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THE ORDER COMPLETION METHOD: A DIFFERENTIAL-ALGEBRAIC REPRESENTATION

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Abstract. This paper deals with an interpretation of the Order Completion Method for systems of nonlinear partial differential equations (PDEs) in terms of suitable differential algebras of generalized functions. In particular, it is shown that certain spaces of generalized functions that appear in the Order Completion Method may be represented as differential algebras of generalized functions. This result is based on a characterization of order convergence of sequences of normal lower semi-continuous functions in terms of pointwise convergence of such sequences. It is further shown how the mentioned differential algebras are related to the nowhere dense algebras introduced by Rosinger, and the almost everywhere algebras considered by Verneave, thus unifying two seemingly different theories of generalized functions. Existence results for generalized solutions of large classes of nonlinear PDEs obtained through the Order Completion Method are interpreted in the context of the earlier nowhere dense and almost everywhere algebras.

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1. Introduction

For over 130 years by now, there has been a general and type independent theorem regarding the existence of solutions of systems of nonlinear partial differential equations (PDEs) [18], see [13] for a more recent presentation. In this regard, consider a system of K nonlinear PDEs of the form

$$D_t^m \mathbf{u}(t,y) = \mathbf{G}\left(t, y, ..., D_t^p D_y^q u_i(t,y), ...\right) \tag{1.1}$$

with $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, $m \geq 1$, $0 \leq p < m$, $q \in \mathbb{N}^{n-1}$, $p + |q| \leq m$ and with analytic Cauchy data

$$D_t^p \mathbf{u}(t_0, y) = \mathbf{g}_p(y), \; 0 \leq p < m, \; (t_0, y) \in S \tag{1.2}$$

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on the hyperplane $S = \{(t_0, y) : y \in \mathbb{R}^{n-1}\}$. If the mapping $\mathbf{G}$ is analytic, then there exists a neighborhood V of t_0 in $\mathbb{R}$, and an analytic function $\mathbf{u} : V \times \mathbb{R}^{n-1} \to \mathbb{C}^K$ that satisfies (1.1) and (1.2). Although this existence result is completely type independent as far as the nonlinear terms that may appear in the equation (1.1), it is deficient on the following two counts.

Firstly, the solution can only be guaranteed to exist on a neighborhood of the hyperplane on which the initial data (1.2) is specified. It should be noted that this *local nature* of the Cauchy-Kovalevskaia Theorem is not due to the particular techniques used in the proof, but may rather be attributed to the very nature of linear and nonlinear PDEs. Indeed, we may recall that the Burgers equation

$$u_t(x,t) + u(x,t)u_x(x,t) = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty) \tag{1.3}$$

with initial condition

$$u(x,0) = u_0(x) \tag{1.4}$$

may fail to have a classical solution on $\mathbb{R} \times (0,\infty)$, even if the initial condition u_0 is analytic on $\mathbb{R}$. If $u_0'(a) < 0$ for any $a \in \mathbb{R}$, then there is no function $u \in \mathcal{C}^1(\mathbb{R} \times (0,\infty))$ that satisfies the initial condition (1.4), and the PDE (1.3) on the *whole domain of definition* of the equation, see for instance [19].

The mentioned local nature of the Cauchy-Kovalevskaia Theorem is unsatisfactory in at least one respect. In many of the physical systems that are supposed to be modeled with systems of PDEs, one may be interested in solutions that exist on domains that are larger than that delivered by the Cauchy-Kovalevskaia Theorem. Thus, as can be seen from rather elementary examples, such as (1.3) to (1.4), classical solutions of a given system of PDEs may fail to exists on the whole domain of physical interest. From a physical point of view, it may be precisely those points in the domain of definition of a system of PDEs where the classical solution fails to exist that are of interest. In particular, these points may represent regions where the underlying physical system experiences shocks and various other types of nonsmooth phenomena.

Secondly, the Cauchy-Kovalevskaia Theorem is limited to the realm of analytic nonlinear PDEs, with analytic initial data. In this regard Lewy [20] showed that, for a large class of functions $f_1, f_2 \in \mathcal{C}^\infty(\mathbb{R}^3)$, the system of equations

$$\begin{array}{rcl} -\frac{\partial}{\partial x_1}U_1 + \frac{\partial}{\partial x_2}U_2 - 2x_1\frac{\partial}{\partial x_3}U_2 - 2x_2\frac{\partial}{\partial x_3}U_1 & = & f_1 \\ -\frac{\partial}{\partial x_1}U_2 + \frac{\partial}{\partial x_2}U_1 + 2x_1\frac{\partial}{\partial x_3}U_1 - 2x_2\frac{\partial}{\partial x_3}U_2 & = & f_2 \end{array}, \tag{1.5}$$

which may be written as a single equation with complex coefficients, has no solution with Hölder continuous first order partial derivatives in any neighborhood of any point in $\mathbb{R}^3$.

On the other hand, in many of the applications of PDEs to mathematics and physics, the assumption of analyticity on the PDE (1.1) and the initial condition (1.2) may not be realistic. Indeed, although many of the equations that appear in applications are analytic, in particular of second order, and with polynomial nonlinear terms, the initial and / or boundary conditions that are specified are often given by functions that are less smooth. In such situations, the Cauchy-Kovalevskaia Theorem does not apply.

As a further motivation for the need to go beyond analytic nonlinear PDEs, we may note that the set of all analytic functions

$$\mathbf{G} : \mathbb{R}^n \times \mathbb{R}^M \to \mathbb{C}^K$$

is of first Baire category in $\mathcal{C}^\infty\left(\mathbb{R}^n \times \mathbb{R}^M, \mathbb{R}^K\right)$, with respect to the usual Fréchet space topology on $\mathcal{C}^\infty\left(\mathbb{R}^n \times \mathbb{R}^M, \mathbb{R}^K\right)$. Thus, the class of analytic nonlinear PDEs is dramatically smaller than the class of all $\mathcal{C}^\infty$-smooth PDEs. In this context, it appears that the Cauchy-Kovalevskaia Theorem applies only to a rather particular class of equations.

In view of these remarks, it is clear that the Cauchy-Kovalevskaia Theorem is deficient on several counts. Thus, there is a particular interest in more general existence and regularity results for the solutions of large classes of PDEs. Furthermore, due to Lewy's nonexistence result, it is clear that such a general existence result must necessarily be formulated in the context of a suitable concept of generalized function. However, Lewy's nonexistence result remains true even in the following stronger form, see for instance [17]: For certain $f_1, f_2 \in \mathcal{C}^\infty\left(\mathbb{R}^3\right)$, the system (1.5) has no distributional solutions in any neighborhood of any point of $\mathbb{R}^3$. Thus, the space of distributions is insufficient from the point of view of existence of solutions, even for linear equations. Furthermore, Schwarz [37] showed that there is no associative multiplication on $\mathcal{D}'(\Omega)$ that extends the pointwise multiplication of continuous functions. That is, nonlinear problems cannot be formulated in terms of distributions alone.

The mentioned Lewy insufficiency result [20], as well as other nonexistence results, see for instance [17] and [38], and the Schwarz impossibility result, are commonly interpreted as implying that a general and type independent theory for the existence and basic regularity properties of (generalized) solutions of nonlinear PDEs is not possible, or at best highly unlikely. Arnold [5] attributes the nonexistence of such a general theory to the more complicated geometry of $\mathbb{R}^n$, for $n > 1$, relative to that of $\mathbb{R}$, which suffices for ordinary differential equations (ODEs) alone.

In the context of the usual linear topological spaces of generalized functions, such as Sobolev spaces and spaces of distributions, it may turn out that a general and type independent theory for the solutions of nonlinear PDEs cannot be obtained. Indeed, in the nearly eighty years since functional analysis was introduced into the study of PDEs [39, 40], the Cauchy-Kovalevskaia Theorem has not been improved upon on its own general and type independent grounds, using the customary linear functional analytic tools. However, the apparent failure of the customary theories of generalized functions to deliver a general existence result for the solutions of large classes of nonlinear PDEs is not due to any failure of mathematics as such, but may rather be attributed to the inherent limitations of the linear functional analytic methods themselves.

In this regard, we may mention the following. Many of the usual concepts of generalized function, and of generalized solutions of PDEs, do not allow sufficiently singular objects to act as solutions of PDEs. Indeed, due to the well know Sobolev Embedding Theorem, see for instance [1], the Sobolev space $W^{k,p}(\mathbb{R}^n)$, with $1 \leq p < \infty$, will for sufficiently large values of k, depending on p and n, contain only continuous functions. Furthermore, even if a given PDE has a solution which is classical, even analytic, everywhere except at a single point of the domain of definition of the equation, this solution may happen not to belong to any of the usual spaces of generalized functions. Recall that, according to the Great Picard Theorem, a function

$$u : \mathbb{C} \setminus \{z_0\} \to \mathbb{C},$$

which is analytic at every point of its domain of definition, and with an essential singularity at z_0, attains every complex number, with possibly one exception, as a value in every neighborhood of z_0. Such a function u_0 cannot satisfy any of the *growth conditions* that are, rather as a rule, imposed on generalized functions near singularities. Indeed, elements of the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ are locally integrable, while distributions satisfy certain polynomial type growth conditions near singularities [34].

The customary theories of generalized solutions of PDEs, and the associated linear topological spaces of generalized functions, appear unable to deliver a general and type independent existence result for the solutions of any significantly large class of PDEs. Moreover, as exemplified by the Schwarz impossibility result, these methods are unable to deal with the nonlinearities which appear in PDEs. However, both the Lewy insufficiency and the Schwarz impossibility have by now been overcome on several occasions, in contexts other than that of the usual functional analytic approach to PDEs.

Already in the late 1960s, see [26, 27, 28, 29, 30], it was shown how the celebrated Schwarz impossibility result may be overcome by embedding $\mathcal{D}'(\Omega)$ as a linear subspace into suitable *algebras of generalized functions*. In this way, one may compute an arbitrary product of distributions, although the result will, in general, no longer be a distribution. This approach to the problem of multiplication of generalized functions has, since its inception in the 1960s, been developed into a comprehensive nonlinear theory of generalized functions, mainly in [31, 32, 33, 34], see also [22], with a variety of applications to the solutions of linear and nonlinear PDEs.

An important particular case of the general theory was introduced, independently, by Colombeau [9, 10]. This version of the theory has gained a certain popularity among analysts, and has seen rapid development and a number of nontrivial application to PDEs and their numerical analysis, geometry and physics, see for instance [8, 11, 12, 16, 22]. The mentioned popularity of the Colombeau theory may be due to the close and rather natural connection with distributions. In particular, the full Colombeau algebra $\mathcal{G}(\Omega)$ was, until recently, the only known differential algebra that admits a *canonical linear em-*

bedding $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}(\Omega)$ that commutes with distributional derivatives. That is, the partial derivatives in $\mathcal{G}(\Omega)$ agree the usual derivatives in $\mathcal{D}'(\Omega)$, when restricted to distributions.

As mentioned, the Colombeau algebras have come to be appreciated as a suitable setting for the study of generalized solutions of large classes of linear and nonlinear PDEs. Furthermore, in view of its close connection with distributions, it may appear that Colombeau's class of generalized functions is the most natural class of differential algebras. However, several deep results obtained within the more general version of the theory have no counterpart in Colombeau's theory. In this regard, we may mention the *global* version of the Cauchy-Kovalevskaia Theorem obtained within the so called nowhere dense algebras of generalized functions [33]: Generalized solutions of all analytic nonlinear PDEs are defined on the whole domain of definition of the respective equations, and are analytic everywhere except on a closed nowhere dense set, which for convenience may be chosen to have zero Lebesgue measure. As mentioned, due to the *polynomial type growth conditions* imposed on Colombeau generalized functions near singularities, one cannot formulate, let alone prove, such a global existence result in $\mathcal{G}(\Omega)$. Furthermore, and also due to the mentioned growth conditions, one cannot define arbitrary Lie groups globally on $\mathcal{G}(\Omega)$.

Here we should mention that a large class of differential algebras that admit a linear embedding of $\mathcal{D}'(\Omega)$ which commutes with distributional derivatives, as well as a global version of the Cauch-Kovalevskaia Theorem, was recently introduced by Vernaeve [51]. In this way, one may come to realize that the way in which products of generalized functions, and in particular distributions, are defined is rather arbitrary. Indeed, multiplication, and in general, nonlinear operations on such singular objects do inevitably branch in infinitely many ways, *without the possibility for the existence of some unique natural or canonical way such nonlinear operations may be performed*, see [36].

As mentioned, the customary theories for generalized functions and (generalized) solutions of PDEs are formulated, almost without exception, in terms of topology and functional analysis. Thus, the differential algebraic nonlinear theories of generalized functions constitute a deescalation to the level of algebra, insofar as these theories may be formulated entirely in algebraic terms. This deescalation has already proven to be particularly powerful, as demonstrated by the two main achievements of these theories. Namely, the circumvention of the Schwarz impossibility result, and a number of existence results for the solutions of classes of PDEs that were previously unsolved, or that were proven to be unsolvable in spaces of distributions or hyperfunctions. In this regard, we may recall the wide ranging, and purely algebraic characterization of the solvability of all polynomial nonlinear PDEs with continuous coefficients [35], as well an existence result for all $\mathcal{C}^{\infty}$-smooth linear PDEs obtained by Todorov [42]. In this way, a generalized solution is obtained, as a particular case, for the Lewy equation (1.5).

A further deescalation in the study of the solutions of PDEs, this time to the level of order, has resulted in a dramatic extension of of the Cauchy-

Kovalevskaia Theorem to a large class of *continuous* PDEs [23]. The Order Completion Method delivers generalized solutions of a large class of nonlinear PDEs, which include all linear and all polynomial type nonlinear PDEs, as elements of the Dedekind order completion of suitable spaces of piecewise smooth functions. In this way, one manages to overcome the Lewy insufficiency within large margins of both *nonlinearity* and *lack of smoothness*. What is more, this method also produces a blanket regularity result for the solutions constructed. Indeed, as shown in [23], the generalized solutions of linear and nonlinear PDEs obtained through the Order Completion Method may be assimilated with usual real measurable functions, or even Hausdorff continuous interval valued functions [3].

Furthermore, in view of the intuitively clear nature of the concept of order, as well as the rather simple constructions involved, one also gains much insight into the mechanisms involved in the solution of PDEs, as well as the structure of such generalized solutions. In this regard, the deescalation from the level of topology and functional analysis to the level of order proves to be particularly useful and relevant. However, when it comes to further properties of the solutions, such as, for instance, regularity of the solutions, functional analysis, or for that matter any mathematics, may yet play an important, but secondary role. Indeed, a recent reformulation of the Order Completion Method [43, 44, 45, 46] in terms of uniform convergence spaces has resulted in significant enrichment of the basic theory. In this regard, a further strengthening of the basic regularity properties of generalized solutions of PDEs, as well as dramatic new insights into the structure of generalized functions have been obtained. These advances are made possible, mainly, by introducing new spaces of generalized functions, much in the original spirit of Sobolev [39, 40].

One of the major problems facing the Order Completion Method is that there does not appear to be any natural correspondence between the new method and the classical theory of PDEs, and the associated spaces of generalized functions. This apparent incompatibility between the Order Completion Method on the one hand, and the customary approach to PDEs on the other, may be attributed to a number of fundamental issues, of which we mention only the following.

The Order Completion Method is essentially based on approximation by sequences of piecewise smooth functions. In particular, generalized functions may be expressed as the limit of sequences of functions

$$U_n \in \mathcal{C}^\infty(\Omega \setminus \Gamma_n)$$

where, for each $n \in \mathbb{N}$, the set $\Gamma_n \subset \Omega$ is closed and nowhere dense in Ω. Such closed nowhere dense singularity sets may happen to be far larger than those that can be handled by the usual concepts of generalized functions. Indeed, a closed nowhere dense set $\Gamma \subset \Omega$ may have arbitrary large positive Lebesgue measure [24]. Furthermore, the approximations constructed in the Order Completion Method do not satisfy any a priori growth conditions near singularities. As mentioned, such growth conditions are typical in the customary constructions of generalized functions. Thus, insofar as the nature of the actual approximations are concerned, the Order Completion Method appears far removed from many of the typical theories for generalized functions and generalized solutions of PDEs.

However, also in the way the very concepts of limit and approximation are understood, the Order Completion Method stands apart from classical PDE theory. In this regard, generalized solutions of PDEs are, rather as a rule, obtained as elements of the completion of a linear topological, often metrizable, space of smooth functions. Thus, generalized solutions of PDEs may be expressed as limits of sequences of classical smooth functions with respect to some (metrizable) linear space topology. On the other hand, the approximation results upon which the Order Completion Method is based cannot be formulated in terms of the usual concepts of topology, see [4, 43, 44, 45, 46], but requires the use of more general notions of topology, such as convergence spaces [7].

In this paper, we give a partial resolution of this problem. In this regard, we will show how the Order Completion Method can be interpreted in the context of the differential algebraic theory of generalized functions. In particular, spaces of generalized function introduced in [45, 47] are interpreted as differential algebras of generalized functions. The mentioned spaces of generalized functions are shown to be closely related to the closed nowhere dense algebras introduced and studied in [32, 33, 34], and the almost everywhere algebras of Vernaeve [51]. This relationship is exploited to give an interpretation of the existence results for solutions of large classes of nonlinear PDEs obtained in the context of the Order Completion Method in the closed nowhere dense and almost everywhere algebras, respectively.

The paper is organized as follows. In Section 2 we recall the basic concepts and results underlying the recent reformulation of the Order Completion Method in terms of convergence spaces. Spaces of generalized functions that appear in the recent pseudo-topological formulation of the Order Completion Method are interpreted as differential algebras of generalized functions in Section 3. Sections 4 and 5 deal with the connections between the algebras constructed in Section 3, and the closed nowhere dense and almost everywhere algebras, respectively. In Section 6 we give an interpretation of the existence results for the solutions of large classes of nonlinear PDEs obtained in [48] in the context of the closed nowhere dense and almost everywhere algebras.

Let us now fix some notation. By Ω we denote an open subset of $\mathbb{R}^n$, and for $x \in \Omega$, $\mathcal{V}_x$ is the set of open neighbourhoods of x. For a function $u : \Omega \to \mathbb{R}$, the sequence of functions on Ω with all terms equal to u is denoted by $\Delta(u)$. The reader who is less familiar with the theory of convergence spaces and uniform convergence spaces is reffered to [7] and [14, 15].

2. The space $\mathcal{NL}^{\infty}(\Omega)$

We recall the main points regarding the construction of the space $\mathcal{NL}^{\infty}(\Omega)$, see [47]. Denote by $\mathcal{NL}(\Omega)$ the set of all nearly finite, normal lower semi-continuous functions $u : \Omega \to \overline{\mathbb{R}}$. That is, $u \in \mathcal{NL}(\Omega)$ if and only if

$$I[S[u]] = u, \quad \{x \in \Omega \ : \ u(x) \in \mathbb{R}\} \text{ is finite,} \tag{2.1}$$

where for $v : \Omega \to \overline{\mathbb{R}}$ we set

$$I[v](x) = \sup\{\inf\{v(y) \ : \ y \in V\} \ : \ V \in \mathcal{V}_x\}, \quad x \in \Omega \tag{2.2}$$

and

$$S[v](x) = \inf\{\sup\{v(y) \ : \ y \in V\} \ : \ V \in \mathcal{V}_x\}, \quad x \in \Omega, \tag{2.3}$$

see for instance [2, 6]. Let

$$\mathcal{ML}^0(\Omega) = \left\{ u \in \mathcal{NL}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense } : \\ & u \in \mathcal{C}^0(\Omega \setminus \Gamma) \end{array} \right\}$$

and

$$\mathcal{ML}^\infty(\Omega) = \left\{ u \in \mathcal{NL}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense } : \\ & u \in \mathcal{C}^\infty(\Omega \setminus \Gamma) \end{array} \right\}.$$

On $\mathcal{ML}^0(\Omega)$, we consider the following uniform convergence structure [43]. Denote by Σ the collection of all order intervals in $\mathcal{ML}^0(\Omega)$, and for $\Sigma' \subset \Sigma$, we denote by $[\Sigma']$ the filter generated by Σ', if this filter exists.

Definition 2.1. A filter $\mathcal{U}$ on $\mathcal{ML}^0(\Omega) \times \mathcal{ML}^0(\Omega)$ belongs to $\mathcal{J}_o$ if there exists $k \in \mathbb{N}$ such that

$$\begin{array}{ll} \forall & i = 1, ..., k \ : \\ \exists & \Sigma_i = \left(I_n^i\right) \subseteq \Sigma : \\ & 1) \quad I_{n+1}^i \subseteq I_n^i, \ n \in \mathbb{N} \\ & 2) \quad ([\Sigma_1] \times [\Sigma_1]) \cap ... \cap ([\Sigma_k] \times [\Sigma_k]) \subseteq \mathcal{U} \end{array} \tag{2.4}$$

and, moreover, for every $i = 1, ..., k$ and $V \subseteq \Omega$ an open set, one has

$$\begin{array}{ll} \exists & u_i \in \mathcal{ML}\left(\Omega\right) \ : \\ & \cap_{n\in\mathbb{N}} I_{n|V}^i = \left\{u_{i|V}\right\} \end{array} \quad \text{or} \quad \cap_{n\in\mathbb{N}} I_{n|V}^i = \emptyset \tag{2.5}$$

where $I_{n|V}^i \subseteq \mathcal{ML}(V)$ denotes the set consisting of the restrictions of members of I_n^i to V.

The uniform convergence structure $\mathcal{J}_o$ is Hausdorff and first countable [43]. Furthermore, $\mathcal{J}_o$ induces the order convergence structure λ_o, see [4, 43, 48]. That is, a filter $\mathcal{F}$ on $\mathcal{ML}^o(\Omega)$ converges to $u \in \mathcal{ML}^0(\Omega)$ if and only if there exist sequences $(\lambda_n), \ (\mu_n) \subset \mathcal{ML}^0(\Omega)$ such that

$$\begin{array}{ll} 1) & \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n, \ n \in \mathbb{N} \\ 2) & \sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n \\ 3) & [\{[\lambda_n, \mu_n] \ : \ n \in \mathbb{N}\}] \subseteq \mathcal{F}. \end{array} \tag{2.6}$$

(2.7)

In particular, a sequence (u_n) converges to $u \in \mathcal{ML}^0(\Omega)$ if and only if it order converges to u, see [21, 43, 48]. That is, (u_n) converges to u if and only if there exist sequences $(\lambda_n),\ (\mu_n) \subset \mathcal{ML}^0(\Omega)$ such that

$$(2.8) \qquad \begin{array}{ll} 1) & \lambda_n \leq \lambda_{n+1} \leq u_{n+1} \leq \mu_{n+1} \leq \mu_n,\ n \in \mathbb{N} \\ 2) & \sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n \end{array}$$

The (Wyler) completion [25, 52] of $\mathcal{ML}^0(\Omega)$ with respect to $\mathcal{J}_o$ is the space $\mathcal{NL}(\Omega)$, equipped with a suitable uniform convergence structure [43].

The space $\mathcal{ML}^\infty(\Omega)$ is equipped with the initial uniform convergence structure $\mathcal{J}_o^\infty$ with respect to the family of mappings

$$(2.9) \qquad \mathcal{D}^p : \mathcal{ML}^\infty(\Omega) \to \mathcal{ML}^0(\Omega), \quad p \in \mathbb{N}^n,$$

where $\mathcal{D}^p(u) = I([S[D^p u]]$ for $p \in \mathbb{N}^n$ and $u \in \mathcal{ML}^\infty(\Omega)$. That is, a filter $\mathcal{U}$ belongs to $\mathcal{J}_o^\infty$ if and only if

$$(2.10) \qquad [\mathcal{D}^p \times \mathcal{D}^p](\mathcal{U}) \in \mathcal{J}_o, \quad p \in \mathbb{N}^n.$$

A filter $\mathcal{F}$ on $\mathcal{ML}^\infty(\Omega)$ converges to $u \in \mathcal{ML}^\infty(\Omega)$ with respect to the convergence structure λ_o^∞ induced by the uniform convergence structure $\mathcal{J}_o^\infty$ if and only if $\mathcal{D}^p(\mathcal{F}) \in \lambda_o(\mathcal{D}^p(u))$ for every $p \in \mathbb{N}^n$. That is, the convergence structure induced by $\mathcal{J}_o^\infty$ is the initial convergence structure with respect to the family of mappings (2.9) and the convergence structure (2.7) on $\mathcal{ML}^0(\Omega)$. In particular, a sequence (u_n) converges to $u \in \mathcal{ML}^\infty(\Omega)$ if and only if $(\mathcal{D}^p u_n)$ order converges to $\mathcal{D}^p u$ in $\mathcal{ML}^0(\Omega)$ for every $p \in \mathbb{N}^n$.

The completion of $\mathcal{ML}^\infty(\Omega)$, which is denoted by $\mathcal{NL}^\infty(\Omega)$, may be represented as a set of nearly finite normal lower semi-continuous functions on Ω. In particular, the mapping

$$\mathbf{D} : \mathcal{NL}^\infty(\Omega) \ni u \mapsto \mathbf{D}(u) = (\mathcal{D}^{p\sharp} u)_{p\in\mathbb{N}^n} \in \mathcal{NL}(\Omega)^\omega$$

is a uniformly continuous injection. Here

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^\infty(\Omega) \to \mathcal{NL}(\Omega), \quad p \in \mathbb{N}^n$$

are the extensions through uniform continuity of the partial derivatives (2.9). In this way, we may represent each generalized function $u \in \mathcal{NL}^\infty(\Omega)$, through its generalized partial derivatives, as an ω-tuple of normal lower semi-continuous functions.

Note that the range of each mapping in (2.9) is contained in $\mathcal{ML}^\infty(\Omega) \subset \mathcal{ML}^0(\Omega)$. In particular, the diagram

$$\begin{tikzcd} \mathcal{ML}^\infty(\Omega) \arrow[rr, "\mathcal{D}^p"] \arrow[dr, "\mathcal{D}^p"'] & & \mathcal{ML}^0(\Omega) \\ & \mathcal{ML}^\infty(\Omega) \arrow[ur, "\subset"'] & \end{tikzcd}$$

commutes for each $p \in \mathbb{N}^n$. Furthermore, the mappings

$$\mathcal{D}^p : \mathcal{ML}^\infty(\Omega) \to \mathcal{ML}^\infty(\Omega), \quad p \in \mathbb{N}^n$$

are clearly uniformly continuous with respect to the uniform convergence structure $\mathcal{J}_o^\infty$, as is the inclusion $\mathcal{ML}^\infty(\Omega) \subset \mathcal{ML}^0(\Omega)$. Therefore we obtain the commutative diagram

$$\begin{tikzcd} \mathcal{NL}^\infty(\Omega) \arrow[rr, "\mathcal{D}^{p\sharp}"] \arrow[dr, "\mathcal{D}^{p\sharp}"'] & & \mathcal{NL}(\Omega) \\ & \mathcal{NL}^\infty(\Omega) \arrow[ur, "\subset^\sharp"'] & \end{tikzcd} \tag{2.11}$$

where $\subset^\sharp$: $\mathcal{NL}^\infty(\Omega) \to \mathcal{NL}(\Omega)$ denotes the extension through uniform continuity of the inclusion $\mathcal{ML}^\infty(\Omega) \subset \mathcal{ML}^0(\Omega)$. The meaning of (2.11) is that the space $\mathcal{NL}^\infty(\Omega)$ is closed under (generalized) partial derivatives.

We now turn to the issue of existence of generalized solutions of nonliner PDEs in the space $\mathcal{NL}^\infty(\Omega)$. In this regard, consider a nonlinear PDE

$$T(x, D)u(x) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^n \tag{2.12}$$

of order at most m, where $f \in \mathcal{C}^\infty(\Omega)$ and the operator $T(x, D)$ is defined through a $\mathcal{C}^\infty$-smooth function

$$F : \Omega \times \mathbb{R}^M \to \mathbb{R} \tag{2.13}$$

by the expression

$$T(x, D)u(x) = F(x, u(x), ..., D^p u(x), ...), \quad x \in \Omega, \ |p| \leq m, \tag{2.14}$$

provided that u has partial derivatives up to order m at x. Clearly the partial differential operator $T(x, D)$ acts as a mapping

$$T(x, D) : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega) \tag{2.15}$$

where, for $u \in \mathcal{C}^\infty(\Omega)$, $T(x, D)u \in \mathcal{C}^\infty(\Omega)$ is defined through (2.14). Since the mapping (2.13) is $\mathcal{C}^\infty$-smooth, it follows that the mapping (2.15) extends to

$$T : \mathcal{ML}^\infty(\Omega) \ni u \mapsto I[S[F(\cdot, u, ..., \mathcal{D}^p(u), ..,)]] \in \mathcal{ML}^\infty(\Omega). \tag{2.16}$$

The mapping (2.16) is uniformly continuous with respect to the uniform convergence structure $\mathcal{J}_o^\infty$, see [47, Theorem 4.1]. Consequently, there exists a unique uniformly continuous extension

$$T^\sharp : \mathcal{NL}^\infty(\Omega) \to \mathcal{NL}^\infty(\Omega) \tag{2.17}$$

of (2.16). A solution $u \in \mathcal{NL}^\infty(\Omega)$ of the equation

$$T^\sharp u = f \tag{2.18}$$

is interpreted as a generalized solution of (2.12).

In order to formulate the existence result for generalized solutions of $\mathcal{C}^\infty$-smooth PDEs in $\mathcal{NL}^\infty(\Omega)$, some additional notations are required. In this regard, assume that (2.13) is $\mathcal{C}^\infty$-smooth. For each $q \in \mathbb{N}^n$, we denote by F^q the mapping

$$F^q : \Omega \times \mathbb{R}^{M_q} \to \mathbb{R} \tag{2.19}$$

such that

$$D^q \left(T\left(x, D\right) u\left(x\right)\right) = F^q \left(x, ..., D^p u\left(x\right), ...\right), \ |p| \leq m + |q|$$

for all functions $u \in \mathcal{C}^\infty\left(\Omega\right)$. Finally, we denote by F^∞ the mapping

$$F^\infty : \Omega \times \mathbb{R}^{\mathbb{N}^n} \ni \left(x, \left(\xi_p\right)_{p \in \mathbb{N}^n}\right) \mapsto \left(F^q\left(x, ..., \xi_p, ...\right)\right)_{q \in \mathbb{N}^n} \in \mathbb{R}^{\mathbb{N}^n}.$$

We equip $\mathbb{R}^{\mathbb{N}^n}$ with the product topology.

Theorem 2.2. *Consider a nonlinear PDE (2.12). Assume that the mapping (2.13) as well as the righthand term f are $\mathcal{C}^\infty$-smooth, and satisfy*

$$\begin{array}{ll} \forall & x_0 \in \Omega : \\ \exists & \xi\left(x_0\right) \in \mathbb{R}^{\mathbb{N}^n}, \ F^\infty\left(x_0, \xi\left(x_0\right)\right) = \left(D^q f\left(x_0\right)\right)_{q \in \mathbb{N}^n} \ : \\ \exists & V \textit{ a neighborhood of } x_0, \ W \textit{ a neighborhood of } \xi\left(x_0\right) \ : \\ & F^\infty : V \times W \to \mathbb{R}^{\mathbb{N}^n} \textit{ is open.} \end{array} \tag{2.20}$$

Then there exists a solution $u \in \mathcal{NL}^\infty(\Omega)$ of (2.18).

3. $\mathcal{NL}^\infty(\Omega)$ as a Differential Algebra

In this section, we show that the space $\mathcal{NL}^\infty(\Omega)$ considered in Section 2 admits a representation as a differential algebra of generalized functions, in the sense of [33, 34]. In this regard, we recall [50] that $\mathcal{NL}(\Omega)$ is an Archimedean, and hence also commutative, *f*-algebra with respect to the usual pointwise order, with the algebraic operations given by

$$[u+v](x) = I[S[u \oplus v]](x), \ [uv](x) = I[S[u \odot v]](x), \quad [\alpha u](x) = \alpha u(x),$$

where $u \oplus v$ and $u \odot v$ denote the pointwise sum and product of functions $u, v \in \mathcal{NL}(\Omega)$. It is easy to see that $\mathcal{ML}^0(\Omega)$ is a subalgebra and a sublattice of $\mathcal{NL}(\Omega)$. Hence the convergence structure induced on $\mathcal{ML}^0(\Omega)$ by the uniform convergence structure $\mathcal{J}_o$, namely, the order convergence structure, is a first countable algebra convergence structure, see [48]. Thus the following properties of the convergence structure λ_o^∞ on $\mathcal{ML}^\infty(\Omega)$ are easily verified.

Proposition 3.1. *The convergence structure λ_o^∞ induced on $\mathcal{ML}^\infty(\Omega)$ is a first countable algebra convergence structure.*

Proof. In view of (2.10), the convergence structure λ_o^∞ is the initial convergence structure with respect to the family of mappings $(\mathcal{D}^p : \mathcal{ML}^\infty(\Omega) \to \mathcal{ML}^0(\Omega))_{p\in\mathbb{N}^n}$, where $\mathcal{ML}^0(\Omega)$ is equipped with the order convergence structure λ_o. Hence λ_o^∞ is a first countable and Hausdorff vector space convergence structure. Furthermore, since λ_o is an algebra convergence structure on $\mathcal{ML}^0(\Omega)$ it follows that

$$\mathcal{D}^p(\mathcal{F}\mathcal{G}) \supseteq \sum_{q\leq p} \binom{p}{q} \mathcal{D}^q(\mathcal{F})\mathcal{D}^{p-q}(\mathcal{G})$$

converges to

$$\sum_{q\leq p} \binom{p}{q} \mathcal{D}^q u \mathcal{D}^{p-q} v = \mathcal{D}^p(uv)$$

in $\mathcal{ML}^0(\Omega)$ for all $p \in \mathbb{N}^n$, $\mathcal{F} \in \lambda_o^\infty(u)$ and $\mathcal{G} \in \lambda_o^\infty(v)$. Hence λ_o^∞ is an algebra convergence structure. □

The set $\mathcal{NL}^\infty(\Omega)$, as the completion of the uniform convergence space $\mathcal{ML}^\infty(\Omega)$, may be constructed concretely in terms of the Cauchy filters on $\mathcal{ML}^\infty(\Omega)$, see for instance [52]. Indeed, if we denote by $C[\mathcal{ML}^\infty(\Omega)]$ the set of Cauchy filters on $C[\mathcal{ML}^\infty(\Omega)]$, then

$$\mathcal{NL}^\infty(\Omega) = C[\mathcal{ML}^\infty(\Omega)]/ \sim_C = \{[\mathcal{F}]_C \ : \ \mathcal{F} \in C[\mathcal{ML}^\infty(\Omega)]\}$$

where $\sim_C$ is the equivalence relation on $\mathcal{ML}^\infty(\Omega)$ defined as

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \mathcal{F} \cap \mathcal{G} \in C[\mathcal{ML}^\infty(\Omega)],$$

and $[\mathcal{F}]_C$ denotes the $\sim_C$-equivalence class generated by $\mathcal{F}$. Thus properties of the Cauchy filters on $\mathcal{ML}^\infty(\Omega)$ will determine the structure of the set $\mathcal{NL}^\infty(\Omega)$. In view of (2.10), a filter $\mathcal{F}$ on $\mathcal{ML}^\infty(\Omega)$ is a Cauchy filter if and only if

$$\mathcal{D}^p(\mathcal{F}) \text{ is a Cauchy filter on } \mathcal{ML}^0(\Omega),\ p \in \mathbb{N}^n. \tag{3.1}$$

Thus the Cauchy filters on $\mathcal{ML}^\infty(\Omega)$ are determined by the Cauchy filters on $\mathcal{ML}^0(\Omega)$. For this reason we proceed by first considering the Cauchy filters on $\mathcal{ML}^0(\Omega)$. In this regard, we have the following.

Proposition 3.2. *A filter $\mathcal{F}$ on $\mathcal{ML}^0(\Omega)$ is a Cauchy filter with respect to $\mathcal{J}_o$ if and only if the filter $\mathcal{F} - \mathcal{F}$ converges to* 0.

Proof. Assume that $\mathcal{F} - \mathcal{F}$ converges to 0. Then (2.7) implies that there exist sequences $(\lambda_n),\ (\mu_n) \subset \mathcal{ML}^0(\Omega)$ such that

$$\begin{array}{ll} 1) & \lambda_n \leq \lambda_{n+1} \leq 0 \leq \mu_{n+1} \leq \mu_n,\ n \in \mathbb{N} \\ 2) & \sup\{\lambda_n \ : \ n \in \mathbb{N}\} = 0 = \inf\{\mu_n \ : \ n \in \mathbb{N}\} \\ 3) & [\{[\lambda_n, \mu_n] \ : \ n \in \mathbb{N}\}] \subseteq \mathcal{F} - \mathcal{F}. \end{array} \tag{3.2}$$

The statement (3.2) implies that for every $n \in \mathbb{N}$ there exists $F_n \in \mathcal{F}$ so that $F_n - F_n \subseteq [\lambda_n, \mu_n]$. In particular, the sets F_n may be selected in such a way that

$$F_{n+1} \subseteq F_n, \ n \in \mathbb{N}. \tag{3.3}$$

Since $F_n - F_n \subseteq [\lambda_n, \mu_n]$ for every $n \in \mathbb{N}$, it follows that each of the sets F_n is order bounded so that the sets $U_n = \left\{u \in \mathcal{ML}^0(\Omega) \ : \ v \leq u, \ v \in F_n\right\}$ and $L_n = \left\{u \in \mathcal{ML}^0(\Omega) \ : \ u \leq v, \ v \in F_n\right\}$ are nonempty. Since $\mathcal{NL}(\Omega) \supset \mathcal{ML}^0(\Omega)$ is Dedekind complete, it follows that the sequences (λ'_n) and (μ'_n) defined as

$$\lambda'_n = \inf F_n, \quad \mu'_n = \sup F_n, \quad n \in \mathbb{N} \tag{3.4}$$

are well-defined in $\mathcal{NL}(\Omega)$. Furthermore, the inclusions in (3.3) imply that

$$\lambda'_n \leq \lambda'_{n+1} \leq \mu'_{n+1} \leq \mu'_n, \quad n \in \mathbb{N}. \tag{3.5}$$

Since $\mathcal{NL}(\Omega)$ is Dedekind complete, $w_0 = \sup_{n\in\mathbb{N}} \lambda'_n$ and $w_1 = \inf_{n\in\mathbb{N}} \mu'_n$ exist in $\mathcal{NL}(\Omega)$. Clearly, $w_0 \leq w_1$. We claim that $w_0 = w_1$. As $F_n - F_n \subseteq [\lambda_n, \mu_n]$ for every $n \in \mathbb{N}$, it follows from (3.4) and [21, Theorem 13.1] that

$$\lambda_n \leq \mu'_n - \lambda'_n \leq \mu_n, \quad n \in \mathbb{N}. \tag{3.6}$$

However, according to (3.2), we have $\sup_{n\in\mathbb{N}} \lambda_n = \inf_{n\in\mathbb{N}} \mu_n = 0$ in $\mathcal{ML}^0(\Omega)$, hence also in $\mathcal{NL}(\Omega)$. Therefore (3.6) and [21, Theorems 13.1 and 15.2] imply that

$$w_1 - w_0 = \inf_{n\in\mathbb{N}} \mu'_n - \sup_{n\in\mathbb{N}} \lambda'_n = \inf_{n\in\mathbb{N}} (\mu'_n - \lambda'_n) = 0$$

which verifies our claim. For each $n \in \mathbb{N}$, let D_n be the open and dense set where λ'_n is finite. Then λ'_n is normal lower semi-continuous on D_n, since D_n is open. According to [4, Proof of Theorem 26], see also Remark 3.3, there exists an increasing sequence $(\lambda_{n,m}) \subset \mathcal{C}^0(D_n)$ with the property that

$$\sup_{m\in\mathbb{N}} \lambda_{n,m}(x) = \lambda'_n(x), \quad x \in D_n. \tag{3.7}$$

For each $m, n \in \mathbb{N}$ set $\lambda'_{n,m} = I(S(\lambda_{n,m}*))$ where

$$\lambda_{n,m}* = \begin{cases} \lambda_{n,m}(x) & \text{if} \quad x \in D_n \\ \\ 0 & \text{if} \quad x \notin D_n \end{cases}$$

Since D_n is open, it follows from (2.2), (2.3) and the continuity of $\lambda_{n,m}$ on D_n that $\lambda'_{m,n}(x) = \lambda_{m,n}(x), \ x \in D_n$. Since $\lambda_{m,n} \in \mathcal{C}^0(D_n)$ it follows that $\lambda'_{m,n} \in \mathcal{ML}^0(\Omega)$. Furthermore, since D_n is dense in Ω, it follows from (2.1), (3.7) and [50, Eq. (3) & Lemma 6] that $\lambda'_n = \sup_{m\in\mathbb{N}} \lambda_{n,m}$. Therefore, the sequence $(\lambda'_{n,m})$ increases to λ'_n for each $n \in \mathbb{N}$. Since (λ'_n) increases to w_0, it now follows from [4, Lemma 36] that the sequence (λ_n^*), where $\lambda_n^* = \sup\{\lambda'_{1,n}, ..., \lambda'_{n,n}\} \in \mathcal{ML}^0(\Omega)$ for each $n \in \mathbb{N}$, increases to w_0 and satisfies

$$\lambda_n^* \leq \lambda'_n, \quad n \in \mathbb{N}. \tag{3.8}$$

In the same way we can construct a sequence (μ_n^*) in $\mathcal{ML}^0(\Omega)$ that decreases to w_0 and satisfies

$$\mu'_n \leq \mu_n^*, \quad n \in \mathbb{N}. \tag{3.9}$$

For each $n \in \mathbb{N}$, let $I_n = [\lambda_n^*, \mu_n^*]$. We verify that (I_n) satisfies the conditions in (2.4) and (2.5). Since (λ_n^*) is an increasing sequence and (μ_n^*) a decreasing sequence, the collection $\Sigma_1 = (I_n)$ satisfies condition 1) in (2.4). Condition 2) in (2.4) follows from (3.4) and the inequalities in (3.8) and (3.9), while (2.5) follows from the fact that (λ_n^*) increases to w_0 and (μ_n^*) decreases to w_0. Since $I_n \supseteq [\lambda_n, \mu_n] \supseteq F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, it follows that $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_o$ so that $\mathcal{F}$ is a Cauchy filter on $\mathcal{ML}^0(\Omega)$.
Conversely, assume that $\mathcal{F}$ is a Cauchy filter on $\mathcal{ML}^0(\Omega)$, and let $\Sigma_j = (I_n^j)$, with $j = 1, ..., k$, be the sequences of order intervals in $\mathcal{ML}^0(\Omega)$ associated with $\mathcal{F}$ through Definition 2.1. For each $j = 1, ..., k$ and $n \in \mathbb{N}$, set $\lambda_n^j = \inf I_n^j$ and $\mu_n^j = \sup I_n^j$. We claim that $(\mu_n^j - \lambda_n^j)$ decreases to 0 in $\mathcal{ML}^0(\Omega)$. Since (λ_n^j) is increasing and (μ_n^j) is decreasing, it follows that $(\mu_n^j - \lambda_n^j)$ is decreasing. Furthermore, since $\lambda_n^j \leq \mu_n^j$ for each $n \in \mathbb{N}$, it follows that $\mu_n^j - \lambda_n^j \geq 0$, $n \in \mathbb{N}$. Now suppose that there is some $u \in \mathcal{ML}^0(\Omega)$ such that $0 < u \leq \mu_n^j - \lambda_n^j$, $n \in \mathbb{N}$. Since $\mathcal{NL}(\Omega)$ is Dedekind complete, there exists $v \in \mathcal{NL}(\Omega)$ so that $v = \sup\{\lambda_n^j \ : \ n \in \mathbb{N}\}$. Hence $\lambda_n^j \leq v < u + v \leq \mu_n^j$, $n \in \mathbb{N}$. It follows from the normal lower semi-continuity of u and $u+v$ that there exists an open set $V \subseteq \Omega$ and $c, d \in \mathbb{R}$ so that $\lambda_n^j(x) \leq v(x) < c < d < [u + v](x) \leq \mu_n^j(x)$ for every $n \in \mathbb{N}$ and $x \in V$. This contradicts (2.5). Hence our claim is verified. Similarly, $(\lambda_n^j - \mu_n^j)$ increases to 0. Furthermore, $[\lambda_n^1 - \mu_n^1, \mu_n^1 - \lambda_n^1] \cup ... \cup [\lambda_n^k - \mu_n^1, \mu_n^1 - \lambda_n^k] \in \mathcal{F} - \mathcal{F}$ and $[\lambda_n^1 - \mu_n^1, \mu_n^1 - \lambda_n^1] \cup ... \cup [\lambda_n^k - \mu_n^1, \mu_n^1 - \lambda_n^k] \in [0]$ for all $n \in \mathbb{N}$, so that $(\mathcal{F} - \mathcal{F}) \times [0] \in \mathcal{J}_o$. Hence $\mathcal{F} - \mathcal{F}$ converges to 0 in $\mathcal{ML}^0(\Omega)$. ◻

Remark 3.3. As mentioned, $\mathcal{ML}^0(\Omega)$ is an Archimedean *f*-algebra. Therefore it follows that the order convergence structure λ_o is an algebra convergence structure on $\mathcal{ML}^0(\Omega)$. The convergence vector space structure of $\mathcal{ML}^0(\Omega)$ induces a uniform convergence structure on $\mathcal{ML}^0(\Omega)$. In particular, a filter $\mathcal{F}$ on $\mathcal{ML}^0(\Omega)$ is a Cauchy filter with respect to this uniform convergence structure if and only if $\mathcal{F} - \mathcal{F}$ converges to 0. Thus Proposition 3.2 states that the Cauchy filters with respect to $\mathcal{J}_o$ are precisely those induced by the convergence vector space structure of $\mathcal{ML}^0(\Omega)$.

Corollary 3.4. *A sequence (u_n) in $\mathcal{ML}^0(\Omega)$ is a Cauchy sequence with respect to the uniform order convergence structure if and only if there exists a sequence $(\mu_n) \subseteq \mathcal{ML}^0(\Omega)$ decreasing to 0 such that $-\mu_n \leq u_n - u_m \leq \mu_n$ whenever $m \geq n$.*

Proof. According to Remark 3.3, the Cauchy sequences with respect to $\mathcal{J}_o$ are precisely those associated with the uniform structure induced on $\mathcal{ML}^0(\Omega)$

through its convergence vector space structure. Since $\mathcal{ML}^0(\Omega)$ is an Archimedean Riesz space, it follows by [50, Proposition 20] that a sequence (u_n) in $\mathcal{ML}^0(\Omega)$ is a Cauchy sequence with respect to $\mathcal{J}_o$ if and only if (u_n) is an order Cauchy sequence [53, Exercise 101.8], which completes the proof. □

Proposition 3.5. *A sequence (u_n) in $\mathcal{ML}^0(\Omega)$ is a Cauchy sequence with respect to the uniform order convergence structure if and only if there exists a residual set $R \subseteq \Omega$ such that $(u_n(x))$ is a convergent sequence in $\mathbb{R}$ for each $x \in R$.*

Proof. Assume that (u_n) is a Cauchy sequence in $\mathcal{ML}^0(\Omega)$, and let (μ_n) be the sequence associated with (u_n) through Corollary 3.4. Since (μ_n) decreases to 0, it order converges to 0 in $\mathcal{ML}^0(\Omega)$, hence also in $\mathcal{NL}(\Omega)$. Since Ω is a Baire space, it follows [49, Theorem 2.4] that there exists a residual set $R \subseteq \Omega$ such that $(\mu_n(x))$ decreases to 0 for all $x \in R$. Thus $(u_n(x))$ is a convergent sequence in $\mathbb{R}$ for all $x \in R$.
Conversely, assume that there is a residual set $R \subseteq \Omega$ such that $(u_n(x))$ converges to some $\alpha_x \in \mathbb{R}$ for all $x \in R$. Since (u_n) is pointwise bounded on R, it follows [49, Lemma 2.1] that there exist $u_0, v_0 \in \mathcal{NL}(\Omega)$ so that $u_0 \leq u_n \leq v_0$ for every $n \in \mathbb{N}$. Hence, since $\mathcal{NL}(\Omega)$ is Dedekind complete, the sequences (λ'_n) and (μ'_n) defined as $\lambda'_n = \inf_{k \geq n} u_k$ and $\mu'_n = \sup_{k \geq n} u_k$ are well defined in $\mathcal{NL}(\Omega)$. Furthermore (λ'_n) is clearly increasing, while (μ'_n) is decreasing and $\lambda'_n \leq u_n \leq \mu'_n$ for all $n \in \mathbb{N}$. We claim that $u = \sup_{n \in \mathbb{N}} \lambda'_n = \inf_{n \in \mathbb{N}} \mu'_n = v$. Suppose that this were not the case, so that $u \neq v$. Since $u \leq v$, it follows from [50, Lemma 6] that there is a nonempty, open set $A \subseteq \Omega$ such that $u(x) < v(x)$ for all $x \in A$. In fact, without loss of generality, we may assume that both u and v are finite on A, and that

$$u(x) < \alpha - \epsilon < \alpha + \epsilon < v(x),\ x \in A \tag{3.10}$$

for some $\alpha \in \mathbb{R}$ and $\epsilon > 0$. For each $n \in \mathbb{N}$, consider the functions φ_n and ψ_n defined as

$$\varphi_n : A \ni x \mapsto \sup_{k \geq n} u_k(x) \in \mathbb{R}, \quad \psi_n : A \ni x \mapsto \inf_{k \geq n} u_k(x) \in \mathbb{R}. \tag{3.11}$$

Since A is open, it follows from [50, Eq. (3)] that $\lambda'_n(x) = I[S[\psi_n]](x)$ and $\mu'_n(x) = I[S[\varphi_n]](x)$ for all $x \in A$ and $n \in \mathbb{N}$. Note that $\mu'_n = I[S[\varphi_n]] \leq S[\varphi_n]$. Hence, since $\lambda'_n \leq u < v \leq \mu'_n$ for all $n \in \mathbb{N}$, it follows from (2.3) and (3.11) that for every $n \in \mathbb{N}$, $x \in A$ and $W \in \mathcal{V}_x$ there exists $x_n \in W$ so that $\varphi_n(x_n) \geq \alpha + \epsilon$. Since φ_n is lower semi-continuous, being the supremum of lower semi-continuous functions, it follows that for each $n \in \mathbb{N}$ there exists $D_n \subseteq A$ open and dense in A so that $\varphi_n(x) \geq \alpha + \epsilon,\ x \in D_n$. It follows from (3.11) that

$$u_n(x) \geq \alpha + \epsilon,\ x \in R' = \bigcap_{n \in \mathbb{N}} D_n,\ n \in \mathbb{N}. \tag{3.12}$$

It follows from (2.2) that for all $x \in A$, $n \in \mathbb{N}$ and $W \in \mathcal{V}_x$ there exists $x_n \in W$ so that $S[\psi_n](x_n) < \alpha - \epsilon$. The upper semi-continuity of $S[\psi_n]$ implies that there exists for each $n \in \mathbb{N}$ a set $E_n \subseteq A$ which is open and dense in A so that $S[\psi_n](x) < \alpha - \epsilon$ for all $x \in E_n$. But $\psi_n \leq S[\psi_n]$ for every $n \in \mathbb{N}$ so that $\psi_n(x) < \alpha - \epsilon$ for every $x \in E_n$ and $n \in \mathbb{N}$. Letting $R'' = \bigcap_{n\in\mathbb{N}} E_n$ we have by (3.11) that

$$(u_n(x)) \text{ is not convergent in } \mathbb{R} \text{ for every } x \in R''. \tag{3.13}$$

Since A is an open subset of $\mathbb{R}^n$, it is a Baire space so that $R' = \bigcap_{n\in\mathbb{N}} D_n$ and R'' are residual sets in A. As $(u_n(x))$ is convergent in $\mathbb{R}$ for every $x \in R$, it follows from (3.12) and (3.13) that $R \cap A$ is of first Baire category in A, a contradiction. Hence the assumption that $u \neq v$ is false, which verifies our claim.
Applying [4, Lemma 36], it follows that there exist sequences (λ_n'') and (μ_n'') in $\mathcal{ML}^0(\Omega)$ that respectively increase and decrease to u, such that $\lambda_n'' \leq u_n \leq \mu_n''$ for all $n \in \mathbb{N}$. The sequence (μ_n), defined by $\mu_n = \mu_n'' - \lambda_n''$ decreases to 0 and satisfies $-\mu_n \leq u_n - u_m \leq \mu_n$ for all $n \in \mathbb{N}$ and $m \geq n$. The result now follows by Corollary 3.4. $\square$

The preceding results on Cauchy filters and Cauchy sequences in $\mathcal{M}^0(\Omega)$ now determine the nature of Cauchy filters on $\mathcal{ML}^\infty(\Omega)$ as follows.

Corollary 3.6. *A filter $\mathcal{F}$ on $\mathcal{ML}^\infty(\Omega)$ is a Cauchy filter with respect to $\mathcal{J}_o^\infty$ if and only if $\mathcal{F} - \mathcal{F} \in \lambda_o^\infty(0)$. That is, the convergence vector space structure of $\mathcal{ML}^\infty(\Omega)$ induces the same Cauchy filters as the uniform convergence structure $\mathcal{J}_o^\infty$.*

Corollary 3.7. *A sequence (u_n) in $\mathcal{ML}^\infty(\Omega)$ is a Cauchy sequence if and only if there exists a residual set $R \subseteq \Omega$ such that $(\mathcal{D}^p u_n(x))$ is a convergent sequence in $\mathbb{R}$ for all $x \in R$ and $p \in \mathbb{N}^n$.*

In view of Corollary 3.6, we may further particularize the construction of $\mathcal{NL}^\infty(\Omega)$. Indeed, since λ_0^∞ is a first countable vector space convergence structure, see Proposition 3.1, we may express $\mathcal{NL}^\infty(\Omega)$ as

$$\mathcal{NL}^\infty(\Omega) = C_s[\mathcal{ML}^\infty(\Omega)]/\sim_s = \{[(u_n)]_s \ : \ (u_n) \in C_s[\mathcal{ML}^\infty(\Omega)]\}, \tag{3.14}$$

where $C_s[\mathcal{ML}^\infty(\Omega)]$ is the set of Cauchy sequences on $\mathcal{ML}^\infty(\Omega)$, $\sim_s$ is the equivalence relation

$$(x_n) \sim_s (y_n) \Leftrightarrow \langle x_n \rangle \sim_C \langle y_n \rangle \tag{3.15}$$

and for $(u_n) \in C_s[\mathcal{ML}^\infty(\Omega)]$, $[(u_n)]_s$ denotes the $\sim_s$-equivalence class generated by (u_n). In view of [7, Proposition 2.5.4] it follows that

$$(u_n) \sim_s (v_n) \Leftrightarrow (u_n - v_n) \in \lambda_0^\infty(0)$$

Thus [49, Theorem 2.4], Proposition 3.1 and (2.8) imply that

$$(u_n) \sim_s (v_n) \Leftrightarrow \left(\begin{array}{ll} \exists & R \subseteq \Omega \text{ a residual set } : \\ \forall & p \in \mathbb{N}^n, \ x \in R \ : \\ & \mathcal{D}^p u_n(x) - \mathcal{D}^p v_n(x) \to 0 \text{ in } \mathbb{R} \end{array} \right).$$

With the preliminary results now settled, we proceed with the mooted description of $\mathcal{NL}^\infty(\Omega)$ as a differential algebra. In order to place the construction of $\mathcal{NL}^\infty(\Omega)$ in the context of differential algebras of generalized functions, we must obtain sequences of smooth functions from the Cauchy sequences in $C_s[\mathcal{ML}^\infty(\Omega)]$. The key mechanism used to achieve this is the well known principle of *Partition of Unity*. In this regard, we may recall the following version of this principle, see for instance [41, Theorem 15].

Theorem 3.8. *Let $\mathcal{O}$ be a locally finite open cover of a smooth manifold M. Then there is a collection*

$$\{\varphi_U : M \to [0,1] \, : \, U \in \mathcal{O}\}$$

of $\mathcal{C}^\infty$-smooth mappings φ_U such that the following hold.

(i) For each $U \in \mathcal{O}$, the support of φ_U is contained in U.

(ii) For each $x \in M$, we have $\sum_{U\in\mathcal{O}} \varphi_U(x) = 1$.

A useful consequence of Theorem 3.8 concerns the separation of disjoint, closed sets by $\mathcal{C}^\infty$-smooth, real valued mappings. In this regard, consider a nonempty, open set $\Omega \subseteq \mathbb{R}^n$. Let A and B be disjoint, nonempty, closed subsets of Ω. Then it follows from Theorem 3.8 that there exists $\varphi \in \mathcal{C}^\infty(\Omega, [0,1])$ so that

$$A \subseteq \varphi^{-1}(1), \quad B \subseteq \varphi^{-1}(0). \tag{3.16}$$

This leads to the following.

Lemma 3.9. *Let (u_n) be a Cauchy sequence in $\mathcal{ML}^\infty(\Omega)$ with respect to $\mathcal{J}_o^\infty$. Then $\mathcal{C}^\infty(\Omega)^{\mathbb{N}} \cap [(u_n)]_s \neq \emptyset$.*

Proof. We claim that for each $n \in \mathbb{N}$ there is a sequence $(u_{n,m}) \subseteq \mathcal{C}^\infty(\Omega)$ that converges to u_n in $\mathcal{ML}^\infty(\Omega)$. Denote by $\Gamma_n \subset \Omega$ the smallest closed nowhere dense set such that $u \in \mathcal{C}^\infty(\Omega \setminus \Gamma_n)$. For each $m \in \mathbb{N}$, we consider the set $\overline{B}_{\frac{1}{m}}(\Gamma_n)$, which is defined as the closure of the set

$$\left\{ x \in \Omega \,\middle|\, \begin{array}{ll} \exists & x_0 \in \Gamma_n : \\ & \|x - x_0\| \leq \frac{1}{2m} \end{array} \right\}$$

and the set

$$\overline{C}_{\frac{1}{m}}(\Gamma_n) = \left\{ x \in \Omega \,\middle|\, \begin{array}{ll} \forall & x_0 \in \Gamma_n : \\ & \|x - x_0\| \geq \frac{1}{m} \end{array} \right\}$$

Each of the sets $\overline{B}_{\frac{1}{m}}(\Gamma_n)$ and $\overline{C}_{\frac{1}{m}}(\Gamma_n)$ is closed, and for each $m \in \mathbb{N}$, $\overline{B}_{\frac{1}{m}}(\Gamma_n)$ and $\overline{C}_{\frac{1}{m}}(\Gamma_n)$ are disjoint. As such, by (3.16), there exists a function $\varphi_m \in \mathcal{C}^\infty(\Omega, [0,1])$ so that

$$\varphi_m(x) = \left\{ \begin{array}{lll} 0 & if & x \in \overline{B}_{\frac{1}{m}}(\Gamma_n) \\ \\ 1 & if & x \in \overline{C}_{\frac{1}{m}}(\Gamma_n) \end{array} \right.$$

Each of the functions $u_{n,m} = \varphi_m u_n$ is $\mathcal{C}^\infty$-smooth and satisfies

$$u_{n,m}(x) = \begin{cases} 0 & if \quad x \in \overline{B}_{\frac{1}{m}}(\Gamma_n) \\ \\ u_n(x) & if \quad x \in \overline{C}_{\frac{1}{m}}(\Gamma_n) \end{cases}$$

Furthermore,

$$\bigcap_{m\in\mathbb{N}} \overline{B}_{\frac{1}{m}}(\Gamma_n) = \Gamma_n, \quad \bigcup_{m\in\mathbb{N}} \overline{C}_{\frac{1}{m}}(\Gamma_n) = \Omega \setminus \Gamma_n$$

which verifies our claim.
Let $R \subseteq \Omega$ be the residual set associated with (u_n) through Corollary 3.7. Set

$$R_0 = R \cap \left(\bigcup_{n\in\mathbb{N}} (\Omega \setminus \Gamma_n) \right)$$

For each $p \in \mathbb{N}^n$ there exists a function $\alpha_p : R_0 \to \mathbb{R}$ so that $(\mathcal{D}^p u_n(x))$ converges to $\alpha_p(x)$ for all $x \in R_0$. Since $(\mathcal{D}^p u_{n,m}(x))$ converges to $\mathcal{D}^p u_n(x)$ for $x \in R_0$ and $p \in \mathbb{N}^n$, it follows that we may select a strictly increasing sequence (m_n) of integers such that $(\mathcal{D}^p u_{n,m_n}(x))$ converges to $\alpha_p(x)$ for all $x \in R_0$ and $p \in \mathbb{N}^n$. According to Corollary 3.7 the sequence (u_{n,m_n}) is a Cauchy sequence in $\mathcal{ML}^\infty(\Omega)$.
It remains to verify that $(u_{n,m_n}) \sim_C (u_n)$. In this regard, it is sufficient to note that the sequence $(u_n(x) - u_{n,m_n}(x))$ converges to 0 in $\mathbb{R}$ for every $x \in R_0$. $\square$

Theorem 3.10. *Let $\mathcal{A}_o^\infty = C_s[\mathcal{ML}^\infty(\Omega)] \cap \mathcal{C}^\infty(\Omega)^{\mathbb{N}}$ and $\mathcal{I}_o^\infty = \lambda_o^\infty(0) \cap \mathcal{C}^\infty(\Omega)^{\mathbb{N}}$. Then the following statements are true:*

(i) $\mathcal{A}_o^\infty$ is a subalgebra of $\mathcal{C}^\infty(\Omega)^{\mathbb{N}}$ and $\mathcal{I}_o^\infty$ is an ideal in $\mathcal{A}_o^\infty$.

(ii) $\Delta(\mathcal{C}^\infty(\Omega)) \subseteq \mathcal{A}_o^\infty$ and $\Delta(\mathcal{C}^\infty(\Omega)) \cap \mathcal{I}_o^\infty = \{0\}$.

(iii) For each $p \in \mathbb{N}^n$ we have $D^p(\mathcal{A}_o^\infty) \subseteq \mathcal{A}_o^\infty$ and $D^p(\mathcal{I}_o^\infty) \subseteq \mathcal{I}_o^\infty$, hence

$$D^p : \mathcal{A}_o^\infty / \mathcal{I}_o^\infty \ni (u_n) + \mathcal{I}_o^\infty \mapsto (D^p u_n) + \mathcal{I}_o^\infty \in \mathcal{A}_o^\infty / \mathcal{I}_o^\infty, \quad p \in \mathbb{N}^n$$

are well defined, linear and satisfy the Leibnitz rule for derivatives of products.

(iv) There exists a bijective mapping $I_o^\infty : \mathcal{NL}^\infty(\Omega) \to \mathcal{A}_o^\infty / \mathcal{I}_o^\infty$ such that the diagram

$$\begin{tikzcd} \mathcal{NL}^\infty(\Omega) \arrow[r, "\mathcal{D}^{p\sharp}"] \arrow[d, "I_o^\infty"'] & \mathcal{NL}^\infty(\Omega) \arrow[d, "I_o^\infty"] \\ \mathcal{A}_o^\infty / \mathcal{I}_o^\infty \arrow[r, "D^p"] & \mathcal{A}_o^\infty / \mathcal{I}_o^\infty \end{tikzcd}$$

commutes for every $p \in \mathbb{N}^n$.

Proof. The result in (i) follows directly from Lemma 3.9, while (ii) follows from the fact that λ_o^∞ is a Hausdorff convergence structure on $\mathcal{ML}^\infty(\Omega)$. Indeed, for each $u \in \mathcal{C}^\infty(\Omega)$, the sequence $\Delta(u)$ with all terms equal to u is convergent to u, and is therefore a Cauchy sequence in $\mathcal{ML}^\infty(\Omega)$. Since λ_o^∞ is Hausdorff, it follows from $\Delta(u) \in \mathcal{I}_o^\infty \subseteq \lambda_o^\infty(0)$ that $u = 0$. The inclusions in (iii) follow from the fact that the partial differential operators are linear and uniformly continuous on $\mathcal{ML}^\infty(\Omega)$, see (2.10). It follows immediately that the mappings D^p in (iii) are well defined. Since $D^p : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ is linear and satisfies the Leibnitz rule for each $p \in \mathbb{N}^n$, the mappings D^p in (iii) satisfy these properties as well, see for instance [34, Chapter 1, §9].
We now prove (iv). Consider the mappings $L : \mathcal{A}_o^\infty \ni u = (u_n) \mapsto u^\sharp \in \mathcal{NL}^\infty(\Omega)$ and $R : \mathcal{A}_o^\infty \ni u = (u_n) \mapsto u + \mathcal{I}_o^\infty \in \mathcal{A}_o^\infty/\mathcal{I}_o^\infty$, where $u^\sharp$ is the limit of (u_n) in $\mathcal{NL}^\infty(\Omega)$, that is, $L(u)$ is the equivalence class generated by $u = (u_n)$. According to Lemma 3.9 the mapping L is a surjection, while R is a surjection by definition of $\mathcal{A}_o^\infty/\mathcal{I}_o^\infty$. Indeed, R is the canonical quotient mapping associated with the ideal $\mathcal{I}_o^\infty$ in the algebra $\mathcal{A}_o^\infty$. Therefore, for $u, v \in \mathcal{A}_o^\infty$, we have $R(u) = R(v)$ if and only if $u - v \in \mathcal{I}_o^\infty$. Now consider the map

$$I_o^\infty : \mathcal{NL}^\infty(\Omega) \ni u^\sharp \mapsto R(L^{-1}(u^\sharp)) \in \mathcal{A}_o^\infty/\mathcal{I}_o^\infty. \tag{3.17}$$

Since L is a surjection, the mapping (3.17) is defined at each $u^\sharp \in \mathcal{NL}^\infty(\Omega)$. Furthermore, according to (3.14), $L(u) = L(v)$ if and only if $u - v \in \mathcal{I}_o^\infty$. Hence $L(u) = L(v)$ if and only if $R(u) = R(v)$, so that $I_o^\infty(u^\sharp)$ is well defined in $\mathcal{A}_o^\infty/\mathcal{I}_o^\infty$ for each $u^\sharp \in \mathcal{NL}^\infty(\Omega)$. Furthermore, since R is a surjection, so is I_o^∞. If $I_o^\infty(u^\sharp) = I_o^\infty(v^\sharp)$ for some $u^\sharp, v^\sharp \in \mathcal{NL}^\infty(\Omega)$, then $u - v \in \mathcal{I}_o^\infty$ for some $u \in L^{-1}(u^\sharp)$ and $v \in L^{-1}(v^\sharp)$, so that $u \sim_s v$. This implies that $u^\sharp = v^\sharp$, since $\mathcal{J}_o^\infty$ is Hausdorff. Hence I_o^∞ is injective. Now observe that the diagram

$$\begin{tikzcd}
\mathcal{A}_o^\infty \arrow[r, "D^p"] \arrow[d, "L"'] & \mathcal{A}_o^\infty \arrow[d, "L"] \\
\mathcal{NL}^\infty(\Omega) \arrow[r, "\mathcal{D}^{p\sharp}"] & \mathcal{NL}^\infty(\Omega)
\end{tikzcd}$$

commutes for every $p \in \mathbb{N}^n$, since the partial differential operators

$\mathcal{D}^p : \mathcal{M}^\infty(\Omega) \to \mathcal{M}^\infty(\Omega)$ are uniformly continuous. Furthermore, the diagram

$$\begin{tikzcd}
\mathcal{A}_o^\infty \arrow[r, "D^p"] \arrow[d, "R"'] & \mathcal{A}_o^\infty \arrow[d, "R"] \\
\mathcal{A}_o^\infty/\mathcal{I}_o^\infty \arrow[r, "D^p"] & \mathcal{A}_o^\infty/\mathcal{I}_o^\infty
\end{tikzcd}$$

also commutes for each $p \in \mathbb{N}^n$. Hence the result follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{NL}^\infty(\Omega) \arrow[rr, "I_o^\infty"] & & \mathcal{A}_o^\infty/\mathcal{I}_o^\infty \\
& \mathcal{A}_o^\infty \arrow[ul, "L"] \arrow[ur, "R"'] &
\end{tikzcd}$$

$\square$

The meaning of Theorem 3.10 is that the space $\mathcal{NL}^\infty(\Omega)$ of generalized functions may be identified with the differential algebra $\mathcal{A}_o^\infty/\mathcal{I}_o^\infty$. Therefore we have a representation of $\mathcal{NL}^\infty(\Omega)$ as a differential algebra of generalized functions. This representation allows us to transfer results obtained through the Order Completion Method to the algebraic theory of generalized functions, and vice versa. In view of the identification of $\mathcal{NL}^\infty(\Omega)$ with the differential algebra $\mathcal{A}_o^\infty/\mathcal{I}_o^\infty$, we will henceforth denote the latter by $\mathcal{NL}^\infty(\Omega)$.

4. $\mathcal{NL}^\infty(\Omega)$ and $\mathcal{A}_{nd}^\infty(\Omega)$

In this section we determine the extent to which the differential algebra $\mathcal{NL}^\infty(\Omega)$ is related to Rosinger's nowhere dense algebra, introduced in [32], see also [33, 34]. We recall briefly the construction of the algebra $\mathcal{A}_{nd}^\infty(\Omega)$.

Let $\mathcal{I}_{nd,\mathbb{N}}^\infty(\Omega)$ denote the subset of $\mathcal{C}^\infty(\Omega)^{\mathbb{N}}$ which is defined through the asymptotic vanishing condition

$$(4.1)\ u = (u_n) \in \mathcal{I}_{nd,\mathbb{N}}^\infty(\Omega) \Leftrightarrow \left\{ \begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense} \quad : \\ \forall & x \in \Omega \setminus \Gamma\ , p \in \mathbb{N}^n \quad : \\ \exists & n_{x,p} \in \mathbb{N} \quad : \\ \forall & n \geq n_{x,p},\ q \in \mathbb{N}^n,\ q \leq p \quad : \\ & D^q u_n(x) = 0 \end{array} \right.$$

The set $\mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega)$ is an ideal in $\mathcal{C}^{\infty}(\Omega)^{\mathbb{N}}$. To see that this is so, fix $u \in \mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega)$ and $v \in \mathcal{C}^{\infty}(\Omega)^{\mathbb{N}}$. The termwise partial derivatives of the product $w = uv = (u_n v_n)_{n\in\mathbb{N}}$ may be expressed as

$$D^p(uv) = \left(\sum_{q\leq p}\begin{pmatrix} p \\ q \end{pmatrix} D^{p-q}u_n D^q v_n\right), \quad p \in \mathbb{N}^n. \tag{4.2}$$

Let $\Gamma \subset \Omega$ be the closed nowhere dense set associated with u through (4.1). For any $x \in \Omega \setminus \Gamma$ and $p \in \mathbb{N}^n$ let $n_{x,p}$ be the natural number associated with u through (4.1) so that $D^q u_n(x) = 0$ for all $q \in \mathbb{N}^n$ and $n \in \mathbb{N}$ so that $q \leq p$ and $n \geq n_{x,p}$. It now follows from (4.2) that $D^q w_n(x) = 0$ for all $q \in \mathbb{N}^n$ and $n \in \mathbb{N}$ so that $q \leq p$ and $n \geq n_{x,p}$. Hence $uv \in \mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega)^{\mathbb{N}}$.

Furthermore, it is easily seen that $\mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega)$ satisfies the inclusions

$$D^p\left(\mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega)\right) \subseteq \mathcal{I}^{\infty}_{nd,\mathbb{N}}, \quad p \in \mathbb{N}^n \tag{4.3}$$

so that

$$\mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega) = \mathcal{C}^{\infty}(\Omega)^{\mathbb{N}} / \mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega) \tag{4.4}$$

is a differential algebra of generalized functions.

In order to establish the relationship that exists between the differential algebras $\mathcal{NL}^{\infty}(\Omega)$ and $\mathcal{A}^{\infty}_{nd}(\Omega)$, we introduce an auxiliary algebra $\mathcal{A}^{\infty}_{nd,o}(\Omega)$. In this regard, we note that $\mathcal{I}^{\infty}_{nd,\mathbb{N}}$ is an ideal in $\mathcal{A}^{\infty}_{o}$. Furthermore,

$$\Delta(\mathcal{C}^{\infty}(\Omega)) \cap \mathcal{I}^{\infty}_{nd,\mathbb{N}} = \{0\}$$

and, due to Theorem 3.10 (ii) and (4.3), the inclusions

$$D^p(\mathcal{A}^{\infty}_{o}) \subseteq \mathcal{A}^{\infty}_{o}, \quad D^p\left(\mathcal{I}^{\infty}_{nd,\mathbb{N}}(\Omega)\right) \subseteq \mathcal{I}^{\infty}_{nd,\mathbb{N}}, \quad p \in \mathbb{N}^n$$

hold. Therefore

$$\mathcal{A}^{\infty}_{nd,o}(\Omega) = \mathcal{A}^{\infty}_{o} / \mathcal{I}^{\infty}_{nd,\mathbb{N}}$$

is a differential algebra of generalized functions, with partial differential operations defined as

$$D^p : \mathcal{A}^{\infty}_{o}/\mathcal{I}^{\infty}_{nd,\mathbb{N}} \ni u + \mathcal{I}^{\infty}_{nd,\mathbb{N}} \mapsto D^p u + \mathcal{I}^{\infty}_{nd,\mathbb{N}} \in \mathcal{A}^{\infty}_{o}/\mathcal{I}^{\infty}_{nd,\mathbb{N}}, \quad p \in \mathbb{N}^n.$$

Theorem 4.1. *There exists an injective algebra homomorphism*

$$\Gamma_{nd,0} : \mathcal{A}^{\infty}_{nd,o}(\Omega) \to \mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega)$$

and a surjective algebra homomorphism

$$\Gamma_{nd} : \mathcal{A}^{\infty}_{nd,o}(\Omega) \to \mathcal{NL}^{\infty}(\Omega)$$

so that the diagram

$$
\begin{tikzcd}
\mathcal{A}^\infty_{nd,\mathbb{N}}(\Omega) \arrow[r, "D^p"] & \mathcal{A}^\infty_{nd,\mathbb{N}}(\Omega) \\
\mathcal{A}^\infty_{nd,o}(\Omega) \arrow[u, "\Gamma_{nd,0}"] \arrow[r, "D^p"] \arrow[d, "\Gamma_{nd}"'] & \mathcal{A}^\infty_{nd,o}(\Omega) \arrow[u, "\Gamma_{nd,0}"'] \arrow[d, "\Gamma_{nd}"] \\
\mathcal{NL}^\infty(\Omega) \arrow[r, "D^p"] & \mathcal{NL}^\infty(\Omega)
\end{tikzcd}
\tag{4.5}
$$

commutes for every $p \in \mathbb{N}^n$*.*

Proof. Consider the mappings

$$\Gamma_{nd,0} : \mathcal{A}^\infty_{nd,o}(\Omega) \ni u + \mathcal{I}^\infty_{nd,\mathbb{N}} \mapsto u + \mathcal{I}^\infty_{nd,\mathbb{N}} \in \mathcal{A}^\infty_{nd,\mathbb{N}}(\Omega)$$

and

$$\Gamma_{nd} : \mathcal{A}^\infty_{nd,o}(\Omega) \ni u + \mathcal{I}^\infty_{nd,\mathbb{N}} \mapsto u + \mathcal{I}^\infty_{o} \in \mathcal{NL}^\infty(\Omega).$$

The mapping $\Gamma_{nd,0}$ is a well defined and injective algebra homomorphism, since $\mathcal{A}^\infty_o \subset \mathcal{C}^\infty(\Omega)^{\mathbb{N}}$. Furthermore, $\mathcal{I}^\infty_{nd,\mathbb{N}} \subseteq \mathcal{I}^\infty_o$ so that Γ_{nd} is a well defined and surjective algebra homomorphism.

For every $u + \mathcal{I}^\infty_{nd,\mathbb{N}} \in \mathcal{A}^\infty_{nd,o}(\Omega)$ and $p \in \mathbb{N}^n$ we have

$$
\begin{aligned}
D^p(\Gamma_{nd,0}(u + \mathcal{I}^\infty_{nd,\mathbb{N}})) &= D^p(u + \mathcal{I}^\infty_{nd,\mathbb{N}}) \\
&= D^p(u) + \mathcal{I}^\infty_{nd,\mathbb{N}} \\
&= \Gamma_{nd,0}(D^p(u) + \mathcal{I}^\infty_{nd,\mathbb{N}}) \\
&= \Gamma_{nd,0}(D^p(u + \mathcal{I}^\infty_{nd,\mathbb{N}}))
\end{aligned}
$$

and

$$
\begin{aligned}
D^p(\Gamma_{nd}(u + \mathcal{I}^\infty_{nd,\mathbb{N}})) &= D^p(u + \mathcal{I}^\infty_{o}) \\
&= D^p(u) + \mathcal{I}^\infty_{o} \\
&= \Gamma_{nd}(D^p(u) + \mathcal{I}^\infty_{nd,\mathbb{N}}) \\
&= \Gamma_{nd}(D^p(u + \mathcal{I}^\infty_{nd,\mathbb{N}})).
\end{aligned}
$$

Therefore, the diagram (4.5) commutes. □

As a consequence of Theorem 4.1, every generalized solution in $\mathcal{NL}^\infty(\Omega)$ of a PDE (2.12), with the map (2.13) and the righthand term f both $\mathcal{C}^\infty$-smooth, corresponds to a generalized solution in the differential algebra $\mathcal{A}^\infty_{nd}(\Omega)$. This result will be discussed in detail in Section 6.

5. $\mathcal{NL}^\infty(\Omega)$ and $\mathcal{A}^\infty_{ae}(\Omega)$

It is known [34, Corollary 1, page 244], that there exists a linear injection

$$E : \mathcal{D}'(\Omega) \to \mathcal{A}^\infty_{nd}(\Omega).$$

However, it is not known whether or not such an embedding exists so that the diagram

$$\begin{tikzcd}
\mathcal{D}'(\Omega) \arrow[r, "D^p"] \arrow[d, "E"'] & \mathcal{D}'(\Omega) \arrow[d, "E"] \\
\mathcal{A}^\infty_{nd}(\Omega) \arrow[r, "D^p"] & \mathcal{A}^\infty_{nd}(\Omega)
\end{tikzcd}$$

commutes for all $p \in \mathbb{N}^n$.

Vernaeve [51] introduced a modified construction of a 'nowhere dense algebra', the so called almost everywhere algebra, which does admit a linear embedding of the distributions which commutes with partial derivatives. In this section, we establish the relationship between Vernaeve's almost everywhere algebra, and the differential algebra $\mathcal{NL}^\infty(\Omega)$.

For the convenience of the reader, we recall briefly the construction of the almost everywhere algebra. Denote by $\mathcal{M}_0$ the set of all closed, nowhere dense subsets of Ω. Let

$$(5.1) \qquad \mathcal{E}^\infty_{ae}(\Omega) = \left\{ (u_n) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & (1)\ u_n : \Omega \longrightarrow \mathbb{R} : \\ & (2)\ u_n \in \mathcal{C}^\infty(\Omega \setminus \Gamma) \end{array} \right\}$$

Clearly, $\mathcal{E}^\infty_{ae}(\Omega)$ is an algebra over $\mathbb{R}$ with respect to the termwise operations on sequences of functions. Consider the ideals

$$(5.2) \qquad \mathcal{I}^\infty_E := \left\{ (u_n) \in \mathcal{E}^\infty_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \end{array} \right\}$$

and

$$(5.3) \qquad \mathcal{I}^\infty_{ae} := \left\{ (u_n) \in \mathcal{E}^\infty_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & u_n(x) = 0, \quad x \in \Omega \setminus \Gamma \end{array} \right\}.$$

As $\mathcal{I}^\infty_{ae}$ and $\mathcal{I}^\infty_E$ are ideals in $\mathcal{E}^\infty_{ae}(\Omega)$, so is

$$\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae} = \{(u_n) + (v_n) \mid (u_n) \in \mathcal{I}^\infty_{ae},\ (v_n) \in \mathcal{I}^\infty_E\}.$$

The almost everywhere algebra $\mathcal{A}^\infty_{ae}(\Omega)$ is defined as

$$\mathcal{A}^\infty_{ae}(\Omega) = \mathcal{E}^\infty_{ae}/(\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}).$$

For $(u_n) \in \mathcal{E}^\infty_{ae}(\Omega)$ and $p \in \mathbb{N}$, set $D^p(u_n) = (D^p u_n)$ where

$$D^p u_n(x) = \begin{cases} D^p u_n(x) & if \quad x \in \Omega \setminus \Gamma \\ 0 & if \quad x \in \Gamma \end{cases}$$

with $\Gamma \in \mathcal{M}_0$ the closed nowhere dense set associated with (u_n) through (5.1). Clearly, the inclusions $D^p(\mathcal{E}^\infty_{ae}(\Omega)) \subseteq \mathcal{E}^\infty_{ae}(\Omega)$ and $D^p(\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}) \subseteq \mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}$ hold. In particular, if $(u_n) - (v_n) \in \mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}$, then $D^p(u_n) - D^p(v_n) \in \mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}$. Therefore

$$(5.4) \qquad D^p : \mathcal{A}^\infty_{ae}(\Omega) \ni (u_n) + (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}) \mapsto D^p(u_n) + (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})$$

is well defined for each $p \in \mathbb{N}^n$. Furthermore, since $D^p : \mathcal{E}^\infty_{ae}(\Omega) \to \mathcal{E}^\infty_{ae}(\Omega)$ is linear and satisfies the Leibnitz rule, the same is true for (5.4). Therefore (5.4) defines a partial differential operator on $\mathcal{A}^\infty_{ae}(\Omega)$ for each $p \in \mathbb{N}^n$.

As in the case of the nowhere dense algebra $\mathcal{A}^\infty_{nd,\mathbb{N}}(\Omega)$, the algebra $\mathcal{A}^\infty_{ae}(\Omega)$ is related to $\mathcal{NL}^\infty(\Omega)$ via an auxiliary algebra $\mathcal{A}^\infty_{ae,o}(\Omega)$, which is defined as follows. Let

$$\mathcal{I}^\infty_{ae,0} = (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}) \cap \mathcal{C}^\infty(\Omega)^\mathbb{N}.$$

Clearly, $\mathcal{I}^\infty_{ae,0} \subseteq \mathcal{I}^\infty_o$ is an ideal in $\mathcal{A}^\infty_o$. Furthermore,

$$D^p(\mathcal{I}^\infty_{ae,0}) \subseteq \mathcal{I}^\infty_{ae,0}, \quad D^p(\mathcal{A}^\infty_o) \subseteq \mathcal{A}^\infty_o$$

so that

$$\mathcal{A}^\infty_{ae,o}(\Omega) = \mathcal{A}^\infty_o/\mathcal{I}^\infty_{ae,0}$$

is a differential algebra, with partial derivatives defined as

$$D^p : \mathcal{A}^\infty_{ae,o}(\Omega) \ni u + \mathcal{I}^\infty_{ae,0} \mapsto D^p u + \mathcal{I}^\infty_{ae,0} \in \mathcal{A}^\infty_{ae,o}(\Omega), \ p \in \mathbb{N}^n.$$

Theorem 5.1. *There exists an injective algebra homomorphism*

$$\Gamma_{ae,0} : \mathcal{A}^\infty_{ae,o}(\Omega) \to \mathcal{A}^\infty_{ae}(\Omega)$$

and a surjective algebra homomorphism

$$\Gamma_{ae} : \mathcal{A}^\infty_{ae,o}(\Omega) \to \mathcal{NL}^\infty(\Omega)$$

so that the diagram

(5.5)
$$\begin{tikzcd}
\mathcal{A}^{\infty}_{ae}(\Omega) \arrow[rr, "D^p"] & & \mathcal{A}^{\infty}_{ae}(\Omega) \\
\mathcal{A}^{\infty}_{ae,o}(\Omega) \arrow[u, "\Gamma_{ae,0}"] \arrow[rr, "D^p"] \arrow[d, "\Gamma_{ae}"'] & & \mathcal{A}^{\infty}_{ae,o}(\Omega) \arrow[u, "\Gamma_{ae,0}"'] \arrow[d, "\Gamma_{ae}"] \\
\mathcal{NL}^{\infty}(\Omega) \arrow[rr, "D^p"] & & \mathcal{NL}^{\infty}(\Omega)
\end{tikzcd}$$

commutes for every $p \in \mathbb{N}^n$*.*

Proof. Define $\Gamma_{ae,0}$ and Γ_{ae} as

$$\Gamma_{ae,0} : \mathcal{A}^{\infty}_{ae,o}(\Omega) \ni u + \mathcal{I}^{\infty}_{ae,o} \mapsto u + (\mathcal{I}^{\infty}_{E} + \mathcal{I}^{\infty}_{ae}) \in \mathcal{A}^{\infty}_{ae}(\Omega)$$

and

$$\Gamma_{ae} : \mathcal{A}^{\infty}_{ae,o}(\Omega) \ni u + \mathcal{I}^{\infty}_{ae,o} \mapsto u + \mathcal{I}^{\infty}_{o} \in \mathcal{NL}^{\infty}(\Omega),$$

respectively. The mapping $\Gamma_{ae,0}$ is a well defined and injective algebra homomorphism, since $\mathcal{A}^{\infty}_{o} \subset \mathcal{E}^{\infty}_{ae}(\Omega)^{\mathbb{N}}$ and $\mathcal{I}^{\infty}_{ae,o} = \mathcal{A}^{\infty}_{o} \cap (\mathcal{I}^{\infty}_{E} + \mathcal{I}^{\infty}_{ae})$. Furthermore, $\mathcal{I}^{\infty}_{ae,o} \subseteq \mathcal{I}^{\infty}_{o}$ so that Γ_{ae} is a well defined and surjective algebra homomorphism.

The commutativity of (5.5) follows in the same way as that of (4.5), see the proof of Theorem 4.1. □

6. Solutions of Nonlinear PDEs

Consider a nonlinear PDE of the form (2.12), with (2.13) and the righthand term f both $\mathcal{C}^{\infty}$-smooth. Since (2.13) is $\mathcal{C}^{\infty}$-smooth, the operator

(6.1)
$$T(x, D) : \mathcal{C}^{\infty}(\Omega) \to \mathcal{C}^{\infty}(\Omega)$$

extends to

$$T(x, D) : \mathcal{C}^{\infty}(\Omega)^{\mathbb{N}} \ni (u_n) \mapsto (T(x, D)u_n) \in \mathcal{C}^{\infty}(\Omega)^{\mathbb{N}}.$$

Furthermore,

$$T(x, D)(\mathcal{A}^{\infty}_{o}) \subseteq \mathcal{A}^{\infty}_{o}$$

and, for $(u_n), (v_n) \in \mathcal{C}^{\infty}(\Omega)$,

$$(u_n) - (v_n) \in \mathcal{I}^{\infty}_{nd,\mathbb{N}} \Rightarrow T(x, D)(u_n) - T(x, D)(v_n) \in \mathcal{I}^{\infty}_{nd,\mathbb{N}}$$

and

$$(u_n) - (v_n) \in \mathcal{I}_o^\infty \Rightarrow T(x,D)(u_n) - T(x,D)(v_n) \in \mathcal{I}_o^\infty.$$

In the same way, we define

$$T(x,D) : \mathcal{E}_{ae}^\infty(\Omega) \ni (u_n) \mapsto (T(x,D)u_n) \in \mathcal{E}_{ae}^\infty(\Omega),$$

where for $u_n \in \mathcal{C}^\infty(\Omega \setminus \Gamma)$ with $\Gamma \in \mathcal{M}_0$,

$$T(x,D)u_n(x) = 0, \quad x \in \Gamma.$$

In addition, for $(u_n), (v_n) \in \mathcal{E}_{ae}^\infty(\Omega)$

$$(u_n) - (v_n) \in \mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty \Rightarrow T(x,D)(u_n) - T(x,D)(v_n) \in \mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty$$

and

$$(u_n) - (v_n) \in \mathcal{I}_{ae,o}^\infty \Rightarrow T(x,D)(u_n) - T(x,D)(v_n) \in \mathcal{I}_{ae,o}^\infty.$$

Therefore the mappings

$$T(x,D) : \mathcal{NL}^\infty(\Omega) \ni u + \mathcal{I}_o^\infty \mapsto T(x,D)u + \mathcal{I}_o^\infty \in \mathcal{NL}^\infty(\Omega),$$

$$T(x,D) : \mathcal{A}_{nd,\mathbb{N}}^\infty(\Omega) \ni u + \mathcal{I}_{nd,\mathbb{N}}^\infty \mapsto T(x,D)u + \mathcal{I}_{nd,\mathbb{N}}^\infty \in \mathcal{A}_{nd,\mathbb{N}}^\infty(\Omega),$$

$$T(x,D) : \mathcal{A}_{nd,o}^\infty(\Omega) \ni u + \mathcal{I}_{nd,o}^\infty \mapsto T(x,D)u + \mathcal{I}_{nd,o}^\infty \in \mathcal{A}_{nd,o}^\infty(\Omega),$$

$$T(x,D) : \mathcal{A}_{ae}^\infty(\Omega) \ni u + (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty) \mapsto T(x,D)u + (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty) \in \mathcal{A}_{ae}^\infty(\Omega)$$

and

$$T(x,D) : \mathcal{A}_{ae,o}^\infty(\Omega) \ni u + \mathcal{I}_{ae,o}^\infty \mapsto T(x,D)u + \mathcal{I}_{ae,o}^\infty \in \mathcal{A}_{ae,o}^\infty(\Omega)$$

are well defined extensions of (6.1). Furthermore, due to Theorems 4.1 and 5.1, in particular the commutativity of the diagrams (4.5) and (5.5), the diagrams

(6.2)

$$\begin{tikzcd}
\mathcal{A}_{nd,\mathbb{N}}^\infty(\Omega) \arrow[r, "T(x,D)"] & \mathcal{A}_{nd,\mathbb{N}}^\infty(\Omega) \\
\mathcal{A}_{nd,o}^\infty(\Omega) \arrow[u, "\Gamma_{nd,0}"] \arrow[r, "T(x,D)"] \arrow[d, "\Gamma_{nd}"'] & \mathcal{A}_{nd,o}^\infty(\Omega) \arrow[u, "\Gamma_{nd,0}"'] \arrow[d, "\Gamma_{nd}"] \\
\mathcal{NL}^\infty(\Omega) \arrow[r, "T(x,D)"] & \mathcal{NL}^\infty(\Omega)
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathcal{A}^{\infty}_{ae}(\Omega) \arrow[rr, "T(x,D)"] & & \mathcal{A}^{\infty}_{ae}(\Omega) \\
\mathcal{A}^{\infty}_{ae,o}(\Omega) \arrow[u, "\Gamma_{ae,0}"] \arrow[d, "\Gamma_{ae}"'] \arrow[rr, "T(x,D)"] & & \mathcal{A}^{\infty}_{ae,o}(\Omega) \arrow[u, "\Gamma_{ae,0}"'] \arrow[d, "\Gamma_{ae}"] \\
\mathcal{NL}^{\infty}(\Omega) \arrow[rr, "T(x,D)"] & & \mathcal{NL}^{\infty}(\Omega)
\end{tikzcd}
\tag{6.3}
$$

commute. We therefore have the following.

Theorem 6.1. *Consider a PDE (2.12) with (2.13) and f both $\mathcal{C}^{\infty}$-smooth. If (2.12) admits a generalized solution $u + \mathcal{I}^{\infty}_{o} \in \mathcal{NL}^{\infty}(\Omega)$, in the sense that*

$$T(x,D)(u + \mathcal{I}^{\infty}_{o}) = \Delta(f) + \mathcal{I}^{\infty}_{o},$$

then there exist $v + \mathcal{I}^{\infty}_{nd,\mathbb{N}} \in \mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega)$ and $w + (\mathcal{I}^{\infty}_{E} + \mathcal{I}^{\infty}_{ae}) \in \mathcal{A}^{\infty}_{ae}(\Omega)$ so that

$$T(x,D)(v + \mathcal{I}^{\infty}_{nd,\mathbb{N}}) = \Delta(f) + \mathcal{I}^{\infty}_{nd,\mathbb{N}} \quad in \quad \mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega)$$

and

$$T(x,D)(w + (\mathcal{I}^{\infty}_{E} + \mathcal{I}^{\infty}_{ae})) = \Delta(f) + (\mathcal{I}^{\infty}_{E} + \mathcal{I}^{\infty}_{ae}), \quad in \quad \mathcal{A}^{\infty}_{ae}(\Omega),$$

respectively.

Proof. The result follows immediately from the commutativity of the diagrams (6.2) and (6.3), and the surjectivity of the mappings Γ_{nd} and Γ_{ae}. □

Theorems 2.2 and 6.1 lead to an existence result for generalized solutions of (2.18) in the algebras $\mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega)$ and $\mathcal{A}^{\infty}_{ae,\mathbb{N}}(\Omega)$, respectively.

Theorem 6.2. *Consider a nonlinear PDE (2.12). Assume that the mapping (2.13) as well as the righthand term f are $\mathcal{C}^{\infty}$-smooth, and satisfy*

$$
\begin{array}{ll}
\forall & x_0 \in \Omega : \\
\exists & \xi(x_0) \in \mathbb{R}^{\mathbb{N}^n},\ F^{\infty}(x_0, \xi(x_0)) = (D^q f(x_0))_{q \in \mathbb{N}^n} : \\
\exists & V \text{ a neighborhood of } x_0,\ W \text{ a neighborhood of } \xi(x_0) \ : \ : \\
 & F^{\infty} : V \times W \to \mathbb{R}^{\mathbb{N}^n} \ \textit{open}
\end{array}
\tag{6.4}
$$

Then there exist $v + \mathcal{I}^{\infty}_{nd,\mathbb{N}} \in \mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega)$ and $w + (\mathcal{I}^{\infty}_{E} + \mathcal{I}^{\infty}_{ae}) \in \mathcal{A}^{\infty}_{ae}(\Omega)$ so that

$$T(x,D)(v + \mathcal{I}^{\infty}_{nd,\mathbb{N}}) = \Delta(f) + \mathcal{I}^{\infty}_{nd,\mathbb{N}} \quad in \quad \mathcal{A}^{\infty}_{nd,\mathbb{N}}(\Omega)$$

and

$$T(x,D)(w+(\mathcal{I}_E^\infty+\mathcal{I}_{ae}^\infty))=\Delta(f)+(\mathcal{I}_E^\infty+\mathcal{I}_{ae}^\infty),\quad in\ \ \mathcal{A}_{ae}^\infty(\Omega),$$

respectively.

Proof. According to Theorem 2.2, there exists $u^\sharp\in\mathcal{NL}^\infty(\Omega)$ so that

$$T^\sharp u^\sharp=f.$$

Then, according to Lemma 3.9, there exists a sequence $(u_n)\in\mathcal{A}_o^\infty$ so that $T(x,D)(u_n)=(T^\sharp u_n)$ converges to f in $\mathcal{NL}^\infty(\Omega)$, hence in $\mathcal{ML}^\infty(\Omega)$. Therefore, by Corollary 3.7, $T(x,D)(u_n)-\Delta(f)\in\mathcal{I}_o^\infty$. Therefore

$$T(x,D)((u_n)+\mathcal{I}_o^\infty)=\Delta(f)+\mathcal{I}_o^\infty\ \text{ in }\ \mathcal{NL}^\infty(\Omega).$$

The result now follows immediately from Theorem 6.1. □

7. Conclusion

It has been shown that the space of generalized functions that underly the recent development of Order Completion Method [23], as presented in [48], may be represented as a differential algebra of generalized functions. Furthermore, this algebra was shown to be closely related to the earlier nowhere dense algebras of Rosinger, see for instance [34], and the almost everywhere algebra of Vernaeve [51]. This result has a twofold power, since each theory, the Order Completion Method and its various recent extensions on the one hand, and the earlier algebraic theory on the other, may benefit from the other's respective strengths.

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