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Advances in Wishart-type modelling of channel capacity

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ABSTRACT

This paper paves the way when the assumption of normality is challenged within the wireless communications systems arena. Innovative results pertaining to the distributions of quadratic forms and their associated eigenvalue density functions for the complex elliptical family are derived, which includes an original Rayleigh-type representation of channels. The presented analytical framework provides computationally convenient forms of these distributions. The results are applied to evaluate an important information-theoretic measure, namely channel capacity. Superior performance in terms of higher capacity of the wireless channel is obtained when considering the underlying complex matrix variate t distribution compared to the usual complex matrix variate normal assumption.

Keywords and phrases: Channel capacity; Complex matrix variate t; Complex Wishart; Eigenvalues; Quadratic form; Rayleigh-type MIMO channel.

2000 *MSC*: 60E05; 62E15; 62H10.

1 Introduction

In wireless communications, systems with multiple-input-multiple-output (MIMO) design have become very popular since they allow higher bit rate and because of their applications in the analysis of signal-to-noise ratio (SNR). In the analysis of channel capacity, the formation of complex channel coefficients play a deterministic role and been taken to be complex matrix variate normal distributed so far, to the best of our knowledge. However, this normality assumption has not been challenged. de Souza and Yacoub (2008) mentioned that the Rayleigh density function is usually derived based on the assumption that from the central limit theorem for large number of partial waves, the resultant process can be decomposed into two orthogonal zero-mean and equal-standard deviation normal random processes. This is an approximation and the restriction of complex normal is unnecessary - it is not always a large number of interfering signals. Thus a more general assumption than complex matrix variate normal may not be that far from reality (see also Ollila et. al. (2011)). This speculative research challenges this assumption of a channel being fed by normal inputs, and sets the platform for introducing our newly proposed models to the MIMO wireless systems arena, and to provide deeper insight into these systems.

The performance of these MIMO systems relies on the quadratic form of the complex normal channel matrix, with n "inputs" and p "outputs", colloquially referred to as "receivers" and "transmitters" respectively. Thus, the distribution of quadratic forms of the underlying complex normal channel matrix is of particular interest. Distributions of quadratic forms of complex normal matrix variates is a topic that has been studied to a wide extent in literature (James (1964), Gupta and Varga (1995), Ratnarajah and Vaillancourt (2005)b). In this paper the distribution of $\mathbf{S} = \mathbf{X}^H \mathbf{A} \mathbf{X}$ is of interest[2], where $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ is taken to be the complex matrix variate *elliptical* distribution to address the criticism against the questionable use of the normal model ($\mathbf{A} \in \mathbb{C}_2^{n\times n}$, where $\mathbb{C}_1^{n\times p}$ denotes the space of $n \times p$ complex matrices, and $\mathbb{C}_2^{p\times p}$ denotes the space of Hermitian positive definite matrices of dimension p). This complex matrix variate elliptical distribution, which contains the well-studied complex matrix variate normal distribution as a special case, is defined next.

The complex matrix variate $\mathbf{X} \in \mathbb{C}_1^{n\times p}$, whose distribution is absolutely continuous, has the complex matrix variate elliptical distribution with parameters $\mathbf{M} \in \mathbb{C}_1^{n\times p}$, $\mathbf{\Phi} \in \mathbb{C}_2^{n\times n}$, $\mathbf{\Sigma} \in \mathbb{C}_2^{p\times p}$, denoted by $\mathbf{X} \sim$

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[2]$\mathbf{X}^H$ denotes the conjugate transpose of $\mathbf{X}$.

$\mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{\Phi}\otimes\mathbf{\Sigma}, g)$, if it has the following density function[3] (see also Micheas et. al. (2006)):

$$h_{\mathbf{X}}(\mathbf{X}) = \frac{1}{|\mathbf{\Sigma}|^n|\mathbf{\Phi}|^p}\, g\left[-\operatorname{tr}\left(\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})\right)\right]. \tag{1}$$

In (1), $g(\cdot)$ denotes the density generator[4] $g:\mathbb{R}^+\to\mathbb{R}^+$, which should be a function of a quadratic form (see also Gupta and Varga (1995)).

Chu (1973) and Gupta and Varga (1995) demonstrates that real elliptical distributions can always be expanded as an integral of a set of normal densities. Similar to Provost and Cheong (2002), we present the following lemma to define the complex matrix variate elliptical distribution as a weighted representation of complex matrix variate normal density functions. This representation can be used to explore the distribution of $\mathbf{S}$ when the distribution of $\mathbf{X}$ can be that of any member of the complex matrix variate elliptical class.

Lemma 1 *If $\mathbf{X}\sim\mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{\Phi}\otimes\mathbf{\Sigma}, g)$ with density function $h_{\mathbf{X}}(\mathbf{X})$, then there exists a scalar weight function $\mathcal{W}(\cdot)$ on $\mathbb{R}^+$ such that*

$$h_{\mathbf{X}}(\mathbf{X}) = \int_{\mathbb{R}^+}\mathcal{W}(t) f_{\mathcal{C}N_{n\times p}(\mathbf{M},\mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})}(\mathbf{X}|t)dt$$

where[5] $f_{\mathcal{C}N_{n\times p}(\mathbf{M},\mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})}(\mathbf{X}|t) = \frac{1}{\pi^{pn}|\mathbf{\Phi}|^p|t^{-1}\mathbf{\Sigma}|^n}\, etr\left[-\left(t\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})\right)\right]$ *is the density function of $\mathbf{X}|t\sim\mathcal{C}N_{n\times p}(\mathbf{M}, \mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})$, with*

$$\mathcal{W}(t) = \pi^{np}t^{-np}\mathcal{L}^{-1}\left\{g\left[-\operatorname{tr}\left(\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})\right)\right]\right\}$$

where $\mathcal{L}$ is the Laplace transform operator.

Proof. Let $s = tr\left(\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})\right)$. Using (1) we have

$$\begin{aligned}
h_{\mathbf{X}}(\mathbf{X}) &= |\mathbf{\Sigma}|^{-n}|\mathbf{\Phi}|^{-p} g[-s] \\
&= |\mathbf{\Sigma}|^{-n}|\mathbf{\Phi}|^{-p}\mathcal{L}\left[\mathcal{W}(t)\pi^{-np}t^{np}\right] \\
&= |\mathbf{\Sigma}|^{-n}|\mathbf{\Phi}|^{-p}\int_{\mathbb{R}^+}\mathcal{W}(t)\pi^{-np}t^{np}e^{-ts}dt \\
&= \int_{\mathbb{R}^+}\mathcal{W}(t)\pi^{-np}\left|t^{-1}\mathbf{\Sigma}\right|^{-n}|\mathbf{\Phi}|^{-p}e^{-ts}dt
\end{aligned}$$

from where the result follows. ∎

Remark 2 *Under the assumptions of Lemma 1, using Fubbini's Theorem, we have*

$$1 = \int_{\mathbb{C}_1^{n\times p}} h_{\mathbf{X}}(\mathbf{X})d\mathbf{X} = \int_{\mathbb{R}^+}\mathcal{W}(t)\left(\int_{\mathbb{C}_1^{n\times p}} f_{\mathbf{X}}(\mathbf{X})d\mathbf{X}\right)dt = \int_{\mathbb{R}^+}\mathcal{W}(t)dt.$$

Thus for a non-negative weight function $\mathcal{W}(\cdot)$, the function $\mathcal{W}(\cdot)$ is a density function of t. Therefore Lemma 1 can only be interpreted as a representation of a scale mixture of complex matrix variate normal distributions. However, $\mathcal{W}(\cdot)$ is not always positive and can be negative on some domains (see Provost and Cheong (2002) for some examples). The only limitation of Lemma 1 is that it defines those complex matrix variate elliptical distributions whose inverse Laplace transform exist. There are some mild sufficient conditions that ensure the inverse Laplace transform exists for most of the well-known complex matrix variate elliptical distributions.

[3] $|\mathbf{X}|$ denotes the determinant of matrix $\mathbf{X}$.
[4] $\mathbb{R}^+$ denotes the positive real line.
[5] $e^{\operatorname{tr}(\cdot)} = \operatorname{etr}(\cdot)$ where $\operatorname{tr}(\mathbf{X})$ denotes the trace of matrix $\mathbf{X}$, and $\mathbf{X}^{-1}$ denotes the inverse of matrix $\mathbf{X}$.

In this paper two special cases of the complex matrix variate elliptical model is of interest. Firstly, the complex random matrix $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ has the complex matrix variate normal distribution with weight function $\mathcal{W}(\cdot)$ in Lemma 1 given by

$$\mathcal{W}(t) = \delta(t-1) \tag{2}$$

where $\delta(\cdot)$ is the dirac delta function (see Chu (1973) and Provost and Cheong (2002)).

Secondly, $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ has the *complex matrix variate t distribution* with the parameters $\mathbf{M} \in \mathbb{C}_1^{n\times p}$, $\mathbf{\Phi} \in \mathbb{C}_2^{n\times n}$, $\mathbf{\Sigma} \in \mathbb{C}_2^{p\times p}$ and degrees of freedom $\upsilon > 0$, denoted by $\mathbf{X} \sim \mathcal{C}t_{n\times p}(\mathbf{M}, \mathbf{\Phi}\otimes\mathbf{\Sigma}, \upsilon)$, with the following density function:

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{\upsilon^{np}\mathcal{C}\Gamma\left(np+\upsilon\right)}{\pi^{np}\mathcal{C}\Gamma_p(\upsilon)}\left\{1+\frac{1}{\upsilon}\operatorname{tr}\left(\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})\right)\right\}^{-(np+\upsilon)} \tag{3}$$

where the complex multivariate gamma function is given by (see James (1964))

$$\mathcal{C}\Gamma_p(a) = \pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{p}\Gamma\left(a-(i-1)\right). \tag{4}$$

In this case the weight function $\mathcal{W}(\cdot)$ in Lemma 1 is given by

$$\mathcal{W}(t) = \frac{(t\upsilon)^{\upsilon}e^{-t\upsilon}}{t\Gamma(\upsilon)} \tag{5}$$

where $\Gamma(\cdot)$ denotes the well-known gamma function.

This paper is organized as follows: in section 2 the distribution of the quadratic form within the complex elliptical class for the nonsingular- and singular case is derived, along with the density functions of the eigenvalues of these quadratic forms. The distribution of the eigenvalues of the quadratic forms are of particular interest in the MIMO environment as it describes the underlying distribution for many of the performance measures for these MIMO systems. In section 3 this newly developed theory in the complex elliptical class is used to evaluate the capacity of MIMO wireless systems for a specific channel environment; by particularly assuming the complex matrix variate t distribution. Furthermore, a Rayleigh-type distribution stemming from the underlying elliptical assumption, is also defined. Section 4 highlights the advantages of the complex matrix variate t distribution in the MIMO environment and includes some conclusions.

2 Distributions of quadratic forms from the complex elliptical class

In this section the necessary theoretical development is presented to set the platform for section 3. The density functions of the nonsingular and singular quadratic forms of complex elliptical random matrices are derived and particular cases of them are of special focus. In addition, the density functions for the joint eigenvalues are also derived; these densities are of particular importance when calculating performance measures of MIMO systems. For the reader's convenience, Remark 3 provides background regarding matrix spaces.

Remark 3 ***Matrix spaces:*** *The set of all $n\times p$ $(n\geq p)$ matrices, $\mathbf{E}$, with orthonormal columns is called the Stiefel manifold, denoted by $\mathcal{C}V_{p,n}$. Thus $\mathcal{C}V_{p,n} = \left\{\mathbf{E}\left(n\times p\right);\mathbf{E}^H\mathbf{E}=\mathbf{I}_p\right\}$. The volume of this manifold is given by $Vol\left(\mathcal{C}V_{p,n}\right) = \int_{\mathcal{C}V_{p,n}}\left(\mathbf{E}^Hd\mathbf{E}\right) = \frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}$. If $n=p$ then a special case of the Stiefel manifold is obtained, the so-called unitary manifold, defined as $\mathcal{C}V_{n,n} = \left\{\mathbf{E}\left(n\times n\right);\mathbf{E}^H\mathbf{E}=\mathbf{I}_n\right\}\equiv U\left(n\right)$ where $U\left(n\right)$ denotes the group of unitary $n\times n$ matrices. The volume of $U\left(n\right)$ is given by $Vol\left(U\left(n\right)\right) = \int_{U(n)}\left(\mathbf{E}^Hd\mathbf{E}\right) = \frac{2^n\pi^{n^2}}{\mathcal{C}\Gamma_n(n)}$.*

2.1 Non-singular case

Theorem 4 *Suppose that $n\geq p$ and $\mathbf{X}\sim\mathcal{C}E_{n\times p}(\mathbf{0},\mathbf{\Phi}\otimes\mathbf{\Sigma},g)$, and let $\mathbf{\Phi},\mathbf{A}\in\mathbb{C}_2^{n\times n}$ and $\mathbf{\Sigma}\in\mathbb{C}_2^{p\times p}$. Then the quadratic form $\mathbf{S}=\mathbf{X}^H\mathbf{A}\mathbf{X}\in\mathbb{C}_2^{p\times p}$ has the integral series complex Wishart-type (ISCW) distribution with density function*

$$f_{\mathbf{S}}(\mathbf{S}) = \frac{|\mathbf{S}|^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_p(n)|\mathbf{\Phi}\mathbf{A}|^p|\mathbf{\Sigma}|^n} \tag{6}$$

where

$$\mathcal{G}(\mathbf{S}) = \int\limits_{\mathbb{R}^+} t^{np} \ {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt$$

and $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$. *This distribution is denoted as* $\mathbf{S} \sim ISCW_p\left(n, \mathbf{\Phi}\otimes\mathbf{\Sigma}, \mathcal{G}(\cdot)\right)$, *where* ${}_0\mathcal{C}F_0^{(p)}(\cdot,\cdot)$ *denotes the complex hypergeometric function with two Hermitian matrix arguments (see James (1964), Koev and Edelman (2006)).*

Proof. From Lemma 1, $\mathbf{X}|t \sim \mathcal{C}N\left(0, \mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma}\right)$. The result follows from Theorem 1 of Ratnarajah and Vaillancourt (2005)b and integrating with respect to the weight function $\mathcal{W}(t)$. ∎

Remark 5 *We know that if* $\mathbf{X} \sim \mathcal{C}N_{n\times p}\left(\mathbf{0}, \mathbf{\Phi}\otimes\mathbf{\Sigma}\right)$ *then* $\mathbf{X}^H\mathbf{A}\mathbf{X}$ *has the complex matrix variate quadratic distribution, denoted by* $\mathcal{C}Q_{n\times p}\left(\mathbf{A}, \mathbf{\Phi}\otimes\mathbf{\Sigma}\right)$. *Assuming that* $\mathbf{X} \sim \mathcal{C}E_{n\times p}\left(\mathbf{0}, \mathbf{\Phi}\otimes\mathbf{\Sigma}, g\right)$, *it then follows from Lemma 1 that*

$$\mathbf{S} = \mathbf{X}^H\mathbf{A}\mathbf{X} \stackrel{d}{=} \mathbf{Z}^H\mathbf{A}\mathbf{Z}, \quad \textit{where } \mathbf{Z}|t \sim \mathcal{C}N_{n\times p}\left(\mathbf{0}, \mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma}\right)$$

with

$$\mathbf{Z}^H\mathbf{A}\mathbf{Z}|t \sim \mathcal{C}Q_{n\times p}\left(\mathbf{A}, \mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma}\right).$$

Therefore

$$f_{\mathbf{S}}\left(\mathbf{S}\right) = \int\limits_{\mathbb{R}^+} \mathcal{W}(t) f_{\mathcal{C}Q_{n\times p}\left(\mathbf{A}, \mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma}\right)}\left(\mathbf{Z}^H\mathbf{A}\mathbf{Z}|t\right)dt.$$

Particular cases of the density function (6) will be focussed on, since they form part of the investigation in Section 3.

Remark 6 *If* $\mathbf{A} = \mathbf{I}_n$ *and* $\mathbf{\Phi} = \mathbf{I}_n$ *then* $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ *has the complex Wishart-type distribution with the following density function*

$$f_{\mathbf{S}}(\mathbf{S}) = \frac{|\mathbf{S}|^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_p(n)|\mathbf{\Sigma}|^n} \tag{7}$$

where

$$\mathcal{G}(\mathbf{S}) = \int\limits_{\mathbb{R}^+} t^{np} \ \mathrm{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt.$$

If $\mathbf{\Sigma} = \sigma^2\mathbf{I}_p$ *(thus, uncorrelated with variance* σ^2*), (7) simplifies to*

$$f_{\mathbf{S}}(\mathbf{S}) = \frac{|\mathbf{S}|^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_p(n)\sigma^{2np}}$$

where

$$\mathcal{G}(\mathbf{S}) = \int\limits_{\mathbb{R}^+} t^{np} \ \mathrm{etr}\left(-t\sigma^{-2}\mathbf{S}\right)\mathcal{W}(t)dt.$$

Next, an expression for the density function of the joint eigenvalues of $\mathbf{S} = \mathbf{X}^H\mathbf{A}\mathbf{X}$ is given, when $\mathbf{S} \sim ISCW_p\left(n, \mathbf{\Phi}\otimes\mathbf{\Sigma}, \mathcal{G}(\cdot)\right)$ (see (6)).

Theorem 7 *Suppose that* $\mathbf{S} \sim ISCW_p\left(n, \mathbf{\Phi}\otimes\mathbf{\Sigma}, \mathcal{G}(\cdot)\right)$, *and let* $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ *represent the ordered eigenvalues of* $\mathbf{S} \in \mathbb{C}_2^{p\times p}$. *Then the eigenvalues of* $\mathbf{S}$, $\mathbf{\Lambda} = diag\left(\lambda_1, \lambda_2, ..., \lambda_p\right)$, *has density function*[6]

$$f(\mathbf{\Lambda}) = K\int\limits_{\mathbb{R}^+} t^{np}\int\limits_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}\mathcal{W}(t)dt \tag{8}$$

$$= K\int\limits_{\mathbb{R}^+} t^{np}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\mathcal{C}C_\kappa\left(\mathbf{B}\right)}{k!C_\kappa\left(\mathbf{I}_n\right)}\frac{\mathcal{C}C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)\mathcal{C}C_\kappa\left(\mathbf{\Lambda}\right)}{C_\kappa\left(\mathbf{I}_p\right)}\mathcal{W}(t)dt \tag{9}$$

where $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$ *and* $K = \frac{\pi^{p(p-1)}\left(\prod_{i=1}^{p}\lambda_i^{n-p}\right)\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)|\mathbf{\Phi}\mathbf{A}|^p|\mathbf{\Sigma}|^n}$.

[6] $\mathcal{C}C_\kappa(\mathbf{Z})$ denotes the complex zonal polynomial of $\mathbf{Z}$ corresponding to the partition $\kappa = (k_1, \ldots, k_p), k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$ and $\sum_\kappa$ denotes summation over all partitions κ.

Proof. Using Eq. 93 of James (1964) and (6), the joint density function of the eigenvalues $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ of $\mathbf{S}$ is given by

$$f(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)} \left(\prod_{k<l}^{p} (\lambda_k - \lambda_l)^2 \right) |\mathbf{\Lambda}|^{n-p}}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)|\mathbf{\Phi A}|^p |\mathbf{\Sigma}|^n} \int\limits_{\mathbf{E}\in U(p)} \mathcal{G}(\mathbf{E\Lambda E}^H) d\mathbf{E}.$$

By using Definition 2.6 from Dubbs and Edelman (2014), (9) follows directly. ■

Particular cases of the density function in (8) are focussed on next, since they form part of the investigation in Section 3.

Remark 8 *If $\mathbf{A} = \mathbf{I}_n$ and $\mathbf{\Phi} = \mathbf{I}_n$ then the joint density function of the eigenvalues of the complex Wishart-type distribution, $f(\mathbf{\Lambda})$, simplifies to*

$$f(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)} \left(\prod_{i=1}^{p} \lambda_i^{n-p} \right) \left(\prod_{k<l}^{p} (\lambda_k - \lambda_l)^2 \right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)|\mathbf{\Sigma}|^n} \int\limits_{\mathbb{R}^+} t^{np} \; {}_0\mathcal{C}F_0^{(p)} \left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1} \right) \mathcal{W}(t) dt. \tag{10}$$

If $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_p$ (thus, uncorrelated with variance σ^2), (10) simplifies to

$$f(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)} \left(\prod_{i=1}^{p} \lambda_i^{n-p} \right) \left(\prod_{k<l}^{p} (\lambda_k - \lambda_l)^2 \right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\sigma^{2np}} \int\limits_{\mathbb{R}^+} t^{np} \; \exp\left(-t\sigma^{-2} \sum_{i=1}^{p} \lambda_i \right) \mathcal{W}(t) dt.$$

Remark 9 *It is known that expressions containing hypergeometric functions of matrix argument and zonal polynomials may be cumbersome to compute, and that software packages have limitations to handle such computations. In this paper only cases with specific interest in MIMO systems will be focussed on. The reader is referred to Alfano et. al. (2014), Gross and Richards (1989), and Koev and Edelman (2006) for some analytical expressions to compute such hypergeometric functions of matrix arguments.*

The following table gives the density function for the special cases (see (7) and (10)) for the complex matrix variate normal and complex matrix variate t distribution (see (5)) case respectively. The expressions for the complex matrix variate normal case reflects the results of James (1964).

Distribution of X	Density function
	$f_{\mathbf{S}}(\mathbf{S})$ (see (7))
Normal	$\left(\mathcal{C}\Gamma_p(n)|\mathbf{\Sigma}|^n\right)^{-1} |\mathbf{S}|^{n-p} \operatorname{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{S}\right)$
t	$\left(\Gamma(\upsilon)\, \mathcal{C}\Gamma_p(n)|\mathbf{\Sigma}|^n\right)^{-1} \upsilon^{\upsilon} |\mathbf{S}|^{n-p} \Gamma(np+\upsilon) \left(\operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{S} + \upsilon\right)^{-(np+\upsilon)}$
	$f(\mathbf{\Lambda})$ (see (10))
Normal	$\left(\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)|\mathbf{\Sigma}|^n\right)^{-1} \pi^{p(p-1)} \left(\prod_{i=1}^{p} \lambda_i^{n-p}\right) \times \left(\prod_{k<l}^{p} (\lambda_k - \lambda_l)^2\right) {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -\mathbf{\Sigma}^{-1}\right)$
t	$\left(\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)|\mathbf{\Sigma}|^n \Gamma(\upsilon)\, \upsilon^{np}\right)^{-1} \pi^{p(p-1)} \left(\prod_{i=1}^{p} \lambda_i^{n-p}\right) \times \left(\prod_{k<l}^{p} (\lambda_k - \lambda_l)^2\right) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\mathcal{C}C_\kappa(-\mathbf{\Sigma}^{-1})\mathcal{C}C_\kappa(\mathbf{\Lambda})}{\upsilon^k k! C_\kappa(\mathbf{I}_p)} \Gamma(np+\upsilon+k)$

Table 2.1: Density functions of certain cases of complex matrix variate elliptical quadratic form

2.2 Singular case

In this section the *singular* case of the quadratic form of the complex matrix variate elliptical distribution is also considered, where $0 < n < p$.

Theorem 10 *Suppose that* $0<n<p$ *and* $\mathbf{X}\sim\mathcal{C}E_{n\times p}(\mathbf{0},\mathbf{\Phi}\otimes\mathbf{\Sigma},g)$*, and let* $\mathbf{\Phi},\mathbf{A}\in\mathbb{C}_2^{n\times n}$ *and* $\mathbf{\Sigma}\in\mathbb{C}_2^{p\times p}$*. Let* $\mathbf{\Lambda}=diag\left(\lambda_1,\lambda_2,...,\lambda_p\right)$*. Then the quadratic form* $\mathbf{S}=\mathbf{X}^H\mathbf{A}\mathbf{X}\in\mathbb{C}_2^{p\times p}$ *has the integral series complex singular Wishart-type (ISCSW) distribution with density function*

$$f_{\mathbf{S}}(\mathbf{S})=\frac{\pi^{n(n-p)}|\mathbf{\Lambda}|^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_n(n)|\mathbf{\Phi}\mathbf{A}|^p|\mathbf{\Sigma}|^n} \tag{11}$$

where

$$\mathcal{G}(\mathbf{S})=\int\limits_{\mathbb{R}^+}t^{np}\ {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt$$

and $\mathbf{B}=\mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$*. This distribution is denoted as* $\mathbf{S}\sim ISCSW_n\left(p,\mathbf{\Phi}\otimes\mathbf{\Sigma},\mathcal{G}(\cdot)\right)$*.*

Proof. See that

$$f(\mathbf{X})=\int\limits_{\mathbb{R}^+}t^{np}|\mathbf{\Phi}\mathbf{A}|^{-p}|\mathbf{\Sigma}|^{-n}\pi^{-np}\operatorname{etr}\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)\mathcal{W}(t)dt$$

where $\mathbf{X}|t\sim\mathcal{C}N\left(\mathbf{0},\mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma}\right)$. Let $\mathbf{X}^H\mathbf{A}^{\frac{1}{2}}=\mathbf{E}_1\mathbf{\Upsilon}\mathbf{H}$ (where $\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}=\mathbf{A}$), and note $\mathbf{S}=\mathbf{X}^H\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}\mathbf{X}=\mathbf{E}_1\mathbf{\Upsilon}\mathbf{H}\mathbf{H}^H\mathbf{\Upsilon}\mathbf{E}_1^H=\mathbf{E}_1\mathbf{\Upsilon}^2\mathbf{E}_1^H=\mathbf{E}_1\mathbf{\Lambda}\mathbf{E}_1^H$ (where $\mathbf{\Upsilon}^2=\mathbf{\Lambda}$). From Remark 3 follows:

$$f(\mathbf{S})=\frac{\pi^{-np}|\mathbf{\Lambda}|^{n-p}}{\mathcal{C}\Gamma_n(n)|\mathbf{\Phi}\mathbf{A}|^p|\mathbf{\Sigma}|^n}\int\limits_{\mathcal{C}V_{n,n}}\int\limits_{\mathbb{R}^+}t^{np}\ {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dtd\mathbf{H}$$

from where the result follows after some simplification. ■

Particular cases of the density function (11) will be focussed on, since they form part of the investigation in Section 3.

Remark 11 *If* $\mathbf{A}=\mathbf{I}_n$ *and* $\mathbf{\Phi}=\mathbf{I}_n$*, then* $\mathbf{S}$ *has the complex singular Wishart-type distribution with the following density function*

$$f_{\mathbf{S}}(\mathbf{S})=\frac{\pi^{n(n-p)}|\mathbf{\Lambda}|^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_n(n)|\mathbf{\Sigma}|^n} \tag{12}$$

where

$$\mathcal{G}(\mathbf{S})=\int\limits_{\mathbb{R}^+}t^{np}\ \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt.$$

If $\mathbf{\Sigma}=\sigma^2\mathbf{I}_p$ *(thus, uncorrelated with variance* σ^2*), (12) simplifies to*

$$f_{\mathbf{S}}(\mathbf{S})=\frac{\pi^{n(n-p)}|\mathbf{\Lambda}|^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_n(n)\sigma^{2np}}$$

where

$$\mathcal{G}(\mathbf{S})=\int\limits_{\mathbb{R}^+}t^{np}\ \operatorname{etr}\left(-t\sigma^{-2}\mathbf{S}\right)\mathcal{W}(t)dt.$$

Next, expressions for the density function of the joint eigenvalues for the singular case are derived.

Theorem 12 *Suppose that* $0<n<p$ *and* $\mathbf{S}\sim ISCSW_n\left(p,\mathbf{\Phi}\otimes\mathbf{\Sigma},\mathcal{G}(\cdot)\right)$ *(see 11), and let* $\lambda_1>\lambda_2>...>\lambda_n>0$ *represent the ordered eigenvalues of* $\mathbf{S}$*. Then the joint distribution of the eigenvalues of* $\mathbf{S}$*,* $\mathbf{\Lambda}=diag\left(\lambda_1,\lambda_2,...,\lambda_p\right)$*, has density function*

$$\begin{aligned} f(\mathbf{\Lambda}) \quad = \quad & \frac{\pi^{n(n-1)}\left(\prod\limits_{i=1}^{n}\lambda_i^{p-n}\right)\left(\prod\limits_{k<l}^{n}\left(\lambda_k-\lambda_l\right)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)|\mathbf{\Phi}\mathbf{A}|^p|\mathbf{\Sigma}|^n} \\ & \times\int\limits_{\mathbb{R}^+}t^{np}\int\limits_{\mathcal{C}V_{p,n}}{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)(d\mathbf{E})\,\mathcal{W}(t)dt \end{aligned} \tag{13}$$

where $\mathbf{B}=\mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$*.*

Proof. Consider a partial spectral decomposition where $\mathbf{S} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^H$, where $\mathbf{E} \in \mathcal{C}V_{p,n}$. The transformation from $\mathbf{S}$ to $\mathbf{E}, \boldsymbol{\Lambda}$ has volume element

$$(d\mathbf{S}) = (2\pi)^{-n} \left|\boldsymbol{\Lambda}^{n-p}\right|^{-2} \prod_{k<l}^{n} (\lambda_k - \lambda_l)^2 \, (d\boldsymbol{\Lambda}) \left(\mathbf{E}^H d\mathbf{E}\right).$$

Therefore, from (11) and Remark 3:

$$\begin{aligned} f(\boldsymbol{\Lambda}) &= \frac{\pi^{n(n-p)}}{\mathcal{C}\Gamma_n(n)|\boldsymbol{\Phi}\mathbf{A}|^p|\boldsymbol{\Sigma}|^n} (2\pi)^{-n} \left|\boldsymbol{\Lambda}^{n-p}\right|^{-2} |\boldsymbol{\Lambda}|^{n-p} \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)\right)^2 \\ &\quad \times \int_{\mathbb{R}^+} t^{np} \int_{\mathcal{C}V_{p,n}} {}_0\mathcal{C}F_0^{(n)} \left(\mathbf{B}, -t\boldsymbol{\Sigma}_2^{-1}\mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^H\right) \left(\mathbf{E}^H d\mathbf{E}\right) \mathcal{W}(t) dt \end{aligned}$$

and the result follows. ■

Some special cases of the density function in (13) are reported next.

Remark 13 *If $\mathbf{A} = \mathbf{I}_n$ and $\boldsymbol{\Phi} = \mathbf{I}_n$, then the joint density function of the eigenvalues of the complex singular Wishart type distribution, $f(\boldsymbol{\Lambda})$, simplifies to the following density function:*

$$f(\boldsymbol{\Lambda}) = \frac{\pi^{n(n-1)} \left(\prod_{i=1}^{n} \lambda_i^{p-n}\right) \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)|\boldsymbol{\Sigma}|^n} \int_{\mathbb{R}^+} t^{np} \; {}_0\mathcal{C}F_0^{(n)} \left(\boldsymbol{\Lambda}, -t\boldsymbol{\Sigma}^{-1}\right) \mathcal{W}(t) dt. \tag{14}$$

If $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_p$ (thus, uncorrelated with variance σ^2), (12) simplifies to

$$f(\boldsymbol{\Lambda}) = \frac{\pi^{n(n-1)} \left(\prod_{i=1}^{n} \lambda_i^{p-n}\right) \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\sigma^{2np}} \int_{\mathbb{R}^+} t^{np} \; \exp\left(-t\sigma^{-2} \sum_{i=1}^{n} \lambda_i\right) \mathcal{W}(t) dt.$$

The following table gives the density function for the special cases (see (12) and (14)) for weight functions (2) and (5) respectively.

Distribution of X	Density function
	$f_{\mathbf{S}}(\mathbf{S})$ (see (12))
Normal	$\left(\mathcal{C}\Gamma_n(n)|\boldsymbol{\Sigma}|^n\right)^{-1} \pi^{n(n-p)} |\boldsymbol{\Lambda}|^{n-p} \operatorname{etr}\left(-\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)$
t	$\left(\Gamma(\upsilon)\mathcal{C}\Gamma_n(n)|\boldsymbol{\Sigma}|^n\right)^{-1} \upsilon^{\upsilon}\pi^{n(n-p)}|\boldsymbol{\Lambda}|^{n-p}\Gamma\left(np+\upsilon\right)\left(\operatorname{tr}\boldsymbol{\Sigma}^{-1}\mathbf{S}+\upsilon\right)^{-(np+\upsilon)}$
	$f(\boldsymbol{\Lambda})$ (see (14))
Normal	$\left(\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)|\boldsymbol{\Sigma}|^n\right)^{-1} \pi^{n(n-1)} \left(\prod_{i=1}^{n} \lambda_i^{p-n}\right)$ $\times \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)^2\right) {}_0\mathcal{C}F_0^{(n)}\left(\boldsymbol{\Lambda}, -\boldsymbol{\Sigma}^{-1}\right)$ (see Eq. 25 in Ratnarajah and Vaillancourt (2005))
t	$\left(\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)|\boldsymbol{\Sigma}|^n\Gamma\left(\upsilon\right)\upsilon^{np}\right)^{-1} \pi^{n(n-1)} \left(\prod_{i=1}^{n} \lambda_i^{p-n}\right)$ $\times \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)^2\right) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\mathcal{C}C_\kappa\left(-\boldsymbol{\Sigma}^{-1}\right)\mathcal{C}C_\kappa(\boldsymbol{\Lambda})}{\upsilon^k k! C_\kappa(\mathbf{I}_p)}\Gamma\left(np+\upsilon+k\right)$

Table 2.2: Density functions of certain cases of complex singular matrix variate elliptical quadratic form

3 Channel capacity

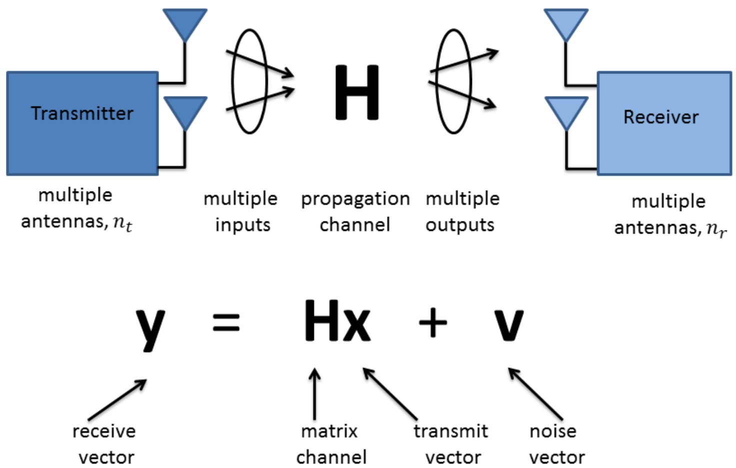

Figure 4.1: MIMO System

Suppose that a communication system is being characterized by the following output relation, as depicted in Figure 4.1:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v},$$

where $\mathbf{y}, \mathbf{v} \in \mathbb{C}_1^{n_r \times 1}$, $\mathbf{x} \in \mathbb{C}_1^{n_t \times 1}$ and $\mathbf{H} \in \mathbb{C}_1^{n_r \times n_t}$. In a correlated Rayleigh channel, the distribution of an $n_r \times n_t$ channel matrix $\mathbf{H}$ is usually given by $\mathbf{H} \sim \mathcal{C}N_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma})$ with $n_r \geq n_t$ (in other words, the channel coefficient from different transmitter antennas to a single receiver antenna is correlated), and note that the off-diagonal elements of $\mathbf{\Sigma} \in \mathbb{C}_2^{n_t \times n_t}$ are nonzero for correlated channels. Suppose that the channel matrix $\mathbf{H}$ and noise vector $\mathbf{v}$ are independently distributed according the complex matrix variate elliptical and complex multivariate normal distributions, respectively, in other words, $\mathbf{H} \sim \mathcal{C}E_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma}, g)$, and $\mathbf{v} \sim \mathcal{C}N_{n_r \times 1}(\mathbf{0}, \sigma^2 \mathbf{I}_{n_r})$. In this section, the focus is to derive the channel capacity capacity if $\mathbf{H} \sim \mathcal{C}t_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma}, \upsilon)$, with the weight function (5).

The input power is distributed equally over all transmitting antennas and is constrained to ρ (the signal to noise ratio) such that (see Ratnarajah and Vaillancourt (2005)b)

$$E\left(\mathbf{x}^H \mathbf{x}\right) \leq \rho.$$

For the purpose of this paper we are particularly interested in Rayleigh distributed channels. However, having an underlying complex matrix variate elliptical distribution for $\mathbf{H}$ results in having to consider a Rayleigh-*type* channel which is defined next.

Proposition 14 *Consider a complex elliptical process, $Z = X + iY$, where X, Y are independent and identically zero-mean elliptical random variates. Let $R = \sqrt{X^2 + Y^2}$ denote an element h_{ij} of $\mathbf{H}$. The density function of R emanating from the complex elliptical class is given by*

$$h(r) = \frac{r}{\sigma^2} \int\limits_{\mathbb{R}^+} t \exp\left(-\frac{r^2}{2\sigma^2 t^{-1}}\right) \mathcal{W}(t) dt$$

where $r > 0$, which is described as a Rayleigh-type density function (see also Miller (1974)).

Moreover, if a block-fading model is assumed together with coding over many independent fading intervals, then the ergodic capacity of the random MIMO channel is given by (see Teletar (1999))

$$\begin{aligned} C &= E_{\mathbf{H}}\left(\log\left|\left(\mathbf{I}_{n_t}+\frac{\rho}{n_t}\mathbf{H}^H\mathbf{H}\right)\right|\right) \\ &= E_{\mathbf{\Lambda}}\left(\log\prod_{k=1}^{n_t}\left(1+\frac{\rho}{n_t}\lambda_k\right)\right) \end{aligned} \tag{15}$$

where $\lambda_1 > ... > \lambda_{n_t}$ are the eigenvalues of $\mathbf{S}$. Hence (15) can be evaluated using the joint density functions of the eigenvalues ((8) and (13) respectively). In the following two sections, the channel capacity is derived for the nonsingular- and singular case, for both correlated- and uncorrelated cases.

3.1 Nonsingular case

In this section the assumption is that the complex channel coefficients are distributed according to the complex matrix variate t distribution. To this end, we first consider the more general complex matrix variate elliptical distribution and subsequently derive the results for the complex matrix variate t distribution. We firstly derive the expressions for the channel capacity of a correlated- and uncorrelated Rayleigh-type $n_r \times 2$ channel environment when the underlying distribution is complex matrix variate elliptical. In particular, a two-input ($n_t = 2$), n_r output communication system is considered and the capacity graphically illustrated.

Theorem 15 *1. For a two-input correlated Rayleigh-type channel* $\mathbf{H} \sim \mathcal{C}E_{n_r\times 2}(\mathbf{0}, \mathbf{I}_{n_r}\otimes\mathbf{\Sigma}, g)$*, with* $n_r \geq 2$*, the capacity* C *is given by*

$$\begin{aligned} C &= \frac{(a_1a_2)^{n_r}}{\Gamma(n_r)\Gamma(n_r-1)(a_1-a_2)}\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) \\ &\times\Bigg\{\lambda_1^{n_r-1}\Gamma(n_r-1)\,a_2^{-(n_r-1)}\int_{\mathbb{R}^+} t^{n_r}\exp(-ta_1\lambda_1)\,\mathcal{W}(t)dt \\ &-\lambda_1^{n_r-1}\Gamma(n_r-1)\,a_1^{-(n_r-1)}\int_{\mathbb{R}^+} t^{n_r}\exp(-ta_2\lambda_1)\,\mathcal{W}(t)dt \\ &-\lambda_1^{n_r-2}\Gamma(n_r)\,a_2^{-n_r}\int_{\mathbb{R}^+} t^{n_r-1}\exp(-ta_1\lambda_1)\,\mathcal{W}(t)dt \\ &+\lambda_1^{n_r-2}\Gamma(n_r)\,a_1^{-n_r}\int_{\mathbb{R}^+} t^{n_r-1}\exp(-ta_2\lambda_1)\,\mathcal{W}(t)dt\Bigg\}d\lambda_1 \end{aligned} \tag{16}$$

where $a_1 > a_2$ *are the ordered eigenvalues of the diagonalized covariance matrix* $\mathbf{\Sigma}$*.*

2. For a two-input uncorrelated Rayleigh-type channel $\mathbf{H} \sim \mathcal{C}E_{n_r\times 2}(\mathbf{0}, \mathbf{I}_{n_r}\otimes\sigma^2\mathbf{I}_2, g)$*, with* $n_r \geq 2$*, the capacity* C *is given by*

$$\begin{aligned} C &= \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\Bigg\{\int_{\mathbb{R}^+}\frac{\lambda_1^{n_r}t^{n_r+1}\exp\left(-t\sigma^{-2}\lambda_1\right)\mathcal{W}(t)}{2\Gamma(n_r)\,\sigma^2}dt \\ &-\int_{\mathbb{R}^+}\frac{\lambda_1^{n_r-1}t^{n_r}\exp\left(-t\sigma^{-2}\lambda_1\right)\mathcal{W}(t)}{\Gamma(n_r-1)}dt \\ &+\int_{\mathbb{R}^+}\frac{\lambda_1^{n_r-2}t^{n_r-1}\Gamma(n_r+1)\exp\left(-t\sigma^{-2}\lambda_1\right)\mathcal{W}(t)}{2\Gamma(n_r-1)\,\sigma^{-2}}dt\Bigg\}d\lambda_1. \end{aligned} \tag{17}$$

Proof. 1. The unordered density function of (10) is obtained by dividing by $p! = n_t! = 2!$:

$$f(\lambda_1, \lambda_2) = \frac{(\lambda_1\lambda_2)^{n_r-2}(\lambda_1-\lambda_2)(a_1a_2)^{n_r}}{2!\Gamma(n_r)\Gamma(n_r-1)(a_2-a_1)}\int_{\mathbb{R}^+} t^{2n_r-1}\left|\exp(-ta_i\lambda_j)\right|\mathcal{W}(t)dt$$

since from (4) we have $\mathcal{C}\Gamma_2(2) = \pi\Gamma(2)\Gamma(1)$, $\mathcal{C}\Gamma_2(n_r) = \pi\Gamma(n_r)\Gamma(n_r-1)$, and using an expression for the complex hypergeometric function by Khatri (1969). Then

$$\begin{aligned}
\left|\exp(-ta_i\lambda_j)\right| &= \begin{vmatrix} \exp(-ta_1\lambda_1) & \exp(-ta_1\lambda_2) \\ \exp(-ta_2\lambda_1) & \exp(-ta_2\lambda_2) \end{vmatrix} \\
&= \exp(-t(a_1\lambda_1 + a_2\lambda_2)) - \exp(-t(a_1\lambda_2 + a_2\lambda_1)).
\end{aligned}$$

From (15) the capacity for a correlated Rayleigh-type fading model of dimension $n_r \times 2$ under the complex matrix variate elliptical distribution is given by

$$\begin{aligned}
C &= 2\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_0^\infty f(\lambda_1,\lambda_2)d\lambda_2 d\lambda_1 \\
&= K\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_0^\infty \left(\lambda_1^{n_r-1}\lambda_2^{n_r-2} - \lambda_1^{n_r-2}\lambda_2^{n_r-1}\right) \\
&\quad \times \int_{\mathbb{R}^+} t^{2n_r-1}\left(\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \exp(-t(a_1\lambda_2+a_2\lambda_1))\right)\mathcal{W}(t)dtd\lambda_2 d\lambda_1 \\
&= K\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_{\mathbb{R}^+} t^{2n_r-1}\int_0^\infty \left(\lambda_1^{n_r-1}\lambda_2^{n_r-2} - \lambda_1^{n_r-2}\lambda_2^{n_r-1}\right) \\
&\quad \times \left(\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \exp(-t(a_1\lambda_2+a_2\lambda_1))\right)d\lambda_2\mathcal{W}(t)dtd\lambda_1
\end{aligned}$$

where $K = \frac{(a_1a_2)^{n_r}}{\Gamma(n_r)\Gamma(n_r-1)(a_2-a_1)}$. The latter integral equals

$$\begin{aligned}
&\lambda_1^{n_r-1}\exp(-ta_1\lambda_1)\Gamma(n_r-1)(ta_2)^{-(n_r-1)} - \lambda_1^{n_r-1}\exp(-ta_2\lambda_1)\Gamma(n_r-1)(ta_1)^{-(n_r-1)} \\
&-\lambda_1^{n_r-2}\exp(-ta_1\lambda_1)\Gamma(n_r)(ta_2)^{-n_r} + \lambda_1^{n_r-2}\exp(-ta_2\lambda_1)\Gamma(n_r)(ta_1)^{-n_r}
\end{aligned}$$

by using Eq. 3.381.4 from Gradshteyn and Rhyzik (2007). Result (16) follows.

2. The proof follows similarly where $\mathbf{\Sigma} = \sigma^2\mathbf{I}_2$. ■

A particular focus is that of an underlying complex matrix variate t distribution, therefore the weight function (5) is substituted into (16) and (17) to obtain the corresponding capacity.

Corollary 16 *1. For a two-input correlated Rayleigh-type channel, $\mathbf{H} \sim \mathcal{C}t_{n_r\times 2}(\mathbf{0}, \mathbf{I}_{n_r}\otimes\mathbf{\Sigma}, \upsilon)$, with $n_r \geq 2$, the capacity is given by*

$$\begin{aligned}
C &= \frac{a_1^{n_r}a_2\upsilon^\upsilon\Gamma(n_r+\upsilon)}{(a_1-a_2)\Gamma(\upsilon)\Gamma(n_r)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}(a_1\lambda_1+\upsilon)^{-(n_r+\upsilon)}d\lambda_1 \qquad (18) \\
&\quad -\frac{a_1a_2^{n_r}\upsilon^\upsilon\Gamma(n_r+\upsilon)}{(a_1-a_2)\Gamma(\upsilon)\Gamma(n_r)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}(a_2\lambda_1+\upsilon)^{-(n_r+\upsilon)}d\lambda_1 \\
&\quad -\frac{a_1^{n_r}\upsilon^\upsilon\Gamma(n_r+\upsilon-1)}{(a_1-a_2)\Gamma(\upsilon)\Gamma(n_r-1)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}(a_1\lambda_1+\upsilon)^{-(n_r+\upsilon-1)}d\lambda_1 \\
&\quad +\frac{a_2^{n_r}\upsilon^\upsilon\Gamma(n_r+\upsilon-1)}{(a_1-a_2)\Gamma(\upsilon)\Gamma(n_r-1)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}(a_2\lambda_1+\upsilon)^{-(n_r+\upsilon-1)}d\lambda_1.
\end{aligned}$$

2. For a two-input uncorrelated Rayleigh-type channel, $\mathbf{H} \sim \mathcal{C}t_{n_r\times 2}(\mathbf{0}, \mathbf{I}_{n_r}\otimes\sigma^2\mathbf{I}_2, \upsilon)$, with $n_r \geq 2$, the capacity

C is given by

$$C = \frac{\upsilon^{\upsilon}\Gamma(n_r+\upsilon+1)}{\sigma^2\Gamma(\upsilon)\Gamma(n_r)}\int_0^{\infty}\log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r}\left(\frac{\lambda_1}{\sigma^2}+\upsilon\right)^{-(n_r+\upsilon+1)}d\lambda_1 \tag{19}$$
$$-\frac{2\upsilon^{\upsilon}\Gamma(n_r+\upsilon)}{\Gamma(\upsilon)\Gamma(n_r-1)}\int_0^{\infty}\log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}\left(\frac{\lambda_1}{\sigma^2}+\upsilon\right)^{-(n_r+\upsilon)}d\lambda_1$$
$$+\frac{\upsilon^{\upsilon}\Gamma(n_r+\upsilon-1)\Gamma(n_r+1)}{\sigma^{-2}\Gamma(\upsilon)\Gamma(n_r-1)}\int_0^{\infty}\log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}\left(\frac{\lambda_1}{\sigma^2}+\upsilon\right)^{-(n_r+\upsilon-1)}d\lambda_1.$$

Figure 4.2 shows the calculated channel capacity (18) versus n_r for different values of ρ, assuming a correlation of 0.9, $\sigma^2 = 1$, and $\upsilon = 10$.

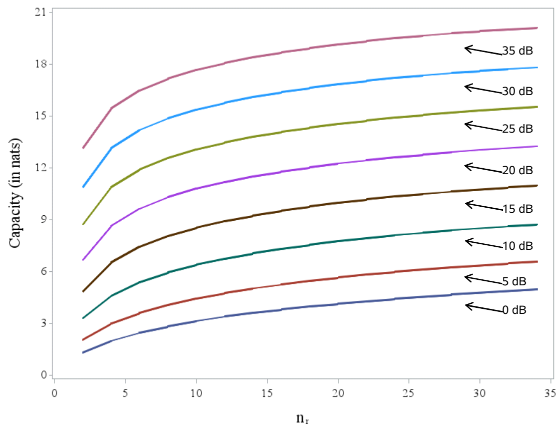

Figure 4.2 (18) against n_r for different values of ρ.

Figure 4.3 shows the calculated channel capacity (19) versus n_r for different values of ρ, assuming a correlation of 0, $\sigma^2 = 1$, and $\upsilon = 10$.

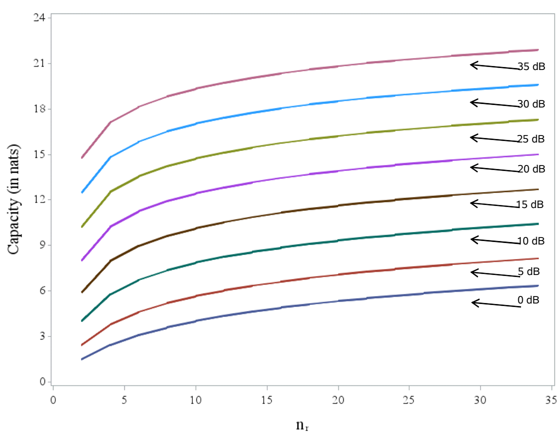

Figure 4.3 (19) against n_r for different values of ρ.

Table 4.1 shows the capacity in nats[7] for this $n_r \times 2$ correlated Rayleigh-type fading channel matrix (as illustrated in Figure 4.2). Table 4.2 shows the capacity in nats for this $n_r \times 2$ uncorrelated Rayleigh-type fading channel matrix (as illustrated in Figure 4.3). Each column represents different levels of SNR, in decibels (dB). Observe how the capacity is increasing in both Tables 4.1 and 4.2 with regards to increasing SNR, as well as increasing number of receivers n_r. Furthermore, note how the capacity for the uncorrelated case (Table 4.2) is higher for all corresponding entries than that of the correlated case (Table 4.1). The same is observed for other arbitrarily chosen υ.

n_r	0 dB	5 dB	10 dB	15 dB	20 dB	25 dB	30 dB	35 dB
2	1.2916	2.0609	3.3057	4.8558	6.6852	8.7821	10.9059	13.1656
4	1.9816	2.9984	4.5956	6.5129	8.6450	10.8836	13.1643	15.4598
6	2.4582	3.6126	5.3811	7.4294	9.6327	11.9010	14.1924	16.4914
8	2.8266	4.0737	5.9455	8.0592	10.2922	12.5715	14.8665	17.1666
10	3.1289	4.4445	6.3856	8.5381	10.7872	13.0721	15.368	17.6696
12	3.3862	4.7550	6.7460	8.9240	11.1831	13.4713	15.7691	18.0700
14	3.6105	5.0222	7.0506	9.2467	11.5125	13.8028	16.1012	18.4021
16	3.8095	5.2564	7.3141	9.5234	11.7939	14.0856	16.3842	18.6850
18	3.9882	5.4646	7.5456	9.7650	12.0988	14.3313	16.6298	18.9303
20	4.1502	5.6515	7.7414	9.9786	12.2547	14.5476	16.8458	19.1458

Table 4.1. Capacity (18) in nats for a $n_r \times 2$ system for different values of ρ and $\upsilon = 10$.

n_r	0 dB	5 dB	10 dB	15 dB	20 dB	25 dB	30 dB	35 dB
2	1.4843	2.4498	4.0298	5.9281	8.0291	10.2403	12.5045	14.7941
4	2.4402	3.7860	5.7830	7.9676	10.2292	12.5184	14.8167	17.1179
6	3.1083	4.6148	6.7334	8.9714	11.2528	13.5486	15.8490	18.1509
8	3.6156	5.2064	7.3788	9.6373	11.9256	14.2237	16.5284	18.8269
10	4.0228	5.6647	7.8668	10.1360	12.4279	14.7270	17.0285	19.3307
12	4.3622	6.0382	8.2591	10.5948	12.8287	15.1285	17.4302	19.7325
14	4.6583	6.3532	8.5869	10.8670	13.1623	15.4625	17.7643	20.0668
16	4.9069	6.6253	8.8684	11.1516	13.4479	14.7484	18.0503	20.3525
18	5.1324	6.8648	9.1149	11.4004	13.6974	15.9981	18.3000	20.6022
20	5.3350	7.0785	9.3342	11.6214	13.9189	16.2197	18.5215	20.8237

Table 4.2. Capacity (19) in nats for a $n_r \times 2$ system for different values of ρ and $\upsilon = 10$.

3.2 Singular case

For the *singular* case, the correlated- and uncorrelated Rayleigh-type $2 \times n_t$ channel matrix is considered, and its corresponding capacity derived.

Theorem 17 *1. For a two-input correlated Rayleigh-type channel,* $\mathbf{H} \sim \mathcal{C}E_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \mathbf{\Sigma}, g)$*, with* $n_t \geq 2$*, the capacity* C *is given by*

$$\begin{aligned} C &= K \int_0^\infty \int_0^{\lambda_1} \left\{ \log\left(1 + \frac{\rho}{n_t}\lambda_1\right) + \log\left(1 + \frac{\rho}{n_t}\lambda_2\right) \right\} (\lambda_1\lambda_2)^{n_t-2} (\lambda_1 - \lambda_2) \\ &\quad \times \int_{\mathbb{R}^+} t^{n_t+1} \det\left(\exp\left(-ta_i\lambda_j\right)\right) \mathcal{W}(t) dt d\lambda_2 d\lambda_1 \end{aligned} \tag{20}$$

[7] In (18) if $\log_e$ is used then the measurement unit for capacity is termed "nats".

where $K = \dfrac{\prod\limits_{i=1}^{n_t} a_i^2}{2\Gamma(n_t)\Gamma(n_t-1)\prod\limits_{k<l}^{n_t}(a_l-a_k)}$, *and* $a_1 > a_2 > ... > a_{n_t} > 0$ *are the eigenvalues of* $\mathbf{\Sigma}^{-1}$.

2. For a two-input uncorrelated Rayleigh-type channel, $\mathbf{H} \sim \mathcal{C}E_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \sigma^2 \mathbf{I}_{n_t}, g)$*, with* $n_t \geq 2$*, the capacity* C *is given by*

$$\begin{aligned} C &= \frac{1}{\sigma^{2n_t+2}\Gamma(n_t)} \int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right]\lambda_1^{n_t}\int_{\mathbb{R}^+} t^{n_t+1}\exp\left(-t\sigma^{-2}\lambda_1\right)\mathcal{W}(t)dtd\lambda_1 \\ &-\frac{2}{\sigma^{2n_t}\Gamma(n_t-1)}\int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right]\lambda_1^{n_t-1}\int_{\mathbb{R}^+} t^{n_t}\exp\left(-t\sigma^{-2}\lambda_1\right)\mathcal{W}(t)dtd\lambda_1 \\ &+\frac{\Gamma(n_t+1)}{\sigma^{2n_t-2}\Gamma(n_t)\Gamma(n_t-1)}\int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right]\lambda_1^{n_t-2}\int_{\mathbb{R}^+} t^{n_t-1}\exp\left(-t\sigma^{-2\cdot}\lambda_1\right)\mathcal{W}(t)dtd\lambda_1. \end{aligned} \tag{21}$$

Proof. 1. The unordered density function of (14) is obtained by dividing by $n! = n_r! = 2!$:

$$f(\lambda_1,\lambda_2) = \frac{\pi^{2(2-1)}\left(\prod\limits_{i=1}^{2}\lambda_i^{n_t-2}\right)\left(\prod\limits_{k<l}^{2}(\lambda_k-\lambda_l)^2\right)}{2\mathcal{C}\Gamma_2(2)\mathcal{C}\Gamma_2(n_t)|\mathbf{\Sigma}|^2}\int_{\mathbb{R}^+} t^{2n_t}\ {}_0\mathcal{C}F_0^{(2)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right)\mathcal{W}(t)dt.$$

In the same way as Theorem 16, integrating with respect to λ_2 and calculating the expectation of (15) leads to the final result.

2. The proof follows similarly where $\mathbf{\Sigma} = \sigma^2\mathbf{I}_2$. ■

Corollary 18 *1. For a two-input correlated Rayleigh-type channel,* $\mathbf{H} \sim \mathcal{C}t_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \mathbf{\Sigma}, \nu)$*, with* $n_t \geq 2$*, the capacity* C *is given by*

$$\begin{aligned} C &= K\frac{\upsilon^\upsilon}{\Gamma(\upsilon)}\int_0^\infty\int_0^{\lambda_1}\left\{\log\left(1+\frac{\rho}{n_t}\lambda_1\right)+\log\left(1+\frac{\rho}{n_t}\lambda_1\right)\right\}(\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2) \\ &\times\int_{\mathbb{R}^+} t^{n_t+\upsilon}e^{-t\upsilon}\det\left(\exp\left(-ta_i\lambda_j\right)\right)dtd\lambda_2 d\lambda_1 \end{aligned} \tag{22}$$

where $K = \dfrac{\prod\limits_{i=1}^{n_t} a_i^2}{2\Gamma(n_t)\Gamma(n_t-1)\prod\limits_{k<l}^{n_t}(a_l-a_k)}$, *and* $a_1 > a_2 > ... > a_{n_t} > 0$ *are the eigenvalues of* $\mathbf{\Sigma}^{-1}$.

2. For a two-input uncorrelated Rayleigh-type channel, $\mathbf{H} \sim \mathcal{C}t_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \sigma^2 \mathbf{I}_{n_t}, \upsilon)$*, with* $n_t \geq 2$*, the capacity* C *is given by*

$$\begin{aligned} C &= \frac{\upsilon^\upsilon\Gamma(n_t+\upsilon+1)}{\sigma^{2n_t+2}\Gamma(\upsilon)\Gamma(n_t)}\int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right]\lambda_1^{n_t}\left(\frac{\lambda_1}{\sigma^2}+\upsilon\right)^{-(n_t+\upsilon+1)}d\lambda_1 \\ &-\frac{2\upsilon^\upsilon\Gamma(n_t+\upsilon)}{\sigma^{2n_t}\Gamma(\upsilon)\Gamma(n_t-1)}\int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right]\lambda_1^{n_t-1}\left(\frac{\lambda_1}{\sigma^2}+\upsilon\right)^{-(n_t+\upsilon)}d\lambda_1 \\ &+\frac{\upsilon^\upsilon\Gamma(n_t+\upsilon-1)\Gamma(n_t+1)}{\sigma^{2n_t-2}\Gamma(\upsilon)\Gamma(n_t)\Gamma(n_t-1)}\int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right]\lambda_1^{n_t-2}\left(\frac{\lambda_1}{\sigma^2}+\upsilon\right)^{-(n_t+\upsilon-1)}d\lambda_1. \end{aligned} \tag{23}$$

Figure 4.4 shows the calculated channel capacity (22) (correlation 0.9) and (23) (no correlation) versus SNR (ρ) for $n_t = 4$ and $\upsilon = 10$. Figure 4.5 illustrates the higher capacity for the underlying complex matrix variate t distribution versus the complex matrix variate normal distribution for the correlated nonsingular case.

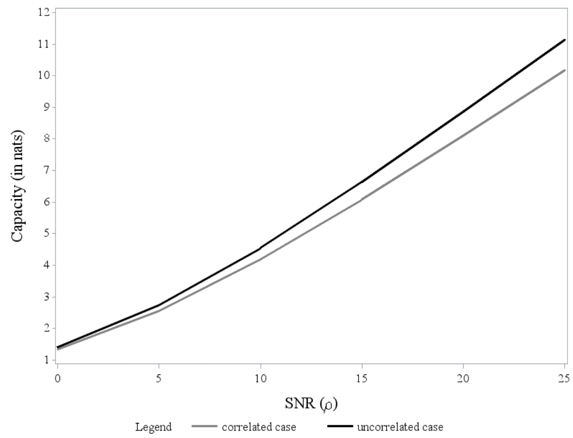

Figure 4.4 (22) and (23) against ρ, for $n_t = 4$.

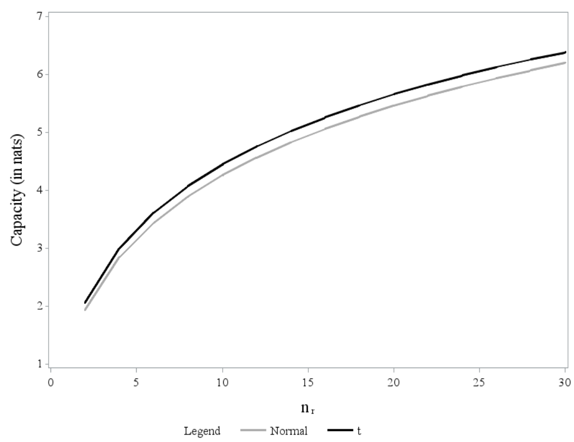

Figure 4.5 (18) and eq. (29) from Ratnarajah and Vaillancourt (2003) against n_r, for $\rho, \upsilon = 10$.

4 Concluding remarks

In this paper the distribution of the quadratic form and its associated joint eigenvalues with an underlying complex matrix variate elliptical model was derived. The proposed methodology is based on an integral representation that provides the researcher with expressions for allowing other underlying models than that of the normal, providing new insightful research possibilities. Some special cases were highlighted with the well-known Wishart distribution as a special case when the complex matrix variate normal distribution is under consideration. Another special case is that of no correlation; this case is of specific interest in the performance measure of channel capacity in the MIMO environment.

In particular the complex matrix variate t distribution was applied and the literature is enriched with its representation. The channel capacity within the MIMO environment is investigated for correlated and uncorrelated scenarios in the nonsingular and singular cases. It is observed that

1. Correlation between transmitters/receivers degrade system capacity; and
2. The capacity of the system is higher in the case of underlying complex matrix variate complex t distribution than that compared to an underlying complex matrix variate normal distribution.

When no correlation exists between receivers, the well-known central limit theorem can be assumed which results in $\mathbf{H} \sim \mathcal{C}N_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma})$. However, this paper provides new possibilities in the wireless communications systems environment with the elliptical platform. In particular, the complex matrix variate t distribution is considered (as the t is a familiar candidate when placed alongside the normal). These numerical examples (see Figure 4.5) of the channel capacity show that the derived expressions under the complex matrix variate t distribution provide significant insights on the behaviour of performance measures when the assumption of the complex matrix variate normal distribution is challenged.

If the receivers and transmitters are correlated simultaneously, i.e. $\mathbf{H} \sim \mathcal{C}N_{n_r \times n_t}(\mathbf{0}, \mathbf{\Phi}_{n_r} \otimes \mathbf{\Sigma}_{n_t})$, then the well-known central limit theorem does not apply. In that case the complex matrix variate elliptical distribution may provide greater flexibility in this regard. Although the results in this paper are presented for the $\mathbf{I}_{n_r} \otimes \mathbf{\Sigma}$ and related cases, in the case of $\mathbf{\Phi}_{n_r} \otimes \mathbf{\Sigma}_{n_t}$ the covariance structure can be adapted to $\mathbf{I}_{n_r} \otimes \mathbf{\Sigma}$ via a transformation.

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