Weighted Distributions of Eigenvalues

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Abstract:

In this article, the weighted version of a probability density function is considered as a mapping of the original distribution. Generally, the properties of the distribution of a random matrix and the distributions of its eigenvalues are closely related. Therefore, the weighted versions of the distributions of the eigenvalues of the Wishart distribution are introduced and their properties are discussed. We propose the concept of rotation invariance for the weighted distributions of the eigenvalues of the Wishart and non-central Wishart distributions. We also introduce here, the concept of a "mirror", meaning, looking at the distribution of a random matrix through the distribution of its eigenvalues. Some graphical representations are given, to visualize the weighted distributions of the eigenvalues for specific cases.

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1 Introduction

Let X be a non-negative random variable (r.v.) having probability density function (p.d.f.) $f(x;\theta)$ where θ is a scalar or a vector of parameters. Let $w(x) > 0$ be a function of x, with $E[w(X)] < \infty$. The $w(x)$ is referred to as a weight function. Then, the weighted version of $f(x;\theta)$ is defined as

$$g(x;\theta) = \frac{w(x)f(x;\theta)}{E[w(X)]}. \tag{1}$$

It is possible to let $w(x)$ have a parameter, say, ψ, of its own. The p.d.f. $f(x;\theta)$ is referred to as the original distribution and $g(x;\theta)$ is called the weighted version of $f(x;\theta)$. When $w(x) = x$ the p.d.f. $g(.;.)$ is referred to as the length-biased version of the original distribution. It is anticipated

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that the properties of $g(.;.)$ will closely resemble the properties of $f(.;.)$. Thus, $g(x;\theta)$ can be used for statistical inference about θ. The factor $w(x)/E[w(X)]$ is an operator that changes $f(x;\theta)$ to $g(x;\theta)$. Alternately, the family of $g(.;.)$ can be thought of as a mapping of family of p.d.f's $f(.;.)$.

During the past thirty years, a number of papers have appeared discussing univariate weighted/ length-biased data and the data analysis using $g(x;\theta)$. In a nutshell, since the eigenvalues explain the most of the variations in the data, the weighted distributions of eigenvalues may have the same importance as the weighted distributions of the random matrix. The importance of distributions of eigenvalues of random matrices is emphasized in literature by Zanella et al. (2008), Wu et al. (2016), Stott et al. (2017) and Zhang et al. (2017), amongst others.

There are indications of the presence of size-biased data random matrices, such as example is the biased mutation matrices as refer to in bioinformatics (Brick and Pizzi, 2008). The bias in the observed values of the eigenvalues is discussed for data collected for face recognition (Hendrikse et al., 2009). The presence of bias in signals associated with the length of the signal in MIMO systems are discussed with regard to the eigenvalues of the related matrix configuration (Shenoy et al., 2008). The above examples provide us a necessary motivation for studying the weighted matrix variate distributions and contribute new distributions to matrix theory.

In (1) instead of a univariate r.v. X we can consider a matrix variate r.v. $\boldsymbol{X} : p \times p$ and an associated weight function with related regularity conditions. In this paper, the interest is the derivation of the weighted version of the distribution of eigenvalues and the study of their properties. The stochastic behavior of $\boldsymbol{X}$ is represented by the distribution of its eigenvalues, $\boldsymbol{\lambda}$. Therefore, given the distribution of $\boldsymbol{\lambda}$, certain properties of the distribution of $\boldsymbol{X}$ can be studied. We also introduce a new concept of "mirror" where the p.d.f. $f(.;.)$ asymptotically approaches the p.d.f. $g(.;.)$.

The organization of the paper is as follows: The notation and the definitions needed for the development of this paper are recorded in Section 2. In Section 3, the properties of the length-biased version of the distribution of the eigenvalues of the Wishart matrix are studied and the related results are obtained for the non-central Wishart distribution. Also, is introduced in Section 3 a concept of rotation invariance. In Section 4, a procedure referred to as "mirror", in this paper, is described for reconstructing the original distribution of a random matrix through the weighted version of the distribution of the eigenvalues of the random matrix. To demonstrate the usefulness of this concept of mirror we have provided the numerical results in Section 5. Section 6 concludes with graphical representations of weighted distributions of eigenvalues.

2 Definitions and Some Useful Results

Throughout this paper, we shall assume the $p \times p$ matrix $\boldsymbol{X}$ is a matrix of real r.v.s or is a derived matrix arising from another $n \times p$ matrix of r.v.s. Let $\mathcal{S}(p)$ be the space of all positive definite matrices of order p. Also denote the space of all orthogonal matrices of order p by

$$\mathcal{O}(p) = \{\boldsymbol{H} | \boldsymbol{H}'\boldsymbol{H} = \boldsymbol{I}_p,\ \boldsymbol{H}\boldsymbol{H}' = \boldsymbol{I}_p\}, \quad \int_{\mathcal{O}(p)} \mathrm{d}\boldsymbol{H} = 1.$$

In this paper, the distribution of the $m \times 1$ vector $\boldsymbol{\lambda}$ of eigenvalues plays a key role for all results obtained and discussed.

Definition 2.1. *Let $f(\boldsymbol{\lambda})$ denote the joint p.d.f of $\boldsymbol{\lambda}$. Then, the weighted version $g(\boldsymbol{\lambda})$ of $f(\boldsymbol{\lambda})$ is given by*

$$g(\boldsymbol{\lambda}) = \frac{w(\boldsymbol{\lambda})f(\boldsymbol{\lambda})}{E[w(\boldsymbol{\lambda})]}, \tag{2}$$

where $w(\boldsymbol{\lambda}) > 0$, is a weight function and $E[w(\boldsymbol{\lambda})] < \infty$.

Note that in $w(\boldsymbol{\lambda})$, it is possible to introduce a vector of parameters as may seem necessary in a given application.

The p.d.f.'s $g(\boldsymbol{\lambda})$ in (2) could be the joint p.d.f.'s of the subsets of vector $\boldsymbol{\lambda}$ or just the marginal distributions of λ_i, $i = 1, 2, \ldots, p$ with corresponding weight functions.

Next, we introduce a concept of a rotation invariant family of distributions. In Section 3, we explore the implications of this concept when combined with Definition 2.2.

Definition 2.2. *Let $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ to be the eigenvalues of the random matrix $\boldsymbol{X} \in \mathcal{S}(p)$ with p.d.f. $f(.)$. Then $g(\boldsymbol{\lambda})$ is said to be rotation invariant with respect to $f(\boldsymbol{\lambda})$, iff $g(\boldsymbol{\lambda})$ can be expressed as a weighted distribution of $f(\boldsymbol{\lambda})$ for a family of weight functions based on the eigenvalues.*

Remark 2.1. *According to Definition 2.2, there exists an order preserving map $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$ for each $f(.)$ and $g(.)$ for which we have*

$$g(\lambda_1, \ldots, \lambda_p) = \mathcal{M}(f(\lambda_1, \ldots, \lambda_p)).$$

Under $\mathcal{M}$ the functional form of $f(.)$ and $g(.)$ is maintained. It allows the embedding of $f(.)$ and $g(.)$ in certain common statistical manifolds. Note that the form of $\mathcal{M}$ is not important, wheras the property of maintaining is.

Definition 2.3. *The random matrix $\boldsymbol{X} \in \mathcal{S}(p)$ is said to have a non-central Wishart distribution with scale matrix $\boldsymbol{\Sigma} \in \mathcal{S}(p)$, $n \geq p$ degrees of freedom and non-centrality parameter $\boldsymbol{\Omega} \in \mathcal{S}(p)$ denoted by $\boldsymbol{X} \sim W_p(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, n)$, if its p.d.f. is*

$$\begin{aligned} f(\boldsymbol{X}) \quad &= \quad \frac{\det(\boldsymbol{\Sigma})^{-\frac{1}{2}n}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)} \det(\boldsymbol{X})^{\frac{1}{2}n-\frac{1}{2}(p+1)} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right) \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \\ &\quad \times {}_0F_1\left(\frac{1}{2}n; \frac{1}{4}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right) \end{aligned}$$

where ${}_0F_1(.;.)$ is defined in Muirhead (2005, p.258).

Theorem 2.1. *(James, 1961) Let $\boldsymbol{X} \sim W_p(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, n)$, $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'$ for some $\boldsymbol{M} \in \mathbb{R}^{p\times n}$, w_i, $i = 1, \ldots, p$ are the eigenvalues of $\det(\boldsymbol{X} - \omega\boldsymbol{\Sigma})$ and $\boldsymbol{\Omega} = \operatorname{diag}(\omega_1, \ldots, \omega_p)$ where ω_i, $i = 1, \ldots, p$ are the eigenvalues of $\det(\boldsymbol{M}\boldsymbol{M}' - \omega\boldsymbol{\Sigma}) = 0$. Then the distribution of $\boldsymbol{W} = \operatorname{diag}(w_1, \ldots, w_p)$ is given by*

$$\begin{aligned} f(\boldsymbol{W}) \quad &= \quad \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) {}_0F_1\left(\frac{1}{2}n; \frac{1}{4}\boldsymbol{\Omega}, \boldsymbol{W}\right) \\ &\quad \times \det(\boldsymbol{W})^{\frac{1}{2}n-\frac{1}{2}(p+1)} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{W}\right) \prod_{i<j}^{p}(w_i - w_j) \end{aligned}$$

Further for any constants a and b,

$$C_{\phi}^{\kappa,\tau}(a\boldsymbol{X}, b\boldsymbol{X}) = a^k b^t C_{\phi}^{\kappa,\tau}(\boldsymbol{X}, \boldsymbol{X}) = a^k b^t \theta_{\phi}^{\kappa,\tau} C_{\phi}^{\kappa,\tau}(\boldsymbol{X}),$$

from Davis (1979).

3 Eigenvalue-based Univariate Weighted Distribution

The theorems stated below originates from Definition 2.2, and is now applied to the distributions of the eigenvalues of Wishart matrices.

Theorem 3.1. *Let* $\boldsymbol{A} \sim \mathcal{W}_2(\boldsymbol{I}_2, n)$, $n > 1$. *Let* λ_i *is the* i*-th eigenvalue of* $\boldsymbol{A}$ *with p.d.f.* $f(\lambda_i)$, $i = 1, 2$. *Then the distribution* $g(.)$ *given by*

$$g(\lambda_i) = \frac{\lambda_i f(\lambda_i)}{E(\lambda_i)}, \quad i = 1, 2,$$

is rotation invariant.

Proof: We give the proof of the distribution of λ_i for $i = 1$; for $i = 2$ it is similar. From Theorem 13.3.2 of Muirhead (2005), the joint p.d.f. of λ_1, λ_2 is given by

$$f(\lambda_1, \lambda_2) \quad = \quad \frac{\pi^{\frac{1}{2}} \lambda_1^{\frac{1}{2}(n-3)} \lambda_2^{\frac{1}{2}(n-3)} (\lambda_1 - \lambda_2) e^{-\frac{1}{2}(\lambda_1 + \lambda_2)}}{2^n \Gamma\left(\frac{1}{2}n\right) \Gamma\left[\frac{1}{2}(n-1)\right]}, \qquad \lambda_1 > \lambda_2.$$

Perform the transformation $y = \frac{\lambda_2}{\lambda_1}$ with the Jacobian $J(\lambda_2 \to y) = \lambda_1$ to get the marginal distribution of the largest eigenvalue as

$$f(\lambda_1) = \frac{\pi^{\frac{1}{2}} K(n, \lambda_1)}{2^n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} \, \lambda_1^{n-1} e^{-\frac{1}{2}\lambda_1}, \tag{3}$$

where $K(.,.)$ is the Kummer function given by $K(n, \lambda_1) = \int_0^1 y^{\frac{1}{2}(n-3)} (1-y) e^{-\frac{1}{2}\lambda_1 y} \mathrm{d}y$.

Further, from (3) we have that

$$\begin{aligned} E(\lambda_1) \quad &= \quad \frac{\pi^{\frac{1}{2}}}{2^n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} \int_0^1 y^{\frac{1}{2}(n-3)} (1-y) \int_0^\infty \lambda_1^n e^{-\frac{1}{2}\lambda_1(1+y)} \mathrm{d}\lambda_1 \mathrm{d}y \\ &= \quad \frac{2\pi^{\frac{1}{2}} \Gamma(n+1)}{2^n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} \Phi(n), \quad \Phi(n) = \int_0^1 y^{\frac{1}{2}(n-3)} (1-y)(1+y)^{-(n+1)} \mathrm{d}y. \end{aligned}$$

Then by making use of (3), it follows that

$$g(\lambda_1) = \frac{K(n, \lambda_1)}{2\Phi(n) \Gamma(n+1)} \, \lambda_1^n e^{-\frac{1}{2}\lambda_1}. \tag{4}$$

Comparing (3) with (4), one can immediately realize that they have the same structure in p.d.f. According to Definition 2.2 the proof is complete. ■

Theorem 3.2. *Let $\boldsymbol{A} \sim \mathcal{W}_2(\boldsymbol{\Sigma}, n)$, $n > 1$ and $\boldsymbol{\Sigma} = \mathrm{diag}(\alpha_1, \alpha_2)$. Then the distribution $g(.)$ given by*

$$g(\lambda_i) = \frac{\lambda_i f(\lambda_i)}{E(\lambda_i)}, \quad i = 1, 2,$$

is rotation invariant.

Proof: We give the proof for $i = 1$; for $i = 2$ it is similar. Using the spectral decomposition on $\boldsymbol{A} = \boldsymbol{H}'\boldsymbol{\Lambda}\boldsymbol{H}$, for $\boldsymbol{H} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $0 < \alpha < 2\pi$ with the Jacobian $J(a_{11}, a_{12}, a_{22} \rightarrow \alpha, \lambda_1, \lambda_2) = (\lambda_1 - \lambda_2)$, where $1 > \lambda_1 > \lambda_2 > 0$ we obtain

$$f(\boldsymbol{H}'\boldsymbol{\Lambda}\boldsymbol{H}) = \frac{\prod_{i=1}^{2} \lambda_i^{\frac{1}{2}(n-3)}(\lambda_1 - \lambda_2)}{2^n \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} e^{-(a\cos^2\alpha + b\sin^2\alpha)},$$

where $a = \frac{1}{2}\left(\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}\right)$, and $b = \frac{1}{2}\left(\frac{\lambda_2}{\alpha_1} + \frac{\lambda_1}{\alpha_2}\right)$. Then it follows that

$$\begin{aligned}
f(\boldsymbol{\Lambda}) &= 4\int_0^{\frac{1}{2}\pi} f(\boldsymbol{H}'\boldsymbol{\Lambda}\boldsymbol{H}) \mathrm{d}\alpha \\
&= \frac{4\prod_{i=1}^{2} \lambda_i^{\frac{1}{2}(n-3)}(\lambda_1 - \lambda_2)}{2^n \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\
&\quad \times \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_0^{\frac{1}{2}\pi} (a\cos^2\alpha + b\sin^2\alpha)^r \mathrm{d}\alpha \\
&= \frac{4\prod_{i=1}^{2} \lambda_i^{\frac{1}{2}(n-3)}(\lambda_1 - \lambda_2)}{2^n \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\
&\quad \times \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{x=0}^{r} C_x^r \int_0^{\frac{1}{2}\pi} (a\cos^2\alpha)^x (b\sin^2\alpha)^{r-x} \mathrm{d}\alpha \\
&= \frac{8\prod_{i=1}^{2} \lambda_i^{\frac{1}{2}(n-3)}(\lambda_1 - \lambda_2)}{2^n \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\
&\quad \times \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{x=0}^{r} C_x^r a^x b^{r-x} B\left(r - x + \frac{1}{2}, x + \frac{1}{2}\right)
\end{aligned}$$

where $B(.,.)$ is the beta function. Therefore, the marginal distribution of the largest eigenvalue is

$$\begin{aligned} f(\lambda_1) &= \int_0^{\lambda_1} f(\mathbf{\Lambda}) \mathrm{d}\lambda_2 \\ &= \frac{8\lambda_1^{\frac{1}{2}(n-3)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{x=0}^{r} C_x^r B\left(r-x+\frac{1}{2}, x+\frac{1}{2}\right)}{2^n \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\ &\quad \times \int_0^{\lambda_1} \lambda_1^{\frac{1}{2}(n-3)} (\lambda_1 - \lambda_2) \left(\frac{\lambda_1}{2\alpha_1} + \frac{\lambda_2}{2\alpha_2}\right)^x \left(\frac{\lambda_2}{2\alpha_1} + \frac{\lambda_1}{2\alpha_2}\right)^{r-x} \mathrm{d}\lambda_2 \\ &= \frac{8\lambda_1^{\frac{1}{2}(n-3)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{x=0}^{r} C_x^r B\left(r-x+\frac{1}{2}, x+\frac{1}{2}\right)}{2^{n+r} \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\ &\quad \times \sum_{k=0}^{x} C_k^x \alpha_1^{k-x} \alpha_2^{-k} \sum_{t=0}^{r-x} C_t^{r-x} \alpha_2^{t+x-r} \alpha_1^{-t} \lambda_1^{r-t-k} \\ &\quad \times \int_0^{\lambda_1} (\lambda_1 - \lambda_2) \lambda_2^{\frac{1}{2}(n-3)+k+t} \mathrm{d}\lambda_2 \\ &= \sum_{r=0}^{\infty} P_1(r) \lambda_1^{n+r-1}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} P_1(r) &= \frac{8\frac{(-1)^r}{r!} \sum_{x=0}^{r} C_x^r B\left(r-x+\frac{1}{2}, x+\frac{1}{2}\right)}{2^{n+r} \pi^{\frac{1}{2}} \prod_{i=1}^{2} \alpha_i^{\frac{1}{2}n} \prod_{i=1}^{2} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\ &\quad \times \sum_{k=0}^{x} \sum_{t=0}^{r-x} \frac{C_k^x C_t^{r-x} \alpha_1^{k-x-t} \alpha_2^{t+x-r-k}}{\left(\frac{1}{2}(n-1)+k+t\right)\left(\frac{1}{2}(n+1)+k+t\right)}. \end{aligned}$$

From (4), after some manipulation, it follows that

$$g(\lambda_1) = \sum_{r=0}^{\infty} P_2(r) \lambda_1^{n+r}, \tag{6}$$

where $P_2(r) = \frac{(n+r)P_1(r)}{\sum_{r=0}^{\infty} P_1(r)}$.

Finally, comparing (5) with (6), f and g has similar distributional structures. ■

4 Eigenvalue-based Matrix-variate Weighted Distribution

Now, we extend the results to a more general case of Wishart distributions. These results are useful for exploring various applications for practical purposes. As an important consequence of these results (see Corollary 4.1.2 below) we define the rotation invariant Wishart distribution, such distribution is associated with the size-biased sampling.

Theorem 4.1. *Let* $\boldsymbol{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$*,* $n > p-1$ *and* $\boldsymbol{X} \in \mathcal{S}(p)$ *be any constant or random matrix independent of* $\boldsymbol{A}$*. Let* $f(.)$ *be the distribution of the eigenvalues of* $\boldsymbol{A}$ *given by* $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$*, and let* h *be a real Borel measurable function. Then the p.d.f.* $g(.)$*, given by*

$$g(\boldsymbol{\Lambda}) = \frac{h(\mathrm{tr}\,\boldsymbol{\Lambda X}) f(\boldsymbol{\Lambda})}{E\left[h(\mathrm{tr}\,\boldsymbol{\Lambda X})\right]},$$

is rotation invariant.

Proof: Using the spectral decomposition on $\boldsymbol{A} = \boldsymbol{H}'\boldsymbol{\Lambda H}$, $\boldsymbol{H} \in \mathcal{O}(p)$ we have

$$f(\boldsymbol{H}'\boldsymbol{\Lambda H}) = \frac{\det(\boldsymbol{\Lambda})^{\frac{1}{2}n - \frac{1}{2}(p+1)} \,\mathrm{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{H}'\boldsymbol{\Lambda H}\right\}}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^p \Gamma\left[\frac{1}{2}(n+1-i)\right]} \prod_{i<j}(\lambda_i - \lambda_j).$$

Then since according to Greenacre (1973), f is not invariant, we have

$$\begin{aligned}
f(\boldsymbol{\Lambda}) &= \frac{\det(\boldsymbol{\Lambda})^{\frac{1}{2}n - \frac{1}{2}(p+1)} \prod_{i<j}(\lambda_i - \lambda_j)}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^p \Gamma\left[\frac{1}{2}(n+1-i)\right]} \int_{\mathcal{O}(p)} \mathrm{etr}\left\{-\frac{1}{2}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{H}'\boldsymbol{\Lambda H}\right\} \mathrm{d}\boldsymbol{H} \\
&= \frac{\det(\boldsymbol{\Lambda})^{\frac{1}{2}n - \frac{1}{2}(p+1)} \prod_{i<j}(\lambda_i - \lambda_j)}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^p \Gamma\left[\frac{1}{2}(n+1-i)\right]} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!} \frac{C_\kappa(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}) C_\kappa(\boldsymbol{\Lambda})}{C_\kappa(\boldsymbol{I}_p)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} P(\kappa) \det(\boldsymbol{\Lambda})^{\frac{1}{2}n - \frac{1}{2}(p+1)} C_\kappa(\boldsymbol{\Lambda}) \prod_{i<j}(\lambda_i - \lambda_j), \qquad (7)
\end{aligned}$$

where $P(\kappa) = \frac{1}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^p \Gamma\left[\frac{1}{2}(n+1-i)\right] k!} \frac{C_\kappa(-\frac{1}{2}\boldsymbol{\Sigma}^{-1})}{C_\kappa(\boldsymbol{I}_p)}$, from Muirhead (2005, p.248).

In order to get the expectation of $h(\mathrm{tr}\,\boldsymbol{\Lambda X})$ one may use the following approach

$$\begin{aligned}
E\left[h(\mathrm{tr}\,\boldsymbol{\Lambda X})\right] &= \frac{1}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^p \Gamma\left[\frac{1}{2}(n+1-i)\right]} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!} \frac{C_\kappa(-\frac{1}{2}\boldsymbol{\Sigma}^{-1})}{C_\kappa(\boldsymbol{I}_p)} \\
&\quad \times \int_{\mathcal{S}(p)} h(\mathrm{tr}\,\boldsymbol{\Lambda X}) \det(\boldsymbol{\Lambda})^{\frac{1}{2}n - \frac{1}{2}(p+1)} C_\kappa(\boldsymbol{\Lambda}) \prod_{i<j}(\lambda_i - \lambda_j) \mathrm{d}\boldsymbol{\Lambda}
\end{aligned}$$

To solve the above integral, we make use of the technique used in Arashi (2013) with $\boldsymbol{\Lambda} = \boldsymbol{HTH}', \boldsymbol{H} \in O(p)$, to get

$$\begin{aligned}
E\left[h(\mathrm{tr}\,\boldsymbol{\Lambda X})\right] &= \int_{\mathcal{O}(p)} \int_{\mathcal{S}(p)} h(\mathrm{tr}\,\boldsymbol{\Lambda X}) f(\boldsymbol{H}'\boldsymbol{\Lambda H}) \mathrm{d}\boldsymbol{\Lambda}\mathrm{d}\boldsymbol{H} \\
&= \int_{\mathcal{O}(p)} \frac{\det(\boldsymbol{\Lambda})^{\frac{1}{2}n - \frac{1}{2}(p+1)} \prod_{i<j}(\lambda_i - \lambda_j)}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^p \Gamma\left[\frac{1}{2}(n+1-i)\right]} \\
&\quad \times \int_{\mathcal{S}(p)} h(\mathrm{tr}\,\boldsymbol{TH}'\boldsymbol{XH}) \det(\boldsymbol{T})^{\frac{1}{2}n - \frac{1}{2}(p+1)}
\end{aligned}$$

$$
\begin{aligned}
& \times \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{T}\right)\mathrm{d}\boldsymbol{T}\mathrm{d}\boldsymbol{H} \\
= & \int_{\mathcal{O}(p)} \frac{1}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \sum_{t=0}^{\infty}\sum_{\tau} \frac{1}{t!} \\
& \times \int_{\mathcal{S}(p)} h(\operatorname{tr} \boldsymbol{T}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H}) \det(\boldsymbol{T})^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\
& \times C_{\tau}\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{T}\right)\mathrm{d}\boldsymbol{T}\mathrm{d}\boldsymbol{H} \qquad (8)
\end{aligned}
$$

Using the result of Teng et al. (1989) for (8) we obtain

$$
E\left[h(\operatorname{tr} \boldsymbol{\Lambda}\boldsymbol{X})\right] = \frac{\sum_{t=0}^{\infty}\sum_{\tau} \frac{\Gamma_p\left[\frac{1}{2}n,\tau\right] C_{\tau}\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\right) C_{\tau}(\boldsymbol{X}^{-1})\gamma}{t!\Gamma\left[\frac{1}{2}np+t\right]C_{\tau}(\boldsymbol{I}_p)} det(\boldsymbol{X})^{-\frac{1}{2}n}}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(n+1-i)\right]}, \qquad (9)
$$

where

$$
\gamma = \int_{\mathbb{R}^+} x^{\left(\frac{1}{2}n\right)p+k-1} h(x)\mathrm{d}x. \qquad (10)
$$

Thus from (7) and (9) one obtains

$$
g(\boldsymbol{\Lambda}) = \sum_{k=0}^{\infty}\sum_{\kappa} P^*(\kappa) h(\operatorname{tr} \boldsymbol{\Lambda}\boldsymbol{X}) \det(\boldsymbol{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)} C_{\kappa}(\boldsymbol{\Lambda}) \prod_{i<j}(\lambda_i - \lambda_j),
$$

where

$$
\begin{aligned}
P^*(\kappa) = & \left\{\sum_{t=0}^{\infty}\sum_{\tau} \frac{1}{t!} \frac{\Gamma_p\left[\frac{1}{2}n,\tau\right]}{\Gamma\left[\frac{1}{2}np+t\right]} \frac{C_{\tau}\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\right) C_{\tau}(\boldsymbol{X}^{-1})}{C_{\tau}(\boldsymbol{I}_p)}\right\}^{-1} \\
& \times \frac{\det(\boldsymbol{X})^{\frac{1}{2}n} C_{\kappa}\left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\right)}{k! C_{\kappa}(\boldsymbol{I}_p)}.
\end{aligned}
$$

Now using Taylor series expansion for $h(x+a) = \sum_{i=1}^{\infty} \frac{h^{(i)}(a)a^i}{i!}$, where $h^{(i)}(a)$ is the i^{th} derivative of h at point a, and taking $\boldsymbol{Y} = \operatorname{diag}\left(\frac{a}{p}, \ldots, \frac{a}{p}\right)$ we can obtain

$$
\begin{aligned}
g(\boldsymbol{\Lambda}) = & \sum_{k=0}^{\infty}\sum_{\kappa} P^*(\kappa) \sum_{s=0}^{\infty} \frac{h^{(s)}(a)}{s!} \left(\operatorname{tr}(\boldsymbol{\Lambda}\boldsymbol{X} - \boldsymbol{Y})\right)^s \det(\boldsymbol{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)} C_{\kappa}(\boldsymbol{\Lambda}) \\
& \times \prod_{i<j}(\lambda_i - \lambda_j) \\
= & \sum_{k=0}^{\infty}\sum_{\kappa} \mathbb{P}(\kappa) \det(\boldsymbol{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)} C_{\kappa}(\boldsymbol{\Lambda}) C_{\phi}(\boldsymbol{\Lambda}\boldsymbol{X} - \boldsymbol{Y}) \prod_{i<j}(\lambda_i - \lambda_j), \qquad (11)
\end{aligned}
$$

where $\mathbb{P}(\kappa) = \sum_{s=0}^{\infty} \frac{h^{(s)}(a)}{s!} \sum_{\phi} P^*(\kappa)$.

Comparing (7) with that of (11), we find both distributions have the same structures. ■

Next, we study certain special cases of Theorem 4.1 arising when $h(.)$ is defined suitably for practical considerations. It is important to note that for constant matrices $\boldsymbol{X} \in \mathcal{S}(p)$ the distribution $g(.)$ is a valid p.d.f. However for random matrices $\boldsymbol{X} \in \mathcal{S}(p)$, for every selection $h(.)$, for $g(.)$ to be a valid p.d.f, the $h(.)$ function has to be integrated out over $\mathcal{S}(p)$.

Corollary 4.1.1. *As a result of Theorem 4.1, we have the following results.*

(i) If $h(x) = x^\alpha$, $\alpha \in \mathbb{C}$, *then*

$$g(\boldsymbol{\Lambda}) = \frac{\operatorname{tr}(\boldsymbol{X}\boldsymbol{\Lambda})^\alpha f(\boldsymbol{\Lambda})}{E\left[\operatorname{tr}(\boldsymbol{X}\boldsymbol{\Lambda})^\alpha\right]}.$$

(ii) If $h(x) = e^{-\beta x}$, $\beta \in \mathbb{C}$, *then*

$$g(\boldsymbol{\Lambda}) = \frac{\operatorname{etr}(-\beta\boldsymbol{X}\boldsymbol{\Lambda}) f(\boldsymbol{\Lambda})}{E\left[\operatorname{etr}(-\beta\boldsymbol{X}\boldsymbol{\Lambda})\right]}.$$

(iii) If $h(x) = x^\alpha e^{-\beta x}$, $\alpha, \beta \in \mathbb{C}$, *then*

$$g(\boldsymbol{\Lambda}) = \frac{\operatorname{tr}(\boldsymbol{X}\boldsymbol{\Lambda})^\alpha \operatorname{etr}(-\beta\boldsymbol{X}\boldsymbol{\Lambda}) f(\boldsymbol{\Lambda})}{E\left[\operatorname{tr}(\boldsymbol{X}\boldsymbol{\Lambda})^\alpha \operatorname{etr}(-\beta\boldsymbol{X}\boldsymbol{\Lambda})\right]}.$$

(iv) If $h(x) = (1+\beta x)^\alpha$, $\alpha \in \mathbb{C}$ *and* $\beta \in \mathbb{R}^+$, *then*

$$g(\boldsymbol{\Lambda}) = \frac{(1+\beta \operatorname{tr}(\boldsymbol{X}\boldsymbol{\Lambda}))^\alpha f(\boldsymbol{\Lambda})}{E\left[(1+\beta \operatorname{tr}(\boldsymbol{X}\boldsymbol{\Lambda}))^\alpha\right]}.$$

Corollary 4.1.2. *From Corollary 4.1.1 (iii), one may think of the following rotation invariant distribution, which will be called as rotation invariant Wishart distribution whose p.d.f. is given by*

$$g(\boldsymbol{\Lambda}) = \frac{h(\operatorname{tr}\boldsymbol{X}\boldsymbol{\Lambda}) f(\boldsymbol{\Lambda})}{E\left[h(\operatorname{tr}\boldsymbol{X}\boldsymbol{\Lambda})\right]},$$

where $\boldsymbol{X} \sim W_p(\boldsymbol{\Sigma}^*, m), \boldsymbol{\Sigma}^* \in \mathcal{S}(p)$ *is independent of* $\boldsymbol{A}$. *The above distribution has an application when the sampling is biased.*

In the following theorem, we prove that the non-central Wishart distribution is also rotation invariant.

Theorem 4.2. *Let* $\boldsymbol{A} \sim W_p(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, n)$. *Let* $f(.)$ *be the distribution of* $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$, $\boldsymbol{X} \in \mathcal{C}^{p\times p}$ *and* h *is a real Borel measurable function. Then the distribution* $g(.)$ *given by*

$$g(\boldsymbol{\Lambda}) = \frac{h(\operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{X}) f(\boldsymbol{\Lambda})}{E\left[h(\operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{X})\right]},$$

is rotation invariant.

Proof: Here we only compute the term $E\left[h(\operatorname{tr}\mathbf{\Lambda}\boldsymbol{X})\right]$; since the rest of the proof is similar to Theorem 4.1.

According to Theorem 2.1 and using Theorem 7.3.3 of Muirhead (2005), the distribution of eigenvalues $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 > \ldots, \lambda_p > 0$ can be rewritten as

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)} \operatorname{etr}\left(-\frac{1}{2}\mathbf{\Omega}\right)\det(\mathbf{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\
&\quad \times \operatorname{etr}\left(-\frac{1}{2}\mathbf{\Lambda}\right)\prod_{i<j}^{p}(\lambda_i-\lambda_j)\prod_{j=1}^{p}\lambda_j \int_{\mathcal{O}(p)} {}_0F_1\left(\frac{1}{2}n;\frac{1}{4}\mathbf{\Omega}\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H} \\
&= \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)} \operatorname{etr}\left(-\frac{1}{2}\mathbf{\Omega}\right)\det(\mathbf{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\frac{1}{2}\mathbf{\Lambda}\right) \\
&\quad \times \prod_{i<j}^{p}(\lambda_i-\lambda_j)\prod_{j=1}^{p}\lambda_j \sum_{t=0}^{\infty}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\tau}\frac{1}{t!}\left(\frac{1}{4}\right)^t\frac{C_\tau(\mathbf{\Omega})C_\tau(\mathbf{\Lambda})}{C_\tau(\boldsymbol{I}_p)} \qquad (12)
\end{aligned}$$

Then, in a similar fashion as in the proof of Theorem 4.1, by taking $\mathbf{\Lambda} = \boldsymbol{H}\boldsymbol{T}\boldsymbol{H}'$, we have

$$\begin{aligned}
E[h(\operatorname{tr}\mathbf{\Lambda}\boldsymbol{X})] &= \int_{\mathcal{O}(p)} \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)}\operatorname{etr}\left(-\frac{1}{2}\mathbf{\Omega}\right) \\
&\quad \times \sum_{t=0}^{\infty}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\tau}\frac{1}{t!}\left(\frac{1}{4}\right)^t\frac{C_\tau(\mathbf{\Omega})}{C_\tau(\boldsymbol{I}_p)} \\
&\quad \times \int_{\boldsymbol{T}} h(\operatorname{tr}\boldsymbol{T}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H})\; C_\tau(\boldsymbol{T}) \\
&\quad \times \det(\boldsymbol{T})^{\frac{1}{2}(n+2)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{T}\right)\mathrm{d}\boldsymbol{T}\mathrm{d}\boldsymbol{H}.
\end{aligned}$$

Make use of the Taylor series expansion

$$h(\operatorname{tr}\boldsymbol{T}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H}) = \sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}(\operatorname{tr}\boldsymbol{T}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H})^k = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{h^{(k)}(0)}{k!}C_\kappa(\boldsymbol{T}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H})$$

and change the order of integration to obtain

$$\begin{aligned}
E[h(\operatorname{tr}\mathbf{\Lambda}\boldsymbol{X})] &= \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)}\operatorname{etr}\left(-\frac{1}{2}\mathbf{\Omega}\right)\sum_{t=0}^{\infty}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\tau}\frac{1}{t!}\left(\frac{1}{4}\right)^t \\
&\quad \times \frac{C_\tau(\mathbf{\Omega})}{C_\tau(\boldsymbol{I}_p)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{h^{(k)}(0)}{k!} \times \int_{\boldsymbol{T}}\det(\boldsymbol{T})^{\frac{1}{2}(n+2)-\frac{1}{2}(p+1)} \\
&\quad \times \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{T}\right)C_\tau(\boldsymbol{T})\int_{\mathcal{O}(p)} C_\kappa(\boldsymbol{T}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H})\;\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{T} \\
&= \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)}\operatorname{etr}\left(-\frac{1}{2}\mathbf{\Omega}\right)\sum_{t=0}^{\infty}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\tau}\frac{1}{t!}\left(\frac{1}{4}\right)^t \\
&\quad \times \frac{C_\tau(\mathbf{\Omega})}{C_\tau(\boldsymbol{I}_p)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{h^{(k)}(0)}{k!}\frac{C_\kappa(\boldsymbol{X})}{C_\kappa(\boldsymbol{I}_p)}\sum_{\phi\in\kappa\cdot\tau}\left(\theta_\phi^{\kappa,\tau}\right)^2
\end{aligned}$$

$$\int_{\boldsymbol{T}} \det(\boldsymbol{T})^{\frac{1}{2}(n+2)-\frac{1}{2}(p+1)} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{T}\right) C_\phi(\boldsymbol{T}) \mathrm{d}\boldsymbol{T}$$

$$= \frac{\pi^{\frac{1}{2}p^2}}{2^{\frac{1}{2}np}\Gamma_p\left(\frac{1}{2}n\right)\Gamma_p\left(\frac{1}{2}p\right)} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \sum_{t=0}^{\infty}\sum_{\tau} \frac{1}{\left(\frac{1}{2}n\right)_\tau} \frac{1}{t!}\left(\frac{1}{4}\right)^t$$

$$\times \frac{C_\tau(\boldsymbol{\Omega})}{C_\tau(\boldsymbol{I}_p)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{h^{(k)}(0)}{k!} \frac{C_\kappa(\boldsymbol{X})}{C_\kappa(\boldsymbol{I}_p)} \sum_{\phi\in\kappa\cdot\tau} \left(\theta_\phi^{\kappa,\tau}\right)^2 2^{\frac{1}{2}(p(n+2)+2k)}$$

$$\times \left(\frac{1}{2}(n+2)\right)_\phi \Gamma_m\left(\frac{1}{2}(n+2)\right) C_\phi(\boldsymbol{I}_p)$$

Accordingly, one may find the desired representation, that completes the proof. ■

5 Mapping illustration

According to (1), it is not possible to rebuild the p.d.f. $f(.)$ from $g(.)$ - the weighted distributions of a population. In other words, it is not possible to assume $h(.) \equiv 1$ to get $g(x) = f(x)$, since the $h(.)$ function should admit the Taylor's series expansion as a regularity condition for the proofs of the theorems in Section 3. Hence, if the researcher is provided with the samples obtained from $g(.)$, there is no mechanism for coming close to $f(.)$. The usefulness of weighted distributions is a well established fact and not having a mechanism in this context is not a weakness of the concept of weighted distribution. However, there is a possibility to come close to this mechanism, which is described below.

If there exists a map $\mathcal{M}$ that concludes $g(x) = f(x)$ or $g(x) \approx f(x)$, then we refer to this map as a mirror. In what follows, we construct a mirror by making use of the weighted distributions of eigenvalues of a random matrix. We consider one special case of weights, (see Theorem 4.1), to show that one may use the map referred to in Definition 2.2 for constructing a mirror.

To be more precise, the comparison of the plots of $g(.)$ and the p.d.f. of the joint distribution of the eigenvalues of the Wishart matrix is of interest. To demonstrate the afore-mentioned concept, we illustrate the performance of the functions $f(\boldsymbol{\Lambda})$ and $g(\boldsymbol{\Lambda})$ based on only one generation from the Wishart distribution.

We note that the matrix $\boldsymbol{X}$ can be a constant or random complex matrix. For our purpose, we assume $\boldsymbol{X} = \boldsymbol{I}_p$, $\boldsymbol{\Sigma} = \alpha\boldsymbol{I}_p$, $\alpha \in \mathbb{R}^+$ and also consider the special weight $h(x) = \exp\left(-\frac{\beta}{2}x\right)$. Thus, according to Corollary 4.1.1, we have

$$g(\boldsymbol{\Lambda}) = w(\boldsymbol{\Lambda})f(\boldsymbol{\Lambda}), \quad \text{where} \quad w(\boldsymbol{\Lambda}) = \frac{\operatorname{etr}\left(-\frac{\beta}{2}\boldsymbol{\Lambda}\right)}{E\left[\operatorname{etr}\left(-\frac{\beta}{2}\boldsymbol{\Lambda}\right)\right]}$$

Using Corollary 3.2.19 of Muirhead (2005),

$$f(\boldsymbol{\Lambda}) \quad = \quad \frac{\pi^{\frac{p}{2}}}{(2\alpha)^{\frac{pn}{2}}\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{n}{2}\right)} \det(\boldsymbol{\Lambda})^{\frac{n}{2}-\frac{1}{2}(p+1)} \operatorname{etr}\left(-\frac{1}{2\alpha}\boldsymbol{\Lambda}\right) \prod_{i<j}^{p}(\lambda_i - \lambda_j).$$

Table 1: Values of p.d.f. $f(\mathbf{\Lambda})$ and $g(\mathbf{\Lambda})$ for different values of parameters (α, β).

		$\beta = 0.01$	$\beta = 0.1$	$\beta = 1$	$\beta = 10$	$\beta = 100$
	$f(\mathbf{\Lambda})$	$g(\mathbf{\Lambda})$	$g(\mathbf{\Lambda})$	$g(\mathbf{\Lambda})$	$g(\mathbf{\Lambda})$	$g(\mathbf{\Lambda})$
$\alpha = 0.01$	0	0	0	0	0	0
$\alpha = 0.1$	5.324068e-52	4.743447e-52	1.677643e-52	5.324068e-52	1.754126e-102	0
$\alpha = 1$	2.175332e-08	1.981957e-08	8.485153e-09	2.175332e-08	5.084515e-57	0
$\alpha = 10$	2.80966e-06	3.168843e-06	4.885186e-06	2.80966e-06	1.677643e-52	0
$\alpha = 100$	2.569014e-08	1.291537e-07	3.168843e-06	2.569014e-08	4.743447e-52	0

Then for $\sigma = \frac{1}{\alpha} + \beta$ we obtain

$$\begin{aligned} E\left[\operatorname{etr}\left(-\frac{\beta}{2}\mathbf{\Lambda}\right)\right] &= \frac{\pi^{\frac{p}{2}}}{(2\alpha)^{\frac{pn}{2}}\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{n}{2}\right)}\int_{\mathbf{\Lambda}} \det(\mathbf{\Lambda})^{\frac{n}{2}-\frac{1}{2}(p+1)} \operatorname{etr}\left(-\frac{\sigma}{2}\mathbf{\Lambda}\right) \\ &\quad \times \prod_{i<j}^{p}(\lambda_i - \lambda_j)\mathrm{d}\mathbf{\Lambda} \\ &= \left(\frac{1}{\alpha\sigma}\right)^{\frac{pn}{2}}. \end{aligned}$$

In conclusion, we have

$$g(\mathbf{\Lambda}) = \left(\frac{\sigma}{2}\right)^{\frac{pn}{2}} \frac{\pi^{\frac{p}{2}}}{\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{n}{2}\right)} \det(\mathbf{\Lambda})^{\frac{n}{2}-\frac{1}{2}(p+1)} \operatorname{etr}\left(-\frac{\sigma}{2}\mathbf{\Lambda}\right)\prod_{i<j}^{p}(\lambda_i - \lambda_j).$$

Table 1 shows the values of $f(\mathbf{\Lambda})$ and $g(\mathbf{\Lambda})$, where the eigenvalues are derived from one generation of $W_5(\alpha \boldsymbol{I}_5, 5)$, for different parameter values α and β. It might be interesting to see the changes based on the parameters p and n. Based on the result of Table 1, as β increases, the values of $f(\mathbf{\Lambda})$ and $g(\mathbf{\Lambda})$ get closer for each value of α. This is apparent from the specific weight $h(x) = \exp\left(-\frac{\beta}{2}x\right)$, since as $\beta \to \infty$, the $h(.)$ function approaches to one and gives $f(\mathbf{\Lambda}) = g(\mathbf{\Lambda})$. This is an important result, since for none of the selections of the quartet (p, n, α, β), we have $f(\mathbf{\Lambda}) = g(\mathbf{\Lambda})$. To see this, it is sufficient to consider that $f(\mathbf{\Lambda}) = g(\mathbf{\Lambda})$ implies $\operatorname{tr}(\mathbf{\Lambda}) = -\frac{(\alpha\sigma)^{\frac{pn}{2}}}{\frac{1}{\alpha}+\sigma}$, which is a contradiction with $\sum_{i=1}^{p}\lambda_i$ to be positive. Thus, for $g(.)$ to be as close to $f(.)$ as possible, it is recommended that β increases. From the view point of the weighted distributions, the larger the β, the more peneplanation for bigger weights.

Remark 5.1. *A mirror for the p.d.f. of main distribution can be constructed through the weighted distribution of eigenvalues where exponential rate is a weight. In other words, according to the result of Section 3, taking exponential weight for the eigenvalue distribution corresponds to a length-biased version of population distribution.*

Table 1 clearly shows that as β increases the weighted distribution approaches to zero, for a fixed value of parameter α. More importantly, although we claimed that as β goes to infinity $g(\mathbf{\Lambda})$ gets close to $f(\mathbf{\Lambda})$, from the result of Table 1, it is clear that this fact needs to be controlled by the parameter α. In other words, the mirror performs well for some specific value of a parameter α. In spite of the well established theoretical performance of the mirror (as $\beta \to \infty$), we always need to select relevant parameters α and β. For example, for all selections of α, when β equals one, a perfect mirror is obtained.

6 Graphical Representations

In this section, we display the graphs of p.d.f. and cumulative distribution function (c.d.f.) of a rotation invariant Wishart distribution.

Under the assumptions of Theorem 4.1, for $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_p^2)$, the p.d.f. of the eigenvalues of $\boldsymbol{A}$ is given by (see Muirhead (2005, p.260))

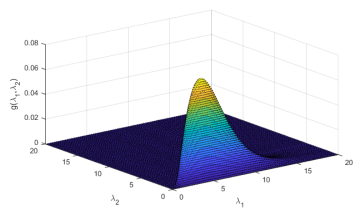
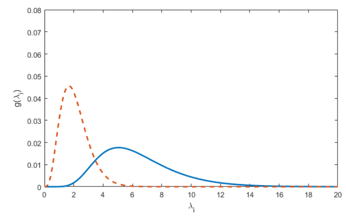
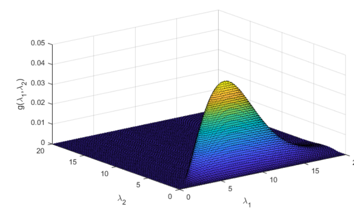
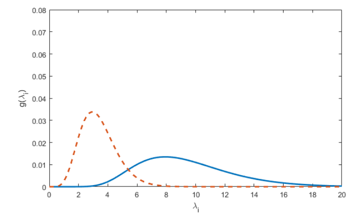
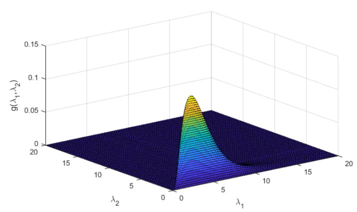
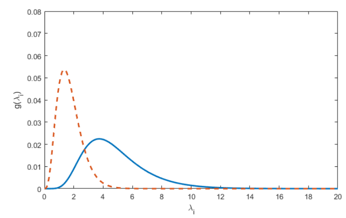
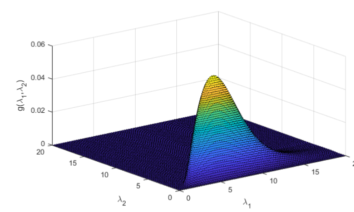
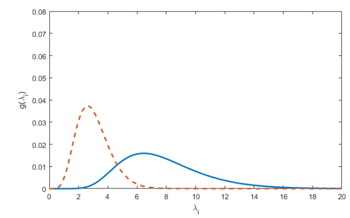
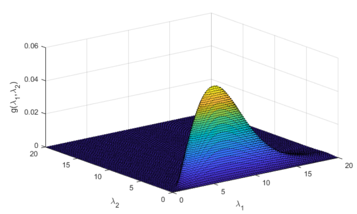
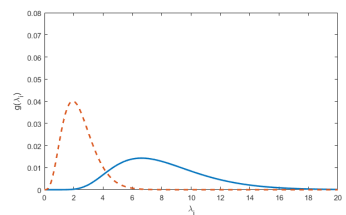
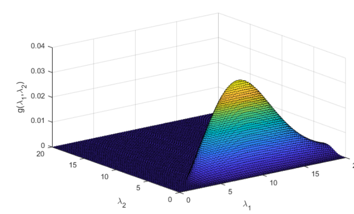
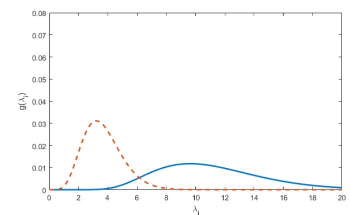
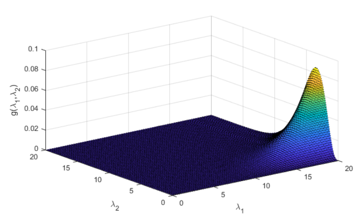
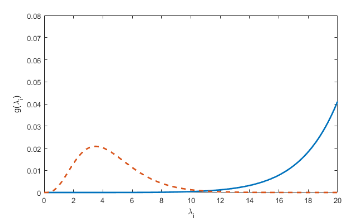
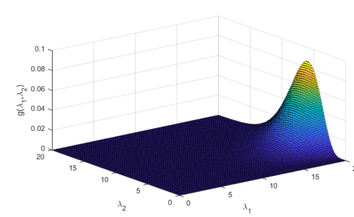
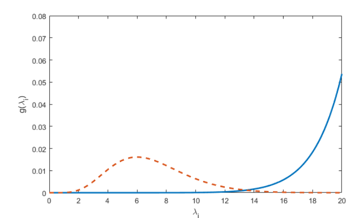
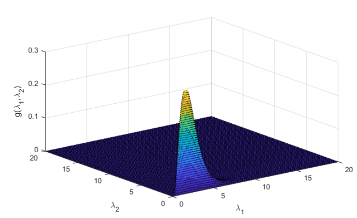
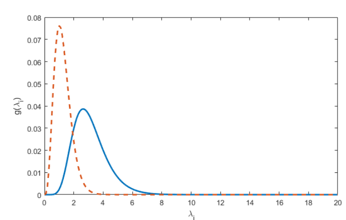
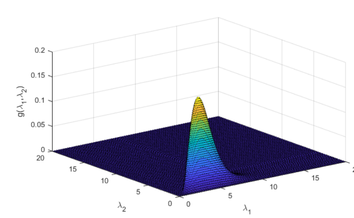
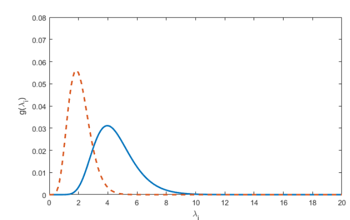
$$\begin{aligned} f(\boldsymbol{\Lambda}) &= \frac{\det(\boldsymbol{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)} \prod_{i<j}(\lambda_i - \lambda_j)}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \int_{\mathcal{O}(p)} {}_0F_0\left(-\frac{1}{2}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{H}'\boldsymbol{\Lambda}\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} \\ &= \frac{\det(\boldsymbol{\Lambda})^{\frac{1}{2}n-\frac{1}{2}(p+1)} \prod_{i<j}(\lambda_i - \lambda_j)}{2^n \pi^{\frac{1}{2}} \det(\boldsymbol{\Sigma})^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \; {}_0F_0^{(p)}\left(-\frac{1}{2}\,\boldsymbol{\Sigma}^{-1}, \boldsymbol{\Lambda}\right) \end{aligned} \tag{13}$$

Using the result for ${}_0F_0^{(p)}$ from Khatri (1968) results in

$$\begin{aligned} f(\boldsymbol{\Lambda}) &= \int_{\mathcal{O}(p)} f(\boldsymbol{H}'\boldsymbol{\Lambda}\boldsymbol{H})\mathrm{d}\boldsymbol{H} \\ &= \frac{\prod_{i=1}^{p} \lambda_i^{\frac{1}{2}n-\frac{1}{2}(p+1)} \prod_{i<j}(\lambda_i - \lambda_j)}{2^n \pi^{\frac{1}{2}} \prod_{j=1}^{n} \sigma_j^n \prod_{i=1}^{p} \Gamma\left[\frac{1}{2}(n+1-i)\right]} \times \frac{\Gamma_p(p) \det\left[\left(\exp\left(-\frac{1}{2}\sigma_i^{-2}\lambda_j\right)\right)\right]}{\pi^{\frac{p(p-1)}{4}} \prod_{k<l}^{p}(\lambda_k - \lambda_l) \prod_{k<l}^{p}\left(-\frac{1}{2}\sigma_k^{-2} - \frac{1}{2}\sigma_1^{-2}\right)} \end{aligned} \tag{14}$$

The effect of the weighting can be observed from Table 2 by comparing the weighted distributions with the standard case, which is the unweighted eigenvalue distribution of a Wishart matrix. The choice of the weight function h will impact the severity of the weighting scheme of the largest and smallest eigenvalues of a random matrix.

Table 2: Graphical displays of $g(\mathbf{\Lambda})$ in Theorem 4.1 for $p = 2$

		$n = 10$		$n = 15$	
		Joint p.d.f. (11)	*Marginal p.d.f's*	*Joint pdf* (11)	*Marginal p.d.f's*
Standard case *					
$h(x) = x^{\alpha}$	$\alpha = -2$				
	$\alpha = 2$				
$h(x) = \exp\left(-\frac{\beta}{2}x\right)$	$\beta = -2$				
	$\beta = 2$				

7 Summary

The literature that is devoted to the study of random matrices is vast, regarding both theory and practice. However, there is an absence of results for the biased version of the matrix variate data, and related parametric models such as the weighted versions of the matrix variate distributions that can be used as models for the analysis of such data when they occur in practice. Possible applications could soon be a reality in bioinformatics (biased mutation matrix), MIMO systems and imaging (diffusion tensor data).

In this paper we defined a map which connected the distribution of eigenvalues to a weighted distribution of eigenvalues. We provided the definition of rotation invariant Wishart distribution. The weight function in the context of rotation invariant distribution can have many interpretations. For example the exponential rate can be a functional choice, since it motivated the idea of mirror. By the mapping illustration, we specifically showed that the mirror enables the researcher to "approximate" the main distribution more closely through the weighted distribution, by using the map $\mathcal{M}$ in our definition. The effect of the weighting of the eigenvalues of a random matrix is evidenced through a graphical illustration. These results can be extended for the ensembles of random matrices that are classified into four categories - (Hermite, Laguerre, Jacobi, Fourier), see Edelman and Rao (2005).

These matrix variate distributions are developments to the distribution theory field similar as Edelman and Koev (2014) and Jones (2015), and should stimulate research and applications.

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