for m, n, q and α. By implementing the simulation algorithm alongside a grid search method, the charting constant $L > 0$ for the chosen (q, α) combination and specified values of m and n, based on the constraint that $ARL_0^* = 370$, was obtained. The values of $L > 0$ are reported for the DGWMA-EX chart (Tables 1 and 2) and the GWMA-EX, and EWMA-EX chart (Tables 3 and 4) along with the attained ARL_0 values.

To ensure our simulation yields reasonable and consistent results and ensure the validity of the algorithm developed in R, we compared our results to those obtained by Chakraborty et al.[23]. For instance, consider two following scenarios:

i. When $m = 49$ and $n = 10$, we have $q_1 = 0.95$, $q_2 = 0$, $\alpha_1 = 0.7$ and $\alpha_2 = 1$, we find from Table 3 that a value of charting constant $L = 0.737$ gives an attained $ARL_0 = 370.03$. In Chakraborty, et al.[23], the GWMA-EX chart with $q = q_1 = 0.95$ and $\alpha = \alpha_1 = 0.7$ and $L = 0.738$ has an attained $ARL_0 = 370.05$.

ii. When $m = 99$ and $n = 5$, we have $q_1 = 0.9$, $q_2 = 0$, $\alpha_1 = 0.7$ and $\alpha_2 = 1$, we find from Table 4 that a value of charting constant $L = 1.805$ gives an attained $ARL_0 = 370.49$. In Chakraborty, et al.[23], the GWMA-EX chart with $q = q_1 = 0.9$ and $\alpha = \alpha_1 = 0.7$ and $L = 1.807$ has an attained $ARL_0 = 370.58$.

The charting constant values in Tables 1, 2, 3 and 4 will be useful for the design and implementation of the DGWMA-EX chart; this includes designing and implementing the GWMA, DEWMA and EWMA exceedance charts.

The main objective of this paper is to focus on the median of the Phase I reference sample i.e. where $X_{(r)}$ is the median of the Phase I sample. However, a short performance analysis is also conducted for the proposed DGWMA-EX using the 25th and 75th percentiles of the Phase I sample. This is discussed in more detail in the next section. However, based on the observed results, the recommendation would be to use the median of the Phase I sample for the DGWMA-EX chart, since the median is a robust measure of the central tendency of distributions and practitioners are more interested in the median. Hence, in this section a general guideline is provided for practitioners on the design of the DGWMA-EX chart.

3.2. The out-of-control (OOC) performance

The preliminary step to evaluate the OOC performance is to ensure that the ARL_0's are close to 370 (when no shift occurs) so that all the charts are at an equal footing. Once different competing charts are designed with equal ARL_0, a chart with the smaller ARL_1 provides better performance for practical applications.

The results for the OOC performance comparisons are shown in Tables 1, 2, 3 and 4 for multiple combinations of the parameters (q, α) as well as for some chosen or specified values of m, n, and δ. Tables 1 and 2 refer to the DGWMA-EX control chart and Tables 3 and 4 correspond to the GWMA-EX and EWMA-EX charts when $m = 49, 99$ and $n = 5, 10$. The optimal design for the proposed DGWMA-EX chart would consists of specifying the desired ARL_0 and ARL_1 values as well as the magnitude of the process shift and then select the combination of design parameters that provides the desired ARL_0 with the minimum ARL_1. For instance, in Table 1, the combination ($q = 0.8$, $\alpha = 0.7$, $L = 1.304$) has the minimum $ARL_1 = 317.48$ among the chosen range of parameters for shift size $\delta = 0.1$, for $m = 49$ and $n = 5$. Since the IC distribution of the exceedance statistic is symmetric when $X_{(r)}$ is selected as the median, for OOC performance (ARL_1), only the positive shifts $\delta = 0.05, 0.1, 0.25, 0.5, 0.75, 1.0, 1.5$ are considered. The main objective of this study is on efficiency of time-weighted/memory-based charts in detecting tiny shifts.

A quick comparison of the results advocates the following points:

i. The DGWMA-EX chart, typically outperforms the GWMA-EX chart when the adjustment parameter ($\alpha < 1$) for $\delta \leq 0.5$. For example, in order to detect a shift of $\delta = 0.1$, a DGWMA chart with $q = 0.9$, $\alpha = 0.8$, $L = 0.924$ has an $ARL_1 = 330.57$ whereas the GWMA-EX with $q_1 = 0.9$, $q_2 = 0$, $\alpha_1 = 0.8$ and $\alpha_2 = 1$ and $L = 1.596$ has an $ARL_1 = 336.84$ when $m = 49$ and $n = 5$. However, in some cases, such as the DGWMA-EX chart with $q = 0.95$ and $\alpha = 0.7, 0.8, 0.9$ are worse than the GWMA-EX chart when $m = 49$ and $n = 5$.

ii. The DGWMA-EX chart, generally performs better than the EWMA-EX chart for $\delta \leq 0.25$. For example, in order to detect a shift of $\delta = 0.05$, a DGWMA-EX with $q =$

0.8, $\alpha = 1.0$, $L = 1.755$ has $ARL_1 = 360.95$ whereas the EWMA-EX chart with $q_1 = 0.8$, $q_2 = 0$, $\alpha_1 = 1$ and $\alpha_2 = 1$ and $L = 2.249$ has $ARL_1 = 366.63$ when $m = 49$ and $n = 5$.

iii. Overall, for small to moderate shift, the DGWMA-EX chart works better than the GWMA-EX and the EWMA-EX charts. For example, when $q = 0.8$, $\alpha = 0.8$, $m = 49$ and $n = 10$, a comparative plot is illustrated in Figure 1 to compare the *ARL* performance and detection ability between the DGWMA-EX, the GWMA-EX and the EWMA-EX charts. One can clearly observe that the DGWMA-EX chart outperforms the other counterparts for small shifts.

Furthermore, the effects of the parameters q, α and n on the OOC performance of the DGWMA-EX are investigated as well. The results for the aforementioned are presented in Figures 2, 3 and 4 respectively. In Figure 2, for $\alpha = 1.3$, $m = 99$ and $n = 10$, three different values for q (0.8, 0.9 and 0.95) are selected and based on the results, larger value of q, has better OOC performance for the DGWMA-EX chart as a consequence. In Figure 3, for $q = 0.8$, $m = 99$ and $n = 5$, three different values for α (0.7, 1 and 1.3) are considered and based on the results, smaller values of α, leads to better OOC performance for the DGWMA-EX chart. In Figure 4, for $q = 0.8$, $\alpha = 1$, $m = 49$, two different values are selected for the test sample size n (5 and 10) and based on the results, the larger the test sample size, the better the OOC performance of the DGWMA-EX chart.

Other characteristics of the run length distribution including the standard deviation (denoted by *SDRL*) and percentile points (denoted by P_i), where i = 5, 25, 50, 75, 95 might be of interest for the practitioners. Results are available upon request from the authors.

In practice one can be interested in selecting an r^{th} order statistic from the Phase I sample other than considering the median. Hence, we conducted a comparative study for the DGWMA-EX chart using the 75$^{\text{th}}$ and 25$^{\text{th}}$ percentiles as well. For $X_{(r)} = 25^{\text{th}}$ percentile, the run length distribution encounters bias, that is ARL_1 is greater than ARL_0 which makes the performance of the control chart worse than in the median case. For $X_{(r)} = 75^{\text{th}}$ percentile, there is a considerable improvement in terms of the run length distribution for each choice of design parameters (q, α, L) and shift size δ. The relative results are presented in Table 5 for the DGWMA-EX chart and in Table 6 for the GWMA-EX and EWMA-EX charts, when $m = 49$ and $n = 5$.

A performance study for the DGWMA-EX chart based on the median run length (MRL) was performed by taking $X_{(r)}$ as the 75th, 50th and 25th percentiles. The reference sample size is taken as $m = 100$, the test sample size is taken as $n = 5$ and a typical value for the MRL is taken as $MRL_0^* = 350$. For given m, n and (q, α), we obtain L values so that the attained $MRL_0^* = 350$ when $X_{(r)}$ is taken as the 75th, 50th and 25th percentiles. These results are reported in Tables 7 and 8 for the DGWMA-EX, GWMA-EX and EWMA-EX charts respectively and they show similar results as in the ARL study. When $X_{(r)}$ is selected as the 25th percentile, it has poorer performance than $X_{(r)}$ =50th percentile and the problem of bias in the run length distribution still remains as a major issue. Hence, there is no significant improvement observed in performance when the study is based on the MRL.

As a conclusion, the median is known to be a better percentile whenever the direction of the shift to be detected, is not specified, and is thus recommended to practitioners.

The DGWMA charts are more sensitive and detect a shift quicker than its main time-weighted counterpart the GWMA chart in the case of a small or tiny shift, see for example, Sheu and Hsieh[9], Huang et al.[12], Lu[18] and the references therein. It is therefore logical to compare the OOC performance of the proposed DGWMA-EX chart with the DGWMA-$\bar{X}$, GWMA-$\bar{X}$, GWMA-EX and EWMA-EX charts under the normal and a number of non-normal distributions when parameter of interest is unknown (Case U). Therefore, three non-normal symmetric (around zero) process distributions are considered which have heavier tails or lighter tails than the normal distribution. We considered the logistic (0, $\sqrt{3}/\pi$) distribution, the uniform (-$\sqrt{3}$, $\sqrt{3}$) distribution and the Laplace (0,1/$\sqrt{2}$) distribution. The parameters of these distribution are selected in such a manner that the variance is 1, which makes the results comparable amongst different distributions. For skewed distributions, we considered the gamma distribution with shape parameters 1, 2 and 3 and scale parameter set equal to 1 in each case.

The OOC performance results are summarized in the following sections.

(a) DGWMA-EX chart vs. GWMA-EX, EWMA-EX, DGWMA-$\bar{X}$ and GWMA-$\bar{X}$ chart under symmetric distributions

From the results in Tables 1, 2, 3 and 4, it is advocated that the DGWMA-EX chart generally outperforms the GWMA-EX and EWMA-EX under the standard normal distribution. However, the rigid assumption of normality might not hold in all cases and hence it is vital to evaluate the performance of the DGWMA-EX chart under non-normal distributions. For comparison purpose, the reference sample size is taken as $m = 49$, the test sample size is $n = 5$, and the design parameters are selected as $q_1 = q_2 = q = 0.8$, $\alpha_1 = \alpha_2 = \alpha = 0.7$ and $L = 1.304$ for the DGWMA-EX chart. Table 9 illustrates that for the aforementioned combination, the DGWMA-EX chart performs better than the GWMA-EX and EWMA-EX charts under non-normal symmetric distributions. For instance, when the process follows a logistic (0, $\sqrt{3}/\pi$) distribution and shift size $\delta = 0.1$, the DGWMA-EX chart with parameters $q = 0.8$, $\alpha =$, 0.7 and $L = 1.304$ has $ARL_1 = 306.82$, while the GWMA-EX chart with parameters $q_1 = 0.8$, $q_2 = 0$, $\alpha_1 = 0.7$, $\alpha_2 = 1$ and $L = 2.032$ has $ARL_1 = 314.89$ and the EWMA-EX chart with parameters $q_1 = 0.8$, $q_2 = 0$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $L = 2.249$ has $ARL_1 = 316.07$. When the process follows a uniform (-$\sqrt{3}$, $\sqrt{3}$) distribution assuming a shift size of $\delta = 0.25$, the DGWMA-EX chart with parameters $q = 0.8$, $\alpha =$, 0.7 and $L = 1.304$ has $ARL_1 = 235.28$, whereas the GWMA-EX chart with parameters $q_1 = 0.8$, $q_2 = 0$, $\alpha_1 = 0.7$, $\alpha_2 = 1$ and $L = 2.032$ has $ARL_1 = 251.50$ and the EWMA-EX chart with parameters $q_1 = 0.8$, $q_2 = 0$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $L = 2.249$ has $ARL_1 = 255.67$. For the Laplace (0,1/$\sqrt{2}$) distribution and same set of parameters considered for the logistic and uniform distributions and shift size $\delta = 0.05$, the OOC ARL (ARL_1) is 319.88, 327.60 and 333.09 for the DGWMA-EX, GWMA-EX and EWMA-EX charts, respectively.

Now, similarly to Sheu and Lin[7], we conducted a comparative study to compare the performance of the DGWMA-EX chart with the DGWMA-$\bar{X}$ and GWMA-$\bar{X}$ charts under the assumption of an underlying normal distribution specifically for Case U.

The parameters q, α for all the time-weighted control charts included in the comparative analysis are taken to be the same, since the main intention is to see whether the same (q, α) combination provides similar robust performance under different non-normal symmetric distributions when $\bar{X}$ is replaced by the exceedance statistic in the DGWMA chart. The mechanism for designing parametric control charts for Case U is to use an IC Phase I sample and obtaining the estimates for the unknown process parameters. Thereafter, these estimates will be used to obtain the control limits and as well as studying the performance of the run length characteristics. Table 9 reveals that, under the normality assumption the DGWMA-$\bar{X}$ chart outperforms DGWMA-EX, GWMA-EX and EWMA-EX charts, which is an expected outcome since the DGWMA-$\bar{X}$ chart is designed under the normality assumption. However, when the process distribution departs from normality, the behavior of the DGWMA-$\bar{X}$ chart is influenced and its attained ARL_0 starts moving further from the standard value 370. For the logistic distribution this does not hold, since the IC ARL does not depart that further from 370 when the underlying process distribution is not normal. For this specific distribution, the attained ARL_0 for DGWMA-$\bar{X}$ chart is 367.04, whereas for the uniform and the Laplace distributions the attained ARL_0 is 396.90 and 391.43, respectively. On the contrary, the nonparametric counterpart DGWMA-EX is IC robust under non-normality. Hence, when the underlying process distribution is either unknown or cannot be identified, the DGWMA-EX chart is a better alternative since it is IC robust under non-normality whereas the DGWMA-$\bar{X}$ chart is non-robust.

Furthermore, the robust IC and OOC performances for the DGWMA-EX chart under normal and symmetric non-normal distributions are presented in Figure 5.

(b) DGWMA-EX chart vs. GWMA-EX, EWMA-EX, DGWMA-$\bar{X}$ and GWMA-$\bar{X}$ under skewed distributions

In this section, we study the performance of the DGWMA-EX, GWMA-EX, EWMA-EX, DGWMA-$\bar{X}$ and GWMA-$\bar{X}$ charts for underlying skewed distributions. For this purpose, the $Gamma(k, \theta)$ distribution is considered as the underlying model. The probability density function (p.d.f.) of X is given by:

$$f(x; k, \theta) = \frac{e^{-x/\theta} x^{k-1}}{\Gamma(k)\theta^k}, \qquad x > 0\,, \theta > 0 \text{ and } k > 0 \tag{19}$$

The following is worth noting regarding the gamma distribution:

i. The parameters k and θ are known as the shape and the scale parameters;

ii. Under the gamma distribution, the mean and the variance are functions of parameters k and θ;

iii. For a given value of the shape parameter k, the scale parameter θ would effect change in both mean and variance. Hence, for the gamma distribution it is not possible to assume mean 0 and variance 1 as in the study pertaining to symmetric distributions.

The IC and OOC scale parameters are denoted as θ_0 and θ_1 respectively. Note that the shift for the gamma distribution is defined as $\delta = \theta_1/\theta_0$ which is different than for symmetric distributions considered in the previous section. The reason is as follows: If X ~ gamma (k, θ), then Y = X/(θ) ~ gamma$(k, 1)$. In other words, the IC scale parameter can be taken as 1 and hence the shift which is defined as the ratio between θ_1 and θ_0 ($\delta = \theta_1/\theta_0$) is equal to the OOC scale parameter ($\delta = \theta_1$). Hence, X/θ_1 and Y/(δ) ~ gamma$(k, 1/(\delta))$, have the same distribution as long as the ratio δ stays the same. However, for the absolute difference between the IC and OOC scale parameters which is defined as $|\theta_1 - \theta_0|$, the effect of the shift depends on the magnitude of θ_0. Therefore, considering $\theta_0 = 1$ would make the chart applicable for any IC θ_0, whereas the OOC performance differs based on different values for θ_0 and θ_1. For the IC process the shift value is considered as 1 ($\delta = 1$) and for the OOC the values are $\delta = 0.975$, 0.95, 0.9, 0.8, 0.7. Note that, as mentioned by Chakraborty et al.[23] for the GWMA-$\bar{X}$ chart, the control limits used for the normal distribution in the case of the DGWMA-$\bar{X}$ chart (Case U) are unsuitable for the gamma distribution since the mean and the variance are no longer 0 and 1, respectively.

In order to calculate the control limits for the DGWMA-$\bar{X}$ chart, the estimation of both the process mean (μ) and the standard deviation (σ) from the IC Phase I sample is required. Thereafter, these estimates denoted by $\hat{\mu}$ and $\hat{\sigma}$ can be used to obtain the control limits. Results for the gamma distribution are presented in Table 10 which reveals that the DGWMA-$\bar{X}$ is not IC robust and the issue related to the bias of the run length distribution exist. For example, for the DGWMA-$\bar{X}$ chart with $q = 0.8$, $\alpha = 0.7$, $L = 2.992$, $m = 49$ and $n = 5$ has $ARL_0 = 436.02$ for gamma(1,1) distribution, $ARL_0 = 441.70$ for gamma(2,1) distribution and $ARL_0 = 432.24$ for gamma(3,1) distribution. Furthermore, when the shape parameters $k = 1$, 2, 3, the DGWMA-EX chart outperforms the GWMA-EX and EWMA-EX chart for all shift $\delta \geq 0.7$. The only exception is for case of $k = 3$ and $\delta = 0.7$ where GWMA-EX chart outperforms the DGWMA-EX and EWMA-EX charts. The IC and OOC *ARL* performance for the DGWMA-EX chart under the gamma distribution with different shape parameters is presented in Figure 6. Based on the illustration, the DGWMA-EX chart with larger shape parameter performs better than others.

4. Illustrative example

In this section, we present a simulated example to demonstrate the applicability of the proposed DGWMA-EX chart. We draw a reference sample of size $m = 49$ from a standard normal (N(0,1)) distribution as a Phase I dataset in order to estimate the process median. Thereafter, we draw 200 Phase II random samples of size $n = 5$, from a N(0.25,1) distribution which can be viewed as an OOC observations following a location shift of $\delta = 0.25$. Two sets of design parameters are used: ($q = 0.8$, $\alpha = 0.7$, $L = 1.304$) and ($q_1 = 0.8$, $q_2 = 0, \alpha_1 = 0.7$, $\alpha_2 = 1$, $L = 2.032$) as in Tables 1 and 3. The first set results in a DGWMA-EX chart whereas the second one results in a GWMA-EX chart. Note that, any other combination can be chosen, however these values are chosen only for the illustration purposes. The in-control ARL (ARL_0) for both charts are close to 370 which put them at equal footing in order to perform a valid comparison. From Table 1, the DGWMA-EX chart has an OOC ARL of 163.35 while from Table 3 the GWMA-EX chart has an OOC ARL of 182.06 when $\delta = 0.25$. Control limits for the DGWMA-EX chart are obtained as UCL = 3.008 and LCL = 1.991, whereas for the GWMA-EX chart these limits are obtained as UCL = 3.437 and LCL = 1.562. The two control charts are displayed in Figure 7. As a conclusion, the DGWMA-EX chart detects the shift $\delta = 0.25$ (small shift) much quicker than the GWMA-EX chart which provides similar results as those presented in Tables 1 and 3.

5. Synopsis and main conclusions

Nonparametric control charts offer an efficient technique to monitor a process, even if the form of the underlying distribution is unknown or not exactly specified. The performance of the DGWMA-$\bar{X}$ and GWMA-$\bar{X}$ charts become worse under skewed distributions when the process distribution is unknown. A new distribution-free (nonparametric) control chart based on an exceedance statistic, denoted as the DGWMA-EX chart, is introduced. This chart provides a method for monitoring when no information is available with regards to the process distribution as well as the process median. A performance comparison of the DGWMA-EX chart is done with its competitors: the GWMA-EX and EWMA-EX charts. The results reveal that the proposed chart is robust to non-normality when the process is IC and in many instances, performs better than the existing GWMA and EWMA charts based on exceedance statistics when the shift is small. This is due to the fact that DGWMA chart take advantage of the sequential (time ordered) accumulation of all the information from the start until the most recent observation, and is known to be more efficient in detecting smaller shifts as showed in this paper as well.

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Table 1: DGWMA-EX IC and OOC ARL for different combinations of (q, α, L) for different δ when m = 49, n = 5,10 with $ARL_0^* = 370$ under a standard normal distribution.

				δ							
m = 49	$q_1 = q_2 = q$	$\alpha_1 = \alpha_2 = \alpha$	L	**0.00**	**0.05**	**0.10**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**
n = 5	0.80	0.70	1.304	368.93	358.68	317.48	163.35	28.39	11.55	8.41	6.29
		0.80	1.480	367.80	368.06	325.90	182.81	28.87	10.92	8.02	6.15
		0.90	1.621	371.58	361.22	323.36	179.03	28.68	10.43	7.62	5.94
		1.00	1.755	369.77	360.95	328.84	183.09	30.99	10.14	7.26	5.63
		1.30	2.056	369.95	360.69	331.75	184.35	35.17	10.38	6.69	5.27
	0.90	0.70	0.741	372.72	354.72	320.52	171.21	35.87	16.89	12.92	10.19
		0.80	0.924	375.32	365.79	330.57	173.23	35.06	15.72	12.14	9.77
		0.90	1.092	371.87	352.77	320.63	169.71	31.63	14.27	11.30	9.19
		1.00	1.255	370.17	355.64	318.89	172.63	31.00	13.36	10.51	8.66
		1.30	1.668	371.38	355.21	319.10	178.82	32.08	11.65	8.86	7.40
	0.95	0.70	0.395	372.29	361.49	337.56	187.68	44.70	25.36	20.18	16.22
		0.80	0.543	366.70	355.16	325.94	176.63	41.76	23.21	18.82	15.40
		0.90	0.706	372.00	362.06	325.40	177.90	37.01	21.26	17.37	14.45
		1.00	0.866	372.19	358.30	332.25	170.47	36.56	19.27	15.91	13.35
		1.30	1.315	370.08	364.00	322.13	173.84	31.62	15.50	12.48	10.69
n = 10	0.80	0.70	1.031	372.19	344.35	327.62	149.35	22.06	8.50	6.22	4.83
		0.80	1.177	372.48	356.12	310.97	156.17	23.76	8.30	6.01	4.80
		0.90	1.311	369.39	353.90	329.35	161.38	23.03	7.74	5.78	4.65
		1.00	1.436	369.62	355.72	322.12	171.27	22.58	7.36	5.58	4.49
		1.30	1.750	370.73	357.21	330.01	181.92	23.53	6.95	5.06	4.17
	0.90	0.70	0.579	367.11	338.97	327.41	161.36	28.11	12.83	10.13	8.06
		0.80	0.721	369.95	359.12	312.87	155.61	26.67	11.98	9.66	7.82
		0.90	0.863	372.19	365.07	326.26	167.24	26.97	11.22	9.06	7.48
		1.00	1.002	368.31	368.67	331.25	164.02	24.82	10.47	8.58	7.19
		1.30	1.370	369.32	359.48	323.10	172.19	25.24	9.04	7.24	6.21
	0.95	0.70	0.315	370.91	349.41	326.96	170.71	36.62	20.63	16.54	13.40
		0.80	0.437	371.53	354.16	325.77	175.02	33.30	19.24	15.74	13.02
		0.90	0.559	372.13	357.84	328.01	173.88	31.64	17.73	14.52	12.18
		1.00	0.682	370.64	353.22	324.58	168.89	29.93	15.72	13.28	11.32
		1.30	1.053	374.46	364.55	327.33	169.36	27.20	12.36	10.56	9.21

Table 2: DGWMA-EX IC and OOC ARL for different combinations of (q, α, L) for different δ when m = 99, n = 5,10 with $ARL_0^* = 370$ under a standard normal distribution.

				δ							
m = 99	$q_1 = q_2 = q$	$\alpha_1 = \alpha_2 = \alpha$	L	**0.00**	**0.05**	**0.10**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**
n = 5	**0.80**	0.7	1.611	369.92	344.78	286.14	103.64	17.99	10.36	7.86	6.09
		0.8	1.780	370.13	346.71	291.70	105.24	17.33	9.88	7.57	5.94
		0.9	1.925	371.34	364.05	304.72	111.66	17.55	9.47	7.21	5.62
		1.0	2.040	371.39	356.12	303.83	114.15	17.20	9.08	6.94	5.53
		1.3	2.300	370.69	353.59	309.00	127.80	19.76	8.63	6.31	5.18
	0.90	0.7	0.939	373.73	352.11	285.50	104.44	23.38	15.05	11.94	9.53
		0.8	1.152	370.01	346.53	290.55	101.78	21.52	14.01	11.32	9.22
		0.9	1.360	372.73	351.93	305.23	101.90	20.58	13.14	10.73	8.83
		1.0	1.529	369.60	348.89	291.79	102.01	19.15	12.20	10.01	8.40
		1.3	1.946	371.96	351.24	304.25	112.07	18.91	10.45	8.51	7.26
	0.95	0.7	0.489	371.20	357.08	298.75	112.26	32.41	22.21	18.12	14.71
		0.8	0.683	372.97	355.06	294.84	108.00	30.08	20.98	17.37	14.40
		0.9	0.884	371.31	347.67	295.12	108.40	27.35	19.39	16.25	13.65
		1.0	1.075	371.80	346.03	291.16	106.49	24.84	17.76	14.97	12.75
		1.3	1.582	370.13	347.04	295.09	106.89	21.02	13.97	11.98	10.41
n = 10	**0.80**	0.7	1.313	370.24	345.21	294.05	82.13	12.55	7.45	5.83	4.52
		0.8	1.472	371.02	344.40	285.89	83.40	11.77	7.13	5.70	4.52
		0.9	1.622	370.24	348.83	289.50	86.90	11.20	6.83	5.51	4.44
		1.0	1.757	371.47	358.09	303.03	91.98	11.14	6.57	5.33	4.34
		1.3	2.069	371.32	359.66	315.28	101.30	11.85	6.01	4.87	4.11
	0.90	0.7	0.742	370.71	350.76	294.16	88.90	17.62	11.59	9.37	7.55
		0.8	0.921	369.46	356.95	287.49	87.61	16.11	10.97	9.02	7.40
		0.9	1.094	372.48	350.18	293.27	87.23	15.08	10.29	8.57	7.18
		1.0	1.256	370.89	351.32	287.41	89.22	14.16	9.67	8.14	6.90
		1.3	1.677	370.64	357.64	298.35	91.44	12.65	8.20	6.99	6.10
	0.95	0.7	0.392	370.28	347.27	287.48	96.89	26.22	18.14	14.87	12.16
		0.8	0.545	372.88	357.67	291.85	92.32	23.88	17.19	14.41	12.00
		0.9	0.708	372.45	357.91	290.34	93.83	21.70	15.97	13.58	11.47
		1.0	0.862	369.38	346.96	297.90	87.25	19.67	14.62	12.55	10.74
		1.3	1.310	372.44	349.89	292.37	87.75	15.72	11.61	10.13	8.94

Table 3: GWMA-EX and EWMA-EX IC and OOC *ARL* for different combinations of (q, α, L) for different δ when m = 49, n = 5,10 with $ARL_0^* = 370$ under a standard normal distribution.

						δ							
m = 49	q_1	q_2	α_1	α_2	L	**0.00**	**0.05**	**0.10**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**
n = 5	**0.80**	**0.00**	0.70	1.00	2.032	369.48	360.13	323.02	182.06	32.07	10.41	6.50	4.14
			0.80	1.00	2.112	369.16	360.33	329.31	185.74	31.77	10.12	6.26	4.06
			0.90	1.00	2.183	370.09	358.99	330.36	184.53	32.17	9.88	6.06	4.03
			1.00	1.00	2.249	370.13	366.63	332.54	187.88	32.80	9.76	5.95	3.92
			1.30	1.00	2.381	371.25	354.82	323.29	194.38	39.23	10.40	5.81	3.84
	0.90	**0.00**	0.70	1.00	1.462	372.80	351.63	322.26	170.07	30.76	11.23	7.66	5.25
			0.80	1.00	1.596	371.98	357.93	336.84	173.33	29.71	10.89	7.37	5.19
			0.90	1.00	1.712	370.10	363.12	333.66	179.81	30.48	10.42	7.08	5.07
			1.00	1.00	1.820	369.01	357.90	337.08	181.37	29.67	10.06	6.86	5.01
			1.30	1.00	2.067	367.08	357.44	328.03	184.24	33.33	9.50	6.22	4.52
	0.95	**0.00**	0.70	1.00	0.950	372.58	351.17	332.17	168.02	31.68	13.03	9.13	6.43
			0.80	1.00	1.089	370.29	347.62	310.26	169.06	30.00	12.65	8.88	6.38
			0.90	1.00	1.228	368.53	349.78	321.38	165.64	30.60	12.08	8.60	6.30
			1.00	1.00	1.365	372.95	360.39	330.96	177.20	31.84	11.52	8.28	6.21
			1.30	1.00	1.715	370.20	360.17	322.88	178.58	29.96	10.25	7.29	5.63
n = 10	**0.80**	**0.00**	0.70	1.00	1.715	370.15	347.30	320.14	164.11	21.12	6.62	4.08	2.71
			0.80	1.00	1.798	367.01	368.84	330.41	171.71	21.64	6.13	4.07	2.86
			0.90	1.00	1.871	370.80	358.76	331.83	169.95	21.93	6.05	3.97	2.83
			1.00	1.00	1.943	369	353.45	328.11	175.56	23.78	6.07	3.89	2.81
			1.30	1.00	2.119	372.48	373.53	335.21	185.60	26.17	5.84	3.68	2.77
	0.90	**0.00**	0.70	1.00	1.164	372.88	353.62	319.78	161.54	22.83	7.49	5.05	3.51
			0.80	1.00	1.274	374.14	357.50	321.47	167.35	21.34	7.30	4.98	3.56
			0.90	1.00	1.378	374.68	357.53	319.04	169.62	21.34	6.85	4.80	3.49
			1.00	1.00	1.478	369.95	359.57	324.07	160.68	19.23	6.87	4.66	3.42
			1.30	1.00	1.750	372.46	371.52	327.48	171.46	21.14	6.04	4.39	3.36
	0.95	**0.00**	0.70	1.00	0.737	370.03	351.18	323.35	153.10	24.60	8.99	6.23	4.42
			0.80	1.00	0.852	370.01	343.67	317.31	161.74	22.63	8.76	6.24	4.53
			0.90	1.00	0.966	369.01	361.94	328.57	170.11	22.86	8.51	6.17	4.56
			1.00	1.00	1.077	369.65	361.64	327.80	159.42	22.33	8.09	5.98	4.52
			1.30	1.00	1.396	372.89	347.82	328.52	175.51	21.78	7.18	5.46	4.32

Table 4: GWMA-EX and EWMA-EX IC and OOC ARL for different combinations of (q, α, L) for different δ when m = 99, n = 5,10 with $ARL_0^* = 370$ under a standard normal distribution.

						δ							
m = 99	q_1	q_2	α_1	α_2	L	**0.00**	**0.05**	**0.10**	**0.25**	**0.50**	**0.75**	**1.00**	**1.50**
n = 5	**0.80**	**0.00**	0.70	1.00	2.325	370.23	341.83	298.54	111.65	18.60	8.91	6.06	4.03
			0.80	1.00	2.397	369.77	355.92	301.38	117.35	18.68	8.58	5.86	3.98
			0.90	1.00	2.456	370.68	355.38	312.78	123.82	19.24	8.42	5.70	3.92
			1.00	1.00	2.503	370.46	360.68	309.11	125.56	20.58	8.32	5.58	3.88
			1.30	1.00	2.589	370.35	364.43	314.70	139.35	23.28	8.44	5.18	3.54
	0.90	**0.00**	0.70	1.00	1.805	370.49	339.03	292.55	103.51	18.55	9.88	7.01	4.91
			0.80	1.00	1.933	371.15	356.68	297.52	106.89	17.61	9.55	6.75	4.82
			0.90	1.00	2.038	369.68	349.85	287.91	106.62	17.14	9.13	6.56	4.80
			1.00	1.00	2.132	370.52	352.12	299.05	109.28	17.14	8.77	6.26	4.62
			1.30	1.00	2.335	370.80	356.22	308.91	121.73	18.29	8.25	5.85	4.45
	0.95	**0.00**	0.70	1.00	1.221	369.83	341.75	292.53	102.49	20.59	11.55	8.33	5.90
			0.80	1.00	1.380	369.03	340.91	289.68	99.19	19.40	11.15	8.13	5.88
			0.90	1.00	1.536	372.14	349.22	293.58	101.51	18.86	10.73	7.93	5.90
			1.00	1.00	1.667	369.62	350.08	295.54	101.53	18.13	10.30	7.69	5.71
			1.30	1.00	2.016	371.55	350.90	309.10	110.90	17.23	9.14	6.95	5.48
n = 10	**0.80**	**0.00**	0.70	1.00	2.077	372.30	348.45	298.02	90.64	11.12	5.54	3.88	2.66
			0.80	1.00	2.144	369.06	346.20	296.15	91.68	11.07	5.36	3.77	2.63
			0.90	1.00	2.213	369.35	356.32	303.21	99.64	10.77	5.19	3.67	2.62
			1.00	1.00	2.272	369.06	355.02	308.95	103.15	11.09	5.11	3.67	2.79
			1.30	1.00	2.409	369.12	353.33	313.84	108.96	12.14	4.92	3.49	2.76
	0.90	**0.00**	0.70	1.00	1.483	370.29	342.42	290.25	87.12	12.15	6.51	4.66	3.34
			0.80	1.00	1.602	371.60	338.28	280.44	84.34	11.58	6.28	4.58	3.34
			0.90	1.00	1.715	368.13	341.65	276.48	78.60	11.33	6.11	4.52	3.35
			1.00	1.00	1.817	369.69	339.07	284.37	86.64	10.96	5.92	4.45	3.35
			1.30	1.00	2.071	372.13	347.89	294.68	96.31	10.85	5.39	4.11	3.22
	0.95	**0.00**	0.70	1.00	0.959	371.00	351.90	298.77	92.72	13.99	7.80	5.63	4.07
			0.80	1.00	1.092	369.65	341.67	281.65	86.87	13.68	7.67	5.67	4.12
			0.90	1.00	1.228	375.11	343.87	287.38	84.76	12.87	7.44	5.65	4.30
			1.00	1.00	1.362	373.73	347.44	285.55	85.54	12.38	7.24	5.57	4.30
			1.30	1.00	1.717	370.07	350.04	285.70	86.75	11.03	6.53	5.18	4.18

Table 5: DGWMA-EX ARL values for $X_{(r)}$ = 75th, 50th and 25th percentiles of the Phase I sample for different shift δ when $m = 49, n = 5$.

				δ							
$X_{(r)}$	q	α	L	0.00	0.05	0.10	0.25	0.50	0.75	1.00	1.50
75th	0.80	0.90	1.622	371.55	357.78	310.67	168.44	31.98	9.01	6.13	4.42
50th	0.80	0.90	1.621	371.58	361.22	323.36	179.03	28.68	10.43	7.62	5.94
25th	0.80	0.90	1.612	369.14	374.21	360.03	258.04	57.09	17.52	11.27	11.23
75th	0.80	1.00	1.739	368.84	340.90	297.79	161.60	30.96	8.78	5.98	4.34
50th	0.80	1.00	1.755	369.77	360.95	328.84	183.09	30.99	10.14	7.26	5.63
25th	0.80	1.00	1.735	369.43	375.55	374.20	264.79	66.44	17.41	11.09	8.68
75th	0.80	1.30	2.020	369.94	337.70	294.49	146.59	27.00	8.32	5.46	4.10
50th	0.80	1.30	2.056	369.95	360.69	331.75	184.35	35.17	10.38	6.69	5.27
25th	0.80	1.30	2.017	369.97	380.13	389.74	306.70	89.51	22.01	11.13	7.72
75th	0.90	0.90	1.093	372.07	344.10	314.04	167.77	33.80	12.79	9.41	7.06
50th	0.90	0.90	1.092	371.87	352.77	320.63	169.71	31.63	14.27	11.30	9.19
25th	0.90	0.90	1.092	370.33	381.35	359.55	223.57	52.60	20.53	15.58	13.04
75th	0.90	1.00	1.255	369.45	343.34	309.66	165.65	33.36	11.91	8.89	6.77
50th	0.90	1.00	1.255	370.17	355.64	318.89	172.63	31.00	13.36	10.51	8.66
25th	0.90	1.00	1.254	370.20	377.19	371.47	238.28	52.49	19.24	14.40	12.38
75th	0.90	1.30	1.660	368.16	344.81	306.91	161.61	31.59	10.45	7.62	5.97
50th	0.90	1.30	1.668	371.38	355.21	319.10	178.82	32.08	11.65	8.86	7.40
25th	0.90	1.30	1.660	371.90	377.28	364.86	264.41	64.37	17.92	12.13	10.27
75th	0.95	0.90	0.706	371.99	355.82	316.40	171.23	39.72	18.81	14.70	11.26
50th	0.95	0.90	0.706	372.00	362.06	325.40	177.90	37.01	21.26	17.37	14.45
25th	0.95	0.90	0.709	369.58	379.19	354.52	233.05	55.80	27.91	23.13	20.18
75th	0.95	1.00	0.866	369.25	342.83	318.63	169.95	38.00	17.51	13.57	10.62
50th	0.95	1.00	0.866	372.19	358.30	332.25	170.47	36.56	19.27	15.91	13.35
25th	0.95	1.00	0.865	368.98	372.37	352.67	214.93	58.19	25.32	20.70	18.12
75th	0.95	1.30	1.314	371.49	349.13	312.93	165.20	32.93	14.00	10.87	8.81
50th	0.95	1.30	1.315	370.08	364.00	322.13	173.84	31.62	15.50	12.48	10.69
25th	0.95	1.30	1.315	372.05	380.05	379.31	249.08	53.35	20.27	15.97	13.93

Table 6: GWMA-EX and EWMA-EX ARL values for $X_{(r)}$ = 75th, 50th and 25th percentiles of the Phase I sample for different shift δ when $m = 49, n = 5$.

						δ							
$X_{(r)}$	q_1	q_2	α_1	α_2	L	0.00	0.05	0.10	0.25	0.50	0.75	1.00	1.50
75th	0.80	0.00	0.90	1.00	2.119	369.89	324.32	268.80	120.02	22.42	7.07	4.01	2.54
50th	0.80	0.00	0.90	1.00	2.183	370.09	358.99	330.36	184.53	32.17	9.88	6.06	4.03
25th	0.80	0.00	0.90	1.00	2.122	369.65	408.05	411.12	346.52	108.23	25.07	12.84	8.92
75th	0.80	0.00	1.00	1.00	2.174	371.01	325.45	265.33	122.28	21.78	7.12	4.07	2.51
50th	0.80	0.00	1.00	1.00	2.249	370.13	366.63	332.54	187.88	32.80	9.76	5.95	3.92
25th	0.80	0.00	1.00	1.00	2.174	366.25	403.30	422.48	354.66	116.03	25.29	12.59	8.10
75th	0.80	0.00	1.30	1.00	2.290	369.09	322.90	266.32	121.85	20.65	6.94	3.87	2.44
50th	0.80	0.00	1.30	1.00	2.381	371.25	354.82	323.29	194.38	39.23	10.40	5.81	3.84
25th	0.80	0.00	1.30	1.00	2.291	369.08	412.64	440.00	410.45	152.92	35.40	14.69	7.58
75th	0.90	0.00	0.90	1.00	1.694	370.25	338.72	302.85	142.90	26.04	8.39	5.26	3.34
50th	0.90	0.00	0.90	1.00	1.712	370.10	363.12	333.66	179.81	30.48	10.42	7.08	5.07
25th	0.90	0.00	0.90	1.00	1.695	370.32	384.67	384.25	271.76	63.41	18.72	12.18	9.09
75th	0.90	0.00	1.00	1.00	1.793	369.38	332.25	295.36	146.08	27.65	8.08	4.99	3.34
50th	0.90	0.00	1.00	1.00	1.820	369.01	357.90	337.08	181.37	29.67	10.06	6.86	5.01
25th	0.90	0.00	1.00	1.00	1.793	370.45	385.25	379.65	273.32	66.45	18.08	11.66	8.82
75th	0.90	0.00	1.30	1.00	2.038	371.96	333.28	295.45	144.22	28.12	8.04	4.56	3.17
50th	0.90	0.00	1.30	1.00	2.067	367.08	357.44	328.03	184.24	33.33	9.50	6.22	4.52
25th	0.90	0.00	1.30	1.00	2.032	374.06	383.29	392.55	302.79	82.67	20.49	10.96	7.77
75th	0.95	0.00	0.90	1.00	1.225	371.01	347.58	303.70	155.50	33.12	10.00	6.35	4.21
50th	0.95	0.00	0.90	1.00	1.228	368.53	349.78	321.38	165.64	30.60	12.08	8.60	6.30
25th	0.95	0.00	0.90	1.00	1.229	368.10	376.60	358.23	235.75	56.76	19.90	14.06	11.05
75th	0.95	0.00	1.00	1.00	1.364	371.00	339.83	310.45	155.85	31.25	9.85	6.42	4.25
50th	0.95	0.00	1.00	1.00	1.365	372.95	360.39	330.96	177.20	31.84	11.52	8.28	6.21
25th	0.95	0.00	1.00	1.00	1.359	368.73	373.37	362.65	236.59	54.07	18.48	13.01	10.12
75th	0.95	0.00	1.30	1.00	1.702	368.90	344.24	301.68	155.67	28.95	8.67	5.58	4.10
50th	0.95	0.00	1.30	1.00	1.715	370.20	360.17	322.88	178.58	29.96	10.25	7.29	5.63
25th	0.95	0.00	1.30	1.00	1.704	371.06	377.53	369.95	236.59	63.10	17.31	11.23	8.77

Table 7: DGWMA-EX MRL values for $X_{(r)} = 75^{th}$, 50^{th} and 25^{th} percentiles of the Phase I sample for different shift δ when $m = 100, n = 5$.

				δ							
$X_{(r)}$	q	α	L	0.00	0.05	0.10	0.25	0.50	0.75	1.00	1.50
75^{th}	0.80	0.90	2.142	349.00	278.00	169.00	39.00	13.00	8.00	6.00	5.00
50^{th}	0.80	0.90	2.140	351.00	320.00	215.00	49.00	15.00	10.00	8.00	6.00
25^{th}	0.80	0.90	2.132	350.00	377.00	321.00	94.00	23.00	14.00	11.00	10.00
75^{th}	0.80	1.00	2.228	349.00	254.00	164.00	39.00	12.00	7.00	6.00	4.00
50^{th}	0.80	1.00	2.253	350.50	328.00	240.00	53.00	14.00	9.00	7.00	6.00
25^{th}	0.80	1.00	2.234	350.00	390.50	341.00	109.00	23.00	13.00	11.00	8.00
75^{th}	0.80	1.30	2.443	348.50	251.00	158.00	39.00	11.00	6.00	5.00	4.00
50^{th}	0.80	1.30	2.472	349.00	322.00	240.00	63.00	14.00	8.00	6.00	5.00
25^{th}	0.80	1.30	2.455	351.00	423.00	416.00	71.00	34.00	14.00	10.00	7.00
75^{th}	0.90	0.90	1.620	348.00	278.00	173.00	44.00	18.00	12.00	10.00	8.00
50^{th}	0.90	0.90	1.618	349.00	301.00	202.00	48.00	20.00	14.00	12.00	10.00
25^{th}	0.90	0.90	1.615	352.00	356.00	274.00	72.00	27.00	19.00	16.00	15.00
75^{th}	0.90	1.00	1.775	350.50	264.00	167.00	42.00	16.00	11.00	9.00	7.00
50^{th}	0.90	1.00	1.782	349.00	310.00	208.00	46.00	18.00	13.00	11.00	9.00
25^{th}	0.90	1.00	1.772	351.00	339.50	285.00	72.00	25.00	17.00	15.00	13.00
75^{th}	0.90	1.30	2.152	349.00	264.00	174.00	41.00	13.00	9.00	7.00	6.00
50^{th}	0.90	1.30	2.165	350.00	314.00	235.00	53.00	15.00	10.00	9.00	8.00
25^{th}	0.90	1.30	2.155	349.00	382.00	335.00	104.00	23.00	14.00	12.00	10.00
75^{th}	0.95	0.90	1.173	350.00	270.00	175.00	57.00	28.00	20.00	16.00	13.00
50^{th}	0.95	0.90	1.171	351.00	306.00	195.00	60.00	31.00	23.00	19.00	16.00
25^{th}	0.95	0.90	1.172	352.00	347.00	259.50	79.00	39.00	30.00	26.00	23.00
75^{th}	0.95	1.00	1.346	347.00	275.00	166.00	49.00	24.00	18.00	14.00	11.00
50^{th}	0.95	1.00	1.361	352.00	314.50	196.50	53.00	27.00	20.00	17.00	15.00
25^{th}	0.95	1.00	1.360	349.5	365.00	274.00	71.00	33.00	26.00	23.00	20.00
75^{th}	0.95	1.30	1.821	352.00	278.00	175.00	40.00	17.00	13.00	11.00	9.00
50^{th}	0.95	1.30	1.836	349.00	310.00	229.00	48.00	19.00	15.00	13.00	11.00
25^{th}	0.95	1.30	1.813	349.00	347.00	283.00	76.00	25.00	18.00	16.00	15.00

Table 8: GWMA-EX and EWMA-EX MRL values for $X_{(r)} = 75^{th}$, 50^{th} and 25^{th} percentiles of the Phase I sample for different shift δ when $m = 100, n = 5$.

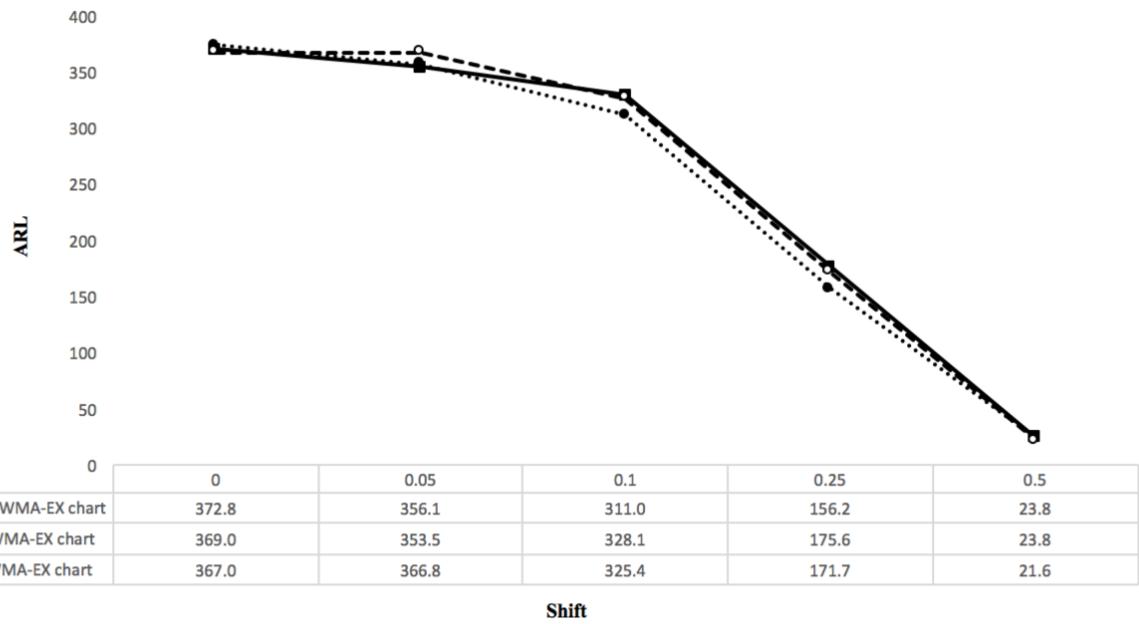
						δ							
$X_{(r)}$	q_1	q_2	α_1	α_2	L	0.00	0.05	0.10	0.25	0.50	0.75	1.00	1.50
75^{th}	0.80	0.00	0.90	1.00	2.596	350.00	217.00	124.00	34.00	10.00	6.00	4.00	3.00
50^{th}	0.80	0.00	0.90	1.00	2.620	350.50	320.00	246.00	58.00	15.00	8.00	6.00	4.00
25^{th}	0.80	0.00	0.90	1.00	2.603	352.00	504.00	581.00	321.00	44.00	19.00	13.00	9.00
75^{th}	0.80	0.00	1.00	1.00	2.625	350.00	219.50	128.00	35.00	10.00	5.00	4.00	3.00
50^{th}	0.80	0.00	1.00	1.00	2.660	349.50	327.00	246.00	62.00	14.00	8.00	5.00	4.00
25^{th}	0.80	0.00	1.00	1.00	2.627	350.00	468.50	576.00	349.00	48.00	18.00	12.00	8.00
75^{th}	0.80	0.00	1.30	1.00	2.677	351.00	220.00	134.00	37.00	10.00	5.00	3.00	3.00
50^{th}	0.80	0.00	1.30	1.00	2.736	352.00	329.00	271.00	79.00	16.00	7.00	5.00	4.00
25^{th}	0.80	0.00	1.30	1.00	2.684	348.00	544.00	691.00	645.00	101.00	28.00	14.00	8.00
75^{th}	0.90	0.00	0.90	1.00	2.250	350.00	253.00	156.00	37.00	13.00	7.00	5.00	3.00
50^{th}	0.90	0.00	0.90	1.00	2.262	350.00	307.00	215.00	50.00	16.00	9.00	7.00	5.00
25^{th}	0.90	0.00	0.90	1.00	2.251	351.00	411.50	359.00	104.00	27.00	16.00	12.00	10.00
75^{th}	0.90	0.00	1.00	1.00	2.324	349.00	256.00	149.00	37.00	12.00	7.00	5.00	3.00
50^{th}	0.90	0.00	1.00	1.00	2.340	351.00	319.00	228.00	50.00	15.00	9.00	7.00	5.00
25^{th}	0.90	0.00	1.00	1.00	2.323	350.00	412.00	358.00	112.00	26.00	15.00	11.00	9.00
75^{th}	0.90	0.00	1.30	1.00	2.491	350.00	246.00	157.00	38.00	11.00	6.00	4.00	3.00
50^{th}	0.90	0.00	1.30	1.00	2.509	350.00	319.50	241.00	60.00	14.00	8.00	6.00	4.00
25^{th}	0.90	0.00	1.30	1.00	2.488	349.00	423.00	412.00	176.00	31.00	14.00	10.00	7.00
75^{th}	0.95	0.00	0.90	1.00	1.798	350.00	273.00	165.00	45.00	16.00	10.00	7.00	5.00
50^{th}	0.95	0.00	0.90	1.00	1.793	351.00	310.00	195.00	51.00	19.00	12.00	9.00	7.00
25^{th}	0.95	0.00	0.90	1.00	1.793	349.00	376.00	298.00	83.00	29.00	19.00	15.00	13.00
75^{th}	0.95	0.00	1.00	1.00	1.921	350.00	251.00	162.00	41.00	15.00	9.00	6.00	4.00
50^{th}	0.95	0.00	1.00	1.00	1.921	350.00	311.00	199.00	48.00	17.00	11.00	9.00	7.00
25^{th}	0.95	0.00	1.00	1.00	1.921	352.00	364.00	294.00	78.00	27.00	17.00	14.00	12.00
75^{th}	0.95	0.00	1.30	1.00	2.211	349.00	260.00	164.00	39.00	12.00	7.00	6.00	4.00
50^{th}	0.95	0.00	1.30	1.00	2.228	350.50	326.00	234.00	51.00	14.00	9.00	7.00	6.00
25^{th}	0.95	0.00	1.30	1.00	2.228	351.00	402.00	361.00	107.00	24.00	14.00	11.00	10.00

Table 9: ARL values for the DGWMA-EX, GWMA-EX, EWMA-EX, DGWMA- $\bar{X}$ and GWMA- $\bar{X}$ charts for various shifts δ when $ARL_0^* = 370$ and $m = 49$, $n = 5$ under symmetric distributions.

δ	Chart	q_1	q_2	α_1	α_2	L	normal(0,1)	logistic$\left(0, \frac{\sqrt{3}}{\pi}\right)$	uniform$(-\sqrt{3}, \sqrt{3})$	Laplace$\left(0, \frac{1}{\sqrt{2}}\right)$
	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	368.93	369.68	368.59	368.04
	GWMA – EX	0.8	0.0	0.7	1.0	2.032	369.48	382.89	386.31	383.38
0.00	EWMA - EX	0.8	0.0	1.0	1.0	2.249	370.13	381.31	381.31	375.63
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	370.33	367.04	396.90	391.43
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	369.33	329.40	492.52	262.20
	DGWMA – EX	0.8	0.8	0.7	0.7	1.304	358.68	345.10	352.68	319.88
	GWMA – EX	0.8	0.0	0.7	1.0	2.032	360.13	360.64	371.26	327.60
0.05	EWMA - EX	0.8	0.0	1.0	1.0	2.249	366.63	367.85	375.50	333.09
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	354.92	350.17	352.36	349.83
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	357.01	314.02	436.42	255.38
	DGWMA – EX	0.8	0.8	0.7	0.7	1.304	317.48	306.82	334.87	235.92
	GWMA – EX	0.8	0.0	0.7	1.0	2.032	323.03	314.89	343.51	248.82
0.10	EWMA - EX	0.8	0.0	1.0	1.0	2.249	332.54	316.07	347.40	258.78
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	298.63	299.50	294.33	302.26
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	324.41	267.67	377.92	219.72
	DGWMA – EX	0.8	0.8	0.7	0.7	1.304	163.35	135.01	235.28	54.32
	GWMA – EX	0.8	0.0	0.7	1.0	2.032	182.06	149.34	251.50	61.12
0.25	EWMA - EX	0.8	0.0	1.0	1.0	2.249	187.88	150.24	255.67	69.18
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	61.34	62.33	64.90	63.25
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	126.57	112.84	146.16	94.68
	DGWMA – EX	0.8	0.8	0.7	0.7	1.304	28.39	20.97	75.06	12.37
	GWMA – EX	0.8	0.0	0.7	1.0	2.032	32.07	21.34	83.41	11.59
0.50	EWMA - EX	0.8	0.0	1.0	1.0	2.249	32.80	22.48	94.50	11.03
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	12.15	12.11	11.94	12.16
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	15.73	15.92	15.92	15.83
	DGWMA – EX	0.8	0.8	0.7	0.7	1.304	8.41	7.81	11.06	7.34
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	6.50	5.82	9.95	5.25
1.00	EWMA - EX	0.8	0.0	1.0	1.0	2.249	5.95	5.34	9.17	4.78
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	5.50	5.46	5.48	5.48
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	4.05	4.17	4.17	4.15

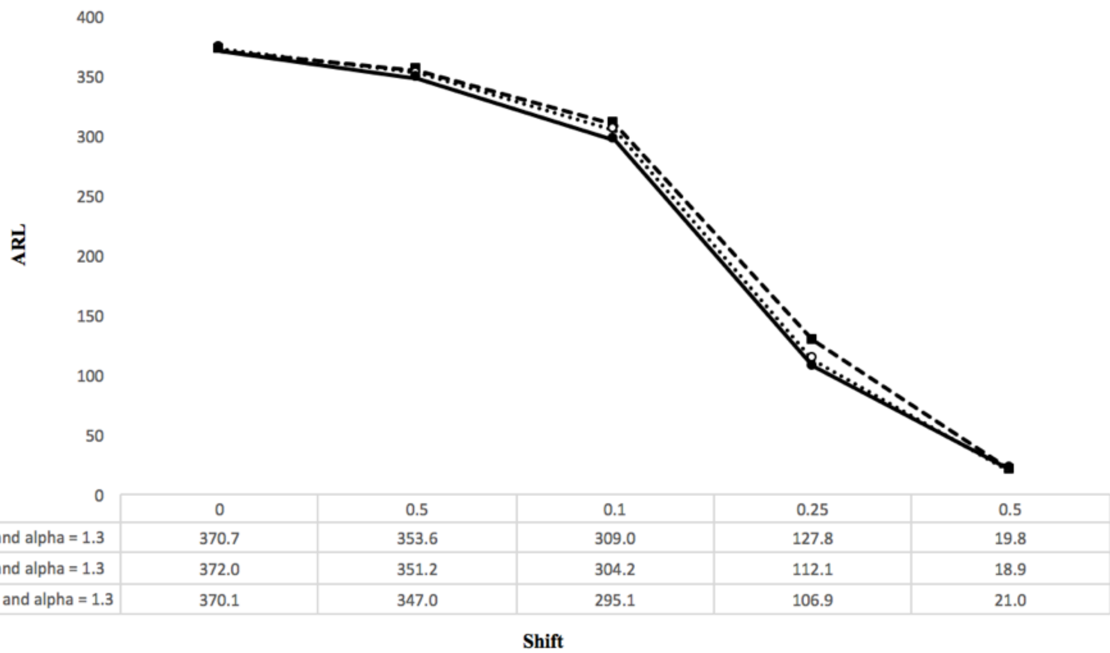
Table 10: ARL values for the DGWMA-EX, GWMA-EX, EWMA-EX, DGWMA- $\bar{X}$ and GWMA- $\bar{X}$ charts for various shifts δ when $ARL_0^* = 370$ and $m = 49$, $n = 5$ under skewed distributions.

δ	Chart	q_1	q_2	α_1	α_2	L	gamma(1,1)	gamma(2,1)	gamma(3,1)
1.000	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	368.89	369.84	369.90
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	367.75	371.50	373.58
	EWMA - EX	0.8	0.0	1.0	1.0	2.249	366.39	374.94	374.45
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	436.02	441.70	432.24
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	347.72	371.54	393.30
0.975	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	359.56	356.10	358.59
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	366.92	367.87	364.50
	EWMA - EX	0.8	0.0	1.0	1.0	2.249	368.74	361.02	361.99
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	515.57	499.04	488.45
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	451.53	490.47	506.47
0.950	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	346.41	340.15	329.70
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	360.40	341.19	342.74
	EWMA - EX	0.8	0.0	1.0	1.0	2.249	361.87	348.35	334.52
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	571.16	551.85	530.00
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	590.12	607.78	594.20
0.900	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	310.19	283.30	246.37
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	329.72	295.07	252.44
	EWMA - EX	0.8	0.0	1.0	1.0	2.249	327.42	293.78	258.99
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	707.54	568.14	469.12
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	864.55	815.73	692.70
0.800	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	222.74	127.23	77.69
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	227.63	142.24	86.47
	EWMA - EX	0.8	0.0	1.0	1.0	2.249	236.89	144.50	89.47
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	659.89	266.98	100.86
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	950.89	648.72	285.70
0.700	DGWMA - EX	0.8	0.8	0.7	0.7	1.304	115.06	35.92	18.30
	GWMA - EX	0.8	0.0	0.7	1.0	2.032	131.60	41.16	18.23
	EWMA - EX	0.8	0.0	1.0	1.0	2.249	129.25	44.54	20.14
	DGWMA- $\bar{X}$	0.8	0.8	0.7	0.7	2.992	309.27	50.14	17.58
	GWMA- $\bar{X}$	0.8	0.0	0.7	1.0	3.263	850.75	190.00	35.23



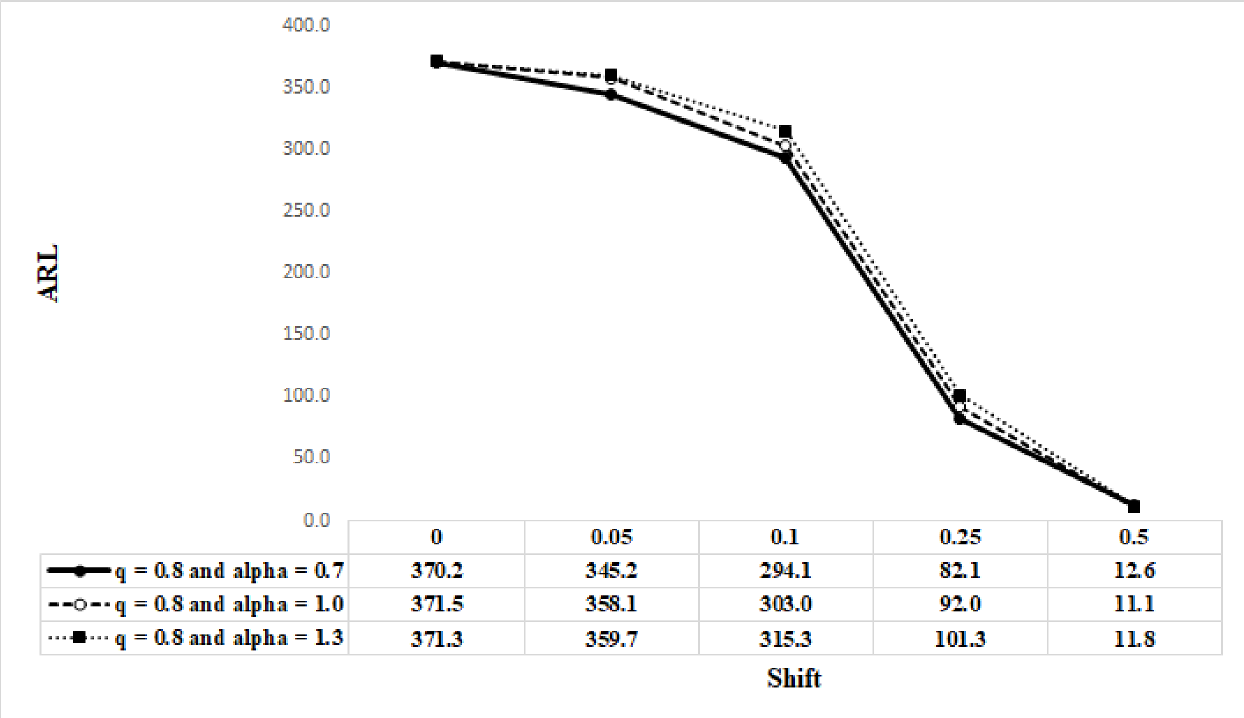
	0	0.05	0.1	0.25	0.5
DGWMA-EX chart	372.8	356.1	311.0	156.2	23.8
GWMA-EX chart	369.0	353.5	328.1	175.6	23.8
EWMA-EX chart	367.0	366.8	325.4	171.7	21.6

Figure 1: DGWMA-EX vs GWMA-EX vs EWMA-EX (q = 0.8, α = 0.8, m = 49 and n = 10).



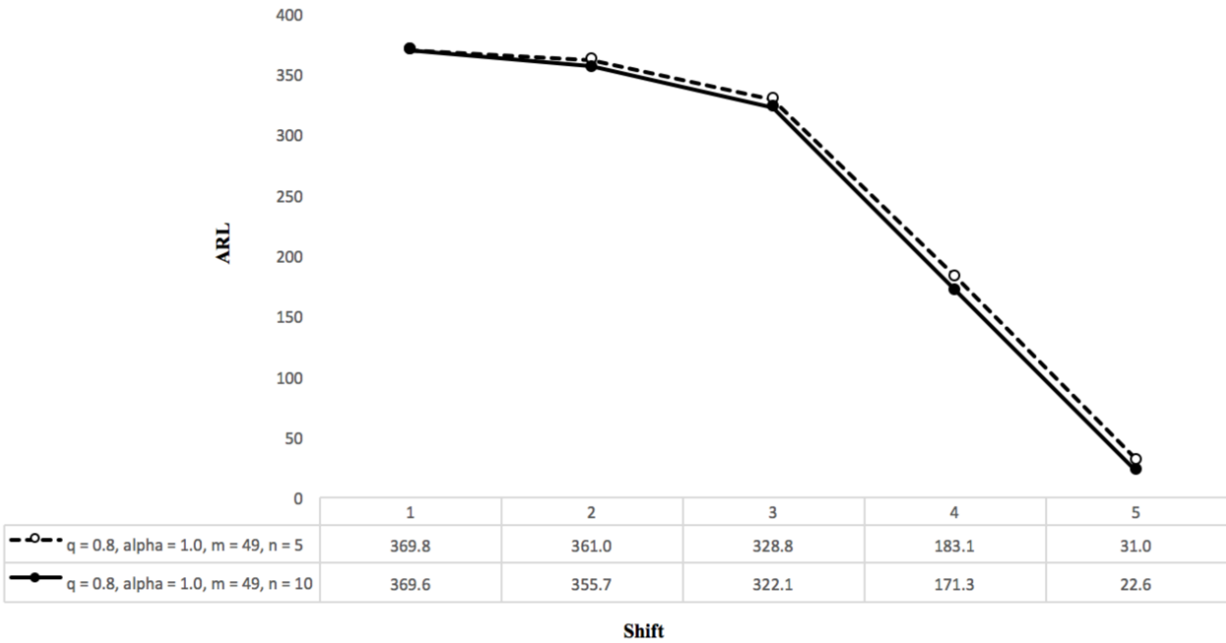
	0	0.5	0.1	0.25	0.5
q = 0.8 and alpha = 1.3	370.7	353.6	309.0	127.8	19.8
q = 0.9 and alpha = 1.3	372.0	351.2	304.2	112.1	18.9
q = 0.95 and alpha = 1.3	370.1	347.0	295.1	106.9	21.0

Figure 2: The effect of q on the performance of the DGWMA-EX chart for m = 99 and n = 10.



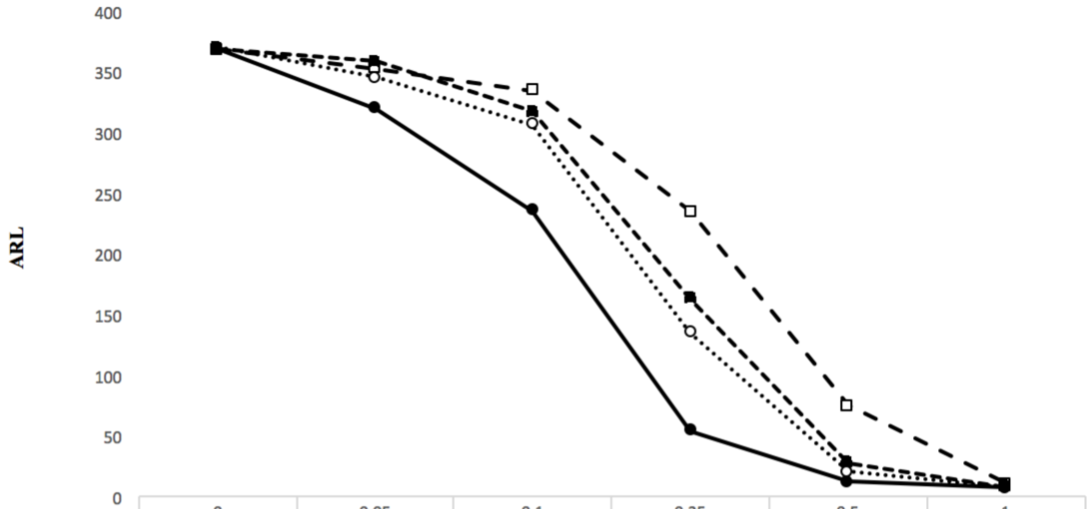
Shift	0	0.05	0.1	0.25	0.5
q = 0.8 and alpha = 0.7	370.2	345.2	294.1	82.1	12.6
q = 0.8 and alpha = 1.0	371.5	358.1	303.0	92.0	11.1
q = 0.8 and alpha = 1.3	371.3	359.7	315.3	101.3	11.8

Figure 3: The effect of α on the performance of DGWMA-EX chart for m = 99 and n = 5.



Shift	1	2	3	4	5
q = 0.8, alpha = 1.0, m = 49, n = 5	369.8	361.0	328.8	183.1	31.0
q = 0.8, alpha = 1.0, m = 49, n = 10	369.6	355.7	322.1	171.3	22.6

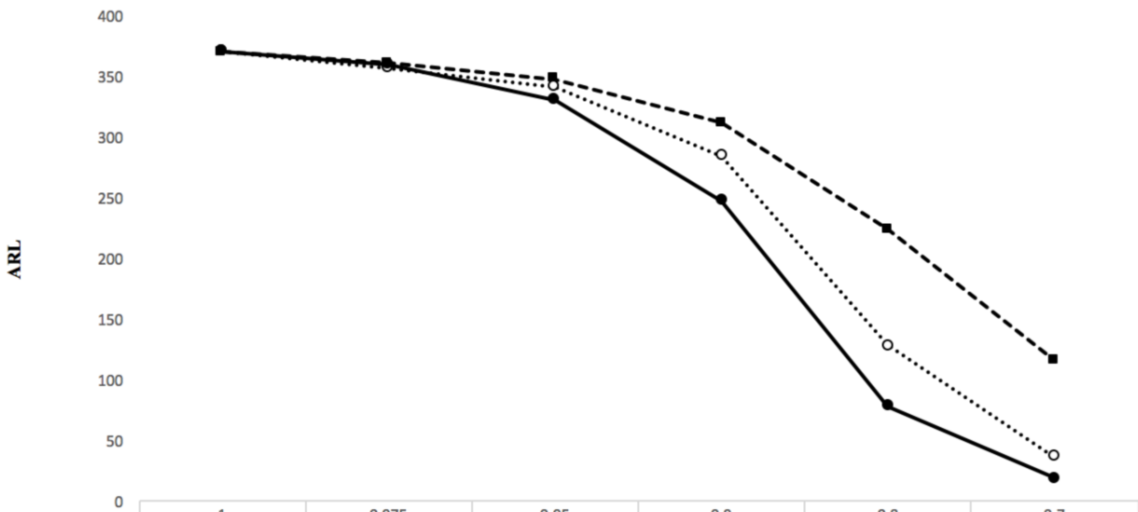
Figure 4: The effect of the test sample size (n) on the performance of the DGWMA-EX chart.



	0	0.05	0.1	0.25	0.5	1
Logistic distribution	369.9	345.1	306.8	135.1	20.9	7.8
Uniform distribution	368.6	352.6	334.8	235.2	75.0	11.0
Laplace distribution	368.0	319.8	235.9	54.3	12.3	7.3
Standard normal distribution	368.9	358.6	317.4	163.3	28.3	8.4

Shift

Figure 5: The DGWMA-EX chart based on different symmetric distributions.



	1	0.975	0.95	0.9	0.8	0.7
Gamma (1,1)	368.8	359.5	346.4	310.1	222.7	115
Gamma (2,1)	369.8	356.1	340.1	283.3	127.2	35.9
Gamma (3,1)	369.9	358.5	329.7	246.3	77.6	18.3

Shift

Figure 6: The DGWMA-EX chart based on different skewed distribution.

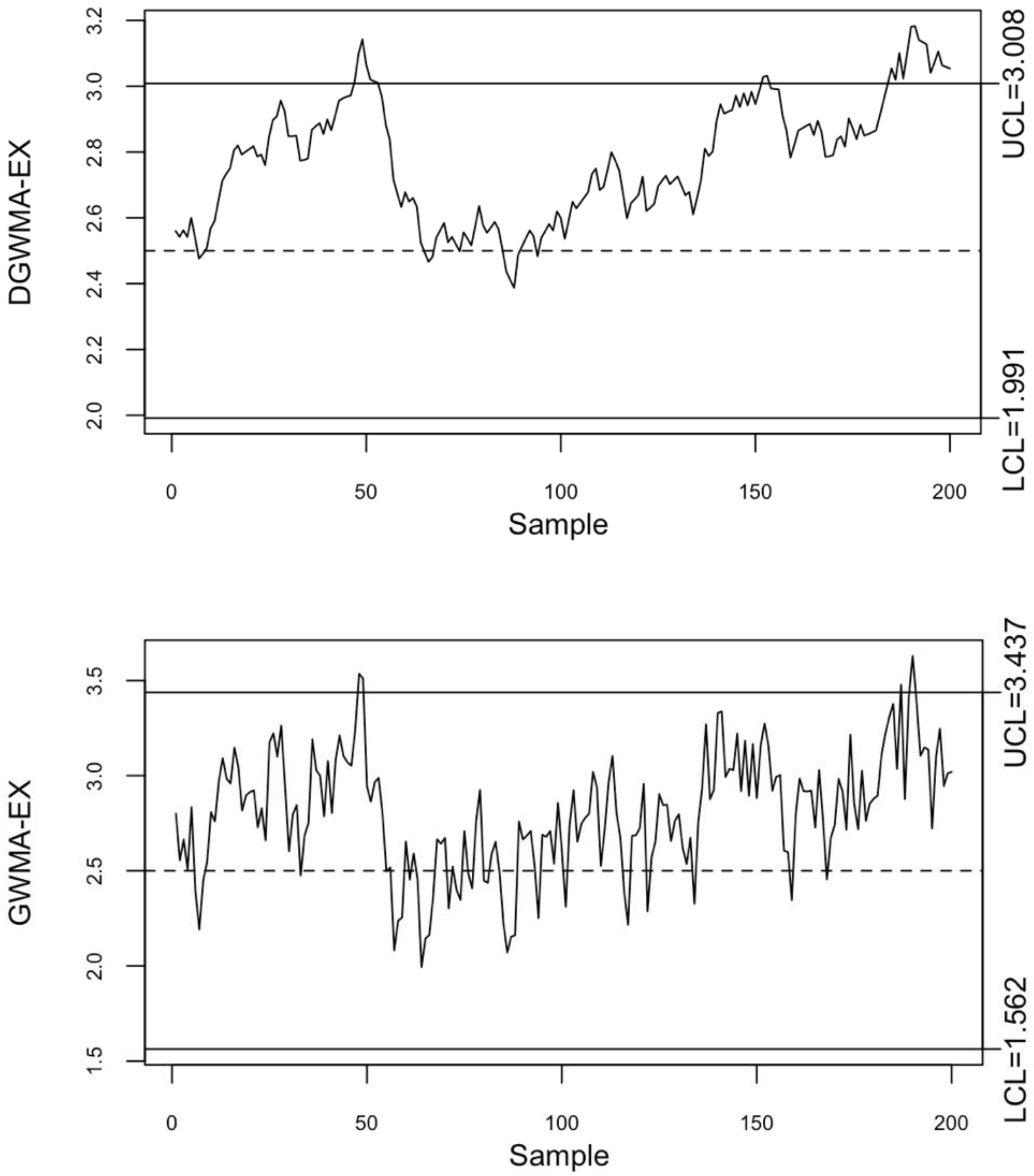


Figure 7: DGWMA-EX and GWMA-EX chart implemented on simulated data.

Appendix

A1. The exceedance statistics U_i, $i = 1,2, \ldots$, are independent and identically distributed binomial with parameters (n, p_r), where n is the sample size and $p_r = 1 - G\left(x_r \middle| X_{(r)} = x_r\right)$, where $G(.)$ is the c.d.f of the Phase II test sample $(Y_{i1}, Y_{i2}, \ldots, Y_{in})$.

Proof: Since every observation Y_{ij} in a test sample has two possible outcomes (smaller or larger than $X_{(r)}$, then the order statistic $X_{(r)}$ follows the properties of a Bernoulli trial. Note that for every Phase II test sample the number of observations smaller or larger than the order statistic are independent. Hence, the random variable U_i referring to the number of exceedances given by the number of observations in the i^{th} test sample that exceed $X_{(r)}$ is following binomial distribution with parameters (n, p_r), given $X_{(r)}$, where the probability of success is $p_r = P\left[Y \geq X_{(r)} \middle| X_{(r)} = x_r\right] = 1 - G\left(x_r \middle| X_{(r)} = x_r\right)$.

A2. The unconditional IC distribution of the exceedance statistic U_i, for all $i = 1, 2, \ldots$, is distribution free and is given by the p.m.f. $P(U_i = u) = \frac{\binom{u+m-r}{u}\binom{n-u+r-1}{n-u}}{\binom{m+n}{n}}$, $u = 0,1,2, \ldots, n$.

Proof: The probability mass function of a exceedance statistic U_i, conditional on $X_{(r)}$ can be written as:

$$P\left[U_i = u \middle| X_{(r)} = x_r\right] = \binom{n}{u} p_r^u (1 - p_r)^{n-u} = \binom{n}{u} \left(1 - G(x_r)\right)^u G(x_r)^{n-u}, u = 0,1,2, \ldots, n$$

By implementing unconditional method we have

$$P[U_i = u] = E_{X_{(r)}}\left(P\left[U_i = u \middle| X_{(r)} = x_r\right]\right)$$

$$= \int_{-\infty}^{\infty} \binom{n}{u} \left(1 - G(x_r)\right)^u G(x_r)^{n-u} \frac{m!}{(r-1)!(m-r)!} F(x_r)^{r-1} \left(1 - F(x_r)\right)^{m-r} f(x_r) dx_r.$$

When the process is IC, $G = F$. Therefore, the IC unconditional distribution of U_i is given by

$$P[U_i = u] = \binom{n}{u} \frac{m!}{(r-1)!(m-r)!} \int_{-\infty}^{\infty} F(x_r)^{n-u+r-1} \left(1 - F(x_r)\right)^{m+u-r} f(x_r) dx_r$$

$$= \frac{n!}{u!\,(n-u)!} \frac{m!}{(r-1)!\,(m-r)!} \frac{(n-u+r-1)!\,(m+u-r)!}{(m+n)!}$$

$$= \frac{\binom{u+m-r}{u}\binom{n-u+r-1}{n-u}}{\binom{m+n}{n}}.$$

A3. $ARL|X_{(r)} = 1 + \sum_{k=1}^{\infty} I_k$ where $I_k = P\left[\bigcap_{i=1}^{k} S_i^c \middle| X_{(r)}\right] = \sum_{LC_1}^{UC_1} \sum_{LC_2}^{UC_2} \ldots \sum_{LC_k}^{UC_k} \left(\prod_{i=1}^{k} P[U_i = u_i | X_{(r)}]\right)$ for $k = 1, 2, 3, \ldots$ and $I_0 = 1$.

Proof:

The conditional run length probabilities can be written as:

$P\left[K = k \middle| X_{(r)}\right] = P\left[\bigcap_{i=1}^{k-1} S_i^c \middle| X_{(r)}\right] - P\left[\bigcap_{i=1}^{k} S_i^c \middle| X_{(r)}\right] = I_{k-1} - I_k$

where $I_k = P\left[\bigcap_{i=1}^{k} S_i^c \middle| X_{(r)}\right] = \sum_{LC_1}^{UC_1} \sum_{LC_2}^{UC_2} \dots \sum_{LC_k}^{UC_k} \left(\prod_{i=1}^{k} P[U_i = u_i | X_{(r)}]\right)$ and $I_0 = 1$.

$P[U_i = u_i | X_{(r)}]$ is given in Appendix A1. Hence:

$ARL|X_{(r)} = P\left[K = 1 \middle| X_{(r)}\right] + \sum_{k=2}^{\infty} kP\left[K = k \middle| X_{(r)}\right] = 1 - P\left[S_1^c \middle| X_{(r)}\right] + \sum_{k=2}^{\infty} k(I_{k-1} - I_k) = 1 - P\left[S_1^c \middle| X_{(r)}\right] + I_1 + \sum_{k=1}^{\infty} I_k$.

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