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A volume-weighted-average-price (VWAP) method for estimating beta in the context of reference-day risk

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Title

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Abstract

The ability to accurately estimate systematic risk (or beta) in the presence of reference-day risk is an ineluctable requirement for all applications of the capital asset pricing model (CAPM).

This research documents evidence of reference-day risk for shares on the Johannesburg All Share Index. In response to the need for greater accuracy when estimating systematic risk, this paper contributes a volume-weighted-average-price (VWAP) method for estimating beta when reference-day risk is exhibited in share betas.

Furthermore, this research applies a graphical time-series approach to test the underlying risk-reward tenet postulated by the CAPM. Using beta as a measure of systematic risk, this research finds that the CAPM appears to imperfectly specify the risk-reward trade-off.

Keywords: Beta, reference-day risk, VWAP

Introduction

The risk-reward tradeoff has been a prominent focus for empirical finance literature and investment practitioners since the 1950s. The parsimony of this risk-reward relationship is intuitively captured by the capital asset pricing model (CAPM), jointly ascribed to Markowitz (1952; 1956; 1959), Treynor (1961), Sharpe (1964), Lintner (1965) and Mossin (1966) for their respective contributions to its development.

The CAPM describes a simple linear model for estimating the expected return on an asset in terms of its systematic risk, and by doing so, provides a formal relationship between risk and return (Ward & Muller, 2012; Fama & French, 2004). The CAPM can be expressed as:

$$E(R_i) = R_f + \beta_i (E(R_m) - R_f)$$

(Equation 1)

Where;

E(R_i) is the expected return on the share (asset),

R_f is the risk-free rate of interest, such as interest arising from government bonds,

E(R_m) is the expected return of the market, and

β_i is the “beta” (a term originally coined by Sharpe (1964, p. 440)) and represents the systematic risk component of an asset, as propagated by the CAPM. In essence, beta describes the sensitivity of the expected excess returns of an asset to the expected excess returns earned by the market, and is expressed as;

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

(Equation 2)

where $\text{Cov}(R_i, R_m)$ is the covariance between the return on asset i and the return on the market portfolio, using historical data (Heymans & Brewer, 2015; Chen & Reeves, 2012). Simply stated, the CAPM emphasizes that the risk of an asset is a function of its beta (Gonzalez, Rodriguez & Stein, 2014). Along with all applications of the CAPM, beta is used extensively for portfolio construction, capital budgeting, investment performance evaluation, risk management and business valuation (Gonzalez *et al*, 2014). Accurate beta measurement is therefore not only desirable, but necessary for prudent financial management.

Since its development, the CAPM in its simplicity and intuition, was not earnestly questioned until the publication of a controversial paper by Fama and French (1992), who suggested that if assets are priced rationally, then share risks are not one-dimensional (i.e. exclusively a function of beta) but rather, multi-dimensional. This assertion drew considerable criticism given the extensive use of beta and the CAPM in academia and practice. Moreover, it spurred significant research aimed at testing the validity and robustness of beta.

Despite the theoretical and practical appeal of the CAPM, the empirical evidence in support of the model is at best, weak (Ward & Muller, 2012). Most of this criticism has emanated from investigations focused on beta. Studies conducted by Brooks, Faff, Gangemi and Lee (1997), Faff and Brooks (1996), Fabozzi and Francis (1978) and Collins, Ledolter and Rayburn (1987) have analysed a variety of financial markets and found evidence of beta instability in individual shares, a finding which has profound ramifications for portfolio construction. Furthermore, Novak (2015) suggests that the traditional method for estimating beta has been criticized for low explanatory power in explaining share returns. Along with

the estimation criticism, Fama and French (1992; 2004), Montier (2009), van Rensburg and Robertson (2003), Strugnell, Gilbert and Kruger (2011) and Ward and Muller (2012), suggest that there is essentially little to no evidence of a positive, linear relationship between systematic risk and returns. In contrast, research has concluded that there is an inverse relationship between systematic risk and return when one approximates systematic risk using beta, a finding which is unequivocally contradictory to the underlying tenet of the CAPM (Ward and Muller, 2012; Montier, 2009). According to Novak (2015), the wave of literature documenting the low explanatory power of beta and the inverse relationship previously documented by some authors has led some to proclaim beta “dead” (p. 168). One of the foremost criticisms against beta is the concept of reference-day risk.

In a topical paper, Acker and Duck (2007) introduced the concept of “reference-day risk” (p. 2) to refer to large sampling variations and estimation risks that can be detected in share returns, variances and share betas, and which is simply attributable to the choice of an initial reference day when calculating monthly share returns. In summary, reference-day risk creates additional uncertainty for investors and finance practitioners and leads to inadvertent or unwarranted outcomes based on beta estimates. As a result, reference-day risk has become one of the most preeminent research areas to gain momentum in recent years, as it has significant implications for accurate beta estimation. Investigations conducted by Baker, Rajaratnam and Flint (2016); Gonzalez *et al* (2014); Acker and Duck (2007) and Dimitrov and Govindaraj (2007) have all evidenced that by using any five-year sample, and selecting one day of the first month as the reference day to construct a series of monthly

returns, different choices for the reference day produce large variations in estimated betas.

The findings from the original investigation by Acker and Duck (2007) have prompted significant research into developing and testing alternative, independent-reference-day methods for estimating beta. Among the beta estimation methods tested, were the Blume (1971) regression method, the Dimson (1979) adjustment for thin trading, the Vasicek (1973) Bayesian beta estimate and the *t*-distribution method for adjusting beta as developed by Cademartori, Romo, Campos and Galea (2003), with only the latter method having any promising result for accurately estimating systematic risk in the presence of reference-day risk.

This paper aims to resurrect the systematic risk parameter, beta, by contributing a volume-weighted-average-price (VWAP) estimate for beta which can be employed in the context of reference day risk. To achieve this, this research first evidences the degree to which reference-day risk leads to variations in beta for shares on the JSE ALSI. Thereafter, estimates of beta using the VWAP methodology are statistically tested to determine the robustness and applicability of applying the VWAP method as a variation to the standard method of estimating beta.

Lastly, this research transcends prior research investigations of a similar nature by determining whether the reference-day-independent beta adheres to the underlying tenets of the CAPM. This is achieved by employing a graphical time-series analysis to observe whether VWAP betas are a more reliable representation of systematic risk in the risk-return tradeoff.

Consequently, an appropriate methodology is derived to test each of these objectives.

This paper is therefore organized as follows. The following section reviews the literature on the subject. Thereafter, the sample, methodology and results are detailed and discussed. The last section offers concluding comments and guidelines for future research.

Literature review

Reference-day risk and the mitigation of it thereof, is one of the most contemporary issues contributing to financial literature intended for accurate beta measurement. The results of the original investigation by Acker and Duck (2007) have prompted significant research into developing and testing alternative, independent-reference-day methods for estimating beta. This section presents some of the empirical literature describing reference-day risk and the variations that different choices of reference day have on beta estimates.

Estimating beta in the context of reference-day risk

Acker and Duck (2007) demonstrate the existence of reference-day risk associated with monthly returns implicit in shares listed on the S&P500 Index and set out to explore the level of reference-day risk implicit in estimates of beta.

Using data from a sample of 459 companies sourced through Datastream, Acker and Duck (2007) examined data for a fifteen-year period, across three five-year sub-intervals. The results from the investigation prove that estimates of beta are highly sensitive to the choice of reference day. In extreme cases, the choice of reference-day can significantly amplify the estimate of a share's beta as well as alter the polarity of beta estimates. Drawing on the sample of listed companies on the S&P500, the authors present evidence of a share's beta falling by 0.931 and rising by 3.454

depending on the choice of reference-day. Perhaps more pertinent to this investigation however, is the finding that approximately 75% of all observed betas estimated using the conventional methodology could be classified as either positive or negative by an appropriate choice of reference day. This finding has profound ramifications for both academia and industry, as it renders risk-adjusted asset selection using conventional beta estimates impractical, inconsequential and insufficient for portfolio construction.

Acker and Duck (2007) thereafter commence with investigating whether the Blume (1971) regression method, the Vasicek (1973) Bayesian beta and the Dimson (1979) adjustment for thin trading reduces the variation and range of beta estimates for which the unadjusted beta is highly sensitive to the choice of reference day.

The authors report that the Blume (1971) regression method reduces only the most severe reference-day variability. Similarly, the Vasicek (1973) Bayesian estimation method also reduces the range and variation of beta estimates for shares which have high sensitivity to the selection of a reference day (Acker & Duck, 2007). However, both estimation methods exhibit sensitivity to the choice of reference day (although this variation is less marked for estimates of Blume (1971) betas). The findings however purport that estimated betas using both the Blume (1971) and Vasicek (1973) adjustments are indeed subject to reference-day risk, causing shares to be incorrectly classified as either aggressive or defensive based on the selection of an initial reference day (Acker & Duck, 2007).

Lastly, even though the investigation evidenced the presence of reference-day risk for both individual shares and equity indices, Acker and Duck (2007) hypothesize that the effect of reference-day risk may be amplified

by thin trading. Accordingly, the authors apply the Dimson (1979) adjustment method for thin trading in estimating betas in the context of reference-day risk. The results of the investigation suggest that the Dimson (1979) adjustment method only slightly reduces the variability of beta estimates for the highest ranges. Furthermore, the authors observe a tendency for estimated betas to exhibit significant variation based on the choice of reference day, indicating that the Dimson (1979) adjustment does not yield any significant difference in estimating systematic risk in the presence of reference-day risk.

Dimitrov and Govindaraj (2007) set out to investigate the findings from Acker and Duck's (2007) study from a data dependency perspective. The authors argue that Acker and Duck's (2007) sample (which was drawn from Datastream) may not be applicable for the United States context and propose replicating the analysis using data sourced from the Centre for Research in Security Prices (CRSP). Furthermore, Dimitrov and Govindaraj (2007) expand the study by Acker and Duck (2007) by investigating the existence of reference-day risk using daily returns (as opposed to monthly returns) over a five-year period.

Using data sourced from the CRSP from the period January 1995 to December 1999, Dimitrov and Govindaraj (2007) use daily dividend-adjusted share returns to construct a series of sixty monthly returns across nineteen different reference days, for each of 439 sample companies. Congruous to the findings of Acker and Duck (2007), the investigation verifies the existence of reference-day risk in companies listed on the S&P500, using data sourced from the CRSP (Dimitrov & Govindaraj, 2007). Moreover, Dimitrov and Govindaraj (2007) also report that betas estimated from the sample exhibit significant variation when different

choices of the initial reference day are used in the computation. Furthermore, the findings from the investigation suggest that reference day risk is implicit for both individual shares and market indices.

Gonzalez *et al* (2014) explicate the research conducted by both Acker and Duck (2007) and Dimitrov and Govindaraj (2007) by expanding the data dependency and methodology parameters employed in the previous studies. Gonzalez *et al* (2014) compare the *t*-distribution method for adjusting beta as developed by Cademartori *et al* (2003) to the OLS method and Blume's (1971) regression method for estimating beta.

According to Gonzalez *et al* (2014), the *t*-distribution method proposes a replacement of the standard normal distribution with the Student's *t*, which is a symmetric distribution with heavier tails than the normal distribution. Gonzalez *et al* (2014) further purport that the heavier tails in the Student's *t* is more appropriate in estimating beta in the context of reference-day risk as it more aptly compensates for the error term in the linear regression used to estimate share betas. Furthermore, Cademartori *et al* (2003) prove that the *t*-distribution method for adjusting beta is better able to incorporate the influence of outliers in estimating beta. Lastly, Blume's (1971) method was selected by Gonzalez *et al* (2014) for comparison to the *t*-distribution method, as it was the method which reduced the effect of reference-day risk the most in Acker and Duck's (2007) investigation.

Gonzalez *et al* (2014) select a sample of 1563 shares, traded on the NYSE, AMEX and NASDAQ exchanges obtained from the CRSP, for the period 2007 to 2011 and report that the choice of reference day results in significant variations in estimated betas. Moreover, Gonzalez *et al* (2014) report that betas estimated using Blume's (1971) regression method also exhibited significant variations across different reference days. Using the

t -distribution method however, the authors record that the choice of reference day becomes less significant with the larger recorded ranges decreasing significantly for the observed betas. The investigation concludes that the t -distribution method for adjusting beta most significantly reduces reference-day variation when compared to Blume's (1971) method and the OLS method.

More recently, Baker *et al* (2016) set out to establish the existence of reference-day risk in the JSE Top 40. Baker *et al* (2016) use daily closing levels of the ALSI and closing prices for shares making up the JSE Top 40 index for the period January 2000 to July 2015, sourced from Datastream.

Congruous to the findings of Acker and Duck (2007), Dimitrov and Govindaraj (2007) and Gonzalez *et al* (2014), the authors conclude that reference-day risk exists and creates additional uncertainty for investors intending to create share portfolios, valuing companies or managing capital (Baker *et al*, 2016). Furthermore, the authors test the Blume (1971), Dimson (1979) and Vasicek (1973) Bayesian methods for adjusting beta to investigate whether the adjusted betas exhibit lower reference-day ranges than unadjusted betas (Baker *et al*, 2016).

In estimating the betas calculated using Blume's (1971) regression method, Baker *et al* (2016) report that the Blume-adjusted betas increase the variation and range of beta estimates for 19 out of 31 companies and conclude that the Blume (1971) regression method for estimating beta does not consistently and considerably reduce the variation and range for betas in the context of reference day risk. Analogous to the results of the Blume (1971) regression method, the average range for the Dimson-adjusted betas was significantly larger than that of the standard betas. Moreover, the range for beta estimates using the Dimson (1979)

adjustment increased for 30 out of the 40 companies sampled, leading the authors to conclude that the Dimson (1979) adjustment for thin trading actually pronounces the reference day variation in systematic risk estimates. Lastly, the authors report that the Vasicek (1973) Bayesian adjustment method was the only estimation method yielding any promising results, with 35 out of the JSE Top 40 companies exhibiting lower reference-day ranges than unadjusted betas.

Baker *et al* (2016) then introduce a nonparametric bootstrap method to determine potential beta estimates which are independent of reference-day risk. Using this method, the authors note that for shares on the JSE Top 40, the expected value of a reference-day independent beta was approximately equal to the average of the 20 unadjusted betas estimated for each day. Baker *et al* (2016) propose that the mean value of the bootstrapped beta distribution therefore provides a reference-day independent estimate of systematic risk for a particular share, even though further investigation is required to recover the mean value of a set of beta standard deviations. Even though the results of the investigation do not yield the desired result, the authors do note some benefits of employing this method in estimating beta (Baker *et al*, 2016).

In summary, the literature discussed above has not revealed a methodology which is independent of the reference-day problem in beta estimation. The final component of this theoretical discussion now focuses on the relevant literature regarding the proposed VWAP methodology.

Empirical evidence of VWAP as a method to eliminate the reference-day problem

There is little to no literature available on estimating beta through employing a VWAP adjustment or VWAP methodology in the context of reference-

day risk. However, perhaps the most pertinent literature is presented by Ting (2006; 2005) who suggests that a VWAP is closer to the equilibrium price of a share than the daily closing price. Ting (2006; 2005) bases this on the premise that VWAP considers all the intra-day prices at which transactions have occurred and evidences that daily returns computed with VWAP have a smaller realized variance than that with the closing price. Moreover, Ting (2006) concludes that the variance spread between VWAP and the closing price is economically significant and has implications for performance measurement and pricing of derivatives.

More critical to this research however, Ting (2006) provides an example which evidences that, relative to the volatility of VWAP returns, the volatility of closing price returns tends to understate the beta risk estimation result. By consequence, the research suggests that by using VWAP along with the closing price, estimation of financial risk and asset pricing can be performed with considerably less noise (Ting, 2006).

The literature presented above has verified the existence of reference-day risk on the S&P500 as well as for the JSE Top 40 index, even after applying various adjustments to the estimation method for beta. Analogous to the South African studies, the question of whether reference-day risk can be observed on the JSE ALSI beckons. Moreover, where estimation methodologies indicate an improvement in the robustness of beta estimates, none of the estimated betas were empirically tested to verify whether beta estimates adhered to the tenets of the CAPM. The empirical evidence presented in the literature also evidences the critical need for a methodology which produces a stable estimate for beta which is independent of the reference day.

This research contributes to this body of literature and proposes a methodology for estimating beta which is intuitively simple, yet differs markedly from the empirical methods tested previously. Additionally, this research aims to redeem the underlying tenet of the CAPM, in attempting to explain a positive risk-reward relationship between the systematic risk parameter, beta, and expected returns.

The sample

This research uses the daily closing level of the JSE ALSI and closing prices of each of the qualifying shares making up the index between 31 December 1992 and 30 June 2017, sourced from Datastream and Ward and Muller's (2015) style engine. The choice of the JSE ALSI as the selected sample, ensured that 99% of South Africa's market capitalization was accounted for in this research (JSE, 2017). Given the longitudinal time-based nature of the research, the qualification of companies to the population applied to any firm which at any stage between 1985 and 2016 had sufficient market capitalisation, irrespective of their eventual state. This eliminated the potential for survivorship bias.

Consistent with the studies by Baker *et al* (2016), Gonzalez *et al* (2014), and Carter, Muller and Ward (2017), all share prices included adjustments for any unbundling, mergers, share splits and dividend pay-outs. For all comparisons of standard betas against the estimated VWAP betas, all qualifying companies on the JSE ALSI during the period 3 January 2012 until 31 January 2017 with at least five years historical share price data were considered. This resulted in a total of 136 shares under observation.

Methodology and results

Investigating the existence of reference-day risk

Every month within the sample period has approximately 20 trading days, as the trading of listed shares on the JSE typically excludes weekends and South African public holidays. The daily closing share prices and JSE ALSI levels can therefore be organized per 20 trading days, with the first trading day corresponding to the first working day of the initial month, in a five-year time series. Monthly log returns for each of the qualifying shares on the JSE ALSI and the index are thereafter calculated for the period 3 January 2012 to 31 January 2017. This resulted in a series of monthly returns (one series for each of the 20 trading days) for each of the 136 sampled shares and the JSE ALSI.

Using this data, 20 different estimates of beta were generated for each share, by approximating the slope of the regression between an individual share's returns and the JSE ALSI returns (or equivalently, by using *Equation 2*).

Consistent with the findings of Baker *et al* (2016) and Gonzalez *et al* (2014), first inspection of the unadjusted betas reveals that there is indeed an effect on beta when the reference day is varied. Using the range of beta across the 20 trading days, reference-day risk is most pronounced in Lonmin (LON), Kumba Iron Ore (KIO) and Royal Bafokeng Platinum (RBP). Similarly, there are also companies for which share betas are relatively constant across the 20 trading days, which includes Remgro (REM), Reboasis Property Fund (REB) and British American Tobacco (BTI).

Alarmingly, 15 out of the 136 sampled companies also exhibited standard betas which could be classified as either positive or negative by an appropriate choice of the reference day. An equally noteworthy observation is that nine out of ten companies with the highest ranges in betas across the 20 trading days were resources companies. This finding potentially indicates a systemic characteristic in the way resources shares exhibit reference-day risk, or is merely a function of the JSE ALSI being more heavily weighted toward resources shares as compared to other stock exchanges.

INSERT TABLE 1 ABOUT HERE.

As suggested by Acker and Duck (2007), Gonzalez *et al* (2014) and Baker *et al* (2016), the differences in betas have profound implications on all applications of beta. One could consider beta-style portfolio construction as such a case.

Consider the investor with a risk-averse profile intending to construct a portfolio of mainly defensive shares. Such a portfolio will have the characteristics of being weakly correlated with the market and having a positive portfolio beta, which is low in value and less than 1. If such an investor were to estimate the betas for KIO or RBP on a trading day which yields a relatively low beta, such a portfolio may have a positive beta ranging between 0.488 (if the portfolio is constructed entirely of KIO shares) or 0.474 (if the portfolio is constructed entirely of RBP shares) – varying according to the weightings of each of the shares held. Applying beta to the CAPM, one could therefore expect returns from such a portfolio

which have a low correlation with the market. This implies that the investor would realise smaller positive returns when the market moves up, but be shielded from all downward movements in the market.

However, the investor has inadvertently constructed a portfolio with a worst-case scenario beta ranging between 1.965 and 3.010, being the largest betas for RBP and KIO, respectively. Such a portfolio would exhibit extreme fluctuations in value when the market moves in either direction. The systematic risk of the investor's portfolio has been severely misestimated due to reference-day risk. Baker *et al* (2016) further indicate that in such a case, an investor may be leisurely to react to a sharp decline in the market, believing his portfolio is weakly correlated with the market.

Equivalent results would be reached for an investor looking to create a portfolio of shares which are negatively correlated with the market. Such an investor may achieve this by including shares with a negative beta in a portfolio. However, the choice of reference day may once again, severely undermine the investor's ability to construct such a portfolio, as some shares yield estimates of beta which can be classified as either positive or negative based on the choice of the reference day (E.g. Brimstone Investment Corporation (BRN)). In similar fashion, managing reference-day risk is made even more complex when one is looking to construct a diversified portfolio of shares or when used in the construction of market-neutral hedge funds. In such cases, variations in beta may yield unintended and potentially severe consequences for the performance of the fund.

To statistically determine the degree to which varying the reference day leads to variations in estimated betas, an ANOVA was performed for the

20 trading days. Even though the preliminary analysis of estimated betas across the 20 trading days suggests that there is indeed a trading day effect when estimating share betas, at both the 5% and 10% level of significance, the ANOVA reveals insufficient evidence to verify a definitive trading day effect of beta, when the reference day is varied (p -value = 0.501). The result is indeed surprising when one considers the large variation in estimated betas exhibited by some shares, suggesting that perhaps the covariance between shares within individual trading days may cancel each other out. Not surprisingly however, a similar result was also reported by Baker *et al* (2016), who could not statistically verify a trading day effect on beta across their 20 estimates of share betas for the JSE Top 40. Different results are however reached when comparing the minimum and maximum estimated betas for each share.

Each share has a trading day corresponding to its highest beta estimate and equivalently, a trading day which yields its lowest beta estimate. Using a (two-tailed) t-test to compare the means of these beta values across firms indicated that the highest and lowest beta values are significantly different at the 5% level ($p < 0.0001$). This result indicates that for some part, reference-day risk does indeed create additional uncertainty when estimating share betas. Baker *et al* (2016) also tested this hypothesis and noted a significant difference in minimum and maximum betas for 22 out of the 40 sampled shares. The variation in beta estimates across reference days implies that any application of share betas (and consequently, the CAPM) may severely undermine one's understanding of risk. It may further lead to inefficient share selection when creating portfolios and ultimately, increase the likelihood of sub-optimal investment decisions, especially when valuing companies or constructing market-neutral hedge funds.

Developing a point estimate of systematic risk for application when reference-day risk is observed

The primary aim of this research was to develop a more robust point estimate of beta when reference-day risk is exhibited in estimates of systematic risk. Baker *et al* (2016) presented the potential for using an average of the betas across the 20 trading days as an estimate of systematic risk for a share, but note the potential introduction of errors due to small sample size. In this subsection, the research investigated the feasibility of using a VWAP method for estimating beta when reference-day risk is observed.

Using the total daily value of shares traded together with the number of shares traded for each share on the JSE ALSI, a daily 60-day ex-ante VWAP was estimated for the sample, as per the computation prescribed by Ting (2006);

$$VWAP = \frac{\sum_{k=1}^n P_k V_k}{V} = \sum_{k=1}^n W_k P_k$$

(Equation 3)

Where;

W_k is the weight and calculated as V_k / V ,

P_k are the n intraday prices at which transactions have occurred during the period,

V is the total share volume traded over the period and is equal to $\sum_{k=1}^n V_k$,

V_k are the subtotals of all shares transacted at the price P_k (Ting, 2006).

The equation simply states that the VWAP is a combination of all intraday prices, which for the case of this research, occurred over the past 60 days

for a share, inclusive of the reference-day. For the JSE ALSI however, the VWAP is a function of the daily movement in share prices for the qualifying shares on the index. Accordingly, the VWAP of the index is weighted on a per share basis, and changes daily with movements in the market capitalisation of firms.

Having computed the daily VWAP for all shares and the JSE ALSI, the log returns for each share and the ALSI were computed over a five-year period, using the 60-day ex-ante VWAP as a substitute for the monthly closing price. VWAP betas were then estimated for each share by approximating the slope of a linear regression of individual share returns on market returns for the period 30 March 2012 to 31 March 2017.

At this point, Gonzalez *et al* (2014) tested the significance of the differences between standard betas against Blume (1971) betas and betas approximated using a student's *t*-distribution method. The authors focused on comparing mean share beta variances for each of the three methods, across 20 trading days. Baker *et al* (2016) employed a different approach and compared adjusted regression coefficients using the Blume (1971), Dimson (1979) and Vasicek (1973) Bayesian adjustment methods.

This research employed a unique approach in estimating beta which was simple and intuitive, yet distinctly different from the previous investigations on reference-day risk. The conventional method for estimating beta was maintained, however monthly closing prices were substituted with the 60-day ex-ante VWAP. This was done prior to regressing individual share returns against the market returns. Reiterating Ting (2006; 2005), the VWAP is statistically more efficient than the closing price in reflecting value as it is closer to the unobservable equilibrium price, resulting in a smaller realized variance, which is the essence of the beta measure.

To understand whether the VWAP beta is a more robust estimate of systematic risk when reference-day risk is observed, VWAP betas were compared to standard betas. To achieve this, both the standard betas and VWAP betas were estimated using the same sample period and tested for statistical difference. This test is critical to the investigation as estimates of VWAP betas employ the same computational methodology as the standard beta. Even though the estimation methods are different, the two measures must produce statistically independent estimates of systematic risk. A positive result is therefore fundamental in proposing VWAP as an alternate estimation method in the context of reference-day risk. A paired t-test at the 5% level of significance confirms that the two measures are statistically different.

Application of the VWAP beta and understanding whether VWAP betas perform better under conditions of reference-day risk

The first step in this analysis was to understand how estimates of VWAP betas for the 136 sampled shares were distributed, and whether tighter estimates of share betas were possible using the conventional method. Furthermore, the research also aimed to understand if there exists a trading day within the 20-day range for which betas tend to be more robust. A visual and statistical comparison was conducted, in analogous fashion to Baker *et al* (2016), when attempting to understand whether systematic risk estimates using the bootstrapped beta method were more robust for shares on the JSE Top 40 index.

VWAP betas for the sample shares were plotted on a histogram to understand the sample characteristics of estimated share betas, particularly focusing on the skewness and tightness of the distribution. Similarly, beta estimates for the sample shares were thereafter plotted

using the median beta for each share from the 20 trading days. This process was then repeated for all share betas estimated using the 20th trading day (as this day yielded the tightest distribution in estimated betas). For control, standard betas were also estimated for the sample and equivalently plotted. The resultant distributions were then visually compared. Figure 1 (below) plots each of the distributions for estimates of share betas for the same sample. A positive result would be a normally distributed set of betas, tightly distributed around the mean value, 1 (which represents the market beta).

INSERT FIGURE 1 ABOUT HERE.

To test the various distributions statistically, the Kolmogorov-Smirnov, Cramer-Von Mises and Anderson-Darling were performed to understand whether the estimated betas were normally distributed. Furthermore, the variance for each of the distributions was analysed to determine which method yielded the tightest share beta estimates around the mean, congruent to the method employed by Gonzalez *et al* (2014). Thereafter, a Levene's test was conducted to statistically verify that the variances were distinctly different across the alternate measures, consequently verifying the visual result.

As evident in Figure 1, betas estimated using the 20th trading day of the month produced the most robust estimates in share betas for the sample period, even more so than the standard betas which use month end share values. Counter-intuitively, the VWAP method produces a less robust set

of share betas for the sample. This was also verified statistically as described above.

Following this result, there may yet be application for the VWAP betas. As per the method conducted by Baker *et al* (2016), beta estimates using the VWAP method were also tested on a per share basis. This is achieved by comparing the distribution of standard beta across the 20 trading days against estimates of beta using the VWAP method. To ensure a robust test of the VWAP beta method on a share by share basis, additional VWAP betas were generated from a non-overlapping, out-of-sample period. As conducted prior, differences were tested both visually and statistically.

The results indicate that for most shares, the distributions of VWAP betas are not statistically different from the standard betas. However, upon closer inspection, some positive results were observed when analysing the shares with the largest distributions in standard betas. For the top three shares with the largest ranges in standard beta across the 20 trading days (LON, KIO and RBP), KIO and RBP had statistically different distributions ($p = 0.004$ and $p = 0.002$, respectively) for which the VWAP estimates were significantly tighter around the mean. This is illustrated in Figure 2. For LON, this was not the case. Further inspection into LON's share price history does however reveal that the share did experience significant volatility during the sample period, potentially influencing estimates of share betas for the company.

INSERT FIGURE 2 ABOUT HERE.

As expected for the three shares with the smallest range in estimated betas (REM, REB and BTI), there was little evidence of a VWAP method improving the range of estimates.

This result suggests that, for cases of extreme reference-day variation in estimated betas, the VWAP beta estimation method does indicate some evidence of a more robust estimate of systematic risk. The VWAP method for estimating beta could consequently be applied alongside other measures of systematic risk. This is especially the case when estimating betas for shares which exhibit a large degree of reference-day variation. Furthermore, future research could contemplate the appropriateness of the VWAP beta, when estimating systematic risk for resources shares, as these were evidenced to have the highest effect when the reference day was varied.

Understanding whether the VWAP betas adhere to the tenets of the CAPM

To consolidate the findings between the standard beta and VWAP beta, the VWAP beta must adhere to the underlying tenet of the CAPM, which states that higher betas would lead to higher returns, and vice versa. This was investigated by means of a graphical time series style analysis, congruent to the methodology employed by Muller and Ward (2013) and Ward and Muller (2012).

To conduct this, each share in the sample was ranked according to the magnitude of the 60-day ex-ante VWAP betas and placed into virtual portfolios in the form of quintiles. Portfolio 1 contained the shares with the highest VWAP betas whereas Portfolio 5 contains the shares with the lowest VWAP betas. The performance of each of the portfolios was thereafter simulated using the Ward and Muller (2015) style engine, which

contained share returns data over the period 1985 to 2017. The simulation was then run over the sample period.

To ensure a robust analysis, each of the VWAP betas for shares within the ALSI are recalculated and ranked every three months. This was done in a similar fashion to that of Carter *et al* (2017) and ensured that the correct shares were placed in the appropriate portfolios according to their updated betas. The portfolios are rebalanced quarterly on this basis.

Consistent with Ward and Muller (2012), the graphical time series analysis also plots a price relative line to indicate the relationship between quintile 1 (which had shares with the highest VWAP betas) and quintile 5 (which had shares with the lowest VWAP betas). The graphical time series analysis also plots a price relative line between quintile 1 and the market. This acts as a control variable would in the investigation, and is used to test whether high VWAP beta shares lead to abnormal returns (i.e. returns more than what the market could provide), as specified by the CAPM. The performance of the portfolios as well as the relationship between each portfolio and the market was then visually analysed.

INSERT FIGURE 3 ABOUT HERE.

The analysis indicates that the quintiles ranked from highest to lowest, according to the simulated annualized returns, were quintile 4, quintile 3, quintile 5, quintile 2 and lastly, quintile 1. The price relative further indicates that since December 1999, the quintile with the lowest VWAP betas (quintile 5) consistently outperformed quintile 1, which contained the highest ranked VWAP betas. The data suggests that, even though the

VWAP betas are statistically different from the standard betas, the VWAP betas do not adhere to the tenet of the CAPM that a higher level of risk (as estimated by beta) will lead to higher returns, and vice versa. Counter intuitively, the analysis has suggested that shares with a lower VWAP beta tend to outperform shares with high VWAP betas. This is consistent with the findings of Ward and Muller (2012), who found that estimates of standard betas also invert the CAPM.

Conclusion

This paper set out to resurrect the systematic risk parameter, beta, by contributing a volume-weighted-average-price (VWAP) point estimate for beta when reference-day risk is observed. This investigation evidenced a significant degree of reference-day risk when estimating share betas on the JSE ALSI, congruous to the findings of Baker *et al* (2016), Gonzalez *et al* (2014), Dimitrov and Govindaraj (2007) and Acker and Duck (2007). As a result, this creates additional uncertainty for investors and practitioners applying beta in portfolio construction, risk management, business valuation and all applications of the CAPM. Moreover, this finding underpins the need for a more accurate estimate of systematic risk, when reference-day risk is observed.

VWAP betas were also found to be statistically different from standard betas. Despite the indication from Ting (2006; 2005), there is however limited evidence to suggest that a VWAP method improves estimates of systematic risk for shares. Even so, the investigation recommends that for cases of extreme reference-day variations, the VWAP beta may be applied alongside other measures to improve financial decision making.

Lastly, this research contributes further insight into the risk-relationship trade-off postulated by the CAPM. A graphical time-series style analysis

revealed that estimates of beta employing a VWAP methodology still inverts the expected returns. In addition to the empirical findings documented by other researchers, this investigation suggests that the CAPM appears to imperfectly specify the risk-reward trade-off (Montier, 2009; van Rensburg & Robertson, 2003; Strugnell *et al*, 2011; Ward & Muller, 2012). Further research is however required to verify this.

For further research, the investigation evidenced a significant degree of reference day risk when estimating share betas for resources shares. Appreciating that the composition of the JSE ALSI is weighted heavily towards resources shares, further research may potentially uncover additional characteristics in the way resources shares exhibit reference-day risk.

To end, this research also evidenced that for shares on the JSE ALSI, VWAP returns produced more robust covariance matrices for shares. Financial practitioners may find this result useful for many financial applications.

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Tables and charts

Table 1: Smallest and largest ranges in standard betas

	Smallest ranges			Largest ranges		
	REM	REB	BTI	LON	KIO	RBP
Max	1.173	0.618	0.589	5.148	3.010	1.965
Min	0.896	0.323	0.288	2.259	0.488	0.474
Range	0.277	0.295	0.301	2.890	2.521	1.491
Mean	1.049	0.482	0.463	3.486	1.776	0.985
Variance	0.005	0.006	0.010	0.571	0.545	0.209
Median	1.052	0.471	0.492	3.458	1.776	0.779

Figure 1: Distribution of estimated betas

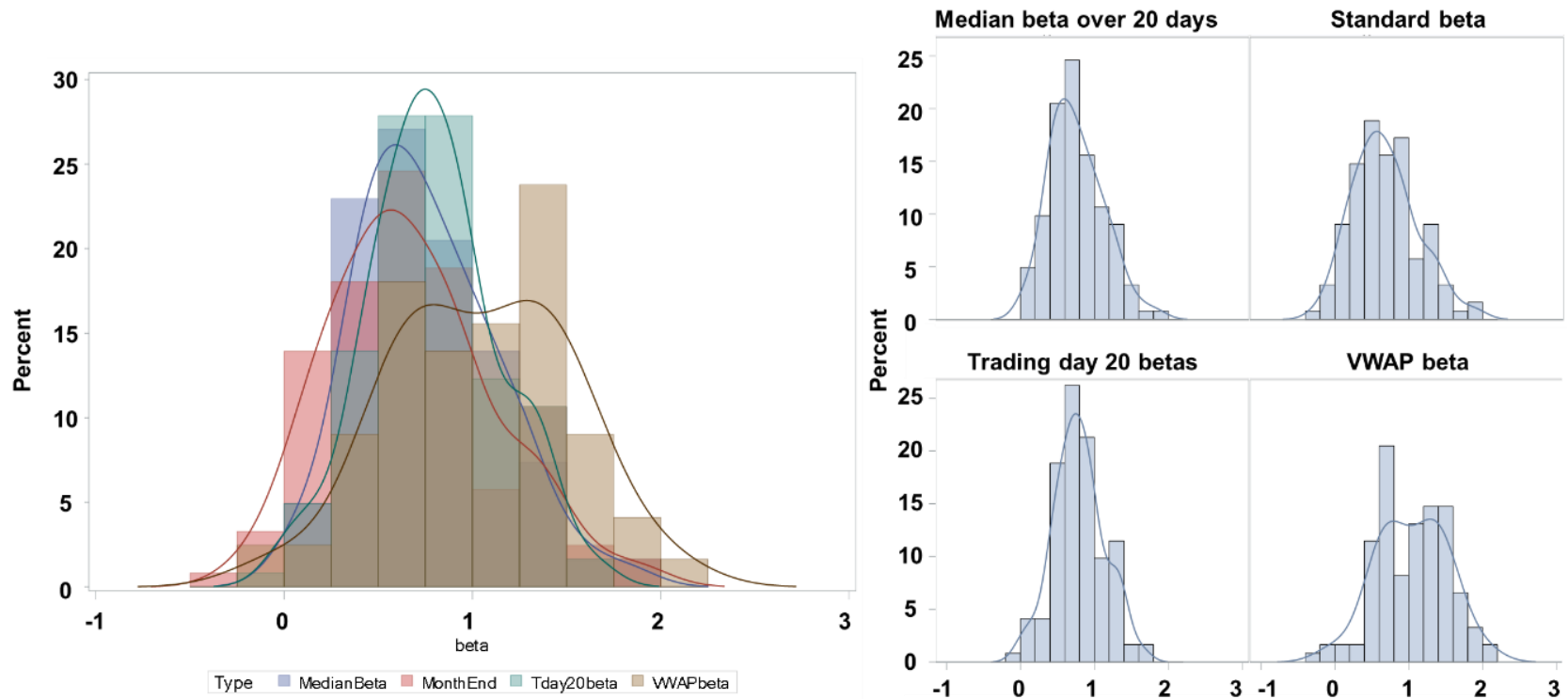


Figure 2: Distribution of estimated betas for Kumba Iron Ore and Royal Bafokeng Platinum

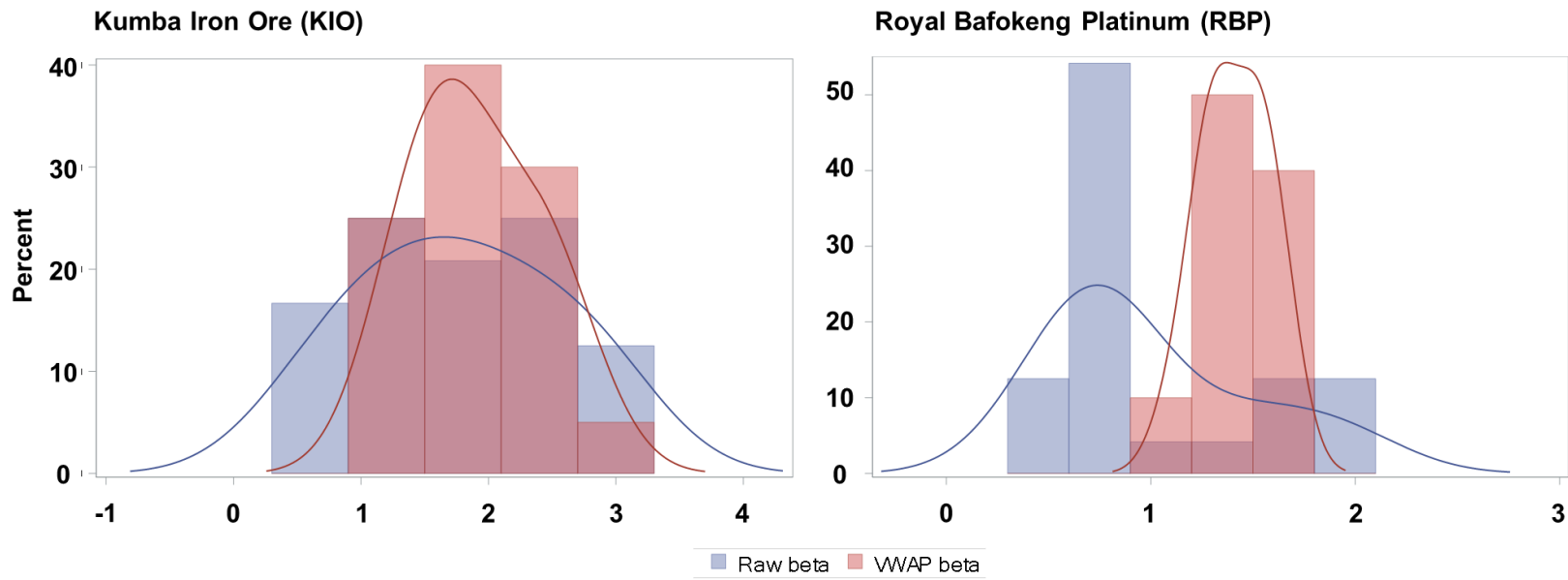
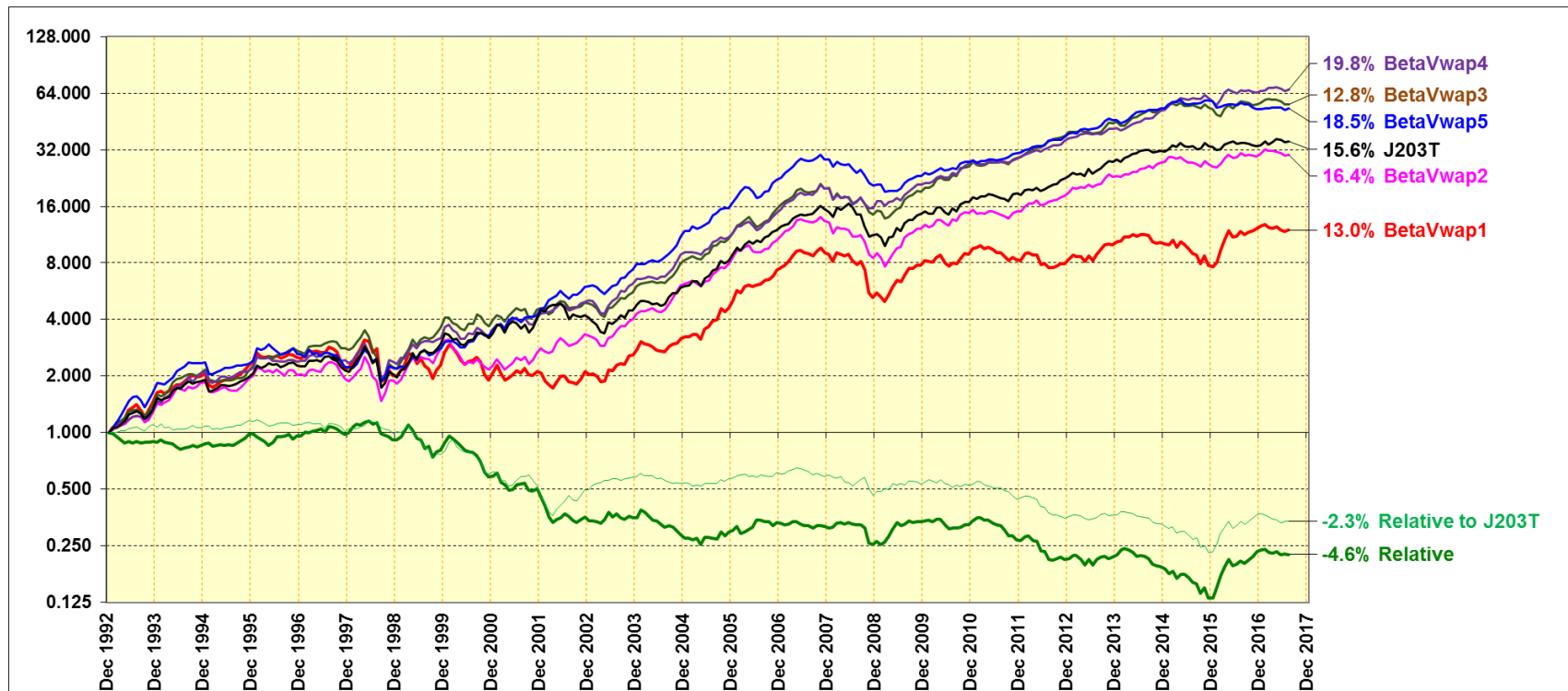


Figure 3: Ranked portfolios in terms of VWAP betas



A volume-weighted-average-price (VWAP) method for estimating beta in the context of reference-day risk

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CHAPTER 2 AND CHAPTER 4

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CHAPTER 2: LITERATURE REVIEW

2.1. Introduction

Empirical finance literature and investment practitioners make extensive use of monthly share returns data. Such data series are commonly used to provide point estimates of, among others, means and variance-covariance matrices of share returns which are input into the calculation of share betas (Acker & Duck, 2007; Dimitrov & Govindaraj, 2007). By convention, this monthly return is estimated by calculating the proportionate change in a share's price between a particular day of the month (termed the reference-day) and the corresponding day of the following month. Investigations conducted by Baker, Rajaratnam and Flint (2016), Acker and Duck (2007) and Dimitrov and Govindaraj (2007) have evidenced that by using any five-year sample, and selecting one day of the first month as the reference day to construct a series of monthly returns, different choices for the reference day produce large variations in the estimated betas.

The following section presents some of the empirical literature describing reference-day risk and the variations that different choices of reference day have on share beta estimates. The discussion will initially focus on the various alternative methods for estimating and adjusting beta. Furthermore, various authors have researched and documented the effectiveness of traditional adjustment methods to beta, following the investigation by Acker and Duck (2007). The results of these investigations are pertinent to this research and will also be focused on. However, it is also imperative that the empirical literature of the CAPM and beta be presented as this fundamentally underpins most risk and reward research. This section begins with an outline of this theory.

2.2. Modern Portfolio Theory and the capital asset pricing model: the birth of beta

Modern Portfolio Theory (MPT), ascribed primarily to Markowitz (1952; 1956; 1959) and various authors in the 1960s and 1970s, most notably Sharpe (1964), remain the foundation for the way in which portfolio managers approach portfolio risk (Rubinstein, 2002). MPT first gave impetus to the suggestion that portfolio risk is determined by the co-variances of assets included within a portfolio (Heymans & Brewer, 2015). Following on from Markowitz' original work on asset diversification, Treynor (1961), Sharpe (1964), Lintner (1965) and Mossin (1966), working independently, were credited for the development of the Capital Asset Pricing Model (CAPM), which remains one of the most widely used asset pricing models in modern portfolio management (Graham & Harvey, 2001).

2.2.1. The conventional method for estimating beta

The CAPM describes a simple linear model for estimating the expected return on an asset in terms of its systematic risk (Ward & Muller, 2012; Fama & French, 2004). The CAPM can be expressed as:

$$E(R_i) = R_f + \beta_i (E(R_m) - R_f) \quad (\text{Equation 1})$$

Where;

$E(R_i)$ is the expected return on the share (asset),

R_f is the risk-free rate of interest, such as interest arising from government bonds,

$E(R_m)$ is the expected return of the market, and

β_i is the “beta” (a term originally coined by Sharpe (1964, p. 440)) and represents the sensitivity of the expected excess asset returns to the expected excess market returns, and is expressed as;

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)} \quad (\text{Equation 2})$$

where $\text{Cov}(R_i, R_m)$ is the covariance between the return on asset i and the return on the market portfolio, using historical data (Heymans & Brewer, 2015; Chen & Reeves, 2012).

Equivalently, Fama and Macbeth (1973) suggest a method for estimating beta by using a rolling linear regression of individual share returns on market returns (monthly), typically over a five-year period. This method is commonly referred to as the ordinary least squares (OLS) regression method and assumes that all the error lies within the asset returns (Baker *et al*, 2016). These two methods are considered the conventional method for estimating beta (Chen & Reeves, 2012). Analogous to this research, the need for greater sophistication and accuracy in estimating systematic risk has spurred several investigations into generating alternative beta measurements.

2.2.2. Estimating beta using symmetric regression: The Total Beta method

In response to the need to greater understand and estimate accurate risk parameters for emerging markets, Laird-Smith, Meyer and Rajaratnam (2016) construct a systematic risk measure using symmetric regression for the case of the Johannesburg Stock Exchange (JSE).

Laird-Smith *et al* (2016) use the methodology proposed by Camp and Eubank (1981) and more recently, Tofallis (2008; 2015), as an alternate estimation methodology for beta. This method has been coined the “Total beta” method (p. 115). According to Laird-Smith *et al* (2016), unlike the standard OLS method, the total beta estimate is less likely to underestimate the magnitude of the beta parameter by explicitly allowing for error in the variables.

Total beta is estimated from the Geometric Mean Function Relationship, which allows for the possibility of error in the independent variable of a regression (Laird-Smith *et al*, 2016). According to Tofallis (2008; 2015), by acknowledging that the market returns are measured completely without error, the standard OLS model for estimating beta makes itself subject to the assumption that rarely reflects reality, that is, it overlooks random fluctuations. Intuitively, the total beta deals with the errors in variables in proportion to their magnitude (Laird-Smith *et al*, 2016). This method is captured in the Geometric Mean Functional Relationship;

$$\beta^* = (\text{sign of } r) \frac{\sigma_i}{\sigma_m} \quad (\text{Equation 3})$$

where,

β^* is a parameter estimated from the function,

r is the correlation between the share’s returns and the market’s returns,

σ_i is the standard deviation of the share’s returns, and

σ_m is the standard deviation of the market’s returns (Laird-Smith *et al*, 2016).

Using a sample of the top 40 shares on the JSE, Laird-Smith *et al* (2016) conclude that the difference in magnitude between the OLS regression and total beta parameters is often significant and that the OLS regression parameter consistently underestimates relevant underlying risk factors. The authors further present the case that in emerging markets like South Africa, the total beta parameter provides a more realistic and stable estimator of market-related risk and return, especially when comparing variations in beta year-on-year, or under different market conditions (Laird-Smith *et al*, 2016). The methodology presented by Laird-Smith *et al* (2016) has however not been tested for the reference-day problem.

In addition to the various standalone methods developed in pursuit of an accurate representation of the systematic risk component, beta, there has been equal impetus applied by academics in investigating varying adjustments to the existing method used to estimate beta.

2.2.3. The Blume regression method for estimating beta

In 1971, Marshall E. Blume set out to examine the stationarity of the conventional beta estimate over time and to propose a method of obtaining improved assessments of systematic risk. By using the conventional OLS estimation method, Blume (1971) created portfolios of shares according to the magnitude of estimated betas, and estimates the betas for the resultant portfolios across two, non-overlapping consecutive periods. The results indicate a tendency for the high-risk portfolios (i.e. portfolios with the high beta estimates) to decline monotonically towards 1, whereas the lower-risk portfolios tend towards 1 over time (Blume, 1971; Baker *et al*, 2016).

According to Blume (1971), if the rate of regression of the estimated betas towards the mean beta is stationary over time, it is possible to correct for the beta regression tendency in estimating a more accurate representation of systematic risk. Blume (1971) uses the betas generated from the two consecutive, non-overlapping periods to estimate “predicted betas” (p. 8) for a third, non-overlapping period. This is done by regressing the estimated values of beta (β_i) in one period on the values estimated in the previous period (β_{i-1}) to yield a modified estimate for the assessment of future systematic risk. This relationship is represented by the equation;

$$\beta_i = a + b\beta_{i-1} + \epsilon_t \quad (\text{Equation 4})$$

where;

β_i is the beta in one period,

β_{i-1} is the beta in the preceding period,

a and b are regression coefficients, and

ϵ_t is a zero-mean error term (Baker *et al*, 2016).

The mean squared errors for the adjusted and unadjusted beta estimates are compared to the beta estimates generated using the conventional estimation method. The research conducted by Blume (1971) concluded that for both individual shares and portfolios of two or more shares, the estimates adjusted for the historical rate of regression are more accurate than the unadjusted estimates, even though the rate of regression is not strictly stationary over time. Since the publishing of his results, Blume’s (1971) method has become widely tested and employed by academics and practitioners alike.

2.2.4. The Dimson adjustment for thin trading

Dimson (1979) uses the findings of Fisher (1966) to postulate an adjustment method to beta estimation when shares are subject to infrequent trading. Fisher (1966) suggests that a major source of bias in estimating beta emanates from the tendency of shares which are subject to thin trading having substantially underestimated covariances. Fisher (1966) further proved that the downward bias in the covariance of frequently traded shares is less significant when compared to infrequently traded shares. The findings of this investigation formed the premise for Dimson's (1979) Aggregate Coefficients (AC) method for estimating the systematic risk of a share.

In essence, Dimson's (1979) AC method asserts that the true systematic risk parameter beta ($\hat{\beta}$) can be obtained from price data which is subject to infrequent trading by regressing observed share returns on lagged, synchronous and leading market returns. The resultant regression equation is given by the expression;

$$\hat{R}_t = \hat{\alpha} + \sum_{k=-n}^n \hat{\beta}_k \hat{M}_{t+k} + \varepsilon_t, \quad (\text{Equation } 5)$$

where;

\hat{R}_t represents the observed share returns,

M_{t+k} represents the lagged, synchronous and leading market returns

$\hat{\alpha}$ and $\hat{\beta}_k$ are the estimated intercept and slope coefficients, respectively and

ε_t is a zero-mean error term (Dimson, 1979).

Subsequently, the systematic risk parameter beta (β) is calculated as the sum of the aggregate coefficients in Equation 5. Thus,

$$\hat{\beta} = \sum_{k=-n}^n \hat{\beta}_k \quad (\text{Equation } 6)$$

Dimson (1979) compares the beta estimates derived using the AC method against the Scholes-Williams (1977) method, the trade-to-trade regression method and the standard OLS regression method. After controlling for and simplifying assumptions to allow comparison across methods,

the authors conclude that the AC method is a more efficient beta estimation method when compared to the Scholes-Williams (1977) method, even though the Scholes-Williams (1977) method does not suffer bias due to infrequent trading. Furthermore, the standard OLS regression method for estimating beta is significantly biased for shares which are thinly traded. Dimson (1979) does note however that the trade-to-trade method was also an efficient method for estimating beta for shares which are thinly traded.

2.2.5. The Vasicek Bayesian method for beta estimation

Using Bayesian Decision Theory, Vasicek (1973) presents a method for generating Bayesian estimates for beta used in the market model. According to Vasicek (1973), Bayesian Decision Theory provides formal procedures which makes use of information available prior to sampling, in conjunction with the sample information, to construct optimal estimates which minimize the expected error.

In deriving the Bayesian beta estimate, Vasicek (1973) argues that the properties of the standard OLS beta parameter do not satisfactorily reflect the required properties of a beta estimator. Specifically, the standard OLS estimator assumes the property;

$$E(b|\beta) = \beta$$

which describes the mean value of the estimator, with the conjoint assumption that the true value for beta is known. Vasicek (1973) challenges the assumption that the true value of beta is known and described by the mean of the estimator. In essence, one would not require an estimator (b) if the true value of beta (β) is known. Vasicek (1973) therefore postulates that the reverse of this is true; it is the sample coefficient (b) which is known and on this basis, one can infer about the distribution of the parameter, beta (β).

Vasicek (1973) therefore postulates that given the normal prior distribution with mean b' and variance s'^2_b , the posterior distribution of β is approximately normal, with mean b'' and variance s''^2_b , and b is the OLS estimate of systematic risk and;

$$b'' = \frac{(b'/s'^2_b + b/s^2_b)}{(1/s'^2_b + 1/s^2_b)}, \quad s^2_b = \frac{s^2}{\sum(M_t - \bar{M})^2},$$

$$s''^2_b = \frac{1}{(1/s'^2_b + 1/s^2_b)}, \quad s^2 = \frac{\sum(R_t - \hat{\alpha} - bM_t)^2}{(T-2)},$$

where t is the time step, with $t = 1, 2, \dots, T$ (Baker *et al*, 2016).

According to Vasicek (1973), the Bayesian estimate for the systematic risk parameter, beta, is given by the mean of the posterior distribution b'' and describes the knowledge about the distribution of the estimated beta, given the information from the sample and the prior information.

To conclude, Vasicek (1973) asserts that Bayesian estimates are preferred to the standard OLS estimates as the Bayesian procedure minimizes the loss of accuracy arising from misestimation. In contrast, the standard OLS estimates minimize the error of sampling. Furthermore, in addition to the sample information, the Bayesian estimates incorporates prior information is the estimation of systematic risk (Vasicek, 1973).

2.3. Empirical performance of beta in explaining the variation in share returns

Much of the aforementioned literature documents the extensive application of the CAPM and beta in financial literature and business. According to Novak (2015) however, empirical evidence suggests that traditional methods for estimating beta have been criticized for low explanatory power.

Black, Jensen and Scholes (1972) tested on average, whether portfolios consisting of shares with high betas generate higher returns. The investigation evidenced that beta does not suffice in explaining the cross section of share returns (Black *et al*, 1972). Similarly, the evidence documented by Basu (1977), Banz (1981), Stattman (1980) and Rosenberg, Reid, and Lanstein (1985) present additional dimensions to the estimation of beta which are culminated by Fama and French (1992), whom suggest that if assets are priced rationally, then share risks are multi-dimensional, unlike that propagated by the CAPM. This assertion was controversial for its time, and drew considerable criticism given the extensive use of beta and the CAPM in academia and practice.

Additionally, Laird-Smith *et al* (2016) note some of the weaknesses underpinning the low explanatory power of asset pricing models using beta. Laird-Smith *et al* (2016) purport that asset pricing models are grounded in the assumption that beta estimates remain constant over time, an observation which differs markedly from the market. Moreover, traditional methods for estimating beta (such as OLS regression) are well established to change over even short periods of time (Laird-Smith *et al*, 2016). Evidence presented by Ferson and Korajczyk (1995) empirically proved the asset pricing models which assumed a static beta perform poorly when compared to asset pricing models which adjusted the time-period over which beta is estimated. By implication, it is therefore imperative to prioritize methodologies which produce stable estimates for beta over time and during different market conditions.

According to Novak (2015), the wave of literature documenting the low explanatory power of beta led some to proclaim beta “dead” (p. 168). The findings from this research have however spurred on further studies seeking to adjust and improve the estimation for beta, most notably Novak (2015) who evidenced that the weak empirical support for beta is caused more by complications with implementation rather than by the weakness of the underlying concept. Similarly, Ward and Muller (2012) used betas estimated from 24-month and 60-month historic share performance data and concluded that for the largest 160 shares on the JSE, there exists minor difference between portfolio cumulative returns for four alternative methods of measuring beta. Furthermore, an OLS estimate of beta using 60-month sample data is suitable for the largest 160 shares on the JSE. The findings from this research corroborate the hypothesis that beta has sufficient application for literature and practice.

2.4. Empirical studies on estimating beta in the context of reference-day risk

Acker and Duck (2007) prove the existence of reference-day risk associated with monthly returns implicit in shares listed on the S&P500 Index and set out to explore the level of reference-day risk implicit in estimates of CAPM beta. Acker and Duck (2007) further investigate the existence of reference-day risk after having adjusted the traditional beta estimates by the Blume (1971), Vasicek Bayesian (1973) and Dimson (1979) adjustment methods.

Using data from a sample of 459 companies sourced through Datastream, Acker and Duck (2007) examined data for a fifteen-year period, across three five-year sub-intervals. The results from the investigation by Acker and Duck (2007) prove that estimates of beta are highly sensitive to the choice of reference day. In extreme cases, Acker and Duck (2007) indicate that the choice of reference-day can double the estimate of a share’s mean monthly return as well as alter the polarity of mean estimates, with observed share return estimates changing from positive to negative (and vice versa) based on the choice of reference-day. Furthermore, estimated medians and variances using different reference days can deviate by over two times, which the authors found to be significant for both individual shares and the market index and consequently having profound implications for the accurate estimation of Beta (Acker & Duck, 2007).

Drawing on the sample of listed companies on the S&P500, the authors illustrate that the mean monthly (non-annualised) return for a particular share ranged between -0.239% and +0.934% based on the choice of reference-day (Acker & Duck, 2007). Moreover, Acker and Duck (2007) present evidence of a share’s Beta falling by 0.931 and rising by 3.454 depending on the choice

of reference-day. Perhaps more pertinent to this investigation however, is the finding that approximately 75% of all observed betas estimated using the conventional methodology could be classified as either positive or negative by an appropriate choice of reference day. This finding has profound ramifications for both academia and industry, as it renders risk-adjusted asset selection using conventional beta estimates impractical, inconsequential and insufficient for portfolio construction. The preliminary findings from the investigation by Acker and Duck (2007) conclude that reference-day sensitivity is inherent within both individual shares and equity indices. Furthermore, the findings from this preliminary investigation by Acker and Duck (2007) set the context for testing the effectiveness of alternative adjustment methods to estimating beta, after having illustrated the influence of reference-day risk on traditional systematic risk measurements.

Acker and Duck (2007) commence with investigating whether Blume's (1971) regression method for estimating beta reduces the variation and range of beta estimates for which the unadjusted beta is highly sensitive to the choice of reference day. The investigation reveals that the Blume (1971) method for estimating beta reduces only the most severe reference-day variability. The authors indicate that shares with the highest estimated range for Blume (1971) beta estimates were reported between a third and a quarter of the equivalent figure for unadjusted beta estimates (Acker & Duck, 2007). The authors however also present evidence that shares with the lowest Blume (1971) beta range exhibited greater variation when compared to the range and variation of unadjusted betas for the same shares. Overall, Acker and Duck (2007) report that the range for betas estimated using Blume's (1971) method are on average, 60% of the mean beta across reference days. Comparatively, the range for unadjusted betas is approximately 70% of the mean beta across reference days (Acker & Duck, 2007).

Similar to the investigation above, Acker and Duck (2007) investigate the level of reference-day risk implicit in shares listed on the S&P500 after applying the Vasicek (1973) Bayesian beta estimation to the sample. Acker and Duck (2007) maintain comparability in the analysis by equating the results of the Vasicek (1973) Bayesian betas to the identical sample of shares analysed using the Blume (1971) regression method. In essence, the investigation reveals that the Vasicek (1973) Bayesian estimation method significantly reduces the range and variation of beta estimates for shares which have high sensitivity to the selection of a reference day (Acker & Duck, 2007). However, the authors also indicate that the Vasicek (1973) Bayesian beta exhibits sensitivity to the choice of reference day with estimated betas varying by over two times across different reference days (Acker & Duck, 2007). The results of the investigation suggest that the Vasicek (1973) Bayesian estimate exhibits results analogous to the Blume (1971) regression

method, although less marked. The findings however purport that estimated betas using both the Blume (1971) and Vasicek (1973) adjustments are indeed subject to reference-day risk, causing shares to be incorrectly classified as either aggressive or defensive based on the selection of an initial reference day (Acker & Duck, 2007).

Lastly, even though the investigation evidenced the presence of reference-day risk for both individual shares and equity indices, Acker and Duck (2007) hypothesize that the effect of reference-day risk may be amplified by thin trading. Accordingly, the authors apply the Dimson (1979) adjustment method for thin trading in estimating betas in the context of reference-day risk. Acker and Duck (2007) estimate the Dimson (1979) beta estimates using a combination of up to two leading and lagged intervals, with the analysis yielding equivalent results across all permutations. The results of the investigation suggest that the Dimson (1979) adjustment method only slightly reduces the variability of beta estimates for the highest ranges. Furthermore, the authors observe a tendency for estimated betas to exhibit significant variation based on the choice of reference day, indicating that the Dimson (1979) adjustment does not yield any significant difference in estimating systematic risk in the presence of reference-day risk.

Dimitrov and Govindaraj (2007) set out to investigate the findings from Acker and Duck's (2007) study from a data dependency perspective. The authors postulate that Acker and Duck's (2007) sample (which was drawn from Datastream) may not be applicable for the United States context and propose replicating the analysis using data sourced from the Centre for Research in Security Prices (CRSP). Furthermore, Dimitrov and Govindaraj (2007) expand the study by Acker and Duck (2007) by investigating the existence of reference-day risk using daily returns (as opposed to monthly returns) over a five-year period. Lastly, the authors explore whether daily returns computed using share prices at different times of the day exhibit "reference-time variation" (p. 560).

Using data sourced from the CRSP from the period January 1995 to December 1999, Dimitrov and Govindaraj (2007) use daily dividend-adjusted share returns to construct a series of sixty monthly returns across nineteen different reference days, for each of 439 sample companies. Congruous to the findings of Acker and Duck (2007), the investigation verifies the existence of reference-day risk in companies listed on the S&P500, using data sourced from the CRSP (Dimitrov & Govindaraj, 2007). Moreover, Dimitrov and Govindaraj (2007) also report that betas estimated from the sample exhibit significant variation when different choices of the initial reference day are used in the computation.

Additionally, Dimitrov and Govindaraj (2007) investigate whether reference-day risk has any effect on daily return series as these series are widely employed to draw inferences about various company announcements such as earnings, dividends, new share issues and repurchases. Analogous to the findings reported earlier, the authors report that the selection of the reference day leads to significant variation in average daily returns, return variances and betas (Dimitrov & Govindaraj, 2007). In particular, daily beta estimates exhibit large variation across different reference days, with the authors reporting a maximum range for a given share as high as 2.31 (Dimitrov & Govindaraj, 2007). Furthermore, the findings from the investigation suggest that reference day risk is implicit for both individual shares and market indices.

Lastly, Dimitrov and Govindaraj (2007) make use of intraday price data to investigate whether daily returns constructed using various times of the day exhibit the effects of “reference-time risk” (p. 568). Consistent with all previous findings, the authors verify the effect of reference-time risk on the variance of share returns, variances and betas. Dimitrov and Govindaraj (2007) conclude by suggesting that betas estimated using monthly, daily and intraday returns are unreliable as these estimates may vary across different reference days and times. The results of the investigation therefore have profound implications for studies using the conventional estimates of beta.

Gonzalez, Rodriguez and Stein (2014) explicate the research conducted by both Acker and Duck (2007) and Dimitrov and Govindaraj (2007) by expanding the data dependency and methodology parameters employed in the previous studies. Gonzalez *et al* (2014) compare the *t*-distribution method for adjusting beta as developed by Cademartori, Romo, Campos and Galea (2003) to the OLS method and Blume’s (1971) regression method for estimating beta.

According to Gonzalez *et al* (2014), the *t*-distribution method proposes a replacement of the standard normal distribution with the Student’s *t*, which is a symmetric distribution with heavier tails than the normal distribution. Gonzalez *et al* (2014) further purport that the heavier tails in the Student’s *t* is more appropriate in estimating beta in the context of reference-day risk as it more aptly compensates for the error term in the linear regression used to estimate company betas. Furthermore, Cademartori *et al* (2003) prove that the *t*-distribution method for adjusting beta is better able to incorporate the influence of outliers in estimating beta. Lastly, Blume’s (1971) method was selected by Gonzalez *et al* (2014) for comparison to the *t*-distribution method, as it was the method which reduced the effect of reference-day risk the most in Acker and Duck’s (2007) investigation.

Expanding on the original sample selected by Acker and Duck (2007), Gonzalez *et al* (2014) select a sample of 1563 shares, traded on the NYSE, AMEX and NASDAQ exchanges obtained from the CRSP, for the period 2007 to 2011. Consistent with the findings from previous investigations, Gonzalez *et al* (2014) report that the choice of reference day results in significant variations in estimated betas. The authors illustrate that in an extreme case, a company with a standard OLS beta estimate of 2.8 using a particular reference day, recorded an OLS beta estimate of 10.4 using another reference day (Gonzalez *et al*, 2014). Similarly, Gonzalez *et al* (2014) report that betas estimated using Blume's (1971) regression method also exhibited significant variations across different reference days, with a maximum range of 6.5. Using the *t*-distribution method, the authors record that the choice of reference day becomes less significant with the larger recorded ranges decreasing from an average of 1.9 to 1.6. The investigation concludes that the *t*-distribution method for adjusting beta most significantly reduces reference-day variation when compared to Blume's (1971) method and the OLS method.

More recently, Baker *et al* (2016) set out to establish the existence of reference-day risk in the JSE Top 40. Baker *et al* (2016) use daily closing levels of the ALSI and closing prices for shares making up the JSE Top 40 index for the period January 2000 to July 2015, sourced from Datastream.

Congruous to the findings of Acker and Duck (2007), Dimitrov and Govindaraj (2007) and Gonzalez *et al* (2014), the authors conclude that reference-day risk exists and creates additional uncertainty for investors intending to create share portfolios, valuing companies or managing capital (Baker *et al*, 2016). Furthermore, the authors test the Blume (1971), Dimson (1979) and Vasicek (1973) methods for adjusting beta to investigate whether the adjusted betas exhibit lower reference-day ranges than unadjusted betas (Baker *et al*, 2016).

In estimating the betas calculated using Blume's (1971) regression method, Baker *et al* (2016) report that the Blume-adjusted betas increase the variation and range of beta estimates for 19 out of 31 companies. Baker *et al* (2016) further illustrate that the range for Old Mutual Plc increased by over 81% when compared to the results obtained through the standard OLS estimation. The authors therefore conclude that the Blume (1971) regression method for estimating beta does not consistently and considerably reduce the variation and range for betas in the context of reference day risk.

Congruently, Baker *et al* (2016) create a multifactor model which estimates the log returns for shares according to leading, synchronous and lagged log returns on the index in order to test the

Dimson (1979) method for estimating beta. Baker *et al* (2016) explain that the Dimson (1979) adjustment method, unlike other beta adjustment methods, produces negative estimates of systematic risk as it adjusts for thin trading. The authors further consider the appropriateness in employing the Dimson (1979) adjustment in the investigation, as the sample comprises shares which are frequently traded and therefore do not suffer thin trading. Analogous to the results of the Blume (1971) regression method, the average range for the Dimson-adjusted betas was significantly larger than that of the standard OLS betas. Moreover, the range for beta estimates using the Dimson (1979) adjustment increased for 30 out of the 40 companies sampled, leading the authors to conclude that the Dimson (1979) adjustment for thin trading actually pronounces the reference day variation in systematic risk estimates when compared to the standard OLS estimate, for shares on the JSE Top 40.

After having tested the Blume (1971) and Dimson (1979) methods for adjusting beta against the standard OLS beta estimate without any promising results, Baker *et al* (2016) then test the Vasicek (1973) Bayesian method under the same conditions. Baker *et al* (2016) report that the Vasicek (1973) Bayesian method reduced the occurrence of extreme variations in estimated betas when compared to the unadjusted betas.

Baker *et al* (2016) conclude that after having tested the Blume (1971), Dimson (1979) and Vasicek (1973) Bayesian adjustment methods, only the latter yielded any promising results, with 35 out of the JSE Top 40 companies exhibiting lower reference-day ranges than unadjusted betas. Baker *et al* (2016) then introduce a nonparametric bootstrap method to determine potential beta estimates which are independent of reference-day risk. Using this method, the authors note that for shares on the JSE Top 40, the expected value of a reference-day independent beta was approximately equal to the average of the 20 unadjusted betas estimated for each day. Baker *et al* (2016) propose that the mean value of the bootstrapped beta distribution therefore provides a reference-day independent estimate of systematic risk for a particular share, even though further investigation is required to recover the mean value of a set of beta standard deviations. Even though the results of the investigation do not yield the desired result, the authors do note some benefits of employing this method in estimating beta (Baker *et al*, 2016, p. 134).

In summary, the literature discussed above has not revealed a methodology which is independent of the reference-day in beta estimation. The final component of this theoretical discussion now focuses on the relevant literature regarding the proposed VWAP methodology.

2.5. Empirical evidence of VWAP as a method to eliminate the reference-day problem

There is little to no literature available on estimating beta through employing a VWAP adjustment or VWAP methodology in the context of reference-day risk. However, perhaps the most relevant literature is presented by Ting (2005; 2006) who suggests that a VWAP is closer to the equilibrium price of a share than the daily closing price. Ting (2005; 2006) bases this on the premise that VWAP considers all the intra-day prices at which transactions have occurred and evidences that daily returns computed with VWAP have a smaller realized variance than that with the closing price. Moreover, Ting (2005; 2006) concludes that the variance spread between VWAP and the closing price is economically significant and has implications for performance measurement and pricing of derivatives. Even though Frei and Westray (2013) aim to investigate the use of a VWAP in trading algorithms (as opposed to estimates of beta), the authors also report a minimisation of mean and variance returns when considering the VWAP.

More critical to the proposed research however, Ting (2005; 2006) provides an example which evidences that, relative to the volatility of VWAP returns, the volatility of closing price returns tends to understate the beta risk estimation result. By consequence, the research suggests that by using VWAP along with the closing price, estimation of financial risk and asset pricing can be performed with considerably less noise (Ting, 2005; 2006).

The literature presented above has verified the existence of reference-day risk on the S&P500 as well as for the JSE Top 40 index, even after applying various adjustments to the estimation methods for beta. Analogous to the South African studies, the question of whether reference-day risk can be observed on the JSE ALSI beckons. Furthermore, the empirical evidence presented above evidences the critical need for a methodology which produces a stable estimate for beta which is independent of the reference day. The next section now assimilates the literature with relevant hypotheses and a prudent methodology for which to conduct the proposed research.

CHAPTER 4: RESEARCH METHODOLOGY AND DESIGN

The literature presented in the previous section documented the empirical performance of beta in explaining share returns, especially within the context of reference-day risk. As per Novak (2015), the combination of the low explanatory power of beta and the inverse relationship observed between beta and expected returns over time, have led some authors to proclaim beta “dead”.

This investigation aims to resurrect the systematic risk parameter, beta, by contributing a volume-VWAP estimate for beta which can be employed under conditions of reference-day risk. To achieve this, the research first evidences the degree to which variations in the reference day leads to variations in share betas for companies on the JSE ALSI. Thereafter, estimates of beta using the VWAP methodology are statistically tested to determine whether these can be applied when reference-day risk is observed in estimates of share betas.

Lastly, this investigation transcends prior research of a similar nature by determining whether the VWAP beta adheres to the underlying tenets of the CAPM. This is achieved by employing a graphical time-series analysis to observe whether VWAP betas are a more reliable representation of systematic risk in the risk-return tradeoff.

Consequently, an appropriate methodology was derived to test each of these objectives and is presented in this section.

4.1. Choice of methodology

As would be expected in a financial research paper, the investigation was designed to be quantitative and causal in nature, congruous to the design of similar reference-day risk investigations (Carter, Muller & Ward, 2017; Baker *et al*, 2016; Gonzalez *et al*, 2014; Dimitrov & Govindaraj, 2007; Acker & Duck, 2007). Moreover, the longitudinal, time-based nature of the financial data used in this investigation presented the opportunity to facilitate a time series analysis, both graphically and statistically. This approach is well documented in investigations empirically testing the risk-reward relationship (Carter *et al*, 2017; Muller & Ward, 2013; Ward & Muller, 2012). In conjunction with the traditional statistical method for which researchers such as Baker *et al* (2016), Gonzalez *et al* (2014), Dimitrov and Govindaraj (2007) and Acker and Duck (2007) have conducted studies of this nature, the graphical time series analysis provided an additional dimension for verifying the applicability of the VWAP beta as an alternative measure of systematic risk. This transcends prior research investigations of this nature by empirically testing

the risk-reward tenet postulated by the CAPM. This approach was deemed both complementary and appropriate given the objectives of the research.

4.2. Population

As noted in the literature, Baker *et al* (2016) verified the existence of reference-day risk for the JSE Top 40 index. In the case of this investigation, the applicable population included all companies listed on the JSE (including dual listed companies), with necessary market capitalisation to be included in the JSE ALSI. Furthermore, given the longitudinal time-based nature of the investigation (1992 to 2016), the qualification of companies to the population applies to any firm, which at any stage between 1992 and 2016, had sufficient market capitalisation irrespective of their eventual state. Furthermore, given that this research focused solely on the relationship between estimates of share betas and expected returns, survivorship bias was not a limiting factor for any results emanating from the investigation, even though measures were taken to reduce this potentially limiting factor.

4.3. Unit of analysis

The primary unit of analysis for the investigation constituted individual share values, for which total returns were estimated for each of the shares. According to Ward and Muller (2012), dividend receipts constitute a significant portion of a shares total returns, and were subsequently included in total returns computation. This was consistent with the recommended procedure prescribed by Bradfield (2003), when estimating the systematic risk parameter, beta. In addition, analogous to Baker *et al* (2016), Gonzalez *et al* (2014) and Carter *et al* (2017), the share values include adjustments for any unbundling, mergers and share splits. Each of the share values comprise the data required for estimating total returns, and eventually, traditional betas and VWAP betas against which the research hypotheses were tested. Given that the JSE ALSI is a dynamic index, the individual shares underpinning the share values were subject to the qualifying characteristics of the JSE ALSI, over a periodical basis, and were amended accordingly.

4.4. Sampling method and size

The quantitative, highly computerised and statistical nature of the analysis, in addition to the availability of information for all companies within the defined population, meant that a purposive sampling technique was appropriate for the research (Zikmund, Babin, Carr & Griffin, 2013; Carter *et al*, 2017). Additionally, the deliberate availability of choice in selecting the JSE ALSI as the

sample further substantiated the purposive sampling technique (Saunders & Lewis, 2012). Furthermore, this was also the sampling method employed by researchers in similar investigations and was deemed the appropriate convention for investigations of this nature (Muller & Ward, 2013; Carter *et al*, 2017; Baker *et al*, 2016; Gonzalez *et al*, 2014).

According to the JSE (2017a), there were approximately 397 companies listed on the exchange during 2017. During the analysis period however, there were typically more than 300 companies listed on the JSE during the entire time series. The JSE ALSI however only comprises the largest 160 companies by market capitalization and was consequently selected as the sample (Ward & Muller, 2012).

According to the JSE (2017b), the choice of the JSE ALSI as the selected sample, ensured that 99% of South Africa's market capitalization was accounted for in the research. As per Ward and Muller (2012) those companies which fall outside of the JSE ALSI were considered too small and illiquid for some investors. Moreover, as suggested by Dimson (1979), shares with illiquid characteristics may already exhibit severely biased estimates of systematic risk. Consequently, it was appropriate to exclude these shares from the sample which further justified selecting the JSE ALSI.

Expanding on the most recent and relevant investigation by Baker *et al* (2016) which sampled the JSE Top 40 over the period January 2000 to July 2015, the research period over which the sample was analysed comprised all qualifying companies on the JSE ALSI between 31 December 1992 and 31 January 2017. Share betas were however estimated using a most recent, single 5-year period during this time frame to understand the degree to which reference-day risk exists on the JSE ALSI. As a result, all qualifying companies on the JSE ALSI during the period 3 January 2012 until 31 January 2017 with a record of at least five years historical share price data were considered. This resulted in a total of 136 shares under observation.

Given the parameters suggested by Kothari (2004) and Saunders and Lewis (2012), the logical selection of the sampling units as well as the proportion of the JSE ALSI sample relative to the entire population of listed shares on the JSE ought to ensure that the sample is representative. This ought to improve the validity and reliability of the research results, and allows for greater generalisability of the findings which stemmed from this research.

4.5. Measurement instrument

The style engine, as developed by Ward and Muller (2015), was utilized as the primary measurement instrument for the research. This was especially the case for empirical tests of the relationship between systematic risk and returns. The style engine contained secondary data collected since 1985 on listed companies on the JSE (Muller & Ward, 2013; Carter *et al*, 2017). According to Carter *et al* (2017), the style engine was constructed using Microsoft Excel and includes associated Visual Basic for Applications (VBA) programming code. Moreover, the style engine adjusts for dividends, share unbundlings, mergers and share splits, as required for investigations of this nature. Pertinent to this research, the style engine also contained VWAP betas for each of the shares and the JSE ALSI for the entire sample period. This markedly simplified the graphical time series analysis component of the investigation.

For all other research components in the investigation, daily share prices and total returns were sourced from Datastream, as done by Baker *et al* (2016) and Acker and Duck (2007). Microsoft Excel was also utilised for all collation, cleaning and processing of the data whereas SAS Studio and IBM SPSS were used for the statistical analyses.

According to Kothari (2004), there are three considerations for researchers to consider when using secondary data of this nature. These include reliability of the data, suitability of the data and adequacy of the data. Given the nature of the financial data under analysis as well as the appropriate sources from which the sample was drawn, it is prudent to assume that the data is indeed accurate and free of bias. Moreover, when conducting the graphical time series analysis, the style engine moderated against survivorship bias by also including companies which have failed during the research period. This is a typical shortcoming of research investigations of this nature (Ward & Muller, 2015) and improved the robustness of the research findings.

4.6. Analysis approach

Having derived a prudent methodology consistent with other investigations of this nature, the research method thereafter focused on a series of specific tests for investigating each of the research objectives. Each of these objectives are briefly outlined, together with a description of the method employed.

4.6.1 Investigating the existence of reference-day risk

Every month within the sample period has approximately 20 trading days, as the trading of listed shares on the JSE typically excludes weekends and South African public holidays. The daily closing share prices and JSE ALSI levels can therefore be organized per 20 trading days, with the first trading day corresponding to the first working day of the initial month, in a five-year time series. Monthly log returns for each of the qualifying shares on the JSE ALSI and the index are thereafter calculated for the period 3 January 2012 to 31 January 2017. This resulted in a series of monthly returns (one series for each of the 20 trading days) for each of the 136 sampled shares and the JSE ALSI.

To illustrate this using the sample data, consider the example where trading day 1 falls on 3 January 2012. In estimating the return for a given share, the first data point in the series is the closing price of a share on 3 January 2012. Given that the trading of listed shares on the JSE is restricted to working days only, the next data point in the series would occur 20 working days later, referenced from 3 January 2012, and corresponds to 31 January 2012. In similar fashion, monthly closing price series were estimated for all 136 sampled shares across 20 trading days. Monthly log returns are then estimated for each of the shares, using each of the 20 trading days as a reference point.

Using this data, 20 different estimates of beta were generated for each share, by approximating the slope of the regression between an individual share's returns and the JSE ALSI returns. Sample statistics were then analysed for the estimated betas across 20 trading days. This analysis included comparing the distributions for estimated betas for each of the 20 trading days. Furthermore, as a subset of their findings, Baker *et al* (2016) propose using the average of the 20 betas as a measure of systematic risk in the context of reference-day risk. This research expanded on this point and considered the distributions of the maximum, minimum and median betas for the sample, as either measure could potentially yield stable estimates of share betas for the 136 sampled companies.

Following a logical top-to-bottom approach, the results of individual shares were also analysed as these observations may lead to key insights in explaining the extent to which estimated betas vary when the reference day is changed.

To statistically determine the degree to which varying the reference day leads to variations in estimated betas, an ANOVA was performed for the 20 trading days. Even though the ANOVA test is robust against violations of normality, the distribution of beta for each trading day was still tested

for normality using Shapiro-Wilk, Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling statistics (Glass, Peckham & Sanders, 1972; Harwell, Rubinstein, Hayes & Olds, 1992; Lix, Keselman & Keselman, 1996). The tests for normality indicated that normal distributions were evident across all trading days, except for trading days 18 and 19. According to Pallant (2011), the adequate sample size of 136 shares would however ensure robustness in the results of the test.

Furthermore, a Levene's test of the homogeneity of variance was run on the sampled data, which did not allow the null hypothesis to be rejected ($p=0.4484$), thus indicating that the distributions of beta for each day were not significantly different.

Consistent with the findings of Baker *et al* (2016), the ANOVA test revealed no evidence of a trading day effect on estimates of systematic risk at both the 5% and 10% level of significance ($p = 0.501$). However, each share has a trading day corresponding to its highest beta estimate and equivalently, a trading day which yields its lowest beta estimate. In an analogous manner to Baker *et al* (2016), the minimum and maximum betas were subsequently tested for statistical difference as investors or practitioners estimating systematic risk could potentially choose a reference day leading to either result. This test was done using a two-tailed independent samples t-test of the means at the 95% confidence level, consistent with the confidence level selected in investigations of a similar nature (Baker *et al*, 2016; Gonzalez *et al*, 2014).

The appropriateness of the independent samples t-test is intuitively described by Pallant (2011), who suggests that the independent samples t-test is best employed when testing two independent measures for statistical differences. In the case of this research, the minimum and maximum betas are distinctly different from each other, and pairing the two measures would not result in a robust analysis as the maximum betas would always be greater than the minimum betas for the sampled shares.

Prior to running the independent samples t-test, the assumptions were checked. Having violated the assumption of equal variances, the Welch-Satterthwaite's method was deemed an appropriate countermeasure as opposed to adjusting the degrees of freedom (Laerd Statistics, 2017). Furthermore, the distributions for the minimum and maximum betas were not normal, but as before, the results were robust given the adequate sample size. The Mann-Whitney U test was run and consequently confirmed the result that the that the highest and lowest beta values are significantly different at the 5% level ($p < 0.0001$).

4.6.2 Developing a point estimate of systematic risk for application when reference-day risk is observed

The primary aim of this research was to develop a more robust point estimate of beta when reference-day risk is exhibited in estimates of systematic risk. Baker *et al* (2016) presented the potential for using an average of the betas across the 20 trading days as an estimate of systematic risk for a share, but note the potential introduction of errors due to small sample size. In this subsection, the research investigated the feasibility of using a VWAP method for estimating beta when reference-day risk is observed.

According to Ting (2005; 2006), the daily VWAP can be estimated by dividing the total value (in this case, using the South African Rand as currency units) by the share volume (i.e. number of shares) which are transacted over a selected number of trading days. Frei and Westray (2013), in attempting to understand optimal trading strategies using a VWAP, report that a 60-day-rolling ex-ante VWAP exhibits the best estimate for stable sample statistics. Furthermore, the authors noted no seasonal dependence in estimated parameters when a VWAP was used.

Using the total daily value of shares traded together with the number of shares traded for each share on the JSE ALSI, a daily 60-day ex-ante VWAP was therefore estimated for the sample, as per the method prescribed by Ting (2006);

$$VWAP = \frac{\sum_{k=1}^n P_k V_k}{V} = \sum_{k=1}^n W_k P_k$$

Where;

W_k is the weight and calculated as V_k / V ,

P_k are the n intraday prices at which transactions have occurred during the period,

V is the total share volume traded over the period and is equal to $\sum_{k=1}^n V_k$,

V_k are the subtotals of all shares transacted at the price P_k (Ting, 2006).

The equation simply states that the VWAP is a combination of all intraday prices, which for the case of this research, occurred over the past 60 days for a share, inclusive of the reference-day. For the JSE ALSI however, the VWAP is a function of the daily movement in share prices for the qualifying shares on the index. Accordingly, the VWAP of the index is weighted on a per share basis, and changes daily with movements in the market capitalisation of firms.

Having computed the daily VWAP for all shares and the JSE ALSI, the log returns for each share and the ALSI were computed over a five-year period, using the 60-day ex-ante VWAP as a substitute for the monthly closing price. VWAP betas were then estimated for each share by approximating the slope of a linear regression of individual share returns on market returns for the period 30 March 2012 to 31 March 2017.

At this point, Gonzalez *et al* (2014) tested the significance of the differences between standard betas against Blume (1971) betas and betas approximated using a student's *t*-distribution method. The authors focused on comparing mean share beta variances for each of the three methods, across 20 trading days. Baker *et al* (2016) employed a different approach and compared adjusted regression coefficients using the Blume (1971), Dimson (1979) and Vasicek (1973) Bayesian adjustment methods.

This research employed a unique approach in estimating beta which was simple and intuitive, yet distinctly different from the previous investigations on reference-day risk. The conventional method for estimating beta was maintained, however monthly closing prices were substituted with the 60-day ex-ante VWAP. This was done prior to regressing individual share returns against the market returns. Reiterating Ting (2006), the VWAP is statistically more efficient than the closing price in reflecting value as it is closer to the unobservable equilibrium price, resulting in a smaller realized variance, which is the essence of the beta measure.

To understand whether the VWAP beta is a more robust estimate of systematic risk when reference-day risk is observed, VWAP betas were compared to standard betas. To achieve this, both the standard betas and VWAP betas were estimated using the same sample period and tested for statistical difference. This test is critical to the investigation as estimates of VWAP betas employ the same computational methodology as the standard beta. Even though the estimation methods are different, the two measures must produce statistically independent estimates of systematic risk. A positive result is therefore fundamental in proposing VWAP as an alternate estimation method in the context of reference-day risk.

To test the difference, a paired-samples t-test was deemed appropriate as the research aimed to compare estimates of systematic risk, using two alternative methods of estimation. This point is further substantiated by Field (2013), who suggests that the paired-sample t-test is a more powerful test than the independent samples t-test, especially in cases where observed differences result from the different treatment of data, on in the case of this research, using two estimation methods for measuring systematic risk.

In a similar method to previous tests, the assumptions of the paired-samples t-test were first checked for any violations. Having failed the assumption of normality after running the Shapiro-Wilk, Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling tests, the Wilcoxon Signed Rank Test was deemed a suitable replacement and was accordingly run (Pallant, 2011). Consequently, at the 5% level of significance, the result confirms that the two measures are statistically different ($p < 0.0001$).

4.6.3 Application of the VWAP beta and understanding whether VWAP betas perform better under reference-day risk

The first step in this analysis was to understand how estimates of VWAP betas for the 136 sampled shares were distributed, and whether tighter estimates of share betas were possible using the conventional method. Furthermore, the research also aimed to understand if there exists a trading day within the 20-day range for which betas tend to be more robust. A visual and statistical comparison was conducted, in analogous fashion to Baker *et al* (2016), when attempting to understand whether systematic risk estimates using the bootstrapped beta method were more robust for shares on the JSE Top 40 index.

VWAP betas for the sample shares were plotted on a histogram to understand the sample characteristics of estimated share betas, particularly focusing on the skewness and tightness of the distribution. Similarly, beta estimates for the sample shares were thereafter plotted using the median beta for each share from the 20 trading days. This process was then repeated for all share betas estimated using the 20th trading day (as this day yielded the tightest distribution in estimated betas). For control, standard betas were also estimated for the sample and equivalently plotted. The resultant distributions were then visually compared. Figure 1 (below) plots each of the distributions for estimates of share betas for the same sample. A positive result would be a normally distributed set of betas, tightly distributed around the mean value, 1 (which represents the market beta).

To test the various distributions statistically, the Kolmogorov-Smirnov, Cramer-Von Mises and Anderson-Darling were performed to understand whether the estimated betas were normally distributed. Furthermore, the variance for each of the distributions was analysed to determine which method yielded the tightest share beta estimates around the mean, congruent to the method employed by Gonzalez *et al* (2014). Thereafter, a Levene's test was conducted to statistically verify that the variances were distinctly different across the alternate measures, consequently verifying the visual result.

As per the method conducted by Baker *et al* (2016), beta estimates using the VWAP method were also tested on a per share basis. This was achieved by comparing the distribution of standard beta across the 20 trading days against estimates of beta using the VWAP method, on the same histogram. To ensure a robust test of the VWAP beta method on a share by share basis, additional VWAP betas were generated from a non-overlapping, out-of-sample period. As conducted prior, differences were tested both visually and statistically. Distinct colours were used to distinguish between the spread of VWAP beta estimates and the standard estimates of beta, and the distributions were then analysed both visually and statistically, as done earlier. Congruent to Baker *et al* (2016), the research then reported on the proportion of the 136 sample shares for which the VWP method yielded more robust estimates of systematic risk.

4.6.4 Understanding whether the VWAP betas adhere to the tenets of the CAPM

To consolidate the findings between the standard beta and VWAP beta, the VWAP beta must adhere to the underlying tenet of the CAPM, which states that higher betas would lead to higher returns, and vice versa. This was investigated by means of a graphical time series style analysis, congruent to the methodology employed by Muller and Ward (2013) and Ward and Muller (2012).

To conduct this, each share in the sample was ranked according to the magnitude of the 60-day ex-ante VWAP betas and placed into virtual portfolios in the form of quintiles. Portfolio 1 contained the shares with the highest VWAP betas whereas Portfolio 5 contains the shares with the lowest VWAP betas. The performance of each of the portfolios was thereafter simulated using the Ward and Muller (2015) style engine, which contained share returns data over the period 1985 to 2017. The simulation was then run over the sample period.

To ensure a robust analysis, each of the VWAP betas for shares within the ALSI are recalculated and ranked every three months. This was done in a similar fashion to that of Carter *et al* (2017) and ensured that the correct shares were placed in the appropriate portfolios according to their updated betas. The portfolios are rebalanced quarterly on this basis.

Consistent with Ward and Muller (2012), the graphical time series analysis also plots a price relative line to indicate the relationship between quintile 1 (which had shares with the highest VWAP betas) and quintile 5 (which had shares with the lowest VWAP betas). The graphical time series analysis also plots a price relative line between quintile 1 and the market. This acts as a control variable would in the investigation, and is used to test whether high VWAP beta shares lead to abnormal returns (i.e. returns more than what the market could provide), as specified by

the CAPM. The performance of the portfolios as well as the relationship between each portfolio and the market was then visually analysed.

4.7. Limitations

According to Singh (2006), the aspects of investigation which could not be controlled for, formed the basis for limitations in the research. As evidenced in similar investigations conducted by Gonzalez *et al* (2014), Acker and Duck (2007) and Dimitrov and Govindaraj (2007) who each independently selected samples from the S&P500 index, the findings emanating from the research may not be wholly generalizable to the population. Gilli and Schumann (2015) indicate that international equity markets have distinctive characteristics and may exhibit different reactions to observed financial phenomenon, which may lead to different results for research investigations across markets. Given that the research focused solely on the JSE, the findings from the research may have limited application for other markets.

Furthermore, companies with a significantly smaller market capitalization (such as those listed on the JSE Alternative Exchange) were not included as part of the sample. The findings regarding the existence of reference-day risk on the JSE ALSI may therefore not be generalizable to this proportion of the population, as these companies may exhibit distinctive characteristics under conditions of reference-day risk.

As much as presenting a constraint on the generalizability of the proposed research results, these limitations presented the case for further research in the future.

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