Properties of a class of generalized Freud polynomials

by

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Declaration

I, Abey Sherif Kelil, declare that the thesis, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE:

DATE: February 15, 2018

This thesis is dedicated to my sister, Hadas Sherif Kelil, and my family.

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Abstract

Semiclassical orthogonal polynomials are polynomials orthogonal with respect to semiclassical weights. The fascinating link between semiclassical orthogonal polynomials and discrete integrable equations can be traced back to the work of Shohat and Freud and later by Bonan and Nevai; orthogonal polynomials with Freud-type exponential weights have three-term recurrence coefficients that satisfy nonlinear second order difference equations. Fokas, Its and Kitaev identified these equations as discrete Painlevé equations.

Magnus related the recurrence coefficients of orthogonal polynomials with respect to the Freud weight and classical solutions of the fourth Painlevé equation. We extend Magnus's results for Freud weight, by considering polynomials orthogonal with respect to a generalized Freud weight, by studying the theory of Painlevé equations. These generalized Freud polynomials arise from a symmetrization of semiclassical Laguerre polynomials.

We prove that the coefficients in the three-term recurrence relation associated with a generalized Freud weight can be expressed in terms of Wronskians of parabolic cylinder functions that appear in the description of special function solutions of the fourth

Painlevé equation. This closed form expression for the recurrence coefficients allows the investigation of certain properties of the generalized Freud polynomials.

We obtain an explicit formulation for the generalized Freud polynomials in terms of the recurrence coefficients, investigate the higher order moments, as well as the Pearson equation satisfied by the generalized Freud weight. We also derive a second-order linear ordinary differential equation and a differential-difference equation satisfied by the generalized Freud polynomials and we use the differential equation to study some properties of the zeros of generalized Freud polynomials.

Furthermore, we obtain limit relations for the recurrence coefficients of the generalized Freud polynomials using Freud's Kunstgriff method. We verify the existence of an asymptotic series for the recurrence coefficient using an extension of the result by Bleher and Its [17] and we provide an asymptotic expansion for the recurrence coefficients of the three-term recurrence relation satisfied by monic generalized Freud polynomials.

Key words: Orthogonal polynomials on the real line, classical, semiclassical, three-term recurrence relation, moments, recurrence coefficients, symmetric, symmetrization, quadratic transformation, semiclassical, Painlevé equations, semiclassical Laguerre polynomials, generalized Freud polynomials, nonlinear difference, asymptotic series, differential-difference, differential equations, zeros

Nomenclature

$\mathrm{L}^2(d\alpha;\mathbb{R})$	The linear space of square integrable functions with respect to $d\alpha$
$\mathbb{P}_n$	The vector space of polynomials in one variable of degree at most n
P_n	Monic orthogonal polynomials in one variable
p_n	Orthonormal polynomials in one variable
α	The Borel measure
$\mu_j(\alpha)$	The j^{th}- moment associated with the measure α
Δ_n	The Hankel determinant
δ_{mn}	The Kronecker delta
α_n, β_n	The recurrence coefficients for monic orthogonal polynomials
a_n, b_n	The recurrence coefficients for orthonormal polynomials
$f^{'}(x)$	The derivative for f with respect to the independent variable x
$\dfrac{\partial f(x;t)}{\partial t} = f_t$	The partial derivative of f with respect to t.
$\mathscr{D}$	The differential operator
$f^{(k)}(x)$	The k^{th}-derivative of f
$\Gamma(x)$	The Gamma function
$\mathfrak{B}(x,y)$	The Euler Beta function
$H_n(x)$	The Hermite polynomials in one variable

$\mathsf{H}_n^{(\lambda)}(x)$	The Sonin-Markov (Generalized Hermite) polynomials
$S_m(x)$	A symmetric real polynomial of degree m in variable x
$D_\nu(z)$	The parabolic cylinder function
$\mu_0(t;\lambda)$	The first moment associated with a semiclassical weight
$\mu_n(t;\lambda)$	The n^{th} moment associated with a semiclassical weight
$\mathcal{W}$	The Wronskian
τ_n	The tau function
$\mathsf{L}_n^{(\lambda)}(x;t)$	Semiclassical Laguerre polynomials of degree n
$S_n(x;t)$	Semiclassical generalized Freud polynomials of degree n
a_n	The Mhaskar-Rahmanov-Saff number
LHS	Left-hand side
RHS	Right-hand side

Contents

Chapter 1

Introduction

1.1 Historical background

The theory of orthogonal polynomials plays an important role in different branches of mathematics, such as approximation theory (best approximation, interpolation, quadrature), special functions, continued fractions and differential and integral equations. The notion of orthogonality originated from the theory of continued fractions, but later became an independent discipline. Contributors to the theory of orthogonal polynomials include outstanding mathematicians such as Abel, Chebyshev, Fourier, Hermite, Laguerre, Laplace, Legendre, Markov and Stieltjes. Beginning with Szegö, Hungarian mathematicians like Erdös, Turán, Freud and Feldheim have made essential contributions to the flourishing theory of orthogonal polynomials in the last century.

The theory of polynomials orthogonal on infinite intervals is significantly different from the theory of polynomials orthogonal on finite intervals. While Szegö did pioneering work in the theory of orthogonality on finite intervals, he didn't carry over his ideas to infinite intervals. Freud founded the now flourishing theory of orthogonal polynomials with respect to exponential weights on $\mathbb{R}$ and the corresponding representative polynomials are named after him. Freud's aim was to extend the theory of best approximation and Jackson-Bernstein type estimates to the real axis. The natural way to do this was to explore properties of orthogonal polynomials, since the expectation was that orthogonal expansions may serve as near-best approximation (cf. [84, 105, 112]).

Jacobi, Laguerre and Hermite polynomials are considered to be classical orthogonal polynomials (cf. [35, 132]). These polynomials were discovered in the 19^{th} century as solutions to interpolation problems and to certain second-order differential equations. The reader may be familiar with these classical polynomials and with the fact that they obey three-term recurrence relations as well as second-order differential equations. It turns out that the second-order linear differential equations are unique to the classical orthogonal polynomials, by a theorem of Bochner [18] (see also [71, Section 20.1]), but that a second-order recurrence relation is a universal property for weight functions supported in $\mathbb{R}$ (cf. [35, 120, 132]).

One of the characterizations of these polynomials was proved in 1972 by Al-Salam and Chihara [4], where the authors showed that the classical orthogonal polynomials are the only orthogonal polynomials satisfying a first-order structure relation of the form

$$\pi(x)P_n'(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \tag{1.1.1}$$

where $\pi(x)$ is a polynomial of degree 2 independent of n. Classical orthogonal polynomials also have their weight function $w(x)$ satisfying *Pearson's differential equation* (cf. [35, Equation 2.25])

$$\frac{d}{dx}\left(\sigma(x)w(x)\right) = \tau(x)w(x), \tag{1.1.2}$$

where $\sigma(x)$ is a monic polynomial with $\deg(\sigma) \leq 2$ and $\tau(x)$ is a polynomial with $\deg(\tau) = 1$. However, when $\deg(\sigma) > 2$ and/or $\deg(\tau) > 1$, the weight function in (1.1.2) produces a class of semiclassical orthogonal polynomials (cf. [67, 93, 95]). The theory of semiclassical orthogonal polynomials is not fully-fledged but the derivation of a differential equation for a general class of orthogonal polynomials by Shohat [127] provides a cornerstone for forming classes of semiclassical orthogonal polynomials. These semiclassical polynomials are also the polynomial solutions of a particular case of second-order linear differential equations known as *holonomic equations* [93, 94, 96]. Holonomic equations can be obtained from the structural relation that is associated with the so-called creation and destruction operators [68, 71]. According to Maroni [94], the monic semiclassical orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ can be defined by the relation

$$A(x)P_n^{'}(x) = \sum_{j=1}^{r+1} c_{n,j}P_{n+m-j}(x), \tag{1.1.3}$$

where $A(x)$ is a polynomial of exactly m^{th} degree, r is a fixed nonnegative integer and $c_{n,j}$ are real coefficients. In a survey of orthogonal polynomials and their applications, Maroni [94] extensively studied semiclassical linear functionals with special emphasis on their structural properties. Belmehdi [13] and Belmehdi and Ronveaux [14] also provided descriptions of semiclassical linear functionals. One can observe from (1.1.1) and (1.1.3) that the classical orthogonal polynomials are a special case of semiclassical orthogonal polynomials (for $m = r = 2$).

The study of a class of semiclassical polynomials orthogonal on unbounded intervals with respect to general exponential weights begun with Géza Freud in the 1970's (for details see Freud [55], Nevai [112], as well as recent monographs by Levin and Lubinsky [84] and Mhaskar [105]).

A function $w : \mathbb{R} \to \mathbb{R}^+$ of the form

$$w(x) = \exp\left(-Q(x)\right) \tag{1.1.4}$$

is said to be a Freud weight if $Q : \mathbb{R} \to \mathbb{R}$ is an even, non-negative and continuous function that satisfies certain conditions involving its derivatives of first and second order (cf. [80, 87]). Specifically, Freud weights are a class of exponential-type weights

$$w_\rho(x) = |x|^\rho \exp(-|x|^m), \quad \rho > -1, \; m = 2k, \; k \in \mathbb{N}, \tag{1.1.5}$$

with an unbounded support on $\mathbb{R}$. Since the Freud weights are even functions, it follows that one of the recurrence coefficients $\alpha_n = 0$, $n \in \mathbb{N}_0$ so that polynomials orthogonal with respect to the weight (1.1.5) satisfy a three-term recurrence relation

$$P_{n+1}(x) = xP_n(x) - \beta_n(\rho)P_{n-1}(x), \tag{1.1.6}$$

with initial conditions $P_{-1} \equiv 0$ and $P_0 \equiv 1$, and $\beta_n(\rho)$ obeys certain nonlinear second-order difference equations (cf. [89, 90, 109, 134]). Since it is not usually possible to determine an explicit formulation of the β_n's, Freud conjectured that $\beta_n \, n^{-\frac{2}{m}}$ converges to some constant depending on m.

Freud [55, 56] investigated the asymptotic behavior of the recurrence coefficients for the weight (1.1.5) , when $m = 2, 4, 6$, by a technique giving rise to an infinite system of nonlinear equations called Freud equations (cf. [89, 90]). For example, for the weight $w(x) = |x|^\rho \exp(-x^4)$ on $\mathbb{R}$, the recurrence coefficients $\beta_n(\rho)$ not only satisfy the three-term recurrence relation (1.1.6), but also a non-linear recurrence relation (cf. [109, 134] for the orthonormal case)

$$4\beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) = n + \rho\,\Omega_n, \tag{1.1.7}$$

with initial conditions

$$\beta_0 = 0, \quad \beta_1 = \frac{\int_{-\infty}^{\infty} x^2 |x|^\rho \exp(-x^4)\, dx}{\int_{-\infty}^{\infty} |x|^\rho \exp(-x^4)\, dx} = \frac{\Gamma(\frac{3+\rho}{4})}{\Gamma(\frac{1+\rho}{4})}. \tag{1.1.8}$$

We note that, for $\rho = 0$, (1.1.7) was first derived by Shohat [127, Equation 39] and it is proved in [109] that there is a unique positive solution to the problem (1.1.7). In [55], Freud gave the limit relation for the recurrence coefficient $\beta_n(\rho)$ and this could be described by

$$\lim_{n\to\infty} \beta_n(\rho)\, n^{-\frac{2}{m}} = \left[\frac{\Gamma(\frac{1}{2}m)\, \Gamma(1+\frac{1}{2}m)}{\Gamma(m+1)}\right]^{\frac{2}{m}}, \quad m = 2k,\ k \in \mathbb{N}.$$

Freud also explored some essential properties, such as the asymptotic behavior for the greatest zeros [56] and of the polynomials themselves, by relying on the recurrence coefficients. The seminal work by Freud on exponential weight functions solved special cases of his conjectures on the asymptotic behavior of the recurrence coefficients for the orthogonal polynomials associated with the weight functions of the form (1.1.5) when $m = 2, 4, 6$ (cf. [5, 55, 84, 90, 91, 110, 112]).

Recent contributions on the asymptotic behavior of the recurrence coefficient associated with Freud-type exponential weights include those of Magnus [91], Mhaskar [87], Rakhmanov [119], Lubinsky [84], Nevai [5, 110, 112], and Nevai and Totik [100, 101].

The connection between recurrence coefficients associated with semiclassical orthogonal polynomials and discrete integrable systems can be traced back to the work of Shohat and Freud [55, 127], and later by Bonan and Nevai [21, 22]. However, it wasn't until the 1990's when the focus within integrable systems shifted from continuous to discrete, that Fokas, Its and Kitaev [51, 52, 53] gave these equations a name: discrete Painlevé equations. It is shown in [19, 134] that the recurrence coefficients associated with the positive weight (1.1.5) satisfy a nonlinear recurrence relation that corresponds to discrete Painlevé $\mathrm{dP_I}$ equation and its hierarchy (This fact was not known to Freud but first pointed out by Magnus [89, 90, 91]) (see also [134]). The dynamics of the recurrence coefficients associated with the weight (1.1.4) can also be described by the differential-difference equations of the Toda lattice (cf. [6, 7, 39, 50, 90]).

Several sequences of monic orthogonal polynomials associated with the weight (1.1.4) and their extensions have been studied in the literature (cf. [88, 89, 109]). For instance, the link between the nonlinear difference equation satisfied by the recurrence coefficients associated with (1.1.4) and discrete Painlevé equations for potentials such as $Q(x) = x^4$, $Q(x) = x^4 - tx^2$ for $t \in \mathbb{R}$ or $Q(x) = x^6$ is well-established (cf. [89, 90, 123]). Magnus [90] showed that the recurrence coefficients in the three-term recurrence relation associated with the Freud weight [55]

$$\exp\left(-x^4 + tx^2\right), \quad t, x \in \mathbb{R}, \tag{1.1.9}$$

can be expressed in terms of simultaneous solutions of the discrete equation

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n, \tag{1.1.10}$$

which is discrete $\mathrm{P_I}$ ($\mathrm{dP_I}$), as shown by Bonan and Nevai [22], and the differential equation

$$\frac{d^2 q_n}{dz^2} = \frac{1}{2q_n}\left(\frac{dq_n}{dz}\right)^2 + \frac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2 - A)q_n + \frac{B}{q_n}, \tag{1.1.11}$$

which is a special case of the fourth Painlevé equation where $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$, with $n \in \mathbb{Z}^+$. This connection between the recurrence coefficients for the Freud weight (1.1.9) and simultaneous solutions of (1.1.10) and (1.1.11) has been shown in [51], see also [53].

With regards to asymptotic expansions for the recurrence coefficients, Shing, Máté and Nevai [76] constructed asymptotic expansions for solutions to recurrence relations of the type which occur in the study of orthogonal polynomials with exponential type weights, in particular the weight $|x|^\rho \exp(-|x|^6)$, $\rho > -1$, $x \in \mathbb{R}$. Bauldry, Máté and Nevai [11] showed that the convergent solutions of a system of smooth recurrence equations, whose Jacobian matrix satisfies a certain non-unimodularity condition, can be approximated by asymptotic expansions and they provide an application to approximate the recurrence coefficients associated with polynomials orthogonal with respect to the weight function (1.1.4) where $Q(x)$ is an even degree polynomial with a positive leading coefficient. Further, Mate, Nevai and Zaslavsky [102] obtained asymptotic expansions for the recurrence coefficients of a larger class of orthogonal polynomials with exponential-type weights (cf. [102, Theorem 1] and [11, Theorem 5.1]).

1.2 Motivation of the study

Semiclassical orthogonal polynomials arise in applications, such as random matrices and integrable systems, in particular, continuous and discrete Painlevé equations. Generalized Freud polynomials (cf. [41]) are semiclassical extensions of the Freud polynomials and they are orthogonal with respect to the positive Borel measure $d\alpha(x) = w_\lambda(x;t)\,dx$ where the weight function

$$w_\lambda(x;t) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \tag{1.2.1}$$

with parameter $\lambda > 0$ and $t \in \mathbb{R}$, is differentiable on the non-compact support $\mathbb{R}$. Monic orthogonal polynomials with respect to the symmetric weight (1.2.1) satisfy the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t), \tag{1.2.2}$$

where $S_{-1} \equiv 0$, $S_0 \equiv 1$. From (1.2.2), using orthogonality, we have

$$\beta_n(t;\lambda) = \frac{1}{h_{n-1}} \int_{\mathbb{R}} xS_n(x;t)S_{n-1}(x;t)|x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right)\,dx > 0. \tag{1.2.3}$$

Since the weight (1.2.1) is even, the polynomial $S_n(x;t)$ is even for n even and odd for n odd (cf. [132, p. 29]).

In view of (1.2.2) and (1.2.3), we see that the sequence of the recurrence coefficients $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ completely determines polynomials orthogonal with respect to the weight (1.2.1). They also have other important features related to the greatest zero of orthogonal polynomials (cf. [54, 55, 56]). The motivation for this research lies in studying recurrence coefficients associated with the generalized Freud polynomials.

It was generally accepted that explicit expressions for the associated coefficients in the three-term recurrence relation and orthogonal polynomials were nonexistent for weights such as the Freud weight. To quote from the Digital Library of Mathematical Functions [117, §18.32]:

" *A Freud weight is a weight function of the form*

$$w(x) = \exp(-Q(x)), \quad -\infty < x < \infty,$$

where $Q(x)$ is real, even, nonnegative, and continuously differentiable. Of special interest are the cases $Q(x) = x^{2m}$, $m = 1, 2, \ldots$. No explicit expressions for the corresponding OP's are available. However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros, see Levin and Lubinsky [84] and Nevai [112]. For a uniform asymptotic expansion in terms of Airy functions for *the OP's in the case x^4 see Bo and Wong [122]. For asymptotic approximations to OP's that correspond to Freud weights with more general functions $Q(x)$ see Deift et al. [47, 48], Bleher and Its [17] and Kriecherbauer and McLaughlin [82]."*

In [41] we provide an explicit expression for the recurrence coefficients $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ associated with the generalized Freud weight (1.2.1) in terms of Wronskians of parabolic cylinder functions that appear in the description of special function solutions of the fourth Painlevé equation. These recurrence coefficients, which can also be viewed as functions of the parameters, are fundamental to study certain properties of the generalized Freud polynomials. The moments of the semiclassical weight provide the link between the weight and the associated Painlevé equation which yields explicit expressions for the recurrence coefficients of the generalized Freud weight (1.2.1) (cf. [41]).

1.3 Objective of the study

The objective of this study is to investigate certain analytic and asymptotic properties of the semiclassical generalized Freud polynomials, by making use of their connection to the theory of Painlevé equations, which extend, improve and generalize the known results in the existing literature.

Properties discussed in this thesis include the higher-order moments and Pearson's equation associated with the generalized Freud weight w_λ in (1.2.1), an explicit formulation for the recurrence coefficient $\beta_n(t;\lambda)$, as well as the generalized Freud polynomials themselves and other related properties such as the differential-difference and differential equations satisfied by the generalized Freud polynomials.

Since recurrence coefficients are fundamental entities in the theory of orthogonal polynomials, we investigate the asymptotic series expansion of the recurrence coefficients $\beta_n(t;\lambda)$ as the degree or, alternatively, the parameter tends to ∞. By proving the existence of an asymptotic expansion by adapting the results of Bleher and Its [17], we investigate the asymptotic behavior of the recurrence coefficient $\beta_n(t;\lambda)$ via the theory of Painlevé equations. We also employ an extension of Freud's conjecture for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight in (1.2.1). We further investigate the asymptotics of the normalized differential equation satisfied by monic generalized Freud polynomials, by using the obtained differential-difference and differential equations.

1.4 Summary of the main results

This section summarizes the main results of this thesis.

(i) An explicit formulation of moments and Pearson's equation associated with the generalized Freud weight (1.2.1) are provided (cf. [41]).

(ii) Symmetrization of the semiclassical Laguerre weight gives rise to the generalized Freud weight (1.2.1) (cf. [41]).

(iii) The relationship between the recurrence coefficients of orthogonal polynomials with respect to the generalized Freud weight (1.2.1) and classical solutions of the fourth Painlevé equation is explored (cf. [41]). One of our main results is that the recurrence coefficient $\beta_n(t;\lambda)$ in the three-term recurrence relation (1.2.2) can be expressed in terms of Wronskians of parabolic cylinder functions that arise in the description of special function solutions of the fourth Painlevé equation.

(iii) A concise formulation of the generalized Freud polynomials in terms of the explicitly obtained recurrence coefficient $\beta_n(t;\lambda)$ is provided (see Theorem 4.4.1 and Corollary 4.4.1) (cf. [41]).

(iv) The differential-difference equation satisfied by the generalized Freud polynomials is derived using two different methods; the classical ladder operator approach and also Shohat's approach based on quasi-orthogonality (cf. [41]).

(v) A second-order linear differential equation with rational-type coefficients satisfied by monic generalized Freud polynomials is derived in Theorem 4.6.2 (cf. [41]).

(vi) Certain results on the asymptotic expansion for the recurrence coefficient $\beta_n(t;\lambda)$ associated with generalized Freud polynomials as the degree, as well as the parameter t, tends to infinity, are given in Proposition 5.3.1 and Theorem 5.4.1.

1.5 Outline of the thesis

The thesis includes five chapters and is focused on generalized Freud semiclassical orthogonal polynomials. In each chapter we give a reasonable amount of relevant references regarding the background of the topic. When a proposition, theorem or lemma is credited to any other author but the proof itself is unreferenced, it means that the proof was done independently. In some cases, the proof is trivial and is not provided.

In Chapter 2 we present some preliminary concepts and notations about orthogonal polynomial sequences. A brief introduction to the connection between semiclassical orthogonal polynomials and Painlevé equations is also included.

Chapter 3 revisits certain properties of the semiclassical Laguerre polynomials given in [39]. Properties studied in [39] include the higher order moments, Pearson's equation associated with the semiclassical Laguerre weight and explicit formulation for the recurrence coefficients in terms of special function solutions of the Painlevé equations. We also provide the differential-difference and differential equations satisfied by the recurrence coefficients as well as the semiclassical Laguerre polynomials themselves, by making a connection to integrable systems. A differential-difference equation and differential equation satisfied by semiclassical Laguerre polynomials, as well as an explicit representation of a 2×2 differential (Lax) system in terms of the recurrence coefficients, is obtained. Further, the Volterra equation for the semiclassical Laguerre weight was derived by differentiating the recurrence coefficients with respect to the parameter $t \in \mathbb{R}$ introduced in the weight function (3.3.1). We also show that generalized Freud polynomials arise from a symmetrization of semiclassical Laguerre polynomials by adapting a symmetrization technique due to Chihara [35].

In Chapter 4 certain analytic properties of monic orthogonal polynomials with respect to the generalized Freud weight $w_\lambda(x;t)$ in (1.2.1) are studied in detail. Properties of interest include the higher order moments and Pearson's equation associated with the generalized Freud weight, the recurrence coefficients and the differential-difference and differential equations satisfied by the polynomials, as well as the concise formulation of the generalized Freud polynomials. As our main result, we show that the coefficients in the three-term recurrence relation satisfied by the generalized Freud polynomials can be expressed in terms of Wronskians of parabolic cylinder functions that arise in the description of special function solutions of the fourth Painlevé equation. We also obtain an explicit formulation for the generalized Freud polynomials in terms of the first moments, where the first moments are given explicitly in terms of parabolic cylinder functions. The results in Chapter 4 have been published in [41].

In Chapter 5 we explore the asymptotic behavior of the generalized Freud polynomials, which are orthogonal with respect to the generalized Freud weight. We first obtain an asymptotic series expansion for the recurrence coefficients $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ as the degree tends to ∞. We also investigate asymptotic results for the polynomials when the parameter t involved in the semiclassical perturbation of the weight (1.2.1) tends to ∞. Further we apply the obtained large n-asymptotics of the recurrence coefficient $\beta_n(t;\lambda)$ to the differential equation satisfied by the generalized Freud polynomials to obtain a normalized differential equation in its asymptotic form, which is valid when x belongs to a fixed, finite interval.

Chapter 6 summarizes the main results obtained in this thesis and provides some insights into future perspectives by suggesting problems for future consideration.

Chapter 2

Preliminaries

In this chapter we provide some definitions and discuss basic concepts that will be used in this thesis.

2.1 A glance at orthogonal polynomials

Denote the linear space of polynomials in one variable with real coefficients of degree at most n by $\mathbb{P}_n$. Let $\mathbb{N}_0, \mathbb{N}, \mathbb{Z}, \mathbb{R}$ denote the set of non-negative integers, the set of natural numbers, the set of integers and the set of real numbers, respectively.

Let α be a positive Borel measure defined on the real line for which the moments

$$\mu_n = \int_{\mathbb{R}} x^n \, d\alpha(x), \quad n \in \mathbb{N},$$

are finite and $\mathrm{L}^2(d\alpha;\mathbb{R})$ the Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_\alpha : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$, associated with the measure α, defined by

$$\langle f, g \rangle_\alpha = \int_{\mathrm{supp}(\alpha)} f(x) g(x) \, d\alpha(x). \tag{2.1.1}$$

By considering a sequence of monomials $\{1, x, x^2, \ldots\}$, which are linearly independent in $\mathrm{L}^2(d\alpha;\mathbb{R})$ and applying the Gram-Schmidt orthogonalization process [73, p. 151], we obtain a sequence of orthogonal polynomials $\{\varphi_n\}_{n=0}^{\infty}$ that can be written as a linear combination of the monomials. Moreover, it holds that

$$\mathbb{P}_n := \mathrm{span}\{1, x, x^2, \ldots, x^n\} = \mathrm{span}\{\varphi_0, \varphi_1, \ldots, \varphi_n\}.$$

Definition 2.1.1. *A sequence of real non-zero polynomials* $\{\varphi_n\}_{n=0}^N$, $N \in \mathbb{N} \cup \{\infty\}$, *where* φ_n *is of exact degree* n, *is orthogonal on the interval* $[a,b]$ *with respect to* α *if*

$$\langle \varphi_m, \varphi_n \rangle_\alpha = \int_{[a,b]} \varphi_m(x)\, \varphi_n(x)\, d\alpha(x) = h_n\, \delta_{mn}; \quad m,n = 0,1,\ldots,N, \tag{2.1.2}$$

where δ_{mn} *is the Kronecker symbol defined by*

$$\delta_{mn} = \begin{cases} 1 & \textit{if} \quad m = n; \\ 0 & \textit{if} \quad m \neq n, \end{cases}$$

and

$$h_n := \langle \varphi_n, \varphi_n \rangle_\alpha = \|\varphi_n\|_\alpha^2. \tag{2.1.3}$$

When $\alpha(x)$ is absolutely continuous, we can write $d\alpha(x) = w(x)\, dx$ with a weight function $w(x) > 0$ and (2.1.2) becomes

$$\int_{[a,b]} \varphi_m(x)\, \varphi_n(x) w(x)\, dx = h_n\, \delta_{mn}, \; m,n = 0,1,\ldots,N,$$

or, equivalently, [120, Theorem 54]

$$\int_{[a,b]} x^k\, \varphi_n(x) w(x)\, dx = 0, \text{ for } n = 1,2,\ldots; \quad k < n.$$

By defining

$$\hat{\varphi}_m(x) = \frac{\varphi_m(x)}{\sqrt{\displaystyle\int_{[a,b]} \varphi_m^2(x) w(x)\, dx}},$$

we have a sequence of orthonormal polynomials with respect to $w(x)$:

$$\int_{[a,b]} \hat{\varphi}_m(x)\, \hat{\varphi}_n(x)\, w(x)\, dx = \delta_{mn}; \quad m,n = 0,1,\ldots,N.$$

Throughout this thesis, we will assume that *orthogonality* refers to orthogonality with respect to a positive weight function, supported on $\mathbb{R}$, and the polynomials we consider are monic polynomials, i.e.,

$$P_n(x) = x^n + c_{n,n-1} x^{n-1} + \ldots + c_{n,1} x + c_{n,0}. \tag{2.1.4}$$

2.2 Properties of orthogonal polynomials

This section provides some of the basic properties of real orthogonal polynomials in one variable.

2.2.1 Three-term recurrence relation

Theorem 2.2.1. *[63, Theorem 1.27]. Let* $\{P_n\}_{n=0}^{\infty}$ *be a sequence of monic orthogonal polynomials with respect to a positive measure* α*. Then,*

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n\, P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{2.2.1}$$
$$P_{-1} \equiv 0, \qquad P_0 \equiv 1,$$

and the recurrence coefficients α_n *and* β_n *are given as:*

$$\alpha_n = \frac{\langle xP_n, P_n\rangle_\alpha}{\langle P_n, P_n\rangle_\alpha}, \quad n \in \mathbb{N}_0, \tag{2.2.2a}$$
$$\beta_n = \frac{\langle P_n, P_n\rangle_\alpha}{\langle P_{n-1}, P_{n-1}\rangle_\alpha}, \; n \in \mathbb{N}. \tag{2.2.2b}$$

The converse of Theorem 2.2.1 is known as the spectral theorem for orthogonal polynomials and states that a set of polynomials satisfying the three-term relation (2.2.1) is orthogonal with respect to a positive measure. This result is often attributed to Favard [49] but was discovered independently, around the same time, by both Shohat [125, 127] and Natanson [108]. A modern proof of the result is given by Beardon [12].

For a sequence of monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$, the sequences of recurrence coefficients $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in (2.2.2) and the sequence of coefficients $\{c_{n,j}\}_{j=0}^{n}$ in (2.1.4) are related by the following recursive relation (cf. [130, p. 5]):

$$c_{n,n-1} = -\sum_{j=0}^{n-1} \alpha_j, \tag{2.2.3a}$$
$$c_{m,m-t} = -\sum_{j=t-1}^{m-1} \left(\alpha_j c_{j,j-t+1} + \beta_j\right), \quad t \geq 2, \tag{2.2.3b}$$

(2.2.3a) implies that $\alpha_n = c_{n,n-1} - c_{n+1,n}$ and β_j is always positive from (2.2.2b).

2.2.2 Orthogonal polynomials in terms of Hankel determinants

Monic orthogonal polynomials of degree $n,\ n=1,2,\ldots$, can be uniquely expressed in terms of the moments $\{\mu_n\}_{n\geq 0}$ (cf. [63, Theorem 2.1]):

$$P_n(x)=\frac{1}{\Delta_n}\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}=x^n-\frac{\tilde{\Delta}_n}{\Delta_n}x^{n-1}+\mathcal{O}(x^{n-2}), \tag{2.2.4}$$

where

$$\Delta_n=\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix},\quad \Delta_0=1, \tag{2.2.5}$$

is the Hankel determinant of moments and the determinant $\tilde{\Delta}_n$ in (2.2.4) is given by

$$\tilde{\Delta}_n=\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix},\quad \tilde{\Delta}_0=0,\ \tilde{\Delta}_1=\mu_1. \tag{2.2.6}$$

The recurrence coefficients α_n and β_n in (2.2.2) can be written (cf. [63]) as

$$\beta_n=\frac{h_n}{h_{n-1}}=\frac{\Delta_{n+1}\,\Delta_{n-1}}{\Delta_n^2},\quad n=1,2,\ldots, \tag{2.2.7a}$$

$$\alpha_n=\frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}}-\frac{\tilde{\Delta}_n}{\Delta_n},\quad n=1,2,\ldots. \tag{2.2.7b}$$

Consequently, the normalization constant h_n in (2.1.3) can be given in terms of the Hankel determinant (2.2.5) by

$$h_n=\prod_{j=1}^{n}\beta_j=\frac{\Delta_{n+1}}{\Delta_n}>0,\quad n\in\mathbb{N}.$$

2.2.3 Zeros of orthogonal polynomials

Theorem 2.2.2. *(cf. [120, Theorem 55]). For $n \in \mathbb{N}$, all zeros of an orthogonal polynomial sequence $\{\varphi_n\}_{n=0}^{\infty}$ are real, simple and located in the interval of orthogonality.*

The Christoffel-Darboux identity [132, Theorem 3.2.2]

$$(x-y)\sum_{k=0}^{n}\frac{P_k(x)P_k(y)}{h_k} = \frac{P_{n+1}(x)P_n(y)-P_{n+1}(y)P_n(x)}{h_n} \tag{2.2.8}$$

is a direct consequence of the three-term recurrence relation (2.2.1) and has numerous applications in the theory of orthogonal polynomials. The confluent form of (2.2.8) is given by

$$\sum_{k=0}^{n}\frac{P_k^2(x)}{h_k} = \frac{P_{n+1}^{'}(x)P_n(x)-P_{n+1}(x)P_n^{'}(x)}{h_n}. \tag{2.2.9}$$

An important consequence of the Christoffel-Darboux identity is that

$$P_{n+1}^{'}(x)P_n(x)-P_{n+1}(x)P_n^{'}(x) > 0, \tag{2.2.10}$$

which is useful in investigating the zeros of orthogonal polynomials (cf. [132, Theorem 3.2.2]).

As a consequence of (2.2.10), the polynomials P_n and P_{n+1} cannot have common zeros. Furthermore, if $x_{n,1} < x_{n,k} < \ldots < x_{n,n}$ are the zeros of P_n, the following interlacing property is satisfied.

Theorem 2.2.3. *[35, Theorem 5.3]. For the zeros of φ_n and φ_{n+1}, we have*

$$x_{n+1,1} < x_{n,1} < x_{n+1,2} < \ldots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1}.$$

2.3 Some special functions

The following results on basic special functions will be used in the thesis.

2.3.1 The Gamma function

Definition 2.3.1. *[2, Chapter 6]. The Pochhammer symbol $(b)_n$ is defined as*

$$(b)_0 = 1, \quad (b)_n = b(b+1)\ldots(b+n-1) = \frac{\Gamma(b+n)}{\Gamma(b)}, \tag{2.3.1}$$

for any $b \in \mathbb{C}$*, where* $\Gamma(z)$ *denotes the Gamma function that can be defined as a definite integral*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt = 2 \int_{-\infty}^{\infty} t^{2z-1} e^{-t^2}\, dt, \tag{2.3.2}$$

with $\mathrm{Re}(z) > 0$.

The Gamma function Γ is continuous and differentiable on $(0, \infty)$ and satisfies the recursion formula

$$\Gamma(z+1) = z\Gamma(z), \tag{2.3.3}$$

which implies that $\Gamma(n+1) = n!$ and we also have that $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Moreover,

$$(1+b)_n = \frac{\Gamma(1+b+n)}{\Gamma(1+b)}.$$

2.3.2 Parabolic cylinder functions

Whittaker and Watson [136, p. 347] define the parabolic cylinder functions $D_\nu(z)$ as solutions to the Weber differential equation

$$\psi''(z) + \left(c + bz + az^2\right) \psi(z) = 0.$$

(i) The parabolic cylinder functions ψ have three distinct standard forms (cf. [117, §12.2 (i)]),

$$\frac{d^2 U(-a)}{dz^2} - \left(\frac{1}{4} z^2 + a\right) U(-a) = 0, \tag{2.3.4a}$$

$$\frac{d^2 W(-a)}{dz^2} - \left(\frac{1}{4} z^2 - a\right) W(a) = 0, \tag{2.3.4b}$$

$$\frac{d^2 D_\nu}{dz^2} - \left(\frac{1}{4} z^2 - \nu - \frac{1}{2}\right) D_\nu = 0. \tag{2.3.4c}$$

Equations (2.3.4a)-(2.3.4c) can be transformed into each other and the solutions of these equations are entire functions of z, a and ν. The form which we will use in this thesis is D_ν given in (2.3.4c), where

$$U(a, z) = D_{-a-\frac{1}{2}}(z).$$

(ii) The parabolic cylinder function $D_\nu(\xi)$, with $\nu \notin \mathbb{Z}$, has an integral representation (cf. [117, §12.2 (i)]),

$$D_\nu(\xi) = \frac{\exp(-\frac{1}{4}\xi^2)}{\Gamma(-\nu)} \int_0^\infty s^{-\nu-1} \exp\left(-\tfrac{1}{2}s^2 - \xi s\right) ds, \quad \mathrm{Re}(\nu) < 0. \tag{2.3.5}$$

(iii) The asymptotic formula for parabolic cylinder function $D_{-a}(x)$ is given by

$$D_{-a}(x) \sim x^{-a} \exp\left(-\frac{x^2}{4}\right) \left[1 + \mathcal{O}(x^{-2})\right],$$

when $x \to \infty$.

Definition 2.3.2. *The parabolic cylinder function D_ν has the following connections to Hermite polynomials [117, §12.7(i)]:*

$$\begin{aligned}
U\left(-\frac{1}{2}, z\right) &= D_0(z) = \exp\left(\frac{z^2}{4}\right), \\
U\left(n - \frac{1}{2}, z\right) &= D_n(z) = \exp\left(-\frac{z^2}{4}\right) He_n(z) = 2^{-\frac{n}{2}} \exp\left(-\frac{z^2}{4}\right) H_n\left(\frac{z}{\sqrt{2}}\right), \\
U\left(n + \frac{1}{2}, z\right) &= \sqrt{\frac{2}{\pi}}(-i)^n He_n(iz) = \sqrt{\frac{2}{\pi}} \exp\left(\frac{z^2}{4}\right) (-i)^n 2^{-\frac{n}{2}} H_n\left(\frac{iz}{\sqrt{2}}\right),
\end{aligned}$$

where $He_n(z)$ denotes polynomials orthogonal with respect to the modified Hermite weight $\exp\left(-\frac{1}{2}x^2\right)$ *and $H_n(z)$ represents the Hermite polynomials.*

2.3.3 The error and complementary error functions

Gauss' error function can be defined as the integral of the Gauss density function

$$\mathrm{erf}\, x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

and has properties [117, Chapter 7]

$$\begin{aligned}
\mathrm{erf}(-\infty) &= -1, & \mathrm{erf}(+\infty) &= 1, \\
\mathrm{erf}(-x) &= -\mathrm{erf}(x), & \mathrm{erf}(x^*) &= [\mathrm{erf}(x)]^*,
\end{aligned}$$

where the asterisk denotes complex conjugation. The complementary error function is defined as

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \mathrm{erf}(x). \tag{2.3.7}$$

We note also that

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-t^2}\, dt = 1 + \mathrm{erf}(x).$$

2.4 Classical orthogonal polynomials

Jacobi, Laguerre and Hermite polynomials are considered to be classical orthogonal polynomials and their weight functions $w(x)$ satisfy *Pearson's differential equation* (1.1.2), where $\sigma(x)$ is a monic polynomial with $\deg(\sigma) \leq 2$ and $\tau(x)$ is a polynomial with $\deg(\tau) = 1$ and these functions are given in Table 2.1.

φ_n	$w(x)$	$\sigma(x)$	$\tau(x)$	Interval
Hermite	$\exp(-x^2)$	1	$-2x$	$\mathbb{R}$
Laguerre	$x^\alpha \exp(-x),\ \alpha > -1$	x	$1+\alpha-x$	$\mathbb{R}^+$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$	$[-1,1]$

Table 2.1: Classical orthogonal polynomials

Furthermore, the classical orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ also satisfy

(a) a Rodrigues formula

$$P_n(x) = \frac{1}{K_n\ w(x)} \mathscr{D}^n[w(x)\sigma^n(x)], \quad n = 0, 1, 2, \ldots,$$

where $w(x)$ is a function which is non-negative on an interval, $\sigma(x)$ is a polynomial in x independent of n and K_n does not depend on x;

(b) a non-linear equation of the form [71, Theorem 20.5.7]

$$\frac{d}{dx}[P_n(x)P_{n-1}(x)] = (a_n x + b_n)\ P_n(x)\ P_{n-1}(x) + \gamma_n\ P_n^2(x) + \delta_n\ P_{n-1}^2(x),$$

where $\{a_n\}, \{b_n\},\ \{\gamma_n\}$ and $\{\delta_n\}$ are sequences of constants;

(c) a second-order linear differential equation

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0, \tag{2.4.1}$$

where λ_n is independent of x and the polynomials $\sigma(x)$ and $\tau(x)$ are given in (1.1.2) ;

(d) a differential-difference relation

$$\sigma(x)P_n'(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x). \tag{2.4.2}$$

where $\sigma(x)$ is given in (1.1.2) and $\{a_n\}, \{b_n\},\ \{c_n\}$ are sequences of constants.

Note that derivatives of classical orthogonal polynomials also form an orthogonal polynomial set (see [71, p. 527]). A polynomial set that satisfies any of the above properties is necessarily a classical orthogonal polynomial set. In particular, Al-Salam and Chihara [4] showed that an orthogonal polynomial set satisfying (2.4.2) is either Hermite, Laguerre or Jacobi polynomials depending on the degree of σ being $0, 1$ or 2 respectively. Since we will refer to the Hermite and Laguerre polynomials later in the thesis, we discuss some of their properties.

2.4.1 Laguerre polynomials

The Laguerre polynomials appear in quantum mechanics as the radial part of the solution of the Schrödinger equation for the hydrogen atom [74]. For $\alpha > -1$, the classical monic Laguerre polynomials $\{\widetilde{L}_n^{(\alpha)}\}_{n=0}^{\infty}$ can be defined (cf. [35, p. 145]) as

$$\widetilde{L}_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^{n+j} n! \binom{n+\alpha}{n-j} \frac{x^j}{j!}, \quad n \in \mathbb{N}_0,$$

and they satisfy the orthogonality relation

$$\langle \widetilde{L}_m^{(\alpha)}, \widetilde{L}_n^{(\alpha)} \rangle_{x^\alpha \exp(-x)} = \int_0^\infty \widetilde{L}_m^{(\alpha)}(x) \widetilde{L}_n^{(\alpha)}(x)\ x^\alpha \exp(-x)\, dx = h_n \delta_{mn}. \tag{2.4.3}$$

The monic normalization constant h_n in (2.4.3) is given by (cf. [35, 132])

$$h_n = \langle \widetilde{L}_n^{(\alpha)}, \widetilde{L}_n^{(\alpha)} \rangle_{x^\alpha \exp(-x)} = \|\widetilde{L}_n^{(\alpha)}\|^2_{x^\alpha \exp(-x)} = n!\ \Gamma(n+\alpha+1), \tag{2.4.4}$$

where Γ is the Gamma function defined by (2.3.2). The structural properties of Laguerre polynomials will be used in the sequel (cf. [35, 68, 92]).

Proposition 2.4.1. *[132, Section 5.5]. Let* $\{\widetilde{L}_n^{(\alpha)}\}_{n=0}^{\infty}$, $\alpha > -1$ *be a sequence of monic Laguerre polynomials. Then the following statements hold for every* $n \in \mathbb{N}$*:*

(i)

$$x\widetilde{L}_n^{(\alpha)}(x) = \widetilde{L}_{n+1}^{(\alpha)}(x) + (2n+\alpha+1)\widetilde{L}_n^{(\alpha)}(x) + n(n+\alpha)\widetilde{L}_{n-1}^{(\alpha)}(x),$$

$$\text{where} \quad \widetilde{L}_0^{(\alpha)} \equiv 1, \quad \widetilde{L}_1^{(\alpha)}(x) = x - (\alpha+1).$$

(ii)

$$\widetilde{L}_n^{(\alpha)}(x) = \widetilde{L}_n^{(\alpha+1)}(x) + n\widetilde{L}_{n-1}^{(\alpha+1)}(x).$$

(iii)

$$\widetilde{L}_n^{(\alpha)}(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{2.4.5}$$

(iv)

$$\left(\widetilde{L}_n^{(\alpha)}\right)'(x) = -\widetilde{L}_{n-1}^{(\alpha+1)}(x).$$

(v) $\widetilde{L}_n^{(\alpha)}(x)$ *is the polynomial eigenfunction of the differential operator*

$$x\mathscr{D}^2 + (\alpha + 1 - x)\mathscr{D}$$

with -n as the corresponding eigenvalue.

(vi) we have the lowering and raising operators.

$$x[\widetilde{L}_n^{(\alpha)}]'(x) - n\widetilde{L}_n^{(\alpha)}(x) = n(n+\alpha)\widetilde{L}_{n-1}^{(\alpha)}(x) \quad \textit{(lowering)},$$
$$x[\widetilde{L}_{n-1}^{(\alpha)}]'(x) + (n+\alpha-x)\widetilde{L}_{n-1}^{(\alpha)}(x) = -\widetilde{L}_n^{(\alpha)}(x) \quad \textit{(raising)}.$$

2.4.2 Hermite polynomials

Hermite polynomials arise in probability theory such as the Edgeworth series [81], in numerical analysis as Gaussian quadrature [63] and in physics, where they give rise to the eigenstates of the quantum harmonic oscillator (cf. [63, 83]). Monic Hermite polynomials can be defined (cf. [35, p. 146]) as

$$\widetilde{H}_n(x) = 2^{-n} n! \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{j!(n-2j)!}(2x)^{n-2j},$$

where $\lfloor x \rfloor$ denotes the greatest integer function and they satisfy the orthogonality relation

$$\langle \widetilde{H}_m, \widetilde{H}_n \rangle_{\exp(-x^2)} = \int_{-\infty}^{+\infty} \widetilde{H}_m(x)\ \widetilde{H}_n(x)\ \exp(-x^2)\,dx = K_n \delta_{mn},$$

with its normalization constant K_n given by

$$K_n = \langle \widetilde{H}_n, \widetilde{H}_n \rangle_{\exp(-x^2)} = \|\widetilde{H}_n\|^2_{\exp(-x^2)} = \frac{n!\sqrt{\pi}}{2^n},$$

and the three-term recurrence relation [81, Equation (1.13.4)] is

$$\widetilde{H}_{n+1}(x) = x\widetilde{H}_n(x) - \frac{n}{2}\widetilde{H}_{n-1}(x), \quad n \in \mathbb{N}, \tag{2.4.6}$$

with initial conditions

$$\widetilde{H}_0 \equiv 1, \quad \widetilde{H}_1(x) = x.$$

Hankel determinants (2.2.5) for monic Hermite polynomials take the following explicit form:

$$\Delta_n = \left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}} \prod_{k=1}^{n-1} k!, \quad \tilde{\Delta}_n = 0, \quad n \in \mathbb{N},$$

where the moments are given by

$$\left.\begin{aligned} \mu_{2k} &= \int_{\mathbb{R}} x^{2k} \exp(-x^2)\, dx = \frac{\sqrt{\pi}(2k)!}{2^{2k}k!} \\ \mu_{2k+1} &= \int_{\mathbb{R}} x^{2k+1} \exp(-x^2)\, dx = 0 \end{aligned}\right\}. \tag{2.4.7}$$

Hermite polynomials satisfy the symmetry condition $\widetilde{H}_n(-x) = (-1)^n \widetilde{H}_n(x)$ and the recurrence coefficients of monic Hermite polynomials $\{\widetilde{H}_n\}_{n=0}^{\infty}$ satisfying (2.4.6) are obtained, using (2.4.7), as $\alpha_n = 0$ and

$$\beta_n = \frac{\left[\left(\frac{1}{2}\right)^{\frac{n(n+1)}{2}} \prod_{k=1}^{n} k!\right]\left[\left(\frac{1}{2}\right)^{\frac{(n-1)(n-2)}{2}} \prod_{k=1}^{n-2} k!\right]}{\left[\left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}} \prod_{k=1}^{n-1} k!\right]^2} = \frac{n}{2}.$$

Monic Hermite polynomials $\{\widetilde{H}_n\}_{n=0}^{\infty}$ are expressed in terms of monic Laguerre polynomials $\{\widetilde{L}_n^{(\alpha)}\}_{n=0}^{\infty}$ with parameters $\alpha = \pm\frac{1}{2}$ by means of the formulas [132, p. 106]

$$\widetilde{H}_{2n}(x) = \widetilde{L}_n^{(-\frac{1}{2})}(x^2); \quad \widetilde{H}_{2n+1}(x) = x\widetilde{L}_n^{(\frac{1}{2})}(x^2), \; n \in \mathbb{N}_0.$$

We also note that Hermite polynomials (and their Freud-weight analogs [117, §18.32]) play an essential role in random matrix theory [60, 117] and also [45, Chapter 5].

Remark 2.4.1. *We note that the respective sequence of classical orthogonal (Hermite, Laguerre or other classical) polynomials, $\{P_n\}_{n=0}^{\infty}$ represents an orthogonal basis in a Hilbert space of the type $\mathcal{H} = \mathrm{L}^2(I, w(x)dx)$ where $I \subset \mathbb{R}$ is an open interval and the weight $w(x) > 0$ is a continuous function on the inteval I.*

2.5 Quasi-orthogonality

In this section we give the definition of quasi-orthogonality of orthogonal polynomials on the real line.

Definition 2.5.1. *[23, Definition 1]. Let R_n be a polynomial of exact degree $n \geq r$. If R_n satisfies the conditions*

$$\int_a^b x^k R_n(x) w(x)\, dx = \begin{cases} 0, & \text{for} \quad k = 0, 1, \ldots, n-r-1, \\ \neq 0, & \text{for} \quad k = n-r. \end{cases} \tag{2.5.1}$$

where w is a positive weight function on $[a, b] \subseteq \mathbb{R}$, then R_n is quasi-orthogonal of order r on $\mathbb{R}$ with respect to w.

Remark 2.5.1. *The quasi-orthogonal polynomials R_n are only defined for $n \geq r$. Thus, equation* (2.5.1) *is equivalent to*

$$\int_a^b R_k(x)\ R_n(x)\ w(x)\, dx = 0, \quad \text{for} \quad k = n-r, \ldots, n+r.$$

When $r = 0$, the usual orthogonality conditions which completely determine R_n (up to a normalization factor) are recovered. If $r > 1$, the polynomials R_n are not uniquely determined by (2.5.1).

Theorem 2.5.1. *[23, Theorem 1]. Let $\{R_n\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials on $[a, b]$ with respect to a positive weight function w. The polynomial*

$$R_n(x) = P_n(x) + c_1 P_1(x) + \ldots + c_r P_{n-r}(x),$$

where the c_j's are numbers which depend on n and $c_r \neq 0$, is quasi-orthogonal of order r on $[a, b]$ with respect to w.

For more information about quasi-orthogonality, consult [23, 34, 77].

2.6 Symmetric orthogonal polynomials

Polynomials orthogonal with respect to an even weight function (i.e, $w(-x) = w(x)$) on a support interval $[-b, b]$ for any given $b \in \mathbb{R}^+ \cup \{\infty\}$ are called symmetric.

The following are properties of symmetric orthogonal polynomials $\{S_n\}_{n=0}^{\infty}$ on the real line (cf. [35, 63, 97, 113, 132]):

(i) Symmetry

$$S_n(-x) = (-1)^n S_n(x), \quad n = 0, 1, 2, \dots, \tag{2.6.1}$$

(ii) Moments

$$\left.\begin{aligned} &\mu_{2k+1} = 0, \quad k \in \mathbb{N}_0, \\ &\mu_{2k} = \int_{-b}^{b} x^{2k} w(x)dx = 2\int_{0}^{b} x^{2k} w(x)dx > 0, \end{aligned}\right\} \tag{2.6.2}$$

(iii) The substitution $x \to -x$ in the recurrence relation (2.2.1) yields that $\alpha_n = 0$ and we have

$$xS_n(x) = S_{n+1}(x) + \beta_n S_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{2.6.3}$$

with initial conditions $S_0 \equiv 1$ and $S_1(x) = x$.

The polynomial $S_n(x)$ in (2.6.3) contains even powers of x when n is even and odd powers of x when n is odd. More precisely, our convention in the case of even weight functions is to write $S_n(x)$ as

$$S_n(x) = x^n + c_{n,n-2}x^{n-2} + c_{n,n-4}x^{n-4} + c_{n,n-6}x^{n-6} + \cdots + S_n(0).$$

2.6.1 Symmetrization and quadratic decomposition

In the sequel, the symmetry property will play a crucial role to construct symmetric orthogonal polynomial sequences based on quadratic transformations [35]. Numerous researchers have dealt with symmetrization problems of orthogonal polynomial sequences on the real line, where a symmetric orthogonal polynomial sequence is decomposed into two nonsymmetric sequences (cf. [32, 35, 94]).

For a symmetric orthogonal sequence $\{S_n\}_{n=0}^{\infty}$ with weight function $w_s(x)$, we have

$$\int_{-b}^{b} S_n(x)S_m(x)w_s(x)\,dx = k_n\delta_{mn}, \quad k_n > 0. \tag{2.6.4}$$

Since $S_n(-x) = (-1)^n S_n(x)$, we may write

$$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = xQ_n(x^2), \tag{2.6.5}$$

to define the polynomials $P_n(x)$ and $Q_n(x)$. Since the integrand is even, (2.6.4) may be replaced by

$$2\int_0^b S_n(x)S_m(x)w_s(x)\,dx = k_n\delta_{mn}, \tag{2.6.6}$$

and hence, by (2.6.5),

$$2\int_0^b P_n(x^2)P_m(x^2)w_s(x)\,dx = k_{2n}\delta_{mn},$$

so that

$$\int_0^{b^2} P_n(x)P_m(x)x^{-\frac{1}{2}}w_s(x^{-\frac{1}{2}})\,dx = k_{2n}\delta_{mn}.$$

Similarly, if we replace m and n in (2.6.6) by $2m+1$ and $2n+1$, respectively, we obtain

$$\int_0^{b^2} Q_n(x)Q_m(x)x^{\frac{1}{2}}w_s(x^{\frac{1}{2}})\,dx = k_{2n+1}\delta_{mn}.$$

The following theorem shows that a quadratic transformation of the symmetric sequence results in two component sequences that are themselves orthogonal polynomial sequences.

Theorem 2.6.1. *[63, Theorem 1.18]. Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials over the interval $(-b, b)$, $0 \le b \le \infty$ with weight function $w_s(x)$ such that $S_n(-x) = (-1)^n S_n(x)$. Then the sequences of polynomials $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$, uniquely determined by (2.6.5), are orthogonal on $[0, b^2]$ with weight functions $x^{-\frac{1}{2}}\, w_s(x^{\frac{1}{2}})$ and $x^{\frac{1}{2}}\, w_s(x^{\frac{1}{2}})$ respectively.*

As an application of Theorem 2.6.1, consider symmetric Legendre polynomials $G_n(x)$ which are orthogonal on $[-1, 1]$ with weight function $w_s(x) = 1$. We may put

$$G_{2n}(x) = P_n(x^2), \quad G_{2n+1}(x) = xQ_n(x^2).$$

Then $P_n(x)$ is orthogonal on $(0, 1)$ with weight function $x^{-\frac{1}{2}}$, while $Q_n(x)$ is orthogonal on $(0, 1)$ with weight function $x^{\frac{1}{2}}$.

For more on symmetrization of orthogonal polynomials, we refer the reader to [63, 97, 104].

2.6.2 Construction of a symmetric orthogonal sequence

Given two sequences of orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ and let

$$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = xQ_n(x^2); \tag{2.6.7}$$

A natural question to ask here is under what conditions the sequence $\{S_n\}_{n=0}^{\infty}$ will be orthogonal?

This question coincides with the converse of Theorem 2.6.1 and it is answered as follows: Assume that

$$\int_0^{b^2} P_m(x)\, P_n(x)\, w_1(x)\ dx = \tilde{k}_n\, \delta_{mn}, \quad \tilde{k}_n > 0. \tag{2.6.8}$$

Then, if there exist sequences of polynomials $\{Q_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ such that (2.6.5) and (2.6.6) are satisfied and

$$\int_0^{b^2} Q_m(x)Q_n(x)w_2(x)\, dx = \hat{k}_n\delta_{mn}, \quad \hat{k}_n > 0, \tag{2.6.9}$$

it follows from Theorem 2.6.1 that

$$w_2(x) = xw_1(x), \quad w_s(x) = xw_1(x^2). \tag{2.6.10}$$

Theorem 2.6.2. *[24, Theorem 3]. Let $\{P_n\}_{n=0}^{\infty}, \{Q_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ be sequences of polynomials that satisfy* (2.6.6), (2.6.8) *and* (2.6.9)*, respectively. Then the polynomials satisfy* (2.6.5) *if and only if the weight functions satisfy* (2.6.10)*. Besides, if any one of these sequences is given, the other two sequences are uniquely determined.*

Proposition 2.6.1. *[35, p. 43]. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials with respect to a positive weight function w supported on $[0, b^2]$, $b \in \mathbb{R}$; i.e.,*

$$\int_0^{b^2} P_m(x)\, P_n(x)\, w(x)\ dx = h_n\, \delta_{mn}.$$

Then the symmetrized form of $\{P_n\}_{n=0}^{\infty}$, i.e., $\{S_n\}_{n=0}^{\infty}$, defined by (2.6.7)*, satisfies the orthogonality condition*

$$\int_{-b}^{b} S_m(x)\, S_n(x)\, |x|\, w(x^2)\ dx = h_n\, \delta_{mn},$$

where h_n is the normalization constant defined in (2.1.3).

As an example, the symmetrization of Laguerre polynomials gives rise to a class of generalized Hermite polynomials (cf. [33, 35, 132]). Generalized Hermite polynomials $\{\mathsf{H}_n^{(\gamma)}\}_{n=0}^{\infty}$ were introduced by Szegö [132, Problem 23] and they satisfy the differential equation (cf. [35])

$$xy''(x) + 2(\gamma - x)y' + (2nx - \theta_n x^{-1})y = 0, \quad y = \mathsf{H}_n^{(\gamma)}(x),$$

where $\theta_{2m} = 0,\ \theta_{2m+1} = 2\gamma$.

Proposition 2.6.2. *[35, p. 157]. The generalized Hermite polynomials* $\{H_n^{(\gamma)}\}_{n=0}^{\infty}$ *defined by*

$$\left.\begin{aligned} \mathsf{H}_{2n}^{(\gamma)}(x) &= (-1)^n\ 2^{2n}\ n!\ L_n^{(\gamma-\frac{1}{2})}(x^2); \\ \mathsf{H}_{2n+1}^{(\gamma)}(x) &= (-1)^n\ 2^{2n+1}\ n!\ xL_n^{(\gamma+\frac{1}{2})}(x^2), \quad n \in \mathbb{N}_0, \end{aligned}\right\} \tag{2.6.11}$$

are orthogonal polynomials corresponding to the weight $w(x) = |x|^{2\gamma}\exp(-x^2)$, *with the parameter* $\gamma > -\frac{1}{2}$ *and* $x \in \mathbb{R}$.

The orthogonality relation of generalized Hermite polynomials is given by

$$\int_{-\infty}^{+\infty} \mathsf{H}_m^{(\gamma)}(x)\mathsf{H}_n^{(\gamma)}(x)|x|^{2\gamma}\exp(-x^2)\,dx = 2^{2n}\ \left[\frac{n}{2}\right]!\ \Gamma\left(\lfloor\frac{n+1}{2}\rfloor + \gamma + \frac{1}{2}\right)\delta_{mn},$$

where $\lfloor x \rfloor$ is the floor function.

Remark 2.6.1. *The technique of symmetrization and quadratic transformation as discussed in this section will be applied to semiclassical weights in Chapter 3 where we use the symmetrization of semiclassical Laguerre polynomials to generate generalized Freud polynomials.*

2.7 Semiclassical orthogonal polynomials

In this section we introduce basic concepts of semiclassical orthogonal polynomials on the real line. These polynomials are orthogonal with respect to a positive weight function w such that $\frac{d}{dx}\ln w(x)$ is a rational function.

A weight function $w(x)$ defined over a bounded or unbounded interval (a, b) can be characterized as *semiclassical* [67, 94] if and only if it satisfies the Pearson differential equation (1.1.2) with the boundary conditions

$$\lim_{x\to a}\tau(x)\ w(x)\ p(x) = 0 = \lim_{x\to b}\tau(x)\ w(x)\ p(x)$$

for every polynomial $p(x)$. The polynomials associated with such a weight w are called *semiclassical orthogonal polynomials*.

The theory of semiclassical orthogonal polynomials is not yet fully developed, but the derivation of a differential relation for a general class of orthogonal polynomials by Shohat [127] provides bases for forming classes of semiclassical orthogonal polynomials. Semiclassical orthogonal polynomials have applications in matrix models [52, 128, 133], soliton theory [7, 128]), random matrices [45] and also in the study of integrable systems [26, 46, 65].

As early as 1929, Bochner [18] solved the problem of determining all families of scalar-valued orthogonal polynomials that are eigenfunctions of some fixed second-order linear differential operator. He identified the fact that classical orthogonal polynomials φ_n are solutions of

$$B[y](x) := \sigma(x)y''(x) - \tau(x)y'(x) = \lambda_n y(x), \quad n = 0, 1, 2, \ldots, \tag{2.7.1}$$

where σ is a polynomial of degree at most 2, τ a polynomial of degree 1 and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of nonzero numbers and $\lambda_0 = 0$. The differential equation (2.7.1) is commonly known as Bochner's differential equation and the associated differential operator as Bochner's operator [71, p. 508]. Equation (2.7.1) is also known as the Sturm-Louiville differential equation. As a consequence of (2.7.1), the weights of classical orthogonal polynomials satisfy the Pearson differential equation given in (1.1.2). However when $\deg(\sigma) > 2$ or $\deg(\tau) > 1$, the weight function produces a class of semiclassical orthogonal polynomials.

Table 2.2 presents examples of semiclassical orthogonal polynomials with their respective weight functions and intervals of orthogonality.

	$w(x;t)$	Parameters	$\sigma(x)$	$\tau(x)$	Interval
Semiclassical Laguerre	$x^{\lambda}\exp(-x^2+tx)$	$\lambda > -1,\ t \in \mathbb{R}$	x	$1+\lambda+tx-2x^2$	$(0,\infty)$
Semiclassical Freud	$\exp(-\frac{1}{4}x^4 - tx^2)$	$t \in \mathbb{R}$	1	$-2tx - x^3$	$(-\infty,\infty)$
Generalized Freud	$\lvert x\rvert^{2\lambda+1}\exp(-x^4+2tx^2)$	$\lambda > 0,\ t \in \mathbb{R}$	x	$2\lambda+2-2tx^2-x^4$	$(-\infty,\infty)$
Semiclassical Airy	$\exp(-\frac{1}{3}x^3+tx)$	$t > 0$	1	$t - x^2$	$(0,\infty)$

Table 2.2: Semiclassical orthogonal polynomials

In 1972 Al-Salam and Chihara [4] showed that polynomials satisfying the differential-difference relation given in (2.4.2) must be either Hermite, Laguerre or Jacobi polynomials. Askey raised the more general question of which orthogonal polynomial sets $\{P_n\}_{n=0}^{\infty}$ have the property that their derivatives satisfy

$$\pi(x)P_n'(x) = \sum_{k=n-t}^{n+s} \alpha_{nk}P_k(x),$$

where $\pi(x)$ is a polynomial and s and t are constants. This problem was solved by Shohat and later independently by Freud [55] and Bonan and Nevai [22].

Maroni [94] stated the problem in a different way, trying to find all orthogonal polynomial sets whose derivatives are quasi-orthogonal and he called these orthogonal polynomial sets semiclassical .

Semiclassical orthogonal polynomials in one variable can be characterized as the only sequences of orthogonal polynomials satisfying one of the following equivalent properties: some special differential-difference equation (the so-called structural relation [94]), the quasi-orthogonality of the derivatives [127] and a second order partial differential-difference relation [13, 67, 71, 94]. Monic semiclassical polynomials $\{P_n\}_{n=0}^{\infty}$ of class s can be characterized by the following structural relation [67, 94, 121]

$$\sigma(x)P_{n+1}'(x) = \sum_{j=n-s}^{n+r} A_{n,j}P_j(x), \quad n \geq s+1, \tag{2.7.2}$$

where

$$s = \max\{\deg(\sigma)-2, \deg(\tau)-1\}, \quad r = \deg(\sigma), \tag{2.7.3}$$

with the polynomials σ and τ defined by (1.1.2).

There is a strong link between semiclassical orthogonal polynomials and Painlevé equations.

2.8 Painlevé equations

The Painlevé equations are second-order ordinary differential equations described by six families $\mathrm{P_I}$ - $\mathrm{P_{VI}}$. $\mathrm{P_I}$ consists of the single equation

$$\frac{d^2q}{dz^2} = 6q^2 + z \tag{2.8.1}$$

while P_{II} - P_{VI} have complex parameters α, β, γ and δ,

$$\begin{aligned}
P_{II}(\alpha) : \frac{d^2q}{dz^2} &= 2q^3 + zq + \alpha, \qquad (2.8.2)\\
P_{III}(\alpha,\beta,\gamma,\delta) : \frac{d^2q}{dz^2} &= \frac{1}{q}\left(\frac{dq}{dz}\right)^2 - \frac{1}{z}\frac{dq}{dz} + \frac{1}{z}(\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q},\\
P_{IV}(\alpha,\beta) : \frac{d^2q}{dz^2} &= \frac{1}{2q}\left(\frac{dq}{dz}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-\alpha)q + \frac{\beta}{q},\\
P_{V}(\alpha,\beta,\gamma,\delta) : \frac{d^2q}{dz^2} &= \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dz}\right)^2 - \frac{1}{z}\frac{dq}{dz} + \frac{(q-1)^2}{z^2}\left(\alpha q + \frac{\beta}{q}\right) + \gamma\frac{q}{z} + \delta\frac{q(q+1)}{q-1},\\
P_{VI}(\alpha,\beta,\gamma,\delta) : \frac{d^2q}{dz^2} &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{dq}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\left(\frac{dq}{dz}\right)\\
&\quad + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left[\alpha - \beta\frac{z}{q^2} + \gamma\frac{z-1}{(q-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{z(z-1)}{(q-z)^2}\right],
\end{aligned}$$

where q and z are complex variables. They were investigated in the early part of the 20^{th} century by Painlevé, with refinements by Gambier and Fuchs [37, 61] as ordinary differential equations of the form

$$q''(z) = F\left(z, q(z), q'(z)\right), \qquad (2.8.3)$$

where $F\left(z, q(z), q'(z)\right)$ is a rational function in q and $q^{'}$ and analytic in z, having the property that the solutions have no movable branch points, i.e., the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the Painlevé property. Painlevé et al. [61] showed that there were 50 canonical ordinary differential equations satisfying the Painlevé property and these were referred to as equations of Painlevé type. Out of these fifty equations, forty-four of them are either integrable in terms of previously known functions (such as elliptic functions or functions that are equivalent to linear equations) or reducible to one of the six nonlinear ordinary differential equations, P_I - P_{VI}, which define new transcendental functions, see (cf. [69, 136]). The Painlevé equations may be thought of as nonlinear analogs of the classical special functions (i.e., Airy, Bessel, Whittaker, Kummer, hypergeometric functions), see [37]. Their general solutions are transcendental, i.e., irreducible in the sense that they cannot be expressed in terms of previously known functions such as rational functions, exponential functions or the classical special functions. For more details, we refer to [64, 66, 69, 75, 136].

The following are some of the properties of Painlevé equations (cf. [37, 42, 66, 115]).

- Each Painlevé equation can be written as a (non-autonomous) *Hamiltonian system* [115].

- $\mathrm{P_{II}}$ - $\mathrm{P_{VI}}$ have rational, algebraic and special function solutions expressed in terms of the classical special functions called *classical* solutions [37, 66], e.g., for $\mathrm{P_{II}}$: Airy $Ai(z), Bi(z)$; $\mathrm{P_{III}}$: Bessel $J_\nu(z), Y_\nu(z)$; $\mathrm{P_{IV}}$: parabolic cylinder $D_\nu(z)$ (see Subsection 2.3.5 for the details of these special functions). These classical solutions can usually be written as *Wronskians* and are often given in terms of *Hankel* determinants or *Toeplitz* determinants.

- Each Painlevé equation possesses a Bäcklund transformation [1]. A Bäcklund transformation is defined as being a system of equations relating one solution of a given equation (in this case $\mathrm{P_{III}}$) either to another solution of the same equation, possibly with different values of the parameters, or to a solution of another equation.

- Each Painlevé equation can be expressed as the compatibility condition of a linear system which is known as an isomonodromy problem or Lax pair [37, 75].

The search for discrete analogues of Painlevé transcendents has been an open problem for many years and only recently has progress was made in this direction. Discretizations of the Painlevé equations have resulted from a variety of methods including applications of orthogonal polynomials [51, 90].

2.8.1 Discrete Painlevé equations

Discrete Painlevé equations are the discrete analogs of Painlevé equations. These equations are not only second-order, nonlinear difference equations which have a continuous Painlevé equation as a continuous limit but they also are mappings that are integrable in the same sense as the continuous Painlevé equation. Grammaticos, Ramani and Papageorgiou [65] suggested the singularity confinement method as an integrability test for discrete equations. This integrability detector is the discrete analog of the Painlevé property for differential equations. The discrete Painlevé equations have the form

$$x_{n+1} = \frac{g_s(x_n;n) + x_{n-1}g_t(x_n;n)}{g_u(x_n;n) + x_{n-1}g_v(x_n;n)},$$

where $g_i(x_n;n)$ is a polynomial of degree i in x_n where $s,t,u,v \in \{0,1,2,3,4\}$. Some examples of discrete Painlevé equations are

$$\begin{aligned}
&(\mathrm{dP_I}) \quad x_{n+1}+x_n+x_{n-1} = \frac{z_n+\gamma(-1)^n}{x_n}+\sigma, \\
&(\mathrm{dP_{II}}) \quad x_{n+1}+x_{n-1} = \frac{x_n z_n+\gamma}{1-x_n^2}, \\
&(\mathrm{dP_{IV}}) \quad (x_{n+1}+x_n)(x_n+x_{n-1}) = \frac{\left(x_n^2-\kappa^2\right)\left(x_n^2-\mu^2\right)}{(x_n+z_n)^2-\gamma^2}, \\
&(\mathrm{dP_V}) \quad \frac{(x_{n+1}+x_n-z_{n+1}-z_n)(x_n+x_{n-1}-z_n-z_{n-1})}{(x_{n+1}+x_n)(x_n+x_{n-1})} \\
&\qquad\qquad = \frac{\left((x_n-z_n)^2-\alpha^2\right)\left((x_n-z_n)^2-\beta^2\right)}{(x_n-\gamma^2)(x_n-\sigma^2)}.
\end{aligned} \tag{2.8.4}$$

where $z_n = \alpha n+\beta$ and $\kappa,\gamma,\beta,\alpha$ and σ are constants (cf. [53, 66, 69, 134]).

2.8.2 Semiclassical weights and discrete Painlevé equations

We examine the connection between recurrence coefficients of the three-term recurrence relation satisfied by semiclassical orthogonal polynomials and the Painlevé equations.

Recurrence coefficients associated with semiclassical orthogonal polynomials satisfy both linear and nonlinear recurrence relations. The relationship between semiclassical orthogonal polynomials and integrable equations dates back to the work of Shohat [127], Freud [55] and Bonan and Nevai [22]. However, it was not until the work of Fokas, Its and Kitaev (cf. [51, 52]) that these equations were identified as discrete Painlevé equations. The discrete Painlevé equations appear in the form of the nonlinear difference relations satisfied by the relevant recurrence coefficients.

Magnus [90] applied ladder operators to nonclassical orthogonal polynomials associated with random matrix theory and the derivation of Painlevé equations, while Tracy and Widom [133] used the associated compatibility conditions in the study of finite n matrix models. Magnus [90] discussed the relationship between semiclassical Freud polynomials

$$w(x;t) = \exp\left(-x^4+tx^2\right), \; x\in\mathbb{R}, \tag{2.8.5}$$

and the Painlevé equations. He showed in [90] that the recurrence coefficients associated with the weight (2.8.5) can be expressed in terms of simultaneous solutions of

(i) the discrete equation

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n, \tag{2.8.6}$$

which is discrete Painlevé I ($\mathrm{dP_I}$), as shown by Bonan and Nevai [22], and

(ii) the differential equation

$$\frac{d^2 q_n}{dz^2} = \frac{1}{2q_n}\left(\frac{dq_n}{dz}\right)^2 + \frac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2 - A)q_n + \frac{B}{q_n}, \tag{2.8.7}$$

which is a special case of the fourth Painlevé equation, where $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$, $n \in \mathbb{Z}^+$. This connection between the recurrence coefficients for the Freud weight (2.8.5) and simultaneous solutions of (2.8.6) and (2.8.7) is due to Kitaev (cf. [51, 52, 53]).

Certain semiclassical orthogonal polynomials have well-established connections with discrete (or Painlevé) integrable systems. Natural questions to ask are

(i) Which semiclassical weights are related to discrete Painlevé equations?

(ii) Which discrete Painlevé equations do we obtain?

We provide some partial answers to the above as follows:

- $w(x) = \exp(\frac{x^3}{3} + tx)$ on $\{x : x^3 < 0\}$ is related to $\mathrm{P_{II}}$ (cf. [90]).
- $w(x) = x^\alpha \exp(-x)\exp(-\frac{s}{x})$ $(\alpha, s > 0)$ on $\mathbb{R}^+$ is related to $\mathrm{P_{III}}$ (cf. [30]).
- The weight $|x - t|^\rho \exp(-x^2)$ is related to $\mathrm{P_{IV}}$ (cf. [27]).
- $w(x) = |x|^\varrho \exp(-x^4)$, $\varrho > -1$ on $\mathbb{R}$ is related to $\mathrm{dP_I}$ (cf. [88]).
- $w(x) = x^\alpha \exp(-x^2)$, $\alpha > -1$ (due to Maxwell/Sonin-type) on $\mathbb{R}^+$ is related to $\mathrm{dP_{IV}}$ (cf. [124]).
- $w(k) = \dfrac{a^k}{(k!)^2}$ (due to Charlier) is related to $\mathrm{dP_{II}}$ (cf. [38, 135]).
- $w(x;t) = x^\alpha \exp(-x^2 + tx)$, $\alpha > -1$ on $\mathbb{R}^+$ is related to $\mathrm{dP_{IV}}$ (cf. [39, 50]).
- $w(x;t) = \exp(-\frac{1}{4}x^4 + tx^2)$ on $x, t \in \mathbb{R}$ is related to $\mathrm{dP_I}$ (cf. [90]).

2.9 Semiclassical Freud-type polynomials

Géza Freud was the pioneer in investigating polynomials orthogonal with respect to certain general exponential weights on the real line (cf. [8, 55, 84, 90, 105, 112]). Freud weights are a class of exponential weights $w(x) = |x|^\rho \exp(-|x|^m)\,dx, \quad \rho > -1,\ x \in \mathbb{R}$. Freud [55] gave the asymptotic behavior of the recurrence coefficients of the three-term recurrence relation for the orthogonal polynomials with the weight functions of the form (1.1.5) for $m = 2, 4, 6$ and he conjectured that such an asymptotic relation is valid for every positive integer m. Freud also investigated some essential properties such as the asymptotics of the largest zeros and the asymptotic behavior of the polynomials themselves (cf. [8, 84, 90, 91, 105, 112, 134]).

The recurrence coefficients associated with the weight of the form (1.1.5) satisfy a non-linear recurrence relation which corresponds to the discrete Painlevé $\mathrm{dP_I}$ equation and its hierarchy and they also satisfy the differential-difference equations of the Toda lattice (cf. [6, 7, 39, 50, 90]). In the following, we discuss and review some known facts about Freud-type weights.

2.9.1 The Freud weight $\exp\left(-\frac{1}{4}x^4 - tx^2\right),\ x, t \in \mathbb{R}$

Relevant properties of orthogonal polynomials $\{P_n(x;t)\}_{n=0}^\infty$ with respect to the semi-classical Freud weight

$$w(x;t) = \exp\left(-\frac{1}{4}x^4 - tx^2\right), \quad x, t \in \mathbb{R}, \tag{2.9.1}$$

were studied in [22, 55, 90]. An important reference for numerous properties about Freud's weight is the paper by Nevai [112] on Freud's mathematical legacy. Properties to be discussed include the higher order moments and Pearson's equation associated with Freud's weight and the recurrence coefficients satisfying the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t), \quad P_0 \equiv 1, \quad P_{-1} \equiv 0.$$

Proposition 2.9.1. *For polynomials orthogonal with respect to the Freud weight* (2.9.1)*, the following statements hold:*

(i) The first moment is

$$\mu_0(t;\lambda) = 2^{\frac{1}{4}}\sqrt{\pi}\,\exp\left(\frac{1}{2}t^2\right)\,D_{-\frac{1}{2}}\left(-\frac{1}{2}\sqrt{2}t\right),$$

where $D_v(\xi)$ is the parabolic cylinder function (2.3.5), whilst the moments are

$$\mu_{2n}(t;\lambda) = \int_{\mathbb{R}} x^{2n}\exp\left(-\frac{1}{4}x^4 - tx^2\right)\,dx = (-1)^n\frac{d^n}{dt^n}\mu_0(t;\lambda),$$
$$\mu_{2n+1}(t;\lambda) = \int_{\mathbb{R}} x^{2n+1}\exp\left(-\frac{1}{4}x^4 - tx^2\right)\,dx = 0, \quad n\in\mathbb{N}_0.$$

(ii) The weight function $w(x;t)$ satisfies Pearson's differential equation (1.1.2)

$$w'(x;t) + (4x^3+2tx)w(x;t) = 0,$$

with $\sigma(x;t)=1$ and $\tau(x;t) = -4x^3-2tx$. Since deg$(\sigma)=0$ and deg$(\tau)=3$, the corresponding polynomial sequence $\{P_n(x;t)\}_{n=0}^{\infty}$ constitute a family of semiclassical orthogonal polynomials(cf. [67, 94]).

(iii) The recurrence coefficient β_n of the monic polynomials orthogonal with respect to the weight w in (2.9.1) *satisfies the discrete Painlevé* $\mathrm{dP_I}$ *equation*

$$\beta_n\left(\beta_{n-1}+\beta_n+\beta_{n+1}\right) = n - 2t\beta_n, \quad n\in\mathbb{N}, \tag{2.9.2}$$

with initial condition $\beta_0=0$(cf. [55, 71, 91]).

Remark 2.9.1. *(i) The recurrence coefficient β_n in* (2.9.2) *can be computed in the same way as in the Hermite case [127]. For Freud polynomials, by letting $q_n=\beta_n$ in* (2.9.2)*, we obtain* (2.8.6) *and* (2.8.7)*.*

(ii) The link between equations (2.8.6) *and* (2.8.7) *is given by*

$$q_{n+1} = \frac{1}{2q_n}\left(n - \frac{dq}{dt} - 2tq_n - q_n^2\right) \quad \textit{and} \quad q_{n-1} = \frac{1}{2q_n}\left(n + \frac{dq}{dt} - 2tq_n - q_n^2\right),$$

which are $\mathrm{P_{IV}}$ *Bäcklund transformations (cf. [19, 39, 90, 91]).*

We note that solutions of $\mathrm{P_{IV}}$ (2.8.7) are known as the "half-integer hierarchy", which arises in quantum gravity [51, 52] and was studied by Bassom, Clarkson and Hicks [9, 37]. In this hierarchy the first solution is given in [37]

$$q\left(t;-\frac{1}{2},-\frac{1}{2}\right) = -2t + \sqrt{2}\frac{C_1D_{-1/2}\left(-\sqrt{2}t\right) - C_2D_{-1/2}\left(-\sqrt{2}t\right)}{C_1D_{-1/2}\left(-\sqrt{2}t\right) + C_2D_{-1/2}\left(-\sqrt{2}t\right)}, \tag{2.9.3}$$

with C_1 and C_2 arbitrary constants. The first few solutions of the hierarchy

$$\beta_n(t) = q_n(t) = \mathbf{w}\left(t; -\frac{1}{2}n, -\frac{1}{2}n^2\right)$$

where $\mathbf{w}(z; A, B)$ satisfies $\mathrm{P_{IV}}$ (2.8.7), which are the first few recurrence coefficients associated with Freud's weight (2.9.1) are:

$$\begin{aligned}
q_1(t) &= \Xi(t) - 2t;\\
q_2(t) &= \frac{1}{\Xi(t) - 2t} - \Xi(t);\\
q_3(t) &= \frac{2(\Xi(t) - 2t)}{\Xi^2(t) - 2t\Xi(t) - 1} - \frac{1}{\Xi(t) - 2t};\\
q_4(t) &= -\frac{4}{3}t + \frac{2(4t^2+3)\Xi(t) - 16t(t^2+1)}{3[3\Xi^2(t) - 10t\Xi(t) + 8t^2 - 1)]} + \frac{2(\Xi(t) - 2t)}{\Xi^2(t) - 2t\Xi(t) - 1},
\end{aligned}$$

where $\Xi(t) = \sqrt{2}\dfrac{D_{1/2}\left(\sqrt{2}t\right)}{D_{-1/2}\left(\sqrt{2}t\right)}$.

2.9.2 The Shohat-Freud weight $|x|^\rho \exp\left(-x^4\right)$

Polynomials orthogonal with respect to the symmetric semiclassical weight

$$w_\rho(x) = |x|^\rho \exp\left(-x^4\right), \quad x \in \mathbb{R}, \quad \rho > -1,$$

were known to satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n\, P_{n-1}(x), \quad P_{-1} \equiv 0,\ P_0 \equiv 1, \tag{2.9.4}$$

where β_n is obtained from the nonlinear difference equation (cf. [55])

$$4\beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) = n + \rho\Omega_n, \quad n = 1, 2, \ldots, \tag{2.9.5}$$

with $\beta_0 = 0$ and

$$\Omega_n = \frac{1 - (-1)^n}{2} = \begin{cases} 0 & \text{if} \quad n \text{ is even},\\ 1 & \text{if} \quad n \text{ is odd}. \end{cases} \tag{2.9.6}$$

Equation (2.9.5) is equal to the discrete Painlevé equation $\mathrm{dP_I}$ (2.8.4) with $x_n = 2\beta_n$; $a = 1,\ b = \frac{\rho}{2},\ c = -\frac{\rho}{2}$ and $d = 0$. Properties of orthogonal polynomials with respect to the weight $w_\rho(x) = |x|^\rho \exp\left(-x^4\right)$ were studied in [22, 55, 134]. For more details about Freud-type weights, one can refer [10, 123].

2.10 Asymptotics for certain Freud type weights

Asymptotic properties of polynomials orthogonal with respect to exponential weight functions have been investigated by, among others, Lew and Quarles, Magnus, Lubinsky, Van Assche, Nevai and his collaborators (cf. [8, 11, 84, 85, 89, 90, 91, 102, 105, 112, 134]).

Geza Freud [55] conjectured the asymptotic behavior of the recurrence coefficient β_n in the three-term recurrence relation (2.9.4) satisfied by the polynomials $\{P_n\}_{n=0}^{\infty}$ orthogonal with respect to the positive weight

$$w(x) = \exp(-x^{2m}), \quad m \in \mathbb{N}, \tag{2.10.1}$$

as follows.

Conjecture 2.10.1. *[55]. Let w be the weight given in* (2.10.1) *and let β_n be the corresponding recurrence coefficient. Then*

$$\lim_{n\to\infty} \beta_n \, n^{-\frac{2}{m}} = \left[\frac{\Gamma(\frac{1}{2}m)\;\Gamma(1+\frac{1}{2}m)}{\Gamma(m+1)}\right]^{\frac{2}{m}}. \tag{2.10.2}$$

Monic polynomials orthogonal with respect to the simplest Freud weight

$$w(x) = \exp(-x^4), \; -\infty < x < \infty, \tag{2.10.3}$$

satisfy the recurrence relation (2.6.3). The recurrence coefficient β_n is determined by the non-linear difference equation [109, p. 266] (see also [111])

$$4\beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) = n, \quad n = 1, 2, \ldots, \tag{2.10.4}$$

where $\beta_0 = 0$ and $\beta_1 = \dfrac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}$. One can show that (2.10.4) follows from Magnus's proof [91] for the case $m = 2$ in (2.10.1) (see also [134]). In the literature, (2.10.4) and its generalizations are often referred to as the Freud equation. In the physics literature, (2.10.4) is known as a discrete string equation, and it was derived and studied in the papers by Bessis, Itzykson and Zuber [16, 72] discussing the problem of enumeration of Feynman graphs in string theory. Despite interesting and deep achievements in the theory of orthogonal polynomials [84, 112], the asymptotics for general exponential weights remains one of the key unsolved problems in the theory of semiclassical orthogonal polynomials since the Freud equation does not have a direct (explicit) solution.

Since $\beta_{n-1} + \beta_{n+1} > 0$, we observe from (2.10.4) that

$$4\beta_n^2 \leq n \Leftrightarrow 0 < \beta_n \leq \frac{1}{2}\sqrt{n}, \tag{2.10.5}$$

Thus, applying (2.10.5) again to (2.10.4) gives

$$n \leq 4\beta_n \left(\frac{\sqrt{n-1}}{2} + \frac{\sqrt{n}}{2} + \frac{\sqrt{n+1}}{2} \right) \leq 6\sqrt{n}\beta_n, \quad n = 1, 2, \ldots,$$

and hence estimates for the recurrence coefficient bounds gives

$$\frac{n}{36} \leq \beta_n^2 \leq \frac{n}{4} \Leftrightarrow \frac{1}{6} \leq \beta_n \, n^{-\frac{1}{2}} \leq \frac{1}{2}. \tag{2.10.6}$$

Note that the bounding values in (2.10.6) assure the validity of Freud's conjecture (2.10.2).

The recurrence coefficient β_n associated with the weight (2.10.3) has an asymptotic expansion

$$\beta_n \sim \left(\frac{n}{12} \right)^{\frac{1}{2}} \sum_{j=0}^{\infty} c_{2j} n^{-2j},$$

with $c_0 = 1$ and $c_1 = \frac{1}{24}$ (cf. [102, Theorem 1] and [101]). The asymptotic expansion of $\beta_n(t)$ satisfying (2.10.4) was studied by Lew and Quarles (cf. [85, 109, 114]) and is

$$\beta_n = \sqrt{\frac{n}{12}} \left(1 + \frac{1}{24n^2} - \frac{7}{576n^4} + \frac{111}{27648n^6} + \mathcal{O}(n^{-8}) \right).$$

An asymptotic expansion for the more general case when $t \in \mathbb{R}$ and $\lambda = -\frac{1}{2}$ in (1.2.1) was given by Clarke and Shizgal (cf. [36]) in the context of bimode polynomials. Bo and Wong [122] also gave a uniform asymptotic formula for polynomials orthogonal with respect to the weight $\exp(-x^4)$. Further, we point out that many other relevant properties of orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ with respect to the weight $\exp(-x^4)$ were studied by Nevai [109, 110] and also in [22, 55, 85, 114].

Asymptotics of more general Freud-type weights are considered in [3, 17, 102, 123].

In Chapter 5, we will discuss asymptotic properties of the recurrence coefficients of monic polynomials $\{S_n(x;t)\}_{n=0}^{\infty}$ orthogonal with respect to the generalized Freud weight (1.2.1).

Chapter 3

Semiclassical Laguerre polynomials

3.1 Introduction

In this chapter we study properties of the semiclassical Laguerre weight

$$w(x;t) = w_0(x)\exp(xt), \quad \lambda > -1,\ t \in \mathbb{R}, \tag{3.1.1}$$

supported on $\mathbb{R}$ where $w_0(x) = x^\lambda \exp(-x^2)$ is a Hermite-type weight. Finding an explicit expression for the recurrence coefficients associated with the semiclassical Laguerre weight is not straightforward. However, these coefficients obey certain non-linear recurrence equations (cf. [19, 50]) that can be identified as discrete Painlevé equations. In [39] the authors provide an explicit formulation of the recurrence coefficients in the three-term recurrence relation associated with the semiclassical Laguerre weight and these coefficients can be expressed in terms of Wronskians of parabolic cylinder functions that arise in the description of special function solutions of the fourth Painlevé equation.

Following the results on certain properties of semiclassical Laguerre polynomials in [39], we determine the differential-difference Toda-type evolution equation satisfied by the recurrence coefficient associated with the semiclassical Laguerre polynomials and derive a second-order differential equation satisfied by polynomials associated with the semiclassical Laguerre polynomials. In this chapter we also show that generalized Freud polynomials arise from semiclassical Laguerre polynomials by the technique of symmetrization discussed in Chapter 2.

In Section 3.2 we give an overview of a general one-parameter family of semiclassical weights of the form (3.1.1). The discussion in Section 3.3 on some (analytic and asymptotic) properties of the semiclassical Laguerre polynomials leads to the later sections of the chapter, where we determine not only the differential-difference equation satisfied by semiclassical Laguerre polynomials but also an explicit representation of a 2×2 differential (Lax) system in terms of the recurrence coefficients associated with the semiclassical Laguerre polynomials. Finally, the chapter ends by providing generalized Freud polynomials as a symmetric form of the semiclassical Laguerre polynomials using the technique of symmetrization due to Chihara [35, Section 3.7] discussed in section 2.6.

3.2 The weight $w(x;t) = w_0(x)\exp(xt)$

In this section we provide fundamental facts about semiclassical polynomials $\{\mathsf{P}_n\}_{n=0}^{\infty}$ which are orthogonal with respect to the weight $w(x;t)$ in (3.1.1). For the general class of semiclassical weights of the form

$$w(x;t) = w_0(x)\exp(xt),$$

where $w_0(x)$ is a classical weight function with finite moments, i.e., $\displaystyle\int_{\mathbb{R}} x^k w_0(x)\exp(xt)\,dx$ exists for all $k \in \mathbb{N}_0$, we have that

(i) the monic polynomials $\{\mathsf{P}_n\}_{n=0}^{\infty}$, the sequence of recurrence coefficients $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and the Hankel determinants Δ_n and the moments $\{\mu_k\}_{k\in\mathbb{N}}$ are all functions of t.

(ii) the three-term recurrence relation for monic orthogonal polynomials is

$$x\mathsf{P}_n(x;t) = \mathsf{P}_{n+1}(x;t) + \alpha_n(t)\,\mathsf{P}_n(x;t) + \beta_n(t)\,\mathsf{P}_{n-1}(x;t), \tag{3.2.1}$$
$$\mathsf{P}_{-1} \equiv 0, \quad \mathsf{P}_0 \equiv 1, \quad n = 1, 2, \ldots,$$

and the coefficients $\{\alpha_n(t)\}_{n=0}^{\infty}$ and $\{\beta_n(t)\}_{n=0}^{\infty}$ satisfy the differential (Toda) system (cf. [7, 58, 118, 128])

$$\left.\begin{aligned} \tfrac{d}{dt}\alpha_n(t) &= \beta_{n+1}(t) - \beta_n(t), \\ \tfrac{d}{dt}\beta_n(t) &= \beta_n(t)\left[\alpha_n(t) - \alpha_{n-1}(t)\right] \end{aligned}\right\}. \tag{3.2.2}$$

(iii)

$$\mu_k(t) = \pm \frac{d}{dt}\mu_{k\pm1}, \tag{3.2.3}$$

as a consequence of (3.1.1) and this is considered as a moment generator for the Toda lattice in the study of integrable systems. (3.2.3) implies that the k^{th} moment, $\mu_k(t)$, associated with the weight function in (3.1.1), takes the form

$$\mu_k(t) = \int_{[a,b]} x^k \, w_0(x) \, \exp(tx) \, dx = \frac{d^k}{dt^k}\int_{[a,b]} w_0(x) \, \exp(tx) \, dx = \frac{d^k\mu_0}{dt^k}. \tag{3.2.4}$$

Applying (3.2.3) to the determinants Δ_n in (2.2.5) and $\tilde{\Delta}_n$ in (2.2.6), and denoting the bidirectional Wronskian by τ_n, we have the following result.

Theorem 3.2.1. *Let*

$$\tau_n(f) = \mathcal{W}\left(f, \frac{df}{dt}, \cdots, \frac{d^{n-2}f}{dt^{n-2}}, \frac{d^{n-1}f}{dt^{n-1}}\right)$$

and suppose that the moment $\mu_k(t)$ satisfies (3.2.4)*. Then the Hankel-Hadamard-type determinants Δ_n and $\tilde{\Delta}_n$ can be written in the form*

$$\Delta_n(t) = \tau_n(\mu_0), \quad \tilde{\Delta}_n(t) = \frac{d\tau_n(\mu_0)}{dt}.$$

Proof. See [38]. □

Corollary 3.2.1. *If the weight has the form* (3.1.1)*, then the determinants $\Delta_n(t)$ and $\tilde{\Delta}_n(t)$ can be expressed as Wronskians*

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \cdots, \frac{d^{n-2}\mu_0}{dt^{n-2}}, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right);$$
$$\tilde{\Delta}_n(t) = \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \cdots, \frac{d^{n-2}\mu_0}{dt^{n-2}}, \frac{d^{n}\mu_0}{dt^{n}}\right) = \frac{d}{dt}\Delta_n(t).$$

Remark 3.2.1. *We note that*

$$\frac{\tilde{\Delta}_n(t)}{\Delta_n(t)} = \frac{d}{dt}\left[\ln \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \cdots, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right)\right].$$

The Hankel determinant (2.2.5) also satisfies the Toda equation

$$\frac{d^2}{dt^2}\ln\Delta_n(t) = \frac{\Delta_{n+1}(t)\;\Delta_{n-1}(t)}{\Delta_n^2(t)},$$

and this is proved in [107, Proposition 1].

Theorem 3.2.2. *[39, Theroem 4.9]. Suppose the condition* (3.2.3) *holds. Then the recurrence coefficients in* (3.2.1) *can be expressed as*

$$\alpha_n(t) = \frac{d}{dt}\ln\left(\frac{\tau_{n+1}(\mu_0)}{\tau_n(\mu_0)}\right), \quad \beta_n(t) = \frac{d^2}{dt^2}\ln\left(\tau_n(\mu_0)\right), \quad n = 1, 2, \ldots. \tag{3.2.5}$$

3.3 The weight $x^\lambda \exp\left(-x^2+tx\right)$, $\lambda > -1$, $t \in \mathbb{R}$

Let $\{\mathsf{L}_n^{(\lambda)}\}_{n=0}^{\infty}$ denote a sequence of semiclassical polynomials orthogonal with respect to the semiclassical Laguerre weight

$$w_\lambda(x;t) = x^\lambda \exp\left(-x^2+tx\right), \ \lambda > -1, \ t \in \mathbb{R}, \ x > 0 \tag{3.3.1}$$

(cf. [19, 39, 57, 59, 129]). Since the weight function (3.3.1) is positive, continuous and integrable on $\mathbb{R}^+$, it follows from the general theory [35, 132], that the orthogonal polynomials $\mathsf{L}_n^{(\lambda)}(x;t)$ exist uniquely for all $n \in \mathbb{N}_0$, $\deg(\mathsf{L}_n^{(\lambda)}) = n$ and they are solutions of the three-term recurrence relation

$$x\mathsf{L}_n^{(\lambda)}(x;t) = \mathsf{L}_{n+1}^{(\lambda)}(x;t) + \alpha_n(t)\mathsf{L}_n^{(\lambda)}(x;t) + \beta_n(t)\mathsf{L}_{n-1}^{(\lambda)}(x;t), \tag{3.3.2}$$

with initial conditions $\mathsf{L}_{-1}^{(\lambda)} \equiv 0$ and $\mathsf{L}_0^{(\lambda)} \equiv 1$ (cf. [19, 39]).

It is shown (cf. [39, 50]) that moments of a semiclassical weight provide the link between the positive weight and the associated Painlevé equation. Hence, explicit expressions for moments of the semiclassical Laguerre weight (3.3.1) were obtained in [39]. The first moment is [39, Theorem 4.6]

$$\mu_0(t;\lambda) = \begin{cases} 2^{\frac{-\lambda-1}{2}}\Gamma(\lambda+1)\,\exp\left(\frac{1}{8}t^2\right)\,D_{-\lambda-1}\left(-\frac{t}{\sqrt{2}}\right), & \text{if} \quad \lambda \notin \mathbb{N}, \\ \dfrac{1}{2}\sqrt{\pi}\dfrac{d^n}{dt^n}\left(\exp\left(\frac{1}{4}t^2\right)\,\left[1+\operatorname{erf}\left(\frac{1}{2}t\right)\right]\right), & \text{if} \quad \lambda = n \in \mathbb{N}, \end{cases} \tag{3.3.3}$$

where $D_v(\xi)$ is the parabolic cylinder function (2.3.5). The first moment $\mu_0(t;\lambda)$ associated with the semiclassical weight (3.3.1) also satisfies the differential equation [39, Theorem 4.6]

$$\frac{d^2\mu_0}{dt^2} - \frac{t}{2}\frac{d\mu_0}{dt} - \frac{1}{2}(\lambda+1)\mu_0 = 0.$$

The following proposition gives certain analytic properties of semiclassical Laguerre polynomials.

Proposition 3.3.1. *[39, Section 5]. Let* $\{\mathsf{L}_n^{(\lambda)}\}_{n=0}^{\infty}$ *be sequence of semiclassical Laguerre polynomials. Then*

(i) Pearson's equation associated with the weight $w_\lambda(x;t)$ *in* (3.3.1) *is satisfied, where*

$$\sigma(x;t) = x \quad \text{and} \quad \tau(x;t) = -2x^2 + tx + \lambda + 1.$$

(ii) The n^{th} *moment* $\mu_n(t;\lambda)$ *for the weight* $w_\lambda(x;t)$ *in* (3.3.1) *is given by*

$$\mu_n(t;\lambda) = 2^{-\frac{(\lambda+n+1)}{2}}\Gamma(\lambda+n+1)\exp\left(\frac{1}{8}t^2\right)D_{-\lambda-n-1}\left(-\frac{t}{\sqrt{2}}\right),$$

where D_ν *is the parabolic cylinder function given in* (2.3.5).

(iii) The recurrence coefficients in (3.3.2) *associated with the weight* $w_\lambda(x;t)$ *in* (3.3.1) *satisfy the discrete system (also called 'String equations') (cf. [39, Lemma 4.2] and [19, Theorem 1.1])*

$$\big(2\alpha_n(t)-t\big)\big(2\alpha_{n-1}(t)-t\big) = \frac{(2\beta_n(t)-n)\,(2\beta_n(t)-n-\lambda)}{\beta_n(t)}, \tag{3.3.4a}$$

$$2\beta_n(t) + 2\beta_{n+1}(t) + \alpha_n(t)\big(2\alpha_n(t)-t\big) = 2n+1+\lambda. \tag{3.3.4b}$$

(iii) The Hankel determinant $\Delta_n(t)$ *takes the form*

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \cdots, \frac{d^{n-2}\mu_0}{dt^{n-2}}, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right),$$

where μ_0 *is given in* (3.3.3).

(iv) The recurrence coefficients $\alpha_n(t)$ *and* $\beta_n(t)$ *in* (3.3.2) *associated with the semiclassical weight* (3.3.1) *satisfy the Toda system (see [50] and [71, p. 41])*

$$\frac{d\alpha_n(t)}{dt} = \beta_{n+1}(t) - \beta_n(t),$$
$$\frac{d\beta_n(t)}{dt} = \beta_n(t)\left(\alpha_n(t) - \alpha_{n-1}(t)\right).$$

Asymptotic results of some properties of the semiclassical Laguerre polynomials as $t \to \infty$ are provided in [39].

Proposition 3.3.2. *[39, Section 5]. For the semiclassical Laguerre weight in* (3.3.1), *the following statements hold:*

(i) As $t \to \infty$, *the first moment* $\mu_0(t;\lambda)$ *has an asymptotic series [39, Lemma 5.1]*

$$\mu_0(t;\lambda) \sim \sqrt{\pi}\left(\frac{1}{2}t\right)^{\lambda}\exp\left(\frac{1}{4}t^2\right)\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)\;n!t^{2n}}.$$

(ii) *The Hankel determinant $\Delta_n(t)$ has a t-asymptotic expansion [39, Lemma 5.2]*

$$\Delta_n(t) = d_n \pi^{n/2}\left(\frac{1}{2}t\right)^{n\lambda}\exp\left(\frac{1}{4}nt^2\right)\left[1+\frac{n\lambda(\lambda-n)}{t^2}+\mathcal{O}(t^{-4})\right],$$

with a constant d_n and the function $H_n(t;\lambda)$ defined by

$$H_n(t;\lambda) = \frac{d}{dt}\ln\Delta_n(t),$$

has the asymptotic expansion

$$H_n(t;\lambda) = \frac{nt}{2}+\frac{n\lambda}{t}+\frac{2n\lambda(n-\lambda)}{t^3}+\mathcal{O}(t^{-5}).$$

(iii) *As $t\to\infty$, the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ associated with the semiclassical weight* (3.3.1) *have the asymptotic expansions [39, Lemma 5.3]*

$$\left.\begin{aligned}\alpha_n(t) &= \frac{t}{2}+\frac{\lambda}{t}+\mathcal{O}\big(t^{-3}\big),\\ \beta_n(t) &= \frac{n}{2}-\frac{n\lambda}{t^2}+\mathcal{O}\big(t^{-4}\big)\end{aligned}\right\}. \tag{3.3.6}$$

Remark 3.3.1. *As $t\to\infty$, it follows from* (3.3.6) *that*

$$\alpha_n(t)\to\frac{1}{2}t \quad \text{and} \quad \beta_n(t)=\frac{1}{2}n.$$

Remark 3.3.2. *The difference between semiclassical orthogonal polynomials, in particular the semiclassical Laguerre polynomials, and the classical orthogonal polynomials is that classical orthogonal polynomials give rise to closed form expressions for the recurrence coefficients but the solutions to the above pair of nonlinear difference equations in* (3.3.4) *are highly transcendental and in fact, it was shown by Boelen and Van Assche [19, 134] that this pair of difference equations* (3.3.4) *can be obtained from an asymmetric Painlevé* $\mathrm{P_{IV}}$ *equation by a limiting process. We also point out that the semiclassical weight in* (3.3.1) *is not the only weight function leading to the difference equations* (3.3.4)*. It can be shown, using the same methods as in [134], that the semiclassical Hermite weight, which is a natural generalization of the Hermite weight,*

$$w_\lambda(x;t) = |x|^\lambda\exp(-x^2+tx),\quad t,x\in\mathbb{R},\quad \lambda>0, \tag{3.3.7}$$

also, leads to the same difference equations for the recurrence coefficients. However, the semiclassical Laguerre and Hermite weight functions are well-connected by the more general weight

$$w(x;t) = \begin{cases} K|x|^\lambda \exp(-x^2+tx), & x<0, \\ M|x|^\lambda \exp(-x^2+tx), & x \geq 0, \end{cases} \tag{3.3.8}$$

which gives rise to the same difference equations (3.3.4). *The difference between the three cases lies in the initial conditions for the difference equation, hence* (3.3.8) *can be thought of as a singular deformation of the classical Hermite weight on* $\mathbb{R}$.

Remark 3.3.3. *Monic generalized Hermite polynomials are a special case of* (3.3.8) *for* $K = M$ *and* $t = 0$. *Their recurrence coefficients are given explicitly by (cf. [33])*

$$\alpha_n = 0, \quad 2\beta_n = n + 2\lambda\Omega_n,$$

where Ω_n *is given in* (2.9.6).

The following result gives expressions for the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the recurrence relation (3.3.2) associated with the semiclassical weight (3.3.1) in terms of solutions of the fourth Painlevé equation $\mathrm{P_{IV}}$.

Theorem 3.3.1. *[39, Theorem 4.13]. Suppose* $\Psi_{n,\lambda}(z)$ *is given by*

$$\Psi_{n,\lambda}(z) = \mathcal{W}\left(\psi_\lambda, \frac{d\psi_\lambda}{dz}, \ldots, \frac{d^{n-1}\psi_\lambda}{dz^{n-1}}\right), \quad \Psi_{0,\lambda}(z) = 1,$$

where

$$\psi_\lambda(z) = \begin{cases} D_{-\lambda-1}\big(-\sqrt{2}\,z\big)\exp\left(\tfrac{1}{2}z^2\right), & \text{if} \quad \lambda \notin \mathbb{N}, \\ \dfrac{d^m}{dz^m}\Big(\big[1+\operatorname{erf}(z)\big]\exp(z^2)\Big), & \text{if} \quad \lambda = m \in \mathbb{N}, \end{cases}$$

with $D_\lambda(\zeta)$ *is the parabolic cylinder function and* $\operatorname{erfc}(z)$ *denotes the complementary error function (see* (2.3.3)*) that is given by*

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)\,dt.$$

Then, the recurrence coefficients $\alpha_n(t)$ *and* $\beta_n(t)$ *in the three-term recurrence relation (3.3.2) associated with the semiclassical weight (3.3.1) are given by*

$$\alpha_n(t) = \tfrac{1}{2}q_n(z) + \tfrac{1}{2}t, \tag{3.3.9a}$$

$$\beta_n(t) = -\tfrac{1}{8}\frac{dq_n}{dz} - \tfrac{1}{8}q_n^2(z) - \tfrac{1}{4}zq_n(z) + \tfrac{1}{4}\lambda + \tfrac{1}{2}n, \tag{3.3.9b}$$

with $z = \dfrac{1}{2}t$ *and* $q_n(z) = -2z + \frac{d}{dz}\ln\dfrac{\Psi_{n+1,\lambda}(z)}{\Psi_{n,\lambda}(z)}$, *which satisfies the fourth Painlevé equation* $\mathrm{P_{IV}}$

$$\frac{d^2q_n}{dz^2} = \frac{1}{2q_n}\left(\frac{dq_n}{dz}\right)^2 + \frac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2-\alpha)q_n + \frac{\beta}{q_n}, \tag{3.3.10}$$

with parameters

$$\alpha = 2n+\lambda+1, \quad \beta = -2\lambda^2. \tag{3.3.11}$$

Remark 3.3.4. *Filipuk, Van Assche and Zhang [50] considered the orthonormal version of the semiclassical Laguerre polynomials* $\{\mathsf{p}_n\}_{n=0}^\infty$ *and proved that the coefficients* $a_n(t)$ *and* $b_n(t)$ *in the three-term recurrence relation*

$$x\mathsf{p}_n(x;t) = b_{n+1}(t)\ \mathsf{p}_{n+1}(x;t) + a_n(t)\mathsf{p}_n(x;t) + b_n(t)\mathsf{p}_{n-1}(x;t), \quad n \in \mathbb{N}_0,$$

satisfy the fourth Painlevé equation (3.3.10) *with* $q_n(z) = a_n(2z) - 2z$. *They also applied different approaches such as the ladder operator formalism and the isomonodromy formation approaches to prove this result. However, they didn't identify the specific solution of* (3.3.10) *[50]. Clarkson and Jordaan [39] explicitly expressed the coefficients in the monic three-term recurrence relation of these polynomials in terms of the Wronskians of parabolic cylinder functions which arise in the description of special function solutions of the fourth Painlevé equation as well as the second degree, second order equation satisfied by the associated Hamiltonian function. The parameters in* (3.3.11) *satisfy the condition of* $\mathrm{P_{IV}}$ *to have solutions expressible in terms of parabolic cylinder functions (see Section* (2.3.2) *for an overview of parabolic cylinder functions).*

In the sequel, using (3.3.9), the first few recurrence coefficients associated with the semiclassical weight (3.3.1) are given as

$$\begin{aligned}
\alpha_0(t) &= \frac{1}{2}t - \frac{D_{-\lambda}(-\frac{1}{2}\sqrt{2}t)}{D_{-\lambda-1}(-\frac{1}{2}\sqrt{2}t)} \equiv \Psi_\lambda(t),\\
\alpha_1(t) &= \frac{1}{2}t - \Psi_\lambda(t) - \frac{\Psi_\lambda(t)}{2\Psi_\lambda^2(t) - t\Psi_\lambda(t) - \lambda - 1},\\
\alpha_2(t) &= \frac{1}{2}t + \frac{2\lambda+4}{t} + \frac{\Psi_\lambda(t)}{2\Psi_\lambda^2(t) - t\Psi_\lambda(t) - \lambda - 1}\\
&\quad - \frac{2\left[(\lambda+1)t^2 + 4(\lambda+2)(2\lambda+3)\right]\Psi_\lambda^2(t)}{2t\left[2t\Psi_\lambda^3(t) - (t^2-4\lambda-6)\Psi_\lambda^2(t) - 3(\lambda+1)t\Psi_\lambda(t)) - 2(\lambda+1)^2\right]}\\
&\quad + \frac{(\lambda+1)t[t^2+2(4\lambda+9)]\Psi_\lambda(t) + (\lambda+1)^2[t^2+8(\lambda+2)]}{2t\left[2t\Psi_\lambda^3(t) + (t^2-4\lambda-6)\Psi_\lambda^2(t) - 3(\lambda+1)t\Psi_\lambda(t)) - 2(\lambda+1)^2\right]},
\end{aligned}$$

$$\beta_1(t) = -\Psi_\lambda^2(t) + \frac{1}{2}t\Psi_\lambda(t) + \frac{1}{2}(\lambda+1),$$
$$\beta_2(t) = -\frac{2t\Psi_\lambda^3(t) - (t^2-4\lambda-6)\Psi_\lambda^2(t) - 3(\lambda+1)t\Psi_\lambda(t) - 2(\lambda+1)^2}{2[\Psi_\lambda^2(t) - \frac{1}{2}t\Psi_\lambda(t) - \frac{1}{2}(\lambda+1)]^2}.$$

By using the three-term recurrence relation (3.3.2), the first few monic polynomials orthogonal with respect to (3.3.1) are given by

$$\mathsf{L}_1^{(\lambda)}(x;t) = x - \Psi_\lambda,$$
$$\mathsf{L}_2^{(\lambda)}(x;t) = x^2 - \frac{2t\Psi_\lambda^2 - (t^2+2)\Psi_\lambda - (\lambda+1)t}{2[\Psi_\lambda^2 - \frac{1}{2}t\Psi_\lambda - \frac{1}{2}(\lambda+1)]}x - \frac{2(\lambda+2)\Psi_\lambda^2 - (\lambda+1)\Psi_\lambda - (\lambda+1)^2}{2[\Psi_\lambda^2 - \frac{1}{2}t\Psi_\lambda - \frac{1}{2}(\lambda+1)]},$$
$$\begin{aligned}&\mathsf{L}_3^{(\lambda)}(x;t)\\
&= x^3 - \left\{\frac{4(t^2+2\lambda+4)\Psi_\lambda^3 - 2t(t^2-\lambda-1)\Psi_\lambda^2 - (\lambda+1)(5t^2+4\lambda+6)\Psi_\lambda - 3(\lambda+1)^2t}{2[2t\Psi_\lambda^3 - (t^2-4\lambda-6)\Psi_\lambda^2 - 3(\lambda+1)t\Psi_\lambda - 2(\lambda+1)^2]}\right\}x^2\\
&\quad + \left\{\frac{2t(t^2+2\lambda+4)\Psi_\lambda^3 - [t^4+4(2\lambda+5)(\lambda+2)]\Psi_\lambda^2 - 2(\lambda+1)t(t^2-\lambda-5))\Psi_\lambda}{4[2t\Psi_\lambda^3 - (t^2-4\lambda-6)\Psi_\lambda^2 - 3(\lambda+1)t\Psi_\lambda - 2(\lambda+1)^2]}\right\}x\\
&\quad - \left\{\frac{(\lambda+1)^2(t^2-4\lambda-12)}{4[2t\Psi_\lambda^3 - (t^2-4\lambda-6)\Psi_\lambda^2 - 3(\lambda+1)t\Psi_\lambda - 2(\lambda+1)^2]}\right\}x\\
&\quad + \left\{\frac{2[(\lambda+1)t^2+4(\lambda+2)^2]\Psi_\lambda^3 - (\lambda+1)t(t^2+2\lambda+8)\Psi_\lambda^2}{4[2t\Psi_\lambda^3 - (t^2-4\lambda-6)\Psi_\lambda^2 - 3(\lambda+1)t\Psi_\lambda - 2(\lambda+1)^2]}\right\}\\
&\quad + \left\{\frac{-2(\lambda+1)^2(t^2+2\lambda+5)\Psi_\lambda - (\lambda+1)^3t}{4[2t\Psi_\lambda^3 - (t^2-4\lambda-6)\Psi_\lambda^2 - 3(\lambda+1)t\Psi_\lambda - 2(\lambda+1)^2]}\right\}.\end{aligned}$$

In the following section we show that semiclassical Laguerre polynomials not only obey the recurrence relation (3.2.1) but also a differential-difference equation.

3.4 A differential-difference equation satisfied by semiclassical Laguerre polynomials

In this section we derive a differential-difference equation satisfied by semiclassical Laguerre polynomials $\mathsf{L}_n^{(\lambda)}(x;t)$.

Theorem 3.4.1. *Let* $\{\mathsf{L}_n^{(\lambda)}\}_{n=0}^\infty$ *be the sequence of semiclassical Laguerre polynomials orthogonal with respect to* (3.3.1). *Then the differential-difference equation satisfied by these polynomials is given by*

$$x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx} = A_n(x;t)\mathsf{L}_n^{(\lambda)}(x;t) + B_n(x;t)\mathsf{L}_{n-1}^{(\lambda)}(x;t), \tag{3.4.1}$$

where $A_n(x;t)$ *and* $B_n(x;t)$ *are given by*

$$\left.\begin{aligned} A_n(x;t) &= n - 2\beta_n(t), \\ B_n(x;t) &= 2x\beta_n(t) + (2\alpha_n(t) - t)\,\beta_n(t) \end{aligned}\right\}. \tag{3.4.2}$$

Proof. For the semiclassical Laguerre weight (3.3.1), an important consequence of Pearson's equation gives

$$x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx} = \sum_{k=n-2}^{n} D_{n,k}\;\mathsf{L}_k^{(\lambda)}(x;t),$$

where

$$D_{n,k}\,h_k = \int_0^\infty x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx}\mathsf{L}_k^{(\lambda)}(x;t)w_\lambda(x;t)\,dx. \tag{3.4.3}$$

By iterating the recurrence relation (3.3.2)

$$\begin{aligned} x^2\mathsf{L}_n^{(\lambda)}(x;t) &= \mathsf{L}_{n+2}^{(\lambda)}(x;t) + (\alpha_n(t)+\alpha_{n+1}(t))\mathsf{L}_{n+1}^{(\lambda)}(x;t) + \big(\alpha_n^2(t)+\beta_{n+1}(t)+\beta_n(t)\big)\,\mathsf{L}_n^{(\lambda)}(x;t) \\ &\quad + (\beta_n(t)\alpha_n(t)+\beta_n(t)\alpha_{n-1}(t))\,\mathsf{L}_{n-1}^{(\lambda)}(x;t) + \beta_n(t)\beta_{n-1}(t)\mathsf{L}_{n-2}^{(\lambda)}(x;t), \end{aligned} \tag{3.4.4}$$

the coefficient $D_{n,k},\ n-2 \le k \le n$, is computed as follows:

For $k = n-2$, we integrate (3.4.3) by parts using (3.4.4) and orthogonality to obtain

$$\begin{aligned} D_{n,n-2} &= \frac{1}{h_{n-2}}\int_0^\infty x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx}\mathsf{L}_{n-2}^{(\lambda)}(x;t)w_\lambda(x;t)\,dx, \quad h_{n-2}\neq 0 \\ &= \frac{1}{h_{n-2}}\Bigg[\int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)\,\big(2x^2 - tx - \lambda - 1\big)\,\mathsf{L}_{n-2}^{(\lambda)}(x;t)w_\lambda(x;t)\,dx \\ &\qquad - \int_0^\infty x\,\frac{d\mathsf{L}_{n-2}^{(\lambda)}(x;t)}{dx}\mathsf{L}_n^{(\lambda)}(x;t)w_\lambda(x;t)\,dx\Bigg] \\ &= 2\beta_n(t)\beta_{n-1}(t). \end{aligned} \tag{3.4.5}$$

For $k = n-1$, using (3.4.3) and (3.4.4) together with orthogonality, yields

$$\begin{aligned} D_{n,n-1} &= \frac{1}{h_{n-1}}\int_0^\infty x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx}\mathsf{L}_{n-1}^{(\lambda)}(x;t)w_\lambda(x;t)\,dx, \quad h_{n-1}\neq 0 \\ &= \frac{1}{h_{n-1}}\Bigg[\int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)\,\big(2x^2 - tx - \lambda - 1\big)\,\mathsf{L}_{n-1}^{(\lambda)}(x;t)w_\lambda(x;t)\,dx \\ &\qquad - \int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)\,\frac{d\mathsf{L}_{n-1}^{(\lambda)}(x;t)}{dx}xw_\lambda(x;t)\,dx\Bigg] \\ &= 2\big(\beta_n(t)\alpha_n(t) + \beta_n(t)\alpha_{n-1}(t)\big) - t\beta_n(t) \\ &= \big[2\alpha_n(t) + 2\alpha_{n-1}(t) - t\big]\beta_n(t). \end{aligned} \tag{3.4.6}$$

For $k = n$, we also employ (3.4.3) and (3.4.4) together with orthogonality, to obtain

$$
\begin{aligned}
D_{n,n} &= \frac{1}{h_n}\int_0^\infty x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx}\mathsf{L}_n^{(\lambda)}(x;t)w_\lambda(x;t)\,dx, \quad h_n \neq 0\\
&= \frac{1}{h_n}\left[\int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)\left(2x^2 - tx - \lambda - 1\right)\mathsf{L}_n^{(\lambda)}(x;t)w_\lambda(x;t)\,dx\right.\\
&\qquad\left. - \int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx}w_\lambda(x;t)\,dx\right]\\
&= \left(2\beta_n(t) + 2\beta_{n+1}(t) + 2\alpha_n^2(t)\right) - t\alpha_n(t) - \lambda - 1 - n\\
&= \left(2\beta_n(t) + 2\beta_{n+1}(t)\right) + \left(2\alpha_n(t) - t\right)\alpha_n(t) - \lambda - 1 - n
\end{aligned}
\tag{3.4.7}
$$

and by using orthogonality, (3.4.3) and the recursion relation

$$
x\,\frac{d\mathsf{L}_n^{(\lambda)}(x;t)}{dx} = n\mathsf{L}_n^{(\lambda)}(x;t) + g(x), \quad g \in \mathbb{P}_{n-1},
$$

we obtain $D_{n,n} = n$. Note that (3.4.7) proves (3.3.4b). Hence, substituting the coefficients in (3.4.5), (3.4.6) and (3.4.7) into (3.4.1), together with expressing $\mathsf{L}_{n-2}^{(\lambda)}$ in terms of $\mathsf{L}_n^{(\lambda)}$ and $\mathsf{L}_{n-1}^{(\lambda)}$, we obtain the required result. □

Remark 3.4.1. *The approach we followed to construct the differential-difference equation* (3.4.1) *is similar to the method of ladder operators (cf. [71]).*

For semiclassical Laguerre polynomials, the following result immediately follows by differentiating (3.4.1) with respect to x and the recurrence relation (3.3.2).

Theorem 3.4.2. *For the semiclassical Laguerre weight* (3.3.1)*, the monic orthogonal polynomials $\mathsf{L}_n^{(\lambda)}(x;t)$ satisfy the second-order differential equation*

$$
x^2\beta_n(t)\theta_n(x;t)\frac{d^2\mathsf{L}_n^{(\lambda)}}{dx^2}(x;t) + U_n(x;t)\frac{d\mathsf{L}_n^{(\lambda)}}{dx}(x;t) + V_n(x;t)\mathsf{L}_n^{(\lambda)}(x;t) = 0, \tag{3.4.8}
$$

where the coefficients $U_n(x;t)$ and $V_n(x;t)$ are given by

$$
\begin{aligned}
U_n(x;t) = \beta_n(t)\theta_n(x;t)&\left[(n - 2\beta_n(t))\left(\frac{2x}{\theta_n(x;t)} + n - 1 - 2\beta_{n-1}(t) + (x - \alpha_{n-1}(t))\theta_{n-1}(x;t)\right)\right.\\
&\left. + \beta_n(t)\theta_n(x;t)\theta_{n-1}(x;t)\right] \quad \textit{and}
\end{aligned}
$$

$$
\begin{aligned}
V_n(x;t) = \beta_n(t)(n - 2\beta_n(t))&\Big(2x + (n - 1 - 2\beta_{n-1}(t))\theta_n(x;t) + (x - \alpha_{n-1}(t))\theta_n(x;t)\theta_{n-1}(x;t)\Big)\\
&+ \beta_n^2(t)\theta_n^2(x;t)\theta_{n-1}(x;t),
\end{aligned}
$$

with $\theta_n(x;t) = 2x + 2\alpha_n(t) - t$.

3.5 The Lax pair of the Toda system

Numerous integrable systems have been shown to be related to orthogonal polynomials through spectral transformations (cf. [6, 26]). One of the interesting properties of the continuous-time Toda lattice, (3.2.2), is that orthogonal polynomials appear as eigenfunctions of their Lax pairs. Lax pairs can exist for both discrete and continuous systems and involve expressing an equation in terms of matrices that satisfy a compatibility condition (cf. [7]).

Lax pairs related to the semiclassical weight (3.3.1) help us to investigate the system of non-linear difference equations satisfied by the recurrence coefficients. It is known (cf. [7, 19, 39, 50, 118]) that the three-term recurrence relation in (3.2.1) satisfied by monic polynomials orthogonal with respect to a semiclassical weight (3.3.1) is one of the Lax pairs, which is also used in scattering problems in discrete soliton theory.

By rewriting the recurrence relation (3.3.2) in matrix form, we have

$$\Psi_{n+1}(x) = \mathcal{L}_n(x)\Psi_n(x), \tag{3.5.1}$$

where

$$\mathcal{L}_n(x) = \begin{pmatrix} x-\alpha_n & -\beta_n \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Psi_n(x) = \begin{pmatrix} \mathsf{L}_n^{(\lambda)}(x;t) & \mathsf{L}_{n-1}^{(\lambda)}(x;t) \end{pmatrix}^T.$$

The differential-difference equation obtained in Theorem 3.4.1 can also be represented in a semi-discrete Lax representation (cf. [129, Subsection 2.2]) as

$$\frac{\partial \Psi_n(x)}{\partial x} = \mathcal{M}_n(x)\Psi_n(x), \tag{3.5.2}$$

where

$$\mathcal{M}_n(x) = \frac{1}{x}\begin{pmatrix} n-2\beta_n & (2x-t+2\alpha_n)\,\beta_n \\ -(2x-t+2\alpha_{n-1}) & 2x^2-xt+2\beta_n-n-\lambda \end{pmatrix}.$$

We observe that the differential system (3.5.2) and the recurrence relation (3.5.1) build the Lax Pairs whose compatibility leads to the semi-discrete Lax equation

$$\frac{\partial \mathcal{L}_n}{\partial x} = \mathcal{M}_{n+1}\mathcal{L}_n - \mathcal{L}_n\mathcal{M}_n. \tag{3.5.3}$$

This yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{x} \begin{pmatrix} (n+1)x - (n+1)\alpha_n + (2\alpha_n - t + 2\alpha_{n+1})\beta_{n+1} & (2\beta_{n+1} - n - 1)\,\beta_n \\ \alpha_n(2\alpha_n - t) + 2\beta_{n+1} - n - \lambda - 1 & (2x - t + 2\alpha_n)\beta_n \end{pmatrix}$$
$$- \frac{1}{x} \begin{pmatrix} nx - n\alpha_n + (2\alpha_n - t + 2\alpha_{n-1})\beta_n & (t\alpha_n - 2\alpha_n^2 - 2\beta_n + n + \lambda)\,\beta_n \\ n - 2\beta_n & (2x - t + 2\alpha_n)\beta_n \end{pmatrix}. \tag{3.5.4}$$

(3.5.4) is given equivalently as

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & 0 \end{pmatrix}, \tag{3.5.5}$$

where

$$\mathsf{A} := (2\alpha_n + 2\alpha_{n+1} - t)\beta_{n+1} - (2\alpha_n - t + 2\alpha_{n-1})\beta_n - \alpha_n + x,$$
$$\mathsf{C} := (2\beta_n + 2\beta_{n+1}) + (2\alpha_n - t)\,\alpha_n - \lambda - 1 - 2n,$$
$$\mathsf{B} := \beta_n\left[(2\beta_n + 2\beta_{n+1}) - 2n - (\lambda+1) + \alpha_n(2\alpha_n - t)\right] \equiv \beta_n\mathsf{C}.$$

Therefore, the entries B and C in (3.5.5) together with $\mathsf{A} - x = 0$ provide an alternative proof to determine the non-linear difference equations given in (3.3.4) satisfied by the recurrence coefficients α_n and β_n associated with the semiclassical Laguerre weight.

Remark 3.5.1. *(i) Semiclassical Laguerre polynomials appear as wave functions of the Lax pair of the Toda lattice due to a one-parameter deformation of the Hermite-type measure [26].*

(ii) For the semiclassical Laguerre weight (3.3.1)*, we observe from Theorem 3.2.2 that the recurrence coefficients, which can be expressed in terms of Hankel determinants* (3.2.5)*, also satisfy the differential-difference equations of the Toda lattice* (3.3.5)*. Besides, it is shown in [50] that the discrete system* (3.3.4) *for the recurrence coefficients associated with the semiclassical Laguerre weight can be obtained from a Bäcklund transformation of the fourth Painlevé equation* $\mathrm{P_{IV}}$.

3.6 Deriving the Volterra evolution equation

In this section we derive an evolution equation satisfied by the semiclassical Laguerre polynomials with respect to the weight $w_\lambda(x;t)$ given in (3.3.1) where λ is fixed and the parameter t varies. The three-term recurrence relation (3.2.1) is one of the Lax pairs and, in order to derive the remaining Lax pair, we consider the derivative $\dfrac{\partial \mathsf{L}_n^{(\lambda)}(x;t)}{\partial t}$, which is a polynomial of degree n and therefore can be written, using the orthogonal basis, as

$$\frac{\partial \mathsf{L}_n^{(\lambda)}(x;t)}{\partial t} = \sum_{k=0}^{n} c_{n,k}(t)\, \mathsf{L}_k^{(\lambda)}(x;t). \tag{3.6.1}$$

Differentiating the normalization relation

$$\int_0^\infty \left[\mathsf{L}_n^{(\lambda)}(x;t)\right]^2\, w_\lambda(x;t)\, dx = h_n > 0$$

with respect to t, we have

$$\int_0^\infty 2\mathsf{L}_n^{(\lambda)} \frac{\partial \mathsf{L}_n^{(\lambda)}(x;t)}{\partial t} w_\lambda(x;t)\, dx + \int_0^\infty \left[\mathsf{L}_n^{(\lambda)}(x;t)\right]^2\, \frac{\partial w_\lambda(x;t)}{\partial t}\, dx = 0,$$
$$\int_0^\infty 2\mathsf{L}_n^{(\lambda)}(x;t) \left[\sum_{k=0}^{n} c_{n,k}(t)\, \mathsf{L}_k^{(\lambda)}(x;t)\right] w_\lambda(x;t)\, dx + \int_0^\infty \left[x\mathsf{L}_n^{(\lambda)}(x;t)\right] \mathsf{L}_n^{\lambda}(x;t)\; w_\lambda(x;t)\, dx = 0,$$

which, using the three-term recurrence relation (3.3.2), reduces to

$$2c_{n,n}h_n + \alpha_n(t)h_n = 0 \Leftrightarrow c_{n,n} = -\frac{1}{2}\alpha_n(t).$$

Similarly, by differentiating the orthogonality relation

$$\int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)\; \mathsf{L}_k^{(\lambda)}(x;t)\; w_\lambda(x;t)\, dx = 0, \quad k < n,$$

with respect to t, we obtain

$$\int_0^\infty \mathsf{L}_k^{(\lambda)}(x;t) \frac{\partial \mathsf{L}_n^{(\lambda)}(x;t)}{\partial t} w_\lambda(x;t)\, dx + \int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t) \frac{\partial \mathsf{L}_k^{(\lambda)}(x;t)}{\partial t} w_\lambda(x;t)\, dx$$
$$+ \int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t)\mathsf{L}_k^{(\lambda)}(x;t) \frac{\partial w_\lambda(x;t)}{\partial t}\, dx = 0.$$

Hence, for $k < n$,

$$\int_0^\infty \mathsf{L}_k^{(\lambda)}(x;t) \frac{\partial \mathsf{L}_n^{(\lambda)}(x;t)}{\partial t} w_\lambda(x;t)\, dx + \int_0^\infty \mathsf{L}_n^{(\lambda)}(x;t) \frac{\partial \mathsf{L}_k^{(\lambda)}(x;t)}{\partial t} w_\lambda(x;t)\, dx$$
$$+ \int_0^\infty [x\mathsf{L}_n^{(\lambda)}(x;t)]\mathsf{L}_k^{(\lambda)}(x;t) w_\lambda(x;t)\, dx = 0. \tag{3.6.2}$$

By using the expansion (3.6.1) with its Fourier coefficients, orthogonality and the recurrence relation (3.3.2), (3.6.2) becomes

$$c_{n,k}h_k + h_n\delta_{k,n-1} = 0, \tag{3.6.3}$$

where $\delta_{k,n}$ is the Kronecker-delta function. Hence, equation (3.6.3) implies that $c_{n,n-1} = -\beta_n(t)$ and $c_{n,k} = 0$ for $k < n-1$. Therefore, the required Lax pair for the Volterra (Toda) equation is given by

$$\frac{\partial \mathsf{L}_n^{(\lambda)}(x;t)}{\partial t} = -\frac{\alpha_n(t)}{2}\mathsf{L}_n^{(\lambda)}(x;t) - \beta_n(t)\mathsf{L}_{n-1}^{(\lambda)}(x;t).$$

3.7 Symmetrizing semiclassical Laguerre polynomials

In this section we show that symmetrizing the semiclassical Laguerre weight (3.3.1) gives rise to the generalized Freud weight (1.2.1) (cf. [41]).

Let $\{\mathsf{L}_n^{(\lambda)}(x;t)\}_{n=0}^{\infty}$ denote the monic semiclassical Laguerre polynomials, orthogonal with respect to the semiclassical weight (3.3.1).

Define

$$S_{2n}(x;t) = \mathsf{L}_n^{(\lambda)}(x^2;t); \quad S_{2n+1}(x;t) = xQ_n^{(\lambda)}(x^2;t) \tag{3.7.1}$$

where

$$Q_n^{\lambda}(x;t) = \frac{1}{x}\left[\mathsf{L}_{n+1}^{(\lambda)}(x;t) - \frac{\mathsf{L}_{n+1}^{(\lambda)}(0;t)}{\mathsf{L}_n^{(\lambda)}(0;t)}\mathsf{L}_n^{(\lambda)}(x;t)\right] \tag{3.7.2}$$

are also monic and of degree n. Then, since the polynomial $x\mathsf{L}_n^{(\lambda+1)} \in \mathbb{P}_{n+1}$, we can write $x\mathsf{L}_n^{(\lambda+1)}$ in terms of the semiclassical Laguerre basis $\{\mathsf{L}_k^{(\lambda)}\}_{k=0}^{n+1}$ as

$$x\mathsf{L}_n^{(\lambda+1)}(x;t) = \sum_{k=0}^{n+1} a_{n+1,k}(t)\ \mathsf{L}_k^{(\lambda)}(x;t),$$

where the coefficients $a_{n+1,k}(t)$, with fixed real parameter t, are given by

$$\begin{aligned} a_{n+1,k}(t)\ \langle \mathsf{L}_k^{(\lambda)}(x;t), \mathsf{L}_k^{(\lambda)}(x;t)\rangle &= \int_0^\infty x\mathsf{L}_k^{(\lambda)}(x;t)\ \mathsf{L}_n^{(\lambda+1)}(x;t)\ x^{\lambda}\exp(-x^2+tx)\,dx \\ &= \int_0^\infty \mathsf{L}_k^{(\lambda)}(x;t)\ \mathsf{L}_n^{(\lambda+1)}(x;t)\ x^{\lambda+1}\exp(-x^2+tx)\,dx \\ &= 0, \quad \text{for} \quad k < n. \end{aligned} \tag{3.7.3}$$

By using (3.7.3), the polynomial $x\mathsf{L}_n^{(\lambda+1)}$ can be written as

$$x\mathsf{L}_n^{(\lambda+1)}(x;t) = a_{n+1,n+1}(t)\mathsf{L}_{n+1}^{(\lambda)}(x;t) + a_{n+1,n}(t)\mathsf{L}_n^{(\lambda)}(x;t). \tag{3.7.4}$$

Now, since $\mathsf{L}_n^{(\lambda+1)}(x;t)$ is monic, we have $a_{n+1,n+1}(t) = 1$ and hence (3.7.4) becomes

$$x\mathsf{L}_n^{(\lambda+1)}(x;t) = \mathsf{L}_{n+1}^{(\lambda)}(x;t) + a_{n+1,n}(t)\mathsf{L}_n^{(\lambda)}(x;t). \tag{3.7.5}$$

Evaluating (3.7.5) at $x = 0$ yields $a_{n+1,n}(t) = -\dfrac{\mathsf{L}_{n+1}^{(\lambda)}(0;t)}{\mathsf{L}_n^{(\lambda)}(0;t)}$ and hence

$$x\mathsf{L}_n^{(\lambda+1)}(x;t) = \mathsf{L}_{n+1}^{(\lambda)}(x;t) - \frac{\mathsf{L}_{n+1}^{(\lambda)}(0;t)}{\mathsf{L}_n^{(\lambda)}(0;t)}\mathsf{L}_n^{(\lambda)}(x;t) = xQ_n^{\lambda}(x;t).$$

Now,

$$\begin{aligned}
&\int_0^\infty \mathsf{L}_m^{(\lambda)}(x;t)\mathsf{L}_n^{(\lambda)}(x;t)\, x^\lambda \exp(-x^2+tx)\, dx \\
&\quad = \int_0^\infty \mathsf{L}_m^{(\lambda)}(x^2;t)\, \mathsf{L}_n^{(\lambda)}(x^2;t)\, x^{2\lambda}\, \exp(-x^4+tx^2)\, 2x\, dx \\
&\quad = 2\int_0^\infty \mathsf{L}_m^{(\lambda)}(x^2;t)\, \mathsf{L}_n^{(\lambda)}(x^2;t)\, |x|^{2\lambda+1}\, \exp(-x^4+tx^2)\, dx \\
&\quad = \int_{-\infty}^\infty \mathsf{L}_m^{(\lambda)}(x^2;t)\, \mathsf{L}_n^{(\lambda)}(x^2;t)\, |x|^{2\lambda+1}\, \exp(-x^4+tx^2)\, dx \\
&\quad = \int_{-\infty}^\infty S_{2m}(x;t)\, S_{2n}(x;t)\, |x|^{2\lambda+1} \exp(-x^4+tx^2)\, dx \\
&\quad = K_n(t)\, \delta_{mn},
\end{aligned}$$

which implies that $\{S_{2m}(x;t)\}_{m=0}^\infty$ is a symmetric orthogonal sequence with respect to the even weight $w(x;t) = |x|^{2\lambda+1}\exp(-x^4+tx^2)$ on $\mathbb{R}$. It is proved in [35, Theorem 7.1] that the kernel polynomials $Q_m^{(\lambda)}(x;t)$ are orthogonal with respect to $xw(x;t) = x^{\lambda+1}\exp(-x^2+tx)$. Hence

$$\begin{aligned}
K_n(t)\, \delta_{mn} &= \int_0^\infty Q_m^{(\lambda)}(x;t)\, Q_n^{(\lambda)}(x;t)\, x^{\lambda+1} \exp(-x^2+tx)\, dx \\
&= 2\int_0^\infty Q_m^{(\lambda)}(x^2;t)\, Q_n^{(\lambda)}(x^2;t)\, x^{2\lambda+3} \exp(-x^4+tx^2)\, dx \\
&= \int_{-\infty}^\infty \left[xQ_m^{(\lambda)}(x^2;t))\right] \left[xQ_n^{(\lambda)}(x^2;t)\right] |x|^{2\lambda+1} \exp(-x^4+tx^2)\, dx \\
&= \int_{-\infty}^\infty S_{2m+1}(x;t)\, S_{2n+1}(x;t)\, |x|^{2\lambda+1} \exp(-x^4+tx^2)\, dx.
\end{aligned}$$

Lastly, since in each case the integrand is odd, we have that

$$\begin{aligned}
&\int_{-\infty}^\infty S_{2m}(x;t)\, S_{2n+1}(x;t)\, |x|^{2\lambda+1} \exp(-x^4+tx^2)\, dx \\
&= \int_{-\infty}^\infty S_{2m+1}(x;t)\, S_{2n}(x;t)\, |x|^{2\lambda+1} \exp(-x^4+tx^2)\, dx = 0,
\end{aligned}$$

and we conclude that $\{S_n(x;t)\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to the generalized Freud weight $w_\lambda(x;t) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \; x \in \mathbb{R}$.

Note that $\widetilde{w}_\lambda(x;t) = |x|^{-1} w_\lambda(x^2;t) = |x|^{2\lambda-1} \exp(-x^4 + tx^2)$ is another symmetric dual weight function for the semiclassical Laguerre weight (cf. [97]).

Remark 3.7.1. *Note that, for $t = 0$, the result of symmetrization of semiclassical Laguerre polynomials given in* (3.7.1) *reduces to the case of half-range generalized Hermite polynomials given in* (2.6.11).

3.8 Conclusion

The classical orthogonal polynomials discussed in Chapter 2 correspond to weights which appear in numerous applications such as mathematical physics, engineering and probability theory. The recurrence coefficients in the three-term recurrence relation (2.2.1) satisfied by classical orthogonal polynomials can be found explicitly and are simple rational expressions in n. Slight modification of classical weights yields semiclassical weights, for instance, the semiclassical Laguerre weight (3.3.1) considered in this chapter.

In Chapter 3, we briefly revisited certain properties of the semiclassical Laguerre polynomials such as the higher order moments, Pearson's equation associated with the semiclassical Laguerre weight, the recurrence coefficients and the differential-difference equations satisfied by the recurrence coefficients as well the semiclassical Laguerre polynomials themselves. As our main results, we obtained a differential-difference equation and differential equation satisfied by semiclassical Laguerre polynomials as well as an explicit representation of a 2×2 differential (Lax) system in terms of the recurrence coefficients. Further, the Volterra equation for the semiclassical Laguerre weight was derived by differentiating the recurrence coefficients with respect to the parameter $t \in \mathbb{R}$ introduced in the weight function (3.3.1).

We concluded by showing (cf. [41]) that semiclassical Laguerre polynomials can be used to construct generalized Freud polynomials using a symmetrization and quadratic transformation described by Chihara in [35]. In the upcoming chapter, we will investigate properties of generalized Freud polynomials.

Chapter 4

Generalized Freud polynomials

4.1 Introduction

In the previous chapter it was shown how semiclassical generalized Freud polynomials arise from semiclassical Laguerre polynomials via a symmetrization of the semiclassical Laguerre weight function. In this chapter we study properties of the semiclassical generalized Freud polynomials. Most of the results obtained in this chapter have been published in [41].

4.2 The generalized Freud weight

Semiclassical generalized Freud polynomials are orthogonal with respect to the semiclassical weight

$$w_\lambda(x;t) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \quad x \in \mathbb{R}, \tag{4.2.1}$$

with parameters $\lambda > 0$ and $t \in \mathbb{R}$. The orthogonality relation for monic generalized Freud polynomials, t being a free real parameter, is given by [41]

$$\int_{\mathbb{R}} S_n(x;t)S_m(x;t)|x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right)\, dx = h_n\delta_{mn}, \quad h_n > 0, \tag{4.2.2}$$

where the normalization constant h_{n-1} is defined by

$$h_n = \int_{\mathbb{R}} xS_n(x;t)S_{n-1}(x;t)|x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right)\, dx. \tag{4.2.3}$$

Monic orthogonal polynomials with respect to the symmetric weight (4.2.1) satisfy the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)\ S_{n-1}(x;t), \tag{4.2.4}$$

where $S_{-1} \equiv 0$ and $S_0 \equiv 1$.

Since the weight w_λ is even, $S_n(x;t)$ is an even polynomial for n even and an odd polynomial for n odd [132, p.29].

Multiplying both sides of (4.2.4) by $S_{n-1}(x;t)w_\lambda(x;t)$ and integrating over the support of the weight, we obtain

$$\beta_n(t;\lambda) = \frac{1}{h_{n-1}} \int_{\mathbb{R}} xS_n(x;t)S_{n-1}(x;t)|x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right)\, dx, \tag{4.2.5}$$

where h_{n-1} is the normalization constant given in (4.2.3). In view of (4.2.4) and (4.2.5), we observe that the sequence of recurrence coefficient $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ completely determines the orthogonal polynomials associated with the weight function w_λ on $\mathbb{R}$.

4.2.1 Pearson's equation for the generalized Freud weight

The weight function w_λ in (4.2.1) is differentiable on the non-compact support $\mathbb{R}$ for $\lambda > 0$ and satisfies the Pearson's differential equation (1.1.2) with $\sigma(x;t) = x$ and $\tau(x;t) = -4x^4 + 2tx^2 + 2\lambda + 2$. Since $\deg(\sigma) = 1$ and $\deg(\tau) = 4$, the polynomial sequence $\{S_n\}_{n=0}^{\infty}$, orthogonal with respect to (4.2.1), is said to constitute a family of *semiclassical orthogonal polynomials* (cf. [13, 14, 39, 67, 96]).

4.2.2 The moments for the generalized Freud weight

Moments of certain semiclassical weights provide the link between the weight function and the associated Painlevé equation (cf. [39, 41]).

The following lemma assures the finiteness of the moments of the semiclassical generalized Freud weight.

Lemma 4.2.1. *Let $x, t \in \mathbb{R}$ and $\lambda > 0$. For the generalized Freud weight w_λ in* (4.2.1), *the first moment $\mu_0(t;\lambda)$ is finite.*

Proof. The first moment $\mu_0(t;\lambda)$ takes the form

$$\mu_0(t;\lambda) = \int_{\mathbb{R}} |x|^{2\lambda+1} \exp(-x^4 + tx^2)\, dx = 2\int_0^\infty x^{2\lambda+1} \exp(-x^4 + tx^2)\, dx. \qquad (4.2.6)$$

Since the integrand $w(x;t) = x^{2\lambda+1}\exp(-x^4 + tx^2)$ on the right hand side of (4.2.6) is continuous on $[0,\infty)$, it is integrable on $[0,K]$ for any $K > 0$.

In order to prove $\displaystyle\int_K^\infty w(x;t)\, dx$ is finite, note that $\displaystyle\lim_{x\to\infty} x^2 w(x;t) = 0$. By definition, there exists an $N > 0$ such that $x^2 w(x;t) < 1$ whenever $x > N$. Since $\displaystyle\int_N^\infty \frac{dx}{x^2} < \infty$, we have that $\displaystyle\int_N^\infty w(x;t)\, dx < \infty$ for $N > 0$ and, in particular, $N = K$. Hence, $\displaystyle\int_0^\infty w(x;t)\, dx < \infty$. $\square$

Our next result proves that the first moment can be explicitly stated in terms of the parabolic cylinder function.

Proposition 4.2.1. *[41, Section 4]. Let $x, t \in \mathbb{R}$ and $\lambda > 0$. For the generalized Freud weight* (4.2.1), *the first moment $\mu_0(t;\lambda)$ is given by*

$$\left.\begin{aligned} \mu_0(t;\lambda) &= 2^{\frac{-\lambda-1}{2}}\Gamma(\lambda+1)\, \exp\left(\tfrac{1}{8}t^2\right)\, D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}t\right), \\ \mu_0(t;n) &= \tfrac{1}{2}\sqrt{\pi}\, \tfrac{d^n}{dt^n}\left(\exp\left(\tfrac{1}{4}t^2\right)\left[1 + \operatorname{erf}\left(\tfrac{1}{2}t\right)\right] \right), \qquad n \in \mathbb{N} \end{aligned}\right\}$$

with $D_v(\xi)$ is the parabolic cylinder function and $\operatorname{erf}(z)$ *is the error function.*

Proof. Let $x^2 = \dfrac{s}{\sqrt{2}}$ and $\xi = \frac{-t}{\sqrt{2}}$ in (4.2.6), to obtain

$$\begin{aligned} \mu_0(t;\lambda) &= 2\int_0^\infty \left(\frac{s}{\sqrt{2}}\right)^{\lambda+\frac{1}{2}} \exp\left[-\frac{1}{2}s^2 - \xi s\right]\left(2^{-\frac{5}{4}}\, s^{-\frac{1}{2}}\, ds\right) \\ &= 2^{\frac{-\lambda-1}{2}} \int_0^\infty s^\lambda\, \exp\left[-\frac{1}{2}s^2 - \xi\, s\right] ds \\ &= 2^{\frac{-\lambda-1}{2}}\, \Gamma(\lambda+1)\, \exp\left(\frac{1}{4}\xi^2\right)\, D_{-\lambda-1}(\xi) \end{aligned}$$

using the integral representation in (2.3.5). Also, when $\lambda = n \in \mathbb{Z}^+$,

$$D_{-n-1}(\xi) = \sqrt{\frac{\pi}{2}}\, \frac{(-1)^n}{n!}\, \exp\left(-\frac{1}{4}\xi^2\right)\, \frac{d^n}{d\xi^n}\left(\exp\left(\frac{1}{2}\xi^2\right)\, \operatorname{erfc}\left(\frac{1}{2}\sqrt{2}\xi\right)\right),$$

with erfc(z) the complementary error function (2.3.7). Since erf$(-z) = 1 + \text{erf}(z)$, we have

$$\mu_0(t;n) = \frac{1}{2}\sqrt{\pi}\frac{d^n}{dt^n}\left(\exp\left(\frac{1}{4}t^2\right) \left[1 + \text{erf}\left(\frac{1}{2}t\right)\right]\right).$$

□

In order to obtain an expression for the higher order moments of the generalized Freud weight, we require the following result which provides conditions under which the order of integration and differentiation for functions of two variables may be reversed.

Lemma 4.2.2. *[79, Theorem 16.11]. Let $I \subset \mathbb{R}$ be an open interval and $f : \mathbb{R}\times I \to \mathbb{R}$. Assume that*

(i) f is integrable with respect to x for every fixed $t \in I$;

(ii) for almost all $x \in \mathbb{R}$, $f(x,t)$ is differentiable on $\mathbb{R}$ with respect to t;

(iii) there exists an integrable function $g : \mathbb{R} \to \mathbb{R}$ with the property that for every $t \in I$,

$$\left|\frac{\partial f(x,t)}{\partial t}\right| \le g(x)$$

holds for almost all $x \in \mathbb{R}$.

Then

$$\frac{d}{dt}\int_{\mathbb{R}} f(x,t)\, dx = \int_{\mathbb{R}} \frac{\partial f(x,t)}{\partial t}\, dx.$$

The higher order moments for the generalized Freud weight are:

Theorem 4.2.1. *[41, Section 4]. Let $t \in I \subset \mathbb{R}$ and $\lambda > 0$. For the generalized Freud weight* (4.2.1)*, the higher order moments satisfy*

$$\mu_{2n}(t;\lambda) = \frac{d^n}{dt^n}\int_{-\infty}^{\infty} |x|^{2\lambda+1}\exp\left(-x^4 + tx^2\right)\, dx = \frac{d^n}{dt^n}\mu_0(t;\lambda), \tag{4.2.7}$$

whilst the odd ones are $\mu_{2n+1}(t;\lambda) = 0,\ n = 1, 2, \ldots$.

Proof. Since the generalized weight function (4.2.1) is symmetric on the non-compact support $\mathbb{R}$, we consider the half-range generalized Freud weight function

$$w(x;t) = x^{2\lambda+1}\exp\left(-x^4 + tx^2\right), \quad x \in \mathbb{R}^+,$$

with $\lambda > 0$ and $t \in \mathbb{R}$. Note that the function w is a rapidly decreasing function. Besides, $w(x;t)$ is differentiable with respect to t and integrable with respect to x for every real t by Lemma 4.2.1. Furthermore,

$$\frac{\partial w(x,t)}{\partial t} = x^{2\lambda+3} \exp\left(-x^4 + tx^2\right)$$

is continuous on $\mathbb{R}^+$. For $t \leq 0$ and $x \in \mathbb{R}^+$, we have $\exp(tx^2) \leq 1$, since $tx^2 \leq 0$. Thus,

$$\left|\frac{\partial w(x,t)}{\partial t}\right| = |x^{2\lambda+1} \exp\left(-x^4 + tx^2\right)| \leq x^{2\lambda+3} \exp\left(-x^4\right) = G(x),$$

with

$$\int_0^\infty G(x)\,dx = \int_0^\infty x^{2\lambda+3} \exp\left(-x^4\right)\,dx = \frac{1}{4}\,\Gamma\left(\frac{\lambda+2}{2}\right) < \infty, \quad \text{for} \quad \lambda > 0,$$

where Γ is defined in (2.3.2). For $t \in [0, A]$, $A \in \mathbb{R}^+$, we have

$$\left|\frac{\partial w(x,t)}{\partial t}\right| = |x^{2\lambda+3} \exp\left(-x^4 + tx^2\right)| \leq x^{2\lambda+3} \exp\left(-x^4 + Ax^2\right) = G(x),$$

with $g(x)$ integrable for $x \in \mathbb{R}^+$ and $\lambda > 0$.

Now, since all the conditions of Lemma 4.2.2 are satisfied, the proof of (4.2.7) follows by mathematical induction. For $n = 1$, we have

$$\begin{aligned}
\frac{d}{dt}\mu_0(t,\lambda) &= \frac{d}{dt}\int_{\mathbb{R}} |x|^{2\lambda+1} \exp(-x^4 + tx^2)\,dx \\
&= 2\frac{d}{dt}\int_0^\infty x^{2\lambda+1} \exp(-x^4 + tx^2)\,dx \\
&= 2\int_0^\infty x^{2+(2\lambda+1)} \exp(-x^4 + tx^2)\,dx \\
&= \int_{-\infty}^\infty x^2|x|^{2\lambda+1} \exp(-x^4 + tx^2)\,dx \\
&= \mu_2(t,\lambda).
\end{aligned}$$

Assume, by inductive argument, that

$$\frac{d^n}{dt^n}\mu_0(t,\lambda) = \mu_{2n}(t,\lambda).$$

It is required to prove that

$$\frac{d^{n+1}}{dt^{n+1}}\mu_0(t,\lambda) = \mu_{2n+2}(t,\lambda).$$

We see that $x^{2n}x^{2\lambda+1}\exp(-x^4+tx^2),\ x\in\mathbb{R}^+,\ t\in I$, also satisfies the conditions of Lemma 4.2.2. Then,

$$\begin{aligned}
\frac{d^{n+1}}{dt^{n+1}}\mu_0(t,\lambda) &= \frac{d}{dt}\left(\frac{d^n}{dt^n}\mu_0(t,\lambda)\right) = \frac{d}{dt}\mu_{2n}(t,\lambda)\\
&= \frac{d}{dt}\int_{\mathbb{R}} x^{2n}|x|^{2\lambda+1}\exp(-x^4+tx^2)\,dx\\
&= 2\int_0^\infty \frac{d}{dt}\left(x^{2(\lambda+n)+1}\exp(-x^4+tx^2)\right)\,dx\\
&= 2\int_0^\infty x^{2(\lambda+n)+1}x^2\exp(-x^4+tx^2)\,dx\\
&= \int_{-\infty}^\infty x^{2n+2}|x|^{2\lambda+1}x^2\exp(-x^4+tx^2)\,dx\\
&= \mu_{2n+2}(t,\lambda)\equiv\mu_0(t;n+\lambda+1),
\end{aligned}$$

and this completes the induction argument.

Finally, for the odd moments μ_{2n+1}, we have

$$\mu_{2n+1}(t;\lambda)=\int_{-\infty}^{\infty}x^{2n+1}\,|x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx=0,\quad n\in\mathbb{N},$$

since the integrand is odd. □

4.3 Recurrence coefficients associated with generalized Freud polynomials

Recurrence coefficients of a three-term recurrence relation associated with certain semi-classical orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations and associated discrete Painlevé equations. We determine explicit expressions for the recurrence coefficient $\beta_n(t;\lambda)$ in the three-term recurrence relation (4.2.4) satisfied by monic generalized Freud polynomials by using their connection with solutions of the fourth Painlevé equation $\mathrm{P_{IV}}$ (3.3.10) and the discrete Painlevé equation $\mathrm{dP_I}$.

Lemma 4.3.1. *[41, Lemma 4]. The recurrence coefficient* $\beta_n(t;\lambda)$ *in* (4.2.4) *satisfies*

(i) the nonlinear difference equation, known as discrete Painlevé $\mathrm{P_I}$ *(*$\mathrm{dP_I}$*)*

$$\beta_{n+1}+\beta_n+\beta_{n-1}=\tfrac{1}{2}t+\frac{2n+(2\lambda+1)[1-(-1)^n]}{8\beta_n},\tag{4.3.1}$$

where $\beta_0 = 0$ *and* β_1 *is given by*

$$\beta_1(t;\lambda) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma\left(\frac{2(\lambda+n)+4}{4}\right)}{\displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma\left(\frac{2(\lambda+n)+2}{4}\right)};$$

(ii) the fourth Painlevé equation $\mathrm{P_{IV}}$

$$\frac{d^2\beta_n}{dt^2} = \frac{1}{2\beta_n}\left(\frac{d\beta_n}{dt}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}A_n)\beta_n + \frac{B_n}{16\beta_n}, \tag{4.3.2}$$

where the parameters A_n *and* B_n *are given by*

$$\begin{pmatrix} A_{2n} \\ B_{2n} \end{pmatrix} = \begin{pmatrix} -2\lambda - n - 1 \\ -2n^2 \end{pmatrix}; \quad \begin{pmatrix} A_{2n+1} \\ B_{2n+1} \end{pmatrix} = \begin{pmatrix} \lambda - n \\ -2(\lambda+n+1)^2 \end{pmatrix}.$$

Proof. (i) We use an approach due to Freud described in [134, Section 2]. Consider, for fixed $t \in \mathbb{R}$,

$$\mathbb{I}_n = \frac{1}{h_n}\int_{-\infty}^{\infty} \left(S_n(x;t)\ S_{n-1}(x;t)\right)'\ w_\lambda(x;t)\ dx, \tag{4.3.3}$$

where $S_n(x;t)$ are the monic polynomials orthogonal with respect to the generalized Freud weight $w_\lambda(x;t)$ given in (4.2.1) and the constant h_n in (2.1.3). Then

$$\begin{aligned} \mathbb{I}_n &= \frac{1}{h_n}\int_{-\infty}^{\infty} \left(S_n'(x;t)\ S_{n-1}(x;t) + S_n(x;t)\ S_{n-1}'(x;t)\right)\ w_\lambda(x;t)\ dx \\ &= \frac{1}{h_n}\int_{-\infty}^{\infty} S_n'(x;t)\ S_{n-1}(x;t)\ w_\lambda(x;t)\ dx \\ &= \frac{1}{h_n}\int_{-\infty}^{\infty} (nx^{n-1} + R_{n-2})\ S_{n-1}(x;t)\ w_\lambda(x;t)\ dx \\ &= \frac{h_{n-1}}{h_n} n, \end{aligned} \tag{4.3.4}$$

where $R_{n-2} \in \mathbb{P}_{n-2}$. Evaluating (4.3.3) using integration by parts, we obtain

$$\begin{aligned} \mathbb{I}_n h_n &= \left[S_n(x;t)\ S_{n-1}(x;t) w_\lambda(x;t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} S_n(x;t)\ S_{n-1}(x;t)\ w_\lambda'(x;t)\ dx \\ &= -\int_{-\infty}^{\infty} S_n(x;t)\ S_{n-1}(x;t)\ \left(\frac{2\lambda+1}{x} - 4x^3 + 2tx\right)\ w_\lambda(x;t)\ dx, \end{aligned} \tag{4.3.5}$$

in view of $\left[S_n(x;t)\ S_{n-1}(x;t) w_\lambda(x;t)\right]_{-\infty}^{\infty} = 0$, where the boundary terms vanish as the expression $-x^4 + tx^2$ in the weight $w_\lambda(x;t)$ only consists of even powers of

x and hence will dominate the limit as $x \to \pm\infty$. Since w_λ is even, the integral expression

$$\int_{-\infty}^{\infty} S_n(x;t)\ S_{n-1}(x;t)\ \frac{1}{x}\ w_\lambda(x;t)\ dx = 0, \tag{4.3.6a}$$

when n is even and, when n is odd, $\frac{S_n(x;t)}{x}$ is a polynomial of degree $n-1$ and hence

$$\int_{-\infty}^{\infty} \frac{S_n(x;t)}{x}\ S_{n-1}(x;t)\ w_\lambda(x;t)\ dx = h_{n-1}. \tag{4.3.6b}$$

Iterating the recurrence relation (4.2.4), yields

$$x^2 S_n(x;t) = S_{n+2}(x;t) + (\beta_n + \beta_{n+1})\, S_n(x;t) + \beta_n\beta_{n-1}S_{n-2}(x;t), \tag{4.3.7a}$$

$$\begin{aligned} x^3 S_n(x;t) &= S_{n+3}(x;t) + (\beta_n + \beta_{n+1} + \beta_{n+2})\, S_n(x;t) + \beta_n\,(\beta_{n-1} + \beta_n + \beta_{n+1})\, S_{n-1}(x;t) \\ &\quad + \beta_n\beta_{n-1}\ \beta_{n-2}S_{n-3}(x;t). \end{aligned} \tag{4.3.7b}$$

Employing the identities (4.3.7) and Pearson's equation for the weight (4.2.1) together with (4.3.6) into (4.3.5), we obtain

$$\mathbb{I}_n h_n = 4\beta_n\ h_{n-1}\left(\beta_{n-1} + \beta_n + \beta_{n+1} - \frac{t}{2}\right) - (2\lambda+1)\Omega_n h_{n-1}, \tag{4.3.8}$$

where Ω_n is given in (2.9.6). Note that (4.3.8) and (4.3.4) yield (4.3.1).

Next, using a Taylor expansion, we obtain, for $j = 0, 1, 2, \ldots$,

$$\begin{aligned} \mu_j(t,\lambda) &= \int_{-\infty}^{\infty} x^j |x|^{2\lambda+1} \exp(-x^4 + tx^2)\, dx \\ &= 2\int_0^{\infty} x^{j+2\lambda+1} \exp(-x^4)\ \left[\sum_{n=0}^{\infty} \frac{(tx^2)^n}{n!}\right]\ dx \\ &= 2\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} x^{j+2\lambda+2n+1} \exp(-x^4)\, dx. \end{aligned} \tag{4.3.9}$$

Using the transformation $y = x^4$ in (4.3.9), we obtain

$$\begin{aligned} \mu_j(t,\lambda) &= 2\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} x^{j+2\lambda+1+2n} \exp(-x^4)\, dx, \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} y^{\frac{j+2\lambda+1+2n}{4}} \exp(-y)\ y^{-\frac{3}{4}}\, dy, \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{t^n}{n!}\ \Gamma\left(\frac{j+2\lambda+2n+2}{4}\right). \end{aligned}$$

Hence, the coefficient β_1 can be given as the ratio of integrals

$$\beta_1(t;\lambda) = \frac{\mu_2(t;\lambda)}{\mu_0(t;\lambda)} = \frac{\displaystyle\int_{-\infty}^{\infty} x^2|x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx}{\displaystyle\int_{-\infty}^{\infty} |x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx}$$

$$= \frac{\displaystyle\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\Gamma\left(\frac{2(\lambda+n)+4}{4}\right)}{\displaystyle\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\Gamma\left(\frac{2(\lambda+n)+2}{4}\right)},$$

and this completes the proof.

(ii) See [50, Theorem 6.1].

□

Remark 4.3.1. *1. Equation (4.3.2) is equivalent to* $\mathrm{P_{IV}}$ *(3.3.10) through the transformation* $\beta_n(t;\lambda) = \frac{1}{2}\nu(z)$*, with* $z = -\frac{1}{2}t$*. Hence*

$$\beta_{2n}(t;\lambda) = \tfrac{1}{2}\nu\big(z; -2\lambda-n-1, -2n^2\big), \tag{4.3.10a}$$

$$\beta_{2n+1}(t;\lambda) = \tfrac{1}{2}\nu\big(z; \lambda-n, -2(\lambda+n+1)^2\big), \tag{4.3.10b}$$

with $z = -\frac{1}{2}t$*, where* $\nu(z;A,B)$ *satisfies* $\mathrm{P_{IV}}$ *(3.3.10). The conditions on the* $\mathrm{P_{IV}}$ *parameters in (4.3.10) are precisely those for which* $\mathrm{P_{IV}}$ *has solutions expressible in terms of the parabolic cylinder function [58]*

$$\psi(z) = \mu_0(-2z;\lambda) = \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}}\exp\left(\tfrac{1}{2}z^2\right)D_{-\lambda-1}\big(\sqrt{2}\,z\big),$$

See also [39, Theorem 3.5].

2. The link between the differential equation (4.3.2) and the difference equation (4.3.1) is given by the Bäcklund transformations

$$\beta_{n+1} = \frac{1}{2\beta_n}\frac{d\beta_n}{dt} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}, \tag{4.3.11a}$$

$$\beta_{n-1} = -\frac{1}{2\beta_n}\frac{d\beta_n}{dt} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}, \tag{4.3.11b}$$

with $c_n = \frac{1}{2}n + \frac{1}{4}(2\lambda+1)[1-(-1)^n]$*. Letting* $n \to n+1$ *in (4.3.11b) and substituting into (4.3.11a) gives the differential equation (4.3.2), while eliminating the derivative yields the difference equation (4.3.1).*

3. *The nonlinear discrete equation (4.3.1) appears in the paper by Freud [55, Equation. 23, p.5]; See [134] and [5, §2] for a historical review of the origin and study of equation (4.3.1).*

Next, we obtain an explicit formulation of the recurrence coefficient $\beta_n(t;\lambda)$ in (4.2.4).

Lemma 4.3.2. *[41, Lemma 5]. Let $x, t \in \mathbb{R}$ and $\lambda > 0$. The sequence of recurrence coefficients $\{\beta_n(t;\lambda)\}_{n\geq 0}$ in the three-term recurrence relation (4.2.4) are explicitly given by*

$$\beta_{2n}(t;\lambda) = \frac{d}{dt}\ln\frac{\tau_n(t;\lambda+1)}{\tau_n(t;\lambda)}, \tag{4.3.12a}$$

$$\beta_{2n+1}(t;\lambda) = \frac{d}{dt}\ln\frac{\tau_{n+1}(t;\lambda)}{\tau_n(t;\lambda+1)}, \tag{4.3.12b}$$

where $\tau_n(t;\lambda)$ is the Wronskian given by

$$\tau_n(t;\lambda) = \mathcal{W}\left(\phi_\lambda, \frac{d\phi_\lambda}{dt}, \ldots, \frac{d^{n-1}\phi_\lambda}{dt^{n-1}}\right) = \det\left[\frac{d^{j+k}}{dt^{j+k}}\mu_0(t;\lambda)\right]_{j,k=0}^{n-1}, \quad \tau_0(t;\lambda) = 1, \tag{4.3.12c}$$

with

$$\phi_\lambda(t) = \mu_0(t;\lambda) = \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}}\exp\left(\tfrac{1}{8}t^2\right) D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right). \tag{4.3.12d}$$

Proof. From the parabolic cylinder solutions of $\mathrm{P_{IV}}$ (3.3.10) given in [39, Theorem 3.5], it is easily shown that the equation

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \tfrac{3}{2}y^3 - ty^2 + \left(\tfrac{1}{8}t^2 - \tfrac{1}{2}A\right)y + \frac{B}{16y}, \tag{4.3.13}$$

has the solutions $\left\{y_n^{[j]}\left(t; A_n^{[j]}, B_n^{[j]}\right)\right\}_{j=1,2,3}$, for $n = 1, 2, \ldots$, and

$$y_n^{[1]}\left(t;\lambda+2n-1,-2\lambda^2\right) = \tfrac{1}{2}t + \frac{d}{dt}\ln\frac{\tau_{n-1}(t;\lambda)}{\tau_n(t;\lambda)}, \tag{4.3.14a}$$

$$y_n^{[2]}\left(t;-2\lambda-n-1,-2n^2\right) = \frac{d}{dt}\ln\frac{\tau_n(t;\lambda+1)}{\tau_n(t;\lambda)}, \tag{4.3.14b}$$

$$y_n^{[3]}\left(t;\lambda-n+1,-2(\lambda+n)^2\right) = \frac{d}{dt}\ln\frac{\tau_n(t;\lambda)}{\tau_{n-1}(t;\lambda+1)}, \tag{4.3.14c}$$

where $\tau_n(t;\lambda)$ is the Wronskian (4.3.12c). Comparing (4.3.10) and (4.3.14) gives the desired result. □

The first few recurrence coefficients $\beta_n(t;\lambda)$ are given by

$$\left.\begin{aligned}
\beta_1(t;\lambda) &= \Phi_\lambda,\\
\beta_2(t;\lambda) &= \frac{t}{2} - \Phi_\lambda + \frac{\lambda+1}{2\Phi_\lambda},\\
\beta_3(t;\lambda) &= -\frac{\lambda+1}{2\Phi_\lambda} - \frac{\Phi_\lambda}{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1},\\
\beta_4(t;\lambda) &= \frac{t}{2(\lambda+2)} + \frac{\Phi_\lambda}{2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1} + \frac{(\lambda+1)\big[(t^2+2\lambda+4)\Phi_\lambda + (\lambda+1)t\big]}{2(\lambda+2)\big[2(\lambda+2)\Phi_\lambda^2 - (\lambda+1)t\Phi_\lambda - (\lambda+1)^2\big]},
\end{aligned}\right\} \tag{4.3.15}$$

where

$$\Phi_\lambda(t) = \frac{d}{dt}\ln\left(D_{-\lambda-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)\exp\left(\tfrac{1}{8}t^2\right)\right) = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}{D_{-\lambda-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}$$

and substituting these into the recurrence relation (4.2.4), we obtain the first few polynomials $S_n(x;t)$:

$$\begin{aligned}
S_1(x;t) &= x,\\
S_2(x;t) &= x^2 - \Phi_\lambda,\\
S_3(x;t) &= x^3 - \frac{t\Phi_\lambda + \lambda + 1}{2\Phi_\lambda}\,x,\\
S_4(x;t) &= x^4 - \frac{2t\Phi_\lambda^2 - (t^2+2)\Phi_\lambda - (\lambda+1)t}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)}\,x^2\\
&\quad - \frac{2(\lambda+2)\Phi_\lambda^2 - (\lambda+1)t\Phi_\lambda - (\lambda+1)^2}{2(2\Phi_\lambda^2 - t\Phi_\lambda - \lambda - 1)}.
\end{aligned}$$

4.4 Coefficients of generalized Freud polynomials

Having explicitly determined the recurrence coefficient $\beta_n(t;\lambda)$ in terms of the first moment and special functions associated with the first moment given in (4.3.12d), the aim of this section is to provide an expression for the coefficients of the monic generalized Freud polynomials following an approach described in [97].

Theorem 4.4.1. *For a fixed $t \in \mathbb{R}$, the sequence of monic polynomials $\{S_n(x;t)\}_{n=0}^{\infty}$, orthogonal with respect to the semiclassical weight $w_\lambda(x;t) = |x|^{2\lambda+1}\exp(-x^4+tx^2)$, $\lambda > 0$, $x \in \mathbb{R}$, is given by*

$$S_n(x;t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)\, x^{n-2k}, \tag{4.4.1a}$$

where $\Psi_0(n) = 1$ *and, for* $k \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$,

$$\Psi_k(n) = (-1)^k \sum_{j_1=1}^{n+1-2k} \beta_{j_1}(t;\lambda) \sum_{j_2=j_1+2}^{n+3-2k} \beta_{j_2}(t;\lambda) \sum_{j_3=j_2+2}^{n+5-2k} \beta_{j_3}(t;\lambda) \cdots \sum_{j_k=j_{k-1}+2}^{n-1} \beta_{j_k}(t;\lambda), \tag{4.4.1b}$$

with $\beta_j(t;\lambda)$ *given by* (4.3.12).

Proof. Since the polynomials $S_n(x;t)$ are monic and symmetric of degree n, i.e.,

$$S_n(-x;t) = (-1)^n S_n(x;t),$$

we obtain for a fixed $t \in \mathbb{R}$

$$\left.\begin{aligned} S_{2n}(x;t) &= \sum_{\ell=0}^{n} d_{2n-2\ell} x^{2n-2\ell} \\ S_{2n+1}(x;t) &= \sum_{\ell=0}^{n} d_{2n-2\ell+1} x^{2n-2\ell+1} \end{aligned}\right\} \tag{4.4.2}$$

where $d_{n-2k} = \Psi_k(n)$ with $\Psi_0(n) = 1$ and $\Psi_k(n) = 0$ for $k > \lfloor \frac{n}{2} \rfloor$. Substituting (4.4.1a) into the three-term recurrence relation (4.2.4) and comparing the coefficients on each side yields

$$\Psi_k(n+1) - \Psi_k(n) = -\beta_n(t;\lambda)\Psi_{k-1}(n-1). \tag{4.4.3}$$

To show (4.4.1b), we apply induction on k. We observe from (4.4.3) that for $k = 1$, we have

$$\Psi_1(n) - \Psi_1(n-1) = -\beta_{n-1},$$

and taking a telescopic sum then gives

$$\Psi_1(n) = -\sum_{j_1=0}^{n-1} \beta_{j_1}(t;\lambda), \text{ for every } n \geq 1.$$

Suppose that (4.4.1b) holds for values up to $k-1$ for every $n \in \mathbb{N}$, i.e.,

$$\Psi_{k-1}(n) = (-1)^{k-1} \sum_{j_1=1}^{n+3-2k} \beta_{j_1}(t;\lambda) \sum_{j_2=j_1+2}^{n+5-2k} \beta_{j_2}(t;\lambda) \sum_{j_3=j_2+2}^{n+7-2k} \beta_{j_3}(t;\lambda) \cdots \sum_{j_{k-1}=j_{k-2}+2}^{n-1} \beta_{j_{k-1}}(t;\lambda). \tag{4.4.4}$$

Now, iterating (4.4.3), we obtain

$$\begin{aligned}
\Psi_k(n) &= \Psi_k(n-1) - \beta_{n-1}\Psi_{k-1}(n-2),\\
&= \Psi_k(n-2) - \beta_{n-2}\Psi_{k-1}(n-3) - \beta_{n-1}\Psi_{k-1}(n-2),\\
&= \Psi_k(n-3) - \beta_{n-3}\Psi_{k-1}(n-4) - \beta_{n-2}\Psi_{k-1}(n-3) - \beta_{n-1}\Psi_{k-1}(n-2),\\
&\quad\vdots\\
&= -\beta_{2k-1}\Psi_{k-1}(2k-2) - \beta_{2k}\Psi_{k-1}(2k-1) - \cdots - \beta_{n-2}\Psi_{k-1}(n-3) - \beta_{n-1}\Psi_{k-1}(n-2).
\end{aligned} \tag{4.4.5}$$

Substituting (4.4.4) into (4.4.5) yields (4.4.1b) and hence the result holds true for $k \in \mathbb{N}$ and this completes the inductive proof. □

An alternative expression for (4.4.1) is given in the following corollary.

Corollary 4.4.1. *For a fixed $t \in \mathbb{R}$, the sequence of monic polynomials $\{S_n(x;t)\}_{n=0}^{\infty}$ orthogonal with respect to the semiclassical weight $w_\lambda(x;t) = |x|^{2\lambda+1}\exp(-x^4+tx^2)$, $\lambda > 0$, $x \in \mathbb{R}$, is given by*

$$S_n(x;t) = x^n + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \left(\sum_{k \in W(n,m)} \beta_{k_1}\beta_{k_2}\cdots\beta_{k_{m-1}}\beta_{k_m} \right) x^{n-2m},$$

where

$$W(n,m) = \{k \in \mathbb{N}^m \mid k_{j+1} \geq k_j + 2 \ \ \textit{for} \ \ 1 \leq j \leq m-1, \quad 1 \leq k_1, k_m < n\}$$

with $\beta_j(t;\lambda)$ given by (4.3.12).

Proof. The result follows using an analogous argument as in the proof of Theorem 4.4.1. □

Proposition 4.4.1. *For the monic generalized Freud polynomials, the normalization constant h_n in* (4.2.2) *is given by*

$$h_n = \langle S_n, S_n \rangle_{|x|^{2\lambda+1}\exp(-x^4+tx^2)} = \|S_n\|^2_{|x|^{2\lambda+1}\exp(-x^4+tx^2)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)\ \mu_{2n-2k}(t;\lambda),$$

where $\Psi_k(n)$ is given in (4.4.1b).

Proof. By using the definition of h_n in (2.1.3) and the fact in (2.6.2), we have

$$\begin{aligned}
h_n &= \langle S_n, S_n \rangle_{|x|^{2\lambda+1}\exp(-x^4+tx^2)} = \langle S_n, x^n \rangle_{|x|^{2\lambda+1}\exp(-x^4+tx^2)} \\
&= \int_{\mathbb{R}} x^n \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n) x^{n-2k} \right) |x|^{2\lambda+1} \exp\left(-x^4+tx^2\right)\, dx \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n) \int_{\mathbb{R}} x^{2n-2k} |x|^{2\lambda+1} \exp\left(-x^4+tx^2\right)\, dx \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)\ \mu_{2n-2k}(t;\lambda). \qquad (4.4.6)
\end{aligned}$$

□

Remark 4.4.1. *The expression for h_n in (4.4.6) is positive since $\Psi_k(n)$ and $\{\mu_{2j}\}_{j\in\mathbb{N}_0}$ are generically positive from the symmetry property of the generalized Freud polynomials.*

Remark 4.4.2. *For $\lambda > -1$ and $t \in \mathbb{R}$, we see from (4.2.7) that*

$$h_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)\ \mu_{2n-2k}(t;\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n) \frac{d^{n-k}}{dt^{n-k}} \mu_0(t;\lambda).$$

4.5 The differential-difference equation satisfied by generalized Freud polynomials

For fixed $t \in \mathbb{R}$, the coefficients $A_n(x;t)$ and $B_n(x;t)$ in the relation

$$\frac{dP_n}{dx}(x;t) = -B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t), \qquad (4.5.1)$$

satisfied by semiclassical orthogonal polynomials are of interest since differentiation of this differential-difference equation yields the second order differential equation satisfied by the orthogonal polynomials.

In this section we study the derivation of a differential-difference equation satisfied by generalized Freud polynomials using two different techniques, one based on ladder operators and the second using Shohat's approach of quasi-orthogonality.

4.5.1 The ladder operator approach

The method of ladder operators was introduced by Chen and Ismail in [28] and a good summary of the technique is provided in [71, Theorem 3.2.1].

In [27], Chen and Feigin adapt the method of ladder operators to the situation where the weight function vanishes at one point. Our next result generalizes the work in [27] by giving a more explicit expression for the coefficients in (4.5.1) when the weight function $w_\gamma(x;t) = |x-k|^\gamma \exp\big(-v_0(x;t)\big),\ x,\ t,\ k \in \mathbb{R}$, is positive on the real line, except for one point.

Theorem 4.5.1. *[41, Theorem 2]. Consider the weight*

$$w_\gamma(x;t) = |x-k|^\gamma \exp\big(-v_0(x;t)\big),\ x,\ t,\ k \in \mathbb{R}, \tag{4.5.2}$$

where $v_0(x,t)$ is a continuously differentiable function on $\mathbb{R}$. Assume that the sequence of polynomials $\{P_n(x;t)\}_{n=0}^{\infty}$ satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} P_n(x;t)P_m(x;t)w_\gamma(x;t)dx = h_n\delta_{nm}.$$

Then, for $\gamma > 1$, $P_n(x;t)$ satisfies the differential-difference equation

$$(x-k)\,\frac{dP_n(x;t)}{dx} = -B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t), \tag{4.5.3}$$

where

$$\begin{aligned}
A_n(x;t) &= \frac{x-k}{h_{n-1}}\int_{-\infty}^{\infty} P_n^2(y;t)\left[\frac{v_0'(x;t)-v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy + a_n(t),\\
B_n(x;t) &= \frac{x-k}{h_{n-1}}\int_{-\infty}^{\infty} P_n(y;t)P_{n-1}(y;t)\left[\frac{v_0'(x;t)-v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy + \frac{\gamma}{h_{n-1}}b_n(t),
\end{aligned}$$

with

$$\begin{aligned}
a_n(t) &= \frac{\gamma}{h_{n-1}}\int_{-\infty}^{\infty} \frac{P_n^2(y;t)}{y-k}\,w_\gamma(y;t)\,dy, \qquad (4.5.4)\\
b_n(t) &= \int_{-\infty}^{\infty} \frac{P_n(y;t)P_{n-1}(y;t)}{y-k}\,w_\gamma(y;t)\,dy.
\end{aligned}$$

Proof. For a fixed $t \in \mathbb{R}$, $P_n'(x;t)$ is a polynomial of degree $n-1$ and can be expressed in terms of the orthogonal basis as

$$P_n'(x;t) = \sum_{j=0}^{n-1} c_{n,j}P_j(x;t). \tag{4.5.5}$$

Applying the orthogonality relation and integrating by parts, we obtain

$$
\begin{aligned}
c_{n,j}\, h_j &= \int_{-\infty}^{\infty} P_n'(y;t)P_j(y;t)w_\gamma(y;t)\,dy \\
&= \Big[P_n(y;t)P_j(y;t)w_\gamma(y;t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} P_n(y;t)\left(P_j'(y;t)w_\gamma(y;t) + P_j(y;t)w_\gamma^{'}(y;t)\right) dy \\
&= -\int_{-\infty}^{\infty} P_n(y;t)P_j(y;t)w_\gamma^{'}(y;t)dy \\
&= \int_{-\infty}^{\infty} P_n(y;t)P_j(y;t)\left[v_0'(y;t) - \frac{\gamma}{y-k}\right] w_\gamma(y;t)\,dy,
\end{aligned}
$$

provided that $\gamma > 1$.

Now, from (4.5.5), we can write

$$
\begin{aligned}
P_n'(x;t) &= \sum_{j=0}^{n-1} \frac{1}{h_j}\left(\int_{-\infty}^{\infty} P_n(y;t)P_j(y;t)\left[v_0'(y;t) - \frac{\gamma}{y-k}\right] w_\gamma(y;t)\,dy\right) P_j(x;t) \\
&= \int_{-\infty}^{\infty} P_n(y;t)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right]\left[v_0'(y;t) - \frac{\gamma}{y-k}\right] w_\gamma(y;t)\,dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} P_n(y;t)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right]\left[v_0'(y;t) - v_0'(x;t)\right] w_\gamma(y;t)\,dy \\
&\quad + v_0'(x;t)\int_{-\infty}^{\infty} P_n(y;t)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right] w_\gamma(y;t)\,dy \\
&\quad - \gamma\int_{-\infty}^{\infty} \frac{P_n(y;t)}{y-k}\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right] w_\gamma(y;t)\,dy \\
&= \int_{-\infty}^{\infty} P_n(y;t)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right]\left[v_0'(y;t) - v_0'(x;t)\right] w_\gamma(y;t)\,dy \\
&\quad - \gamma\int_{-\infty}^{\infty} \frac{P_n(y;t)}{y-k}\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right] w_\gamma(y;t)\,dy.
\end{aligned}
$$

Next, using the orthogonality relation again, we obtain

$$
\begin{aligned}
&(x-k)P_n'(x;t)\\
&= (x-k)\int_{-\infty}^{\infty} P_n(y;t)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right][v_0'(y;t)-v_0'(x;t)]\,w_\gamma(y;t)\,dy\\
&\quad -\gamma\int_{-\infty}^{\infty} P_n(y;t)\left(\frac{x-y}{y-k}+1\right)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right]w_\gamma(y;t)\,dy.\\
&= (x-k)\int_{-\infty}^{\infty} P_n(y;t)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right][v_0'(y;t)-v_0'(x;t)]\,w_\gamma(y;t)\,dy\\
&\quad -\gamma\int_{-\infty}^{\infty} P_n(y;t)\left(\frac{x-y}{y-k}\right)\left[\sum_{j=0}^{n-1}\frac{P_j(y;t)P_j(x;t)}{h_j}\right]w_\gamma(y;t)\,dy.
\end{aligned}
$$

Thus, (4.5.3) now follows from the Christoffel-Darboux formula (2.2.8). □

In the sequel, we consider the symmetric weight

$$w_\gamma(x;t) = |x|^\gamma \exp\big(-v_0(x;t)\big),\ \ x,\, t\in\mathbb{R}, \tag{4.5.6}$$

with $v_0(x;t)$ assumed to be an even, continuously differentiable function on $\mathbb{R}$. The weight (4.5.6) is a symmetric generalization of the semiclassical weight (4.2.1); i.e., $k=0$ in (4.5.2).

Lemma 4.5.1. *[41, Lemma 1]. Consider the weight $w_\gamma(x;t)$ defined by (4.5.6). Assume that the polynomials $\{P_n(x;t)\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to $w_\gamma(x;t)$ and that they satisfy the three-term recurrence relation*

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t;\gamma)P_{n-1}(x;t), \tag{4.5.7}$$

with $P_0\equiv 1$ and $P_1(x;t)=x$. Then the polynomials $P_n(x;t)$ satisfy

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{P_n^2(y;t)}{y}w_\gamma(y;t)\,dy &= 0, \\
\int_{-\infty}^{\infty}\frac{P_n(y;t)P_{n-1}(y;t)}{y}w_\gamma(y;t)\,dy &= \tfrac{1}{2}[1-(-1)^n]\,h_{n-1},
\end{aligned}
\tag{4.5.8}
$$

where $n\in\mathbb{Z}^+$ and

$$h_n = \int_{-\infty}^{\infty} P_n^2(y;t)w_\gamma(y;t)\,dy > 0.$$

Proof. Since the weight $w_\gamma(x;t)$ in (4.5.6) is even, the integrand in (4.5.8) is odd and hence

$$\int_{-\infty}^{\infty}\frac{P_n^2(y;t)}{y}w_\gamma(y;t)\,dy = 0.$$

Furthermore, the monic orthogonal polynomials $P_n(x;t)$ satisfy the three-term recurrence relation (4.5.7), hence

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{P_n(y;t)P_{n-1}(y;t)}{y} w_\gamma(y;t)\, dy\\ &= \int_{-\infty}^{\infty} \frac{\big[yP_{n-1}(y;t) - \beta_{n-1}P_{n-2}(y;t)\big]P_{n-1}(y;t)}{y} w_\gamma(y;t)\, dy\\ &= \int_{-\infty}^{\infty} P_{n-1}^2(y;t)w_\gamma(y;t)\, dy - \beta_{n-1}\int_{-\infty}^{\infty} \frac{P_{n-1}(y;t)P_{n-2}(y;t)}{y} w_\gamma(y;t)\, dy\\ &= h_{n-1} - \beta_{n-1}\int_{-\infty}^{\infty} \frac{P_{n-1}(y;t)P_{n-2}(y;t)}{y} w_\gamma(y;t)\, dy\end{aligned}$$

using (4.5.8). Hence, if we define

$$J_n = \int_{-\infty}^{\infty} \frac{P_n(y;t)P_{n-1}(y;t)}{y} w_\gamma(y;t)\, dy,$$

then J_n satisfies the recurrence relation

$$J_n = h_{n-1} - \beta_{n-1}J_{n-1} = h_{n-1} - \frac{h_{n-1}}{h_{n-2}}J_{n-1},$$

since $\beta_n = h_n/h_{n-1}$. Iterating this gives

$$J_n = \frac{h_{n-1}}{h_{n-3}}J_{n-2} = h_{n-1} - \frac{h_{n-1}}{h_{n-4}}J_{n-3} = \frac{h_{n-1}}{h_{n-5}}J_{n-4} = h_{n-1} - \frac{h_{n-1}}{h_{n-6}}J_{n-5},$$

and so on. Hence, by induction,

$$J_{2N} = \frac{h_{2N-1}}{h_1}J_2, \qquad J_{2N+1} = h_{2N} - \frac{h_{2N}}{h_1}J_2,$$

and since

$$J_2 = \int_{-\infty}^{\infty} \frac{P_2(y;t)P_1(y;t)}{y} w_\gamma(y;t)\, dy = \int_{-\infty}^{\infty} P_2(y;t)w_\gamma(y;t)\, dy = 0,$$

we have that $J_{2N} = 0$ and $J_{2N+1} = h_{2N}$, as required. □

Corollary 4.5.1. *[41, Corollary 1]. Let $w_\gamma(x;t)$ be the weight defined by* (4.5.6). *Assume that the sequence of polynomials $\{P_n(x;t)\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to $w_\gamma(x;t)$. Then, for $\gamma > 1$, $P_n(x;t)$ satisfies the differential-difference equation*

$$x\,\frac{dP_n(x;t)}{dx} = -B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t), \tag{4.5.9}$$

where

$$A_n(x;t) = \frac{x}{h_{n-1}}\int_{-\infty}^{\infty} P_n^2(y;t)\left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\, dy, \tag{4.5.10a}$$

$$B_n(x;t) = \frac{x}{h_{n-1}}\int_{-\infty}^{\infty} P_n(y;t)P_{n-1}(y;t)\left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\, dy + \frac{\gamma}{2}[1-(-1)^n]. \tag{4.5.10b}$$

Proof. The result is an immediate consequence of Theorem 4.5.1 and Lemma 4.5.1. □

Lemma 4.5.2. *[41, Lemma 2]. Let $w_\gamma(x;t)$ be the weight defined by (4.5.6) and the coefficients $A_n(x;t)$ and $B_n(x;t)$ are defined in* (4.5.10)*. Then, when $\gamma > 1$,*

$$B_n(x;t) + B_{n+1}(x;t) = \frac{xA_n(x;t)}{\beta_n} + \gamma - xv_0'(x;t).$$

Proof. From (4.5.10), (4.5.7) and the fact that $h_n = h_{n-1}\beta_n$, we have

$$\begin{aligned}
&B_n(x;t) + B_{n+1}(x;t)\\
&= \frac{x}{h_{n-1}} \int_{-\infty}^{\infty} P_n(y;t)P_{n-1}(y;t) \left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy\\
&\quad + \frac{x}{h_n} \int_{-\infty}^{\infty} P_n(y;t)P_{n+1}(y;t) \left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy + \frac{\gamma}{2}[1-(-1)^n] + \frac{\gamma}{2}[1-(-1)^{n+1}]\\
&= \frac{x}{h_n}\left(\int_{-\infty}^{\infty} P_n(y;t)\left[\beta_n P_{n-1}(y;t) + P_{n+1}(y;t)\right] \times \left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy\right) + \gamma\\
&= \frac{x}{h_n} \int_{-\infty}^{\infty} yP_n^2(y;t) \left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy + \gamma\\
&= \frac{x}{h_n} \int_{-\infty}^{\infty} P_n^2(y;t)\left[v_0'(y;t) - v_0'(x;t)\right] w_\gamma(y;t)\,dy\\
&\quad + \frac{x^2}{h_n} \int_{-\infty}^{\infty} P_n^2(y;t) \left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\gamma(y;t)\,dy + \gamma\\
&= \frac{x}{h_n} \int_{-\infty}^{\infty} P_n^2(y,t)v_0'(y;t)w_\gamma(y;t)\,dy - \frac{xv_0'(x;t)}{h_n}\int_{-\infty}^{\infty} P_n^2(y;t)w_\gamma(y;t)\,dy\\
&\quad + \frac{x}{h_n}h_{n-1}A_n(x;t) + \gamma\\
&= \frac{x\gamma}{h_n}\int_{-\infty}^{\infty} P_n^2(y;t)\frac{w_\gamma(y;t)}{y}\,dy - \frac{x}{h_n}\int_{-\infty}^{\infty} P_n^2(y,t)w_\gamma'(y;t)\,dy\\
&\quad - xv_0'(x;t) + \frac{xA_n(x;t)}{\beta_n} + \gamma,
\end{aligned} \tag{4.5.11}$$

since $w_\gamma'(y;t) = \left(-v_0'(y;t) + \dfrac{\gamma}{y}\right) w_\gamma(y;t)$. The first integral in (4.5.11) vanishes since the integrand is odd, hence it follows, using integration by parts, that

$$\begin{aligned}
B_n(x;t) + B_{n+1}(x;t) = -\frac{x}{h_n}\left(\left[w_\gamma(y;t)P_n^2(y;t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2P_n(y;t)P_n'(y;t)w_\gamma(y;t)\,dy\right)\\
- xv_0'(x;t) + \frac{xA_n(x;t)}{\beta_n} + \gamma,
\end{aligned}$$

and the result follows from the orthogonality of P_n. □

Theorem 4.5.2. *[41, Lemma 3]. Let $w_\gamma(x;t)$ be the weight defined by (4.5.6) and the coefficients $A_n(x;t)$ and $B_n(x;t)$ are defined by (4.5.10). Then*

$$\left(-x\frac{d}{dx} + B_n(x;t) + xv_0'(x;t) - \gamma\right)P_{n-1}(x;t) = \frac{A_{n-1}(x;t)}{\beta_{n-1}}P_n(x;t). \tag{4.5.12}$$

Proof. From the differential-difference equation (4.5.9), we have

$$\begin{aligned} xP_{n-1}'(x;t) &= -B_{n-1}(x;t)P_{n-1}(x;t) + A_{n-1}(x;t)P_{n-2}(x;t) \\ &= -B_{n-1}(x;t)P_{n-1}(x;t) + \frac{A_{n-1}(x;t)}{\beta_{n-1}}\left[xP_{n-1}(x;t) - P_n(x;t)\right], \end{aligned}$$

using the recurrence relation (4.5.7). Hence, using Lemma 4.5.2,

$$\begin{aligned} \frac{A_{n-1}(x;t)}{\beta_{n-1}}P_n(x;t) &= -xP_{n-1}'(x;t) - B_{n-1}(x;t)P_{n-1}(x;t) + \frac{xA_{n-1}(x;t)}{\beta_{n-1}}P_{n-1}(x;t) \\ &= -xP_{n-1}'(x;t) - B_{n-1}(x;t)P_{n-1}(x;t) \\ &\quad + \left(B_n(x;t) + B_{n-1}(x;t) - \gamma + xv_0'(x;t)\right)P_{n-1}(x;t) \\ &= -xP_{n-1}'(x;t) + B_n(x;t)P_{n-1}(x;t) + \left[xv_0'(x;t) - \gamma\right]P_{n-1}(x;t) \\ &= \left(-x\frac{d}{dx} + B_n(x;t) + xv_0'(x;t) - \gamma\right)P_{n-1}(x;t). \end{aligned}$$

□

Finally, we derive the differential-difference equation satisfied by the generalized Freud polynomials associated with the weight (4.2.1).

Lemma 4.5.3. *[41, Lemma 9]. For the generalized Freud weight (4.2.1), the monic orthogonal polynomials $S_n(x;t)$ satisfy*

$$\int_{-\infty}^{\infty}\left[\frac{v'(x;t) - v'(y;t)}{x-y}\right]S_n^2(y;t)\,w_\lambda(y;t)\,dy = 4\left[x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}\right]h_n, \tag{4.5.13}$$

$$\int_{-\infty}^{\infty}\left[\frac{v'(x;t) - v'(y;t)}{x-y}\right]S_n(y;t)S_{n-1}(y;t)\,w_\lambda(y;t)\,dy = 4xh_n, \tag{4.5.14}$$

where $n \in \mathbb{Z}$, $v(x;t) = x^4 - tx^2$ and

$$h_n = \int_{-\infty}^{\infty} S_n^2(y;t)\,w_\lambda(y;t)\,dy > 0.$$

Proof. For the weight (4.2.1), we have $w_\lambda(x;t) = |x|^{2\lambda+1}\exp\bigl(-v(x;t)\bigr)$, with

$$v(x;t) = x^4 - tx^2,$$

and so

$$\frac{v'(x;t) - v'(y;t)}{x-y} = 4x^2 + 4xy + 4y^2 - 2t.$$

Hence the left-hand side of (4.5.13) is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\frac{v'(x;t) - v'(y;t)}{x-y}\right] S_n^2(y;t)\, w_\lambda(y;t)\, dy \\
&\qquad = (4x^2 - 2t)\int_{-\infty}^{\infty} S_n^2(y;t)\, w_\lambda(y;t)\, dy \\
&\qquad + 4x\int_{-\infty}^{\infty} yS_n^2(y;t)\, w_\lambda(y;t)\, dy + 4\int_{-\infty}^{\infty} y^2S_n^2(y;t)\, w_\lambda(y;t)\, dy \\
&\qquad = (4x^2 - 2t)h_n + 4x\int_{-\infty}^{\infty} S_n(y;t)\bigl[S_{n+1}(y;t) + \beta_n S_{n-1}(y;t)\bigr]\, w_\lambda(y;t)\, dy \\
&\qquad\qquad + 4\int_{-\infty}^{\infty} \bigl[S_{n+1}(y;t) + \beta_n S_{n-1}(y;t)\bigr]^2\, w_\lambda(y;t)\, dy \\
&\qquad = (4x^2 - 2t)h_n + 4h_{n+1} + 4\beta_n^2 h_{n-1} \\
&\qquad = 4\bigl[x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}\bigr]h_n,
\end{aligned} \tag{4.5.15}$$

since $\beta_n = \frac{h_n}{h_{n-1}}$, the monic orthogonal polynomials $S_n(x;t)$ satisfy the three-term recurrence relation (4.2.4) and are orthogonal, i.e.,

$$\int_{-\infty}^{\infty} S_m(y;t)S_n(y;t)\, w_\lambda(y;t)\, dy = 0 \qquad \text{if} \quad m \neq n. \tag{4.5.16}$$

The left-hand side of (4.5.14) is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\frac{v'(x;t) - v'(y;t)}{x-y}\right] S_n(y;t)S_{n-1}(y;t)\, w_\lambda(y;t)\, dy \\
&= (4x^2 - 2t)\int_{-\infty}^{\infty} S_n(y;t)S_{n-1}(y;t)\, w_\lambda(y;t)\, dy + 4x\int_{-\infty}^{\infty} yS_n(y;t)S_{n-1}(y;t)\, w_\lambda(y;t)\, dy \\
&\qquad + 4\int_{-\infty}^{\infty} y^2S_n(y;t)S_{n-1}(y;t)\, w_\lambda(y;t)\, dy \\
&= 4x\int_{-\infty}^{\infty} S_n(y;t)\bigl[S_n(y;t) + \beta_{n-1}S_{n-2}(y;t)\bigr]\, w_\lambda(y;t)\, dy \\
&\qquad + 4\int_{-\infty}^{\infty} \bigl[S_{n+1}(y;t) + \beta_n S_{n-1}(y;t)\bigr] \times \bigl[S_n(y;t) + \beta_{n-2}S_{n-2}(y;t)\bigr]\, w_\lambda(y;t)\, dy \\
&= 4xh_n,
\end{aligned}$$

using the recurrence relation (4.2.4) and orthogonality (4.5.16). $\square$

Theorem 4.5.3. *[41, Theorem 5]. For the generalized Freud weight (4.2.1), the monic orthogonal polynomials $S_n(x;t)$ satisfy the differential-difference equation*

$$x\frac{dS_n}{dx}(x;t) = -B_n(x;t)S_n(x;t) + A_n(x;t)S_{n-1}(x;t), \tag{4.5.17}$$

where

$$A_n(x;t) = 4x\beta_n(x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}), \tag{4.5.18a}$$

$$B_n(x;t) = 4x^2\beta_n + \frac{(2\lambda+1)[1-(-1)^n]}{2}, \tag{4.5.18b}$$

with β_n the recurrence coefficient in the three-term recurrence relation (4.2.4).

Proof. Corollary 4.5.1 shows that monic orthogonal polynomials $S_n(x;t)$ with respect to the weight $w(x;t) = |x|^{2\lambda+1}\exp\bigl(-v(x;t)\bigr)$ satisfy the differential-difference equation (4.5.17), where

$$A_n(x;t) = \frac{x}{h_{n-1}}\int_{-\infty}^{\infty}\left[\frac{v'(x;t)-v'(y;t)}{x-y}\right]S_n^2(y;t)w(y,t)\,dy,$$

$$B_n(x;t) = \frac{x}{h_{n-1}}\int_{-\infty}^{\infty}\left[\frac{v'(x;t)-v'(y;t)}{x-y}\right]S_n(y;t)S_{n-1}(y;t)w(y,t)\,dy + \frac{2\lambda+1}{2}[1+(-1)^n].$$

For the generalized Freud weight (4.2.1), using Lemma 4.5.3 yields the result. □

4.5.2 An approach using quasi-orthogonality

Shohat [127] gave a procedure using quasi-orthogonality to derive (4.5.1) for weights $w(x;t)$ such that $w'(x;t)/w(x;t)$ is a rational function, which we apply to the generalized Freud weight in (4.2.1) [41, Section 4.5]. This technique was rediscovered by several authors including Bonan, Freud, Mhaskar and Nevai approximately 40 years later, see [112, p. 126 - 132] and the references therein for more detail. The concept of quasi-orthogonality is discussed in Section 2.5 (see also [127]).

Derivatives of monic polynomials $S_n(x;t)$ that are orthogonal with respect to the generalized Freud weight (4.2.1), are quasi-orthogonal of order $m = 5$ [112, Subsection 4.20] and hence we can write

$$x\frac{dS_n}{dx}(x;t) = \sum_{k=n-4}^{n} c_{n,k}S_k(x;t), \tag{4.5.19}$$

where the coefficient $c_{n,k}$ is given by

$$c_{n,k} = \frac{1}{h_k}\int_{-\infty}^{\infty} x\frac{dS_n}{dx}(x;t)\ S_k(x;t)\ w_\lambda(x;t)\,dx, \tag{4.5.20}$$

for $n-4 \le k \le n$ and $h_k > 0$.

Integrating by parts, we obtain for $n-4 \le k \le n-1$,

$$\begin{aligned}
h_k c_{n,k} &= \Big[xS_k(x;t)S_n(x;t)w_\lambda(x;t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left[xS_k(x;t)w_\lambda(x;t)\right]S_n(x;t)\,dx \\
&= -\int_{-\infty}^{\infty}\left[S_n(x;t)S_k(x;t) + xS_n(x;t)\frac{dS_k}{dx}(x;t)\right]w_\lambda(x;t)\,dx \\
&\quad -\int_{-\infty}^{\infty} xS_n(x;t)S_k(x;t)\frac{dw_\lambda}{dx}(x;t)\,dx \\
&= -\int_{-\infty}^{\infty} xS_n(x;t)S_k(x;t)\frac{dw_\lambda}{dx}(x;t)\,dx \\
&= -\int_{-\infty}^{\infty} S_n(x;t)S_k(x;t)\left(-4x^4+2tx^2+2\lambda+1\right)w_\lambda(x;t)\,dx \\
&= \int_{-\infty}^{\infty}\left(4x^4-2tx^2\right)S_n(x;t)S_k(x;t)w_\lambda(x;t)\,dx,
\end{aligned} \tag{4.5.21}$$

since

$$x\frac{dw_\lambda}{dx}(x;t) = \big(-4x^4+2x^2+2\lambda+1\big)w_\lambda(x;t).$$

Iterating the three-term recurrence relation (4.2.4), the following relations are obtained:

$$x^2S_n = S_{n+2} + (\beta_n+\beta_{n+1})S_n + \beta_n\beta_{n-1}S_{n-2}, \tag{4.5.22a}$$

$$\begin{aligned}
x^4S_n = S_{n+4} &+ (\beta_n+\beta_{n+1}+\beta_{n+2}+\beta_{n+3})S_{n+2} \\
&+ \big[\beta_n(\beta_{n-1}+\beta_n+\beta_{n+1}) + \beta_{n+1}(\beta_n+\beta_{n+1}+\beta_{n+2})\big]S_n \\
&+ \beta_n\beta_{n-1}(\beta_{n-2}+\beta_{n-1}+\beta_n+\beta_{n+1})S_{n-2} \\
&+ (\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3})S_{n-4}.
\end{aligned} \tag{4.5.22b}$$

Substituting (4.5.22a) and (4.5.22b) into (4.5.21), yields the coefficients $\{c_{n,k}\}_{k=n-4}^{n-1}$ in (4.5.19) as follows:

$$c_{n,n-4} = 4\beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}, \tag{4.5.23a}$$

$$c_{n,n-3} = 0, \tag{4.5.23b}$$

$$c_{n,n-2} = 4\beta_n\beta_{n-1}(\beta_{n-2}+\beta_{n-1}+\beta_n+\beta_{n+1}-\tfrac{1}{2}t), \tag{4.5.23c}$$

$$c_{n,n-1} = 0. \tag{4.5.23d}$$

Lastly, we consider the case when $k = n$. Integration by parts in (4.5.20) yields

$$\begin{aligned}
h_n c_{n,n} &= \int_{-\infty}^{\infty} x \frac{dS_n}{dx}(x;t) S_n(x;t) w_\lambda(x;t)\, dx, \\
&= -\tfrac{1}{2} \int_{-\infty}^{\infty} S_n^2(x;t) \left[w_\lambda(x;t) + x \frac{dw_\lambda}{dx}(x;t) \right] dx, \\
&= -\tfrac{1}{2} h_n + \int_{-\infty}^{\infty} S_n^2(x;t) \big(2x^4 - tx^2 - \lambda - \tfrac{1}{2} \big) w_\lambda(x;t)\, dx, \\
&= \int_{-\infty}^{\infty} \big(2x^4 - tx^2 \big) S_n^2(x;t)\, w_\lambda(x;t)\, dx - (\lambda+1) h_n. \qquad (4.5.24)
\end{aligned}$$

From the three-term recurrence relation (4.2.4), we have

$$\begin{aligned}
x^2 S_n^2 &= (S_{n+1} + \beta_n S_{n-1})^2 = S_{n+1}^2 + 2\beta_n S_{n+1} S_{n-1} + \beta_n^2 S_{n-1}^2, \\
x^4 S_n^2 &= x^2 \big(S_{n+1}^2 + 2\beta_n S_{n+1} S_{n-1} + \beta_n^2 S_{n-1}^2 \big) \\
&= x^2 S_{n+1}^2 + 2\beta_n (x S_{n+1})(x S_{n-1}) + \beta_n^2 x^2 S_{n-1}^2 \\
&= \big(S_{n+2} + \beta_{n+1} S_n \big)^2 + 2\beta_n \big(S_{n+2} + \beta_{n+1} S_n \big)\big(S_n + \beta_{n-1} S_{n-2} \big) + \beta_n^2 \big(S_n + \beta_{n-1} S_{n-2} \big)^2 \\
&= S_{n+2}^2 + 2(\beta_{n+1} + \beta_n) S_{n+2} S_n + (\beta_{n+1} + \beta_n)^2 S_n^2 + 2\beta_n \beta_{n-1} S_{n+2} S_{n-2} \\
&\qquad + 2\beta_n \beta_{n-1} (\beta_n + \beta_{n+1}) S_n S_{n-2} + \beta_n^2 \beta_{n-1}^2 S_{n-2}^2,
\end{aligned}$$

and, by orthogonality,

$$\int_{-\infty}^{\infty} x^2 S_n^2(x;t)\, w_\lambda(x;t)\, dx \ = h_{n+1} + \beta_n^2 h_{n-1} = (\beta_{n+1} + \beta_n) h_n, \qquad (4.5.25)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} x^4 S_n^2(x;t) w_\lambda(x;t)\, dx &= h_{n+2} + (\beta_{n+1} + \beta_n)^2 h_n + \beta_n^2 \beta_{n-1}^2 h_{n-2} \\
&= \beta_{n+2} \beta_{n+1} h_n + (\beta_{n+1} + \beta_n)^2 h_n + \beta_n \beta_{n-1} h_n \\
&= \big[(\beta_{n+1} + \beta_n + \beta_{n-1}) \beta_n + (\beta_{n+2} + \beta_{n+1} + \beta_n) \beta_{n+1} \big] h_n \\
&= \tfrac{1}{2} \big[t(\beta_{n+1} + \beta_n) + n + \lambda + 1 \big] h_n, \qquad (4.5.26)
\end{aligned}$$

using $h_{n+1} = \beta_{n+1} h_n$ and dP$_\mathrm{I}$ (4.3.1). Hence from (4.5.24), (4.5.25) and (4.5.26) we have

$$\begin{aligned}
c_{n,n} &= t(\beta_{n+1} + \beta_n) + n + \lambda + 1 - t(\beta_{n+1} + \beta_n) - (\lambda + 1) \\
&= n. \qquad (4.5.27)
\end{aligned}$$

Combining (4.5.23) with (4.5.19), we write

$$x \frac{dS_n}{dx}(x;t) = c_{n,n-4} S_{n-4}(x;t) + c_{n,n-2} S_{n-2}(x;t) + c_{n,n} S_n(x;t). \qquad (4.5.28)$$

In order to express S_{n-4} and S_{n-2} in (4.5.28) in terms of S_n and S_{n-1}, we iterate the recurrence (4.2.4) to obtain

$$S_{n-2} = \frac{xS_{n-1} - S_n}{\beta_{n-1}}, \tag{4.5.29}$$

$$S_{n-3} = \frac{xS_{n-2} - S_{n-1}}{\beta_{n-2}} = \frac{x^2 - \beta_{n-1}}{\beta_{n-1}\beta_{n-2}}\, S_{n-1} - \frac{x}{\beta_{n-1}\beta_{n-2}}\, S_n,$$

$$\begin{aligned} S_{n-4} &= \frac{xS_{n-3} - S_{n-2}}{\beta_{n-3}} \\ &= \frac{x^3 - (\beta_{n-1} + \beta_{n-2})x}{\beta_{n-1}\beta_{n-2}\beta_{n-3}}\, S_{n-1} - \frac{x^2 - \beta_{n-2}}{\beta_{n-1}\beta_{n-2}\beta_{n-3}}\, S_n. \end{aligned} \tag{4.5.30}$$

Substituting (4.5.23), (4.5.27), (4.5.29) and (4.5.30) into (4.5.28) yields

$$x\frac{dS_n}{dx}(x;t) = -B_n(x;t)S_n(x;t) + A_n(x;t)S_{n-1}(x;t),$$

where $A_n(x;t)$ and $B_n(x;t)$ are given by (4.5.18).

4.6 The differential equation satisfied by generalized Freud polynomials

We first provide a derivation of a differential equation satisfied by orthogonal polynomials associated with the weight $w_\gamma(x;t) = |x|^\gamma \exp\big(-v_0(x;t)\big)$ as in (4.5.6). The differential equation satisfied by polynomials associated with the generalized Freud weight (4.2.1) follows as a special case.

4.6.1 The differential equation related to the weight (4.5.6)

Theorem 4.6.1. *[41, Theorem 3]. For the weight defined by (4.5.6), the associated monic orthogonal polynomials $P_n(x;t)$ satisfy the differential equation*

$$x\frac{d^2P_n}{dx^2}(x;t) + R_n(x;t)\frac{dP_n}{dx}(x;t) + T_n(x;t)P_n(x;t) = 0, \tag{4.6.1a}$$

where

$$R_n(x;t) = \gamma - xv_0'(x;t) - \frac{xA_n'(x)}{A_n(x)} + 1, \tag{4.6.1b}$$

$$\begin{aligned} T_n(x;t) &= \frac{A_n A_{n-1}}{x\beta_{n-1}} + B_n'(x) + \frac{\gamma B_n}{x} \\ &\quad - B_n\left(v_0'(x;t) + \frac{B_n}{x}\right) - \frac{A_n'(x)B_n(x)}{A_n(x)}, \end{aligned} \tag{4.6.1c}$$

with

$$A_n(x;t) = \frac{x}{h_{n-1}}\int_{-\infty}^{\infty} P_n^2(y;t)\left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\lambda(y;t)\,dy,$$

$$B_n(x;t) = \frac{x}{h_{n-1}}\int_{-\infty}^{\infty} P_n(y;t)P_{n-1}(y;t)\left[\frac{v_0'(x;t) - v_0'(y;t)}{x-y}\right] w_\lambda(y;t)\,dy + \frac{\gamma}{2}[1-(-1)^n].$$

Proof. Differentiating both sides of (4.5.9) with respect to x, we obtain

$$\begin{aligned} xP_n''(x;t) &= (-B_n(x) - 1)P_n'(x,t) + A_n'(x)P_{n-1}(x;t) \\ &\quad - B_n'(x)P_n(x;t) + A_n(x)P_{n-1}(x;t). \end{aligned} \tag{4.6.2}$$

Substituting (4.5.12) into (4.6.2) yields

$$\begin{aligned} xP_n''(x;t) &= (-B_n(x) - 1)P_n'(x;t) - \left(B_n'(x) + \frac{A_n(x)A_{n-1}(x)}{x\beta_{n-1}}\right) P_n(x;t) \\ &\quad + \left(A_n'(x) + \frac{A_n(x)B_n(x)}{x} + A_n(x)v_0'(x;t) - \frac{\gamma A_n(x)}{x}\right) P_{n-1}(x;t) \end{aligned} \tag{4.6.3}$$

and the result follows by substituting $P_{n-1}(x;t)$ in (4.6.3) using (4.5.9). □

4.6.2 The differential equation related to the weight (4.2.1)

A differential equation satisfied by generalized Freud polynomials associated with the weight (4.2.1) is given in the following theorem.

Theorem 4.6.2. *[41, Theorem 6]. For the generalized Freud weight (4.2.1), the monic orthogonal polynomials $S_n(x;t)$ satisfy the differential equation*

$$x\frac{d^2 S_n}{dx^2}(x;t) + R_n(x;t)\frac{dS_n}{dx}(x;t) + T_n(x;t)S_n(x;t) = 0, \tag{4.6.4}$$

where

$$R_n(x;t) = x\left(-4x^3 + 2tx + \frac{2\lambda+1}{x} - \frac{2x}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}}\right), \tag{4.6.5a}$$

$$\begin{aligned} T_n(x;t) = x\Big[&4nx^2 + 4\beta_n + 16\beta_n(\beta_n + \beta_{n+1} - \tfrac{t}{2})(\beta_n + \beta_{n-1} - \tfrac{t}{2}) \\ &- \frac{8\beta_n x^2 + (2\lambda+1)[1-(-1)^n]}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}} + 4(2\lambda+1)(-1)^n\beta_n \\ &+(2\lambda+1)[1-(-1)^n]\left(t - \frac{1}{2x^2}\right)\Big]. \end{aligned} \tag{4.6.5b}$$

Proof. In Theorem 4.6.1 we proved that the coefficients in the differential equation (4.6.1a) satisfied by polynomials orthogonal with respect to $w_\gamma(x;t) = |x|^\gamma \exp(-v_0(x;t))$ are given by (4.6.1b) and (4.6.1c). For the generalized Freud weight (4.2.1), we use (4.6.1b) and (4.6.1c) with $\gamma = 2\lambda + 1$, $v_0'(x;t) = 4x^3 - 2tx$ and A_n and B_n given by (4.5.18) to obtain the stated result. □

Remark 4.6.1. *The differential equation* (4.6.4) *for the special case where* $\lambda = -\frac{1}{2}$ *and* t *is replaced by* $2t$ *is given in [8, Equation 6] though here the statement on p. 104 needs to be corrected to read*

$$\begin{aligned} S_n^t(x) = 4a_n^2(t)\Big[&4x^2\left(a_{n-1}^2(t) + a_n^2(t) + a_{n+1}^2(t) - t - \frac{2}{x^2 - t + a_n^2(t) + a_{n+1}^2(t)}\right) \\ &+4\left(a_n^2(t) + a_{n+1}^2(t) - t\right)\left(a_{n-1}^2(t) + a_n^2(t) - t\right) + 1\Big]. \end{aligned}$$

4.7 Conclusion

In this chapter we described some analytic properties of the generalized Freud polynomials that were published in [41]. Properties discussed include the higher-order moments, Pearson's equation related to the weight w_λ in (4.2.1), an explicit formulation for the recurrence coefficient $\beta_n(t;\lambda)$, as well as the generalized Freud polynomials themselves and other related properties such as the differential-difference and differential equations satisfied by the generalized Freud polynomials.

The connection of special function solutions of the fourth Painlevé equation to the recurrence coefficients associated with generalized Freud polynomials orthogonal with respect to the weight (4.2.1) played a fundamental role in determining explicit formulations for the polynomials and recurrence coefficients associated with the generalized Freud weight.

In the next chapter we explore additional properties of the generalized Freud polynomials and their recurrence coefficients, investigating, in particular, aymptotic properties of the polynomials and recurrence coefficients as the parameters t and/or the degree n tend to infinity.

Chapter 5

Asymptotic properties satisfied by generalized Freud polynomials

5.1 Introduction

In Chapter 4 we have found an explicit formulation of the recurrence coefficient $\beta_n(t;\lambda)$ and the monic polynomials $\{S_n(x;t)\}_{n=0}^{\infty}$, orthogonal with respect to the generalized Freud weight (4.2.1), using the connection between the moments and special function solutions of the fourth Painlevé equation.

In this chapter we study asymptotic properties of the polynomials $\{S_n(x;t)\}_{n=0}^{\infty}$. The investigation of estimates and asymptotic expansions of the recurrence coefficients of the three-term recurrence relation (4.2.4) satisfied by Freud-type polynomials is important in the context of practical applications (cf. [86, 87, 110, 111, 112, 119]).

We provide an extension of Freud's conjecture for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight (4.2.1). In particular, following [55], we provide the asymptotic behavior of the recurrence coefficient $\beta_n(t;\lambda)$ as the degree or, alternatively, the parameter tends to infinity. Using a new approach, we also investigate the asymptotic behavior of the recurrence coefficient $\beta_n(t;\lambda)$ via the theory of Painlevé equations. We further investigate, by using the differential-difference and differential equations obtained in Chapter 4, the asymptotics of the normalized differential equation satisfied by monic generalized Freud polynomials.

5.2 Limit relations for the coefficient $\boldsymbol{\beta_n(t;\lambda)}$

Freud [55], via the Freud equations, proved the conjecture that the asymptotic behavior of the recurrence coefficient $\beta_n(t;\lambda)$ in the three-term recurrence relation (4.2.4) satisfied by the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ orthogonal with respect to the positive weight

$$w(x) = |x|^{\lambda}\exp(-|x|^m), \quad m \in \mathbb{N},$$

with $\lambda > -1$, could be described by

$$\lim_{n\to\infty} \beta_n\, n^{-\frac{2}{m}} = \left[\frac{\Gamma(\frac{1}{2}m)\,\Gamma(1+\frac{1}{2}m)}{\Gamma(m+1)}\right]^{\frac{2}{m}}. \tag{5.2.1}$$

An equivalent statement for equation (5.2.1) is (cf. [55, 88])

$$\lim_{n\to\infty} \beta_n\, n^{-\frac{2}{m}} = \frac{1}{4}\left[\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{m}{2})}{\Gamma(\frac{m+1}{2})}\right]^{-\frac{2}{m}} = \left[2\frac{\Gamma(m)}{\big(\Gamma(\frac{m}{2})\big)^2}\right]^{-\frac{2}{m}}, \tag{5.2.2}$$

that follows from the recursive property of the Gamma function given in (2.3.3). Note that Freud proved the result for orthonormal polynomials while (5.2.1) is stated for monic orthogonal polynomials. Freud proved that the existence of the limit for $m = 2n,\ n \in \mathbb{N}$, implied that the limit is given by the expression in (5.2.1) but he only managed to prove the existence of the limit (5.2.1) for $m = 2, 4, 6$. Significant progress was made by Magnus (cf. [88, 89]), when he proved (5.2.1) for m an even positive integer and also extended Freud's conjecture to the recurrence coefficients associated with the weight

$$w(x) = \exp\big(-Q(x)\big) \tag{5.2.3}$$

where $Q(x)$ is a polynomial of even degree with a positive leading coefficient. A proof of Freud's conjecture for recurrence coefficients associated with exponential weights (5.2.3) where $Q(x)$ is more general, is due to Lubinsky, Mhaskar and Saff [87], see also [44, 55, 112].

The objective of this section is to adapt existing techniques (cf. [56, 89, 109, 134]) to extend the Freud conjecture to the recurrence coefficients $\beta_n(t;\lambda)$ associated with the generalized Freud weight (4.2.1).

Lemma 5.2.1. *For a fixed parameter $t \in \mathbb{R}$ and $\lambda > 0$, let $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ be a real, positive sequence satisfying the discrete Painlevé* $\mathrm{P_I}$ *equation in* (4.3.1)*. Then, the sequence $\{\frac{\beta_n(t;\lambda)}{\sqrt{n}}\}_{n\in\mathbb{N}}$ is bounded.*

Proof. To prove this lemma we first divide the discrete Painlevé equation (4.3.1) by n to obtain

$$\frac{1}{4}+\frac{(2\lambda+1)\Omega_n}{4n}=(Y_n+X_n)+\frac{\beta_n^2(t;\lambda)}{n}-\frac{t}{2}\frac{\beta_n(t;\lambda)}{n}, \tag{5.2.4}$$

where

$$Y_n=\frac{\beta_n(t;\lambda)\beta_{n+1}(t;\lambda)}{n}, \quad X_n=\frac{\beta_n(t;\lambda)\beta_{n-1}(t;\lambda)}{n}$$

and Ω_n is given in (2.9.6).

Note that both X_n and Y_n are positive since $\beta_n(t;\lambda)>0$ for all $n\in\mathbb{N}$.

By letting $\omega_n=\frac{\beta_n(t;\lambda)}{\sqrt{n}}$, (5.2.4) takes the form

$$\frac{1}{4}+\frac{(2\lambda+1)\Omega_n}{4n}=(Y_n+X_n)+\omega_n^2-\frac{t\omega_n}{2\sqrt{n}},$$

and we have $\frac{1}{4}+\frac{(2\lambda+1)\Omega_n}{4n}\le\frac{1}{4}+2\lambda+1,\quad \forall n\in\mathbb{N}$. Consequently,

$$(Y_n+X_n)+\omega_n^2-\frac{t\omega_n}{2\sqrt{n}}\le\frac{1}{4}+2\lambda+1:=M. \tag{5.2.5}$$

Since $Y_n+X_n>0$, (5.2.5) implies that

$$\omega_n^2-\frac{t}{2\sqrt{n}}\omega_n-M\le 0,$$

and hence

$$\frac{\frac{t}{2\sqrt{n}}-\sqrt{\frac{t^2}{4n}+4M}}{2}\le\omega_n\le\frac{\frac{t}{2\sqrt{n}}+\sqrt{\frac{t^2}{4n}+4M}}{2},$$

for any fixed $t\in\mathbb{R}$. Thus, the sequence $\{\frac{\beta_n(t;\lambda)}{\sqrt{n}}\}_{n\in\mathbb{N}}$ is bounded for a fixed $t\in\mathbb{R}$ and $\lambda>0$. □

Next we give the limit relation satisfied by the recurrence coefficient $\beta_n(t;\lambda)$ in (4.2.4).

Theorem 5.2.1. *Let $t\in\mathbb{R}$ be fixed, $\lambda>0$ and suppose $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ is a real, positive sequence satisfying the discrete Painlevé equation* $\mathrm{P_I}$ (4.3.1)*. Then*

$$\lim_{n\to\infty}\frac{\beta_n(t;\lambda)}{\sqrt{n}}=\frac{\sqrt{3}}{6}. \tag{5.2.6}$$

Proof. In Lemma 5.2.1 we have seen that the sequence $\{\frac{\beta_n(t;\lambda)}{\sqrt{n}}\}_{n\in\mathbb{N}}$ is bounded and positive, that is,

$$0<\frac{\beta_n(t;\lambda)}{\sqrt{n}}<R, \quad \text{where} \quad R=\frac{\frac{t}{2\sqrt{n}}+\sqrt{\frac{t^2}{4n}+4M}}{2}. \tag{5.2.7}$$

Consequently, $0 \le |\frac{t}{\sqrt{n}}\frac{\beta_n(t;\lambda)}{\sqrt{n}}| \le |\frac{t}{\sqrt{n}}|\ R$. Hence, $\frac{t}{\sqrt{n}}\frac{\beta_n(t;\lambda)}{\sqrt{n}} \to 0$ as $n \to \infty$. On the other hand, dividing both sides of (4.3.1) by n yields

$$\underbrace{\frac{1}{4} + \frac{(2\lambda+1)\Omega_n}{4n} + \frac{t}{2\sqrt{n}}\frac{\beta_n(t;\lambda)}{\sqrt{n}}}_{g(n)} = \frac{\beta_n(t;\lambda)}{\sqrt{n}}\left(\frac{\beta_{n+1}(t;\lambda)}{\sqrt{n}} + \frac{\beta_n(t;\lambda)}{\sqrt{n}} + \frac{\beta_{n-1}(t;\lambda)}{\sqrt{n}}\right). \tag{5.2.8}$$

Note that $g(n) \to \dfrac{1}{4}$ as $n \to \infty$ and by setting $A = \liminf\limits_{n\to\infty} \dfrac{\beta_n(t;\lambda)}{\sqrt{n}}$ and $B = \limsup\limits_{n\to\infty} \dfrac{\beta_n(t;\lambda)}{\sqrt{n}}$ and taking $\liminf\limits_{n\to\infty}$ on both sides of (5.2.8), we obtain

$$\frac{1}{4} = A\,(A + A + A) \le A\,(B + A + B) = A^2 + 2AB. \tag{5.2.9}$$

Similarly, by taking lim sup on both sides of (5.2.8), we have

$$\frac{1}{4} = B(B + B + B) \ge B(A + B + A) = B^2 + 2AB. \tag{5.2.10}$$

Combining (5.2.9) and (5.2.10), we obtain

$$B^2 + 2AB \le A^2 + 2AB \Longrightarrow B^2 \le A^2,$$

which implies $B \le A$. Hence $A = B$ and $3A^2 = \frac{1}{4}$, which means $A = B = \dfrac{\sqrt{3}}{6}$. □

Remark 5.2.1. *Nevai [109] and later Freud [55] proved that the recurrence coefficient $\beta_n(t;\lambda)$ associated with a special case of the symmetric weight* (4.2.1) *with $\lambda = -\frac{1}{2}$ and $t = 0$ has the same limit as the one in Theorem 5.2.1 (see also [62, 84, 85, 87, 88, 89, 91, 114] for detailed information). Van Assche [134] obtained the same limit as in Theorem 5.2.1 for the case where $\lambda > -1$ and $t = 0$.*

Theorem 5.2.2. *Let $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ be a real sequence satisfying the recurrence relation* (4.2.4) *and the discrete Painlevé equation* $\mathrm{P_I}$ (4.3.1). *Then, for $t \in \mathbb{R}$, $\lambda > 0$,*

$$\lim_{n\to\infty} \frac{\beta_n(t;\lambda)}{\sqrt{n + (2\lambda+1)\Omega_n}} = \frac{\sqrt{3}}{6}.$$

Proof. The result follows from [5, Theorem 6.5]. □

The following corollary is an extension of [109, Theorem 2] and is a consequence of the limit relation (5.2.6) for the recurrence coefficient $\beta_n(t;\lambda)$.

Corollary 5.2.1. *Let the sequence* $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ *be a real solution of the recurrence* (4.2.4) *and the discrete Painlevé equation* $\mathrm{P_I}$ (4.3.1). *Then, for* $t\in\mathbb{R}$,

$$\sum_{n=1}^{\infty}\left|\frac{2\beta_n(t;\lambda)}{\sqrt{n}}-\frac{1}{\sqrt{3}}\right|<\infty.$$

It is of interest to mention the contribution by Damelin in [44] where he considers asymptotics of recurrence coefficients associated with weights $|x|^{\rho}\exp(-Q)$ when Q is an even polynomial of fixed degree.

Proposition 5.2.1. *[44, Theorem 2.1]. For the generalized Freud weight* $w_\lambda(x;t)$ *in* (4.2.1), *with* $t\in\mathbb{R}$, *the recurrence coefficients* $\beta_n(t;\lambda)$ *satisfy*

$$\begin{aligned}\frac{\beta_{n+1}(t;\lambda)}{\beta_n(t;\lambda)}&=1+\mathcal{O}\left(\frac{1}{n}\right)\quad as\, n\to\infty,\\ \frac{\beta_n(t;\lambda)}{\mathsf{a}_n^2(t)}&=\frac{1}{4}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right]\quad as\, n\to\infty,\end{aligned}$$

where a_n *is the Mhaskar-Rahmanov-Saff number (cf. [84, Equation 1.11]) which is the unique positive solution of the equation*

$$n=\frac{2}{\pi}\int_0^1 \mathsf{a}_n\; t\; Q'(\mathsf{a}_n t)(1-t^2)^{-\frac{1}{2}}\,dt,$$

for $Q(x)=x^4-tx^2$.

5.3 Asymptotics of the recurrence coefficient $\beta_n(t;\lambda)$ as $t\to\infty$

In this section we provide an asymptotic analysis of the recurrence coefficient $\beta_n(t;\lambda)$ associated with polynomials orthogonal with respect to the generalized Freud weight (4.2.1) using the theory of Painlevé equations.

As shown in Chapter 4, the first few recurrence coefficients $\beta_n(t;\lambda)$ for the generalized Freud polynomials are given by (4.3.15) in terms of parabolic cylinder functions

$$\begin{aligned}\Phi_\lambda(t)&=\frac{d}{dt}\ln\Big(D_{-\lambda-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)\exp\big(\tfrac{1}{8}t^2\big)\Big)\\ &=\tfrac{1}{2}t+\tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}{D_{-\lambda-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}.\end{aligned}\tag{5.3.1}$$

We note that

$$\Phi_\lambda(t)=\frac{d}{dt}\ln\phi_\lambda(t),$$

where $\phi_\lambda(t)=\exp(\tfrac18 t^2)D_{-\lambda-1}\left(-\tfrac12\sqrt{2}\,t\right)$ satisfies

$$\frac{d^2\phi_\lambda}{dt^2}-\tfrac12 t\frac{d\phi_\lambda}{dt}-\tfrac12(\lambda+1)\phi_\lambda=0,$$

and in the next lemma we will prove that $\Phi_\lambda(t)$ satisfies the Riccati equation

$$\frac{d\Phi_\lambda}{dt}=-\Phi_\lambda^2+\tfrac12 t\Phi_\lambda+\tfrac12(\lambda+1). \tag{5.3.2}$$

Lemma 5.3.1. *[41, Lemma 6]. The function $\Phi_\lambda(t)$ defined by* (5.3.1) *satisfies the Riccati equation* (5.3.2) *and has the asymptotic expansion*

$$\Phi_\lambda(t)=\tfrac12 t+\sum_{n=1}^{\infty}\frac{a_n}{t^{2n-1}}\quad \text{as}\quad t\to\infty, \tag{5.3.3}$$

where the constants a_n are given by the nonlinear recurrence relation

$$a_{n+1}=2(2n-1)a_n-2\sum_{j=1}^{n}a_j a_{n+1-j},$$

with $a_1=\lambda$. In particular, as $t\to\infty$,

$$\Phi_\lambda(t)=\tfrac12 t+\frac{\lambda}{t}+\frac{2\lambda(1-\lambda)}{t^3}+\frac{4\lambda(\lambda-1)(2\lambda-3)}{t^5}+\mathcal{O}\big(t^{-7}\big). \tag{5.3.4}$$

Proof. Letting $\Phi_\lambda(t)=\dfrac{d}{dt}\ln\phi_\lambda(t)$ in (5.3.2) yields

$$\frac{d^2\phi_\lambda}{dt^2}-\tfrac12 t\frac{d\phi_\lambda}{dt}-\tfrac12(\lambda+1)\phi_\lambda=0,$$

which has solution [117, §32.10(iv)]

$$\phi_\lambda(t)=\left(C_1D_{-\lambda-1}\big(-\tfrac12\sqrt{2}\,t\big)+C_2D_{-\lambda-1}\big(\tfrac12\sqrt{2}\,t\big)\right)\exp(\tfrac18 t^2),$$

with C_1 and C_2 arbitrary constants. Hence setting $C_1=1$ and $C_2=0$ gives the solution (5.3.1) and shows that $\Phi_\lambda(t)$ satisfies (5.3.2).

Substituting (5.3.3) into (5.3.2) gives

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(2n-1)a_n}{t^{2n}}&=\tfrac12 t\sum_{n=1}^{\infty}\frac{a_n}{t^{2n-1}}+\left(\sum_{n=1}^{\infty}\frac{a_n}{t^{2n-1}}\right)^2-\tfrac12\lambda\\&=a_1-\lambda+\tfrac12\sum_{n=1}^{\infty}\frac{a_{n+1}}{t^{2n}}+\sum_{n=1}^{\infty}\frac{1}{t^{2n}}\sum_{j=1}^{n}a_ja_{n+1-j},\end{aligned}$$

hence, comparing coefficients of powers of t gives $a_1=\lambda$ and

$$a_{n+1}=2(2n-1)a_n-2\sum_{j=1}^{n}a_ja_{n+1-j},$$

as required. Hence

$$a_1=\lambda,\qquad a_2=-2\lambda(\lambda-1),\qquad a_3=4\lambda(\lambda-1)(2\lambda-3),$$

which gives (5.3.4) as required. □

It was shown in (5.3.4) that, as $t\to\infty$,

$$\Phi_\lambda(t)=\tfrac{1}{2}t+\frac{\lambda}{t}+\frac{2\lambda(1-\lambda)}{t^3}+\frac{4\lambda(\lambda-1)(2\lambda-3)}{t^5}+\mathcal{O}\big(t^{-7}\big).$$

Hence, as $t\to\infty$,

$$\frac{1}{\Phi_\lambda(t)}=\frac{2}{t}-\frac{4\lambda}{t^3}+\frac{8\lambda(2\lambda-1)}{t^5}+\mathcal{O}\big(t^{-7}\big).$$

The following result motivates the strong connection between the recurrence coefficient $\beta_n(t;\lambda)$ and the Painlevé σ-equations. An essential ingredient here is the fact that the recurrence coefficients $\beta_n(t;\lambda)$ can be expressed as Hankel determinants which arise in the solution of the Painlevé equations and the Painlevé σ-equations, the second-order, second-degree equations associated with the Hamiltonian representation of the Painlevé equations.

Lemma 5.3.2. *[41, Lemma 7]. Let $H_n(t;\lambda)$ be defined by*

$$H_n(t;\lambda)=\frac{d}{dt}\ln\tau_n(t;\lambda), \tag{5.3.5}$$

where $\tau_n(t;\lambda)$ is the Wronskian given by

$$\tau_n(t;\lambda)=\mathcal{W}\left(\phi_\lambda,\frac{d\phi_\lambda}{dt},\ldots,\frac{d^{n-1}\phi_\lambda}{dt^{n-1}}\right),$$

with

$$\phi_\lambda(t)=\frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}}\exp\left(\tfrac{1}{8}t^2\right)D_{-\lambda-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big).$$

Then $H_n(t;\lambda)$ satisfies the second-order, second-degree equation

$$\left(\frac{d^2H_n}{dt^2}\right)^2-\frac{1}{4}\left(t\frac{dH_n}{dt}-H_n\right)^2+\frac{dH_n}{dt}\left(2\frac{dH_n}{dt}-n\right)\left(2\frac{dH_n}{dt}-n-\lambda\right)=0. \tag{5.3.6}$$

Proof. Equation (5.3.6) is equivalent to $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right)^2 + 4\frac{d\sigma}{dz}\left(\frac{d\sigma}{dz} + 2\theta_0\right)\left(\frac{d\sigma}{dz} + 2\theta_\infty\right) = 0, \tag{5.3.7}$$

as shown in [39, Theorem 4.11]. Equation (5.3.6) is the same as [41, Equation 46]. Special function solutions of $\mathrm{S_{IV}}$ (5.3.7) in terms of parabolic cylinder functions have been classified in [58, 116]; see also [39, Theorem 3.5]. □

We remark that (5.3.7) and hence also (5.3.6), is equivalent to equation SD-I.c in the classification of second order, second-degree equations with the Painlevé property by Cosgrove and Scoufis [43], an equation first derived and solved by Chazy [25].

Proposition 5.3.1. *[41, Lemma 8] As $t \to \infty$, the recurrence coefficient $\beta_n(t;\lambda)$ has the asymptotic expansion*

$$\left.\begin{aligned} \beta_{2n}(t;\lambda) &= \tfrac{n}{t} - \tfrac{2n(2\lambda-n+1)}{t^3} + \mathcal{O}\big(t^{-5}\big), \\ \beta_{2n+1}(t;\lambda) &= \tfrac{t}{2} + \tfrac{\lambda-n}{t} - \tfrac{2(\lambda^2-4\lambda n+n^2-\lambda-n)}{t^3} + \mathcal{O}\big(t^{-5}\big), \end{aligned}\right\} \tag{5.3.8}$$

for $n \in \mathbb{N}$. Further, as $t \to -\infty$,

$$\begin{aligned} \beta_{2n}(t;\lambda) &= -\frac{n}{t} + \frac{2n(2\lambda+3n+1)}{t^3} + \mathcal{O}\big(t^{-5}\big), \\ \beta_{2n+1}(t;\lambda) &= -\frac{\lambda+n+1}{t} + \frac{2(\lambda+n+1)(\lambda+3n+2)}{t^3} + \mathcal{O}\big(t^{-5}\big). \end{aligned}$$

Proof. In terms of the function $H_n(t;\lambda)$, defined by (5.3.5), the recurrence coefficients are given as

$$\beta_{2n}(t;\lambda) = H_n(t;\lambda+1) - H_n(t;\lambda), \tag{5.3.10a}$$

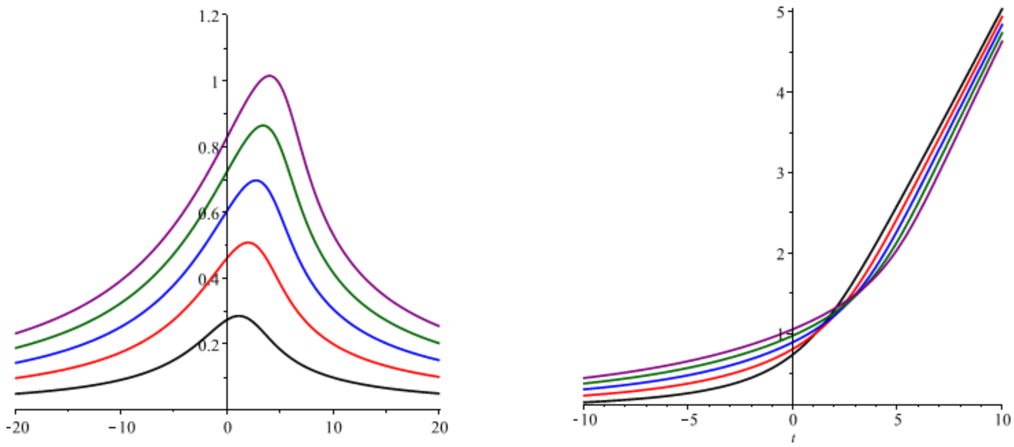
$$\beta_{2n+1}(t;\lambda) = H_{n+1}(t;\lambda) - H_n(t;\lambda+1). \tag{5.3.10b}$$

As $t \to \infty$, $H_n(t;\lambda)$ has the asymptotic expansion

$$H_n(t;\lambda) = \frac{nt}{2} + \frac{n\lambda}{t} + \frac{2n\lambda(n-\lambda)}{t^3} + \mathcal{O}\big(t^{-5}\big), \tag{5.3.11}$$

for $n = 0, 1, 2, \ldots$, [39, Lemma 5.2]. Note that the functions $\Omega_n(t)$ and $S_n(t)$ in [39] are the same as our functions $\tau_n(t;\lambda)$ and $H_n(t;\lambda)$, respectively. Substituting (5.3.11) in (5.3.10) immediately gives the result. □

In Figures 5.1a and 5.1b we show that there is a completely different behavior for $\beta_n(t;\lambda)$ as $t\to\infty$, depending on whether n is even or odd, which is also reflected in Proposition 5.3.1, i.e., the even recurrence coefficients $\beta_{2n}(t;\lambda)$ undergo decaying as $t\to\infty$ and the odd recurrence coefficients $\beta_{2n+1}(t;\lambda)$ exhibit an algebraic growth as $t\to\infty$.



(a) $\beta_{2n}(t;\frac{1}{2})$

(b) $\beta_{2n-1}(t;\frac{1}{2})$

Figure 5.1: Plots of the recurrence coefficients for $n=1$(Black), $n=2$(Red), $n=3$(Blue), $n=4$(Green), $n=5$(Purple).

Based on (5.3.8), it follows that for the generalized Freud weight (4.2.1),

$$\beta_{2n}(t;\lambda)\to 0 \quad \text{and} \quad \beta_{2n+1}(t;\lambda)\to \frac{t}{2} \quad \text{as} \quad t\to\infty.$$

5.4 Large n-asymptotics of the recurrence coefficient $\beta_n(t;\lambda)$

The asymptotic expansion for orthogonal polynomials with Freud weight $\exp(-|x|^\alpha)$, for $\alpha>0$ on $\mathbb{R}$, has been studied by several authors (cf. [11, 85, 98, 109, 110, 112, 114]). Lew and Quarles [85] provided the asymptotic expansion for the recurrence coefficient associated with the semiclassical weight $|x|^\rho\exp\left(-x^4\right)$, $x\in\mathbb{R}$, $\rho>-1$ following work by Nevai [5, 109, 110, 111] for the weight $\exp\left(-x^4\right)$ (see also [114] for the asymptotic series related to the weight $\exp\left(-x^4\right)$). An asymptotic series expansion for the recurrence coefficient associated with the semiclassical weight $\exp(-x^4+tx^2)$

was investigated by Clarke and Shizgal [36] in the context of bimode polynomials (see also the recent work in [40] for more details). In this section we prove the existence of an asymptotic series expansion and provide this expansion for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight (4.2.1).

Theorem 5.4.1. *The recurrence coefficient $\beta_n(t;\lambda)$, associated with the generalized Freud weight* (4.2.1)*, satisfying the nonlinear difference equation*

$$\beta_n(t;\lambda)\left(\beta_{n+1}(t;\lambda)+\beta_n(t;\lambda)+\beta_{n-1}(t;\lambda)-\tfrac{1}{2}t\right)=\frac{1}{4}(n+(2\lambda+1)\Omega_n), \tag{5.4.1}$$

where $\Omega_n=\frac{1-(-1)^n}{2}$, has the asymptotic expansion

$$\beta_n(t;\lambda)=\sqrt{\frac{n}{12}}\left(1+\frac{t}{2\sqrt{3n}}+\frac{t^2}{24n}+\frac{48-t^4}{1152n^2}+\frac{t}{48\sqrt{3n^5}}+\mathcal{O}(n^{-3})\right)$$

when n is even and

$$\begin{aligned}\beta_n(t;\lambda)=&\sqrt{\frac{n}{12}}\left(1+\frac{t}{2\sqrt{3n}}+\frac{24\lambda+t^2+12}{24n}+\frac{-t^4-24t^2(1+2\lambda)-96(1+6\lambda(1+\lambda))}{1152n^2}\right.\\&\left.+\frac{t}{48\sqrt{3n^5}}+\mathcal{O}(n^{-3})\right),\end{aligned}$$

when n is odd.

Proof. Following the approach in [17], we obtain the existence of an asymptotic expansion of the coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight (4.2.1). $\beta_n(t;\lambda)\left(\beta_{n-1}(t;\lambda)+\beta_{n+1}(t;\lambda)\right)>0$ since

$$\beta_n(t;\lambda)=\frac{h_n}{h_{n-1}}=\frac{\|S_n\|^2_{w_\lambda}}{\|S_{n-1}\|^2_{w_\lambda}}>0.$$

Hence, it follows from the discrete equation (5.4.1) that

$$4\beta_n^2(t;\lambda)<2t\beta_n(t;\lambda)+[n+(2\lambda+1)\Omega_n], \tag{5.4.2}$$

where $\lambda>0$ and Ω_n is given in (2.9.6).

(5.4.2) is quadratic in $\beta_n(t;\lambda)$, hence

$$0<\beta_n(t;\lambda)<\frac{1}{4}t+\frac{1}{2}\sqrt{\frac{1}{4}t^2+n+(2\lambda+1)\Omega_n}. \tag{5.4.3}$$

Furthermore, the binomial expansion of $\left[1+\dfrac{\left(\frac{1}{4}t^2+(2\lambda+1)\Omega_n\right)}{n}\right]^{\frac{1}{2}}$ yields

$$
\begin{aligned}
0<\beta_n(t;\lambda)&<\frac{1}{4}t+\frac{1}{2}\sqrt{\frac{1}{4}t^2+n+(2\lambda+1)\Omega_n}\\
&=\frac{1}{4}t+\frac{1}{2}\sqrt{n}\left(1+\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{n}\right)^{\frac{1}{2}}\\
&=\frac{1}{4}t+\frac{1}{2}\sqrt{n}\left[1+\frac{1}{2}\left(\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{n}\right)+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}(2k-3)!}{2^{2k-2}(k!)(k-2)!}\left(\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{n}\right)^k\right]\\
&=\frac{1}{4}t+\frac{1}{2}\sqrt{n}+\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{2\sqrt{n}}+\frac{1}{2}\sqrt{n}\left[\sum_{k=2}^{\infty}c_k\left(\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{n}\right)^k\right]\\
&=\frac{1}{4}t+\frac{1}{2}\sqrt{n}+\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{2\sqrt{n}}+\frac{1}{2}\sum_{k=2}^{\infty}c_k\frac{(\frac{1}{4}t^2+(2\lambda+1)\Omega_n)^k}{n^{\frac{2k-1}{2}}}\\
&=\frac{1}{4}t+\frac{1}{2}\sqrt{n}+\frac{\frac{1}{4}t^2+(2\lambda+1)\Omega_n}{2\sqrt{n}}+\frac{1}{2}\sum_{k=2}^{\infty}c_k\left(\frac{1}{4}t^2+(2\lambda+1)\Omega_n\right)^k n^{-\frac{2k-1}{2}} \qquad (5.4.4)
\end{aligned}
$$

where $c_k=\dfrac{(-1)^{k-1}(2k-3)!}{2^{2k-2}(k!)(k-2)!}=\dfrac{(-1)^{k-1}\;\Gamma(2k-2)}{2^{2k-2}\Gamma(k+1)\;\Gamma(k-1)}$.

The recurrence coefficient β_n associated with the weight (4.2.1) is positive and diverges as $n\to\infty$, which suggests that we can write

$$\beta_n=n^r\widehat{\beta}_n,$$

where the quantity $\hat{\beta}_n$ approaches some positive constant, say B, as $n\to\infty$. Hence

$$\beta_n\sim Bn^r \qquad (5.4.5)$$

as $n\to\infty$, where r is an unknown positive constant. Substituting the asymptotic form (5.4.5) into (5.4.1), we obtain

$$12B^2n^{2r}-2tBn^r=n+1+(2\lambda+1)\Omega_{n+1}.$$

Since we require this equation to hold for all $n=1,2,\ldots$, it follows that $r=\frac{1}{2}$, $B=\frac{1}{2\sqrt{3}}$ and the leading behavior is given by $\beta_n\sim\frac{\sqrt{n}}{2\sqrt{3}}$.

Based on (5.4.4), we now consider

$$\beta_n=n^{\frac{1}{2}}\widehat{\beta}_n, \qquad (5.4.6)$$

where the asymptotic expansion of (5.4.6) is assumed to be

$$\widehat{\beta}_n=\sum_{k=0}^{\infty}b_k\varepsilon^k, \qquad (5.4.7)$$

with

$$\varepsilon = \frac{1}{n^z}, \tag{5.4.8}$$

z is an unknown positive constant and the first term in the series given by $b_0 = B = \frac{1}{2\sqrt{3}}$. Substituting (5.4.6) into (5.4.1), we obtain

$$n^{\frac{1}{2}}\widehat{\beta}_n\left((n+1)^{\frac{1}{2}}\widehat{\beta}_{n+1} + n^{\frac{1}{2}}\widehat{\beta}_n + (n-1)^{\frac{1}{2}}\widehat{\beta}_{n-1} - \tfrac{t}{2}\right) = \frac{1}{4}\left(n + (2\lambda+1)\Omega_n\right). \tag{5.4.9}$$

Dividing both sides of (5.4.9) by n yields

$$\widehat{\beta}_n\left[\left(1-\frac{1}{n}\right)^{\frac{1}{2}}\widehat{\beta}_{n-1} + \left(1+\frac{1}{n}\right)^{\frac{1}{2}}\widehat{\beta}_{n+1} + \widehat{\beta}_n - \frac{t}{2n^{\frac{1}{2}}}\right] = \frac{1}{4}\left(1 + \frac{(2\lambda+1)\Omega_n}{n}\right)$$

and by letting $z = \frac{1}{2}$ in (5.4.8), we can write this as

$$\widehat{\beta}_n\left(\sqrt{\varepsilon^2-1}\widehat{\beta}_{n-1} + \sqrt{\varepsilon^2+1}\widehat{\beta}_{n+1} + \widehat{\beta}_n - \frac{t}{2}\varepsilon\right) = \frac{1}{4}\left(1 + \varepsilon^2(2\lambda+1)\Omega_n\right). \tag{5.4.10}$$

In order to evaluate the coefficients $b_k,\ k = 0,1,2,\ldots,5$, in the asymptotic series for $\widehat{\beta}_n$ given in (5.4.7), we note that

$$\begin{aligned}
\widehat{\beta}_n &= A \simeq b_0 + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + b_4\varepsilon^4 + b_5\varepsilon^5,\\
\widehat{\beta}_{n+1} &= \sum_{k=0}^{\infty} b_k\left(\frac{\varepsilon}{\sqrt{\varepsilon^2+1}}\right)^k \simeq \frac{B}{\sqrt{\varepsilon^2+1}(\varepsilon^2+1)^2},\\
\widehat{\beta}_{n-1} &= \sum_{k=0}^{\infty} b_k\left(\frac{\varepsilon}{\sqrt{\varepsilon^2-1}}\right)^k \simeq \frac{C}{\sqrt{\varepsilon^2-1}(\varepsilon^2-1)^2},
\end{aligned}$$

where the approximations

$$B = (b_1+b_3+b_5)\varepsilon^5 + \left(\frac{15b_0}{8} + \frac{3b_2}{2} + b_4\right)\varepsilon^4 + (2b_1+b_3)\varepsilon^3 + \left(\frac{5b_0}{2} + b_2\right)\varepsilon^2 + b_1\varepsilon + b_0$$

$$C = (b_1-b_3+b_5)\,\varepsilon^5 + \left(\frac{15b_0}{8} + b_4 - \frac{3b_2}{2}\right)\varepsilon^4 + (b_3-2b_1)\,\varepsilon^3 + \left(b_2 - \frac{5b_0}{2}\right)\varepsilon^2 + b_1\varepsilon + b_0$$

are obtained using the series expansions of $\widehat{\beta}_{n+1}$, $\widehat{\beta}_{n-1}$ and the binomial expansions of $(\varepsilon^2 \pm 1)^{\ell}$ for $\ell \in \{\frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}, 2\}$ in powers of ε, keeping only the terms of order ε^5 and lower.

Substituting the expressions (5.4.11) into (5.4.10), we obtain

$$\begin{aligned}
&A\left(\left(1-\varepsilon^2\right)^2\left(\varepsilon^2+1\right)^2\left(A - \frac{t\varepsilon}{2}\right) + \left(B\left(1-\varepsilon^2\right)\right)^2 + \left(C\left(\varepsilon^2+1\right)\right)^2\right)\\
&= \frac{1}{4}\left(\varepsilon^2+1\right)^2\left(\varepsilon^2-1\right)^2\left((2\lambda+1)\varepsilon^2\Omega_{n+1}\right) \qquad (5.4.12)\\
&= \begin{cases} \frac{1}{4}\left(\varepsilon^2+1\right)^2\left(\varepsilon^2-1\right)^2 & \text{if } n \text{ is even},\\ \frac{1}{4}\left(\varepsilon^2+1\right)^2\left(\varepsilon^2-1\right)^2\left((2\lambda+1)\varepsilon^2+1\right) & \text{if } n \text{ is odd}, \end{cases} \qquad (5.4.13)
\end{aligned}$$

Expanding all terms in (5.4.12) in powers of ε, retaining only terms of order ε^5 and lower, yields

$$\begin{aligned}&\varepsilon^5\left(b_0t-\frac{b_4t}{2}+6b_5b_0-\frac{49b_1b_0}{4}+6b_2b_3+6b_1b_4\right)+\varepsilon^4\left(-\frac{b_3t}{2}-\frac{1}{4}25b_0^2+6b_4b_0+3b_2^2+6b_1b_3\right)\\&+\varepsilon^3\left(-\frac{b_2t}{2}+6b_1b_2+6b_0b_3\right)+\varepsilon^2\left(-\frac{b_1t}{2}+3b_1^2+6b_0b_2\right)+\varepsilon\left(6b_0b_1-\frac{b_0t}{2}\right)+3b_0^2\\&=\begin{cases}\frac{\varepsilon^8}{4}-\frac{\varepsilon^4}{2}+\frac{1}{4} & \text{if } n \text{ is even,}\\ -\frac{\epsilon^4}{2}+\left(\frac{\lambda}{2}+\frac{1}{4}\right)\epsilon^2+\frac{1}{4} & \text{if } n \text{ is odd.}\end{cases}\end{aligned} \tag{5.4.14}$$

Equating the coefficients of ε on both sides of this equation yields the coefficients $b_k,\ k=1,2,\ldots,5$ in (5.4.7). $\square$

Remark 5.4.1. *Computational algorithms to determine the coefficients involved in the asymptotic expansion for the recurrence coefficients of polynomials orthogonal with respect to exponential-type weights are given in [91, 123] and a Fortran source code can be found in [91, p. 231].*

Remark 5.4.2. *The existence of an asymptotic series for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight* (4.2.1) *could probably also be obtained via a Riemann-Hilbert approach (cf. [15, 17, 46, 82]) but an investigation of this is beyond the scope of this thesis.*

5.5 n-asymptotics of the differential equation

Linear second-order differential equations, which are at the heart of special function theory, can be used in various ways to obtain asymptotic approximations and inequalities. This section focuses on exploring asymptotic results for generalized Freud polynomials via the differential equation satisfied by monic generalized Freud polynomials in chapter 4 (see also [40, 41]).

Monic polynomials $S_n(x;t)$, orthogonal with respect to the generalized Freud weight (4.2.1), satisfy the differential equation (4.6.4), that can be transformed into normal form through the change of the dependent variable

$$Z_n(x;t)=S_n(x;t)\exp\left(\frac{1}{2}\int^x R_n(s;t)ds\right) \tag{5.5.1}$$

and the corresponding equation for $Z_n(x;t)$ is of the form

$$\frac{d^2 Z_n}{dx^2}(x;t) + B_n(x;t) Z_n(x;t) = 0, \tag{5.5.2}$$

where

$$B_n(x;t) = T_n(x;t) - \frac{1}{4}(R_n(x;t))^2 - \frac{1}{2}\frac{dR_n(x;t)}{dx}.$$

The advantage of this transformation is that it does not change the independent variable and the zeros of $Z_n(x;t)$ remain the same as those of $S_n(x;t)$.

Theorem 5.5.1. *Suppose $\{S_n(x;t)\}_{n=0}^{\infty}$ is a sequence of monic polynomials orthogonal with respect to the semiclassical weight $w_\lambda(x;t)$ in* (4.2.1) *and let $Z_n(x;t)$ be given by* (5.5.1)*. Then $Z_n(x;t)$ satisfies the differential equation* (5.5.2) *with*

$$\begin{aligned} B_n(x;t) =& 4\beta_n[(t-2\beta_n-2\beta_{n+1})(t-2\beta_n-2\beta_{n-1}) + (2\lambda+1)(-1)^n + 1] - t[1+(2\lambda+1)(-1)^n] \\ &+ 4tx^4 - 4x^6 + x^2\left(4\lambda+4n-t^2+8\right) - \frac{2(\lambda+1)\lambda - (2\lambda+1)(-1)^n + \frac{1}{2}}{2x^2} \\ &+ \frac{1-2x^2(4\beta_n - t + 2x^2) + (2\lambda+1)(-1)^n}{\beta_n+\beta_{n+1}-\frac{t}{2}+x^2} - \frac{3x^2}{(\beta_n+\beta_{n+1}-\frac{t}{2}+x^2)^2}. \end{aligned} \tag{5.5.3}$$

Remark 5.5.1. *We observe from Theorem 5.4.1 that, as $n \to \infty$,*

$$\beta_n(t;\lambda) = \left(\frac{n}{12}\right)^{\frac{1}{2}} + \mathcal{O}(1),$$

and hence it follows that

$$\begin{aligned} R_n(x;t) &= -4x^3 + 2tx + \frac{2\lambda+1}{x} + \mathcal{O}(n^{-\frac{1}{2}}), \\ T_n(x;t) &= \left(2\sqrt{\frac{n}{3}}\right)^3 + \mathcal{O}(n) = \left(\frac{4}{3}n\right)^{\frac{3}{2}} + \mathcal{O}(n). \end{aligned}$$

The following corollary gives an asymptotic equivalence for the differential equation (5.5.2).

Corollary 5.5.1. *Fix an interval $\Delta \subset \mathbb{R}$. For all $x \in \Delta$ and n sufficiently large, equation* (5.5.2) *is asymptotically equivalent to*

$$\frac{d^2 Z_n}{dx^2}(x;t) + (4nx^2) Z_n(x;t) = \mathcal{O}(n^{\frac{3}{2}}) Z_n(x;t). \tag{5.5.4}$$

Proof. Fix $x \in \Delta$. We let $\psi_n(x) = \beta_n + \beta_{n+1} - \frac{t}{2} + x^2$. Then, $\psi_n'(x) = 2x$ is $\mathcal{O}(1)$ because x is bounded. If n is sufficiently large, then $\dfrac{1}{\psi_n(x)}$ is $\mathcal{O}(1)$ since it follows from

Theorem 5.4.1 that $\beta_n \sim n^{\frac{1}{2}}$. Observe inside the brackets of (5.5.3) that there are 4 terms and we will examine their asymptotic behavior individually.

In the first line of equation (5.5.3), we have

$$4\beta_n[(t-2\beta_n-2\beta_{n+1})(t-2\beta_n-2\beta_{n-1})+(2\lambda+1)(-1)^n+1]-t(1+(2\lambda+1)(-1)^n),$$

which is of $\mathcal{O}(n^{\frac{3}{2}})$ since $\beta_n(t;\lambda) \sim n^{\frac{1}{2}}$ and $x \in \Delta$. The term $x^4 - 4x^6$ in the second line of (5.5.3) is of $\mathcal{O}(1)$ when n is sufficiently large since $x, t \in \Delta$ and λ is real. We can also observe that the term $x^2\left(4\lambda + 4n - t^2 + 8\right)$ in the second line of (5.5.3) is $\mathcal{O}(1)$. Besides, the term $\frac{1-2x^2(4\beta_n-t+2x^2)+(2\lambda+1)(-1)^n}{\beta_n+\beta_{n+1}-\frac{t}{2}+x^2}$ is $\mathsf{o}(1)$ (which is also $\mathcal{O}(1)$), since $\beta_n(t;\lambda) \sim n^{\frac{1}{2}}$ and $x \in \Delta$. Further, we can see that the term $-\frac{3x^2}{(\beta_n+\beta_{n+1}-\frac{t}{2}+x^2)^2}$ is $\mathcal{O}(n^{-1})$ and hence combining all these facts complete the proof. □

Remark 5.5.2. (5.5.4) *can be written as*

$$2x^2\psi_n^2(x)\frac{d^2Z_n}{dx^2}(x;t) + C(x)Z_n(x;t) = \mathcal{O}(n^{\frac{3}{2}})Z_n(x;t), \tag{5.5.5}$$

where $C(x) = 2x^2\psi_n^2(x)B_n(x)$ *and* $B_n(x) = 2\left(\frac{n}{3}\right)^{\frac{3}{2}} + \mathcal{O}(n)$.

5.6 Conclusion

In Chapter 5 we have obtained asymptotic properties of the generalized Freud polynomials, as well as the recurrence coefficients, as the degree, or alternatively, the parameter, tends to infinity. An asymptotic expansion for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight using the theory of Painlevé equations was also obtained. By applying the asymptotics of the recurrence coefficient $\beta_n(t;\lambda)$ in Theorem 5.4.1 to the differential equation satisfied by the generalized Freud polynomials, we obtained a normalized differential equation in its asymptotic form which is valid when x belongs to a fixed, finite interval. The results in this chapter, together with those in [40], provide a framework for possible future investigation of the asymptotic behavior of the generalized Freud polynomials $S_n(x;t)$, as will be discussed in the concluding chapter.

Chapter 6

Summary and future perspectives

Semiclassical orthogonal polynomials, in particular generalized Freud polynomials, have been the main focus of this thesis. Generalized Freud polynomials are orthogonal with respect to the so-called generalized Freud inner product

$$\langle p, q\rangle_{w_\lambda} = \int_{\mathbb{R}} p(x)\ q(x)\ |x|^{2\lambda+1} \exp(-x^4 + tx^2)\, dx, \quad \lambda > 0, \quad t, x \in \mathbb{R}. \tag{6.0.1}$$

We employed the theory of Painlevé equations as our main tool to study recurrence coefficients of the three-term recurrence relation satisfied by the semiclassical generalized Freud polynomials.

In Chapter 4 we focused on analytic properties of the generalized Freud polynomials. It had been believed that for Freud-type weight functions, explicit expressions for the recurrence coefficients in the three-term recurrence relation and the polynomials orthogonal with respect to this weight were non-existent (cf. [117, §18.32]). As one of our pioneering results, we obtained an explicit formulation for the recurrence coefficient $\beta_n(t;\lambda)$ in the three-term recurrence relation (4.2.4) associated with the semiclassical generalized Freud weight (4.2.1), showing that $\beta_n(t;\lambda)$ can be expressed in terms of Wronskians of parabolic cylinder functions that arise in the description of special function solutions of the fourth Painlevé equation $\mathrm{P_{IV}}$. We also obtain an explicit formulation for the generalized Freud polynomials in terms of the recurrence coefficients $\beta_n(t;\lambda)$.

The higher order moments associated with the generalized Freud weight, which are expressible in terms of the derivatives of the first moments, where the first moments are given explicitly in terms of parabolic cylinder functions (cf. Proposition 4.2.1) have also been explored. It is given in [41] that these moments of the semiclassical generalized Freud weight provide the link between the semiclassical generalized Freud weight and the associated Painlevé equation. Future work involves extending our research to establish links between recurrence coefficients associated with a broader class of Shohat-Freud type exponential weights and the theory of Painlevé-type equations.

Continuing our emphasis on the links between the theory of Painlevé equations and semiclassical generalized Freud polynomials, we studied certain (analytic and asymptotic) properties of polynomials orthogonal with respect to the generalized Freud weight. Specifically, we found that the recurrence coefficients satisfy a non-linear difference equation, which is called discrete Painlevé equation $\mathrm{dP_I}$ and also a non-linear special function known to be continuous fourth Painlevé equation $\mathrm{P_{IV}}$ (cf. Lemma 4.3.1). One of the analytic properties we derived is a differential-difference equation (cf. Theorem 4.5.3) satisfied by generalized Freud polynomials. We used two different techniques, one based on the approach of ladder operators and the other using Shohat's approach of quasi-orthogonality. We also derived a second-order linear ordinary differential equation satisfied by generalized Freud polynomials. The coefficients in this differential equation are rational functions that depend on the recurrence coefficient $\beta_n(t;\lambda)$ associated with the weight (4.2.1) (cf. Theorem 4.6.2). In future work, we plan to investigate the recurrence coefficients associated with semiclassical orthogonal polynomials by using alternative methods, such as the method of ladder operators via the large n -asymptotics of the Hankel determinant.

Since recurrence coefficients are fundamental entities in the theory of orthogonal polynomials with respect to exponential weights (1.1.4), the asymptotic expansion of the sequence of recurrence coefficients $\{\beta_n(t;\lambda)\}_{n=0}^{\infty}$ associated with the generalized Freud weight is of interest for applications. In Chapter 5 we obtained certain asymptotic properties of the generalized Freud polynomials, as well as the recurrence coefficients, as the degree, or alternatively, the parameter tends to infinity. We have employed an extension of Freud's conjecture for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud weight (cf. Theorem 5.2.2). We proved the existence of an asymptotic expansion by adapting results by Bleher and Its [17] (see Theorem 5.4.1) and we explored the asymptotic behavior of the recurrence coefficient $\beta_n(t;\lambda)$ via the theory of Painlevé equations. Further, we applied the large n-asymptotics of the recurrence coefficient $\beta_n(t;\lambda)$ in Theorem 5.4.1 to the differential equation (4.6.4) satisfied by the generalized Freud polynomials to obtain a normalized differential equation in its asymptotic form, which is valid when x belongs to a fixed, finite interval (cf. Theorem 5.5.1).

Following the work in [35], we showed in [41] that generalized Freud polynomials arise from the semiclassical Laguerre polynomials by symmetrization. In Chapter 3 we discussed the link between semiclassical Laguerre polynomials and Painlevé equations which involved incorporating orthogonality with certain compatibility relations (3.5.3) of Laguerre-Freud type weights that are governed by the Pearson equation (1.1.2). For the semiclassical Laguerre weight (3.3.1), the associated Pearson equation (1.1.2) will generate a polynomial for $\sigma(x)$ and $\tau(x)$, which in turn controls the outcome of the key entries in the differential-difference equation (Theorem 3.4.1) and differential structure (Theorem 3.4.2) governing the semiclassical Laguerre polynomials.

For the semiclassical Laguerre weight (3.3.1), we also derived two coupled non-linear difference equations (3.3.4) for the recurrence coefficients α_n and β_n. An alternative formal way to derive these non-linear difference equations via the Laguerre method is given in [57, 59, 129]. We also note that not only the semiclassical Laguerre weight (3.3.1), but also the semiclassical Hermite weight (3.3.7), leads to the same difference equations (3.3.4) for the recurrence coefficients.

We remark here that analyzing other semiclassical weight functions with the approach described in Chapter 3, may yield new discrete Painlevé type systems, that might not have yet appeared in the literature.

The overall theme of this thesis has been the investigation of certain properties of the generalized Freud polynomials by making use of their connection to the Painlevé equations. Particularly, Painlevé equations appear when we are studying the recurrence coefficient of the semiclassical generalized Freud polynomials. The work in this thesis illustrates the increasing significance of the Painlevé equations in the field of semiclassical orthogonal polynomials and special functions.

Many aspects and problems originating from the present work deserve further investigation. Here we mention several possible lines of investigation:

(i) Analyzing the existence of an asymptotic expansion for the recurrence coefficient $\beta_n(t;\lambda)$ associated with the generalized Freud polynomials using Riemann-Hilbert's technique.

(ii) Investigating properties of the zeros of generalized Freud polynomials by using the second-order differential equation we have obtained in Theorem 4.6.2.

(iii) Exploring Plancherel-Rotach type asymptotics for polynomials orthogonal with respect to the generalized Freud weight (4.2.1). Alternatively, determine whether a generating function for the generalized Freud polynomials provide us with asymptotic results by using a Darboux-type method.

(iv) Determining a Lax formulation for the generalized Freud polynomials, by adapting the approach used for semiclassical Laguerre polynomials. This may perhaps shed some light on the close relation between the non-linearizability of the corresponding difference equation satisfied by recurrence coefficients and the notion of integrability of the associated discrete integrable system.

(v) Investigating a class of polynomials orthogonal with respect to a more general Shohat-Freud type weight using the techniques described in this thesis.

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