ANALYTICITY AND SPECTRAL PROPERTIES OF NONCOMMUTATIVE RICCI FLOW IN A MATRIX GEOMETRY

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ABSTRACT. We study a first variation formula for the eigenvalues of the Laplacian evolving under the Ricci flow in a simple example of a noncommutative matrix geometry, namely a finite dimensional representation of a noncommutative torus. In order to do so, we first show that the Ricci flow in this matrix geometry is analytic.

1. INTRODUCTION

In [8], Ricci flow was defined and studied in a simple example of a matrix geometry, namely in a finite dimensional representation of a noncommutative (or quantum) 2-torus. This was motivated by [2], which attempts to define Ricci flow in the usual infinite dimensional representation of the noncommutative 2-torus by using a first variation formula for the eigenvalues of the Laplace-Beltrami operator obtained in the classical case in [7].

In [8], however, the Ricci flow was defined more directly by a non-commutative version of the Ricci flow equation, with no reference to the spectrum of the Laplace-Beltrami operator or a first variation formula. In this paper the aim is to show that a first variation formula can in fact also be obtained for the Ricci flow as defined in [8].

The formula is obtained in Section 5. This is actually the second of the two main results of the paper. In order to prove it, we first need to show that the Ricci flow in [8] is analytic, which is the first of our main results, and is obtained in Section 3.

Section 2 briefly reviews the noncommutative Ricci flow from [8] in preparation for Section 3. The central object in Section 5 is the non-commutative Laplace-Beltrami operator. Section 4 presents this operator in the finite dimensional representation in analogy to the known Laplace-Beltrami operator in the infinite dimensional representation of the noncommutative torus.

The results of this paper contribute to showing that various properties of Ricci flow in classical (i.e. commutative) differential geometry can be systematically extended to a noncommutative example, indicating that Ricci flow can be sensibly studied in the noncommutative case. Secondly, the paper to some extent clarifies the similarities and differences between the approaches taken in [8] and [2] respectively.

As Ricci flow is of importance in differential geometry and related areas, it seems plausible that extensions of results on Ricci flow to the noncommutative case can ultimately be of value in noncommutative geometry and its applications. Keep in mind that Ricci flow originated as part of Hamilton's programme to prove the Poincaré conjecture [12], and that this programme was indeed later completed by Perelman in [24, 25, 26]. In Friedan's work at about the same time as [12], Ricci flow essentially also appeared as part of a low order approximation to the renormalization group equation of nonlinear sigma models in physics; see [10] for the initial paper, but for a clearer formulation see in particular [11, Section II.1]. These remarks clearly illustrate the power and range of applications of classical Ricci flow. Refer to [2, Section 6] for a brief discussion of the possibility of corresponding applications in the noncommutative case.

Since in the formulation used in [8] the usual partial differential equation in fact becomes a system of ordinary differential equations, one can use tools from linear algebra, systems of ordinary differential equations (including the case on a complex domain, rather than just on a real interval), complex analysis and perturbation theory of linear operators, to obtain results that in the infinite dimensional representation are far more difficult to prove or are, as yet, not accessible. An example of this is the convergence of the Ricci flow to the flat metric, shown in [8] using techniques from systems of ordinary differential equations, but which had not yet been obtained in [2]. In this paper we can use these techniques to derive the analyticity of the Ricci flow, and consequently of the eigenvalues and eigenvectors. Even in the classical case in [7], on the other hand, the existence of sufficiently smoothly parametrized eigenvalues and eigenvectors had to be assumed in order to obtain the first variation formula. The setting in [8] is also very concrete and should for example be amenable to numerical methods.

One should keep in mind though that the finite dimensional representation is in many ways far simpler than the infinite dimensional case, so we would not expect all these methods to extend easily to solve the corresponding problems in the latter case. Nevertheless, one can learn a lot about noncommutative geometry by studying simple examples, in particular in this case about noncommutative partial differential equations. It seems plausible that some of the ideas and techniques could be extended to more general situations, in either infinite dimensional

representations or other matrix geometries. This could include non-commutative versions of partial differential equations other than the Ricci flow. See for example the noncommutative heat equations subsequently studied in [20] and [21], in the same matrix geometry as in [8]. Also, see [28] for partial differential equations in the infinite dimensional representation of a noncommutative torus.

To conclude this introduction, we mention that the basic ideas regarding matrix geometries originated in [14] and [22]. Noncommutative geometry is, however, much broader (and older) than just the matrix case; see for example [4]. It remains an active field with much work still to be done. In particular, Ricci flow has not yet been extensively studied in the noncommutative case. Aside from the two papers on noncommutative Ricci flow mentioned above, we can only point to [5] and [30].

2. The Ricci flow

Here we review the definition of Ricci flow as studied in [8], including notation to be used throughout the paper.

Note that in the classical case, Ricci flow is given by [12]

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu},$$

where $g_{\mu\nu}$ is a metric on a differentiable manifold, $R_{\mu\nu}$ is the Ricci tensor, and t is a real variable ("time"). Restricting ourselves to surfaces and to metrics of the form $g_{\mu\nu} = c\delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Kronecker delta and c is some strictly positive function on the surface (a conformal rescaling factor), it can be shown that this equation becomes

$$\frac{\partial c}{\partial t} = (\partial_1\partial_1 + \partial_2\partial_2)\log c,$$

where ∂_1 and ∂_2 are the partial derivatives with respect to the coordinates x^1 and x^2 on the surface. This paper studies a noncommutative version of the latter equation.

First, we recall the matrix geometry we are going to work with. We consider two unitary $n \times n$ matrices u and v generating the algebra M_n of $n \times n$ complex matrices, and satisfying

$$vu = quv,$$

where

$$q = e^{2\pi i m/n}$$

for an $m \in \{1, 2, ..., n-1\}$ such that m and n are relatively prime. Note that $q^n = 1$, but $q^j \neq 1$ for $j = 1, ..., n-1$. We can for example

use

$$u = \begin{bmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{n-1} \end{bmatrix}$$

and

$$v = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix},$$

where the blank spaces are filled with zeroes. It is straightforward to check that the commutant of the set $\{u, v\}$ only consists of scalar multiples of the identity matrix I, so u and v indeed generate M_n. These matrices appeared in physics at least as far back as [29], in relation to quantum mechanics, but the geometric interpretation in terms of a noncommutative (or "fuzzy") torus seems to have only come later; see for example [22, Section 2].

For any two Hermitian matrices x and y such that

$$u = e^{\frac{2\pi i}{n} x}$$

and

$$v = e^{\frac{2\pi i}{n} y},$$

we define derivations δ_1 and δ_2 on M_n by the commutators

$$\delta_1 := [y, \cdot]$$

and

$$\delta_2 := -[x, \cdot],$$

which are analogues of the partial derivatives $\frac{1}{i}\partial_\mu$ in the classical case (please refer to [8, Section 2] for a discussion of this). Note that such x and y exist but are not uniquely determined by u and v. However, our analysis will not depend on the choices made. The existence of x is clear for the diagonal u above. For v above, we first diagonalize using the Fourier transform for the group $\mathbb{Z}_n$, obtain a y in this basis, and then transform back to the original basis.

Note that M_n is an involutive algebra, i.e. a $*$-algebra, with the involution given by the usual Hermitian adjoint of a matrix a, denoted by a^*. In the operator norm, M_n is a unital C*-algebra. Although the theory of C*-algebras will not appear in this paper, the C*-algebraic point of view helps to connect what we do in this paper with the setting used in [2] and [6]. Furthermore, we for the most part only need abstract properties of the derivations (like those listed in [8, Proposition 2.1]), rather than the explicit definitions of u and v, and δ_1 and δ_2, given above. The exceptions are one point in each of the proofs of Theorems 3.1 and 5.1, where we do use the fact that the derivations

are given by commutators with some fixed matrices. To emphasize this abstract point of view, we usually rather use the notation

$$A = M_n$$

for our unital $*$-algebra. The unit of A is the $n \times n$ identity matrix I.

Using the derivations above, we define a noncommutative analogue of a Laplacian as an operator on A:

(1) $$\triangle := \delta_1^2 + \delta_2^2,$$

i.e. $\triangle a = [y, [y, a]] + [x, [x, a]]$ for all $a \in A$.

The Hilbert-Schmidt inner product

$$\langle a, b \rangle := \tau \left(a^* b \right)$$

on A then becomes relevant. Here τ denotes the usual trace on M_n, i.e. the sum of the diagonal entries of a matrix. One can interpret $\tau(a)$ as a noncommutative integral of the complex-valued "function" a, corresponding in the classical case to the integral over the flat torus.

We denote the Hilbert space $(A, \langle \cdot, \cdot \rangle)$ by H.

Note that the Laplacian

$$\triangle : H \to H$$

is a positive operator (see [8, Proposition 2.1]), so it corresponds to $-(\partial_1 \partial_1 + \partial_2 \partial_2)$ in the classical case.

Now we turn to noncommutative metrics.

For $a \in A$, we write

$$a > 0$$

if a is a positive operator, i.e. if it can be written as $a = b^* b$ for some $b \in A$, and in addition 0 is not an eigenvalue of a. In other words, $a > 0$ means that a is a Hermitian $n \times n$ matrix whose eigenvalues are strictly positive. We also write

$$P = \{a \in A : a > 0\}$$

to denote the set of all these elements.

A *noncommutative metric* is any $c \in P$. This is the noncommutative version of the metric $c\delta_{\mu\nu}$ in the classical case above. Given any $c \in P$, we also consider the Hilbert space H_c given by the inner product

(2) $$\langle a, b \rangle_c := \varphi \left(a^* b \right)$$

on the vector space A, where

(3) $$\varphi(a) := \tau(ca)$$

for all $a \in A$. The positive linear functional φ is a noncommutative version of the integral over a curved surface (with metric given by a conformal rescaling factor as above) in the classical case, a point which becomes relevant in Section 5.

This setup for noncommutative metrics is adapted from the infinite dimensional representation as studied in [6]. The Hilbert space H_c

will play a central role in Sections 4 and 5, where we work with the noncommutative version of the Laplace-Beltrami operator, which is essentially a Laplacian on a curved noncommutative space.

A noncommutative version of the classical Ricci flow equation above is the following:

$$(4) \qquad \frac{d}{dt}c(t) = -\triangle \log c(t),$$

where $c(t) \in P$ denotes the metric at time t. Here we define $\log a$ of an $a \in P$ by diagonalizing a, applying log to each of the diagonal entries, and then returning to the original basis (this is the Borel functional calculus in finite dimensions). In this sense we can view log as the real logarithm applied to elements of P. Below, when studying the analyticity of this Ricci flow, we also use the principal complex logarithm on $D := \mathbb{C}\backslash(-\infty, 0]$, denoted by Log, and applied to the larger set of matrices whose spectra are in D.

3. Analyticity of the Ricci flow

In [8, Section 3], we used the theory of systems of differential equations to show that, given initial conditions, the Ricci flow equation (4) has a unique solution c and that this solution is C^1, i.e. it is differentiable and its derivative is continuous. Here we go further and obtain the first of the main results of this paper, namely that the Ricci flow is in fact analytic. We use the theory of systems of differential equations on a complex domain to do so. We collect these results as follows:

Theorem 3.1. *Let $c_{t_0} \in P$ be any initial noncommutative metric at the initial time $t_0 \in \mathbb{R}$. Then the noncommutative Ricci flow, Eq. (4), has a unique C^1 solution c on any interval $[t_0, t_1]$ with $t_1 \geq t_0$, and also on the interval $[t_0, \infty)$, such that $c(t_0) = c_{t_0}$. In addition, such a solution is necessarily analytic, i.e. at each point $t_2 \in [t_0, \infty)$ there is a number $\varepsilon > 0$ such that each entry in the matrix $c(t) \in P$ is a power series in $t - t_2$ for all t in the interval $[t_0, \infty) \cap (t_2 - \varepsilon, t_2 + \varepsilon)$.*

Proof. We first consider the C^1 property globally, and afterwards we study analyticity locally. Although the C^1 property was considered in [8], we again look at certain aspects carefully and in more detail here, since the results we obtain on the way are subsequently used in proving analyticity.

We start by looking at the properties of Log when applied to matrices, and viewed as a function of several complex variables. Below we define $\text{Log}\, a$ for all a in the open set $B := \{a \in A : \sigma(a) \subset D\}$, where $\sigma(a)$ is the spectrum of a, and $D = \mathbb{C}\backslash(-\infty, 0]$. That B is indeed open in A, follows from fact that the eigenvalues of a matrix depend continuously on the matrix, since the roots of a polynomial depend continuously on the coefficients of the polynomial (see for example [23,

Section I.1]). By using the analytic functional calculus, we set

$$(5) \qquad \mathrm{Log}\, a := \frac{1}{2\pi i} \int_{\Gamma} (zI - a)^{-1} \mathrm{Log}\, z\, dz$$

for any positively oriented simple closed smooth contour Γ in D, surrounding $\sigma(a)$, for all $a \in B$. This definition is independent of Γ. See for example [1, Section III.3] for further details. Note that because of the Cauchy integral formula, for $a > 0$ this definition corresponds to the definition for $\log a$ discussed at the end of the previous section.

The inverse $\cdot^{-1} : \mathrm{Inv}(A) \to \mathrm{Inv}(A) : a \mapsto a^{-1}$, on the open set $\mathrm{Inv}(A)$ of invertible matrices, is differentiable in each entry $a_{jk} \in \mathbb{C}$ of the matrix a being inverted. One can see this from the formula for the inverse obtained from Cramer's rule. So each entry in a^{-1} is an analytic complex function of each entry of a separately. Therefore, by the Hartogs theorem (see for example [15, Theorem 2.2.8]), each entry in the matrix a^{-1} is an analytic function in several complex variables, the variables being the entries of the matrix a. The derivatives of these analytic functions are therefore themselves analytic, and hence continuous functions. Because of this we can differentiate Eq. (5) under the integral with respect to the entries of a (refer for example to [19, Theorem VIII.6.A3]), to see that each entry in the matrix $\mathrm{Log}\, a$ is (again by the Hartogs theorem) an analytic function in several complex variables. Since $\triangle$ is given by commutators with certain matrices, we immediately also know that the entries of the matrix $\triangle \mathrm{Log}\, a$ are each analytic functions in several complex variables, the variables still being the entries of a.

In particular, $a \mapsto \triangle \log a$ is a C^1 function on P, which means (by the theory of systems of ordinary differential equations; see for example [3, Section 1.1]) that the initial value problem for Eq. (4) has a unique C^1 solution on any interval for which the solution stays in P. This has already been discussed in [8, Section 3], where in particular it was shown that the solution remains in P for all $t \geq t_0$, i.e. it exists (and is necessarily unique) on $[t_0, \infty)$.

However, we are now interested in the analyticity of this solution c. We approach this problem using the theory of systems of ordinary differential equations on a complex domain.

For any $t_2 \geq t_0$, consider the system of differential equations given in matrix form by

$$\frac{d}{dz} w(z) = -\triangle \mathrm{Log}\, w(z)$$

where w is required to be a function on some neighbourhood of t_2 in $\mathbb{C}$, with values in the open set B consisting of matrices $a \in A$ such that $\sigma(a)$ lies in D. I.e. the values of w should be in the domain of Log viewed as a matrix function as defined above.

Because of the analyticity of $a \mapsto \triangle \operatorname{Log} a$ shown above, and the fact that it is consequently locally Lipschitz (see for example [18, Section 6.3]), it follows by [13, Theorem 2.2.2] that, on a small enough open disc in $\mathbb{C}$ of radius ε around t_2, this system has a unique analytic solution such that $w(t_2) = c(t_2)$. I.e. on this disc the entries of $w(z)$ are analytic functions, and therefore have power series expansions on this disc.

Restricting such a solution to real elements $[t_0, \infty) \cap (t_2 - \varepsilon, t_2 + \varepsilon)$ in the disc, we necessarily obtain our solution c with values in P discussed above on this interval around t_2. The reason for this is that the system

$$\frac{d}{dt} w(t) = -\triangle \operatorname{Log} w(t)$$

for real t, has a unique solution by the theory of systems of ordinary differential equations on a real interval, just as for the case of c above. So given the condition $w(t_2) = c(t_2)$, such a solution of w must in fact be the solution we already have, namely c, on the interval in question. But the restriction of the solution on a complex domain to $[t_0, \infty) \cap (t_2 - \varepsilon, t_2 + \varepsilon)$ mentioned above is exactly such a solution, hence on this interval it is indeed c.

This means that the entries of c have power series expansions at each t_2, and therefore c is analytic. $\qquad\square$

In addition, [8, Theorem 3.2] also showed the convergence of c to the flat metric (proportional to the identity matrix I) as t goes to infinity, as well as monotonicity of the determinant and preservation of the trace under the flow. However, analyticity of the Ricci flow is the property of fundamental importance in obtaining a first variation formula for the eigenvalues of the Laplace-Beltrami operator in Section 5.

4. The Laplace-Beltrami operator

In the classical case, one can define a Laplacian that incorporates the metric, the so-called Laplace-Beltrami operator. For the classical metric $g_{\mu\nu} = c\delta_{\mu\nu}$ mentioned in Section 2, the Laplace-Beltrami operator is of the form

$$-\frac{1}{c}(\partial_1\partial_1 + \partial_2\partial_2),$$

if we use the convention that it should be a positive operator. (Often the minus sign is dropped, then minus the Laplace-Beltrami operator would be a positive operator.) Note that it reduces to the Laplacian $-(\partial_1\partial_1 + \partial_2\partial_2)$ for the flat metric $c = I$.

We need to write down a suitable noncommutative version of this operator, which should have a similar form and similar properties. In particular, it should also be a positive operator, and it should reduce to the noncommutative Laplacian $\triangle$ for the flat metric $c = I$. A natural choice is the operator

$$(6) \qquad\qquad \triangle_c : H_c \to H_c : a \mapsto (\triangle a)c^{-1}$$

for any metric $c \in P$, where the product of $\triangle a$ and c^{-1} is taken in A. Keep in mind from Section 2 that $H_c = (A, \langle \cdot, \cdot \rangle_c)$. Note that the operator $\triangle_c : H_c \to H_c$ is indeed positive, since

$$\langle a, \triangle_c a \rangle_c = \tau(ca^*(\triangle a)c^{-1}) = \tau(a^* \triangle a) = \langle a, \triangle a \rangle \geq 0$$

by the fact that $\triangle : H \to H$ is positive (see [8, Proposition 2.1]), where $H = H_I$ as in Section 2. The operator $\triangle_c$ also clearly reduces to $\triangle$ when $c = I$. We therefore use (6) as our definition of the Laplace-Beltrami operator.

Note that the alternative definition $c^{-1}\triangle a$ for $\triangle_c a$, which may at first seem to be the more obvious choice, would fail, since it would not guarantee positivity of $\triangle_c$.

The right multiplication by c^{-1} in the definition of the Laplace-Beltrami operator also appears in the infinite dimensional representation of the noncommutative 2-torus. See for example [9, Remark 2.2].

In the next section it will also be convenient to represent $\triangle_c$ on H instead of H_c. To do this we define a unitary operator

$$U_c : H_c \to H : a \mapsto ac^{1/2}$$

where the product $ac^{1/2}$ is taken in the algebra A. Note that this is indeed unitary, since

$$\langle U_c a, U_c b \rangle = \tau((ac^{1/2})^* bc^{1/2}) = \tau(ca^* b) = \langle a, b \rangle_c$$

for all $a, b \in H_c$, and U_c is invertible, since $c^{1/2}$ is. We can therefore represent $\triangle_c$ on H by the positive operator

$$\bar{\triangle}_c := U_c \triangle_c U_c^* : H \to H,$$

for which we have

(7) $$\bar{\triangle}_c a = (\triangle(ac^{-1/2}))c^{-1/2}$$

for all $a \in H$.

5. The first variation formula

This section presents our second main result, namely a version in our context of the classical first variation formula obtained in [7, Corollary 2.3] for the eigenvalues of the time-dependent Laplace-Beltrami operator given by the classical Ricci flow. The analyticity of the noncommutative Ricci flow, given by Theorem 3.1, will be used in proving this result. We then discuss this first variation formula in relation to [2] and the classical case.

Keep in mind from Section 2 that τ is the trace on M_n, that $\triangle$ is the flat Laplacian given by Eq. (1), and that the Hilbert space H_c is defined by the inner product in Eq. (2). To make some expressions easier to read, we denote $c(t)$ also by

$$c_t := c(t).$$

In terms of this notation, we can formulate our second main result as follows:

Theorem 5.1. *Let c be the Ricci flow on $[t_0, \infty)$ as given by Theorem 3.1. Then the eigenvalues and normalized eigenvectors of $\triangle_{c_t}$ can be obtained as analytic functions of t, and for each such eigenvalue λ_t and a corresponding normalized eigenvector $a_t \in H_{c_t}$, we have the first variation formula*

$$\frac{d\lambda_t}{dt} = \lambda_t \tau(|a_t|^2 \triangle \log c_t),$$

where $|a_t|^2 := a_t^* a_t$, for all $t \in [t_0, \infty)$. (At $t = t_0$, this derivative can be viewed as the right-hand derivative.)

Proof. We are going to apply perturbation theory of linear operators to $\triangle_{c(t)}$. In order to do so, we work via $\bar{\triangle}_{c(t)} : H \to H$ as defined in the previous section, since then we have an operator on the same space H for all t. In order to apply perturbation theory (see [16, Chap. VII, Sections 2 and 3]), we show that $t \mapsto \bar{\triangle}_{c(t)}$ can be extended to a neighbourhood of $[t_0, \infty)$ in $\mathbb{C}$, i.e. to an open set in $\mathbb{C}$ containing $[t_0, \infty)$. More precisely, we need the extension of $t \mapsto \bar{\triangle}_{c(t)} a$ to be analytic for all $a \in A$.

Because of Eq. (7), we start by showing that $t \mapsto c(t)^{-1/2}$ is analytic: $t \mapsto c(t)$ is analytic by Theorem 3.1, and therefore at each point t_1 in $[t_0, \infty)$ has a power series expansion for each of its entries which can be used to extend these entries to analytic complex functions on a neighbourhood (a disc) N of t_1 in $\mathbb{C}$, giving us a complex matrix valued function $z \mapsto w(z)$ extending $t \mapsto c(t)$ to N. Secondly, the entries of a^{-1} are complex analytic functions of several complex variables (the entries of a), for a in the set of invertible elements $\text{Inv}(A)$ of A, as mentioned in the proof of Theorem 3.1. Thirdly, the entries of the square root $a^{1/2}$, for $a \in A$ whose spectrum $\sigma(a)$ lies in $D = \mathbb{C} \backslash (-\infty, 0]$, are complex analytic functions of several complex variables (the entries of a), using the same argument as for $\text{Log}\, a$ in the proof of Theorem 3.1. (Here we are using the branch of the square root given by $e^{\frac{1}{2} \text{Log}}$, and we express $a^{1/2}$ in terms of the analytic functional calculus.) Choosing the radius of N small enough, we can ensure that $\sigma(w(z)) \subset D$, so $w(z)^{-1}$ exists and $\sigma(w(z)^{-1}) \subset D$, for $z \in N$. This is because $\sigma(c(t_1)) \subset (0, \infty) \subset D$, and for N small enough, each element of $\sigma(w(z))$ will be as close to some element of $\sigma(c(t_1))$ as we require, simply because w is continuous and $w(t_1) = c(t_1)$. Here we have again used the fact that the eigenvalues of a matrix depend continuously on the entries of the matrix, as mentioned in the proof of Theorem 3.1. Hence the composition of the above mentioned three functions, namely $z \mapsto (w(z)^{-1})^{1/2} = w(z)^{-1/2}$ on N, is well-defined, and its entries are complex differentiable (because of the above mentioned analyticity of the three functions) and therefore analytic.

It follows that the entries of $z \mapsto \bar{\triangle}_{w(z)} a := (\triangle(aw(z)^{-1/2}))w(z)^{-1/2}$ are differentiable with respect to z, and therefore analytic on N. Thus its restriction $t \mapsto \bar{\triangle}_{c(t)} a$ to the real line is also analytic.

Because we are working in finite dimensions, the fact that $t \mapsto \bar{\triangle}_{c(t)} a$ is analytic for all a, is equivalent to the entries of $\bar{\triangle}_{c(t)}$ (viewed as an $n^2 \times n^2$ matrix acting on an n^2 dimensional space) being analytic. One can now analytically extend it to a neighbourhood of $[t_0, \infty)$. To do this we consider power series expansions for the entries of the $n^2 \times n^2$ matrix $\bar{\triangle}_{c(t)}$ on a neighbourhood in $\mathbb{R}$ of each $t_1 \in [t_0, \infty)$, allowing us to define an analytic extension $z \mapsto T(z)$ of $t \mapsto \bar{\triangle}_{c(t)}$ to a disc in $\mathbb{C}$ around t_1, for each $t_1 \in [t_0, \infty)$. Since the extensions on two overlapping discs are equal on a non-empty open interval in the real line, they are equal on the overlap because of analyticity. Hence we have an analytic extension $z \mapsto T(z)$ of $t \mapsto \bar{\triangle}_{c(t)}$ to a neighbourhood of $[t_0, \infty)$, namely to the union D_0 of all these discs. Note that D_0 is symmetric around the real axis.

To apply the results from [16, Chap. VII, Sections 2 and 3], we furthermore need $T(z)^* = T(\bar{z})$ to hold. To see that this is indeed true, represent H as $\mathbb{C}^{n^2}$ with its usual inner product, in other words we choose some orthonormal basis in H. Then we can represent $T(z)$ as an $n^2 \times n^2$ matrix, such that $T(z)^*$ is simply the usual Hermitian adjoint of $T(z)$, i.e. transpose and entrywise complex conjugation. By analyticity, at each $t_1 \in [t_0, \infty)$ each entry of $T(z)$ is a power series in $z - t_1$ for all z in some disc N centered at t_1, with radius only depending on t_1, since we can use the smallest radius that still works for all entries. Say the power series for the (k, l) entry of $T(z)$ is given by

$$T(z)_{kl} = \sum_{j=0}^{\infty} m_{jkl}(z - t_1)^j$$

for all $z \in N$ and all (k, l). Then in particular the (k, l) entry of $\bar{\triangle}_{c(t)}$ in the same representation is $(\bar{\triangle}_{c(t)})_{kl} = \sum_{j=0}^{\infty} m_{jkl}(t - t_1)^j$ for $t \in N \cap [t_0, \infty)$. But $\bar{\triangle}_{c(t)}$ is self-adjoint, since it is a positive operator because of its definition in the previous section. It follows that

$$\bar{m}_{jlk} = m_{jkl},$$

from which it in turn follows that $T(z)^* = T(\bar{z})$, as required.

By the perturbation theory of linear operators, in particular [16, Chap. VII, Theorem 3.9], we now conclude that the eigenvalues and normalized eigenvectors of $\bar{\triangle}_{c(t)} : H \to H$ can be parametrized as analytic functions of $t \in [t_0, \infty)$. (Also see [17, Theorem (A)] for a review, and [27] for the original literature.) Given such an eigenvalue λ_t and a corresponding normalized eigenvector $\bar{a}_t$, it follows that λ_t is also an eigenvalue of $\triangle_{c(t)}$, and that $a_t := U_{c(t)}^* \bar{a}_t = \bar{a}_t c_t^{-1/2} \in H_{c(t)}$ is a corresponding normalized eigenvector. Since both $\bar{a}_t$ and $c_t^{-1/2}$ are

analytic functions of t, the same is true of a_t. So, we have transformed back to the representation on $H_{c(t)}$, to see that the eigenvalues and normalized eigenvectors of $\triangle_{c(t)} : H_{c(t)} \to H_{c(t)}$ can be parametrized as analytic functions of $t \in [t_0, \infty)$. (In [7, Section 1] a result of this form was assumed without proof.)

Ignoring time dependence for the moment, consider a metric $c \in P$, and let λ be any eigenvalue of $\triangle_c$, with $a \in H_c$ a corresponding, but not necessarily normalized, eigenvector, i.e. $\triangle_c a = \lambda a$, then we have

$$a^*(\triangle a)c^{-1} = \lambda a^* a$$

from which

$$\lambda = \frac{\tau(a^* \triangle a)}{\tau(ca^* a)}$$

follows. Here $\tau(ca^* a) = \langle a, a \rangle_c = \|a\|_c^2 \neq 0$ is the norm squared of the eigenvector a in the Hilbert space H_c. (This step in our proof is closely related to the approach taken in [7]. The rest of our proof, however, is rather different from the proof of the classical first variation formula given in [7].)

Now consider any eigenvalue λ_t and corresponding eigenvector a_t of $\triangle_{c(t)}$, both analytic in t, but with the eigenvector not necessarily normalized. Denoting time derivatives by $\dot{\lambda}_t = d\lambda_t/dt$, and similarly for other time-dependent objects, we obtain

$$(8) \quad \dot{\lambda}_t = \frac{\tau(\dot{a}_t^* \triangle a_t + a_t^* \triangle \dot{a}_t)}{\tau(c_t a_t^* a_t)} - \frac{\tau(a_t^* \triangle a_t)}{\tau(c_t a_t^* a_t)^2} \tau(\dot{c}_t a_t^* a_t + c_t \dot{a}_t^* a_t + c_t a_t^* \dot{a}_t)$$

where we have used the fact that $\triangle$ is defined in terms of commutators with some fixed operators, and therefore the time derivative can be taken over $\triangle$.

Let us now specialize to the case of a normalized eigenvector a_t, as obtained above. Since Eq. (8) holds for an eigenvector in general, it in particular holds when it is normalized, i.e. when $\tau(c_t a_t^* a_t) = 1$. So, using the fact that $\triangle : H \to H$ is Hermitian, as well as the eigenvalue equation for $\triangle_{c(t)}$ in the form $\triangle a_t = \lambda_t a_t c_t$, and lastly the Ricci flow equation, namely Eq. (4), we obtain

$$\begin{aligned}
\dot{\lambda}_t &= \tau(\dot{a}_t^* \triangle a_t + a_t^* \triangle \dot{a}_t) - \tau(a_t^* \triangle a_t)\tau(\dot{c}_t a_t^* a_t + c_t \dot{a}_t^* a_t + c_t a_t^* \dot{a}_t) \\
&= \tau(\dot{a}_t^* \triangle a_t + (\triangle a_t)^* \dot{a}_t) - \lambda_t \tau(\dot{c}_t a_t^* a_t + c_t \dot{a}_t^* a_t + c_t a_t^* \dot{a}_t) \\
&= \lambda_t \tau(\dot{a}_t^* a_t c_t + c_t a_t^* \dot{a}_t) - \lambda_t \tau(\dot{c}_t a_t^* a_t + c_t \dot{a}_t^* a_t + c_t a_t^* \dot{a}_t) \\
&= -\lambda_t \tau(\dot{c}_t a_t^* a_t) \\
&= \lambda_t \tau(a_t^* a_t \triangle \log c_t),
\end{aligned}$$

as required. $\qquad\square$

We note that this formula can also be written as

$$(9) \qquad \frac{d\lambda_t}{dt} = \lambda_t \varphi_t(|a_t|^2 \triangle_{c_t} \log c_t),$$

where $\varphi_t(a) := \tau(c_t a)$ for all $a \in A$. The significance of this is that $\triangle_{c_t} \log c_t = (\triangle \log c_t) c_t^{-1}$ corresponds exactly to the formula for scalar curvature in the classical case. Therefore Eq. (9) appears to be a reasonable analogue of the classical formula

$$\frac{d\lambda_t}{dt} = \lambda_t \int f_t^2 R_t d\mu_t$$

obtained in [7, Corollary 2.3], where R_t is the classical scalar curvature, f_t is an eigenfunction of the classical Laplace-Beltrami operator at time t, and the integral $\int (\cdot) d\mu_t$ over the surface in question corresponds to the positive linear functional φ_t. However, it should also be pointed out that $(\triangle \log c_t) c_t^{-1}$ is not a sensible noncommutative scalar curvature. It can for example not even be expected to be a Hermitian element of A. See for example [8, Section 5] for how one can define a more sensible noncommutative scalar curvature in this context from the noncommutative Ricci flow. Therefore the analogy between Eq. (9) and the classical case is not perfect, since Eq. (9) is not in terms of scalar curvature, but rather in terms of a noncommutative object having a form similar to the classical scalar curvature.

This result also indicates some similarity between our setting for Ricci flow, and that of [2], where the first variation formula is used as the basis for the noncommutative Ricci flow. However, there a much more complicated object is used in the place of the classical R; see in particular [2, Theorem 3.5].

In [7] the existence of eigenvalues and eigenvectors which are sufficiently smooth (namely C^1) in t is assumed, rather than proved. It should be added, though, that in that paper classical manifolds more general than just surfaces are considered, so such assumptions may be unavoidable there. In [7, Section 1], it is also mentioned that for the classical Ricci flow one can not in general expect analyticity in t.

In [2] it appears that an analogous assumption of sufficient smoothness is made in the case of the noncommutative torus (in the infinite dimensional representation). It is however not clear how the time-dependent eigenvectors in [2] arise in the first place, since there the Ricci flow is defined and studied by expressing the time-derivative of an eigenvalue as a type of first variation formula involving the corresponding eigenvector, without further equations from which the time-dependence of the eigenvector can be obtained.

We now conclude with further general remarks on Theorem 5.1 in relation to the classical case:

In the classical case, one can work with real-valued eigenfunctions f, which in the noncommutative case correspond to Hermitian elements of A. However, noncommutativity appears to prohibit restricting ourselves to Hermitian elements of A as eigenvectors. The reason for this is that if we split a not necessarily Hermitian eigenvector a

of $\triangle_c$ into its Hermitian parts, i.e. $a = a_1 + ia_2$, with $a_1, a_2 \in A$ Hermitian, the noncommutativity makes it impossible to show that a_1 and a_2 (or whichever of them are not zero) are eigenvectors, since $(\triangle_c a)^* = c^{-1}\triangle a_1 - ic^{-1}\triangle a_2$, which need not be equal to $\triangle_c a_1 - i\triangle_c a_2$. This is unlike the classical case, where the real and imaginary parts (at least whichever of them are not zero) of an eigenfunction are in fact themselves eigenfunctions, as is easily verified. This is why we have to work with $|a_t|^2$ rather than a_t^2 in the place of f_t^2.

In the proof of the classical first variation formula in [7], the eigenfunctions (other than 1) of the Laplace-Beltrami operator for a given metric average to zero, i.e.

$$\int f d\mu = 0,$$

where $\int (\cdot) d\mu$ is the integral over the manifold. We did not explicitly use the noncommutative version of this fact in the proof above, but it is interesting to note that it is indeed true. To see this, consider any eigenvalue λ and corresponding eigenvector $a \in H_c$ of $\triangle_c$, for any metric $c \in P$. Then $\triangle a = \lambda ac$, from which (see Eq. (3)) we have $\lambda\varphi(a) = \tau(\triangle a) = 0$. When $\lambda = 0$, we necessarily have that a is proportional to I, since $\ker\triangle = \mathbb{C}I$ by [8, Proposition 2.1], showing that

$$\ker\triangle_c = \mathbb{C}I$$

as well. On the other hand, for all other eigenvectors a, i.e. those not proportional to I, we have $\lambda \neq 0$, so

$$\varphi(a) = 0$$

in exact analogy to the classical case.

Acknowledgments

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