Iterative methods for Stokes flow under nonlinear slip boundary condition coupled with the heat equation

J.K. Djoko* V. Konlack Socgnia† M. Mbehou‡

September 26, 2018

Abstract

We study two iterative schemes for the finite element approximation of the heat equation coupled with Stokes flow under nonlinear slip boundary conditions of friction type. The iterative schemes are based on Uzawa's algorithm in which we decouple the computation of the velocity and pressure from that of the temperature by means of linearization. We derive some a priori estimates and prove convergence of these schemes. The theoretical results obtained are validated by means of numerical simulations.

Keywords: Nonlinear slip boundary condition,variational inequalities, iterative schemes,convergence

AMS subject classification: 65N30, 76M10, 35J85

1 Introduction

In this work, we are interested in the finite element computation of the solution of the system of equations:

$$\begin{aligned} &-2\,\mathrm{div}\,\nu(\theta)D\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \\ &\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega\ , \\ &-\kappa\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta = \mathrm{b} \quad in\ \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded simply connected open domain in $\mathbb{R}^d$ (d=2,3) with a Lipschitz-continuous boundary $\partial\Omega$ divided in two parts S and $\Gamma = \partial\Omega\backslash\overline{S}$ with $\overline{\Gamma}\cap\overline{S} = \emptyset$. In (1.1), $\boldsymbol{u}$ is the velocity and θ the temperature, while p stands for the pressure. b is the external heat source, while $\boldsymbol{f}$ is the external body force per unit volume acting on the fluid. κ is positive and stands for the thermal conductivity. The Cauchy stress tensor $\boldsymbol{T}$ is

$$\boldsymbol{T} = 2\nu(\theta)D\boldsymbol{u} - p\boldsymbol{I}\ ,$$

with ν, positive and representing the viscosity of the fluid and depending on temperature [1]. $\boldsymbol{I}$ is the identity tensor, and $D\boldsymbol{u}$ is the symmetric part of the velocity gradient defined as follows

$$2D\boldsymbol{u} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\boldsymbol{T}}\ .$$

The first two equations of (1.1) are the Boussinesq approximation of a steady approximation, while the third equation in (1.1) is a "perturbation" of the heat equation. The system (1.1) is a simplified model for a number of incompressible fluids when some variations are observed in the temperature and we refer to [2] for one of the first analysis of this simplification. A reader interested in a rigorous derivation of (1.1) can consult [3, 4]. The coupling in (1.1) are represented through the convective term $(\boldsymbol{u}\cdot\nabla)\theta$ and the expression $\nu(\theta)D\boldsymbol{u}$. The system of equations (1.1)

*Departement of Mathematics and Applied Mathematics, University of Pretoria, Private bag X20, Hatfield 0028, Pretoria, South Africa; Email: jules.djokokamdem@up.ac.za

†Department of Mathematics, University of Yaounde I, Cameroon;Email: ksognia@gmail.com

‡Department of Mathematics, University of Yaounde I, Cameroon;Email: mbehoumoh@uy1.uninet.cm

is supplemented by the boundary conditions on the velocity and temperature. For that purpose, we assume that

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \quad \text{and} \quad \theta = \theta_0 \quad \text{on } \partial\Omega \ , \tag{1.2}$$

θ_0 being given, and non-negative. On the other part of the boundary S, the velocity is decomposed following its normal and tangential part; that is

$$\boldsymbol{u} = u_{\boldsymbol{n}} + u_{\boldsymbol{\tau}} = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{u} \cdot \boldsymbol{\tau})\boldsymbol{\tau},$$

where $\boldsymbol{n}$ is the normal outward unit vector to S and $\boldsymbol{\tau}$ is the tangent vector orthogonal to $\boldsymbol{n}$. We assume the impermeability of the fluid on S; that is,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } S \ . \tag{1.3}$$

Just like the velocity, the traction $\boldsymbol{Tn}$ on S is decomposed following its normal and tangential part; that is

$$\begin{aligned} \boldsymbol{Tn} &= (\boldsymbol{Tn} \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{Tn} \cdot \boldsymbol{\tau})\boldsymbol{\tau} \\ &= (-p + 2\nu(\theta)\boldsymbol{n} \cdot \boldsymbol{D}(\boldsymbol{u})\boldsymbol{n})\boldsymbol{n} + 2\nu(\theta)(\boldsymbol{\tau} \cdot \boldsymbol{D}(\boldsymbol{u})\boldsymbol{n})\boldsymbol{\tau} \\ &= (\boldsymbol{Tn})_{\boldsymbol{n}} + (\boldsymbol{Tn})_{\boldsymbol{\tau}} \ . \end{aligned}$$

The traction $(\boldsymbol{Tn})_{\boldsymbol{\tau}}$ is related to $\boldsymbol{u}_{\boldsymbol{\tau}}$ following the nonlinear slip condition introduced by C. Leroux [5] which states that

$$\left.\begin{aligned} &|(\boldsymbol{Tn})_{\boldsymbol{\tau}}| \leq g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ &|(\boldsymbol{Tn})_{\boldsymbol{\tau}}| > g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0} \ , \ -(\boldsymbol{Tn})_{\boldsymbol{\tau}} = g\frac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{ on } \quad S \ ,$$

where $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ is the Euclidean norm and $g : S \longrightarrow (0, \infty)$ is a given non-negative function called threshold slip or barrier function. Following [6], it is readily shown that the nonlinear slip boundary condition is equivalent to

$$-(\boldsymbol{Tn})_{\boldsymbol{\tau}} \in g\partial|\boldsymbol{u}_{\boldsymbol{\tau}}| \qquad \text{on} \quad S \, , \tag{1.4}$$

where $\partial|\cdot|$ is the sub-differential of the real-valued function $|\cdot|$. We recall that if $\mathcal{X}$ is a Hilbert space with scalar product $(\cdot,\cdot)$, and for $x_0 \in \mathcal{X}$, and $y \in \mathcal{X}'$,

$$y \in \partial\Psi(x_0) \text{ means that } \Psi(x) - \Psi(x_0) \geq (y, x - x_0) \quad \forall x \in \mathcal{X} \, .$$

At this point it is worth noting that the motion of a fluid under such boundary condition had been formulated first by H. Fujita [7] with "Tresca's friction law" which reads; if $|(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g$ then $\boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}$, and slip occurs, but if $|(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g$ then $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}$ and no-slip.

Some practical assumptions are needed on $\nu(\theta)$ for the mathematical analysis of (1.1)...(1.4). Thus we assume that $\nu(\cdot)$ is a bounded continuous function defined on $(0, \infty)$ satisfying, for some ν_0, ν_1, ν_2,

$$\nu \in \mathcal{C}^1(\boldsymbol{R}^+) \ \text{ and for } \quad s \in \boldsymbol{R}^+, \ \ 0 < \nu_0 \leq \nu(s) \leq \nu_1 \text{ and } \ |\nu'(s)| \leq \nu_2 \, . \tag{1.5}$$

It is important to note that the model used for the viscosity and diffusion functions for many applications are not necessarily bounded over $\boldsymbol{R}$, but the mathematical analysis of the resulting problem is very complex.

The study of heat convection in a liquid medium whose motion is described by the Navier Stokes or Darcy equations coupled with the heat equation under Dirichlet boundary condition have been investigated in many publications (see [8, 9, 10, 11] just to mentioned a few). In [8, 9, 11], the continuous and approximate variational formulations for the Navier-Stokes/Darcy equations coupled with the heat equation are formulated and thorough discussion about the existence and uniqueness for both formulations are provided. The convergence of the approximate solution is also made clear. In [10], the focus is in the convergence analysis via the contraction principle of the iterative scheme proposed. Our work focus on the analysis and validation of iterative schemes for the nonlinear problem (1.1)...(1.5) whose weak formulation is a variational inequality of second kind. Our work is inspired from [12] and differs from it in many respect among others; the number of unknowns we are dealing with, the presence of the nonlinear slip condition (1.4) which is responsible of the inequality in the variational formulation, the nonlinear terms $\nu(\theta)D\boldsymbol{u}$ and $(\theta\cdot\nabla)\boldsymbol{u}$.

In this work we did not considered the Navier Stokes equations because the nonlinearity of the later has much in common with the expression $(\boldsymbol{u}\cdot\nabla)\theta$. From the analysis point of view the difficulties encountered in both problems are the same. Thus our work combined the difficulties of the "Navier Stokes equations" and the ones generated by the non classical boundary condition (1.4) together with $\nu(\theta)D\boldsymbol{u}$. It should be noted that the unique weak solution of (1.1)...(1.5) can be constructed following closely the arguments in [8](see theorem 2.2 and proposition 2.3), while the convergence of the finite element approximation associated to it has been studied recently and convergence of the following Uzawa's type algorithm established in [13]

Initialization: Given $\{\boldsymbol{u}_h^n, p_h^n, \boldsymbol{\lambda}_h^n, \theta_h^n\}$, we compute $\{\theta_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}\}$ by solving

Step 1: for all $(\boldsymbol{v}, q)\in \boldsymbol{V}_h\times M_h$

$$\begin{aligned}\int_\Omega \nu(\theta_h^n+\widetilde{\theta}_0)D\boldsymbol{u}_h^{n+1} : D\boldsymbol{v} - \int_\Omega \operatorname{div}\boldsymbol{v}\, p_h^{n+1} &= \langle \boldsymbol{f},\boldsymbol{v}\rangle - (\boldsymbol{\lambda}_h^n, g\boldsymbol{v}_{\boldsymbol{\tau}})_S\,,\\ \int_\Omega \operatorname{div}\boldsymbol{u}_h^{n+1}\, q &= 0\,.\end{aligned} \tag{1.6}$$

Step 2: For all $\rho\in H^1_{0h}$

$$\kappa\int_\Omega \nabla\theta_h^{n+1}\cdot\nabla\rho + d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1}+\widetilde{\theta}_0, \rho) = \langle \mathrm{b},\rho\rangle - \kappa\int_\Omega \nabla\widetilde{\theta}_0\cdot\nabla\rho\,. \tag{1.7}$$

Step 3:

$$\text{for all } \gamma>0,\quad \boldsymbol{\lambda}_h^{n+1} = P_\Lambda(\boldsymbol{\lambda}_h^n + \gamma g\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})\,, \tag{1.8}$$

with $P_\Lambda : \boldsymbol{L}^2(S)\longrightarrow \Lambda = \{\boldsymbol{\alpha}\in\boldsymbol{L}^2(S);\quad |\boldsymbol{\alpha}|\le 1 \text{ a.e. on } S\}$ the projection operator given as follows

$$P_\Lambda(\boldsymbol{\alpha}) = \frac{\boldsymbol{\alpha}}{\sup(1,|\boldsymbol{\alpha}|)}\,.$$

The trilinear form $d_h(\cdot,\cdot,\cdot)$ in (1.7) is

$$d_h(\boldsymbol{v}_h,\theta_h,\rho_h) = d(\boldsymbol{v}_h,\theta_h,\rho_h) + \frac{1}{2}\left((\operatorname{div}\boldsymbol{v}_h)\theta_h,\rho_h\right) \text{ with } \quad d(\boldsymbol{v},\theta,\rho) = ((\boldsymbol{v}\cdot\nabla)\theta,\rho)\,,$$

and $\widetilde{\theta}_0$ is the lifting of θ_0 (see [14], Chap 4, Lemma 2.3). In (1.6), (1.7) and (1.8) we have used standard notations and $\mathcal{T}_h$ is the triangulation of Ω consisting of closed non degenerate triangles/tetrahedra which are regular in the sense of Ciarlet [15]. So

- for each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$
- the intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge or face
- the ratio of the diameter of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or ball is bounded by a constant independent of h.

As usual h is the maximum of the diameters of the elements of $\mathcal{T}_h$. For each non-negative integer n and any K in $\mathcal{T}_h$, we denote $\mathcal{P}_l(K)$ the space of restrictions to K of polynomials with d variables and total degree less than or equal to l. The finite dimensional spaces $\boldsymbol{V}_h, M_h$ and $H^1_h(\Omega)$, are defined as follows

$$\begin{aligned}&\boldsymbol{V} = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)\ :\quad \boldsymbol{v}|_\Gamma = 0\ \ ,\ \ \boldsymbol{v}\cdot\boldsymbol{n}|_S = 0\}\,,\\ &\boldsymbol{V}_h = \{\boldsymbol{v}_h\in\mathcal{C}(\overline{\Omega})^2\cap\boldsymbol{V},\ \text{for all } K\in\mathcal{T}_h,\ \boldsymbol{v}_h|_K\in\mathcal{P}_2(K)^d\}\,,\\ &M_h = \{q_h\in L^2(\Omega)\cap\mathcal{C}(\overline{\Omega}),\ \ \text{for all } K\in\mathcal{T}_h,\ \ q_h|_K\in\mathcal{P}_1(K)\}\,,\\ &H^1_h = \{v_h\in H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}),\ \text{for all } K\in\mathcal{T}_h,\ \ v_h|_K\in\mathcal{P}_1(K)\}\,,\\ &H^1_{0h} = H^1_h\cap H^1_0(\Omega)\,.\end{aligned} \tag{1.9}$$

In this contribution we study different iterative schemes for the numerical solution of (1.1)....(1.5). The idea is to add an additional term in the algorithm (1.6), (1.7) and (1.8) by considering the expression $\frac{1}{\alpha}(\cdot,\cdot)$, where α is an artificial time step that shall be chosen to speed up the computation and achieve convergence of the new algorithm. It is worth mentioning that the new schemes have been formulated and studied recently in [12] for a nonlinear elliptic problem. The first method we would like to analyze reads as follows:

Initialization: Given $\{\boldsymbol{u}_h^n, p_h^n, \boldsymbol{\lambda}_h^n, \theta_h^n\}$, we compute $\{\theta_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}\}$ by solving
Step 1: for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h$

$$\begin{aligned} \frac{1}{\alpha}(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{v}) + \int_\Omega \nu(\theta_h^n + \widetilde{\theta}_0) D\boldsymbol{u}_h^{n+1} : D\boldsymbol{v} - \int_\Omega \operatorname{div} \boldsymbol{v}\, p_h^{n+1} &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle - (\boldsymbol{\lambda}_h^n, g\boldsymbol{v}_{\boldsymbol{\tau}})_S\,, \\ \int_\Omega \operatorname{div} \boldsymbol{u}_h^{n+1}\, q &= 0\ . \end{aligned} \tag{1.10}$$

Step 2: For all $\rho \in H^1_{0h}$

$$\frac{1}{\alpha}(\theta_h^{n+1} - \theta_h^n, \rho) + \kappa \int_\Omega \nabla\theta_h^{n+1} \cdot \nabla\rho + d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \rho) = \langle \mathrm{b}, \rho\rangle - \kappa \int_\Omega \nabla\widetilde{\theta}_0 \cdot \nabla\rho\,. \tag{1.11}$$

Step 3:

$$\text{for all } \gamma > 0, \quad \boldsymbol{\lambda}_h^{n+1} = P_\Lambda(\boldsymbol{\lambda}_h^n + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})\ . \tag{1.12}$$

The additional term added relax the computation of the finite element solution in the sense that in the condensation process, the stiffness matrix obtained can easily be inverted. Furthermore, there is more flexibility to achieving convergence in the system (1.10)...(1.12) because we have now two parameters to play with, namely γ and α.
The second iterative scheme we explore in this work is the following:

Initialization: Given $\{\boldsymbol{u}_h^n, p_h^n, \boldsymbol{\lambda}_h^n, \theta_h^n\}$, we compute $\{\theta_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}\}$ by solving
Step 1: for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h$

$$\begin{aligned} \frac{1}{\alpha}(D\boldsymbol{u}_h^{n+1} - D\boldsymbol{u}_h^n, D\boldsymbol{v}) + \int_\Omega \nu(\theta_h^n + \widetilde{\theta}_0) D\boldsymbol{u}_h^{n+1} : D\boldsymbol{v} - \int_\Omega \operatorname{div} \boldsymbol{v} p_h^{n+1} &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle - (\boldsymbol{\lambda}_h^n, g\boldsymbol{v}_{\boldsymbol{\tau}})_S\,, \\ \int_\Omega \operatorname{div} \boldsymbol{u}_h^{n+1} q &= 0\ . \end{aligned} \tag{1.13}$$

Step 2: For all $\rho \in H^1_{0h}$

$$\frac{1}{\alpha}(\nabla\theta_h^{n+1} - \nabla\theta_h^n, \nabla\rho) + \kappa \int_\Omega \nabla\theta_h^{n+1} \cdot \nabla\rho + d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \rho) = \langle \mathrm{b}, \rho\rangle - \kappa \int_\Omega \nabla\widetilde{\theta}_0 \cdot \nabla\rho\,. \tag{1.14}$$

Step 3:

$$\text{for all } \gamma > 0, \quad \boldsymbol{\lambda}_h^{n+1} = P_\Lambda(\boldsymbol{\lambda}_h^n + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})\ . \tag{1.15}$$

The main difference between the algorithms (1.13), (1.14) and (1.10), (1.11) is the stabilization term considered in both schemes. Our objectives in this work are as follows; study the feasibility of these two iterative schemes, their convergence and compare them numerically. It should be noted that similar additional terms have been incorporated to numerical schemes in different contexts in [16, 17]. This work is the follow up of the other contribution [13], and we believe that these contributions are to the best of our knowledge the first ones towards the numerically analysis from the mathematical viewpoint of fluid flows under nonlinear slip boundary condition coupled with the heat equation. The rest of the paper is organized as follows:

(a) Section 2 presents the variational setting and finite element approximations.

(b) Section 3 is devoted to the analysis of the first iterative scheme.

(c) Section 4 is devoted to the analysis of the second iterative scheme.

(d) Numerical experiments are discussed in Section 5.

2 Preliminaries

2.1 Variational formulation

To write the system (1.1)...(1.4) in a variational form, we need some preliminaries. We adopt the standard definitions from [18] for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{H^s(D)}$, and semi-norms $|\cdot|_{H^s(D)}$ for $s \geq 0$, and their subspaces $H^s_0(D)$. For

each $s \geq 0$, $H^{-s}(D)$ denotes the dual space of $H_0^s(D)$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and inner product are denoted as $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. If $D = \Omega$, we drop D.
Throughout this work, boldface characters denote vector quantities, and $\boldsymbol{H}^1(\Omega) = H^1(\Omega)^d$ and $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$.
The following space is important in the analysis of (1.1)...(1.5):

$$M = \left\{ q \in L^2(\Omega) \ : \quad \int_\Omega q dx = 0 \right\} \ . \tag{2.1}$$

The following Poincaré-Friedrich's inequality holds: there is a positive constant c depending on the domain Ω such that

$$\text{for all} \ \ \boldsymbol{v} \in \boldsymbol{V} \ \ , \quad \|\boldsymbol{v}\| \leq c|\boldsymbol{v}|_{H^1(\Omega)} \ \ , \tag{2.2}$$

which ensures that the norms $\|\cdot\|_{H^1(\Omega)}$ and $|\cdot|_{H^1(\Omega)}$ are equivalent on $\boldsymbol{V}$. We introduce the following functionals that will be used to write the weak form on the problem in abstract setting.

$$\begin{array}{rccl}
a_1: & \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\
& (\boldsymbol{v}, \boldsymbol{u}) & \longrightarrow & a_1(\theta : \boldsymbol{v}, \boldsymbol{u}) = 2(\nu(\theta) D\boldsymbol{v}, D\boldsymbol{u}) \\
a_2: & H_0^1(\Omega) \times H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\
& (\theta, \rho) & \longrightarrow & a_2(\theta, \rho) = \kappa(\nabla\theta, \nabla\rho) \\
b: & \boldsymbol{H}^1(\Omega) \times M & \longrightarrow & \mathbb{R} \\
& (\boldsymbol{v}, q) & \longrightarrow & b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q) \ , \\
j: & \boldsymbol{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\
& \boldsymbol{v} & \longrightarrow & j(\boldsymbol{v}) = (g, |\boldsymbol{v_\tau}|)_S, \\
d: & \boldsymbol{H}^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) & \longrightarrow & \mathbb{R} \\
& (\boldsymbol{v}, \theta, \rho) & \longrightarrow & d(\boldsymbol{v}, \theta, \rho) = ((\boldsymbol{v} \cdot \nabla)\theta, \rho) \, .
\end{array}$$

We consider the variational problem: for $\theta_0 \in H^{1/2}(\partial\Omega)$, $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $\mathrm{b} \in H^{-1}(\Omega)$

$$\begin{cases}
\text{Find} \ \ (\boldsymbol{u}, p, \theta) \in \boldsymbol{V} \times M \times H^1(\Omega), \ \ \text{such that} \\
\theta = \theta_0 \ \ \text{on} \ \ \partial\Omega, \\
\text{and for all} \ \ (\boldsymbol{v}, q, \rho) \in \boldsymbol{V} \times M \times H_0^1(\Omega), \\
a_1(\theta; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle, \\
b(\boldsymbol{u}, q) = 0 \, , \\
a_2(\theta, \rho) + d(\boldsymbol{u}, \theta, \rho) = \langle \mathrm{b}, \rho \rangle \, ,
\end{cases} \tag{2.3}$$

with $\langle \cdot, \cdot \rangle$ being the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It can be shown that (see [6])

Proposition 2.1. *Problems (2.3) and (1.1),....,(1.4) are equivalent. Indeed, any triplet $(\boldsymbol{u}, p, \theta) \in \boldsymbol{V} \times M \times H^1(\Omega)$ is a solution of (1.1),....,(1.4) in the sense of distribution if and only if it is a solution of (2.3).*

The following standard results will be used for the analysis of problem (2.3) and its corresponding finite element discretization.

Lemma 2.1. *[18, 19] Given Ω as described, then there exists $c(\Omega)$ such that*

$$\begin{aligned}
&\textit{for all} \ \ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ \ \|\boldsymbol{v}\|_{L^4(\Omega)}^4 \leq c(\Omega) \|\boldsymbol{v}\|_{L^2(\Omega)}^2 \|\boldsymbol{v}\|_{H^1(\Omega)}^2 \quad \textit{if} \ \ d = 2 \\
&\textit{for all} \ \ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ \ \|\boldsymbol{v}\|_{L^4(\Omega)}^4 \leq c(\Omega) \|\boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)}^3 \quad \textit{if} \ \ d = 3 \ .
\end{aligned}$$

We also recall the Korn's inequality from [6]: there exists $c(\Omega)$ such that

$$\|D\boldsymbol{v}\| \geq c(\Omega) \|\nabla \boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V} \ . \tag{2.4}$$

It is also important to recall Poincaré-Friedrichs's inequality which reads;there exists $c(\Omega)$ such that

$$c(\Omega) \|\boldsymbol{v}\| \leq \|\nabla \boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V} \ . \tag{2.5}$$

Since ν is bounded from below and above (see (1.5)), we deduce that $a_1(\cdot,\cdot)$ is continuous and elliptic on $\boldsymbol{V}$; this means that for $(\boldsymbol{v}, \boldsymbol{w})$ element of $\boldsymbol{V} \times \boldsymbol{V}$,

$$a_1(\theta; \boldsymbol{v}, \boldsymbol{w}) \leq \nu_1 \|\boldsymbol{v}\|_{H^1(\Omega)} \|\boldsymbol{w}\|_{H^1(\Omega)} \ , \quad a_1(\theta; \boldsymbol{v}, \boldsymbol{v}) \geq \nu_0 c \|\boldsymbol{v}\|_{H^1(\Omega)}^2 \, . \tag{2.6}$$

From (2.2), we deduce that $a_2(\cdot,\cdot)$ is continuous and elliptic on $H_0^1(\Omega)$; this means that for (θ,ρ) element of $H_0^1(\Omega)\times H_0^1(\Omega)$,

$$a_2(\theta,\rho)\le \kappa c\|\theta\|_{H^1(\Omega)}\|\rho\|_{H^1(\Omega)}\ ,\quad a_2(\rho,\rho)\ge \kappa c\|\rho\|^2_{H^1(\Omega)}\,. \tag{2.7}$$

The trilinear form $d(\cdot,\cdot,\cdot)$ enjoys the standard properties (see R. Temam [19]): for all $(\boldsymbol{v},\theta,\rho)\in \boldsymbol{H}^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$

$$d(\boldsymbol{v},\theta,\rho)\le c\|\boldsymbol{v}\|^{1/2}\|\nabla\boldsymbol{v}\|^{1/2}\|\rho\|^{1/2}\|\nabla\rho\|^{1/2}\|\nabla\theta\| \tag{2.8}$$

and $\boldsymbol{v}$ is such that $\operatorname{div}\boldsymbol{v}|_\Omega=0$, then

$$\begin{aligned}&d(\boldsymbol{v},\theta,\rho)=-d(\boldsymbol{v},\rho,\theta)\ ,\\&d(\boldsymbol{v},\rho,\rho)=0\,.\end{aligned} \tag{2.9}$$

One of key points for the study of (2.1) is the inf-sup condition, its proof can be seen in [14, 23]: there exists $c(\Omega)$ such that

$$c\|q\|\le \sup_{\boldsymbol{v}\in\boldsymbol{V}}\frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{H^1(\Omega)}}\quad\text{for all}\quad \boldsymbol{v}\in\boldsymbol{V}. \tag{2.10}$$

The kernel of $b(\cdot,\cdot)$ in $\boldsymbol{V}$ is

$$\boldsymbol{V}_{\text{div}}=\big\{\boldsymbol{v}\in\boldsymbol{V}:\ \ b(\boldsymbol{v},q)=0\ \text{ for all }\ q\in L^2(\Omega)\big\}\,,$$

which is characterized by

$$\boldsymbol{V}_{\text{div}}=\{\boldsymbol{v}\in\boldsymbol{V}\ :\quad \operatorname{div}\boldsymbol{v}|_\Omega=0\ \}\,.$$

One easily verifies that $b(\cdot,\cdot)$ is continuous; that is

$$\text{for all}\ \ (\boldsymbol{v},q)\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\quad,\qquad b(\boldsymbol{v},q)\le\|\boldsymbol{v}\|_{H^1(\Omega)}\|q\|\ . \tag{2.11}$$

The functional $j(\cdot)$ is convex, lower semi continuous (continuous) on $\boldsymbol{V}$ but not differentiable at zero.
The following formulation equivalent to (2.3) will be useful later

$$\begin{cases}\text{Find}\ \ (\boldsymbol{u},p,\theta,\lambda)\in\boldsymbol{V}\times M\times H^1(\Omega)\times\Lambda,\ \ \text{such that}\\ \theta=\theta_0\ \ \text{on}\ \ \partial\Omega,\\ \text{and for all}\ \ (\boldsymbol{v},q,\rho)\in\boldsymbol{V}\times M\times H_0^1(\Omega),\\ a_1(\theta;\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)+(\boldsymbol{\lambda},g\boldsymbol{v}_{\boldsymbol{T}})_S=\langle\boldsymbol{f},\boldsymbol{v}\rangle,\\ b(\boldsymbol{u},q)=0\,,\\ \boldsymbol{\lambda}\cdot\boldsymbol{u}_{\boldsymbol{T}}=|\boldsymbol{u}_{\boldsymbol{T}}|\ \ \text{a.e. in}\ S,\\ a_2(\theta,\rho)+d(\boldsymbol{u},\theta,\rho)=\langle\text{b},\rho\rangle\,,\end{cases} \tag{2.12}$$

with

$$\Lambda=\big\{\boldsymbol{\alpha}|\boldsymbol{\alpha}\in\boldsymbol{L}^2(S),\ \ |\boldsymbol{\alpha}|\le 1\ \text{a.e. in}\ S\big\}\ .$$

The existence of $\boldsymbol{\lambda}$ in the formulation (2.12) can be proved either by using the Hahn-Banach Theorem (see [6]), or one can make use of a more constructive approach based on regularization (see [20]). It should be noted that $\boldsymbol{\lambda}\cdot\boldsymbol{u}_{\boldsymbol{T}}=|\boldsymbol{u}_{\boldsymbol{T}}|$ a.e. in S is equivalent to;

$$\text{for all}\ \gamma>0\quad \boldsymbol{\lambda}=P_\Lambda(\boldsymbol{\lambda}+\gamma g\boldsymbol{u}_{\boldsymbol{T}})\ ,$$

with P_Λ being the projection operator $\boldsymbol{L}^2(S)\longrightarrow\Lambda$ defined above. The equivalent formulation (2.12) has many numerical advantages, and is the one uses to design numerical strategies. The new unknown $\boldsymbol{\lambda}$ is not strictly speaking a Lagrange nor Kuhn Tucker multiplier but has some common properties with such vectors. Hence it is called multiplier by many researchers.

In what follows, c is a positive constant that may vary from one line to the next. We assume that

$$\boldsymbol{f}\in\boldsymbol{H}^{-1}(\Omega),\quad g\in L^\infty(S),\quad b\in H^{-1}(\Omega)\ \ \text{and}\ \ \theta_0\in H^{1/2}(\partial\Omega)\ . \tag{2.13}$$

Following [8] we claim that

Proposition 2.2. *For any data $(\boldsymbol{f}, b, \theta_0, g)$ satisfying (2.13), the problem (2.3) admits at least one weak solution $(\boldsymbol{u}, p, \theta) \in \boldsymbol{H}^1(\Omega) \times M \times H^1(\Omega)$, and there exist three positive constants c_1, c_2, c_3 such the the following hold*

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \frac{c_1}{\nu_0} \|\boldsymbol{f}\|_{H^{-1}(\Omega)} \ ,$$
$$\|\theta\|_{H^1(\Omega)} \leq c_2 \left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_2}{\kappa} \|\mathrm{b}\|_{H^{-1}(\Omega)} \ ,$$
$$\|p\| \leq c_3 \|\boldsymbol{f}\|_{H^{-1}(\Omega)} \, .$$

Let p_1, p_2 two positive constants greater than one such that $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{2}$.
Assume that $\boldsymbol{u} \in \boldsymbol{W}^{1,p_2}(\Omega)$, and choose ν_0 or κ such that the relation

$$\nu_0 - c\frac{\nu_2}{\kappa} \left[\left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa} \|\mathrm{b}\|_{H^{-1}(\Omega)} \right] \|\boldsymbol{u}\|_{W^{1,p_2}(\Omega)} \geq 0 \ , \tag{2.14}$$

is satisfied for a positive constant c depending only on Ω. Then the solution of (2.3) is unique.

Remark 2.1. *With the change of variable on the temperature, the problem (2.12) reads*

$$\begin{cases} \text{Find } \ (\boldsymbol{u}, p, \theta, \lambda) \in \boldsymbol{V} \times M \times H_0^1(\Omega) \times \Lambda, \ \text{ such that} \\ \text{for all } \ (\boldsymbol{v}, q, \rho) \in \boldsymbol{V} \times M \times H_0^1(\Omega), \\ a_1(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + (\boldsymbol{\lambda}, g\boldsymbol{v}_{\boldsymbol{T}})_S = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}, q) = 0 \, , \\ \boldsymbol{\lambda} \cdot \boldsymbol{u}_{\boldsymbol{T}} = |\boldsymbol{u}_{\boldsymbol{T}}| \ \text{ a.e. in } S, \\ a_2(\theta, \rho) + d(\boldsymbol{u}, \theta + \widetilde{\theta}_0, \rho) = \langle \mathrm{b}, \rho \rangle - a_2(\widetilde{\theta}_0, \rho) \, . \end{cases} \tag{2.15}$$

The analysis here can be extended to the following situations:

(a) *one replaces (1.3) and (1.4) by the leak boundary conditions [21]*

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ on } \ S \ \text{ and } \quad -(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{n}} \in g\partial|\boldsymbol{u}_{\boldsymbol{n}}| \qquad \text{on} \quad S \, .$$

(b) *The Dirichlet boundary condition on θ is replaced by the mixed one*

$$\theta|_\Gamma = \theta_0 \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = \theta_1 \ \text{ on } \ S \, .$$

2.2 Finite element approximation

From now on, we assume that Ω is a polygon or polyhedron. In order to approximate the problem (2.3), one considers a regular family $(\mathcal{T}_h)_h$ of triangulations of Ω introduced in section 1. We consider the finite dimensional spaces defined in (1.9), and the finite element approximation of the problem (2.3) reads;

$$\begin{cases} \text{Find } \ (\boldsymbol{u}_h, p_h, \theta_h) \in \boldsymbol{V}_h \times M_h \times H_h^1, \ \text{ such that} \\ \theta_h = \theta_0 \ \text{ on } \ \partial\Omega, \\ \text{and for all } \ (\boldsymbol{v}_h, q_h, \rho_h) \in \boldsymbol{V}_h \times M_h \times H_{0h}^1, \\ a_1(\theta_h; \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + b(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) + j(\boldsymbol{v}_h) - j(\boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle, \\ b(\boldsymbol{u}_h, q_h) = 0 \, , \\ a_2(\theta_h, \rho_h) + d_h(\boldsymbol{u}_h, \theta_h, \rho_h) = \langle \mathrm{b}, \rho_h \rangle \, . \end{cases} \tag{2.16}$$

The trilinear form $d_h(\cdot,\cdot,\cdot)$ enjoys the properties (2.8) and (2.9) (see R.Temam [19]). We recall that the discrete version of inf-sup condition (2.11) holds: there exists β (independent of h) such that

$$\beta\|q_h\| \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \ \ \text{for all} \quad q_h \in M_h \, . \tag{2.17}$$

The unique solvability of (2.16) can be obtained exactly in the same way as for the continuous problem (2.3), and we have the following

Proposition 2.3. *Let $(\boldsymbol{f}, \mathrm{b}, \theta_0, g)$ satisfying (2.13). Then the problem (2.16) has at least one solution $(\boldsymbol{u}_h, \theta_h, p_h) \in \boldsymbol{V}_h \times H^1_h(\Omega) \times M_h$ such that*

$$\begin{aligned}
&\|\boldsymbol{u}_h\|_{H^1(\Omega)} \leq \frac{c_1}{\nu_0} \|\boldsymbol{f}\|_{H^{-1}(\Omega)} \ , \\
&\|\theta_h\|_{H^1(\Omega)} \leq c_2 \left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_2}{\kappa} \|\mathrm{b}\|_{H^{-1}(\Omega)} \ , \\
&\|p_h\| \leq c_3 \|\boldsymbol{f}\|_{H^{-1}(\Omega)} \, .
\end{aligned} \tag{2.18}$$

If furthermore $\boldsymbol{u}_h \in \boldsymbol{W}^{1,p_2}(\Omega)$, and we take ν_0 or κ such that the relation

$$\nu_0 - c\frac{\nu_2}{\kappa} \left[\left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa} \|\mathrm{b}\|_{H^{-1}(\Omega)} \right] \|\boldsymbol{u}_h\|_{W^{1,p_2}(\Omega)} \geq 0 \ , \tag{2.19}$$

is satisfied for a positive constant c depending only on Ω, then the solution of (2.16) is unique.

Before the formulation of the discrete version of (2.15), we need the following spaces

$$\boldsymbol{L}_h = \left\{ \boldsymbol{\alpha}_h | \boldsymbol{\alpha}_h, \ \ \boldsymbol{\alpha}_h = \sum_{K \in \mathcal{T}_h} \boldsymbol{\alpha}_K \mathcal{I}_K, \ \ \boldsymbol{\alpha}_K \in \mathbb{R}^2, \ \ \forall\ K \in \mathcal{T}_h \right\} \ ,$$

where $\mathcal{I}_K$ is the characteristic function of K. We let $\Lambda_h = \Lambda \cap \boldsymbol{L}_h$.
The finite element approximation of problem (2.15) reads:

$$\begin{cases}
\text{Find } \ (\boldsymbol{u}_h, p_h, \theta_h, \lambda_h) \in \boldsymbol{V}_h \times M_h \times H^1_{0h} \times \Lambda_h, \ \text{ such that} \\
\text{for all } \ (\boldsymbol{v}, q, \rho) \in \boldsymbol{V}_h \times M_h \times H^1_{0h}, \ \text{ and all } \gamma > 0 \\
a_1(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) + (\boldsymbol{\lambda}_h, g\boldsymbol{v}_{\boldsymbol{\tau}})_S = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\
b(\boldsymbol{u}_h, q) = 0 \, , \\
\boldsymbol{\lambda}_h = P_\Lambda(\boldsymbol{\lambda}_h + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}) \ \text{ a.e. in } S, \\
a_2(\theta_h, \rho) + d_h(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, \rho) = \langle \mathrm{b}, \rho \rangle - a_2(\widetilde{\theta}_0, \rho) \, .
\end{cases} \tag{2.20}$$

We conclude this section by recalling the following convergence result obtained recently [13]

Theorem 2.1. *Let $(\boldsymbol{u}, p, \theta)$ be the solution of (2.3) such that $\boldsymbol{u} \in \boldsymbol{W}^{1,p_2}(\Omega)$ and (2.14) is valid. Let $(\boldsymbol{u}_h, p_h, \theta_h)$ the solution of (2.16) such that $\boldsymbol{u}_h \in \boldsymbol{W}^{1,p_2}(\Omega)$ and (2.19) is valid. There is c independent of h such that for all $(\boldsymbol{v}_h, q_h, s_h) \in \boldsymbol{V}_h \times M_h \times H^1_h$*

$$\begin{aligned}
&\|\boldsymbol{u}_h - \boldsymbol{u}\|_{H^1(\Omega)} \leq c\|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1(\Omega)} + c\|\boldsymbol{v}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_S^{1/2} + c\|p - q_h\| + c\|\theta - s_h\|_{H^1(\Omega)} \, , \\
&\|\theta - \theta_h\|_{H^1(\Omega)} \leq c\|\theta - s_h\|_{H^1(\Omega)} + c\|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1(\Omega)} + c\|\boldsymbol{v}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_S^{1/2} + c\|p - q_h\| \, , \\
&\|p - p_h\| \leq c\|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1(\Omega)} + c\|\boldsymbol{v}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_S^{1/2} + c\|p - q_h\| + c\|\theta - s_h\|_{H^1(\Omega)} \, .
\end{aligned}$$

3 First iterative scheme

3.1 Formulation

Since the discrete problem (2.20) is nonlinear, it can be solved by an iterative scheme. Considering that we are dealing with two physical processes (fluid flow and convection diffusion), the need to separate them for easy computation is very important. Hence we shall consider the following Uzawa's type iterative scheme

Initialization: Given $\{\boldsymbol{u}_h^n, p_h^n, \boldsymbol{\lambda}_h^n, \theta_h^n\}$, we compute $\{\theta_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}\}$ by solving
Step 1: for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h$

$$\begin{aligned}
\frac{1}{\alpha}(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{v}) + a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^{n+1}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle - (\boldsymbol{\lambda}_h^n, g\boldsymbol{v}_{\boldsymbol{\tau}})_S \, , \\
b(\boldsymbol{u}_h^{n+1}, q) = 0 \ .
\end{aligned} \tag{3.1}$$

Step 2: For all $\rho \in H^1_{0h}$

$$\frac{1}{\alpha}(\theta_h^{n+1} - \theta_h^n, \rho) + a_2(\theta_h^{n+1}, \rho) + d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \rho) = \langle \mathrm{b}, \rho \rangle - a_2(\widetilde{\theta}_0, \rho) \, . \tag{3.2}$$

Step 3:

$$\text{for all } \gamma > 0, \;\; \boldsymbol{\lambda}_h^{n+1} = P_\Lambda(\boldsymbol{\lambda}_h^n + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}) \,. \tag{3.3}$$

Remark 3.1. *In the algorithm (3.1)...(3.3), α is a positive parameter suitably chosen to ensure the convergence of the process. It plays the role of an artificial time step. It can be observed that the fluid has been decoupled from the heat convection. We shall prove that this Uzawa's type algorithm is convergent by applying to it the arguments in R.Glowinski [16, 20] introduced first by Lions-Mercier in [22] for scalar equation.*

For the iterative scheme (3.1)...(3.3), we take θ^0 solution of;

$$\textit{for all} \;\; \rho \in H^1_{0h}, \qquad a_2(\theta_h^0, \rho) = \langle \mathrm{b}, \rho \rangle - a_2(\widetilde{\theta}_0, \rho) \,, \tag{3.4}$$

and $\lambda^0 = (0, 1)$ with $(\boldsymbol{u}_h^0, p_h^0)$ solution of

$$\begin{cases} \textit{for all} \;\; (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h, \\ a_1(\theta_h^0 + \widetilde{\theta}_0; \boldsymbol{u}_h^0, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^0) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \;, \\ b(\boldsymbol{u}_h^0, q) = 0 \,. \end{cases} \tag{3.5}$$

For the analysis of the iterative scheme, we will need the following result [14] (see Chap 4, Lemma 2.3): For any $\delta > 0$, there exists a lifting $\widetilde{\theta}_0 \in H^1(\Omega)$ of $\theta_0 \in H^{1/2}(\partial\Omega)$ which satisfies

$$\|\widetilde{\theta}_0\|_{L^4(\Omega)} \le \delta \|\theta_0\|_{H^{1/2}(\partial\Omega)} \;\; \text{and} \;\; \|\widetilde{\theta}_0\|_{H^1(\Omega)} \le c \|\theta_0\|_{H^{1/2}(\partial\Omega)} \;, \tag{3.6}$$

where c is a positive constant independent of δ.

3.2 A priori estimates

The analysis of the iterative scheme formulated will require the following;

$$\begin{aligned} &2(\boldsymbol{a} - \boldsymbol{b}, \boldsymbol{a}) = \|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2 + \|\boldsymbol{a} - \boldsymbol{b}\|^2 \;, \quad \text{for all} \;\; \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L}^2(\Omega), \\ &ab \le \frac{\varepsilon}{p} a^p + \frac{1}{q \varepsilon^{q/p}} b^q \;\;, \;\; \text{for all } a, b, \varepsilon > 0, \;\; \text{with} \;\; \frac{1}{p} + \frac{1}{q} = 1 \,. \end{aligned} \tag{3.7}$$

The first step can be recast as follows; given $(\boldsymbol{u}_h^n, \boldsymbol{\lambda}_h^n, \theta_h^n)$, find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ such that for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h$

$$\begin{aligned} a_1^1(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^{n+1}) &= \ell_1^1(\boldsymbol{v}) \,, \\ b(\boldsymbol{u}_h^{n+1}, q) &= 0 \;, \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} a_1^1(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) &= \frac{1}{\alpha} (\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}), \\ \ell_1^1(\boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle - (\boldsymbol{\lambda}_h^n, g \boldsymbol{v}_{\boldsymbol{\tau}})_S + \frac{1}{\alpha} (\boldsymbol{u}_h^n, \boldsymbol{v}) \,. \end{aligned} \tag{3.9}$$

The variational problem (3.8) is a perturbed Stokes equations. Hence the existence and uniqueness of solution is obtained from the same conditions needed for the Stokes equations (see [14, 23]), and we claim that

Proposition 3.1. *Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ be the solution of (3.8). There are c_1, c_2, c_3, c_4 (independent of both h and n) such that if*

$$\|\boldsymbol{u}_h^0\|^2 \le \frac{c_1}{\nu_0^2} \left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S \right) \,,$$

then the following a priori estimates hold:

$$\|\boldsymbol{u}_h^n\|^2 \le \frac{c_1}{\nu_0^2} \left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S \right) \;\; \textit{for all } n \ge 1 \,,$$

$$\|\boldsymbol{u}_h^n\|^2_{H^1(\Omega)} \le \left(\frac{1}{\alpha \nu_0} + 1 \right) \frac{c_2}{\nu_0^2} \left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S \right) \,,$$

$$\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 \le \left(\frac{1}{\nu_0} + \alpha \right) \frac{c_3}{\nu_0} \left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S \right) \,,$$

$$\|p_h^n\| \le c_4 \left(1 + \frac{1}{\alpha} \left(\left(\left(\frac{1}{\nu_0} + \alpha \right) \frac{1}{\nu_0} \right)^{1/2} + \frac{\nu_1}{\nu_0} \right) \left(\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \|g\|_S \right) \,.$$

Proof. We take $\boldsymbol{v} = 2\boldsymbol{u}_h^{n+1}$ in (3.8), use the first relation in (3.7). We find

$$\begin{aligned}\|\boldsymbol{u}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^n\|^2 + \|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 &+ 2\alpha a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}) \\ &= 2\alpha\langle \boldsymbol{f}, \boldsymbol{u}_h^{n+1}\rangle - 2\alpha(\boldsymbol{\lambda}_h^n, g\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})_S \,.\end{aligned} \tag{3.10}$$

We now would like to find an upper bound of the right hand side of (3.10). For that purpose, we use the inequalities of Cauchy-Shwartz, Holder, trace, (2.4) and $(3.7)_2$ to obtain

$$\begin{aligned}&2\langle \boldsymbol{f}, \boldsymbol{u}_h^{n+1}\rangle \le 2c\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|D\boldsymbol{u}_h^{n+1}\| \le \frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \frac{\nu_0}{2}\|D\boldsymbol{u}_h^{n+1}\|^2, \\ &2(\boldsymbol{\lambda}_h^n, g\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})_S \le 2\|g\|_S\|\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}\|_S \le 2c\|g\|_S\|D\boldsymbol{u}_h^{n+1}\| \le \frac{c}{\nu_0}\|g\|_S^2 + \frac{\nu_0}{2}\|D\boldsymbol{u}_h^{n+1}\|^2 \,.\end{aligned} \tag{3.11}$$

Hence we deduce from (3.10), (1.5) and (3.11) that

$$\|\boldsymbol{u}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^n\|^2 + \|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \alpha\nu_0\|D\boldsymbol{u}_h^{n+1}\|^2 \le \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|_S^2 \,. \tag{3.12}$$

Thus to obtain the first inequality announced, we use the induction as follows.

We assume that $\|\boldsymbol{u}_h^m\|^2 \le \frac{c_1}{\nu_0^2}\left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|_S^2\right)$ is verified for all $m \le n$ and we would like to show the relation for $m = n+1$.

If $\|\boldsymbol{u}_h^{n+1}\| \le \|\boldsymbol{u}_h^n\|$, then by induction hypothesis we deduce that it is true for $m = n+1$.
But, if $\|\boldsymbol{u}_h^{n+1}\| \ge \|\boldsymbol{u}_h^n\|$, the first three terms in left hand side of (3.12) can be ignored, and one obtains

$$\alpha\nu_0\|D\boldsymbol{u}_h^{n+1}\|^2 \le \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|_S^2 \,.$$

But the application (2.4) and (2.5) yields

$$\nu_0\|\boldsymbol{u}_h^{n+1}\|^2 \le \frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \frac{c}{\nu_0}\|g\|_S^2 \,,$$

which is the desired result.

Now using (2.4) in (3.12), one obtains

$$\alpha\nu_0 c\|\boldsymbol{u}_h^{n+1}\|^2_{H^1(\Omega)} \le \|\boldsymbol{u}_h^n\|^2 + \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|_S^2 \,.$$

We then deduce the second inequality announced by replacing the bound on $\|\boldsymbol{u}_h^n\|$.
From (3.12), we deduce that

$$\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 \le \|\boldsymbol{u}_h^n\|^2 + \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|_S^2 \,.$$

Whence the third inequality is obtained by replacing the bound on $\|\boldsymbol{u}_h^n\|$.
Finally, from the inf-sup condition (2.20) and the first relation in (3.1), one has

$$\begin{aligned}\beta\|p_h^{n+1}\| &\le \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \\ &\le \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{\langle \boldsymbol{f}, \boldsymbol{v}\rangle - (\boldsymbol{\lambda}_h^n, g\boldsymbol{v}_{\boldsymbol{\tau}})_S - \frac{1}{\alpha}(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{v}) - a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{v})}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \\ &\le \|\boldsymbol{f}\|_{H^{-1}(\Omega)} + c\|g\|_S + \frac{c}{\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\| + \nu_1\|\boldsymbol{u}_h^{n+1}\|_{H^1(\Omega)} \,,\end{aligned}$$

thus the inequality is obtained by replacing the estimates on $\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|$ and $\|\boldsymbol{u}_h^{n+1}\|_{H^1(\Omega)}$.
□

The second step within the first iterative scheme is re-written as follows:

$$\begin{aligned}&\text{Find } \theta_h^{n+1} \in H^1_{0h} \text{ such that or all } \rho \in H^1_{0h}, \\ &a_2^1(\theta^{n+1}, \rho) = \ell_2^1(\rho) \,,\end{aligned} \tag{3.13}$$

with

$$
\begin{aligned}
a_2^1(\theta^{n+1},\rho) &= \frac{1}{\alpha}(\theta_h^{n+1},\rho) + a_2(\theta_h^{n+1},\rho) + d_h(\boldsymbol{u}_h^{n+1},\theta_h^{n+1},\rho),\\
\ell_2^1(\rho) &= \langle \mathrm{b},\rho\rangle - a_2(\widetilde{\theta}_0,\rho) + \frac{1}{\alpha}(\theta_h^n,\rho) - d_h(\boldsymbol{u}_h^{n+1},\widetilde{\theta}_0,\rho)\,.
\end{aligned} \tag{3.14}
$$

It is readily checked that problem (3.13) has a unique solution as a consequence of the properties of the trilinear form $d_h(\cdot,\cdot,\cdot)$ and the bilinear form $a_2(\cdot,\cdot)$.

Proposition 3.2. *Let θ_h^{n+1} be the solution of (3.13). There are c_1, c_2 (both independent of h and n) such that if*

$$
\|\theta_h^0\|^2 \le c_1\left(\frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right),
$$

then the following a priori estimates hold

$$
\begin{aligned}
&\|\theta_h^n\|^2 \le c_1\left(\frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right) \textit{ for all } n \ge 1\,,\\
&\|\theta_h^n\|^2_{H^1(\Omega)} \le c_2\left(\frac{1}{\alpha\kappa}\|\theta_h^n\|^2 + \frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right).
\end{aligned}
$$

Proof. We take $\rho = 2\theta_h^{n+1}$ in (3.13) and using $(3.7)_1$ we obtain

$$
\|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \|\theta_h^{n+1}-\theta_h^n\|^2 + 2\alpha\kappa\|\nabla\theta_h^{n+1}\|^2 = 2\alpha\langle \mathrm{b},\theta_h^{n+1}\rangle - 2\alpha a_2(\widetilde{\theta}_0,\theta_h^{n+1}) - 2\alpha d_h(\boldsymbol{u}_h^{n+1},\widetilde{\theta}_0,\theta_h^{n+1})\,.
$$

By using the standard inequalities we have

$$
\begin{aligned}
&2\alpha\langle \mathrm{b},\theta_h^{n+1}\rangle \le 2\alpha\|\mathrm{b}\|_{H^{-1}(\Omega)}\|\nabla\theta_h^{n+1}\| \le \frac{\alpha}{\varepsilon_1}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \alpha\varepsilon_1\|\nabla\theta_h^{n+1}\|^2,\\
&2\alpha a_2(\widetilde{\theta}_0,\theta_h^{n+1}) \le 2\alpha\kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\theta_h^{n+1}\| \le 2c\alpha\kappa\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\theta_h^{n+1}\| \le \frac{c\alpha\kappa}{\varepsilon_2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \alpha\kappa\varepsilon_2\|\nabla\theta_h^{n+1}\|^2.
\end{aligned}
$$

From the properties of $d_h(\cdot,\cdot,\cdot)$ and (3.6)

$$
\begin{aligned}
2\alpha d_h(\boldsymbol{u}_h^{n+1},\widetilde{\theta}_0,\theta_h^{n+1}) &\le 2\alpha\|\boldsymbol{u}_h^{n+1}\widetilde{\theta}_0\|\|\nabla\theta_h^{n+1}\| \le 2\alpha\|\boldsymbol{u}_h^{n+1}\|_{L^4(\Omega)}\|\widetilde{\theta}_0\|_{L^4(\Omega)}\|\nabla\theta_h^{n+1}\|\\
&\le 2\alpha\delta\|\boldsymbol{u}_h^{n+1}\|_{L^4(\Omega)}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\theta_h^{n+1}\|\\
&\le \frac{\alpha}{\varepsilon_3}\delta^2\|\boldsymbol{u}_h^{n+1}\|^2_{L^4(\Omega)}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \alpha\varepsilon_3\|\nabla\theta_h^{n+1}\|^2\,.
\end{aligned}
$$

We deduce for $\delta = 1/\|\boldsymbol{u}_h^{n+1}\|_{L^4(\Omega)}$, $\varepsilon_1 = \varepsilon_3 = \kappa/3$ and $\varepsilon_2 = 1/3$ that

$$
\begin{aligned}
&\|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \|\theta_h^{n+1}-\theta_h^n\|^2 + \alpha\kappa\|\nabla\theta_h^{n+1}\|^2\\
&\qquad\le \frac{3\alpha}{\kappa}\|b\|^2_{H^{-1}(\Omega)} + c\alpha\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{\alpha}{\kappa}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.
\end{aligned} \tag{3.15}
$$

We now use the induction to obtain the first inequality.
We assume that $\|\theta_h^m\|^2 \le c_1\left(\frac{1}{\kappa^2}\|b\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right)$ is verified for all $m \le n$ and we would like to show the relation for $m = n+1$.
If $\|\theta_h^{n+1}\| \le \|\theta_h^n\|$, then by induction hypothesis we deduce that it is true for $m = n+1$.
But, if $\|\theta_h^{n+1}\| \ge \|\theta_h^n\|$, the first three terms in left hand side of (3.15) can be ignored and one obtains

$$
\kappa\|\nabla\theta_h^{n+1}\|^2 \le \frac{3}{\kappa}\|b\|^2_{H^{-1}(\Omega)} + c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.
$$

Thus the first inequality follows. Next, from (3.15) we have

$$
\alpha\kappa\|\nabla\theta_h^{n+1}\|^2 \le \|\theta_h^n\|^2 + \frac{3\alpha}{\kappa}\|b\|^2_{H^{-1}(\Omega)} + c\alpha\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{\alpha}{\kappa}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.
$$

Hence the second inequality is obtained by re-arranging terms. □

3.3 Convergence

In this paragraph, we are interested in the convergence analysis of the algorithm (3.1)...(3.3) when n tends to infinity. We claim that

Theorem 3.1. *Assume that the triangulations $\mathcal{T}_h$ is uniformly regular. Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, \theta_h^{n+1}, \boldsymbol{\lambda}_h^{n+1})$ be the solution of (3.1), (3.2), and (3.3). Assume that there are c_1, c_2 (independent of both h and n) such that*

$$\|\boldsymbol{u}_h^0\|^2 \leq \frac{c_1}{\nu_0^2}\left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S\right),$$
$$\|\theta_h^0\|^2 \leq c_2\left(\frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right).$$

Let $(\boldsymbol{u}_h, \theta_h, p_h, \boldsymbol{\lambda}_h)$ be the solution of (2.20) such that (2.19) holds. Then there is c independent of h and n such that if

$$\begin{aligned}&\frac{\nu_2^2}{\kappa}\|D\boldsymbol{u}_h\|^2_{L^{p_2}(\Omega)} + \gamma\|g\|^2_{L^\infty(S)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)}\\ &+\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}}+1\right) \leq \nu_0\end{aligned} \tag{3.16}$$

and

$$c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}}+1\right) + c\|\theta_0\|_{H^{1/2}(\partial\Omega)} - \frac{\kappa}{2} \leq 0,$$

then the following properties hold:

$$\boldsymbol{u}_h^n \to \boldsymbol{u}_h \quad \textit{strongly in } \boldsymbol{V} \quad , \quad \theta_h^n \to \theta_h \quad \textit{strongly in } H^1(\Omega),$$
$$\textit{and the sequence } \{\|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S\}_n \textit{ converges}.$$

Proof. First we take the difference between the velocities equations in (3.1) and (2.20) for $\boldsymbol{v} = \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h$. Using the first relation in (3.7), one obtains

$$\begin{aligned}&\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\|^2 - \frac{1}{2\alpha}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|^2 + \frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \int \nu(\theta_h^n + \widetilde{\theta}_0)|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)|^2\\ &= \int\left(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\right) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) + \int_S g(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}_h^n)\cdot(\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h}).\end{aligned} \tag{3.17}$$

Secondly, we take the difference between the temperatures equations in (3.2) and (2.20) for $\rho = \theta_h^{n+1} - \theta_h$. We use the first relation in (3.7), the properties of $d_h(\cdot,\cdot,\cdot)$ to find

$$\begin{aligned}&\frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h\|^2 - \frac{1}{2\alpha}\|\theta_h^n - \theta_h\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h^n\|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2\\ &= d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \theta_h - \theta_h^{n+1}) + d_h(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h)\\ &= d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h).\end{aligned} \tag{3.18}$$

Putting together (3.17) and (3.18), one has

$$\begin{aligned}&\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\|^2 - \frac{1}{2\alpha}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|^2 + \frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \int \nu(\theta_h^n + \widetilde{\theta}_0)|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)|^2\\ &+ \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h\|^2 - \frac{1}{2\alpha}\|\theta_h^n - \theta_h\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h^n\|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2\\ &= \int\left(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\right) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) - \int_S g(\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h)\cdot(\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma\\ &+ d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \theta_h^{n+1} - \theta_h) + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h).\end{aligned} \tag{3.19}$$

Thirdly, we recall that for $\gamma > 0$, one has

$$\boldsymbol{\lambda}_h = P_\Lambda(\boldsymbol{\lambda}_h + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}) \text{ and } \boldsymbol{\lambda}_h^{n+1} = P_\Lambda(\boldsymbol{\lambda}_h^n + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}).$$

Thus

$$\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h = P_\Lambda(\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h + \gamma g(\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})) \ .$$

From the fact that P_Λ is a contraction mapping, we obtain

$$\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S \leq \|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h + \gamma g(\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})\|_S \ ,$$

from which we deduce that

$$\begin{aligned}
&\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 - \|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S^2 \\
\leq\ & \gamma^2 \|g(\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})\|_S^2 + 2\gamma \int_S g(\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h) \cdot (\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})\, d\sigma \\
\leq\ & c\gamma^2 \|g\|_{L^\infty(S)}^2 \|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 + 2\gamma \int_S g(\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h) \cdot (\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})\, d\sigma \, .
\end{aligned}$$

Hence

$$\begin{aligned}
&- \int_S g(\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h) \cdot (\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})\, d\sigma \\
&\leq -\frac{1}{2\gamma} \|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 + \frac{1}{2\gamma} \|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S^2 + \frac{c}{2}\gamma \|g\|_{L^\infty(S)}^2 \|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \, .
\end{aligned} \tag{3.20}$$

Inserting (3.20) in (3.19), we obtain

$$\begin{aligned}
&\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\|^2 - \frac{1}{2\alpha}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|^2 + \frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \int \nu(\theta_h^n + \widetilde{\theta}_0)|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)|^2 \\
&+ \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h\|^2 - \frac{1}{2\alpha}\|\theta_h^n - \theta_h\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h^n\|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \\
&\leq \int \Big(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\Big) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) \\
&- \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 + \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S^2 + \frac{c}{2}\gamma\|g\|_{L^\infty(S)}^2 \|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\
&+ d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \theta_h^{n+1} - \theta_h) + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h)\, ,
\end{aligned}$$

which by re-arranging gives

$$\begin{aligned}
&\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h\|^2 + \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 - \frac{1}{2\alpha}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|^2 - \frac{1}{2\alpha}\|\theta_h^n - \theta_h\|^2 - \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S^2 \\
&+ \frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h^n\|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 + \int \nu(\theta_h^n + \widetilde{\theta}_0)|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)|^2 \\
&\leq \int \Big(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\Big) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) + \frac{c}{2}\gamma\|g\|_{L^\infty(S)}^2 \|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\
&+ d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \theta_h^{n+1} - \theta_h) + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h)\, .
\end{aligned} \tag{3.21}$$

Now we need to estimate the terms on the right hand side of (3.21). For that purpose we recall the following properties.

Generalized Holder's inequality. Let $1 \leq i \leq n$, $1 < p_i < \infty$ with $f_i \in L^{p_i}(\Omega)$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{p} \leq 1,$$

then $\prod_{i=1}^n f_i \in L^p(\Omega)$ and

$$\left\| \prod_{i=1}^n f_i \right\|_{L^p(\Omega)} \leq \prod_{i=1}^n \|f_i\|_{L^{p_i}(\Omega)} \ . \tag{3.22}$$

Sobolev inequalities. If Ω is an open set of class C^1 with Γ bounded then

$$W^{1,p}(\Omega) \text{ is embedded in } L^q(\Omega) \begin{cases} \text{for } \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{d} \text{ if } p < d, \\ \qquad\text{or} \\ \text{for all } q \in [p, \infty) \text{ if } p = d \ . \end{cases} \tag{3.23}$$

Thus using the mean value theorem, (1.5), the generalized Holder's inequality (3.22) with $p_3 = 2$ and $p = 1$, so that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, one finds

$$\begin{aligned}\int \Big(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\Big) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) \quad &\leq \quad \nu_2 \int |\theta_h^n - \theta_h| \, |D\boldsymbol{u}_h| \, |D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)| \\ &\leq \quad \nu_2 \, \|\theta_h^n - \theta_h\|_{L^{p_1}(\Omega)} \, \|D\boldsymbol{u}_h\|_{L^{p_2}(\Omega)} \, \left\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\right\| \, .\end{aligned}$$

Now using the Sobolev inequality (3.23) with $p = 2$ and $d = 2$ or $d = 3$, one has

$$\|\theta_h^n - \theta_h\|_{L^{p_1}(\Omega)} \leq c\|\nabla(\theta_h^n - \theta_h)\| \, .$$

Thus

$$\begin{aligned}&\int_\Omega \Big(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\Big) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) \\ &\qquad \leq \nu_2 c \|D\boldsymbol{u}_h\|_{L^{p_2}(\Omega)} \|\nabla(\theta^n - \theta_h)\| \, \|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\| \, .\end{aligned} \tag{3.24}$$

Secondly from the properties of $d_h(\cdot,\cdot,\cdot)$, proposition 3.2, and (3.6) we find

$$\begin{aligned}&d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \theta_h^{n+1} - \theta_h) + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h) \\ &\leq \left\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\right\|_{L^4(\Omega)} \left\|\nabla\theta_h^{n+1}\right\| \left\|\theta_h^{n+1} - \theta_h\right\|_{L^4(\Omega)} + \left\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\right\|_{L^4(\Omega)} \left\|\nabla\widetilde{\theta}_0\right\| \left\|\theta_h^{n+1} - \theta_h\right\|_{L^4(\Omega)} \\ &\leq c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}} + 1\right) \left\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\right\| \left\|\nabla(\theta_h^{n+1} - \theta_h)\right\| \\ &+ c\|\theta_0\|_{H^{1/2}(\partial\Omega)} \left\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\right\| \left\|\nabla(\theta_h^{n+1} - \theta_h)\right\| \; .\end{aligned} \tag{3.25}$$

Returning to (3.21) with (3.24) and (3.25) and using Young's inequality, one gets

$$\begin{aligned}&\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h\|^2 + \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 - \frac{1}{2\alpha}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|^2 - \frac{1}{2\alpha}\|\theta_h^n - \theta_h\|^2 - \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S^2 \\ &+ \frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h^n\|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 + \nu_0\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\ &\leq \frac{\kappa}{2}\|\nabla(\theta_h^n - \theta_h)\|^2 + \left[c\frac{\nu_2^2}{\kappa}\|D\boldsymbol{u}_h\|_{L^{p_2}(\Omega)}^2 + \frac{c}{2}\gamma\|g\|_{L^\infty(S)}^2 + c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\ &+ c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}} + 1\right)\left\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\right\|^2 \\ &+ c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}} + 1\right)\left\|\nabla(\theta_h^{n+1} - \theta_h)\right\|^2 \\ &+ c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\left\|\nabla(\theta_h^{n+1} - \theta_h)\right\|^2\end{aligned}$$

which is re-written as follows

$$\begin{aligned}&\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h\|^2 + \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 + \frac{\kappa}{2}\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \\ &- \frac{1}{2\alpha}\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|^2 - \frac{1}{2\alpha}\|\theta_h^n - \theta_h\|^2 - \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S^2 - \frac{\kappa}{2}\|\nabla(\theta_h^n - \theta_h)\|^2 \\ &+ \frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \frac{1}{2\alpha}\|\theta_h^{n+1} - \theta_h^n\|^2 \\ &\leq c\left[\frac{\nu_2^2}{\kappa}\|D\boldsymbol{u}_h\|_{L^{p_2}(\Omega)}^2 + \gamma\|g\|_{L^\infty(S)}^2 + \|\theta_0\|_{H^{1/2}(\partial\Omega)} - \nu_0\right]\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\ &+ c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}} + 1\right)\left\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\right\|^2 \\ &+ c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}} + 1\right)\left\|\nabla(\theta_h^{n+1} - \theta_h)\right\|^2 \\ &+ c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\left\|\nabla(\theta_h^{n+1} - \theta_h)\right\|^2 - \frac{\kappa}{2}\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \, .\end{aligned} \tag{3.26}$$

From (3.16), the right hand side of (3.26) is non-positive. Hence the sequence with general expression $\frac{1}{2\alpha}\|\boldsymbol{u}_h^n-\boldsymbol{u}_h\|^2+\frac{1}{2\alpha}\|\theta_h^n-\theta_h\|^2+\frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^n-\boldsymbol{\lambda}_h\|_S^2+\frac{\kappa}{2}\|\nabla(\theta_h^n-\theta_h)\|^2$ is non-increasing and and positive, therefore converges. So

$$\lim_{n\to\infty}\left[\begin{array}{l}\frac{1}{2\alpha}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h\|^2+\frac{1}{2\alpha}\|\theta_h^{n+1}-\theta_h\|^2+\frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1}-\boldsymbol{\lambda}_h\|_S^2+\frac{\kappa}{2}\|\nabla(\theta_h^{n+1}-\theta_h)\|^2\\ -\frac{1}{2\alpha}\|\boldsymbol{u}_h^n-\boldsymbol{u}_h\|^2-\frac{1}{2\alpha}\|\theta_h^n-\theta_h\|^2-\frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^n-\boldsymbol{\lambda}_h\|_S^2-\frac{\kappa}{2}\|\nabla(\theta_h^n-\theta_h)\|^2\end{array}\right]=0\ . \tag{3.27}$$

So returning to (3.26), with (3.27) and (3.16), we deduce that

$$0\le\lim_{n\to\infty}\left[\begin{array}{l}c\left[\frac{\nu_2^2}{\kappa}\|D\boldsymbol{u}_h\|_{L^{p_2}(\Omega)}^2+\gamma\|g\|_{L^\infty(S)}^2+\|\theta_0\|_{H^{1/2}(\partial\Omega)}-\nu_0\right]\|D(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)\|^2\\ +c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)}+\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}}+1\right)\left\|D(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)\right\|^2\\ +c\left[\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)}+\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right]\left(\frac{1}{\sqrt{\alpha\kappa}}+1\right)\left\|\nabla(\theta_h^{n+1}-\theta_h)\right\|^2\\ +c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\left\|\nabla(\theta_h^{n+1}-\theta_h)\right\|^2-\frac{\kappa}{2}\|\nabla(\theta_h^{n+1}-\theta_h)\|^2\end{array}\right]\le 0\,. \tag{3.28}$$

Thus one deduces from (3.28) that $\boldsymbol{u}_h^n$ converges to $\boldsymbol{u}_h$ strongly in $\boldsymbol{V}$ and θ_h^n converges to θ_h strongly in $H^1(\Omega)$. Returning to (3.27), one deduces that

$$\lim_{n\to\infty}\left[\,\|\boldsymbol{\lambda}_h^{n+1}-\boldsymbol{\lambda}_h\|_S-\|\boldsymbol{\lambda}_h^n-\boldsymbol{\lambda}_h\|_S\right]=0\ . \tag{3.29}$$

□

Remark 3.2. *Several observations about the convergence of the algorithm (3.1), (3.2) and (3.3) are in order.*
First, the assumption (2.19) is needed here to ensure the unique solvability of (2.20). It is important to include that condition because convergence only make sense when there is a unique solution.
Secondly, as mentioned before the proof of the above theorem is inspired from the original work in [22], and the conditions (3.16) we obtained via an energy method is not necessary and sufficient.
Thirdly, choosing α and γ via (3.16) is very hard to implement because the constant c and p_2 appearing in $\|D\boldsymbol{u}_h\|_{L^{p_2}(\Omega)}$ are not known.
Finally the convergence of $(\boldsymbol{\lambda}_h^n)_n$ is more complicated and we refer the interested reader to [16] (Chapter 4).

4 Second iterative scheme

The objective in this section is to study the algorithm (1.13), (1.14), (1.15) by; deriving some a priori estimates and study the convergence of the iterative solution.

4.1 Formulation

Initialization: Having $\{\boldsymbol{u}_h^0,p_h^0,\boldsymbol{\lambda}_h^0,\theta_h^0\}$ from (3.4) and (3.5), we compute $\{\theta_h^{n+1},\boldsymbol{u}_h^{n+1},p_h^{n+1},\boldsymbol{\lambda}_h^{n+1}\}$ from $\{\boldsymbol{u}_h^n,p_h^n,\boldsymbol{\lambda}_h^n,\theta_h^n\}$ by solving

Step 1: for all $(\boldsymbol{v},q)\in\boldsymbol{V}_h\times M_h$

$$\begin{aligned}\frac{1}{\alpha}(D\boldsymbol{u}_h^{n+1}-D\boldsymbol{u}_h^n,D\boldsymbol{v})+a_1(\theta_h^n+\widetilde{\theta}_0;\boldsymbol{u}_h^{n+1},\boldsymbol{v})+b(\boldsymbol{v},p_h^{n+1})&=\langle\boldsymbol{f},\boldsymbol{v}\rangle-(\boldsymbol{\lambda}_h^n,g\boldsymbol{v}_{\boldsymbol{\tau}})_S\,,\\ b(\boldsymbol{u}_h^{n+1},q)&=0\ .\end{aligned} \tag{4.1}$$

Step 2: For all $\rho\in H^1_{0h}$

$$\frac{1}{\alpha}(\nabla\theta_h^{n+1}-\nabla\theta_h^n,\nabla\rho)+a_2(\theta_h^{n+1},\rho)+d_h(\boldsymbol{u}_h^{n+1},\theta_h^{n+1}+\widetilde{\theta}_0,\rho)=\langle\mathrm{b},\rho\rangle-a_2(\widetilde{\theta}_0,\rho)\,. \tag{4.2}$$

Step 3:

$$\text{for all }\gamma>0,\quad\boldsymbol{\lambda}_h^{n+1}=P_\Lambda(\boldsymbol{\lambda}_h^n+\gamma g\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})\ . \tag{4.3}$$

4.2 A priori estimates

The variational problem (4.1) is a perturbed Stokes equations. Hence the existence and uniqueness of solution is obtained following to the line the Babuska-Brezzi's approach for mixed problems (see [14, 23]). Furthermore we claim that

Proposition 4.1. *Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ be the solution of (4.1). There are c_1, c_2, c_3 (independent of both h and n) such that if*

$$\|\nabla \boldsymbol{u}_h^0\|^2 \leq \frac{c_1}{\nu_0^2}\left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S\right),$$

then the following a priori estimates hold

$$\|\nabla \boldsymbol{u}_h^n\|^2 \leq \frac{c_1}{\nu_0^2}\left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S\right) \quad \text{for all } n \geq 1,$$

$$\|\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|^2 \leq \frac{c_2}{\nu_0^2}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \frac{c_2}{\nu_0^2}\|g\|^2_S + \alpha\frac{c_2}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c_2}{\nu_0}\|g\|^2_S,$$

$$\|p_h^{n+1}\| \leq c_3\left(1 + \frac{1}{\alpha\nu_0} + \frac{1}{\alpha^{1/2}\nu_0^{1/2}} + \frac{\nu_1}{\nu_0}\right)\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + c_3\left(1 + \frac{1}{\alpha\nu_0} + \frac{\nu_1}{\nu_0} + \frac{1}{\alpha^{1/2}\nu_0^{1/2}}\right)\|g\|_S.$$

Remark 4.1. *The H^1-estimate of $\boldsymbol{u}_h^n$ is independent of α, which is in net contradiction of similar estimate (see Proposition 3.1).*

Proof. We follow the proof of Proposition 3.1.
We take $\boldsymbol{v} = 2\boldsymbol{u}_h^{n+1}$ in (4.1) and obtain

$$\|D\boldsymbol{u}_h^{n+1}\|^2 - \|D\boldsymbol{u}_h^n\|^2 + \|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|^2 + 2\alpha a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}) = 2\alpha\langle \boldsymbol{f}, \boldsymbol{u}_h^{n+1}\rangle - 2\alpha(\boldsymbol{\lambda}_h^n, g\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1})_S.$$

We deduce using (2.6), (2.4) and $(3.7)_2$ that

$$\|\nabla \boldsymbol{u}_h^{n+1}\|^2 - \|\nabla \boldsymbol{u}_h^n\|^2 + \|\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|^2 + \alpha\nu_0\|D\boldsymbol{u}_h^{n+1}\|^2 \leq \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|^2_S. \quad (4.4)$$

We now use the induction to obtain the first inequality.
We assume that the relation is verified for all $m \leq n$ and we would like to show the relation for $m = n + 1$.
If $\|\nabla \boldsymbol{u}_h^{n+1}\| \leq \|\nabla \boldsymbol{u}_h^n\|$, then by induction assumption we deduce that it is true for $m = n + 1$.
But, if $\|\nabla \boldsymbol{u}_h^{n+1}\| \geq \|\nabla \boldsymbol{u}_h^n\|$, the first three terms in the left hand side of (4.4) can be ignored and one obtains

$$\|D\boldsymbol{u}_h^{n+1}\|^2 \leq \frac{c}{\nu_0^2}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \frac{c}{\nu_0^2}\|g\|^2_S.$$

Hence application of the Poincaré-Friedrichs's inequality conclude with the first inequality.
From (4.4), we deduce that

$$\begin{aligned}\|\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|^2 &\leq \|\nabla \boldsymbol{u}_h^n\|^2 + \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|^2_S \\ &\leq \frac{c}{\nu_0^2}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \frac{c}{\nu_0^2}\|g\|^2_S + \alpha\frac{c}{\nu_0}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \alpha\frac{c}{\nu_0}\|g\|^2_S\end{aligned}$$

which is the second bound. Finally, from the inf-sup condition (2.20) and the first relation in (4.1), one has

$$\begin{aligned}\beta\|p_h^{n+1}\| &\leq \sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \\ &\leq \sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{\langle \boldsymbol{f}, \boldsymbol{v}\rangle - (\boldsymbol{\lambda}_h^n, g\boldsymbol{v}_{\boldsymbol{\tau}})_S - \frac{1}{\alpha}(\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n), \nabla\boldsymbol{v}) - a_1(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \\ &\leq \|\boldsymbol{f}\|_{H^{-1}(\Omega)} + c\|g\|_S + \frac{c}{\alpha}\|\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\| + \nu_1\|\boldsymbol{u}_h^{n+1}\|_{H^1(\Omega)},\end{aligned}$$

thus the inequality is obtained by replacing the estimates on $\|\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|$ and $\|\boldsymbol{u}_h^{n+1}\|_{H^1(\Omega)}$.

□

The second step within the second iterative scheme is a linear problem and reads as follows: Find $\theta_h^{n+1} \in H^1_{0h}$ such that

$$\text{for all } \rho \in H^1_{0h}, \qquad a_2^2(\theta^{n+1}, \rho) = \ell_2^2(\rho)\,. \tag{4.5}$$

with

$$a_2^2(\theta^{n+1}, \rho) = \frac{1}{\alpha}(\nabla\theta_h^{n+1}, \nabla\rho) + a_2(\theta_h^{n+1}, \rho) + d_h(\boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \rho),$$
$$\ell_2^2(\rho) = \langle b, \rho\rangle - a_2(\widetilde{\theta}_0, \rho) + \frac{1}{\alpha}(\nabla\theta_h^n, \nabla\rho) - d_h(\boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \rho)\,.$$

It is readily checked that problem (4.5) has a unique solution as a consequence of the properties of the trilinear form $d_h(\cdot,\cdot,\cdot)$ and the bilinear form $a_2^2(\cdot,\cdot)$.

Proposition 4.2. *Let θ_h^{n+1} be the solution of (4.2). There are c_1 (independent of both h and n) such that if*

$$\|\theta_h^0\|^2_{H^1(\Omega)} \le c_1 \left(\frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right),$$

then the following a priori estimate holds:

$$\|\theta_h^{n+1}\|^2_{H^1(\Omega)} \le c_1 \left(\frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right).$$

Remark 4.2. *The H^1-estimate of θ_h^n is independent of α, which is in net contradiction of similar estimate (see Proposition 3.2).*

Proof. We take $\rho = 2\theta_h^{n+1}$ in (4.2) and obtain

$$\|\nabla\theta_h^{n+1}\|^2 - \|\nabla\theta_h^n\|^2 + \|\nabla(\theta_h^{n+1} - \theta_h^n)\|^2 + 2\alpha\kappa\|\nabla\theta_h^{n+1}\|^2$$
$$= 2\alpha\langle \mathrm{b}, \theta_h^{n+1}\rangle - 2\alpha a_2(\widetilde{\theta}_0, \theta_h^{n+1}) - 2\alpha d_h(\boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1})\,.$$

$$\begin{aligned}\|\nabla\theta_h^{n+1}\|^2 - \|\nabla\theta_h^n\|^2 + \|\nabla(\theta_h^{n+1} - \theta_h^n)\|^2 + \alpha\kappa\|\nabla\theta_h^{n+1}\|^2&\\ \le \frac{3\alpha}{\kappa}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \alpha\left(c\kappa + \kappa^{-1}\right)\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}&\,.\end{aligned} \tag{4.6}$$

We now use the induction to obtain the first inequality.
We assume that the relation is verified for all $m \le n$ and we would like to show the relation for $m = n+1$.
If $\|\nabla\theta_h^{n+1}\| \le \|\nabla\theta_h^n\|$, then by induction hypothesis we deduce that it is true for $m = n+1$.
But, if $\|\nabla\theta_h^{n+1}\| \ge \|\nabla\theta_h^n\|$, the first three terms in the left hand side of (4.6) can be ignored and one obtains

$$\kappa\|\nabla\theta_h^{n+1}\|^2 \le \frac{3}{\kappa}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \left(c\kappa + \kappa^{-1}\right)\|\theta_0\|^2_{H^{1/2}(\partial\Omega)},$$

Thus the first inequality follows. □

4.3 Convergence

In this paragraph, we are interested in the convergence of the algorithm (4.1), (4.2) and (4.3) when n tends to infinity. We claim that

Theorem 4.1. *Assume that the triangulations $\mathcal{T}_h$ is uniformly regular. Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, \theta_h^{n+1}, \boldsymbol{\lambda}_h^{n+1})$ be the solution of (4.1), (4.2), and (4.3), and assume that there are c_1, c_2 (independent of both h and n) such that*

$$\|\nabla\boldsymbol{u}_h^0\|^2 \le \frac{c_1}{\nu_0^2}\left(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)} + \|g\|^2_S\right),$$
$$\|\nabla\theta_h^0\|^2 \le c_2 \left(\frac{1}{\kappa^2}\|\mathrm{b}\|^2_{H^{-1}(\Omega)} + \|\theta_0\|^2_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa^2}\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\right).$$

Let $(\boldsymbol{u}_h, \theta_h, p_h, \boldsymbol{\lambda}_h)$ *be the solution of (2.20) such that (2.19) holds. Then there is* c *independent of* h *and* n *such that if*

$$c\frac{\nu_2^2}{\kappa}\|D\boldsymbol{u}_h\|^2_{L^{p_2}(\Omega)} + c\gamma\|g\|^2_{L^\infty(S)} + c\left(\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right) \le \nu_0$$

and

$$c\left(\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right) \le \frac{\kappa}{2}\ , \tag{4.7}$$

then the following properties hold:

$$\boldsymbol{u}_h^n \to \boldsymbol{u}_h \quad \textit{strongly in} \ \ \boldsymbol{V} \quad , \quad \theta_h^n \to \theta_h \ \ \textit{strongly in} \ \ H^1(\Omega)\ ,$$
$$\textit{and the sequence}\ \ \{\|\boldsymbol{\lambda}_h^n - \boldsymbol{\lambda}_h\|_S\}_n \ \ \textit{converges}\ .$$

Proof. We follow the proof of Theorem 3.1.

We take the difference between the velocities equations in (4.1) and (2.20) for $\boldsymbol{v} = \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h$. We find

$$\frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 - \frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n} - \boldsymbol{u}_h)\|^2 + \frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|^2 + \int \nu(\theta_h^n + \widetilde{\theta}_0)|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)|^2$$
$$= \int \left(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\right) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) + \int_S g(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}_h^n)\cdot(\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - \boldsymbol{u}_{\boldsymbol{\tau},h})\,. \tag{4.8}$$

We take the difference between the temperatures equations in (4.2) and (2.20) for $\rho = \theta_h^{n+1} - \theta_h$ and find

$$\begin{aligned}
&\frac{1}{2\alpha}\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 - \frac{1}{2\alpha}\|\nabla(\theta_h^{n} - \theta_h)\|^2 + \frac{1}{2\alpha}\|\nabla(\theta_h^{n+1} - \theta_h^n)\|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \\
&= d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1} + \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h) \\
&= d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \theta_h^{n+1} - \theta_h) + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h)\,.
\end{aligned} \tag{4.9}$$

Adding (4.8) and (4.9), and using (3.20) in the resulting relation we find

$$\begin{aligned}
&\frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 + \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 + \frac{1}{2\alpha}\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \\
&- \frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n} - \boldsymbol{u}_h)\|^2 - \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n} - \boldsymbol{\lambda}_h\|_S^2 - \frac{1}{2\alpha}\|\nabla(\theta_h^{n} - \theta_h)\|^2 \\
&+ \int \nu(\theta_h^n + \widetilde{\theta}_0)|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)|^2 + \kappa\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \\
&\le \int \left(\nu(\theta_h + \widetilde{\theta}_0) - \nu(\theta_h^n + \widetilde{\theta}_0)\right) D\boldsymbol{u}_h : D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) + \frac{c}{2}\gamma\|g\|^2_{L^\infty(S)}\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\
&+ d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \theta_h^{n+1}, \theta_h^{n+1} - \theta_h) + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}, \widetilde{\theta}_0, \theta_h^{n+1} - \theta_h)\,.
\end{aligned} \tag{4.10}$$

Now using (3.24), properties of $d_h(\cdot,\cdot,\cdot)$, Proposition 4.2 and making use of Young's inequality, one gets

$$\begin{aligned}
&\frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 + \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n+1} - \boldsymbol{\lambda}_h\|_S^2 + \left(\frac{1}{2\alpha} + \frac{\kappa}{2}\right)\|\nabla(\theta_h^{n+1} - \theta_h)\|^2 \\
&- \frac{1}{2\alpha}\|D(\boldsymbol{u}_h^{n} - \boldsymbol{u}_h)\|^2 - \frac{1}{2\gamma}\|\boldsymbol{\lambda}_h^{n} - \boldsymbol{\lambda}_h\|_S^2 - \left(\frac{1}{2\alpha} + \frac{\kappa}{2}\right)\|\nabla(\theta_h^{n} - \theta_h)\|^2 \\
&\le \left(-\nu_0 + c\frac{\nu_2^2}{\kappa}\|D\boldsymbol{u}_h\|^2_{L^{p_2}(\Omega)} + c\gamma\|g\|^2_{L^\infty(S)} + c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\|^2 \\
&+ c\left(\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\left\|D(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\right\|^2 \\
&+ c\left(\frac{1}{\kappa}\|\mathrm{b}\|_{H^{-1}(\Omega)} + \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\left\|\nabla(\theta_h^{n+1} - \theta_h)\right\|^2 \\
&+ \left(c\ \|\theta_0\|_{H^{1/2}(\partial\Omega)} - \frac{\kappa}{2}\right)\|\nabla(\theta_h^{n+1} - \theta_h)\|^2\,.
\end{aligned}$$

We continue as in the proof of Theorem 3.1. □

Remark 4.3. *In the first iterative scheme, there is a restriction on α to achieve convergence (see Theorem 3.1), but in the second iterative scheme there is no restriction on α (see Theorem 4.1). This observation will is also be confirmed computationally.*

5 Numerical experiments and Conclusion

All computations were performed using Matlab on Desktop DELL i3 with 8 GB RAM. The test problems used are designed to illustrate the behavior of the algorithms more than to model an actual phenomena. We study, computationally the following algorithms;

algorithm 1 : represented via (3.1),(3.2) and (3.3).

algorithm 2 : represented via (4.1), (4.2) and (4.3).

We stop the computations when the following condition is satisfied:

$$\frac{\|\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n)\|^2 + \|\nabla(\theta_h^{n+1} - \theta_h^n)\|^2}{\|\nabla \boldsymbol{u}_h^n\|^2 + \|\nabla \theta_h^n\|^2} \leq tol \equiv 3.60e-9\,.$$

5.1 Example 1: choice of α

This simulation has been considered by Bernardi, Dakroub, Mansour, Rafei, Sayah [12] in a different context. The objective is to compute the relative error with respect to α in order to achieve the convergence of algorithm 1 and algorithm 2.
We consider the domain $\Omega = (0,1)^2$ with its boundary $\partial\Omega$ consisting of two portions Γ and S defined as follows

$$S = (0,1) \times \{1\} \quad \text{and} \quad \Gamma = \partial\Omega \backslash S\,.$$

We consider

$$\begin{cases} u_1(x,y) = 20x^2(1-x)^2y(2-3y), \\ u_2(x,y) = -20(2x-6x^2+4x^3)y^2(1-y), \\ p(x,y) = (2x-1)(2y-1), \\ \theta(x,y) = xy(1-x)(1-y), \end{cases} \tag{5.1}$$

and

$$\nu(\theta) = \frac{1}{\theta^2+1} + \frac{1}{8} \text{ for which } \frac{1}{8} \leq \nu(\theta) \leq \frac{9}{8}\,.$$

The right hand side of (1.1) is adjusted for (5.1) to be the solution. Considering $\boldsymbol{n} = (0,1)^{\boldsymbol{T}}$ and $\boldsymbol{\tau} = (1,0)^{\boldsymbol{T}}$, a direct computation gives

$$(\boldsymbol{Tn})_{\boldsymbol{\tau}} = -80\nu(\theta)\, x^2(1-x)^2\, \boldsymbol{\tau} \quad \text{on} \quad S \quad \text{and} \quad \max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = 5.625.$$

We consider $\gamma = 0.5$, $\kappa = 1$, $g = 3$ and $h = 1/10$, and compute the relative error

$$RE = \frac{\|\nabla(\boldsymbol{u}_h^n - \boldsymbol{u})\|^2 + \|\nabla(\theta_h^n - \theta)\|^2}{\|\nabla \boldsymbol{u}\|^2 + \|\nabla\theta\|^2}$$

for different values of α. Here $(\boldsymbol{u}, \theta) = (\boldsymbol{u}_{ref}, \theta_{ref})$ with $(\boldsymbol{u}_{ref}, \theta_{ref})$ being the finite element solution when $h = 1/128$. The results reported in Table 1 shows the convergence of algorithm 1 and algorithm 2 with respect to α.

α	20	10	5	1	0.75	0.5	0.1	0.02	0.01
RE (algorithm 1)	–	–	–	–	–	0.0193	0.0192	0.0192	0.0192
CPU time (algorithm 1)	–	–	–	–	–	3.22	3.09	2.63	2.79
RE (algorithm 2)	0.0192	0.0192	0.0192	0.0192	0.0192	0.0192	0.0192	0.0192	0.0192
CPU time (algorithm 2)	7.74	7.55	7.02	4.67	5.24	6.99	17.93	53.98	105.45

Table 1: Relative error for algorithm 1 and algorithm 2

We note that algorithm 1 converges for $0.01 \leq \alpha \leq 0.5$ while algorithm 2 is convergent for all values of α. The restriction on α for algorithm 1 was predicted in theorem 3.1, while the non-restriction on α for algorithm 2 is supported by theorem 4.1. Finally we note that when both algorithms converge, algorithm 1 is faster.

5.2 Example 2: Driven cavity flow

We consider the geometry and the solution (5.1) defined in example 1, with $\alpha = 0.01$, $h = 1/10$, $\gamma = 0.5$, $\kappa = 1$. This is classical example that has been studied by among others [24, 25] using classical Tresca's condition. The nonlinear slip condition we use is different and we would like to show by means of numerical simulations the existence of slip/stick zone.
First we report in Table 2 and Table 3 the number of iterations and CPU time needed to achieve the convergence for different values of h in algorithm 1 and algorithm 2 respectively. It appears that there is no direct correlation between the number of iterations required for convergence and the mesh size. It is important to note that the number of iterations is declining when g is increasing.

h	$g = 1$		$g = 3$		$g = 6$	
	Iter	CPU	Iter	CPU	Iter	CPU
1/4	39	0.36	41	0.4	21	0.24
1/8	57	4.53	43	3.34	28	1.77
1/16	54	34.6	37	24	24	15.60
1/32	40	345	36	301	26	214.6

Table 2: Number of iterations and CPU (seconds) for algorithm 1

h	$g = 1$		$g = 3$		$g = 6$	
	Iter	CPU	Iter	CPU	Iter	CPU
1/4	90	1.08	79	0.90	93	1.04
1/8	80	7.16	85	8.01	101	8.56
1/16	83	83.0	81	83.51	99	100.34
1/32	83	1036.73	80	980.0	99	989.56

Table 3: Number of iterations and CPU (seconds) for algorithm 2

Secondly, the velocity, streamlines, pressure, and temperature are represented in Figures 1–6 for indicated values of g. In Figure 1 (obtained using algorithm 1) and Figure 4 (obtained using algorithm 2), one has $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| > g = 1$. It is apparent from the graphs showing streamlines or velocity field that $\boldsymbol{u_\tau}|_S \neq \mathbf{0}$ (recall that $S = (0,1) \times \{1\}$). Hence the nonlinear slip occurs here and (u_1, u_2, p, θ) defined in (5.1) is not the solution of (1.1)...(1.5).
In Figure 2 and Figure 3 obtained using algorithm 1, or Figure 5 and Figure 6 obtained using algorithm 2, $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| \leq g$, and $\boldsymbol{u_\tau} = \mathbf{0}$ (there is no velocity field on S). Hence no slip is observed which implies that (u_1, u_2, p, θ) defined in (5.1) is the solution of (1.1)...(1.5).
One notes through Figures 1–6 that the temperature is non-negative and bounded from above (this observation is not supported by the theory discussed.)

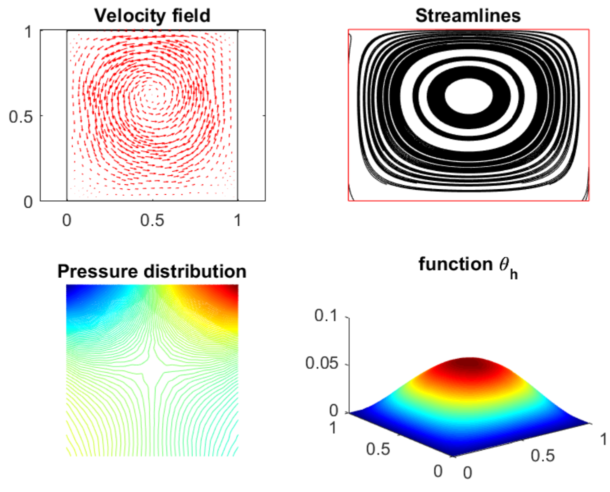

Figure 1: Algorithm 1 for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| > g = 1$

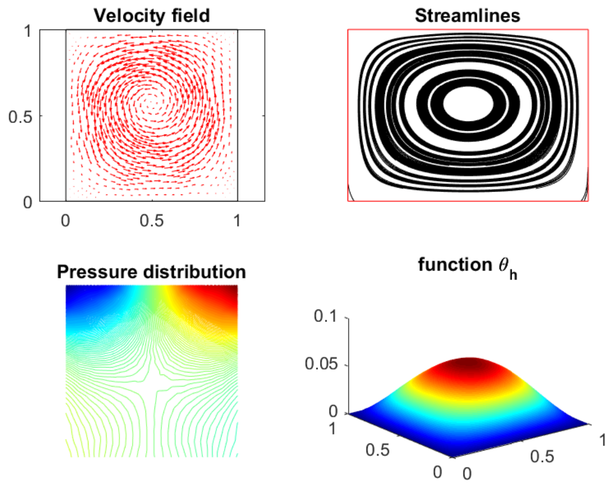

Figure 2: Algorithm 1 for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g = 5.625$

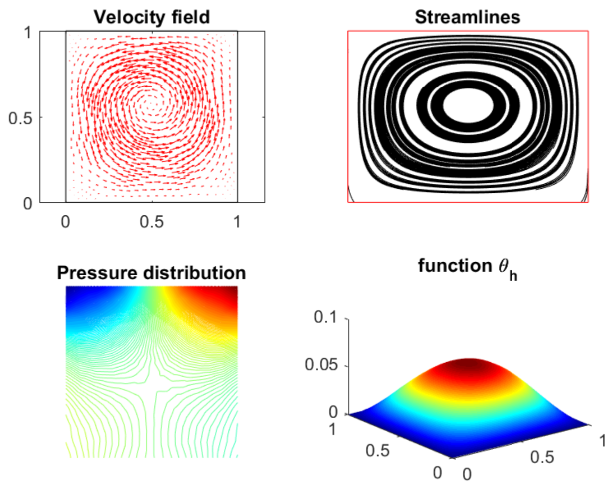

Figure 3: Algorithm 1 for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g = 6$

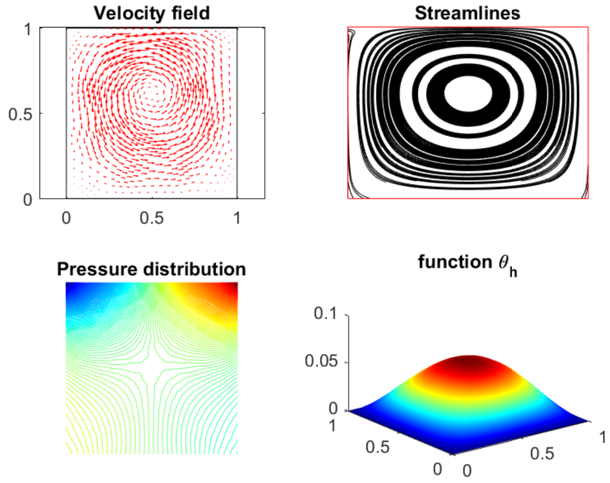

Figure 4: Algorithm 2 for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| > g = 1$

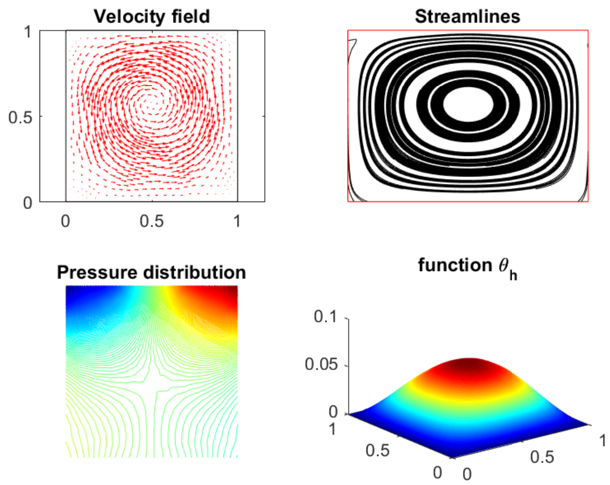

Figure 5: Algorithm 2 for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g = 5.625$

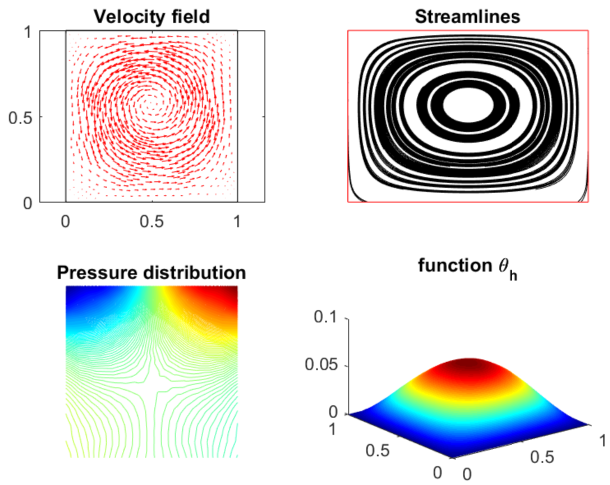

Figure 6: Algorithm 2 for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g = 6$

5.3 Convergence Check

To analyze the convergence, we compute the rates of convergence. For that purpose, we consider the same test example on uniform meshes with $\alpha = 0.01$, $\gamma = 0.5$, and $\kappa = 1$ and $(\boldsymbol{f}, \mathrm{b})$ is obtained by replacing (5.1) in (1.1). Since we do not have the exact solution for $g = 1$, we consider the reference solution $(\boldsymbol{u}_{ref}, p_{ref}, \theta_{ref})$ computed on a refine mesh with $h = 1/128$.
In Table 4, Table 5 we report the results using algorithm 1, while Table 6 and Table 7 are concerned with the convergence using algorithm 2. One notes a linear convergence rate for the quantity $\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1 + \|\theta_{ref} - \theta_h\|_1 + \|p_{ref} - p_h\|$.

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|$	Rate	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1$	Rate	$\|p_{ref} - p_h\|$	Rate
1/4	1.391e-4		4.961e-3		2.317e-2	
1/8	3.423e-5	2.02	2.433e-3	1.02	9.976e-3	1.21
1/16	8.149e-6	2.07	9.782e-4	1.31	5.122e-3	0.96
1/32	1.792e-6	2.18	4.694e-4	1.05	2.862e-3	0.83
1/64	4.991e-7	1.84	2.741e-4	0.77	1.421e-3	1.01

Table 4: Convergence rates with function $g = 1$ and Algorithm 1

h	$\|\theta_{ref} - \theta_h\|$	Rate	$\|\theta_{ref} - \theta_h\|_1$	Rate
1/4	6.535e-6		6.281e-4	
1/8	3.209e-6	1.02	3.963e-4	0.76
1/16	9.956e-7	1.54	2.082e-4	0.93
1/32	3.867e-7	1.36	9.251e-5	1.17
1/64	2.015e-7	0.95	4.315e-5	1.10

Table 5: Convergence rates with function $g = 1$ and Algorithm 1

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|$	Rate	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1$	Rate	$\|p_{ref} - p_h\|$	Rate
1/4	1.392e-2		4.560e-2		1.807e-2	
1/8	3.421e-3	2.02	1.733e-2	1.26	7.976e-3	1.18
1/16	7.149e-4	2.25	8.382e-3	1.04	4.102e-3	0.95
1/32	2.012e-4	1.82	3.291e-3	1.31	1.668e-3	1.13
1/64	4.591e-5	2.13	1.946e-3	0.77	8.822e-4	1.07

Table 6: Convergence rates with function $g = 1$ and Algorithm 2

5.4 Concluding Remarks

We have formulated and analyzed two numerical schemes for the Stokes equations under non-linear slip boundary conditions coupled with the heat equation. We have shown the feasibility of these numerical schemes and established their convergence. Finally, we have validated our theoretical findings by presenting solid numerical experiments. Our next challenge is to study a posteriori error control for flows under nonlinear slip boundary condition.

Acknowledgements. We are grateful to the comments and suggestions of the referees which have contributed to improve this study.

References

[1] A.M. Rashad. Effects of radiation and variable viscosity on unsteady MHD flow of a rotating fluid from stretching surface in porous medium. *J. Egypt. Math. Soc.*, **22**, 134–142, 2014.

[2] F. Brezzi, C. Canuto & A. Russo. A self-adaptive formulation for the Euler/Navier Stokes coupling. *Comput. Methods Appl. Mech. Engrg.*, **73**, 317–330, 1989.

[3] K. R. Rajagopal, G. Saccomandi & L. Vergori. On the Oberbeck-Boussinesq approximation for fluids with pressure dependent viscosities. *Nonlinear Analysis: Real World Applications.*, **10**, 1139–1150, 2009.

h	$\|\theta_{ref} - \theta_h\|$	Rate	$\|\theta_{ref} - \theta_h\|_1$	Rate
1/4	6.035e-5		2.971e-3	
1/8	2.318e-5	1.45	1.962e-3	0.62
1/16	9.656e-6	1.19	1.083e-3	0.85
1/32	4.017e-6	1.25	4.626e-4	1.22
1/64	1.995e-6	1.01	2.314e-4	0.99

Table 7: Convergence rates with function $g = 1$ and Algorithm 2

[4] M. Bulíček, E. Feireisl, & J. Málek. A Navier Stokes-Fourier system for imcompressible fluids with temperature dependent material coefficients. *Nonlinear Analysis: Real World Applications.*, **10**, 992–1015, 2009.

[5] C. Le Roux, Steady Stokes flows with threshold slip boundary conditions, *Math. Models Methods Appl. Sci.*, **15**, No 8, 1141–1168, 2005.

[6] G. Duvaut & J.-L. Lions. Inequalities in Mechanics and Physics. Grundlehren der Mathematischen Wissenschaften. **219**, Springer-Verlag, Berlin, 1976.

[7] H. Fujita. A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions. In:*Mathematical Fluid Mechanics and Modeling.*, RIMS Kōkyūroko, **888**, Kyoto University, Kyoto, 199–216, 1994.

[8] A. Rahma, M. A. Saloua, C. Bernardi, & J. Satouri. Spectral discretization of the Navier-Stokes equations coupled with the heat equation. *ESAIM Math. Model. Numer. Anal.*, **49**, no. 3, 621–639, 2015.

[9] A. Rahma, C. Bernardi, & J. Satouri. Spectral discretization of the time dependant Navier Stokes problem coupled with the heat equation. *Appl. Math. Comput.*, **268**, 59–82, 2015.

[10] J. Deteix, A. Jendoubi, & D. Yakoubi. A coupled prediction scheme for solving the Navier Stokes and convection-diffusion equations. *SIAM J. Numer. Anal.*, **52**, no. 5, 2415–2439, 2014.

[11] C. Bernardi, Sarra Maarouf, & Driss Yakoubi. Spectral discretization of Darcy's equations coupled with the heat equation. *IMA J. Numer. Anal.*, **36**, 1193–1216, 2016.

[12] C. Bernardi, J. Dakroub, G. Mansour, F. Rafei, T. Sayah. Convergence analysis of two numerical schemes applied to a nonlinear elliptic problem. *J. Sci. Comput.*, **71**, no. 1, 329-347, 2017.

[13] J.K. Djoko, V. Konlack Socgnia & M. Mbehou. Stokes equations under nonlinear slip boundary conditions coupled with the heat equation: A priori error analysis. *submitted.*

[14] V. Girault & P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms.* Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo 1986.

[15] P. Ciarlet. Finite Element Method for Elliptic Problems, North Holland, Amstedam, 1978.

[16] R. Glowinski. Finite element methods for incompressible viscous flow. In Handbook of Numerical Analysis, P.G Ciarlet and J.L Lions, eds, vol. IX, North Holland; Amsterdam, 3–1176, 2003.

[17] V. Girault, M.F. Wheeler. Numerical discretization of a Darcy-Forchheimer model. *Numer. Math.*, **110**, no. 2, 161-198, 2008.

[18] H. Brezis. *Functional Analysis, Sobolev Spaces and partial differential equations.* Springer, 2010.

[19] R. Temam. *Navier-Stokes equations: Theory and Numerical Analysis*, 2nd ed., AMS Chelsea publishing, 2001.

[20] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems*, Springer Series in Computational Physics, Springer-Verlag, Berlin Heidelberg, 2008.

[21] H. Fujita. Non-stationary Stokes flows under leak boundary conditions of friction type. *J. Comput. Appl. Math.*, **19**, 1–8, 2001.

[22] J.L. Lions, M. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.*, **16**, no. 6, 964–979, 1979.

[23] D. Boffi , F. Brezzi & M. Fortin. *Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics*, Springer Verlag Berlin, 2013.

[24] T. Kashiwabara. On a finite element approximation of the Stokes equations under a slip boundary condition of the friction type. *J. Indust. Appl. Math*, **30**, 227–261, 2013.

[25] Y. Li and R. An. Penalty finite element method for Navier-Stokes equations with nonlinear slip boundary conditions. *Int. J. Numer. Methods Fluids.*, **69**, 550-566, 2011.