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Portfolio risk measures and option pricing under a Hybrid Brownian motion model

by

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae in Financial Engineering at the University of Pretoria, is my own work and has not been submitted by me for a degree at this or any other tertiary institution.

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Abstract

The 2008/9 financial crisis intensified the search for realistic return models, that capture real market movements. The assumed underlying statistical distribution of financial returns plays a crucial role in the evaluation of risk measures, and pricing of financial instruments. In this dissertation, we discuss an empirical study on the evaluation of the traditional portfolio risk measures, and option pricing under the hybrid Brownian motion model, developed by Shaw and Schofield. Under this model, we derive probability density functions that have a fat-tailed property, such that "25σ" or worse events are more probable. We then estimate Value-at-Risk (VaR) and Expected Shortfall (ES) using four equity stocks listed on the Johannesburg Stock Exchange, including the FTSE/JSE Top 40 index. We apply the historical method and Variance-Covariance method (VC) in the valuation of VaR. Under the VC method, we adopt the GARCH(1,1) model to deal with the *volatility clustering* phenomenon. We backtest the VaR results and discuss our findings for each probability density function.

Furthermore, we apply the hybrid model to price European style options. We compare the pricing performance of the hybrid model to the classical Black-Scholes model.

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Symbol / Abbreviation	Informal Definition
$\mathbb{E}$	expectation operator under some measure.
$\sim, \approx$	equivalent, approximately.
$\mathcal{O}$	order of approximation.
$\mathbb{P}$	real world probability measure.
$\mathbb{Q}$	martingale measure or risk-adjusted measure.
C	closed contour.
$i \in \mathbb{C}$	complex number.
Ω	set of all possible outcomes.
$\mathcal{F}$	$\sigma-$algebra.
W_t	standard Wiener or Brownian process at time t.
S_t	share price at time t.
exp	exponential function.
$\text{Var}(X)$	variance of random variable X.
$\text{cov}(X, Y)$	covariance of random variables X and Y.
$\ln x$	natural logarithm of x.
$\mathbb{P}(A)$	probability of random event A.
$\ll$ or $\gg$	very small, and very large respectively.
$\mathcal{L}$ & $\mathcal{L}^{-1}$	Laplace transform and inverse Laplace transform respectively.
$\mathscr{F}$ & $\mathscr{F}^{-1}$	Fourier transform and inverse Fourier transform respectively.
χ	Chi-square distribution.
σ_1	the fundamental volatility related to the economy.
σ_2	the technical volatility related to the economy.

Table 1: List of Symbols and Abbreviations.

Symbol / Abbreviation	Informal Definition
Γ	Gamma function.
$\sigma(X)$	standard deviation of a random variable X.
μ_1	the fundamental drift.
μ_2	the technical drift.
K	strike (exercise price).
r	risk-free rate.
T	maturity (expiry date).
V^h	the value process corresponding to the portfolio h.
λ	the market price of risk, unless if specified otherwise.
min	minimum.
max	maximum.
$P\&L$	profit and loss.
$Corr(X,Y)$	correlation between random variables X and Y.
GARCH	Generalised Autoregressive Conditional Heteroscedasticity.
ODE	Ordinary Differential Equation.
SDE	Stochastic Differential Equation.
PDF	Probability Distribution Function.
CDF	Cumulative Distribution Function.
PDE	Partial Differential Equation.

Table 2: List of Symbols and Abbreviations.

Glossary

- The random variables $X_1, X_2, \ldots, X_n$ are said to be independent and identically distributed (i.i.d), if they share the same probability density function and are independent of each other.

- Probability generating function (PGF)
  Let X be a discrete random variable taking non-negative integers $\{0, 1, 2, \ldots\}$. The PGF of X is given by

$$\begin{aligned} G_X(z) &= \mathbb{E}[z^X] \\ &= \sum_{x=0}^{\infty} z^x \mathbb{P}(X = x) = \sum_{x=0}^{\infty} z^x p(x) \end{aligned}$$

  where $\mathbb{P}(X = x) = p(x)$.

  Properties of a PGF

  - $G_X(0) = p(0)$.
  - $G_X(1) = 1$.
  - $p(n) = \left(\frac{1}{n!}\right)\frac{d^n}{dz^n}\left(G_X(z)\right)\Big|_{z=0}$.
  - $\mathbb{E}[X] = G'_X(1)$.
  - $\mathrm{Var}[X] = G''_X(1) + G'_X(1) - (G'_X(1))^2$.

- Supremum (sup) definition: Let X be an ordered set, and let $A \subseteq X$ be non-empty and bounded above. We say A has a smallest upper bound or an supremum, if there is a point in X, denoted by $\sup(A)$ such that the $\sup(A)$ is an upper bound for A, and if there exist another upper bound for A say b, then $\sup(A) \leq b$.

- Infimum (inf) definition: Let X be an ordered set, and let $A \subseteq X$ be non-empty and bounded below. We say A has a greatest lower bound or an infimum, if there is a point in X, denoted by $\inf(A)$ such that the $\inf(A)$ is a lower bound, and if there exist another lower bound for A say b, then $\inf(A) \geq b$.

- The k-period (say daily) log-return at time t is given by, $\log\left(\frac{S_t}{S_{t-k}}\right)$.
- The distribution function of a random variable X is defined by $F_X(x) = \mathbb{P}(X \leq x)$, where F_X is a cumulative function of X.
- Value-at-Risk (VaR)

  Given some confidence level $\alpha \in (0,1)$. The VaR of a portfolio at the confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is not larger than $(1-\alpha)$. Mathematically we write

$$VaR_\alpha = \inf\{l \in \mathbb{R} : \mathbb{P}(L > l) \leq 1-\alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\},$$

  where F_L is the loss distribution.

- Expected Shortfall (ES)

  For a loss L with $\mathbb{E}(|L|) < \infty$ and the distribution function F_L, the expected shortfall at the confidence level $\alpha \in (0,1)$ is defined as

$$\begin{aligned} ES_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du \\ &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du \end{aligned}$$

  where $q_u(F_L)$ is the quantile function of F_L.

- Backtesting is a method used to validate the VaR model, by periodically comparing the estimated VaR value to the observed P& L.
- Central moments of a continuous random variable:

  Let X be a continuous random variable with a density function $f(x)$. We define the r^{th} central moments as

$$\mu_r = \mathbb{E}[(X-\mu_X)^r] = \int_{-\infty}^{\infty}(x-\mu_X)^r f(x)dx,$$

  where r is a positive integer greater than one, and $\mu_X = \mathbb{E}[X]$.

- The skewness of a random variable X is a measure of symmetry of the probability distribution, defined by the ratio

$$\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma_X^3}.$$

- The kurtosis of a random variable X is a measure of heaviness of the tails of the distribution function, defined by the ratio

$$\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma_X^4}.$$

- A leptokurtic distribution is a statistical distribution with heavier or fatter tails than the Normal distribution.

- One dimensional Laplace Transform

  The Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}\{f(s)\} = \int_0^\infty e^{-st} f(t)dt,$$

  where t is a real variable, and $s \in \mathbb{C}$ is a complex variable.

- A characteristic function: Let X be a random variable with a CDF F_X and a PDF $f_X(x)$. The characteristic function is defined as

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx = \int_{-\infty}^{\infty} e^{itx} dF_X(x)\cdot$$

- Fourier Transform

  A Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ (integrable) is defined as

$$\mathscr{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(x)dt\cdot$$

  and the inverse Fourier transform is defined as

$$\mathscr{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} F(\omega)d\omega\cdot$$

  for any real number ω and x.

- Unit step function or Heaviside step function is a discontinuous function, defined as

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- The Residue theorem

  Let $f(z)$ be analytic on a simply connected set G at a finite number of poles. Then it follows for each closed curve C in G, which does not go through a pole of f that

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} Res(f; a_j),$$

  where $a_1, a_2, \ldots, a_n$ are the poles of $f(z)$ in the interior of C.

- Laurent expansion

  If f is analytic for $|z-a| < \mathbb{R}$, then we have the following for each such z that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

  where $a_n = \dfrac{1}{2\pi i} \oint_{|\xi - a| < r} \dfrac{f(\xi)}{(\xi - a)^{n+1}} d\xi = \dfrac{f^n(a)}{n!}$, where $0 < r < \mathbb{R}$.

- The Convolution theorem

  Let $F(s)$ and $G(s)$ denote the Laplace transforms of $f(t)$ and $g(t)$ respectively. Then the product given by $H(s) = F(s)G(s)$ is the Laplace transform of the convolution of f and g, denoted by

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

Chapter 1

Introduction

1.1 Overview

It is well-known that equity returns are not well captured by a Gaussian probability density function, see [EFM05, Dan11]. In general, the distribution of equity returns exhibits a high excess of kurtosis and skewness, see [Fam65, Man63]. A distribution that displays a high excess of kurtosis has a higher peak and heavier tails, compared to the Gaussian distribution, see [RMF05, Fam65]. If the underlying asset is not well captured by a statistical distribution, then risk measures and pricing of derivatives are not well estimated, see [EFM05, SS15]. One major factor that contributed to the failure of Long-Term Capital Management, was the underestimation of risk measures (e.g. VaR) and mispricing of derivatives. They applied Gaussian models which did not anticipate fat-tails, see [Jor00]. Two essential features of equity returns are volatility clustering and the statistical distribution of equity returns, see [RMF05, Fam65, Con01]. The GARCH (1,1) model has been widely used to model volatility clustering, see [FZ+04]. VaR and ES are tools used by financial institutions to estimate market and credit risk, see [EFM05, C+10].

Statistical distributions that capture greater likelihoods of extreme price movement are the most recommended distributions for evaluating risk measures, see [EFM05, MF00]. A number of authors have proposed different statistical distributions, that attempt to capture the real market distribution of equity returns. The distributions range from simple ones to more complex ones, see [MF00, Dan11]. The Student-t and skew-Student-t distributions have been studied extensively in risk management, see [EFM05, MLR14]. [BM+12, BCY08, BMK09] were the first ones to apply the Pearson type IV distribution, in conjunction with GARCH(1,1) model to evaluate VaR and ES.

Pricing and hedging of financial instruments are of foremost importance for financial institutions, see [Hul06]. There is progress in developing option pricing models that are realistic. Many models have been developed that aim to improve the Black-Scholes (1973) model, see [BCC97, Bjö09, Miy11]. Examples of such models include the Merton model (1973) (stochastic interest rate), Bates model (1991) (jump-diffusion), Heston model (1993) (stochastic volatility) etc. Every option pricing model is based on some assumptions. The common assumptions include: (i) how to model the underlying price process, (ii) modelling volatility and interest rate process. According to [BCC97], an option pricing model is assessed by pricing errors and hedging performance.

In this research, we aim to implement the hybrid model developed by [SS15]. We compute standard risk measures under the distributions we obtain under this model, and we price European style options.

1.2 Objectives

The main objective of this research is to study the model of returns developed by [SS15], that results in fat tails distributions, and its implications for portfolio risk measures, option pricing, hedging and calibration.

The minor objectives are the following:

To show the derivation of the arithmetic-geometric hybrid SDE following the work of [SS15]. To derive fat-tailed distributions and show that the probability of large losses is more probable under this model, other than in the Gaussian distribution. To investigate implications for portfolio risk evaluation under the distributions we obtain from this model. To fit our model to four shares listed on the JSE and compute VaR and ES. To backtest all VaR results by using the Success-Failure ratio and the Kupiec Likelihood ratio test. We also study the properties of a coherent risk measure. The final objective of the dissertation is to apply the hybrid SDE to price European style options, discuss hedging and calibration.

Highlights of the arithmetic-geometric model:

1. Addresses the risk posed by technical trading in a market.
2. This model is a simple SDE with constant parameters, that produces fat-tailed distributions, without resorting to Lévy Processes nor stochastic volatility models.
3. The probability of a "25σ" or worse event is much more probable under this model, other than in the Black-Scholes model.

1.3 Structure of the dissertation

In Chapter 2, we introduce and derive the arithmetic-geometric hybrid Brownian Motion model following the work of [SS15]. We provide an explicit solution of the model and its moments. In Chapter 3, we derive the quantilised Fokker-Planck equation (QFPE), associated with the hybrid SDE, following the work of [SS08]. The QFPE is then applied to derive *equilibrium* fat-tailed probability distributions, i.e. the Pearson type IV and the Student-t distribution. Under *non-equilibrium* market conditions, we apply the Fokker-Planck equation on the hybrid SDE, to derive fat-tailed distributions.

In Chapter 4, we introduce the empirical data we use in this dissertation. We analyse the first four moments of our data and apply the QQ-plot to test for normality. We apply the method of maximum likelihood, to fit the Pearson type IV and Student-t distributions to our data. We test how well the Pearson type IV and Student-t distributions fitted our data, by applying the QQ-plot method. In Chapter 5, we present the definitions of the most common risk measures and different methods of evaluating them. We discuss the main two methods of estimating VaR which includes, the Historical method and Variance-covariance (VC) method. Under the VC method, we adopt the GARCH model to compute VaR, following the work of [EFM05, BM+12, BCY08, BMK09, SMNZ12, SZ13]. We apply the success-failure ratio, and the Kupiec likelihood method, to backtest VaR results. We discuss the properties of a coherent risk measure, introduced by [ADEH99].

Furthermore, we fit the non-equilibrium probability density distributions to our data. Once more we compute VaR estimates and backtest our results. In Chapter 6, we apply the hybrid SDE to price European style call options. We calibrate the model and apply the Crank-Nicolson finite difference scheme to compute European call options. Finally, in Chapter 7, we discuss our findings and possible future research.

Chapter 2

The Arithmetic-Geometric Hybrid Brownian Motion Model

We remarked earlier that the share price returns are not well captured by the Normal distribution. In the 1960's authors like Mandelbrot (1963) and Fama (1965) observed an excess of kurtosis and skewness in financial asset returns. A distribution that displays excess in kurtosis has heavier tails and higher peak than the Normal distribution, see [EFM05, Dan11]. In this chapter, we present a model for asset returns developed by [SS15]. In the next section, we derive the arithmetic-geometric hybrid SDE and its solution.

2.1 Model derivation

2.1.1 Fundamental and technical traders

In this model, we consider two types of market traders namely, the fundamental and technical traders. We consider a liquid market containing an asset with share price S_t at time $t \geq 0$, and the trading period $[0, T]$. The price S_t will continuously fluctuate up and down, due to the demand and supply of S_t. The buyer-initiated trades will force prices to increase, while the seller-initiated trades will force prices to decrease. The net effect on the market prices over a small period is described by the log-returns.

We define the log-return X_t on the asset S_t over a period $[0,t]$ by :

$$X_t = \log\left(\frac{S_t}{S_0}\right). \tag{2.1}$$

We assume that (X_t) (real-valued stochastic process) describes the current state of the market. We study the influence of fundamental and technical traders in a market. Both types of traders are seeking information about the general movement of an asset price, and trade based on that information but their techniques of obtaining such information differs. In this model, one set of traders consist those who are trading independently of the current value of (X_t), these will include fundamental traders. The second set of traders consist those who trade based on the historical performance of (X_t), these will include technical traders (i.e. $\{X_s \,|0 \leq s \leq t\}$).

The market state (X_t) is driven by the buy and sell orders to the market. We are interested in knowing the waiting time between the trade orders, and the size of the trade order. We first consider buy orders based on fundamental trading. At some time Δ_t, we consider a time interval $(t, t+\Delta_t) \subset [0,T]$ with $\Delta_t \ll T$, where buy orders occur in many sizes of M. Let $Y(t)$ be a discrete random variable taking non-negative integers $\{0,1,2,\cdots\}$, to denote the number of buy trades arriving at time t, and define N_i to be a discrete integer-valued r.v. denoting the number of sizes in each buy order for $i = 1,\ldots,Y$. The N_i's are assumed to be i.i.d. with a common density function N. In total the number of buy orders based on fundamental trading is

$$I = M \times \sum_{i=0}^{Y} N_i = M \times Z \tag{2.2}$$

where $Z = \sum_{i=0}^{Y} N_i$. We apply the probability generating function (PGF) of Z to compute the mean and variance of equation (2.2), [SS15] use a probabilistic argument to show the following results. The PGF of Z is given by

$$\begin{aligned}
G_Z(z) &= \mathbb{E}[z^Z] \\
G_Z(z) &= \mathbb{E}[z^{\sum_{i=0}^{Y} N_i}] \\
&= \mathbb{E}(\mathbb{E}(z^{\sum_{i=0}^{Y} N_i} | Y)) && \text{(by the Tower property)} \\
&= \mathbb{E}[\mathbb{E}[z^{N_0} z^{N_1} \cdots z^{N_Y} | Y]] && N_i's \text{ are i.i.d.} \\
&= \mathbb{E}((G_N(z))^Y) \\
&= G_Y(G_N(z)) && \text{by definition of } G_Y
\end{aligned} \tag{2.3}$$

By the property of a PGF,

$$\begin{aligned}\mathbb{E}[Z] &= G_Z'(1)\\ &= G_Y'(G_N(z))G_N'(z)|_{z=1}\\ &= G_Y'(G_N(1))G_N'(1)\\ &= G_Y'(1)G_N'(1)\\ &= \mathbb{E}[Y]\mathbb{E}[N] = \mathbb{E}[Y]\bar{n}\end{aligned} \tag{2.4}$$

where $\bar{n} = \mathbb{E}[N]$, and

$$\begin{aligned}\mathrm{Var}[Z] &= G_Z''(1) + G_Z'(1) - [G_Z'(1)]^2\\ &= G_Y''(G_N(z))[G_N'(z)]^2 + G_Y'(G_N(z))G_N''(z)|_{z=1} + \mathbb{E}[Z] - [\mathbb{E}[Z]]^2\\ &= G_Y''(1)\bar{n}^2 + \mathbb{E}[Y]G_N''(1) + \mathbb{E}[Y]\bar{n} - (\mathbb{E}[Y])^2\bar{n}^2\\ &= \mathrm{Var}[Y]\bar{n}^2 + \mathbb{E}[Y]\mathrm{Var}[N]\,\cdot\end{aligned} \tag{2.5}$$

We analyse the distributions of Y and N_i, and their first two moments, in order to make progress. We assume that the inter-trade arrival times follow a renewal process[1].

We derive the mean and variance of $Y(t)$ and N_i, from a characteristic function between the time trades, following the work of [SS15] and adding some steps. Let the time between the trades be denoted by $P(t)$ and let $f_P(t)$ denote the PDF of $P(t)$. Finally let

$$S_n = \sum_{i=1}^{n} P_i, \text{ where } n \in \mathbb{R}. \tag{2.6}$$

The relationship between $Y(t)$(the number of buy trades arriving at time t) and $P(t)$ is given by

$$\mathbb{P}[Y(t) < n] = \mathbb{P}(S_n > t) = 1 - F_{S_n}(t) \tag{$\star$}$$

where $F_{S_n}(t)$ is the cumulative distribution function of S_n. We relate $Y(t)$ to $P(t)$ by considering the characteristic function of P, denoted by

$$\psi_P(\omega) = \mathbb{E}[e^{i\omega P}] = \int_0^{\infty} e^{iws} f_P(s)ds\,\cdot \tag{2.7}$$

From equation (2.7), it follows that, the characteristic function of S_n is given by $\psi_{S_n}(\omega) = [\psi_P(\omega)]^n$. Our task now is to find the $F_{S_n}(t)$ function in order to solve ($\star$). The CDF $F_{S_n}(t)$

[1]See Appendix A for a formal definition.

is given by the convolution of the density function $f_{S_n}(t) = F'_{S_n}(t)$ and the Heaviside step function (see the Glossary) $\theta(x)$ that is one for $x > 0$ and zero elsewhere.

$$\begin{aligned} F_{S_n}(t) &= \int_{-\infty}^{t} f_{S_n}(s)ds \\ &= \int_{-\infty}^{\infty} \theta(t-s) f_{S_n}(s)ds \\ &= (\theta * f_{S_n})(t). \end{aligned} \tag{2.8}$$

The Fourier transform of the convolution, is the product of the Fourier transforms, see [AS64]. The Fourier transform of $F_{S_n}(t)$ is given by

$$\mathscr{F}[F_{S_n}(\omega)] = \psi_{S_n}(\omega)\mathscr{F}(\theta(\omega)),$$

where

$$\mathscr{F}(\theta(\omega)) = \int_{-\infty}^{\infty} \theta(t)e^{i\omega t}dt = \int_{0}^{\infty} e^{i\omega t}dt \tag{2.9}$$

since $\theta(t)$ is one for $t > 0$. Equation (2.9) diverges if ω is real. Hence, $\omega = a + ib$ and $\bar{w} = a - ib$ (complex conjugate). The solution of equation (2.9) is in [SS15], but we add some steps. From equation (2.9) we have

$$\begin{aligned} \int_{0}^{\infty} e^{i\omega t}dt &= \int_{0}^{\infty} e^{it(a+ib)}dt \\ &= \int_{0}^{\infty} e^{ait}e^{-bt}dt \\ &= \int_{0}^{\infty} e^{-bt}\cos(at)dt + i\int_{0}^{\infty} e^{-bt}\sin(at)dt \\ &= \lim_{k\to\infty}\int_{0}^{k} e^{-bt}\cos(at)dt + i\lim_{k\to\infty}\int_{0}^{k} e^{-bt}\sin(at)dt. \end{aligned}$$

Applying integration by parts twice we have

$$\begin{aligned} \int_{0}^{k} e^{-bt}\cos(at)dt &= \frac{e^{-bt}}{b^2+a^2}\Big(a\sin(at) - b\cos(at)\Big)\Big|_{0}^{k} \\ &= \frac{ae^{-bk}sin(ak) - be^{-bk}\cos(ak) + b}{b^2+a^2} \end{aligned}$$

$$\begin{aligned} \therefore \lim_{k\to\infty}\int_{0}^{k} e^{-bt}\cos(at)dt &= \frac{1}{b^2+a^2}\lim_{k\to\infty}\Big(ae^{-bk}\sin(ak) - be^{-bk}\cos(ak) + b\Big) \\ &= \frac{b}{b^2+a^2} \qquad \text{by the Squeeze theorem.} \end{aligned}$$

Similarly

$$\lim_{k\to\infty}\int_{0}^{k} e^{-bt}\sin(at)dt = \frac{a}{b^2+a^2}.$$

Therefore

$$\int_0^\infty e^{i\omega t}dt = \frac{b}{b^2+a^2} + i\left(\frac{a}{b^2+a^2}\right)$$
$$= \frac{ai+b}{b^2+a^2} \times \frac{i}{i} = \frac{bi-a}{i(a^2+b^2)} = -\frac{\bar{\omega}}{i(a^2+b^2)}\cdot$$

Substituting

$$a = \frac{\omega+\bar{\omega}}{2} \qquad \text{and} \qquad b = \frac{\omega-\bar{\omega}}{2i}\cdot$$

Therefore

$$a^2+b^2 = \left(\frac{\omega+\bar{\omega}}{2}\right)^2 + \left(\frac{\omega-\bar{\omega}}{2i}\right)^2$$
$$= \frac{1}{4}(\omega+\bar{\omega})^2 - \frac{1}{4}(\omega-\bar{\omega})^2 = \omega\bar{\omega}.$$

Finally

$$\mathscr{F}(\theta(\omega)) = \int_0^\infty e^{i\omega t}dt = -\frac{\bar{\omega}}{i(a^2+b^2)} = -\frac{\bar{\omega}}{i\omega\bar{\omega}} = -\frac{1}{i\omega} = \frac{i}{\omega}\cdot$$

Thus

$$\begin{aligned} F_{S_n(t)} &= \mathscr{F}^{-1}[F_{S_n}(\omega)] \\ &= \mathscr{F}^{-1}[\psi_{S_n}(\omega)\frac{i}{\omega}] \\ &= \frac{i}{2\pi}\oint_C \frac{\psi_{S_n}(\omega)e^{-i\omega t}}{\omega}d\omega\ \cdot \end{aligned} \tag{2.10}$$

From ($\star$) we have

$$\mathbb{P}[Y(t)<n] = \mathbb{P}(S_n>t) = 1 - \frac{i}{2\pi}\oint_C \frac{[\psi_P(\omega)]^n e^{-i\omega t}}{\omega}d\omega\ \cdot$$

From probability theory we know that

$$\begin{aligned} \mathbb{P}[Y(t)=n] &= \mathbb{P}[Y(t)<n+1] - \mathbb{P}[Y(t)<n] \\ &= \frac{1}{2\pi i}\oint_C e^{-i\omega t}\frac{[\psi_P(\omega)]^n}{\omega}d\omega - \frac{1}{2\pi i}\oint_C e^{-i\omega t}\frac{[\psi_P(\omega)]^{n+1}}{\omega}d\omega \\ &= \frac{1}{2\pi i}\oint_C e^{-i\omega t}\frac{(\psi_P(\omega)-1)}{\omega}[\psi_P(\omega)]^n d\omega\ \cdot \end{aligned}$$

The PGF $G_{Y(t)}(p)$ of $Y(t)$ is obtained by taking the geometric series expansion of $[\psi_P(\omega)]^n$. Thus

$$G_{Y(t)}(p) = \frac{i}{2\pi}\oint_C e^{-i\omega t}\frac{(1-\psi_P(\omega))}{\omega(1-p\psi_P(\omega))}d\omega\ \cdot \tag{2.11}$$

We estimate the mean and variance by

$$\mathbb{E}[Y(t)] = G'_{Y_t}(1) = \frac{i}{2\pi}\oint_C e^{-i\omega t}\frac{(\psi_P(\omega))}{\omega(1-\psi_P(\omega))}d\omega, \tag{2.12}$$

$$\text{Var}[Y(t)] = G''_{Y_t}(1) + (G'_{Y_t}(1)) - (G'_{Y_t}(1))^2 \tag{2.13}$$

where

$$G'_{Y_t}(p) = \frac{i}{2\pi}\oint_C e^{-i\omega t}\frac{\omega\psi_P(\omega)(1-\psi_P(\omega))}{\omega^2(1-\psi_P(\omega))^2}d\omega \cdot$$
$$G''_{Y_t}(1) = \frac{i}{\pi}\oint_C e^{-i\omega t}\frac{(\psi_P^2(\omega))}{\omega(1-\psi_P(\omega))^2}d\omega \cdot \tag{2.14}$$

To calculate the moments we need to simplify $\psi_P(\omega)$, [SS15] applies the expansion of the form

$$\psi_P(\omega) = 1 + i\omega m_{1P} - \frac{1}{2}\omega^2 m_{2P} - \frac{i}{6}\omega^3 m_{3P} + \cdots \tag{2.15}$$

We consider the integrand of equations (2.12- 2.14) and apply the residue theorem to simplify the mean and variance. From the equations of the mean and variance, we have the roots at $\omega = 0$ and $\psi_P(\omega) = 0$. By the Laurent expansion we have

$$f(\omega_0) = \frac{1}{2\pi i}\oint_C \frac{f(\omega)}{\omega - \omega_0}d\omega \cdot \tag{2.16}$$

So

$$\oint_C \frac{f(\omega)}{\omega}d\omega = 2\pi i \cdot f(0) = 2\pi i \cdot \frac{\psi_P(0)}{1-\psi_P(0)} \text{ from equation (2.12)}, \tag{2.17}$$

where

$$f(\omega) = \frac{e^{-i\omega t}\psi_P(\omega)}{1-\psi_P(\omega)} \cdot \tag{2.18}$$

We observe, by definition

$$f'(0) = \frac{1}{2\pi i}\underbrace{\oint_C \frac{f(\omega)}{\omega^2}d\omega}_{=I} = \frac{1}{2\pi i}\oint_C \frac{e^{-i\omega t}\psi_P(\omega)}{\omega^2(-im_{1P} + 0.5\omega m_{2P} + \cdots)}dw, \tag{2.19}$$

so, $I = 2\pi i f'(0)$.

Differentiating the integrand of equation (2.19) with the quotient rule w.r.t. ω and substituting $\omega = 0$, we obtain

$$\frac{d}{d\omega}\left(\frac{e^{-i\omega t}\psi_P(\omega)}{(-im_{1P}+0.5\omega m_{2P}+\cdots)}\right)$$
$$=\frac{(-im_{1P}+0.5\omega m_{2P}+\cdots)[-ite^{-i\omega t}\psi_P(\omega)+e^{-i\omega t}\psi_P'(\omega)]-[e^{-i\omega t}\psi_P(\omega)][0.5m_{2P}+\cdots]}{(-im_{1P}+0.5\omega m_{2P}+\cdots)^2}.$$

Substituting $\omega = 0$,

$$=\frac{(-im_{1P})[-it\psi_P(0)+\psi_P'(0)]-\psi_P(0)(0.5m_{2P}+\cdots)}{(-im_{1P})^2}.\ \text{From (2.15)}, \psi_P(0)=1 \text{ and } \psi_P'(0)=im_{1P}$$
$$=\frac{(-im_{1P})[-it+im_{1P}]-(0.5m_{2P}+\cdots)}{(-im_{1P})^2}$$
$$=\frac{t}{m_{1P}}+1-\frac{0.5m_{2P}+\cdots}{(m_{1P})^2}$$
$$\therefore \mathbb{E}[Y(t)]=\frac{t}{m_{1P}}+\mathcal{O}(1)=\lambda t+\mathcal{O}(1), \text{ where } \lambda = 1/\mathbb{E}[P]. \tag{2.20}$$

Similarly,

$$\text{Var}[Y(t)]=\frac{(m_{2P}-m_{1P}^2)t}{m_{1P}^3}+\mathcal{O}(1)=\frac{\text{Var}[P]}{\mathbb{E}[P]^3}+\mathcal{O}(1)=\lambda^3\text{Var}[P]+\mathcal{O}(1)\cdot \tag{2.21}$$

It follows from equation (2.20) that the mean of $Y(t)$ is given by

$$\mathbb{E}[Y(t)]\sim\lambda_P\Delta_t, \tag{2.22}$$

where $\lambda_P = 1/\mathbb{E}[P]$, and $P(t)$ is a non-negative r.v. giving the inter-arrival times.

The variance follows from equation (2.21)

$$\text{Var}[Y(t)]\sim\text{Var}[P]\lambda_P^3\Delta_t \ \text{ or } \text{Var}[Y(t)]\sim\gamma_P\lambda_P\Delta_t \tag{2.23}$$

where $\gamma_P = \text{Var}[P]\lambda_P^2$.

We return to fundamental buy orders moments, and we denote λ_B and γ_B in a similar manner as λ_P and γ_P. So from equation (2.4) we have

$$\mathbb{E}[Z]=\lambda_B\Delta_t\bar{n}, \tag{2.24}$$

and from equation (2.5) we have

$$\text{Var}[Z]=\lambda_B\Delta_t(\gamma_B\bar{n}^2+\text{Var}[N]). \tag{2.25}$$

It then follows from equations (2.24) that the expectation of total collection of buy orders (M_B) under fundamental trading is given by

$$\mathbb{E}[M_B]=M\lambda_B\Delta_t\bar{n} \tag{2.26}$$

and standard deviation (sd) of (M_B) follows from equation (2.25)

$$\mathrm{sd}(M_B) = M\sqrt{\lambda_B \Delta_t(\gamma_B \bar{n}^2 + \mathrm{Var}[N])}\cdot \tag{2.27}$$

We follow the same procedure with the fundamental sell (M_S) orders and obtain an expected value of

$$\mathbb{E}[M_S] = M\lambda_S \Delta_t \bar{n}, \tag{2.28}$$

and a standard deviation of

$$\mathrm{sd}(M_S) = M\sqrt{\lambda_S \Delta_t(\gamma_S \bar{n}^2 + \mathrm{Var}[N])}\cdot \tag{2.29}$$

The net effect of fundamental trades is given by

$$M_F = M_B - M_S \cdot \tag{2.30}$$

The mean of M_F is given by

$$\mathbb{E}[M_F] = M(\lambda_B - \lambda_S)\Delta_t \bar{n}\cdot \tag{2.31}$$

We assume that the fundamental buyers and sellers are trading based on different strategy, so we assume independence. Hence, the variance of M_F is given by

$$\mathrm{Var}[M_F] = M^2 \Delta_t[\lambda_B(\gamma_B \bar{n}^2 + \mathrm{Var}[N]) + \lambda_S(\gamma_S \bar{n}^2 + \mathrm{Var}[N])]\cdot \tag{2.32}$$

We assume that the above parameters under fundamental trading are not functions of X_t. Now we focus on technical traders, which we model in a similar style as for the fundamental traders. We denote the arrival rates for buying and selling as μ_B and μ_S respectively, and ρ as the correlation between arrival rates of buy and sell trades. The net effect of technical trades is denoted by (M_T). Following the same procedure as above, the mean and the variance is respectively given by

$$\mathbb{E}[M_T] = M(\mu_B - \mu_S)\Delta_t \bar{n}_T \tag{2.33}$$

$$\mathrm{Var}[M_T] = M^2 \Delta_t(\mu_B \alpha_T + \mu_S \beta_T - 2\rho^T \sqrt{\mu_B \mu_S \alpha_T \beta_T}) \tag{2.34}$$

where
$\alpha_T = \gamma_B^T \bar{n}_T^2 + \mathrm{Var}[N_T]$ and $\beta_T = \gamma_S^T \bar{n}_T^2 + \mathrm{Var}[N_T]\cdot$

The buying and selling of stocks cause price changes which are described by the price impact function. Let us suppose the current state of the market (X_t) is known at some $t > 0$, and it will change at time $(t+\Delta_t)$ by a quantity $\Delta_t X = X_{t+\Delta_t} - X_t$. We define a log-return impact function $\mathcal{I}(q)$ (which is non-decreasing), that is a function with $\mathcal{I}(0) = 0$ (i.e. no orders implies no price impact), such that the aggregate of buy and sell orders of both trades have a log-return impact of the form

$$\Delta_t X = \mathcal{I}(\Delta_t M_F + \Delta_t M_T)\cdot \tag{2.35}$$

The return impact function can be complicated[2], in this dissertation we use a linear impact function. For some fixed constant $\kappa > 0$, we write:

$$\Delta_t X = \kappa \times (\Delta_t M_F + \Delta_t M_T)\cdot \tag{2.36}$$

The return impact function $\mathcal{I}$ is just a description of the order book [Sch15], and we assume it is linear. By linearising the return impact function we enforce the Markov property, so that the process $\{X_s | 0 \le s \le t\}$ is now represented by the current value of X_t. So we have

$$\begin{aligned} &\mathbb{E}[\Delta_t M_F] = \mu_F \Delta_t \ \&\ \mathrm{Var}(\Delta_t M_F) = \sigma_F^2 \Delta_t \text{ and} \\ &\mathbb{E}[\Delta_t M_T] = -\mu_T(X_t)\Delta_t \ \&\ \mathrm{Var}(\Delta_t M_F) = \sigma_T^2(X_t)\Delta_t \end{aligned} \tag{2.37}$$

The minus sign is included without any loss of in generality.

We assume that fundamental and technical traders are independent of each other. From equations (2.37) and (2.31) we obtain a discrete-time SDE, and we approximate the noises of the trade arrival models by two independent Brownian motions.

$$\Delta_t X = \kappa M([\bar{n}(\lambda_B - \lambda_S) - \mu \bar{n}_T X]\Delta_t + s_1 \Delta W_t^1 + s_2 \Delta W_t^2) \tag{2.38}$$

where

$$s_1 = \sqrt{\frac{\mathrm{Var}[M_F]}{\Delta_t M^2}}, \quad s_2 = \sqrt{\frac{\mathrm{Var}[M_T]}{\Delta_t M^2}}. \tag{2.39}$$

We re-write equation (2.38) as

$$\Delta_t X = (\mu_1 - \mu_2 X_t)\Delta_t + \Sigma_1 \Delta_t W^1 + \Sigma_2 \Delta_t W^2$$

[2]The return impact function can be of the form $\mathcal{I}(x) = (\kappa x)^\alpha$ with $0 < \alpha < 1$ or $\mathcal{I}(x) \propto x^\alpha$.

where

$$
\begin{aligned}
\alpha &= \kappa M \\
\mu_1 &= \alpha \bar{n}(\lambda_B - \lambda_S) \\
\mu_2 &= \alpha \mu \bar{n}_T \\
\Sigma_1 &= \alpha s_1 \\
\Sigma_2 &= \alpha s_2
\end{aligned}
$$

A SDE is obtained from the equation above by taking limits as $\Delta_t \to 0$, replacing $\Delta_t = t_{n+1} - t_n$ by dt, $\Delta_t W = W_{t+\Delta_t} - W_t$ by dW_t and $\Delta_t X = X_{t+\Delta_t} - X_t$ by dX_t.

$$
\begin{aligned}
dX_t &= (\mu_1 - \mu_2 X_t)dt + \Sigma_1 dW_t^1 + \Sigma_2 dW_t^2 \\
X_t &= X_0 + \int_0^t (\mu_1 - \mu_2 X_s)ds + \int_0^t \Sigma_1 dW_s^1 + \int_0^t \Sigma_2 dW_s^2.
\end{aligned}
\tag{2.40}
$$

To complete the model we assume that the fundamental trading parameters are independent of X_t, hence we assume these parameters to be constant over a single trading period. However, they may change daily. Conversely, we assume technical trading parameters to be dependent on X_t. The general hybrid model takes the form

$$dX_t = (\mu_1 - f(X_t))dt + \sigma_1 dW_t^1 + g(X_t)dW_t^2 \cdot \tag{2.41}$$

In this dissertation, we consider an SDE of the form[3]

$$dX_t = (\mu_1 - \mu_2 X_t)dt + \sigma_1 dW_t^1 + \sigma_2 X_t dW_t^2 \ , t > 0, \ X_0 = x \cdot \tag{2.42}$$

In the SDE above we note the following, the parameters (μ_1 and σ_1 that are independent of X_t) represent fundamental trades (Arithmetic Brownian motion), and the parameters (μ_2 and σ_2 that are dependent of X_t) represent technical trades (Geometric Brownian motion).

We reduce equation (2.42) to an SDE with a single noise term, see [Sch15, Chapter 5]. Let $\rho \in [-1, 1]$ be the correlation between W_t^1 and W_t^2 (i.e $\rho = Corr(W_t^1, W_t^2)$), then we re-write the SDE for (X_t) as follows

$$dX_t = (\mu_1 - \mu_2 X_t)dt + \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t} dW_t \ , t > 0, \ X_0 = x \cdot \tag{2.43}$$

The existence of a strong solution of the SDE (2.42) is guaranteed by the following theorem.

[3]Another interesting hybrid SDE is the Arithmetic-CIR model of the form $dX_t = (\mu_1 - \mu_2 X_t)dt + \sigma_1 dW_t^1 + \sigma_2 \sqrt{|X_t|} dW_t^2$, see [SS15, Sch15]

Theorem 1 ***Existence and Uniqueness***
Let (X_t) satisfy the following SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ t > 0,\ \ X_0 = x\cdot$$

Let $\mu(x),\ \sigma(x)\ : \mathbb{R} \to \mathbb{R}$ be functions satisfying the Lipschitz condition and a Hölder condition of order α, $\alpha > 1/2$ respectively, that there exist a constant K such that

$$|\sigma(x) - \sigma(y)| \le K|x-y|^{\alpha} \quad \forall x, y \in \mathbb{R}\cdot$$

Then the strong solution of the SDE (X_t) exists and is unique.
The proof of this theorem is given by [Kle05, Theorem 5.4].

Definition 1 ***A Lipschitz and a Hölder condition*** *A function f satisfies a Hölder condition of order α, $0 < \alpha < 1$, on $[a,b]$ $(\mathbb{R})$ if there is a constant $K > 0$, so that for all $x, y \in [a,b]$*

$$|f(x) - f(y)| \le K|x-y|^{\alpha} \quad \forall x, y \in \mathbb{R}\cdot$$

A Lipschitz condition is a Hölder condition with $\alpha = 1$,

$$|f(x) - f(y)| \le K|x-y| \quad \forall x, y \in \mathbb{R}\cdot$$

2.2 Explicit solution of the hybrid SDE

We give an explicit solution of the SDE (2.42). We introduce the integrating factor I_t

$$I_t = \exp\left[-\sigma_2 W_t^2 + \left(\mu_2 + \frac{1}{2}\sigma_2^2\right)t\right]\cdot \tag{2.44}$$

By Itô's lemma

$$\begin{aligned} \mathrm{d}I_t &= I_t(\mu_2 + 0.5\sigma_2^2)dt + I_t(-\sigma_2)dW_t^2 + \frac{1}{2}I_t\sigma_2^2 dt \\ &= (\mu_2 + \sigma_2^2)I_t dt - \sigma_2 I_t dW_t^2\ \cdot \end{aligned} \tag{2.45}$$

We define a new random variable $V_t = X_t I_t$ and apply Itô's lemma on $V(t)$, i.e.

$$\begin{aligned} dV_t &= X_t dI_t + I_t dX_t + dX_t dI_t \\ &= X_t((\mu_2 + \sigma_2^2)I_t dt - \sigma_2 I_t dW_t^2) + I_t((\mu_1 - \mu_2 X_t)dt + \sigma_1 dW_t^1 + \sigma_2 X_t dW_t^2) \\ &- \rho\sigma_1\sigma_2 I_t dt - \sigma_2^2 I_t X_t dt \\ &= I_t((\mu_1 - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_t^1)\ \cdot \end{aligned} \tag{2.46}$$

We integrate (2.46) using condition $V_0 = 0 = X_0$, and obtain

$$V_t = \int_0^t I_s((\mu_1 - \rho\sigma_1\sigma_2)ds + \sigma_1 dW_s^1) \cdot \tag{2.47}$$

So we obtain

$$X_t = \int_0^t (I_s \cdot I_t^{-1})((\mu_1 - \rho\sigma_1\sigma_2)ds + \sigma_1 dW_s^1) \tag{2.48}$$

where

$$(I_s \cdot I_t^{-1}) = \exp\left[\sigma_2(W_t^2 - W_s^2) + (\mu_2 + \frac{1}{2}\sigma_2^2)(s-t)\right] \cdot \tag{2.49}$$

Lastly, we define a time reversal $u = t - s$ with the associated Brownian motions $\tilde{W}_i$, and finally, we have

$$X_t = \int_0^t ((\mu_1 - \rho\sigma_1\sigma_2)du + \sigma_1 d\tilde{W}_u^1) \times \exp\left[\sigma_2\tilde{W}_u^2 - (\mu_2 + \frac{1}{2}\sigma_2^2)u\right] \cdot \tag{2.50}$$

We re-write equation (2.50) in terms variable, $\nu = 1 + \frac{2\mu_2}{\sigma_2^2}$ i.e. d.o.f

$$X_t = \int_0^t ((\mu_1 - \rho\sigma_1\sigma_2)du + \sigma_1 d\tilde{W}_u^1) \times \exp\left[\sigma_2\tilde{W}_u^2 - \frac{\nu}{2}\sigma_2^2 u\right] \cdot \tag{2.51}$$

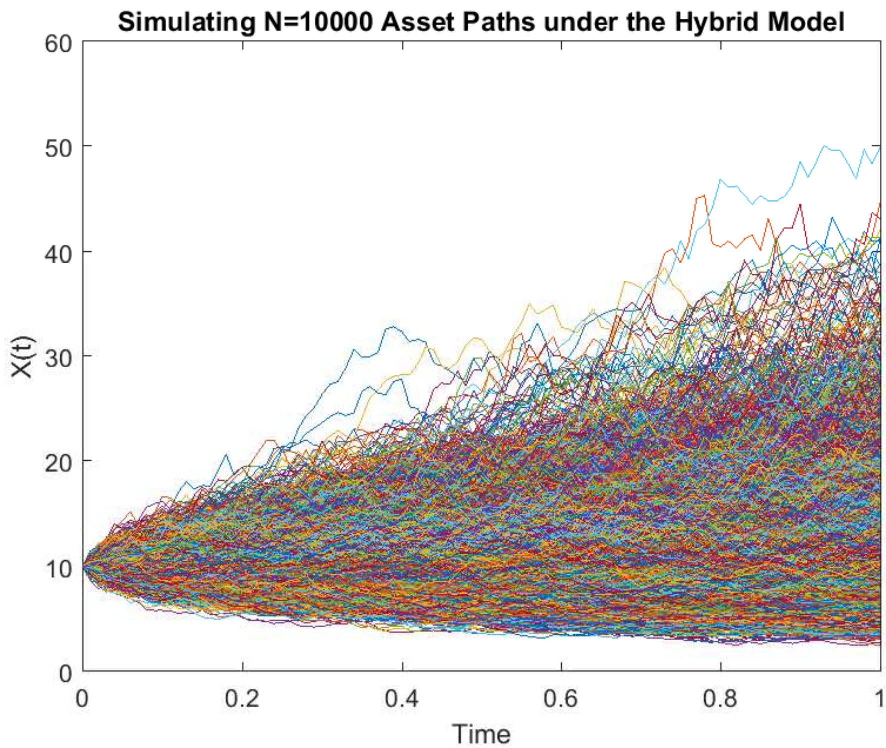


Figure 2.1: Simulating the hybrid Geometric model sample paths with $X_0 = 10, T = 1, \sigma_1 = 0, \sigma_2 = 0.3, \rho = 0$.

2.3 Dynamic moments of the hybrid SDE

We analyse the moment evolution of the hybrid SDE. We define

$$m_k(t) = \mathbb{E}[X_t^k],\ t \geq 0,\ k \in \mathbb{N} \tag{2.52}$$

and apply Itô's lemma on X_t^k.

$$\begin{aligned} dX_t^k &= kX_t^{k-1}dX_t + \frac{1}{2}k(k-1)X_t^{k-2}(dX_t)^2 \\ &= kX_t^{k-1}((\mu_1 - \mu_2 X_t)dt + \sigma_1 dW_t^1 + \sigma_2 X_t dW_t^2) \\ &+ \frac{1}{2}k(k-1)X_t^{k-2}(\sigma_1^2 dt + \sigma_2^2 X_t^2 dt + 2\rho\sigma_1\sigma_2 X_t dt), \end{aligned}$$

so we have

$$\frac{dX_t^k}{dt} = k\mu_1 X_t^{k-1} - k\mu_2 X_t^k + \frac{1}{2}k(k-1)\sigma_1^2 X_t^{k-2} + \frac{1}{2}k(k-1)\sigma_2^2 X_t^k + k(k-1)\rho\sigma_1\sigma_2 X_t^{k-1}\cdot \tag{2.53}$$

Taking expectation, we obtain

$$\frac{dm_k(t)}{dt} + (k\mu_2 - \frac{1}{2}k(k-1)\sigma_2^2)m_k(t) = \frac{1}{2}k(k-1)\sigma_1^2 m_{k-2}(t) + (k\mu_1 + k(k-1)\rho\sigma_1\sigma_2)m_{k-1}(t) \tag{2.54}$$

with $m_0 = 1$ and $m_k(0) = 0,\ k \geq 1$.

For the mean we set $k = 1$. So

$$\frac{dm_1(t)}{dt} + \mu_2 m_1(t) = \mu_1\cdot \tag{2.55}$$

To find the explicit formula for the mean, we solve the ODE (2.55). We multiply equation (2.55) by the integrating factor $e^{\mu_2 t}$, so we have

$$e^{\mu_2 t}\frac{dm_1(t)}{dt} + \mu_2 m_1 e^{\mu_2 t} = \mu_1 e^{\mu_2 t}\cdot \tag{2.56}$$

We solve equation (2.56), and obtain

$$\mathbb{E}[X_t] = m_1 = \frac{\mu_1}{\mu_2}(1 - e^{-\mu_2 t})\cdot \tag{2.57}$$

The mean settles down to μ_1/μ_2 if μ_2 is strictly positive and grows exponentially if negative.

For the variance, we consider the special case where $\rho = 0 = \mu_1$ (we will explore this special in detail later on).

From equation (2.51) with $\rho = 0 = \mu_1$ we have

$$X_t = \int_0^t \sigma_1 d\tilde{W}_{1u} e^{(\sigma_2 \tilde{W}_{2u} - \frac{\nu}{2}\sigma_2^2 u)}, \tag{2.58}$$

so we calculate

$$\begin{aligned}\mathbb{E}[X_t^2] &= \sigma_1^2 \int_0^t \mathbb{E}(e^{2(\sigma_2 \tilde{W}_{2u} - \frac{\nu}{2}\sigma_2^2 u)})du \\ &= \sigma_1^2 \int_0^t e^{-\nu\sigma_2^2 u} e^{2\sigma_2^2 u} du \quad \text{using the fact that } \mathbb{E}[e^{\alpha W_t}] = e^{\frac{\alpha^2}{2}t},\end{aligned}$$

hence the variance is given by

$$\mathrm{Var}(X_t) = \frac{\sigma_1^2}{\sigma_2^2(\nu-2)}[1 - e^{-\sigma_2^2(\nu-2)t}]\,. \tag{2.59}$$

In general, for the moments we have

$$m(x) = \mathbb{E}^x[X_t] = \frac{\mu_1}{\mu_2} + \left(x - \frac{\mu_1}{\mu_2}\right)e^{-\mu_2 t},$$

assuming that $\nu > 2$

$$\mathbb{E}^x[X_t^2] = h(x) + (x^2 - h(x))e^{-(2\mu_2 - \sigma_2^2)t}$$

where $h(x) = \frac{2\mu_1^2/\mu_2 + \sigma_1^2}{2\mu_2 - \sigma_2^2} - \frac{2(\mu_1 - \mu_2 x)}{\mu_2 - \sigma_2^2}$, see [Sch15, Chapter 5].

If $\mu_2 > \sigma_2^2/2$ the market settles down otherwise it explodes exponentially, which gives rise to large price movements so that the 25σ (extreme events), or worse events are significantly probable.

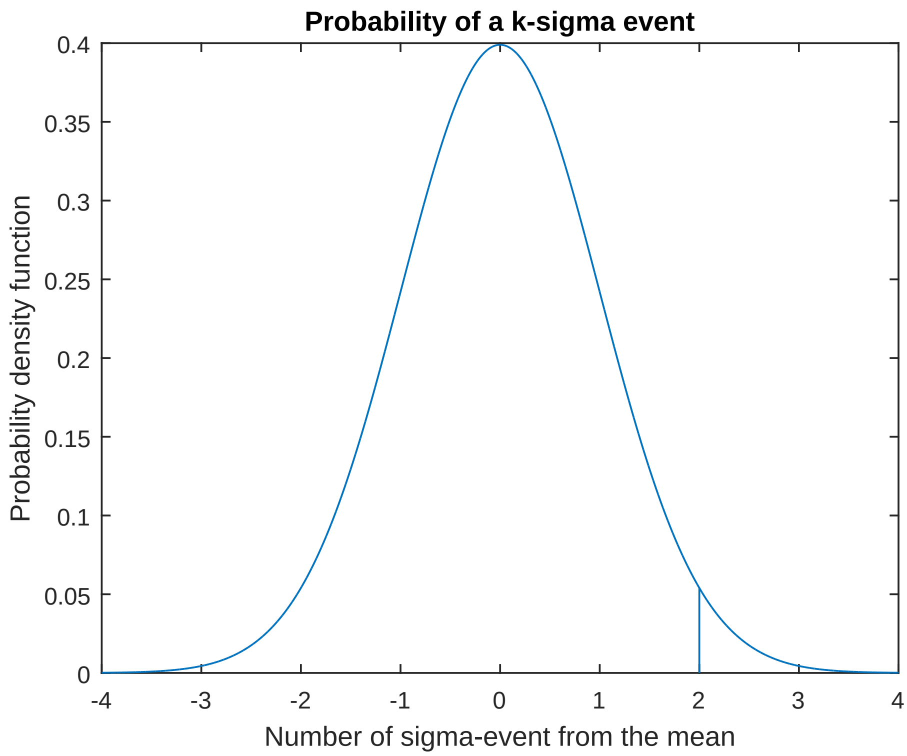


Figure 2.2: Gaussian density function with $k-$sigma events.

Let us assume that, losses are normally distributed and we are interested in the probability of a 25σ or worse events. Following the work of [DCHW08]. we first consider a 2σ event (i.e. 2 standard deviations or more away from the mean), the probability of such event under the standard Gaussian density function is approximately 2.275%. We expect to have a 2σ loss event on one trading day[4] out of 1/2.275%=43.96 (i.e. approximately one day out of 44 days). Let us now assume that losses obey the Student-t distribution with 4 d.o.f. Under the Student-t distribution, the probability of a 2σ event is approximately 26.52%, we expect to see such event once out of 4 trading days. In the following table, we estimate the probabilities for 3,4 and 5 sigma events.

Number of sigma event	Probability of such event		Expected occurrence (days)	
	Gaussian	Student-t	Gaussian	Student-t
2	0.02275	0.26517	43.96	3.77
3	0.00135	0.07877	740.76	12.69
4	3.20455×10^{-5}	0.02683	31205.61	37.27
5	2.98336×10^{-6}	0.01059	3351920.30	94.35

Table 2.1: Various sigma events and their probabilities, under the Gaussian and the Student-t (4 d.o.f) distributions.

We observe as the number of sigma events gets bigger their respective probabilities decline. In the Gaussian distribution, the probabilities decrease significantly compared to the Student-t distribution. This implies that extreme movements are more likely to occur in the Student-t distribution, other than in the Gaussian distribution. We now look at the famous quote by the CFO of Goldman Sachs David Vinar when he said, " We were seeing things that were 25-standard deviation moves, several days in a row" after they suffered huge losses.

We are interested in the question: how likely is a 25σ event? Following the same calculations, as in Table 2.1, the probability of a 25σ or worse event is 3.057×10^{-138} in the Gaussian distribution. We expect to have a 25σ event one trading day out of $1/3.057 \times 10^{-138} = 3.057 \times 10^{136}$ days. Under the Student-t distribution with 4 d.o.f the probability of a 25σ event is 4.83×10^{-6}, which we expect to see in once in 2.06×10^{5} trading days.

[4] We assume that there are 250 trading days in a year.

2.4 Chapter Summary

In this chapter, we introduced the Arithmetic-Geometric hybrid Brownian motion model, that was developed by [SS15]. The model is based on two types of market participants namely, the fundamental and technical traders. The fundamental traders trade independently of the current value of (X_t), while technical traders trade based on the historical performance of (X_t). We derived the SDE of the model in equation (2.42), where the Arithmetic Brownian motion part is related to the fundamental traders, and the Geometric Brownian motion part is related to the technical traders. The model has five parameters $\sigma_1, \sigma_2, \mu_1, \mu_2$ and ρ where, σ_1 is the fundamental volatility related to the economy, σ_2 is the technical volatility, μ_1 and μ_2 is the fundamental and technical drift respectively and ρ represent the relationship between W_t^1 and W_t^2. In Section 2.2 we explicitly derived the solution of the SDE, and in Section 2.3 we derived the dynamic moments of the SDE. In the next two chapters, we use this model to derive fat-tailed distributions.

Chapter 3

Probability Density Distributions

In this chapter, we first derive the quantilised Fokker-Planck equation (QFPE) related to the SDE in equation (3.1), following the work of [SS08]. We use the QFPE to derive probability distributions from an SDE. In simple terms, a quantile function $Q(\cdot)$ of a probability distribution, is the inverse of its CDF (F), provided that F is continuous and monotonic increasing, see [Csö83].

Definition 2 ***Quantile function***
Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a cumulative distribution function
$F(x) = \mathbb{P}\{\omega \in \Omega : X(\omega \leq x)\}, x \in \mathbb{R}$. *A quantile function is defined as* $Q(u) = F^{-1}(u) = \inf\{x : F(x) = u\}$ *for* $0 \leq u \leq 1$ *and F is continuous, i.e.* $F(Q(u)) = u \in [0,1]$.

3.1 The Quantilised Fokker-Planck Equation

We analyse a time-dependent quantile function, where (X_t) is a stochastic process defined as

$$dX_t = \mu(t, X_t)dt + \Sigma(t, X_t)dW_t, \tag{3.1}$$

where μ and Σ are deterministic functions of X and t. We define $f(x,t)$ to be a PDF consistent with equation (3.1). Let $Q(u,t)$ denote a quantile function associated with the SDE in equation (3.1), where $0 \leq u \leq 1$.

The quantile function is denoted by the integral condition:

$$F(Q(u,t),t) = \int_{-\infty}^{Q(u,t)} f(x,t)dx = u \cdot \tag{3.2}$$

Differentiating equation (3.2) with respect to (w.r.t.) u, we obtain

$$f(Q(u,t),t)\frac{\partial Q(u,t)}{\partial u} = 1 \cdot \tag{3.3}$$

We differentiate equation (3.3) and obtain,

$$\frac{\partial f(Q(u,t),t)}{\partial Q}\left(\frac{\partial Q(u,t)}{\partial u}\right)^2 + f(Q(u,t),t)\frac{\partial^2 Q(u,t)}{\partial u^2} = 0 \cdot \tag{3.4}$$

We re-write equation (3.4) as

$$\frac{\partial^2 Q(u,t)}{\partial u^2} = -\frac{\partial \log(f(Q(u,t),t))}{\partial Q}\left(\frac{\partial Q(u,t)}{\partial u}\right)^2 \cdot \tag{3.5}$$

Since the quantile function is time-dependent, we go back to equation (3.2) and differentiate it w.r.t. t, and we obtain

$$f(Q(u,t),t)\frac{\partial Q(u,t)}{\partial t} + \int_{-\infty}^{Q(u,t)} \frac{\partial f(x,t)}{\partial t} dx = 0 \cdot \tag{3.6}$$

By equation (3.3), we re-write equation (3.6) as follows

$$\frac{\partial Q(u,t)}{\partial t} = -\frac{\partial Q}{\partial u}\int_{-\infty}^{Q(u,t)} \frac{\partial f(x,t)}{\partial t} dx \cdot \tag{3.7}$$

We apply the Fokker-Planck equation[1] applied to equation (3.1) which gives,

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[-\mu(x_t,t) f(x_t,t) + \frac{1}{2}\frac{\partial}{\partial x}(\Sigma^2(x_t,t) f(x_t,t))\right] \cdot \tag{3.8}$$

We substitute equation (3.8) into equation (3.7) and integrate to obtain

$$\frac{\partial Q(u,t)}{\partial t} = -\frac{\partial Q}{\partial u}\left[-\mu(Q,t) f(Q,t) + \frac{1}{2}\left(\Sigma^2(Q,t)\frac{\partial f}{\partial Q} + f\frac{\partial \Sigma^2(Q,t)}{\partial Q}\right)\right] \cdot \tag{3.9}$$

We use equations (3.3) and (3.4) to completely eliminate f and its derivative, and obtain

$$\frac{\partial Q}{\partial t} = \mu(Q,t) - \frac{1}{2}\frac{\partial \Sigma^2(Q,t)}{\partial Q} + \frac{\Sigma^2(Q,t)}{2}\left(\frac{\partial Q}{\partial u}\right)^{-2}\frac{\partial^2 Q}{\partial u^2} \cdot \tag{3.10}$$

Equation (3.10) is known as the quantilised Fokker-Planck equation (QFPE) associated with the SDE in equation (3.1). This is the second order non-linear PDE.

[1] The Fokker-Planck equation is a PDE that describes the time evolution of the probability density function of a stochastic process, see [Bjö09, Proposition 5.12].

In the next section, we apply the QFPE to derive fat-tailed distributions namely, the Pearson Type IV and the Student-t distributions under equilibrium conditions i.e. when equation (3.10) is equal to zero.

$$\frac{1}{2}\frac{\partial \Sigma^2(Q,t)}{\partial Q} - \mu(Q,t) = \frac{\Sigma^2(Q,t)}{2}\left(\frac{\partial Q}{\partial u}\right)^{-2}\frac{\partial^2 Q}{\partial u^2}\,. \tag{3.11}$$

Comparing equation (2.43) with equation (3.1) we observe that,

$$\mu = (\mu_1 - \mu_2 Q) \;\text{ and }\; \Sigma^2 = \sigma_1^2 + \sigma_2^2 Q^2 + 2\rho\sigma_1\sigma_2 Q, \tag{3.12}$$

note that $\frac{\partial \Sigma^2}{\partial Q} = 2\sigma_2^2 Q + 2\rho\sigma_1\sigma_2$.

Plugging equation (3.12) into equation (3.11) we obtain

$$\frac{\partial^2 Q}{\partial u^2}\left(\frac{\partial Q}{\partial u}\right)^{-2} = \frac{2[(\rho\sigma_1\sigma_2 - \mu_1) + (\sigma_2^2 + \mu_2)Q]}{(\sigma_1^2 + \sigma_2^2 Q^2 + 2\rho\sigma_1\sigma_2 Q)}\,. \tag{3.13}$$

In order to obtain equilibrium PDF's we need to solve equation (3.13), but before we do that, we study the Pearson family distributions.

3.2 Pearson Family Distributions

The Pearson distribution types are obtained by considering a PDF f such that,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log f) = \frac{a + x - \lambda}{B_0 + B_1(x-\lambda) + B_2(x-\lambda)^2} \tag{3.14}$$

for various constants a, λ, B_0, B_1 and B_2, see [Pea94, KSO48].

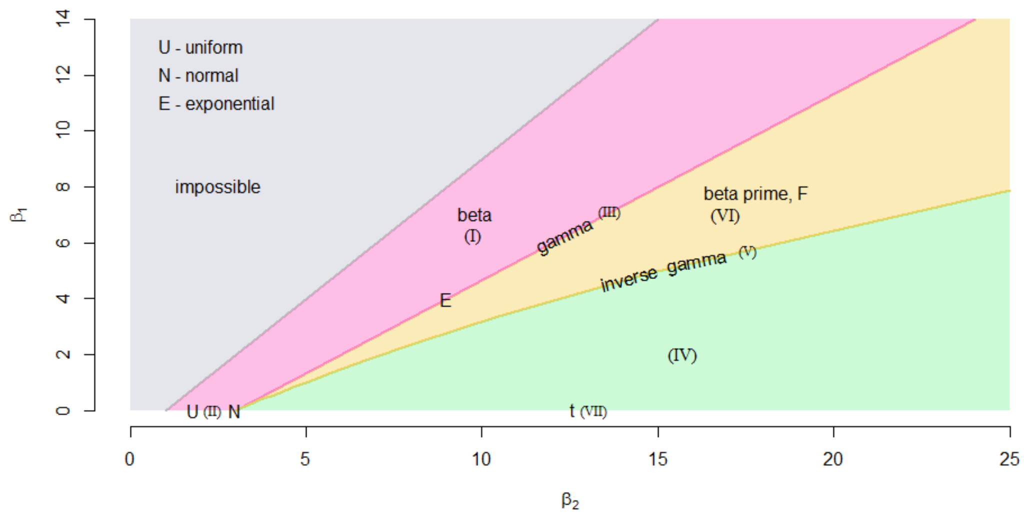


Figure 3.1: The diagram of the Pearson curve family (downloaded from http://stats.stackexchange.com/questions/236118/fitting-distribution-for-data-in-r)

Figure (3.1) represents the Pearson family curve for each range of skewness and kurtosis. The x-axis is β_2 = kurtosis and β_1 = squared skewness. Type 0: Normal distribution, Type I: Beta distribution, Type II: Student-t distribution, Type III: Gamma distribution, Type IV: Not related to any standard distribution, Type V: Inverse gamma distribution, Type VI: F-distribution and Type VII: Student's t-distribution/t-location-scale distribution.

Remark: From Figure (3.1) we observe that the Pearson Type IV distribution is suitable for a financial data that exhibits a high excess of kurtosis and a moderate skewness. Empirical studies of financial data indicate that a very high excess kurtosis but only moderate skewness is usually observed, see [Pea94, Man63, SS15]. Therefore, we expect the Pearson type IV distribution to model our financial data well, see [BM+12, SS15, SMNZ12].

In equation (3.14) the parameter a determines a stationary point. We set $a = 0$, set $X = x - \lambda$, see [KSO48]. So we re-write equation (3.14) as

$$\frac{\mathrm{d}}{\mathrm{d}X}(\log f) = \frac{X}{B_0 + B_1 X + B_2 X^2}\,. \tag{3.15}$$

The explicit solution of the density function f is found by integrating equation (3.15), [Pea94] distinguish three main types of distributions according to the roots of the denominator on the right-hand side of equation (3.15). We have real roots of opposite sign, real roots of the same sign and imaginary roots. Our interest is the imaginary roots, which will lead us to the Pearson type IV distribution, other cases will be investigated elsewhere.

So we consider a quadratic equation

$$h(X) = B_0 + B_1 X + B_2 X^2. \tag{3.16}$$

The roots of equation (3.16) are given by

$$h(X) = \frac{-B_1 \pm \sqrt{B_1^2 - 4B_0 B_2}}{2B_2} \quad \text{for} \quad B_2 > 0\cdot$$

We have imaginary roots if $B_1^2 - 4B_0B_2 < 0$, so we assume that $4B_0B_2 - B_1^2 > 0$ and $B_2 \neq 0$. We complete the square and transform equation (3.16) into

$$h(X) = B_2\left\{\left(X + \frac{B_1}{2B_2}\right)^2 + \frac{B_0}{B_2} - \frac{B_1^2}{4B_2^2}\right\}\cdot \tag{3.17}$$

Going back to equation (3.15) we have

$$\frac{\mathrm{d}}{\mathrm{d}X}(\log f) = \frac{X}{B_2\left\{\left(X + \frac{B_1}{2B_2}\right)^2 + \frac{B_0}{B_2} - \frac{B_1^2}{4B_2^2}\right\}} = \frac{X}{B_2\left\{(X+\gamma)^2 + \delta^2\right\}}, \tag{3.18}$$

where $\gamma = \frac{B_1}{2B_2}$ and $\delta^2 = \frac{B_0}{B_2} - \frac{B_1^2}{4B_2^2}\cdot$

We integrate equation (3.18) to obtain the explicit solution of the density function f.

$$\int \frac{X}{B_2\left\{(X+\gamma)^2 + \delta^2\right\}} dX = \frac{1}{B_2}\int \frac{X}{\left\{(X+\gamma)^2 + \delta^2\right\}} dX$$

$$= \frac{1}{B_2}\int \frac{(u-\gamma)}{\delta^2 + u^2} du \quad \text{by substituting } u = (X+\gamma) \therefore du = dX$$

$$= \frac{1}{B_2}\int \frac{u}{\delta^2 + u^2} du - \frac{1}{B_2}\int \frac{\gamma}{\delta^2 + u^2} du$$

$$\int \frac{u}{\delta^2 + u^2} du = \frac{1}{2}\ln|\delta^2 + u^2| + k_1 \quad \text{and} \quad \int \frac{\gamma}{\delta^2 + u^2} du = \frac{\gamma \arctan(u/\delta)}{\delta} + k_2$$

so by substituting back $u = (X+\gamma)$ and simplify we obtain

$$= \frac{\frac{1}{2}\ln|(X+\gamma)^2 + \delta^2| - \frac{\gamma \arctan(\frac{X+\gamma}{\delta})}{\delta}}{B_2} + k, \quad \text{where } k = k_1 + k_2 \cdot$$

Hence,

$$\log f = \log k + \frac{1}{2B_2}\log\left\{(X+\gamma)^2+\delta^2\right\} - \frac{\gamma}{B_2\delta}\arctan\left(\frac{X+\gamma}{\delta}\right). \tag{3.19}$$

Therefore,

$$f(X) = k\left[(X+\gamma)^2+\delta^2\right]^{\frac{1}{2B_2}}\exp\left[-\frac{\gamma}{B_2\delta}\arctan\left(\frac{X+\gamma}{\delta}\right)\right]. \tag{3.20}$$

Equation (3.20) is the Pearson Type IV distribution that is characterised by five parameters (λ, a, ν, m and k) . The derivation of the Pearson type IV was adopted from [Pea94] and [KSO48, Chapter 6].

Parameter	Definition
λ :	Translation from the centre of the distribution function.
a :	Scale of the distribution function (always > 0).
ν :	Measure of skewness.
m :	Shape factor (always > 0).
k :	Normalisation factor.

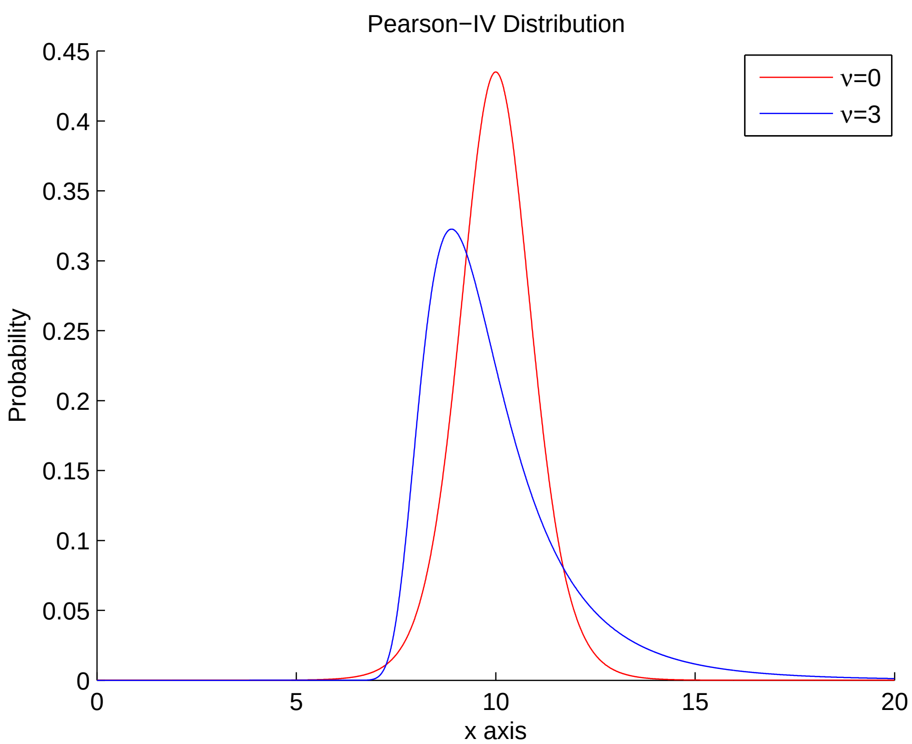


Figure 3.2: Illustrates the Pearson-IV distribution function demonstrating the effect of skewness and kurtosis, with parameters $\lambda = 10$, $a = 2$.

3.2.1 The Pearson Type IV Distribution PDF

Equation (3.20) is usually written in the form

$$f(x) = k\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{-m} \exp\left[-\nu \arctan\left(\frac{x-\lambda}{a}\right)\right] \tag{3.21}$$

where

$$m = \frac{1}{2B_2}, \; \nu = \frac{\gamma}{2B_2\delta} \text{ and } a = \delta.$$

Unfortunately, both [Pea94, KSO48] do not show how to find the constant k. The derivation of the constant k is done by [Hei04], where he shows how to calculate the normalisation factor k. It is given by

$$k = \frac{2^{2m-2}|\Gamma(m+i\nu/2)|^2}{\pi a\Gamma(2m-1)} = \frac{\Gamma(m)}{\sqrt{\pi}a\Gamma(m-1/2)}\left|\frac{\Gamma(m+i\nu/2)}{\Gamma(m)}\right|^2. \tag{3.22}$$

Now it becomes an easy task to solve (3.13), we transform it using equation (3.5) to

$$\frac{\mathrm{d}}{\mathrm{d}Q}(\log f(Q)) = -\frac{2[(\rho\sigma_1\sigma_2-\mu_1)+(\sigma_2^2+\mu_2)Q]}{(\sigma_1^2+\sigma_2^2Q^2+2\rho\sigma_1\sigma_2Q)}. \tag{3.23}$$

Comparing equation (3.23) with equation (3.15) we read-off

$$B_0 = \sigma_1^2, \; B_1 = 2\rho\sigma_1\sigma_2 \text{ and } B_2 = \sigma_2^2.$$

To obtain parameters that are consistent with [SS15], we need to calculate γ, δ, m and ν, note they write ν_2 in place for ν. So we have that

$$\gamma = \frac{B_1}{2B_2} = \frac{2\rho\sigma_1\sigma_2}{2\sigma_2^2} \therefore \lambda = -\rho\frac{\sigma_1}{\sigma_2}\cdot$$

$$\delta = \sqrt{\frac{B_0}{B_2}-\frac{B_1^2}{4B_2^2}} = \sqrt{\frac{\sigma_1^2}{\sigma_2^2}-\frac{4\rho^2\sigma_1^2\sigma_2^2}{4\sigma_2^4}} \therefore a = \frac{\sigma_1}{\sigma_2}\sqrt{1-\rho^2}.$$

$$\nu = 1+2\frac{\mu_2}{\sigma_2^2} \text{ (degress of freedom)}\cdot$$

$$m = \frac{1}{2}(\nu+1) = 1+\frac{\mu_2}{\sigma_2^2} \text{ and,}$$

$$\nu_2 = \frac{2(\mu_1\sigma_2+\rho\sigma_1\mu_2)}{\sigma_1\sigma_2^2\sqrt{1-\rho^2}}\cdot$$

3.2.2 The Pearson Type IV Distribution Moments

After we have derived the PDF for the Pearson type IV distribution, the next step is to find the moments. We are mainly interested in finding the third and fourth moments, which measure the skewness and kurtosis of a PDF respectively. We re-write equation (3.21) in a simpler form as,

$$y = y_0\left[1+\left(\frac{x}{a}\right)^2\right]^{-m}\exp\left[-\nu\arctan\left(\frac{x}{a}\right)\right]\cdot \tag{3.24}$$

Let $x = a\tan(\theta) \therefore \tan(\theta) = \frac{x}{a} \therefore \theta = \tan^{-1}(\frac{x}{a})$. Hence

$$\begin{aligned} y &= y_0\frac{1}{(1+\tan^2(\theta))^m}e^{-\nu\theta} \\ &= y_0\frac{1}{\sec^{2m}(\theta)}e^{-\nu\theta} \\ &= y_0\cos^{2m}(\theta)e^{-\nu\theta}\cdot \end{aligned}$$

Changing the limits of integration such that $x = \tan^{-1}(\infty) = \frac{\pi}{2}$ and $x = \tan^{-1}(-\infty) = -\frac{\pi}{2}$. We define the moments

$$\begin{aligned} \mu_n &= \int_{-\infty}^{\infty} y_0\cos^{2m}(\theta)e^{-\nu\theta}x^n dx \\ &= \int_{-\infty}^{\infty} y_0\cos^{2m}(\theta)e^{-\nu\theta}a^n\tan^n(\theta)a\sec^2(\theta)d\theta \quad \text{where} \quad dx = a\sec^2(\theta)d\theta \\ &= y_0a^{n+1}\int_{-\pi/2}^{\pi/2}\cos^{2m-n-2}(\theta)\sin^n(\theta)e^{-\nu\theta}d\theta, \\ &= y_0a^{n+1}\int_{-\pi/2}^{\pi/2}\cos^{r-n}(\theta)\sin^n(\theta)e^{-\nu\theta}d\theta \quad \text{where} \quad r = 2m-2. \end{aligned}$$

Hence integrating by parts with $\cos^{r-n}(\theta)\sin^n(\theta)$ as one term, we obtain

$$\int u dv = uv - \int v du \text{ where } u = e^{-\nu\theta} \text{ and } dv = \cos^{r-n}(\theta)\sin^n(\theta).$$

$$\mu_n = \frac{y_0a^{n+1}}{r-n+1}\left\{(n-1)\int_{-\pi/2}^{\pi/2}\cos^{r-n+2}(\theta)\sin^{n-2}(\theta)e^{-\nu\theta}d\theta - \nu\int_{-\pi/2}^{\pi/2}\cos^{r-n+1}(\theta)\sin^{n-1}(\theta)e^{-\nu\theta}d\theta\right\}\cdot \tag{3.25}$$

Hence,

$$\mu_n = \frac{a}{r-n+1}\left\{(n-1)a\mu_{n-2} - \nu\mu_{n-1}\right\} \text{ provided } r > n-1 \text{ with } \mu_0 = 1,\ \mu_1 = 0\cdot \tag{3.26}$$

So we have:

The mean

$$\mu_1 = \lambda - \frac{a\nu}{r} \qquad (m > 1)\cdot \tag{3.27}$$

The variance

$$\mu_2 = \frac{a^2}{r^2(r-1)}(r^2+\nu^2) \qquad (m > 3/2)\cdot \tag{3.28}$$

The third moment

$$\mu_3 = -\frac{4a^3\nu(r^2+\nu^2)}{r^3(r-1)(r-2)} \qquad (m>2)\,. \tag{3.29}$$

The fourth moment

$$\mu_4 = \frac{3a^4(r^2+\nu^2)[(r+6)(r^2+\nu^2)-8r^2]}{r^4(r-1)(r-2)(r-3)} \qquad (m>5/2)\,. \tag{3.30}$$

Equations (3.27 - 3.30) are used to fit financial data to the Pearson type IV distribution, by using the method of moments, see [BCY08, SZ13].

3.3 The Student-t Distribution PDF

We consider the special case when $\mu_1 = 0 = \rho$ in the SDE (2.43), so we obtain the SDE

$$dX_t = -\mu_2 X_t dt + \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2}\, dW_t, \tag{3.31}$$

as before we obtain the quantile ODE from equation (3.13)

$$\frac{\partial^2 Q}{\partial u^2}\left(\frac{\partial Q}{\partial u}\right)^{-2} = \frac{2(\sigma_2^2+\mu_2)Q}{(\sigma_1^2+\sigma_2^2 Q^2)} \tag{3.32}$$

where

$$a = \frac{\sigma_1}{\sigma_2}$$

$$\lambda = 0$$

$$\nu_2 = 0$$

$$k = \frac{\Gamma(\frac{\nu+1}{2})}{a\sqrt{\pi}\Gamma(\frac{\nu}{2})}$$

m and ν remain the same as in the Pearson Type IV distribution. So we obtain the Student-t PDF

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{a\pi}\Gamma(\frac{\nu}{2})}\left(1 + x^2/a\right)^{-\frac{\nu+1}{2}}. \tag{3.33}$$

3.3.1 The Student-t Distribution Moments

We do not prove the following results since they are very well-known. See for example [Sha11] where he derives the moments from the first principle. The expected value and the skewness of the Student-t distribution are equal to zero (i.e. just like a Normal distribution). The variance is given by $\frac{\nu}{\nu-2}$ if $\nu > 2$, and the excess kurtosis is given by $\frac{6}{\nu-4}$ if $\nu > 4$.

We have shown above that under equilibrium conditions we are able to derive the Pearson Type IV and the Student-t distributions from a simple SDE model. However, it is not common for a market to remain at equilibrium for a long period especially during financial panics, see [SS15, Sch15]. Thus in the next section, we explore non-equilibrium density functions, from the SDE point of view and the full Fokker-Planck equation following the work by [SS15].

3.4 Non-equilibrium Density Function: A Special Case

In this section, we next analyse the full Fokker-Planck equation on the general SDE (2.43) to obtain the density function. We have

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[-(\mu_1-\mu_2 x)f(x,t)+\frac{1}{2}\frac{\partial}{\partial x}[(\sigma_1^2+\sigma_2^2x^2+2\rho\sigma_1\sigma_2 x)f(x,t)]\right] \tag{3.34}$$

with the initial condition $f(x,0)=\delta(x)$,where $\delta(x)$ is the Dirac delta function. We state the property of the $\delta(x)$ function, see [AS64] for proof,

$$\int_a^b \varphi(x)\delta(x)dx = \begin{cases} \varphi(0) & \text{if } a<0<b \\ 0 & \text{if } 0\notin[a,b] \\ \frac{1}{2}\varphi(0+) & \text{if } a=0 \\ \frac{1}{2}\varphi(0-) & \text{if } b=0 \end{cases}$$

We introduce the Laplace transform w.r.t. time, so we define

$$\tilde{f}(x,s) = \int_0^\infty f(x,t)e^{-st}dt\,\cdot \tag{3.35}$$

By the definition of Laplace transform we have,

$$s\tilde{f}-\delta(x,0) = \frac{\partial}{\partial x}\left[-(\mu_1-\mu_2 x)\tilde{f}+\frac{1}{2}\frac{\partial}{\partial x}[(\sigma_1^2+\sigma_2^2x^2+2\rho\sigma_1\sigma_2 x)\tilde{f}]\right]\cdot \tag{3.36}$$

We consider two independent solutions for equation (3.36), where $x>0$ and $x<0$

$$s\tilde{f} = \frac{\partial}{\partial x}\left[-(\mu_1-\mu_2 x)\tilde{f}+\frac{1}{2}\frac{\partial}{\partial x}[(\sigma_1^2+\sigma_2^2x^2+2\rho\sigma_1\sigma_2 x)\tilde{f}]\right]\cdot \tag{3.37}$$

such that $\tilde{f}$ is continuous at $x = 0$ and differentiable. Integrating equation (3.36) with limits of $-a$ to a and setting $a \to 0$ we have

$$\int_{-a}^{a} (s\tilde{f} - \delta(x,0))dx = \int_{-a}^{a} \frac{\partial}{\partial x}\left[-(\mu_1 - \mu_2 x)\tilde{f} + \frac{1}{2}\frac{\partial}{\partial x}[(\sigma_1^2 + \sigma_2^2 x^2 + 2\rho\sigma_1\sigma_2 x)\tilde{f}]\right] dx$$

$$\int_{-a}^{a} s\tilde{f}(x,s)dx \to 0 \ \text{ and } \ \int_{-a}^{a} \delta(x,0)dx = 1\cdot$$

So we have the condition about zero

$$-1 = \frac{\sigma_1^2}{2}\left(\frac{\partial \tilde{f}}{\partial x}(0+,s) - \frac{\partial \tilde{f}}{\partial x}(0-,s)\right) \Rightarrow \frac{\partial \tilde{f}}{\partial x}(0+,s) - \frac{\partial \tilde{f}}{\partial x}(0-,s) = -\frac{2}{\sigma_1^2}\cdot \tag{3.38}$$

We first consider the case where $\sigma_2 = 0 = \mu_2$ in equation (3.37),

$$\begin{aligned} s\tilde{f} &= \frac{\partial}{\partial x}[-\mu_1 \tilde{f} + \frac{1}{2}\frac{\partial}{\partial x}(\sigma_1^2 \tilde{f})] \\ &= \frac{\partial}{\partial x}[-\mu_1 \tilde{f} + \frac{1}{2}\sigma_1^2 \tilde{f}'] \\ &= -\mu_1 \tilde{f}' + \frac{1}{2}\sigma_1^2 \tilde{f}''. \end{aligned}$$

So we solve the ODE

$$\frac{1}{2}\sigma_1^2 \tilde{f}'' - \mu_1 \tilde{f}' - s\tilde{f} = 0 \tag{3.39}$$

with the roots: $\frac{\mu_1}{\sigma_1^2} \pm \frac{\sqrt{\mu_1^2 + 2s\sigma_1^2}}{\sigma_1^2}$ and the junction condition at zero we obtain

$$\tilde{f}(x,s) = \frac{e^{\mu_1 x/\sigma_1^2}}{\sqrt{\mu_1^2 + 2s\sigma_1^2}} \begin{cases} \exp\left[-\frac{x}{\sigma_1^2}\sqrt{\mu_1^2 + 2s\sigma_1^2}\right] & \text{if } x > 0 \\ \exp\left[+\frac{x}{\sigma_1^2}\sqrt{\mu_1^2 + 2s\sigma_1^2}\right] & \text{if } x < 0 \end{cases}$$

Inversion of the two cases[2] leads to the Gaussian density function, i.e. $f(x,t) = \mathcal{L}^{-1}\{\tilde{f}(x,s)\}$

$$f(x,t) = \frac{1}{\sqrt{2\pi\sigma_1^2 t}} \exp[-(x-\mu_1 t)^2/(2\sigma_1^2 t)]\cdot \tag{3.40}$$

We next consider a very interesting case when $\rho = 0 = \mu_1$.

$$\begin{aligned} s\tilde{f} &= \frac{\partial}{\partial x}\left[\mu_2 x\tilde{f} + \frac{1}{2}\frac{\partial}{\partial x}(\sigma_1^2 \tilde{f} + \sigma_2^2 x^2 \tilde{f})\right] \\ &= \mu_2 \tilde{f} + \mu_2 x\tilde{f}' + \frac{\sigma_1^2}{2}\tilde{f}'' + \sigma_2^2 \tilde{f} + \sigma_2^2 x\tilde{f}' + \sigma_2^2 x\tilde{f}' + \frac{\sigma_2^2}{2}x^2\tilde{f}'' \end{aligned}$$

[2] Using the identity 29.3.84 in [AS64].

$$\therefore \frac{1}{2}(\sigma_1^2 + \sigma_2^2 x^2)\tilde{f}''(x,s) + (\mu_2 + 2\sigma_2^2)x\tilde{f}'(x,s) + (\mu_2 + \sigma_2^2 - s)\tilde{f}(x,s) = 0\cdot \tag{3.41}$$

Equation (3.41) is transformed by setting

$$\tilde{f}(x,s) = (\sigma_1^2 + \sigma_2^2 x^2)^{-(1+\mu_2/\sigma_2^2)} g(x,s), \tag{3.42}$$

computing $\tilde{f}'(x,s)$ and $\tilde{f}''(x,s)$ on equation (3.42) and simplify, then the ODE (in terms of $g(x,s)$) is given by

$$(\sigma_1^2 + \sigma_2^2 x^2) g''(x,s) - 2\mu_2 x g'(x,s) - 2sg(x,s) = 0\cdot \tag{3.43}$$

We solve the ODE by using the change of independent variables, we introduce $z(x)$ such that

$$\frac{dz}{dx} = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 x^2}}, \tag{3.44}$$

where

$$z(x) = \frac{1}{\sigma_2}\sinh^{-1}\left(\frac{\sigma_2 x}{\sigma_1}\right)\cdot \tag{3.45}$$

Re-writing equation (3.43) in terms of z, we have

$$\frac{d^2 g}{dz^2} - (2\mu_2 + \sigma_2^2)\frac{1}{\sigma_2}\tanh(\sigma_2 z)\frac{dg}{dz} - 2sg = 0, \tag{3.46}$$

expressing equation (3.46) using $\nu = 1 + 2\mu_2/\sigma_2^2$, we obtain

$$\frac{d^2 g}{dz^2} - \nu\sigma_2 \tanh(\sigma_2 z)\frac{dg}{dz} - 2sg = 0\cdot \tag{3.47}$$

We consider the case when $\nu = 0$, then we solve equation (3.47) (in terms of g) together with equation (3.42) to obtain

$$\tilde{f}(x,s) = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 x^2}}\frac{1}{\sqrt{2s}}\begin{cases} e^{-\sqrt{2s}z(x)} & \text{if } x,z > 0 \\ e^{+\sqrt{2s}z(x)} & \text{if } x,z < 0 \end{cases}$$

The inversion of the two cases[3] leads to the Bougerol identity function, i.e. $f(x,t) = \mathcal{L}^{-1}\{\tilde{f}(x,s)\}$

$$f(x,t) = \frac{1}{\sqrt{2\pi t(\sigma_1^2 + \sigma_2^2 x^2)}}\exp\left\{\frac{-1}{2\sigma_2^2 t}[\sinh^{-1}(\sigma_2 x/\sigma_1)]^2\right\}\cdot \tag{3.48}$$

[3]Using the identity 29.3.84 in [AS64].

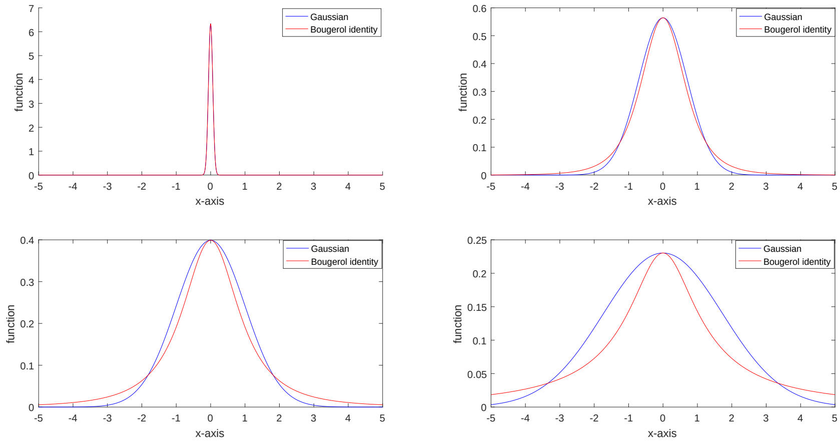


Figure 3.3: Gaussian vs Bougerol identity PDFs, with $\sigma_1 = \sigma_2 = 1$ and $t = 1/252, 1/2, 1$ and 3 respectively. For small t the Gaussian and the Bougerol PDFs look identical, however as t gets large they differ considerably.

In Figure (3.4) we plot the log-PDF's for the Gaussian (3.40), Student-t (3.33) with 4 d.o.f, Pearson-IV (3.21) and the special case hybrid model (Bougerol) in equation (3.48) (for parameters $\sigma_1 = \sigma_2 = 0.5$ and $t = 2$). We observe that as time passes the Bougerol, Student-t and Pearson-IV distributions spreads out more in a manner that is consistent with the phenomenon of variance explosion [SS15].

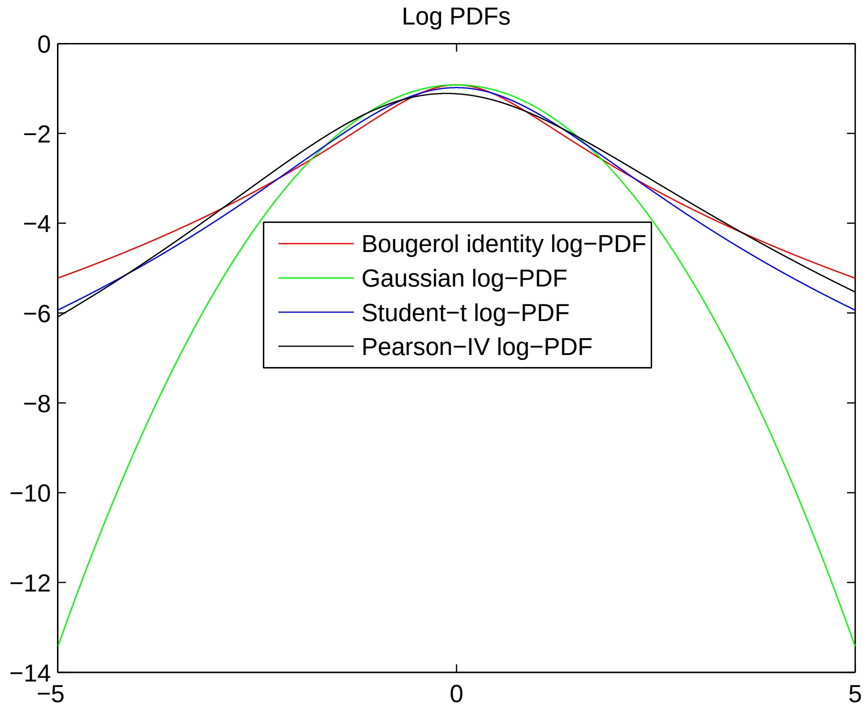


Figure 3.4: Illustrates the Log-PDF's for the Gaussian, Student-t (4 d.o.f), Bougerol identity and Pearson-IV with a mean = 0, sigma = 1.5, skewness = 0.7 and a kurtosis = 10.

3.5 Non-equilibrium Density Function: A General Case

In a special case we solved the ODE in equation (3.47) for $\nu = 0$, in a general case we solve the ODE for $\nu \geq 0$. From equations (3.45 - 3.47), we change the independent variable to

$$u = \sinh^{-1}(\sigma_2 x/\sigma_1) = \sigma_2 z(x)\cdot \tag{3.49}$$

We set $k = 2s/\sigma_2^2$ and use the identity $\tanh(u) = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$ to re-write equation (3.47) as follows

$$(e^u + e^{-u})\left(\frac{d^2 g}{du^2} - kg(u)\right) = \nu(e^u - e^{-u})\frac{dg}{du}\cdot \tag{3.50}$$

We simplify equation (3.50) by setting

$$w = e^{-u} = \left(\frac{\sigma_2 x}{\sigma_1} + \sqrt{1 + \frac{\sigma_2^2 x^2}{\sigma_1^2}}\right)^{-1}. \tag{3.51}$$

So we have $u = -\log w$ and $d/du = -w\, d/dw$, and the ODE in equation (3.50) can be written as follows

$$\left(\frac{1}{w} + w\right)\left[w^2 \frac{dg^2}{dw^2} + w\frac{dg}{dw} - kg(w)\right] = \nu\left(w - \frac{1}{w}\right)w\frac{dg}{dw}\cdot \tag{3.52}$$

To solve the ODE in equation (3.52) we seek the Frobenius power series solution, which is written in the form

$$g(w) = \sum_{n=0}^{\infty} a_n w^{n+r} \ \ (a_0 \neq 0), \ r > 0 \tag{3.53}$$

then we compute

$$\frac{dg}{dw} = \sum_{n=0}^{\infty} a_n(n+r)w^{n+r-1}, \ r > 1 \tag{3.54}$$

$$\frac{dg^2}{dw^2} = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)w^{n+r-2}, \ r > 2 \tag{3.55}$$

Plugging equations (3.53 - 3.55) in equation (3.52), we obtain

$$\left(\frac{1}{w}+w\right)\left[w^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)w^{n+r-2} + w\sum_{n=0}^{\infty} a_n(n+r)w^{n+r-1} - k\sum_{n=0}^{\infty} a_n w^{n+r}\right]$$
$$= \nu\left(w^2-1\right)\sum_{n=0}^{\infty} a_n(n+r)w^{n+r-1} \ .$$

Grouping the terms:

w^{r-1} : $(r^2+\nu r-k)a_0 = 0 \therefore r^2+\nu r-k = 0$ since $a_0 \neq 0$. So we have

$$r = \sqrt{\frac{\nu^2}{4}+k} - \frac{\nu}{2} \tag{3.56}$$

w^r : $\underbrace{(r^2+2r+1-k+\nu(1+r))}_{\star} a_1 = 0 \therefore a_1 = 0$ since $(\star)$ is non-zero for $r > 0$ in equation (3.56).

Since $a_1 = 0$ we ignore all the odd terms in the general sequence.

w^{r+1} : $a_2(4r+4+2\nu) = 2\nu r a_0$, so we obtain the following recurrence relation

$$a_{n+2} = -a_n\frac{(n-\nu)(n+2r)}{(n+2)(n+2+2r+\nu)}, \ n = 0, 2, 4, \cdots \tag{3.57}$$

We compute the coefficients a_n for $n = 0, 2, 4, \cdots$

For $n = 0$

$$a_2 = \frac{a_0\nu r}{(2+2r+\nu)} \ . \tag{3.58}$$

For $n = 2$

$$a_4 = -\frac{a_2(2-\nu)(2+2r)}{4(4+2r+\nu)} = -\frac{a_0\nu r(2+2r)(2-\nu)}{4(2+2r+\nu)(4+2r+\nu)} \tag{3.59}$$

and so on. So we have

$$\begin{aligned} g(w) &= w^r(a_0w^0 + a_2w^2 + a_4w^4 + \cdots) \\ &= a_0w^r\left(1 + \frac{\nu r}{(2+2r+\nu)}w^2 - \frac{\nu r(2+2r)(2-\nu)}{4(2+2r+\nu)(4+2r+\nu)}w^4 + \cdots\right). \end{aligned} \tag{3.60}$$

[SS15] show that we can write equation (3.60) in a compact form using the hypergeometric function. So we write equation (3.60) as follows,

$$g(w) = a_0 w^r {}_2F_1(r, (-\nu/2); r + (\nu/2) + 1; -w^2)\,. \tag{3.61}$$

The Gauss hypergeometric series is defined as follows

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{3.62}$$

where $(a)_n$, $(b)_n$ and $(c)_n$ are defined by the Pochhammer symbol. The Pochhammer symbol $(\alpha)_n$ is defined as

$$\begin{aligned} (\alpha)_n &= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) \quad \text{where } (\alpha)_0 \equiv 1 \text{ and } n \text{ is a positive integer.} \\ &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,. \end{aligned} \tag{3.63}$$

In equation (3.61) we have $a = r$, $b = -\nu/2$, $c = r+(\nu/2)+1$ and $z = -w^2$. So the hypergeometric series in equation (3.61) is given by

$$\begin{aligned} {}_2F_1(r,(-\nu/2); r+(\nu/2)+1; -w^2) &= \sum_{n=0}^{\infty} \frac{(r)_n(-\nu/2)_n}{(r+(\nu/2)+1)_n}\frac{(-w^2)^n}{n!} \\ &= 1 + \frac{r(\nu/2)}{(r+(\nu/2)+1)}w^2 - \frac{r(r+1)(-\nu/2)(-\nu/2+1)}{(r+\nu/2+1)(r+\nu/2+2)}\frac{w^4}{2} + \cdots \\ &= 1 + \frac{\nu r}{(2r+\nu+2)}w^2 - \frac{\nu r(2r+2)(2-\nu)}{4(2r+\nu+2)(2r+\nu+4)}w^4 + \cdots \end{aligned} \tag{3.64}$$

Clearly equation (3.61) is the same as equation (3.60).

We re-write equation (3.62) using equation (3.63) as follows

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}\,. \tag{3.65}$$

The hypergeometric function was first introduced by John Wallis in 1655, which was further studied by, Gauss (1812), Kummer (1836), Reimann (1857) and Thomae (1879). Hypergeometric functions are special functions which arise from a solution of ODE's, see [Sla].

If $a = 1$ and $b = c$ in equation (3.65), then the hypergeometric series is reduced to a geometric series.

To complete our solution we determine a_0, which is given by

$$a_0 = \frac{\sigma_1^{\nu}}{\sigma_2 \Psi(\nu, r)} \tag{3.66}$$

where

$$\Psi(\nu, r) = \frac{d}{dw}\left[w^r {}_2F_1(r, -(\nu/2); r + (\nu/2) + 1; -w^2)\right]\Bigg|_{w=1} . \tag{3.67}$$

Using the following identities from [AS64].

$${}_2F_1(a, b; a - b + 1; -1) = 2^{-a}\sqrt{\pi}\frac{\Gamma(1 + a - b)}{\Gamma(1 + (a/2) - b)(\Gamma(1/2 + (a/2)))} \quad \text{for} \quad (1 + a - b \neq 0, -1, -2, \cdots) \tag{3.68}$$

$${}_2F_1(a, b; a - b + 2; -1) = 2^{-a}\sqrt{\pi}(b - 1)^{-1}\Gamma(a - b + 2) \times \tag{3.69}$$
$$\left[\frac{1}{\Gamma(a/2)\Gamma(3/2 + (a/2) - b)} - \frac{1}{\Gamma(1/2 + (a/2))\Gamma(a + (a/2) - b)}\right] \text{ for } (a - b + 2 \neq 0, -1, -2, \cdots)$$

so with $a = r$ and $b = -\nu/2$ we have

$$\Psi(\nu, r) = \frac{2^{1-r}\sqrt{\pi}\Gamma(r + (\nu/2 + 1))}{\Gamma(r/2)\Gamma(0.5(r + \nu + 1))} . \tag{3.70}$$

So the Laplace transform for the general case is given by

$$\tilde{f}(x, s) = \frac{\left\{\sigma_1^{\nu} 2^{r-1} w^r \Gamma(r/2)\Gamma(0.5(r + \nu + 1)) \times {}_2F_1(r, -(\nu/2); r + (\nu/2) + 1; -w^2)\right\}}{\sqrt{\pi}\sigma_2\Gamma(r + (\nu/2) + 1)(\sigma_1^2 + \sigma_2^2 x^2)^{0.5(\nu+1)}} . \tag{3.71}$$

To obtain a PDF for the general case we need to apply the Laplace inverse transform on equation (3.71) (which is difficult).

We test the general case by substituting values for ν, where ν is an even integer. We start with $\nu = 0$ (in equation 3.71) to obtain the special case Laplace transform. We obtain the following

$$\begin{aligned}
\tilde{f}(x, s) &= \frac{2^{r-1} w^r \Gamma(r/2)\Gamma(0.5(r + 1) \times {}_2F_1(r, 0; r + 1; -1))}{\sigma_2\sqrt{\pi}\Gamma(r + 1)\sqrt{\sigma_1^2 + \sigma_2^2 x^2}} \\
&= \frac{w^r \Gamma(r/2)}{2\sigma_2\Gamma(r + 1)\sqrt{\sigma_1^2 + \sigma_2^2 x^2}} \quad \text{using the identity in equation (3.68)} .
\end{aligned}$$

Using the gamma property $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 0$, we obtain

$$\Rightarrow \tilde{f}(x,s) = \frac{w^r}{r\sigma_2\sqrt{\sigma_1^2 + \sigma_2^2 x^2}} \tag{3.72}$$

where

$$k = \frac{2s}{\sigma_2^2},\ r = \sqrt{k} = \sqrt{\frac{2s}{\sigma_2^2}} = \frac{\sqrt{2s}}{\sigma_2}$$

$$w = e^{-u} \therefore w^r = e^{-ur} = e^{-\sqrt{2s}z(x)}\ .$$

Equation (3.72) is the same as the one we obtained for a special case above.

Similarly for $\nu = 2$

$$\begin{aligned}\tilde{f}(x,s) &= \frac{\sigma_1^2}{2\sigma_2(\sigma_1^2 + \sigma_2^2 x^2)^{3/2}}\left(\frac{w^r}{r} + \frac{w^{r+2}}{r+2}\right) \\ &= \frac{\sigma_1^2}{2\sigma_2(\sigma_1^2 + \sigma_2^2 x^2)^{3/2}} \times \left[\frac{e^{-[\sqrt{k+1}-1][u(x)]}}{\sqrt{k+1}-1} + \frac{e^{-[\sqrt{k+1}+1][u(x)]}}{\sqrt{k+1}+1}\right]\end{aligned} \tag{3.73}$$

where

$$k = \frac{2s}{\sigma_2^2},\ r = \sqrt{k+1} - 1,\ u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1) = \sigma_2 z(x)\ \cdot$$

[SS15, Sch15] show that for the case $\nu = 2$, the Laplace inverse transform of equation (3.73) gives the following PDF

$$\begin{aligned}f(x,t) &= \frac{\sigma_1 \exp[-(u(x)^2/2t\sigma_2^2) - (t\sigma_2^2/2)]}{\sqrt{2\pi t}(\sigma_1^2 + \sigma_2^2 x^2)} \\ &+ \frac{\sigma_1^2\sigma_2}{2(\sigma_1^2 + \sigma_2^2 x^2)^{3/2}} \times \left[\Phi\left(\frac{|u(x)| + t\sigma_2^2}{\sqrt{t}\sigma_2}\right) - \Phi\left(\frac{|u(x)| - t\sigma_2^2}{\sqrt{t}\sigma_2}\right)\right]\end{aligned} \tag{3.74}$$

where Φ is the standard Normal CDF. This PDF has the following properties:

- Zero mean.
- Variance is $\sigma_1^2 t,\ \forall \sigma_2, t$.
- As $t \to 0$ the asymptotic behavior is of the Gaussian form

$$f(x,t) \sim \frac{\sigma_1 \exp[-u(x)^2/2t\sigma_2^2]}{\sqrt{2\pi t}(\sigma_1^2 + \sigma_2^2 x^2)} \sim \frac{\exp[-x^2/2t\sigma_1^2]}{\sqrt{2\pi t}\sigma_1}\ .$$

- as $t \to \infty$ the density takes the form of the scaled Student-t distribution with 2 d.o.f .

$$f(x,t) \sim \frac{\sigma_1^2\sigma_2}{2(\sigma_1^2 + \sigma_2^2 x^2)^{3/2}}\ .$$

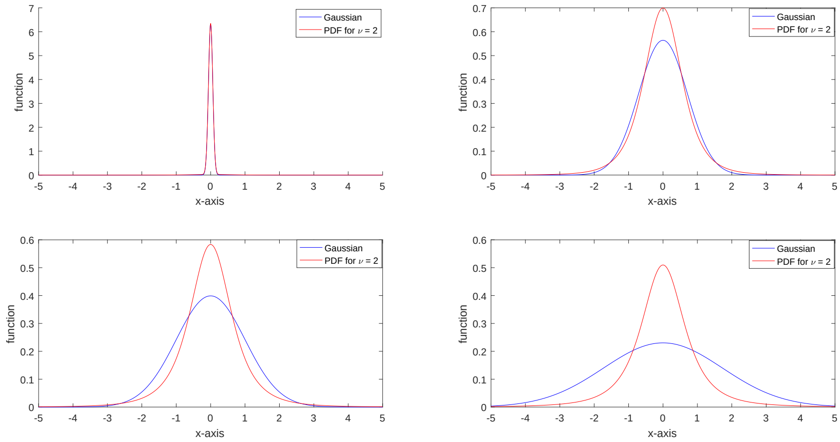


Figure 3.5: Gaussian vs PDF $\nu = 2$, with $\sigma_1 = \sigma_2 = 1$ and $t = 1/252, 1/2, 1$ and 3 respectively.

For $\nu = 4$

$$\begin{aligned}\tilde{f}(x,s) &= \frac{\sigma_1^4}{4\sigma_2(\sigma_1^2 + \sigma_2^2 x^2)^{5/2}}\left(\frac{w^r(r+3)}{r(r+2)} + \frac{2w^{r+2}}{r+2} + \frac{w^{r+4}(r+1)}{(r+2)(r+4)}\right) \\ &= \frac{\sigma_1^4}{4\sigma_2(\sigma_1^2 + \sigma_2^2 x^2)^{5/2}} \times \left[\frac{(\sqrt{k+4}+1)e^{-[\sqrt{k+4}-2][u(x)]}}{\sqrt{k+4}(\sqrt{k+4}-2)} + \frac{2e^{-[\sqrt{k+4}+1][u(x)]}}{\sqrt{k+4}} + \right. \\ &\left. \frac{(\sqrt{k+4}-1)e^{-[\sqrt{k+4}+2][u(x)]}}{\sqrt{k+4}(\sqrt{k+4}+2)}\right]\end{aligned} \tag{3.75}$$

where

$$k = \frac{2s}{\sigma_2^2},\ r = \sqrt{k+4} - 2,\ u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1) = \sigma_2 z(x)\,\cdot$$

We cannot find an explicit inverse Laplace transform for the case $\nu = 4$ (equation 3.75), neither does [SS15] nor [Sch15], give explicit PDF for this case. One has to employ numerical methods to find the inverse Laplace transform, see [AW92].

We have already examined the cases when $\nu = 0$ (special case) and $\nu \geq 0$ (general case). Lastly, we examine the case when $\nu < 0$. For this case, we apply the Legendre symmetry in [AS64].

$${}_2F_1(c-a, c-b; c; z) = (1-z)^{a+b-c}\,{}_2F_1(a,b;c;z)\,\cdot \tag{3.76}$$

Recall that we have $a = r,\ b = -\nu/2,\ c = r + (\nu/2) + 1$ and $z = -w^2$. So we have

$${}_2F_1(r, (-\nu/2); r + (\nu/2) + 1; -w^2) = (1+w^2)^{\nu+1}\,{}_2F_1((r/2)+1, r+\nu+1; r+(\nu/2)+1; -w^2)\,\cdot \tag{3.77}$$

Also recall that

$$
\begin{aligned}
w &= \left(\frac{\sigma_2 x}{\sigma_1} + \sqrt{1 + \frac{\sigma_2^2 x^2}{\sigma_1^2}} \right)^{-1} \\
\therefore w^2 + 1 &= \frac{2w}{\sigma_1} \sqrt{\sigma_1^2 + \sigma_2^2 x^2} \cdot
\end{aligned}
\tag{3.78}
$$

So the Laplace transform for the case $\nu < 0$ is given by

$$
\tilde{f}(x,s) = \frac{2^{r+\nu} w^{r+\nu+1} \Gamma(r/2)\Gamma(0.5(r+\nu+1))}{\sqrt{\pi}\sigma_1\sigma_2\Gamma(r+(\nu/2)+1)} \times {}_2F_1((r/2)+1, r+\nu+1; r+(\nu/2)+1; -w^2) \tag{3.79}
$$

where $r = \sqrt{\frac{\nu^2}{4} + k} - \frac{\nu}{2} \cdot$

For $\nu = -2$

$$
\begin{aligned}
\tilde{f}(x,s) &= \frac{w^{r-1}}{\sigma_1\sigma_2(r-1)} \\
&= \frac{w^{\sqrt{k+1}}}{\sigma_1\sigma_2\sqrt{k+1}}
\end{aligned}
\tag{3.80}
$$

where

$$
r = \sqrt{k+1} + 1 \cdot
$$

The Laplace inverse transform of equation (3.80) leads to the Generalized Bougerol Identity,

$$
f(x,t) = \frac{1}{\sqrt{2\pi t(\sigma_1^2 + \sigma_2^2 x^2)}} \exp\left\{ -\frac{\sigma_2^2 t}{2} - \frac{u(x)^2}{2\sigma_2^2 t} \right\}, \text{ where } u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1) \cdot \tag{3.81}
$$

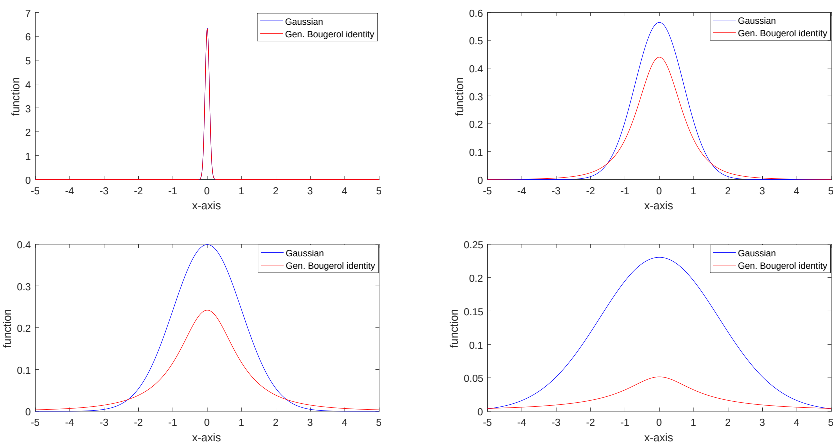


Figure 3.6: Gaussian vs Generalized Bougerol identity PDFs, with $\sigma_1 = \sigma_2 = 1$ and $t = 1/252, 1/2, 1$ and 3 respectively.

For $\nu = -4$

$$\begin{aligned}\tilde{f}(x,s) &= \frac{1}{2\sigma_1\sigma_2}\left(\frac{w^{r-1}}{r-1} + \frac{w^{r-3}}{r-3}\right) \\ &= \frac{1}{2\sigma_1\sigma_2}\left(\frac{w^{\sqrt{k+4}+1}}{\sqrt{k+4}+1} + \frac{w^{\sqrt{k+4}-1}}{\sqrt{k+4}-1}\right)\end{aligned} \tag{3.82}$$

where

$$r = \sqrt{k+4} + 2\,\cdot$$

[Sch15] show that the Laplace inverse transform is given by

$$\begin{aligned}f(x,t) &= e^{-2\sigma_2^2 t}\left(\frac{\sqrt{(1+\sigma_2 x/\sigma_1)^2}}{\sqrt{2\pi\sigma_1^2 t}} e^{-u(x)^2/2t}\right) \\ &+ \frac{e^{\frac{\sigma_2^2 t}{2}}}{2a} \times \left[\Phi\left(\frac{|u(x)| + t\sigma_2}{\sqrt{t}}\right) - \Phi\left(\frac{|u(x)| - t\sigma_2}{\sqrt{t}}\right)\right]\end{aligned} \tag{3.83}$$

where

$$a = \frac{\sigma_1}{\sigma_2}\sqrt{1-\rho^2},\ u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1) = \sigma_2 z(x)$$

where Φ is the standard Normal CDF.

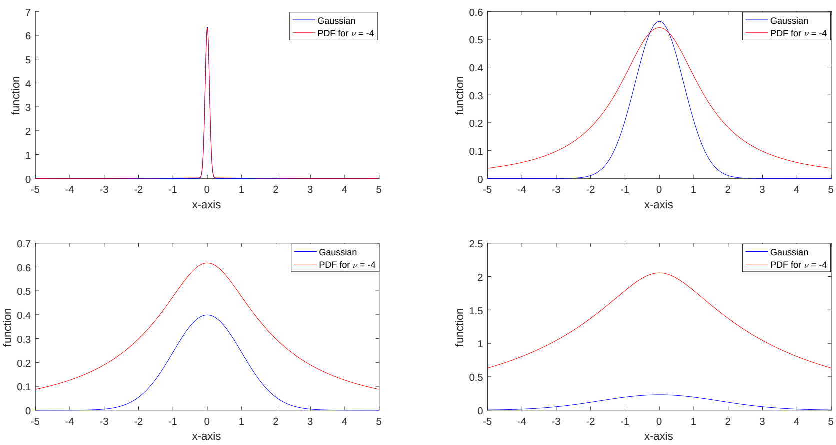


Figure 3.7: Gaussian vs PDF $\nu = -4$, with $\sigma_1 = \sigma_2 = 1$ and $t = 1/252, 1/2, 1$ and 3 respectively.

3.6 Chapter Summary

In this chapter we have derived the Pearson type-IV distribution subsequently, we derived the Student-t distribution based on the hybrid SDE. In Figure (3.2) we illustrated the flexibility of the Pearson type-IV distribution, in terms of being able to capture skewness and kurtosis, based on some choices of parameters. Both the Person type-IV and the Student-t distributions were derived under equilibrium conditions in the QFPE. However, as we remarked earlier it not usual for a market to remain at equilibrium especially during financial turbulence, hence we explored the full Fokker-Planck equation under non-equilibrium conditions. Under special cases where $\nu = 0, 2, -2$ and $\nu = -4$ we explicitly derived the PDF's for the non-equilibrium cases. For large values of ν, it becomes difficult to explicitly compute the Laplace inverse transform, as a consequence it becomes difficult to explicitly derive a PDF.

In the next chapter, we fit our model to the market data using the density functions derived in this chapter.

Chapter 4

Empirical Data and Distribution Fitting

4.1 Empirical Data

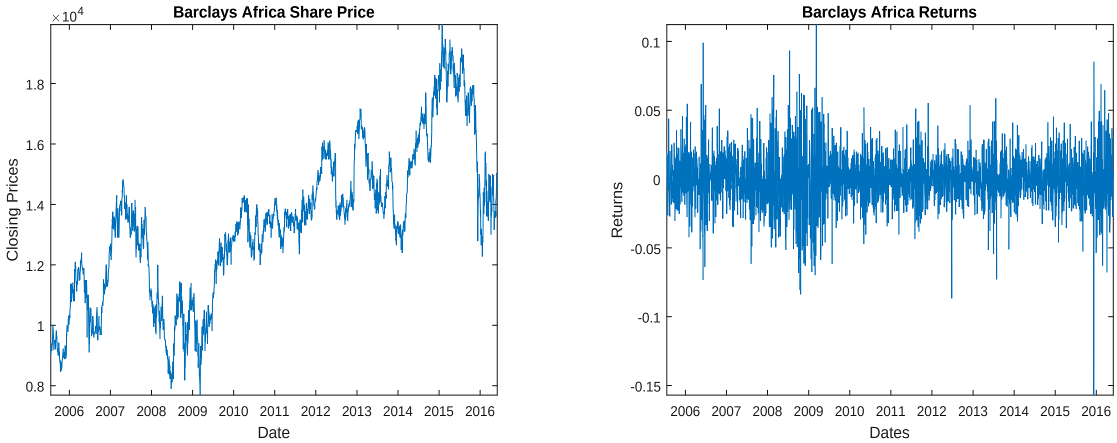
In this chapter, we investigate empirically the distributions obtained from the hybrid SDE. We analyse four South African shares and the FTSE/JSE Top 40 index. We investigate the normality assumption and fit our distributions to the market data. We randomly selected four shares which include Barclays Africa, Nedbank, Old Mutual and Standard Bank. We include the FTSE/JSE Top 40 index which consists of the top 40 largest companies listed on the Johannesburg Stock Exchange (JSE). The closing prices for each share was obtained from google finance (https://www.google.com/finance). We start by calculating daily log returns using equation (2.1), all shares exclude dividends.

Table (4.1) summarises the first four statistical moments of our shares, Figures (4.1a - 4.5a) displays the daily closing returns of our shares on different dates. The dates for Barclays Africa (20-07-2005 to 01-06-2016), FTSE/JSE Top 40 (05-06-2006 to 01-06-2016), Nedbank (29-09-2003 to 01-06-2016), Old Mutual (28-08-2003 to 01-06-2016) and Standard Bank (30-09-2003 to 01-06-2016).

Company	Mean	Variance	Skewness	Excess Kurtosis	No. of Obs.
Barclays Africa	1.7609×10^{-4}	0.0194	-0.0623	6.6555	2716
FTSE/JSE Top 40	7.9264×10^{-4}	0.0182	0.3101	9.1825	2494
Nedbank	3.0500×10^{-4}	0.0187	0.0027	5.9124	3167
Old Mutual	1.5755×10^{-4}	0.0259	-0.2338	17.4072	3193
Standard Bank	4.2719×10^{-4}	0.0187	-0.0267	6.2421	3168

Table 4.1: Statistical data summary for each share and the index.

Remarks: the mean and variance of the log returns is relatively small for all shares. We observe that the kurtosis is greater than three for all shares, which indicate that our data deviate significantly the Normal distribution[1]. In general, we can already observe that the Normal distribution does not fit our data well (we expected this from the literature) since the skewness is non-zero and the kurtosis is greater than three. We next apply the quantile-quantile (QQ) plot technique to further test our data for normality.

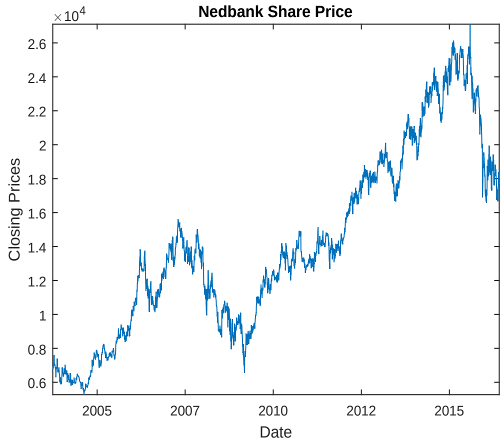


(a) Barclays Africa share price history.

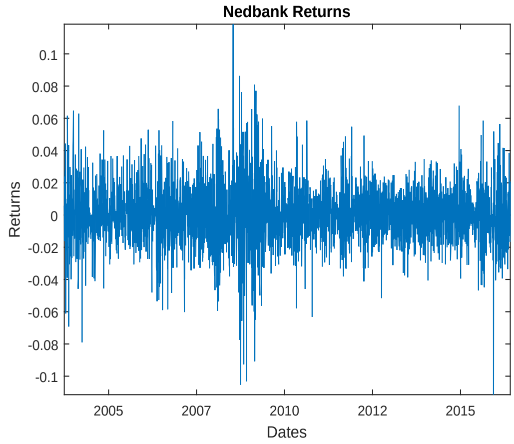
(b) Barclays Africa log returns.

Figure 4.1: Barclays Africa share price and daily log returns.

[1]The excess kurtosis for a standard Normal distribution is three and the skewness is zero.

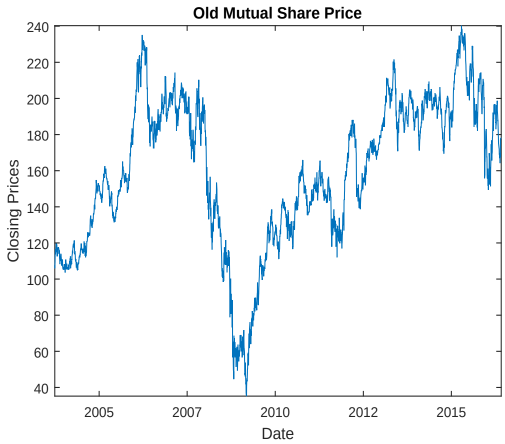


(a) Nedbank share price history.

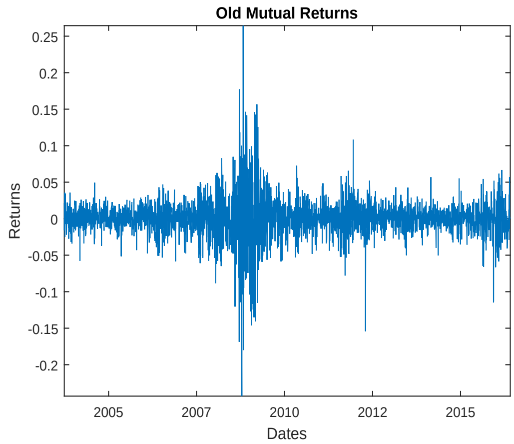


(b) Nedbank log returns.

Figure 4.2: Nedbank share price and daily log returns.

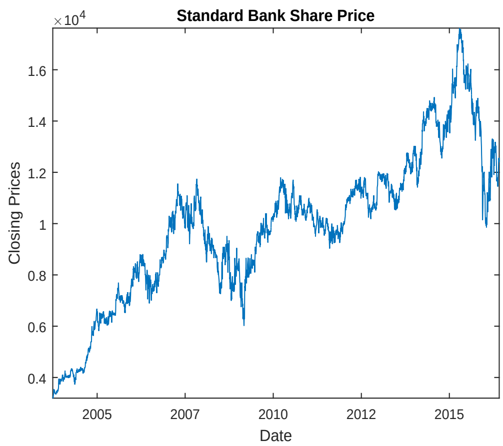


(a) Old Mutual share price history.

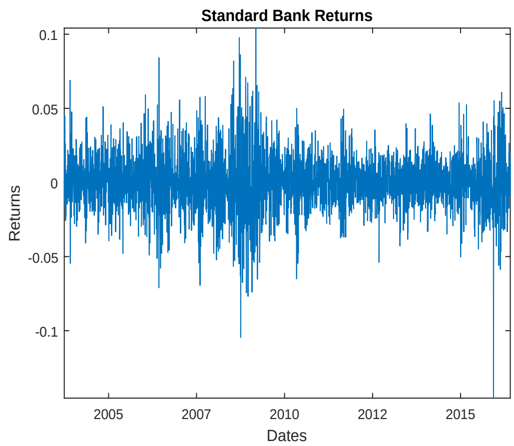


(b) Old Mutual log returns.

Figure 4.3: Old Mutual share price and daily log returns.

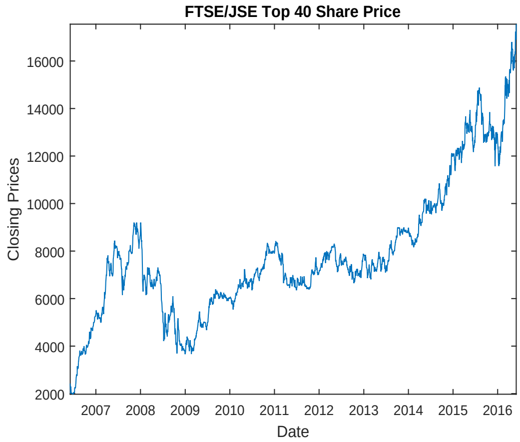


(a) Standard Bank share price history.

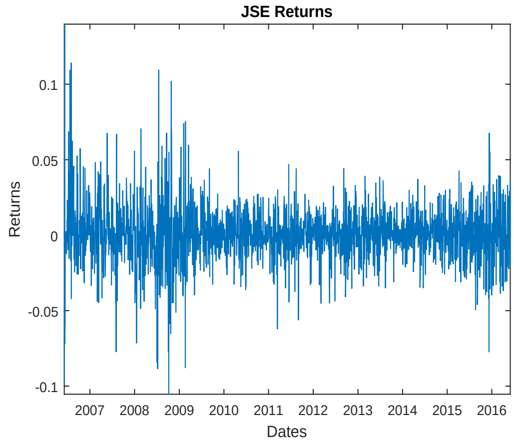


(b) Standard Bank log returns.

Figure 4.4: Standard Bank share price and daily log returns.



(a) FTSE/JSE Top40 share price history.



(b) FTSE/JSE Top40 log returns.

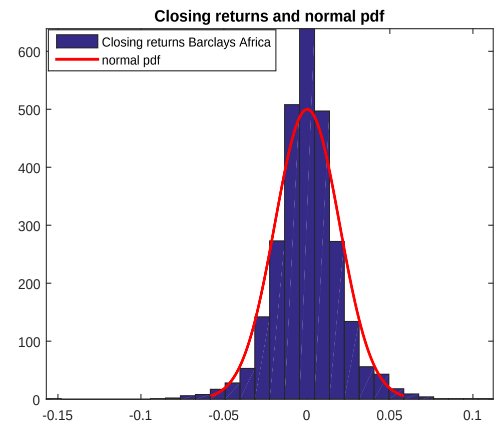
Figure 4.5: FTSE/JSE Top40 share price and daily log returns.

4.2 Normality Test

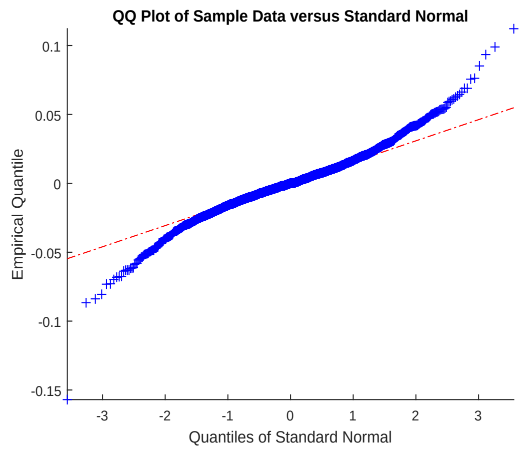
We noted in Table (4.1) that the log-returns of all the shares deviate significantly from the Normal distribution. There are many different methods that can be used to test for normality. One simple method is to use the QQ-plot[2] that we have already highlighted, other methods include hypothesis testing technique such as Jarque-Bera, Shapiro-Wilk test etc, see [MLR14].

Remarks: Figures (4.6a - 4.10a) represent the QQ-plot test for normality. The histogram figures represent the log-returns fitted with the Normal distribution, we observe the excess in kurtosis throughout our shares as indicated by our results in Table (4.1). If the fitted data is normally distributed then, the returns would lie in the straight line in the QQ-plot. In this case, it clearly does not fit, it deviates significantly from the straight line. The skewness does not seem to be a huge problem in this case, but we have a huge deviation in the tails. In the next section, we fit the PDF's we derived in equation (3.48) (Student-t), and the Pearson type-IV distribution in equation (3.50) to see if it can fit our data better.

[2]The QQ-plot is a simple graphical method of testing the goodness of fit of observed returns to the Normal distribution.

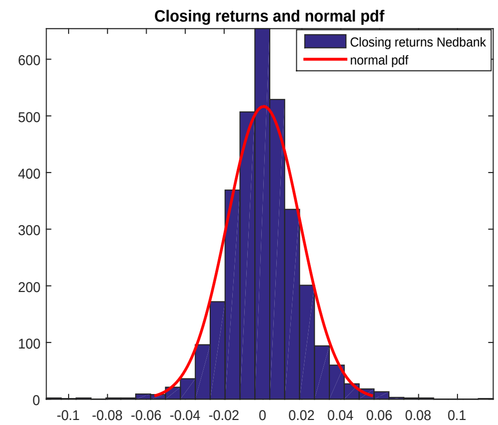


(a) Barclays Africa log return histogram.

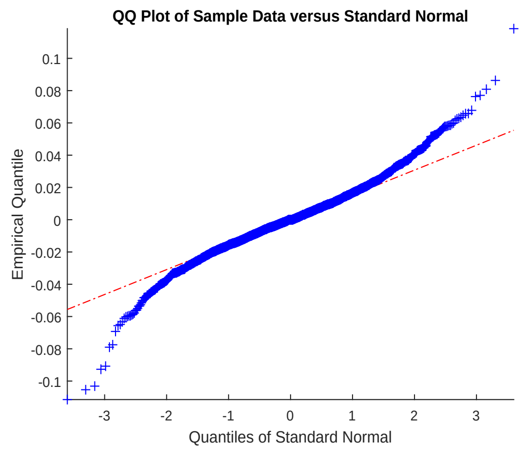


(b) Barclays Africa qq-plot.

Figure 4.6: Barclays Africa qq-plot normality test.

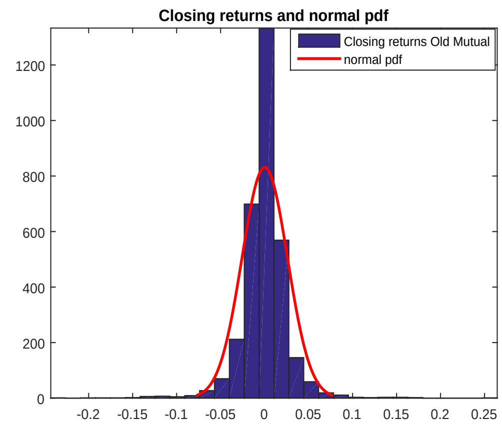


(a) Nedbank log return histogram.

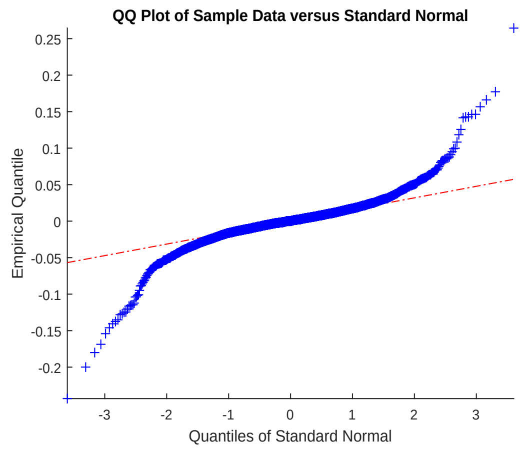


(b) Nedbank qq-plot.

Figure 4.7: Nedbank qq-plot normality test.

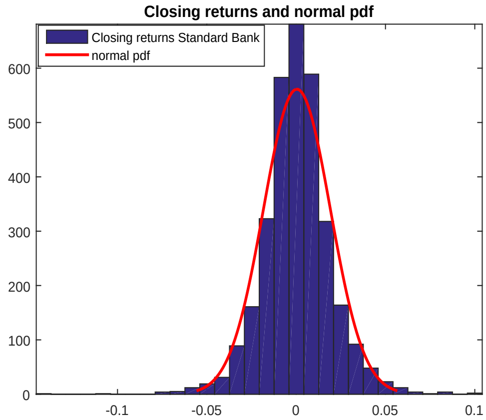


(a) Old Mutual log return histogram.

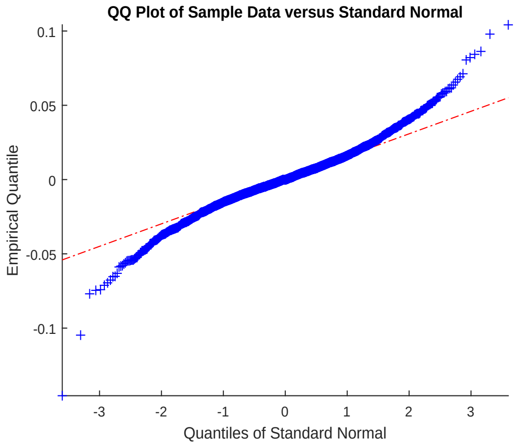


(b) Old Mutual qq-plot.

Figure 4.8: Old Mutual share qq-plot normality test.

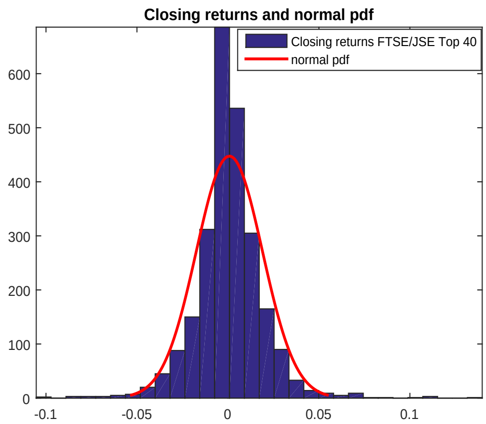


(a) Standard Bank share price log return histogram.

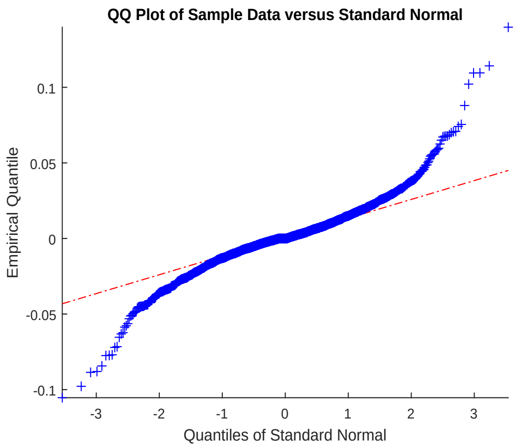


(b) Standard Bank qq-plot.

Figure 4.9: Standard Bank qq-plot normality test.



(a) FTSE/JSE Top40 share price log return histogram.



(b) FTSE/JSE Top40 qq-plot.

Figure 4.10: FTSE/JSE Top40 qq-plot normality test.

4.3 Fitting Equilibrium Distributions

We now fit the Pearson type-IV and Student-t distributions to the market data, to attempt to address the issue of fat-tails. The Pearson type-IV distribution is characterised by five parameters we discussed in equation (3.20). There are many different methods that can be used to estimate the parameters of a distribution, see [Pea94]. We apply the method of maximum likelihood estimation (MLE), which is the most recommended method, see [Hei04, Pea94].

The MLE method involves evaluating the maximum likelihood function, and the estimates are obtained by maximising the log-likelihood.

Suppose X is a continuous r.v. with a PDF $f(X;\vec{\theta}_i)$ where $\vec{\theta}_i = (\theta_1, \ldots, \theta_k)$ are unknown parameters for the N independent observations $X_1, X_2, \cdots, X_N$. The likelihood function is given by

$$L(X_1, X_2, \cdots, X_n | \theta_1, \theta_2, \cdots, \theta_k) = \prod_{i=1}^{N} f(X_i; \vec{\theta}_i)$$

and the logarithmic function is

$$\Lambda = \ln L = \sum_{i=1}^{N} \ln f(X_i; \vec{\theta}_i)$$

so the MLE of $\theta_1, \theta_2, \cdots, \theta_k$ are obtained by maximising L or by maximising Λ which is usually much easier to work with than L, see [Hei04, Pea94].

In the case of the Pearson-IV (in 3.20) we optimise the log likelihood given by

$$-\ln L = m \sum_{i=1}^{N} \ln\left[1 + \left(\frac{x_i - \lambda}{a}\right)^2\right] + \nu \sum_{i=1}^{N} \tan^{-1}\left(\frac{x_i - \lambda}{a}\right) - N \ln k$$

where there are N sample size and x_i data points, see [Hei04].

Since there is no analytical expression for the MLE of the Pearson-IV distribution, we use the statistical program in R (*pearsonMSC*), it has the built-in function for computing MLE, and we obtain results that are summarised in the table below.

Company	$\hat{m}$	$\hat{\nu}$	$\hat{\lambda}$	$\hat{a}$
Barclays Africa	2.63036	0.15458	0.00118	0.02979
FTSE/JSE Top 40	1.96488	0.04756	-0.00029	0.01944
Nedbank	2.92261	0.27027	0.00188	0.03180
Old Mutual	1.71884	-0.07387	-0.00126	0.02113
Standard Bank	2.76866	0.13890	0.00073	0.03027

Table 4.2: Maximum likelihood parameter estimators of the Pearson type-IV distribution.

The log likelihood for the Student-t (3.33) is given by

$$\ln L = \sum_{i=1}^{N} \ln\left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + x_i^2/a\right)^{-\frac{\nu+1}{2}}\right).$$

Again no explicit solution exist, so we use the R statistical program (*stdFit*), and the results are summarised in the table below.

Company	Mean	Standard deviation	d.o.f
Barclays Africa	1.5455×10^{-5}	0.019828	4.2665
FTSE/JSE Top 40	6.408×10^{-4}	0.020157	2.9319
Nedbank	9.05037×10^{-5}	0.018883	4.8550
Old Mutual	5.462×10^{-4}	0.032034	2.4344
Standard Bank	2.944×10^{-4}	0.019013	4.5387

Table 4.3: Maximum likelihood parameter estimators of the Student-t distribution.

The MLE for the Gaussian distribution is well known is Statistics, see [Pea94]. The estimate are

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{N} x_i \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 \cdot$$

Company	$\hat{\mu}$	$\hat{\sigma}$
Barclays Africa	0.0001760867	0.0194334260
FTSE/JSE Top 40	0.0007926401	0.0181831017
Nedbank	0.0003050014	0.0187247156
Old Mutual	0.000157551	0.025905585
Standard Bank	0.0004271856	0.0187251489

Table 4.4: Maximum likelihood parameter estimators of the Normal distribution.

In Figures (4.11 - 4.15) we fit the Pearson type IV and the Student-t distribution to the market data, using the parameters we have estimated in Tables (4.2 - 4.4). Our results indicate that both the Pearson-IV and Student-t distributions fit the market data much better (compared to the Normal distribution).

In Figures (4.16a - 4.20a) we further verify that the Pearson-IV and Student-t distributions fit the market data well, by using the QQ-plot.

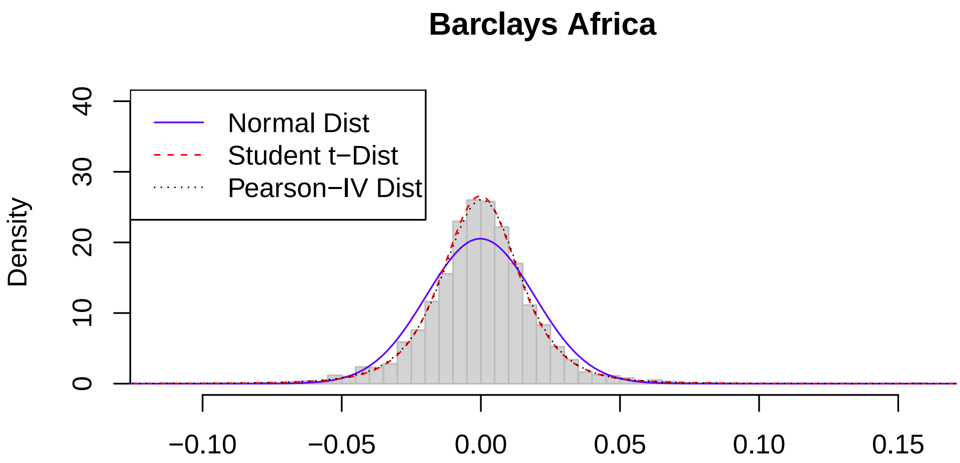


Figure 4.11: Barclays Africa distribution fit.

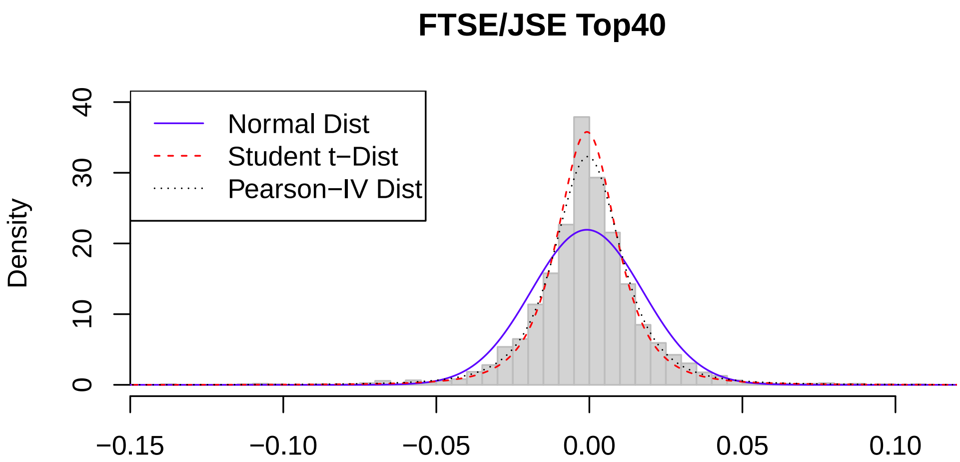


Figure 4.12: FTSE/JSE Top40 distribution fit.

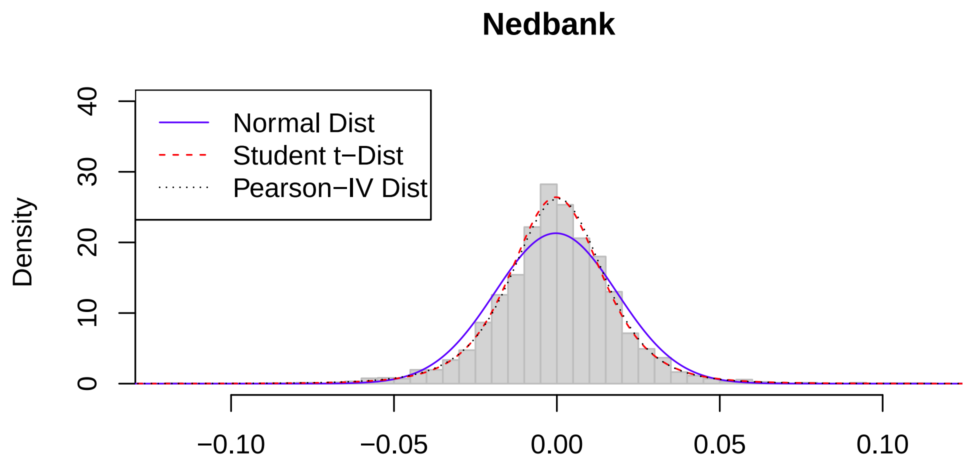


Figure 4.13: Nedbank distribution fit.

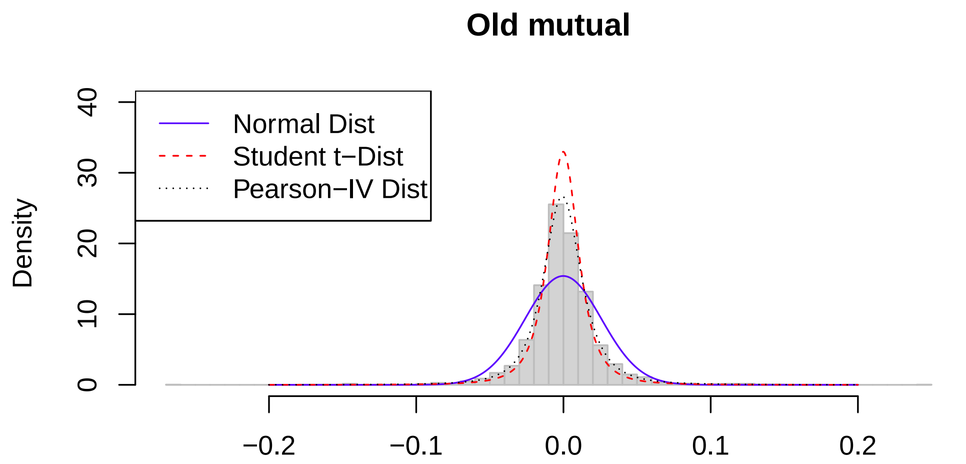


Figure 4.14: Old Mutual distribution fit.

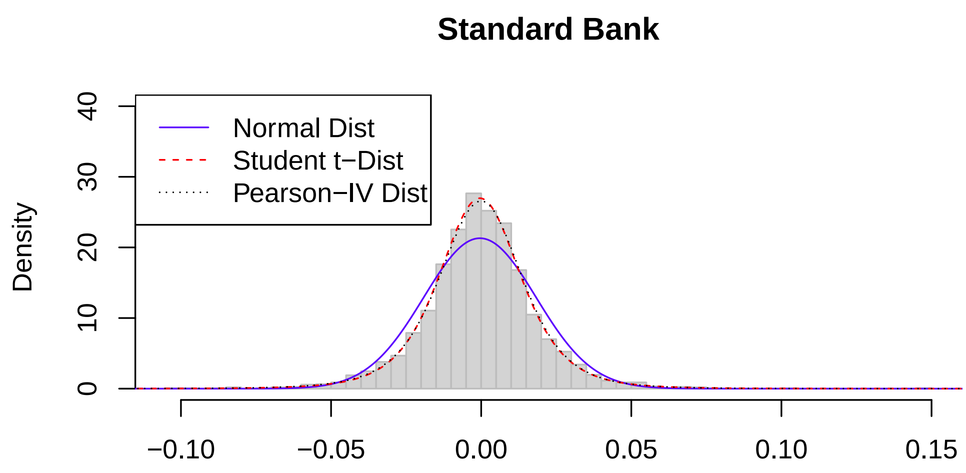
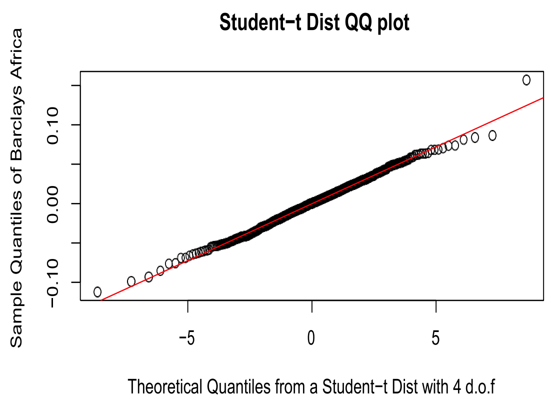
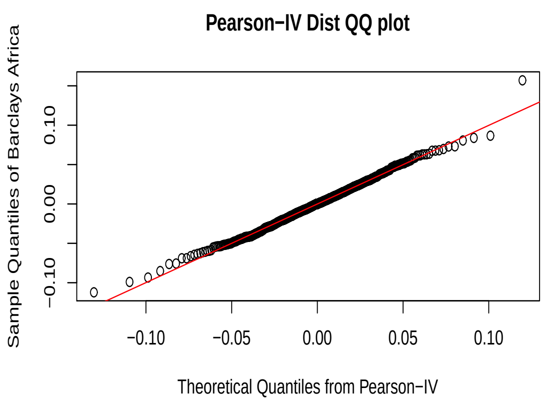


Figure 4.15: Standard Bank distribution fit.

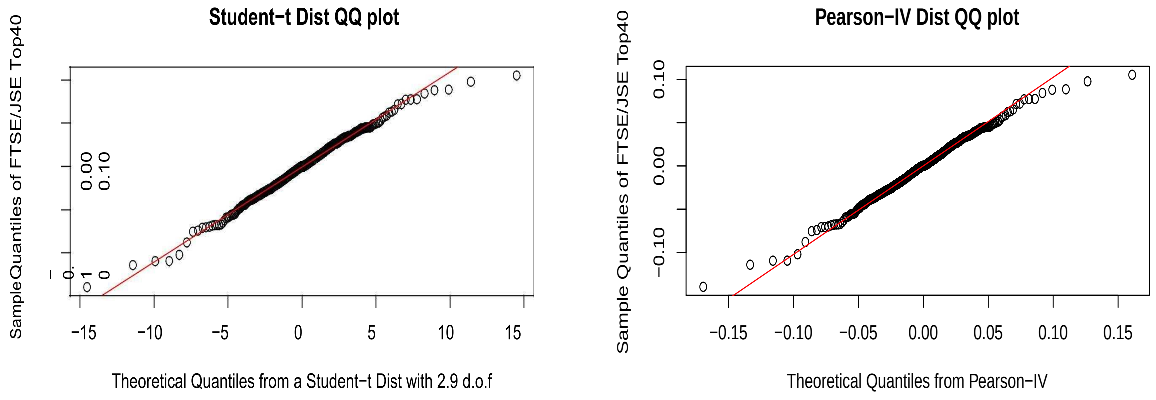


(a) Barclays Africa Student-t QQ plot.



(b) Barclays Africa Pearson-IV QQ plot.

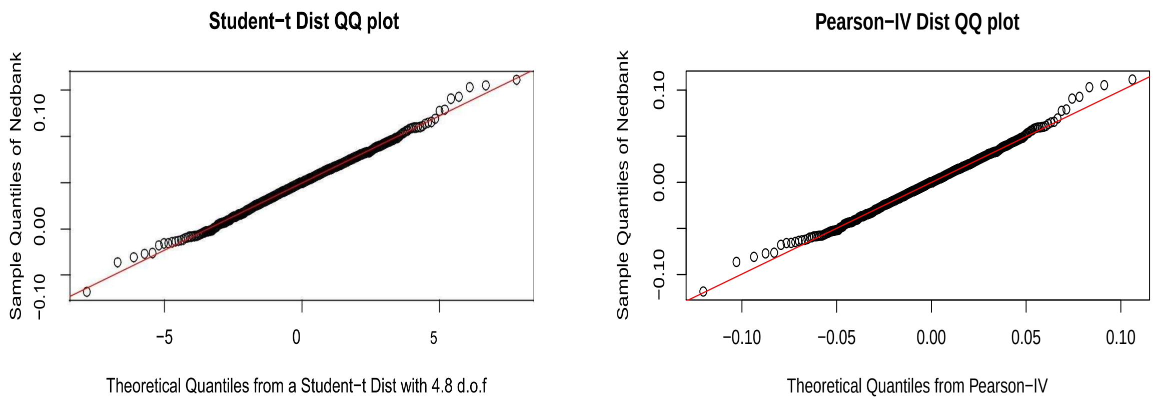
Figure 4.16: Barclays Africa Student-t and Pearson-IV QQ plots.



(a) FTSE/JSE Top40 Student-t QQ plot.

(b) FTSE/JSE Top40 Pearson-IV QQ.

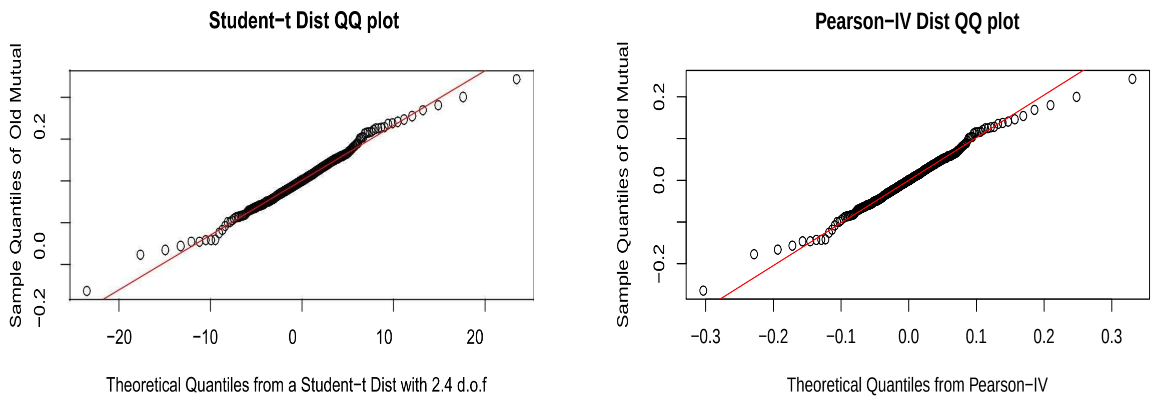
Figure 4.17: FTSE/JSE Top40 Student-t and Pearson-IV QQ plots.



(a) Nedbank Student-t QQ plot.

(b) Nedbank Pearson-IV QQ plot.

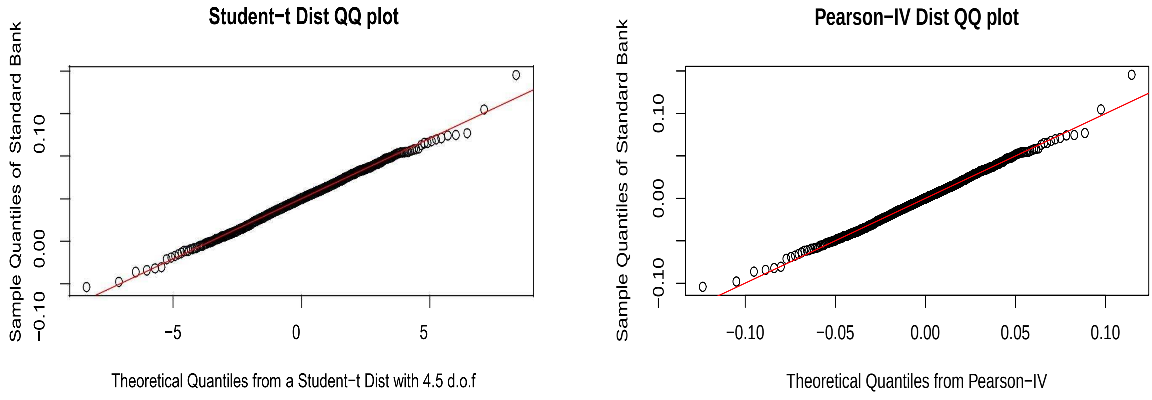
Figure 4.18: Nedbank Student-t and Pearson-IV QQ plots.



(a) Old Mutual Student-t QQ plot.

(b) Old Mutual Pearson-IV QQ plot.

Figure 4.19: Old Mutual Student-t and Pearson-IV QQ plots.



(a) Standard Bank Student-t QQ plot. (b) Standard Bank Pearson-IV QQ plot.

Figure 4.20: Standard Bank Student-t and Pearson-IV QQ plots.

4.4 Chapter Summary

We have introduced and analysed our empirical data that we will work with in our computation of risk measures. Table (4.1) summarises the first four moments of our empirical data. In all shares, the skewness is relatively small (close to zero), but the excess kurtosis is greater than three in all shares. Indicating that our data exhibit heavier tails than the Gaussian distribution. We then applied the QQ-plot method to test for normality. Figures (4.6a - 4.10a) represent our empirical data fitted with the Normal distribution. From the QQ-plots, it is clear that the Normal distribution does not capture the data very well, especially in the tails, which is something we anticipated.

We then fitted the Pearson type-IV and the Student-t distributions to the market data, using the method of maximum likelihood. Pearson type-IV distribution is characterised by four parameters i.e. λ, a, ν and m which we described their meaning in Section 3.2. The estimates of these parameters are summarised in Table (4.2). Similarly, we fitted the Student-t distribution which is characterised the mean, standard deviation and degrees of freedom. The estimates for Student-t distribution are summarised in Table (4.3).

Finally, we re-fitted the Pearson type-IV and the Student-t distribution using the estimates in Tables (4.3) and (4.4) respectively. Figures (4.11 - 4.15) represents the market data fitted by the above distributions. Both distributions seem to capture the excess kurtosis better than the Normal distribution. To verify our results we did the QQ-plot test once more, and the results are captured in Figures (4.16a - 4.20a). In the next chapter, we compute portfolio risk measures based on the above distributions.

Chapter 5

Portfolio Risk Measures

In this chapter, we introduce theoretical definitions of the general risk measures and methods of evaluating them. The general measures of risk are volatility, Value-at-Risk (VaR) and expected shortfall (ES), see [EFM05, MF00, Dan11].

VaR is the most popular risk measure, despite its well-known flaws, see [Hul06, ADEH99]. Intuitively VaR is defined as the worst expected loss over a given period at a specified confidence level i.e. VaR attempts to provide a single number summarising the total risk in a portfolio of financial assets, see [Hul06]. We define VaR in terms of P&L using quantiles.

Definition 3 ***Quantiles***

Given $\alpha \in (0,1)$. The number q is an $\alpha-$quantile of a random variable X if one of the following equivalent properties is satisfied:

1. $\mathbb{P}(X \leq q) \geq \alpha \geq \mathbb{P}(X < q)$.
2. $\mathbb{P}(X \leq q) \geq \alpha$ *and* $\mathbb{P}(X \geq q) \geq 1 - \alpha$.
3. $F_X(q) \geq \alpha$ *and* $F_X(q-) \leq \alpha$*, where* F_X *is the cumulative distribution of* X*, see [ADEH99].*

Since Ω is finite, then there is a finite left quantile (q_α^-) and a finite right quantile (q_α^+) s.t.

$$\begin{aligned} q_\alpha^- &= \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq \alpha\} = \inf\{x \in \mathbb{R} | F_X(x) \geq \alpha\} \\ &= \sup\{x \in \mathbb{R} | \mathbb{P}(X \leq x) < \alpha\} = \sup\{x \in \mathbb{R} | F_X(x) < \alpha\} \\ q_\alpha^+ &= \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) > \alpha\} = \inf\{x \in \mathbb{R} | F_X(x) > \alpha\} \\ &= \sup\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \leq \alpha\} = \sup\{x \in \mathbb{R} | F_X(x) \leq \alpha\} \end{aligned}$$

See [ADEH99].

Definition 4 ***Value-at-Risk (VaR)***

Given $\alpha \in (0,1)$, the VaR_α at a level α of the P&L X distribution is the negative of the q_α^+ i.e.

$$VaR_\alpha(X) = -\inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) > \alpha\} = -\inf\{x \in \mathbb{R} | F_X(x) > \alpha\}\cdot$$

Note: If X is a continuous r.v. with the density function of the P&L denoted by $f_X(x)$, we have

$$\mathbb{P}(X \leq -VaR_\alpha(X)) = \alpha \quad \text{or} \quad \alpha = \int_{-\infty}^{-VaR_\alpha(X)} f_X(x)dx\cdot$$

The figure below illustrates the VaR equation for a continuous r.v. There are two main parameters in the computation of VaR, that is the confidence level α (usually set at 95% or 99%) and the time period over which VaR is estimated (usually in days).

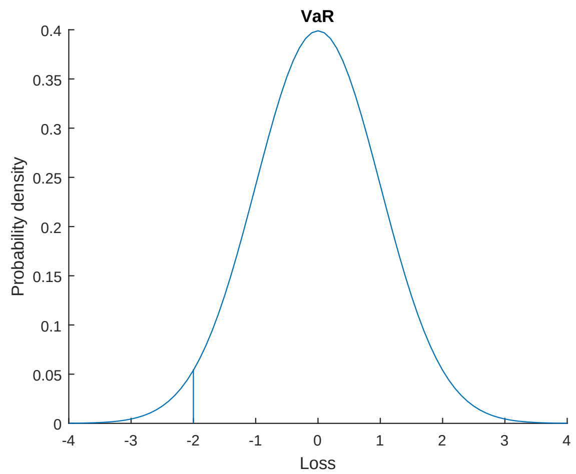


Figure 5.1: An example of α% VaR marked by a vertical line.

5.1 Methods for Evaluating VaR

There are numerous methods of calculating VaR, each method has it own *pros and cons*, see [EFM05, Hul06]. VaR methods can be classified into two distinct groups, one group is parametric and the other is non-parametric, see [JR09]. Parametric methods assume that financial returns follow a particular statistical distribution, and non-parametric methods do not assume any distribution in the valuation of VaR. In this dissertation, we apply two traditional methods of calculating VaR i.e. the Historical method (HS) and Variance-Covariance (VC) method. Under the VC method, we adopt the GARCH(1,1) model to cope with the volatility clustering phenomenon[1], that is usually present in financial returns, see [Man63, BM+12].

5.1.1 Historical Method

The HS method is very popular in practice as it is very simple to understand and implement, see [EFM05, MLR14]. This method applies empirical quantiles to compute VaR, whereby theoretical quantiles of a loss distribution are computed from the historical data, see [EFM05]. We first sort the financial returns in an ascending order and for some given confidence level $(1-\alpha)$ we estimate VaR by determining the $(1-\alpha)$ quantile of the returns distribution.

The advantage of the HS method is that it does not assume any statistical distribution, therefore it avoids the biases caused by estimating the parameters of a statistical distribution, see [EFM05, JR09]. [HW98] proposed a procedure for using GARCH or EWMA (exponentially weighted moving average), in conjunction with historical method when computing VaR. The HS method can be modified for a particular distribution by using Monte Carlo simulation, see [EFM05, Dan11].

[1]Volatility clustering phenomenon is best described [Man63], he defines it as the observation where "large changes tend to be followed by large changes, and small changes tend to be followed by small changes". This means we observe many days with high volatility, followed by many days with low volatility.

5.1.2 Analytic Variance-Covariance Approach

The VC method is also known as the delta-normal or model-building method. This method originally assumed that returns are jointly normally distributed i.e. $X_t \sim N(\mu, \Sigma)$ where μ is the mean vector and Σ is the variance-covariance matrix, and that the P&L in a portfolio value is linearly dependent on all risk factor returns, see [EFM05, MF00, Dan11]. The VC method can be adjusted for non-normal distributions, see [EFM05, Dan11, BM+12]. The weakness of this method is the assumption of a linear relationship between actual losses distribution, and the risk factor changes. The advantage of this method is that it has an analytical formula for computing VaR, and the volatility of financial data can be modelled by volatility models e.g. EWMA or GARCH model, see [EFM05, Dan11]. In this paper, we use the GARCH(1,1) model, following the work of [BM+12, BCY08, SMNZ12, SZ13]. A summary of the GARCH model is provided in Appendix B and the code is found in [Dan11, Chapter 5].

The GARCH(1,1) is described by the following equation:

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2, \tag{5.1}$$

where R_{t-1} is actual return on day $t-1$, σ_{t-1}^2 is the volatility of the return on day $t-1$, α and β are the expected return and variance constants respectively. There are two parameter restrictions on the GARCH(1,1) model, (i) we require $\omega, \alpha, \beta > 0$ (to ensure positive volatility forecast), (ii) $\alpha + \beta < 1$ (to ensure covariance stationarity). Under the VC approach we follow the following steps, firstly a distribution (e.g. Normal, Student-t, or Pearson) is fitted to the daily returns data, to estimate model parameters. Secondly the GARCH(1,1) model is fitted to the returns to estimate the parameters in equation (5.1). Lastly, we calculate daily VaR as described below.

We use the Gaussian results for comparison purposes.

Gaussian Approach

Suppose that the loss distribution is normally distributed with the conditional mean μ_t and the conditional variance σ_t^2. We fix an $\alpha \in (0,1)$ then VaR is computed by the following analytical formula:

$$\text{VaR}_\alpha(t) = \mu_t + \sigma_t \Phi^{-1}(\alpha), \tag{5.2}$$

where Φ denotes the standard Normal distribution and Φ^{-1} is the $\alpha-$quantile of Φ, and μ_t, σ_t^2 are modelled by the GARCH(1,1) model. See [EFM05, Chapter 2] for proof.

Student-t Approach

We obtain similar results for the Student-t distribution. VaR is computed by the following analytical formula:

$$\text{VaR}_\alpha(t) = \mu_t + \sigma_t T_\nu^{-1}(\alpha), \tag{5.3}$$

where T_ν^{-1} denotes the $\alpha-$quantile of the Student-t distribution. See [EFM05, Chapter 2] for proof.

Pearson Type-IV Approach

The VaR computation under the Pearson type-IV distribution is given by the following analytical formula:

$$\text{VaR}_\alpha(t) = \mu_t + \sigma_t F^{-1}_{\text{Pearson-IV}}(\alpha), \tag{5.4}$$

where the $F^{-1}_{\text{Pearson-IV}}$ denotes the $\alpha-$quantile of the Pearson type-IV distribution. See [SMNZ12] for proof.

5.2 Expected Shortfall

The alternative risk measure to VaR is the expected shortfall (ES) also known as Conditional VaR (CVaR), Expected Tail Loss (ETL) or Average VaR (AVaR), see [MF00, Dan11]. VaR answers the question, "How much can I lose with α probability over a certain period?", while ES answers the question, "What is the expected loss when losses exceed VaR?", see [Hul06, Dan11].

[Dan11, EFM05] defines expected shortfall as follows:

Definition 5 *(Expected Shortfall)*

The Expected Shortfall (ES) is the expected value of our losses (X), if the losses exceeds VaR:

$$ES_\alpha = \mathbb{E}[X|X > VaR_\alpha]\cdot \tag{5.5}$$

5.3 Backtesting Methods

Backtesting is a procedure of comparing the estimated losses with the actual observed P&L. The purpose of backtesting is to validate the forecasting risk model (i.e. VaR), see [Dan11, BM+12]. Daily losses that exceed the model estimate are referred to as *violations.* Whenever daily losses exceed the VaR estimate, a VaR violation is said to have occurred. VaR models should always be backtested so that financial institutions do not underestimate nor overestimate their risk, see [BM+12, BCY08, SMNZ12].

The main question is, how do you decide to accept or reject results produced by a VaR model given some confidence level. We discuss two popular methods of backtesting VaR models. One method is to record the number of violations and evaluate failure rate which is the proportion of VaR exceeds in a total number of observations. The second method was introduced by [Kup95]. The Kupiec test assists in making a decision whether the number of violations is acceptable or not, so we can accept or reject the model, see [Kup95, SMNZ12].

Success-Failure Ratio (LR) Test

Let N be a total number of observations and z be a number of violations for a given α confidence level. We denote the failure rate by $f = z/N$. The test is conducted by comparing the failure rate with the confidence level. An ideal model should have $f = \alpha$ otherwise we might overestimate or underestimate the risk for a given confidence level. If the failure rate is greater than the α-value, then a model underestimate the risk conversely if the failure rate is less than the α-value, then a model overestimates the risk. An acceptable model should have a failure rate close to the α-value, see [SMNZ12].

Kupiec Likelihood Ration Test

We apply the Kupiec Likelihood (LR) method to test whether we can accept or reject a model, based on the number of violations. The LR test assumes that, the number of violations (z) in a sample size of N is binomially distributed as $z \sim B(N, \alpha)$. Hence, the probability of z excess occurring over the N period is given by $\alpha^z(1-\alpha)^{N-z}$, where α is the probability of exceeding VaR on a given day. The Kupiec LR is given by

$$\text{LR} = 2\log[f^z(1-f)^{N-z}] - 2\log[\alpha^z(1-\alpha)^{N-z}], \tag{5.6}$$

which is distributed by χ^2 with one degree of freedom. The LR tests the null hypothesis, the failure rate is equal to the respecified VaR level α.

Statistically we write

$$H_0 : f = \alpha \quad \text{vs} \quad H_1 : f \neq \alpha \cdot$$

The test statistic rejects H_0 if LR $> \chi^2(1)$ or if the α-value:= $\mathbb{P}(\text{LR} > \chi^2(1)) < \alpha$, if H_0 is rejected then the model is considered to be inaccurate.

5.4 Properties of a Coherent Risk Measure

[ADEH99] study market risks in an incomplete markets and present four properties that a risk measure should satisfy. A risk measure that satisfies the properties is called *coherent*. Let a risk measure be denoted by $\varphi(\cdot) : \mathcal{G} \to \mathbb{R}$, where $\mathcal{G}$ is the set of all risks (i.e. set of all real-valued functions on Ω), $\mathcal{G}$ can be identified with $\mathbb{R}^n$, where $n = card(\Omega)$. We list the following properties for a coherent risk measure as in the paper [ADEH99] with finite a Ω. The properties can be extended to infinite Ω, see [Del02].

Definition 6 *(**Coherent risk measure**)*
A risk measure $\varphi(\cdot)$ is called a coherent risk measure if it satisfies the following conditions:

1. ***Monotonicity*** *For all X and $Y \in \mathcal{G}$ with $X \leq Y$, we have $\varphi(Y) \leq \varphi(X)$.*
   Interpretation: if portfolio Y produces higher returns than portfolio X, then the risk of portfolio Y must be less than or equal to the portfolio X (the least returns).

2. ***Subadditivity*** *For all X and $Y \in \mathcal{G}$, then $\varphi(X+Y) \leq \varphi(X)+\varphi(Y)$.*
   Interpretation: the portfolio risk of the sum $(X+Y)$, should be less than or equal to the individual risk. This property is also known as diversification.

3. ***Positive homogeneity*** *For all $c \geq 0$ and all $X \in \mathcal{G}$ then $\varphi(cX) = c\varphi(X)$.*
   Interpretation: this simply means if the portfolio doubles then the risk doubles by the same factor.

4. ***Translation invariance*** *For all $X \in \mathcal{G}$ and a real number c we have, $\varphi(X+c) = \varphi(X) - c$.*
   Interpretation: adding c to the portfolio is like adding cash, so the risk of $X+c$ is less than the risk of X by the amount of cash.

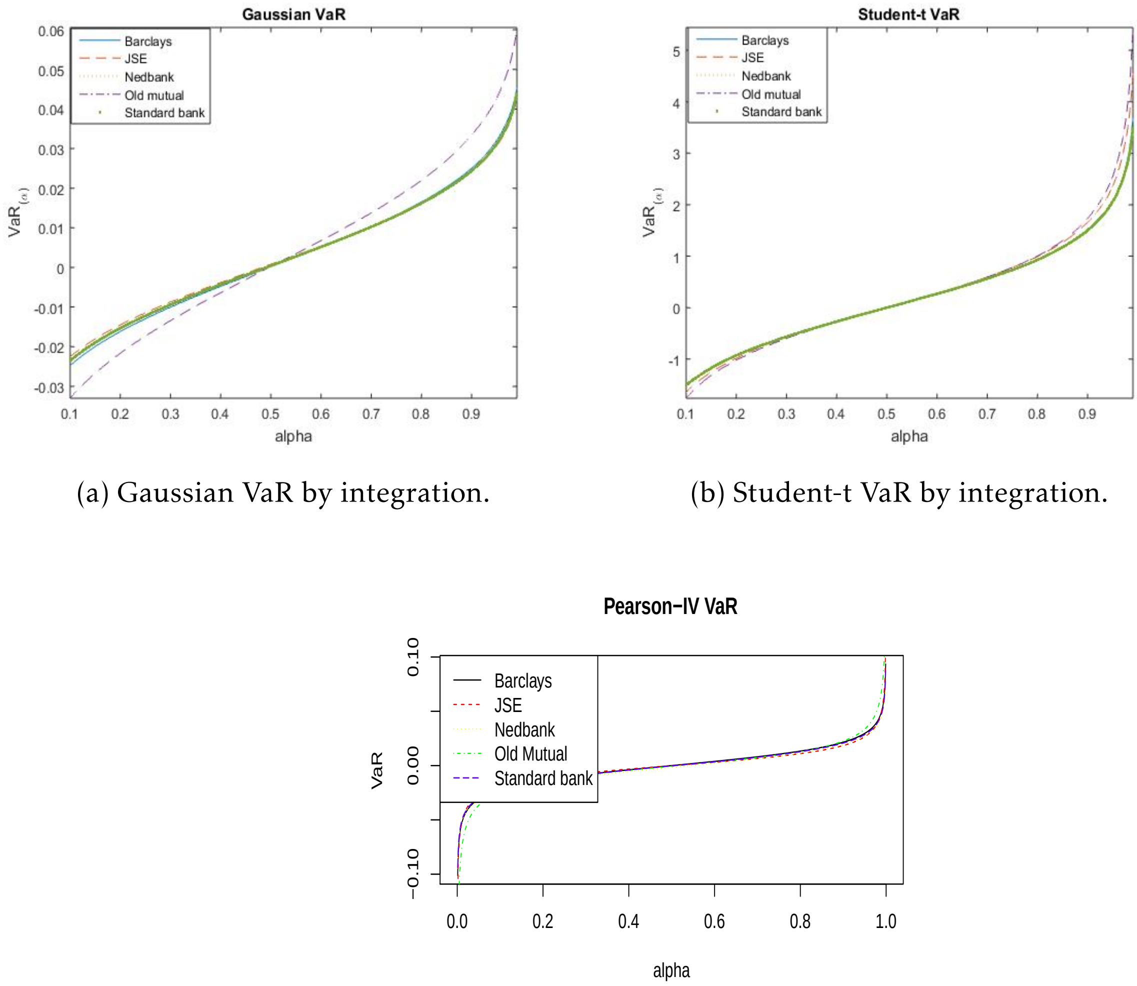
We comment on property number 2 (subadditivity) in relation to volatility, VaR and ES. If subadditivity property holds for some risk measure then, the risk of the portfolio (sum) is less than or equal to the sum of the individual assets. [ADEH99, Dan11] show that the volatility and ES are both coherent risk measures but VaR is not since it generally violates the subadditivity property. The subadditivity property only holds under the Gaussian distribution, where VaR is proportional to volatility, see [Dan11] for examples. In the case of non-Gaussian distributions the subadditivity property of VaR is violated, see [Dan11].

5.5 Risk Measures using Standard Probability Density Functions

The VaR, ES and backtesting estimates were computed under the Normal, Student-t and Pearson-IV distribution for a one-day holding period at 99% confidence level. Backtesting was done throughout the sample for each company.

5.5.1 VaR, ES and Backtesting Results

VaR Results



(a) Gaussian VaR by integration.

(b) Student-t VaR by integration.

(c) Pearson type-IV VaR by integration.

Figure 5.2: Standard distributions VaR by numerical integration.

Company	Historical
Barclays Africa	5.18%
FTSE/JSE Top 40	5.53%
Nedbank	5.23%
Old Mutual	6.94%
Standard Bank	5.06%

Table 5.1: One day VaR at a 99% confidence level using Historical method.

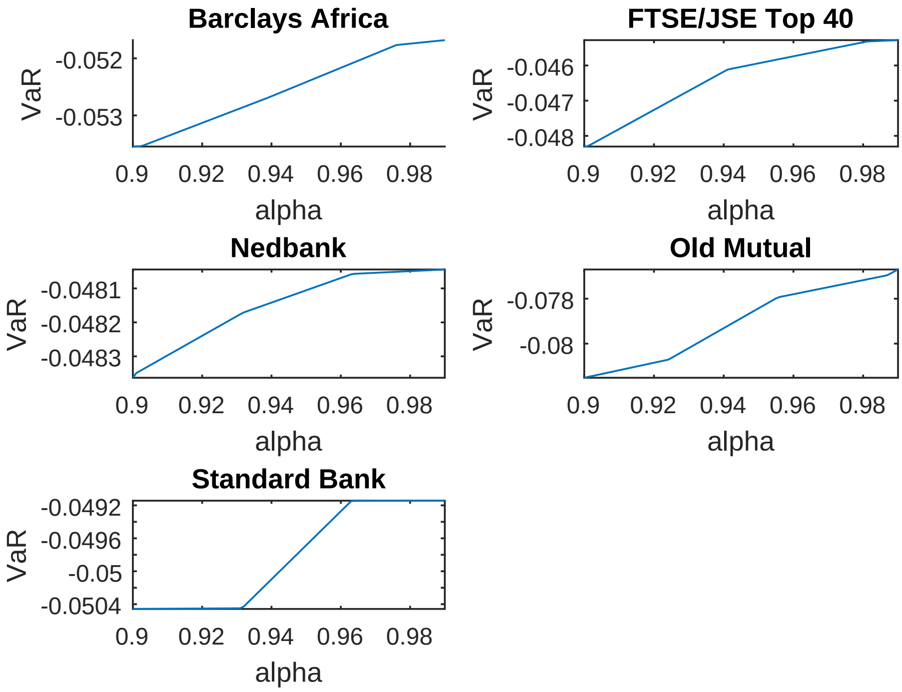
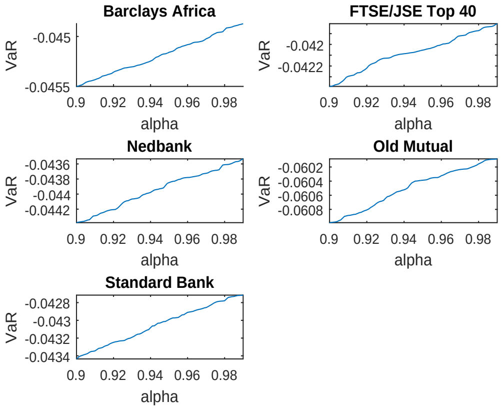


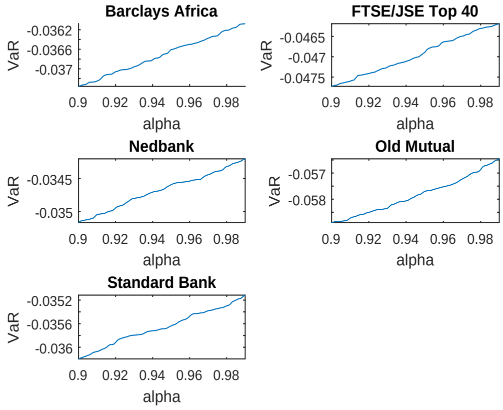
Figure 5.3: Historical VaR for different alpha's.

Company	Normal Distr	Student-t Distr	Pearson-IV Distr
Barclays Africa	3.62%	5.5%	5.3%
FTSE/JSE Top 40	4.61%	6.08%	6.02%
Nedbank	4.63%	6.72%	5.01%
Old Mutual	4.49%	7.84%	7.41%
Standard Bank	4.00%	5.80%	5.20%

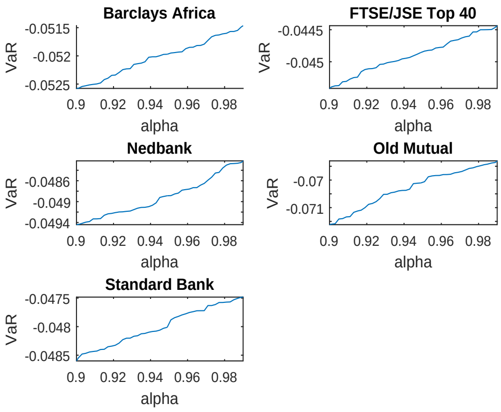
Table 5.2: One day VaR at a 99% confidence level using Variance-Covariance method.



(a) Normal VaR by CV.



(b) Student-t VaR by CV.



(c) Pearson-IV VaR by CV.

Figure 5.4: Variance-Covariance VaR for different alpha's.

ES Results

Company	Historical
Barclays Africa	6.79%
FTSE/JSE Top 40	7.52%
Nedbank	6.90%
Old Mutual	11.43%
Standard Bank	6.46%

Table 5.3: One day ES at a 99% confidence level using Historical method.

Company	Normal Distr	Student-t Distr	Pearson-IV Distr
Barclays Africa	5.10%	7.12%	7.12%
FTSE/JSE Top 40	6.67%	7.93%	7.93%
Nedbank	6.31%	9.11%	9.11%
Old Mutual	7.53%	12.39%	12.39%
Standard Bank	5.36%	7.65%	7.66%

Table 5.4: One day ES at a 99% confidence level using Variance-Covariance method.

Backtesting VaR results[2]

Company	Failure rate	LR Test	Decision
Barclays Africa	0.037234	125	Reject H_0.
FTSE/JSE Top 40	0.013611	2.95632	Do not reject H_0.
Nedbank	0.014209	5.01309	Do not reject H_0.
Old Mutual	0.030692	89.0492	Reject H_0.
Standard Bank	0.024305	46.78851	Reject H_0.

Table 5.5: Backtesting Variance-Covariance (Normal distribution) VaR at 99%.

Company	Failure rate	LR Test	Decision
Barclays Africa	0.006259	4.428255	Do not reject H_0.
FTSE/JSE Top 40	0.007206	2.18234	Do not reject H_0.
Nedbank	0.001194	3.86272	Do not reject H_0.
Old Mutual	0.007829	1.641672	Do not Reject H_0.
Standard Bank	0.011568	1.06828	Do not reject H_0.

Table 5.6: Backtesting Variance-Covariance (Student-t distribution) VaR at 99%.

[2] The $\chi^2(1) = 6.635$.

Company	Failure rate	LR Test	Decision
Barclays Africa	0.008837	0.38652	Do not reject H_0.
FTSE/JSE Top 40	0.007206	2.18234	Do not reject H_0.
Nedbank	0.027102	2.60491	Do not reject H_0.
Old Mutual	0.008456	0.81139	Do not reject H_0.
Standard Bank	0.008523	0.73516	Do not reject H_0.

Table 5.7: Backtesting Variance-Covariance (Pearson-IV distribution) VaR at 99%.

In graphs (5.5 - 5.9) we plot the backtesting results for VaR and daily log returns (black). We calculate VaR (using variance-covariance approach) under Student-t (green), Normal (purple) and PearsonIV (blue), and historical method (red).

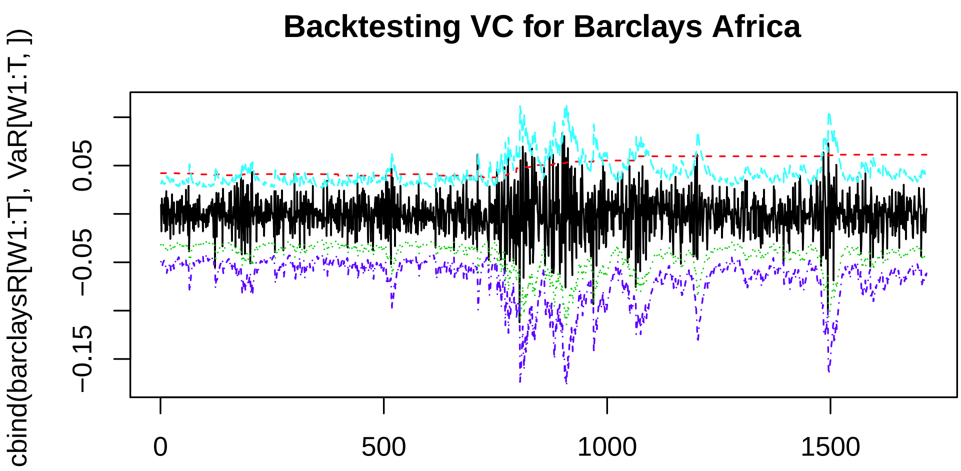


Figure 5.5: Backtesting Barclays Africa using the HS, Normal, Student-t and PearsonIV VC-GARCH(1,1)

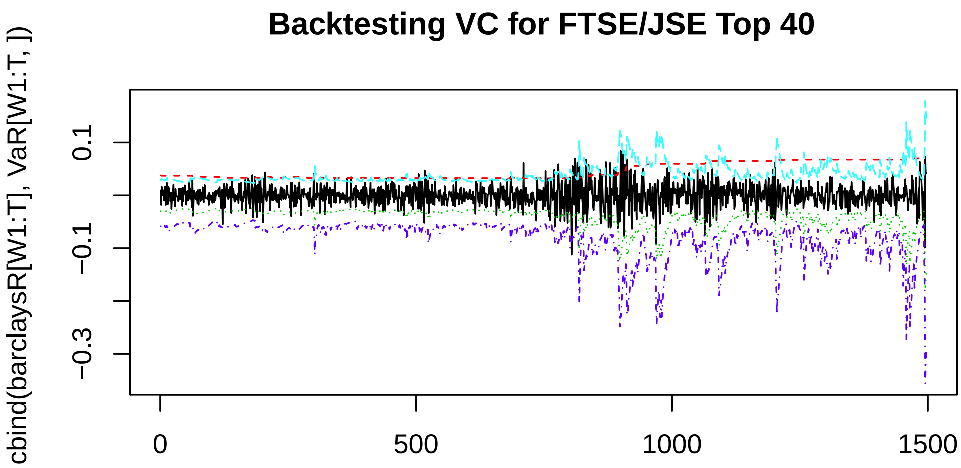


Figure 5.6: Backtesting FTSE/JSE Top 40 using the HS, Normal, Student-t and PearsonIV VC-GARCH(1,1)

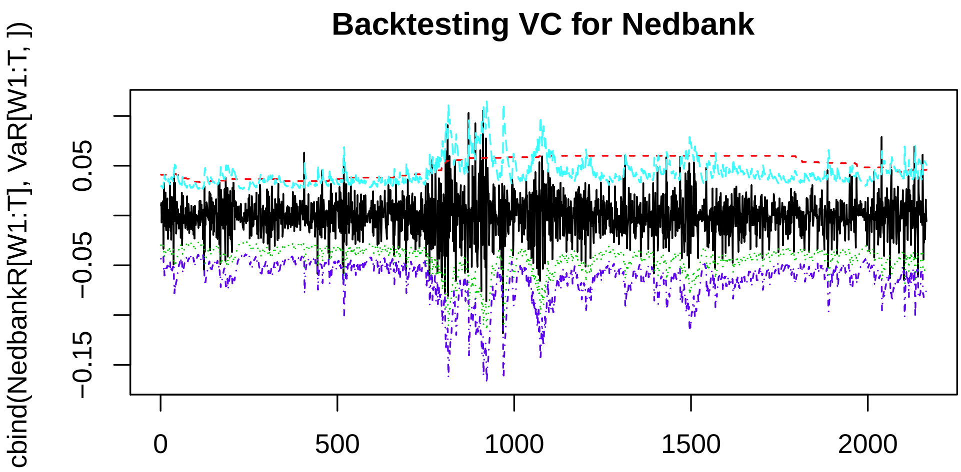


Figure 5.7: Backtesting Nedbank using the HS, Normal, Student-t and PearsonIV VC-GARCH(1,1)

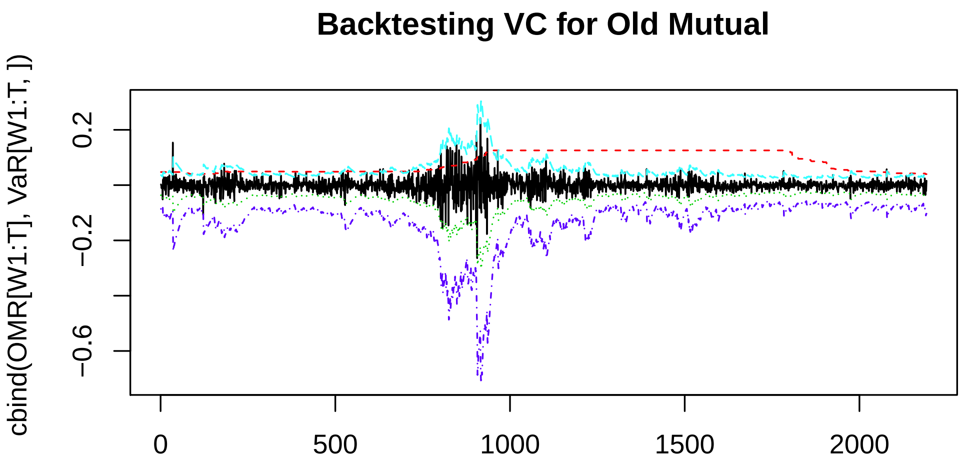


Figure 5.8: Backtesting Old Mutual using the HS, Normal, Student-t and PearsonIV VC-GARCH(1,1)

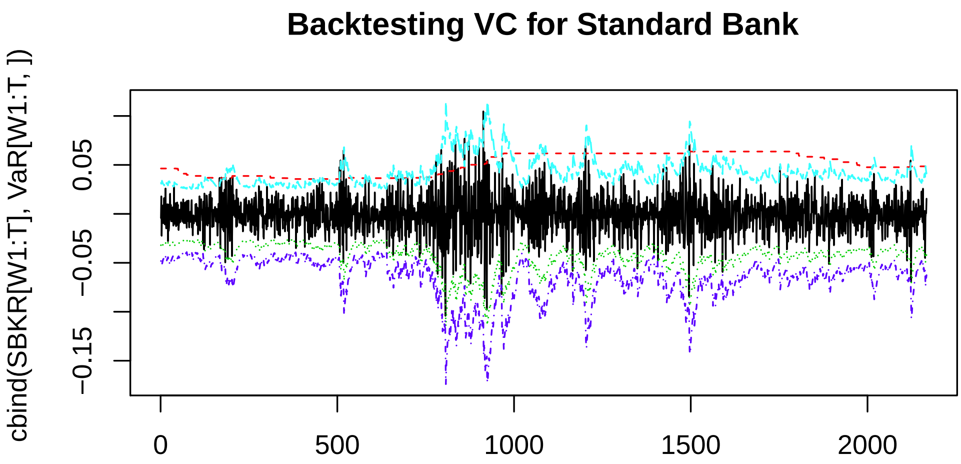


Figure 5.9: Backtesting Standard Bank using the HS, Normal, Student-t and PearsonIV VC-GARCH(1,1)

5.6 Risk Measures using Non-Standard Probability Density Functions

In this section we extend the previous section, by computing risk measures using non-equilibrium density functions, we derived in section (3.4) and (3.5). We derived a special case PDF for $\nu = 0$ in equation (3.48), and the general case for $\nu \geq 0$ in equation (3.71) and $\nu \leq 0$ in equation (3.79). We aim to investigate the effect of ν on VaR. We have the following PDF's for $\nu = \{0, 2, -2, -4\}$. The effect of ν on the PDF's is given by the following figure.

For $\nu = 0$ (Bougerol Identity)

$$f(x,t) = \frac{1}{\sqrt{2\pi t(\sigma_1^2 + \sigma_2^2 x^2)}} \exp\left\{ \frac{-1}{2\sigma_2^2 t} [\sinh^{-1}(\sigma_2 x/\sigma_1)]^2 \right\} \cdot \tag{5.7}$$

For $\nu = 2$

$$\begin{aligned} f(x,t) &= \frac{\sigma_1 \exp[-(u(x)^2/2t\sigma_2^2) - (t\sigma_2^2/2)]}{\sqrt{2\pi t}(\sigma_1^2 + \sigma_2^2 x^2)} \\ &+ \frac{\sigma_1^2 \sigma_2}{2(\sigma_1^2 + \sigma_2^2 x^2)^{3/2}} \times \left[\Phi\left(\frac{|u(x)| + t\sigma_2^2}{\sqrt{t}\sigma_2} \right) - \Phi\left(\frac{|u(x)| - t\sigma_2^2}{\sqrt{t}\sigma_2} \right) \right], \end{aligned} \tag{5.8}$$

where Φ is the standard Normal CDF.

For $\nu = -2$ (Generalized Bougerol Identity)

$$f(x,t) = \frac{1}{\sqrt{2\pi t(\sigma_1^2 + \sigma_2^2 x^2)}} \exp\left\{ -\frac{\sigma_2^2 t}{2} - \frac{u(x)^2}{2\sigma_2^2 t} \right\}, \text{ where } u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1) \cdot \tag{5.9}$$

For $\nu = -4$

$$\begin{aligned} f(x,t) &= e^{-2\sigma_2^2 t} \left(\frac{\sqrt{(1 + \sigma_2 x/\sigma_1)^2}}{\sqrt{2\pi\sigma_1^2 t}} e^{-u(x)^2/2t} \right) \\ &+ \frac{e^{\frac{\sigma_2^2 t}{2}}}{2a} \times \left[\Phi\left(\frac{|u(x)| + t\sigma_2}{\sqrt{t}} \right) - \Phi\left(\frac{|u(x)| - t\sigma_2}{\sqrt{t}} \right) \right], \end{aligned} \tag{5.10}$$

where

$$a = \frac{\sigma_1}{\sigma_2}\sqrt{1 - \rho^2}, \; u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1) = \sigma_2 z(x)$$

and Φ is the standard Normal CDF.

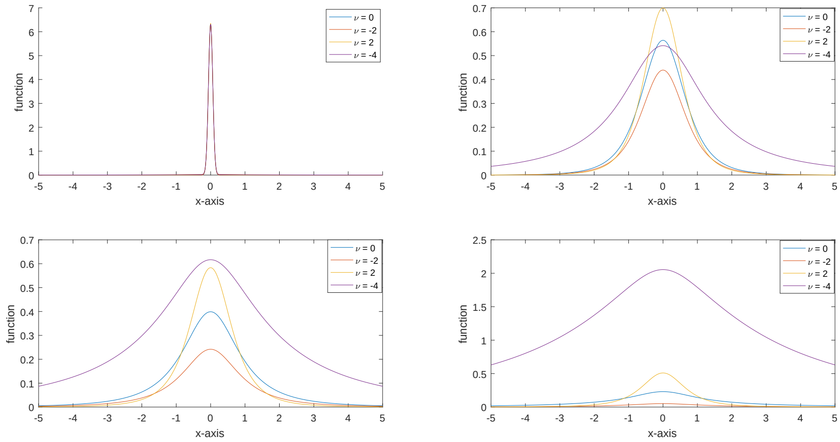


Figure 5.10: Shows the effect of ν on the PDF's, with $\sigma_1 = \sigma_2 = 1$ and $t = 1/252, 1/2, 1$ and 3 respectively.

We fit the non-equilibrium distributions using the MLE function in Matlab.

5.6.1 Distribution Fitting

ν	$\hat{\sigma_1}$	$\hat{\sigma_2}$
0	0.499815905743740	1.409327029349859
-2	0.498825190695306	1.406533517818738
2	0.513886541366686	1.408474043575879
-4	0.519828582202129	1.401727420337593

Table 5.8: Barclays Africa: maximum likelihood parameter estimators of non-equilibrium distributions.

ν	$\hat{\sigma_1}$	$\hat{\sigma_2}$
0	0.499815905743740	1.075469954027268
-2	0.497323063514015	1.070106025253909
2	0.520318821309506	1.074373160480211
-4	0.529851907174183	1.060581410621951

Table 5.9: FTSE/JSE Top 40: maximum likelihood parameter estimators of non-equilibrium distributions.

ν	$\hat{\sigma_1}$	$\hat{\sigma_2}$
0	0.499815905739469	0.915328485061770
-2	0.495668079803303	0.907732441824942
2	0.524449601301280	0.914382634621584
-4	0.536901184641203	0.893854835282527

Table 5.10: Nedbank: maximum likelihood parameter estimators of non-equilibrium distributions.

ν	$\hat{\sigma_1}$	$\hat{\sigma_2}$
0	0.499815905743661	0.962112665600021
-2	0.496258512232988	0.955264917631601
2	0.523178653450586	0.961078795916083
-4	0.534658680948590	0.942876319191007

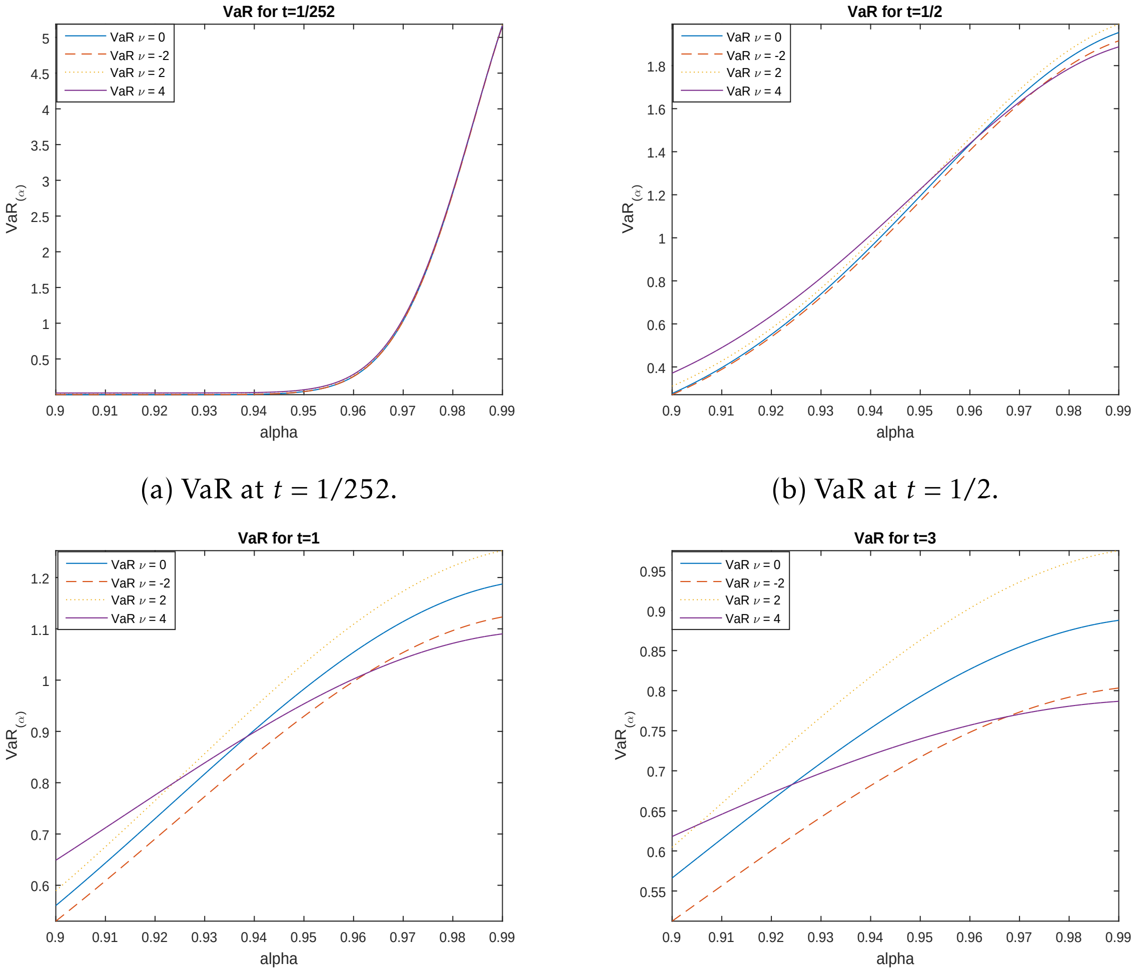
Table 5.11: Old Mutual: maximum likelihood parameter estimators of non-equilibrium distributions.

ν	$\hat{\sigma_1}$	$\hat{\sigma_2}$
0	0.499815905743739	1.163917053676783
-2	0.497897229003689	1.159449047451732
2	0.518330100892248	1.162853669722373
-4	0.526657411586139	1.151602007088079

Table 5.12: Standard Bank: maximum likelihood parameter estimators of non-equilibrium distributions.

5.6.2 VaR and Backtesting Results

For the non-equilibrium cases we could not derive an analytical formula for VaR as in the three cases, and we are not aware of any such formula. Hence, we apply direct numerical integration to compute VaR.



(a) VaR at $t = 1/252$.

(b) VaR at $t = 1/2$.

(c) VaR at $t = 1$.

(d) VaR at $t = 3$.

Figure 5.11: Non-Standard distributions VaR by numerical integration with $\sigma_1 = \sigma_2 = 1$.

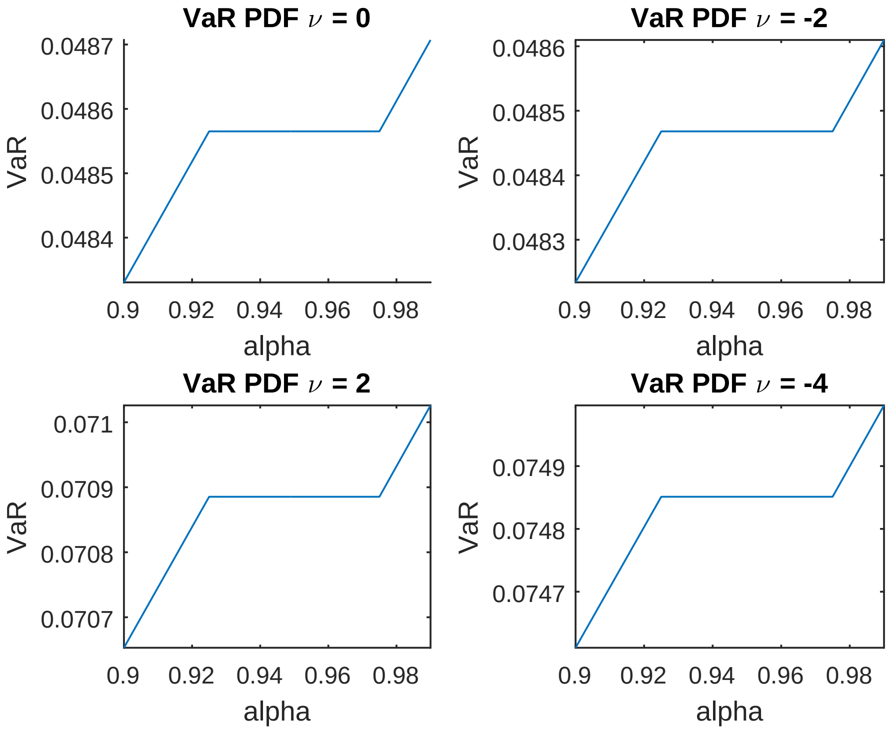


Figure 5.12: Barclays non-equilibrium distributions VaR.

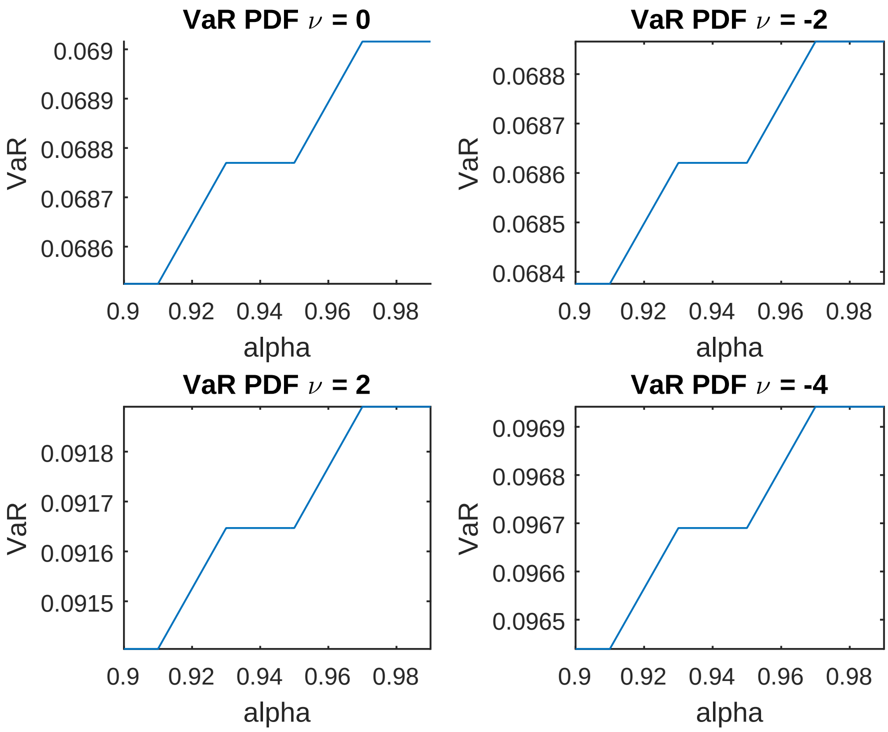


Figure 5.13: FTSE/JSE Top 40 non-equilibrium distributions VaR.

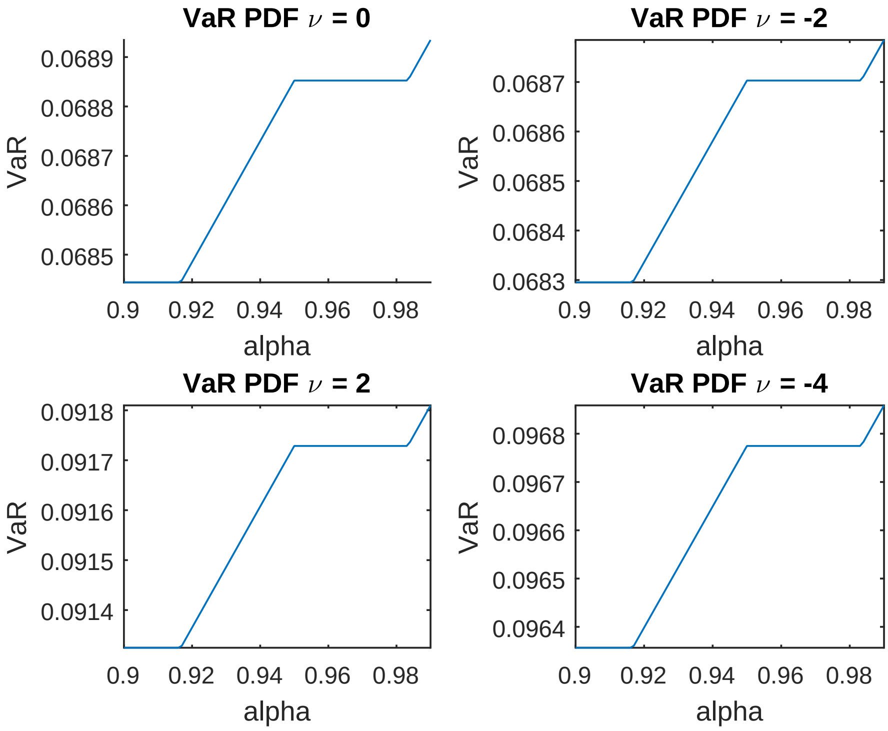


Figure 5.14: Nedbank non-equilibrium distributions VaR.

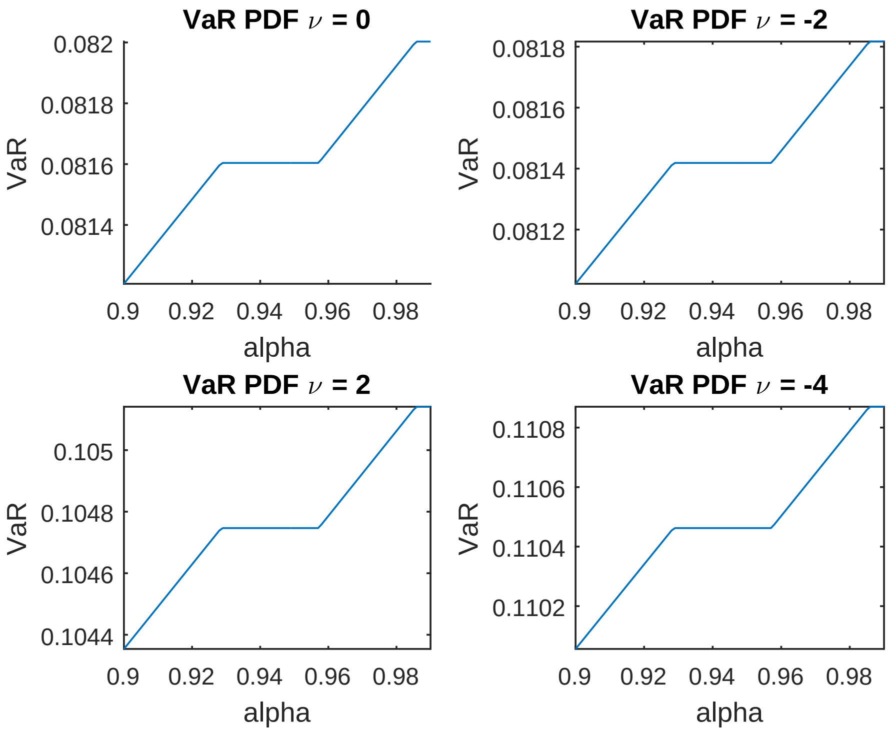


Figure 5.15: Old Mutual non-equilibrium distributions VaR.

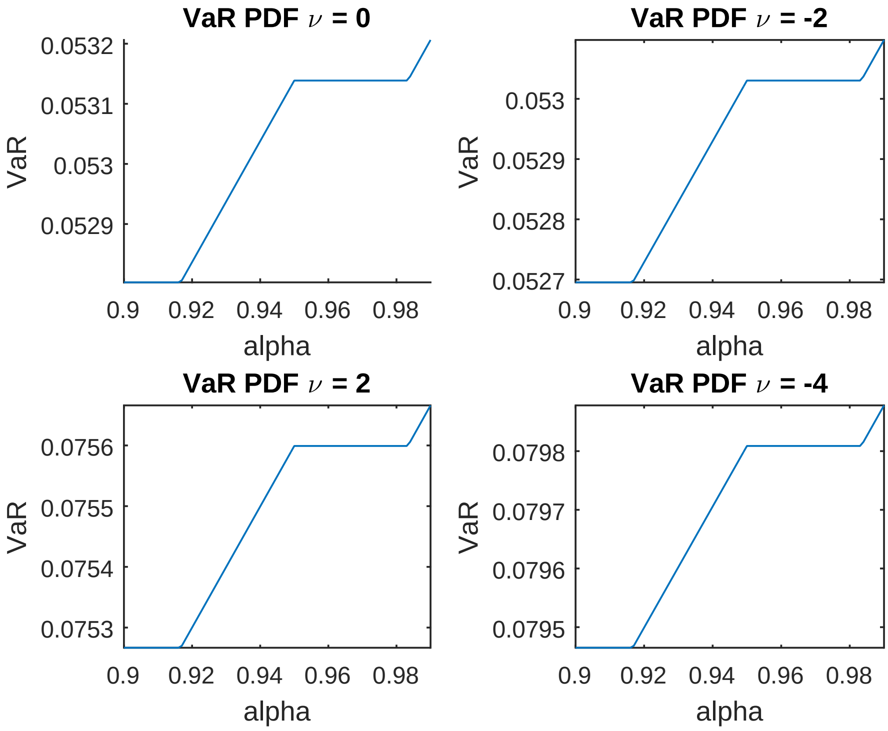


Figure 5.16: Standard bank non-equilibrium distributions VaR.

PDF	Failure rate	Decision
$\nu = 0$	0.008869	Do not reject H_0.
$\nu = -2$	0.008863	Do not reject H_0.
$\nu = 2$	0.09783	Reject H_0.
$\nu = -4$	0.09819	Reject H_0.

Table 5.13: Backtesting Barclays non-equilibrium distributions at 99%.

PDF	Failure rate	Decision
$\nu = 0$	0.009087	Do not reject H_0.
$\nu = -2$	0.009083	Do not reject H_0.
$\nu = 2$	0.09832	Reject H_0.
$\nu = -4$	0.09848	Reject H_0.

Table 5.14: Backtesting FTSE/JSE Top 40 non-equilibrium distributions at 99%.

PDF	Failure rate	Decision
$\nu = 0$	0.008983	Do not reject H_0.
$\nu = -2$	0.008990	Do not reject H_0.
$\nu = 2$	0.09836	Reject H_0.
$\nu = -4$	0.09858	Reject H_0.

Table 5.15: Backtesting Nedbank non-equilibrium distributions at 99%.

PDF	Failure rate	Decision
$\nu = 0$	0.008750	Do not reject H_0.
$\nu = -2$	0.008747	Do not reject H_0.
$\nu = 2$	0.09615	Reject H_0.
$\nu = -4$	0.09649	Reject H_0.

Table 5.16: Backtesting Old Mutual non-equilibrium distributions at 99%.

PDF	Failure rate	Decision
$\nu = 0$	0.008946	Do not reject H_0.
$\nu = -2$	0.008957	Do not reject H_0.
$\nu = 2$	0.09826	Reject H_0.
$\nu = -4$	0.09855	Reject H_0.

Table 5.17: Backtesting Standard bank non-equilibrium distributions at 99%.

5.7 Chapter Summary

We have introduced the most common risk measures i.e. VaR, ES and volatility. We discussed two methods of estimating VaR namely, the Historical and Variance-Covariance method. Under the Variance-Covariance method, we applied the GARCH(1,1) model to deal with the phenomenon of volatility clustering. We compute VaR and ES using the methods we have discussed here. We backtest each method, to verify the accuracy of each model. Our results indicate that the Pearson type IV and Student-t distributions perform well.

We also studied the properties of a coherent risk measure introduced by [ADEH99]. VaR is a coherent risk measure under Gaussian distribution, see [ADEH99, Dan11]. Statistical distributions that exhibit fat-tails violets the subaddictivity property of VaR, see [ADEH99, Dan11]. On the other hand, volatility and ES are coherent risk measures, see [ADEH99, Dan11] for proof. Furthermore we computed VaR under non-equilibrium distributions. We estimate σ_1 and σ_2 constants for the non-equilibrium distributions using the MLE in Matlab. The non-equilibrium distributions produced reasonable results but no better than the Student-t and the Pearson-IV distributions. The PDFs for $\nu = 0$ and $\nu = -2$ seem to produce good results based on our backtesting results. We observed that the VaR estimates get large, as time increases. In Figure (5.10) we observed the effect of ν on the densities at different times. For small t the densities are identical, but as t increases they deviate from each other significantly. This might justify the deviation of VaR estimates for each density function.

Chapter 6

Option Pricing and Calibration

In this chapter, we apply the hybrid SDE to price European style options. We derive a PDE for the model and calibrate it. Lastly, we apply the Crank-Nicolson finite difference (FD) scheme to price options. If a model has more sources of noise (e.g. Wiener process) than risky assets, then by the Meta-theorem it is considered to be incomplete[1]. The hybrid SDE in equation (2.42) it consists of one risky asset (share) and two sources of noise (i.e. W_t^1 and W_t^2), which makes it incomplete. In a special case where W_t^1 and W_t^2 are perfectly correlated, the model will be complete. Here we consider both cases.

The Meta-theorem[2] is a general rule of thumb of determining if a model is complete or not, it is stated as follows.

The Meta-theorem: Let N denote the number of risky assets in the model, excluding the risk-free asset (money account), and R denote the number of random sources of noise in the model. Then we generically have the following relations:

1. The model is arbitrage-free if and only if $N \leq R$.
2. The model is complete if and only if $N \geq R$.
3. The model is complete and arbitrage-free if and only if $N = R$.

[1] In practise markets are generically incomplete due to market imperfections like transition cost, stochastic interest rates and volatility, see [Bra13, Bjö09].

[2] See [Bjö09, Chapter 8].

An arbitrage portfolio/strategy is a trading strategy with zero initial cost, zero chance of making a loss, and a non-zero chance of making a profit. We formally define it below.

Definition 7 *An arbitrage strategy/portfolio h is a trading strategy, such that*

- $V_0^h = 0$
- $V_T^h \geq 0$
- $\mathbb{E}[V_T^h] > 0$

See [Bjö09, Definition 2.2].

6.1 Option Pricing

Definition 8 ***Market Completeness***
A market is said to be complete if every contingent claim in the market can be replicated. Alternatively, assume that the model is arbitrage-free. Then the market is complete if and only if the martingale measure is unique. See, [Bjö09, Definition 3.12 and Proposition 3.14].

Let $h_t = (\phi_t^1, \phi_t^2)$ be a portfolio vector where ϕ_t^1 is the number of the units of the bond (B_t) held at time t, and ϕ_t^2 is the number of the units of the stock (S_t) held at time t. The value process of the portfolio h is defined by

$$V_t^h = \phi_t^1 B_t + \phi_t^2 S_t, \quad t = 0, 1, \cdots$$

We next define a self-financing portfolio.

Definition 9 *A trading strategy/portfolio h is said to be self-financing if the following condition is satisfied* $\forall t = 0, 1, \cdots, T-1$

$$\phi_t^1(1+R) + \phi_t^2 S_t = \phi_{t+1}^1 + \phi_{t+1}^2 S_t$$

where R is the spot rate, see [Bjö09, Chapter 2 & 6].

Intuitively, a portfolio h is said to be self-financing if the portfolio $(\phi_{t+1}^1, \phi_{t+1}^2)$ at time $t+1$, is solely financed by a portfolio (ϕ_t^1, ϕ_t^2) at time t.

Definition 10 *A contingent claim $\mathcal{X}$ is said to be hedgeable or replicable, if there exists a self-financing portfolio h, such that $V^h(T) = \mathcal{X}$, $\mathbb{P} - a.s.$ [Bjö09, Definition 2.8].*

Definition 11 *A European call (put) option gives the holder the right to buy (sell) an asset (the underlying) for an agreed amount K (the strike price) on a specified future date T (maturity). See [Bjö09, Definition 7.3].*

Proposition 1 *Suppose that there exists a self-financing portfolio h such that, the value process V^h has the dynamics $dV^h(t) = k(t)V^h(t)dt$ where $k(t)$ is an adapted process. Then it must hold that $k(t) = r(t)\ \forall t$, where $r(t)$ is the risk-free rate. Otherwise there exist an arbitrage opportunity. See [Bjö09, Proposition 7.6.1] for proof.*

We recall the arithmetic-geometric hybrid Brownian motion model,

$$dS_t = (\mu_1 - \mu_2 S_t)dt + \sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}dW_t\ , t > 0,\ S_0 = x\cdot \tag{6.1}$$

The parameter ρ in equation (6.1) represents the correlation between the two noises in equation (2.42). In the special case where $\rho = \pm 1$ i.e. if there is perfect correlation between the two sources of noise, then the underlying model becomes complete, see [BC12, Sch15, Bjö09].

Case: $\rho \pm 1$ **(Complete)**

In this case, we follow the classical derivation of the Black-Scholes PDE[3]. Let $V(S,t)$ be any European style derivative, where the value of the derivative depends on both the underlying asset S_t and time t. Let B_t denote the riskless bank account with dynamics

$$\frac{dB_t}{B_t} = rdt\cdot \tag{6.2}$$

We construct a portfolio Π which consists of one derivative and g shares i.e.

$$\Pi_t = V_t + gS_t\cdot \tag{6.3}$$

Taking the differentials and by the self-financing assumption we obtain,

$$d\Pi_t = dV_t + gdS_t\cdot \tag{6.4}$$

[3]The general results (PDE) for any SDE of the form $dS(t) = \mu(t,S(t))S(t)dt + \sigma(t,S(t))S(t)dW(t)$ is given by [Bjö09, Chapter 7].

Then by Itô's Lemma on $V(S,t)$

$$\begin{aligned}
dV_t &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left((\mu_1 - \mu_2 S_t)dt + \sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}dW_t\right) \\
&+ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t\right)dt \\
&= \left(\frac{\partial V}{\partial t} + (\mu_1 - \mu_2 S_t)\frac{\partial V}{\partial S} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 V}{\partial S^2}\right)dt \\
&+ \sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\frac{\partial V}{\partial S}dW_t
\end{aligned} \tag{6.5}$$

Hence,

$$\begin{aligned}
d\Pi_t &= \left(\frac{\partial V}{\partial t} + (\mu_1 - \mu_2 S_t)\frac{\partial V}{\partial S} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 V}{\partial S^2}\right)dt \\
&+ \sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\frac{\partial V}{\partial S}dW_t + g\left((\mu_1 - \mu_2 S_t)dt + \sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}dW_t\right) \\
&= \left(\frac{\partial V}{\partial t} + (\mu_1 - \mu_2 S_t)\left[\frac{\partial V}{\partial S} + g\right] + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 V}{\partial S^2}\right)dt \\
&+ \sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\left[\frac{\partial V}{\partial S} + g\right]dW_t.
\end{aligned} \tag{6.6}$$

We take $g = -\frac{\partial V}{\partial S}$ i.e. short $\frac{\partial V}{\partial S}$ shares. Hence,

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 V}{\partial S^2}\right)dt \tag{6.7}$$

At this stage our portfolio is riskless. However, as S and t change, $\frac{\partial V}{\partial S}$ will also change. Hence, to keep the portfolio riskless, $\frac{\partial V}{\partial S}$ must be adjusted accordingly, see [Bjö09, Hul06]. By the assumption of no-arbitrage, the portfolio must earn the same as the risk-free account.

$$d\Pi_t = r\Pi_t dt = r(V_t - \frac{\partial V}{\partial S}S)dt. \tag{6.8}$$

Equating equation (6.7) and (6.8), we obtain the PDE for the hybrid SDE

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{6.9}$$

Remark: The quantity $\frac{\partial V}{\partial S}$ is called the delta of the derivative. One can thus replicate any European style derivative with the underlying share S by holding *delta*-many shares, at any time. This procedure of hedging is called *delta*-hedging. Also note that μ_1 and μ_2 do not appear in the PDE (6.9) therefore they are insignificant to pricing European style derivatives, see [Bjö09].

To find the price of a European call option ($C(S,t)$), we must solve the following initial-boundary value problem (IBVP)[4]:

$$\begin{cases} \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 C}{\partial S^2} - rC = 0\cdot \\ C(0,t) = 0, \ \forall t\cdot \\ C(S,t) \sim S \text{ as } S \to \infty\cdot \\ C(S,T) = \max(S_T - K, 0)\cdot \end{cases}$$

Case : $\rho \neq \pm 1$ **(Incomplete)**

If the market is incomplete, then from Definition 8, we have the following:

- It is impossible to perfectly hedge or to replicate a contingent claim.
- There is no unique martingale measure to price a contingent claim, as a consequence, there is no unique price for a contingent claim.

The fundamental challenge of pricing a contingent claim in an incomplete market is the uniqueness of a risk-neutral measure, see[Bjö09, Miy11]. The best thing we can do in this case is to build an internally coherent price system, which is consistent with the market prices and is arbitrage-free, see [Bra13, Bjö09]. In other words, we need to fix a risk-neutral measure, which will be consistent with the market prices. Our goal is to find a "fair" (arbitrage-free) price for a European call/put option, we have the following:

- The underlying is modelled by the hybrid model.
- The European call/put option is a function of the underlying and is evaluated at time T.

The problem set up here is very similar to that of the Black-Scholes model (complete and arbitrage-free). So we mimic the Black-Scholes model.

If incompleteness of a market is caused by having fewer underlying assets, than the random sources of noise, [Bjö09] proposes the following methods to mitigate for incompleteness:

[4]We show in the case of incomplete market how to solve such IBVP using the Crank-Nicolson scheme.

1. We can enlarge the market by adding one more asset (not money account) in the market without introducing, any new random noise. Then we can expect the market to be complete, by the Meta-theorem.

2. We can set a benchmark price for a particular contingent claim, then other contingent claims will be uniquely be determined by the price of a benchmark. Then by the Meta-theorem, we expect the market to be complete.

Since we cannot set up a perfect and self-financing hedging portfolio in an incomplete market, that replicates the contingent claim. The best thing we can do is to set up a portfolio that replicates the final pay-off of the claim on average, see [KW12, Bra13, Bjö09]. The portfolio we set up will not offset all the risk, and this will have significant consequences on the variance of the hedging portfolio. In fact, the variance of a hedging portfolio in an incomplete market is never zero, see [BC12, BC10].

We proceed as follows, we consider two fixed derivative securities Y and Z[5] (written in terms of the process X_t) of the form:

$$Y = \Phi(X(T))\cdot$$
$$Z = \Psi(X(T)),$$

where Φ and Ψ are deterministic real-valued functions (e.g. payoff function). Our objective is to study the price process of Y and Z. In order to make progress, we make the following assumptions:

- The market is frictionless and liquid for each of the derivative securities Y and Z.

- The market price process of the derivative securities are of the form:

$$\Lambda(t, Y) = F(t, X_t)\cdot$$
$$\Lambda(t, Z) = G(t, X_t),$$

[5][Bra13, Chapter 2] uses zero-coupon bonds to derive similar results.

where F and G are real-valued functions based on the underlying X_t. We apply Itô's Lemma on F and G.

$$\begin{aligned} dF(t,X_t) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}\left((\mu_1 - \mu_2 X_t)dt + \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t}dW_t\right) \\ &+ \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\left(\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t\right)dt \\ &= \left(\frac{\partial F}{\partial t} + (\mu_1 - \mu_2 X_t)\frac{\partial F}{\partial X} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t)\frac{\partial^2 F}{\partial X^2}\right)dt \\ &+ \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t}\frac{\partial F}{\partial X}dW_t \\ &= \mu_F F(t,X_t)dt + \sigma_F F(t,X_t)dW_t \cdot \end{aligned} \tag{6.10}$$

Similarly,

$$dG(t,X_t) = \mu_G G(t,X_t)dt + \sigma_G G(t,X_t)dW_t \tag{6.11}$$

where

$$\begin{aligned} \mu_F &= \frac{\partial F}{\partial t} + (\mu_1 - \mu_2 X_t)\frac{\partial F}{\partial X} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t)\frac{\partial^2 F}{\partial X^2}\Big/F(t,X_t) \\ \sigma_F &= \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t}\frac{\partial F}{\partial X}\Big/F(t,X_t) \end{aligned}$$

and μ_G is similar to μ_F as well as σ_G to σ_F. We proceed in a similar fashion as in the Black-Scholes model (hedging portfolio), by setting up a portfolio that replicates the final pay-off of the claims Y and Z. Hence, we set up a self-financing portfolio based on F and G, with portfolio weights w_F and w_G respectively. The portfolio weights must be chosen appropriately so that the portfolio is riskless. We state the following Lemma [Bjö09, Lemma 6.4] for a self-financing portfolio.

Lemma 1 *A portfolio-consumption pair (h,c) is self-financing if and only if:*

$$dV^h(t) = V^h(t)\sum_{i=1}^{N} u_i(t)\frac{dS_i}{S_i(t)} - c(t)dt$$

where V^h is the value process corresponding to the portfolio h, $c(t)$ is the amount of cash spent on consumption per unit time, $S_i(t)$ is the share price at time t and $u_i(t)$ is the relative portfolio with $\sum_{i=1}^{N} u_i(t) = 1$. See [Bjö09, Chapter 6] for proof.

So by Lemma 1 we have

$$\begin{aligned} dV &= V\left(w_F \cdot \frac{dF}{F} + w_G \cdot \frac{dG}{G}\right) \\ &= V(w_F \cdot (\mu_F dt + \sigma_F dW_t) + w_G \cdot (\mu_G dt + \sigma_G dW_t)) \\ &= V(w_F \cdot \mu_F + w_G \cdot \mu_G)dt + V(w_F\sigma_F + w_G\sigma_G)dW_t. \end{aligned} \tag{6.12}$$

In order to make this portfolio riskless we need to eliminate the dW_t-term[6], i.e. we must choose w_F and w_G such that $w_F\sigma_F + w_G\sigma_G = 0$, and the weights w_F and w_G must sum up to one. So we need to solve the following equations:

$$\begin{cases} w_F + w_G = 1. \\ w_F\sigma_F + w_G\sigma_G = 0. \end{cases}$$

This simplifies to

$$\begin{cases} w_F = -\frac{\sigma_G}{\sigma_F - \sigma_G}. \\ w_G = \frac{\sigma_F}{\sigma_F - \sigma_G}. \end{cases}$$

Now plugging w_F and w_G into equation (6.12) we obtain

$$dV = V \underbrace{\left(\frac{\mu_G \cdot \sigma_F - \mu_F \cdot \sigma_G}{\sigma_F - \sigma_G}\right)}_{k(t)} dt. \tag{6.13}$$

At this stage our portfolio is riskless. Then by Proposition 1:

$$\begin{aligned} \frac{\mu_G \cdot \sigma_F - \mu_F \cdot \sigma_G}{\sigma_F - \sigma_G} &= r. \\ \therefore \frac{\mu_F - r}{\sigma_F} &= \frac{\mu_G - r}{\sigma_G}. \end{aligned} \tag{6.14}$$

The left-hand side is independent of G and the right-hand side is independent of F, which means that equation (6.14) is independent of F nor G. Equation (6.14) is formalised by the following Proposition.

Proposition 2 *Assume that the market for derivatives is arbitrage-free. Then there exists a universal process $\lambda(t)$ such that, with probability one and for all t, we have*

$$\frac{\mu_F(t) - r(t)}{\sigma_F(t)} = \lambda(t) \tag{6.15}$$

regardless of the choice of the derivative F.

[6] The hedging error is the cost of making this portfolio riskless.

Remarks: the quantity $\mu_F(t) - r(t)$ is the excess rate of return of the asset F. Thus the ratio $\dfrac{\mu_F(t) - r(t)}{\sigma_F(t)}$ is interpreted as the excess rate of return per unit of volatility. The variable $\lambda(t)$ is commonly known as "the market price of risk" or the "the price volatility risk", see [Bjö09, Bra13]. If the market is arbitrage-free, then all derivative securities will have the same market price of risk, regardless of the choice of the derivative contract [Bjö09]. However, it is possible to have a different drift and volatility for two assets, but equation (6.14) must still hold. This means we can use real-world observations to estimate risk-neutral volatility. Substituting equations for μ_F and σ_F in equation (6.15), we obtain the following PDE:

$$\frac{\partial F}{\partial t} + (\mu_1 - \mu_2 X_t)\frac{\partial F}{\partial X} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t)\frac{\partial^2 F}{\partial X^2} - rF = \lambda\sqrt{\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t}\,\frac{\partial F}{\partial X}\cdot$$
$$\frac{\partial F}{\partial t} + \left((\mu_1 - \mu_2 X_t) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t}\right)\frac{\partial F}{\partial X} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 X_t^2 + 2\rho\sigma_1\sigma_2 X_t)\frac{\partial^2 F}{\partial X^2} - rF = 0\cdot \tag{6.16}$$

Proposition 3 *Assume the absence of arbitrage, the pricing function $F(t,x)$ of the T-claim $\Phi(X(T))$ solves the following boundary value problem:*

$$\frac{\partial F}{\partial t}(t,x) + \mathcal{A}F(t,x) - rF(t,x) = 0,\ (t,x) \in (0,T)\times\mathbb{R},$$
$$F(T,x) = \Phi(x),\ x \in \mathbb{R},$$

where
$$\mathcal{A}F(t,x) = \left((\mu_1 - \mu_2 x_t) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 x_t^2 + 2\rho\sigma_1\sigma_2 x_t}\right)\tfrac{\partial F}{\partial x} + \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2 x_t^2 + 2\rho\sigma_1\sigma_2 x_t)\tfrac{\partial^2 F}{\partial x^2}\cdot$$

So the value of the derivative security F is given by

$$F(t,x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,x}[\Phi(X(T))]\cdot \tag{6.17}$$

In this case, the market price of risk (λ) acts as a martingale measure $\mathbb{Q}$, see [Bjö09].

Earlier we mentioned that in an incomplete market there is no unique arbitrage-free price for derivative security since there is no unique martingale measure. However, in Proposition 3, we derived a PDE, which when solved will give a particular value. Also, recall Proposition 2 which states that in an arbitrage-free market all derivative securities must have the same market price of risk.

In a complete market, the price of a derivative is unique because the martingale measure

($\mathbb{Q}$), or equivalently the market price of risk (λ) is obtained within the model, see [Bjö09]. In an incomplete market there are multiple martingale measures, so to price derivatives we need to fix a martingale measure i.e. fix a market price of risk. There is a lot of literature on how to determine the market price of risk, see for example [Hul06]. The market for traded derivative instruments determines the market price of risk. The PDE is dependent on r, μ_1& μ_2, σ_1& $\sigma_2, \Phi(x)$ and λ. The parameters $\lambda, \sigma_1, \sigma_2$ are not directly observable from a market, they must be estimated.

Suppose from Proposition 2 we know the exact form of μ and σ (we assume these are constants), and in order to price a claim we need to know λ. We assume there exist a market for claims $\Phi_i(X(T)), i = 1, \cdots, n$ in equation (6.17), and that λ takes the form

$$\lambda = \lambda(t, x, \beta), \quad \beta \in \mathbb{R}^k.$$

The vector β must be chosen in a way that the theoretical prices are close as possible to the observed/market prices. We estimate σ_1, σ_2 via a process called calibration. In the next section, we discuss calibration in detail.

To find the price of a European call option ($C(S,t)$), we must solve the following IBVP:

$$\begin{cases} \frac{\partial C}{\partial t} + \left((\mu_1 - \mu_2 S_t) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\right)\frac{\partial C}{\partial S} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 C}{\partial S^2} - rC = 0. \\ C(0,t) = 0, \ \forall t. \\ C(S,t) \sim S \text{ as } S \to \infty. \\ C(S,T) = \max(S_T - K, 0). \end{cases}$$

The finite difference scheme is the most common method used to solve PDE's. There are three main schemes:

- Explicit method.
- Implicit method.
- Crank-Nicolson method.

The Crank-Nicolson (CN) method is an average of the explicit and implicit method, which makes it more superior in terms of stability, convergence and consistency, see [FN13, AP05].

6.1.1 The Crank-Nicolson Method

We now solve the PDE for the hybrid model using the CN scheme:

$$rC = \frac{\partial C}{\partial t} + \left((\mu_1 - \mu_2 S_t) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\,\right)\frac{\partial C}{\partial S} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\frac{\partial^2 C}{\partial S^2} \quad (6.18)$$

over the grid, $0 \le t \le T$ and $S_{\min} \le S \le S_{\max}$. The parameters $r, \sigma_1, \sigma_2, \mu_1, \mu_2, \rho > 0$. The PDE is solved by approximating the partial derivatives with finite differences. $S_{\max}$ is a sufficiently large asset price, and $S_{\min}$ is a smallest asset price (usually set at zero). The domain for the PDE (6.10) is unbounded with respect to the asset price, so we bound it by $S_{\max}$ for computational purposes , see [AP05]. The grid on the domain consists of points (S, t) such that

$$S = 0, \Delta S, 2\Delta S, \ldots, M\Delta S \equiv S_{\max}\cdot$$

$$t = 0, \Delta_t, 2\Delta_t, \ldots, N\Delta_t \equiv T\cdot$$

with $C_{nm} = C(n\Delta_t, m\Delta S)$. There are three basic FD schemes to approximate the partial derivatives:

The forward difference

$$\frac{\partial C}{\partial S} = \frac{C_{m+1,n} - C_{n,m}}{\Delta S} + \mathcal{O}(\Delta S),\ \frac{\partial C}{\partial t} = \frac{C_{m,n+1} - C_{n,m}}{\Delta_t} + \mathcal{O}(\Delta_t)\cdot$$

The backward difference

$$\frac{\partial C}{\partial S} = \frac{C_{n,m} - C_{n,m-1}}{\Delta S} + \mathcal{O}(\Delta S),\ \frac{\partial C}{\partial t} = \frac{C_{n,m} - C_{n-1,m}}{\Delta_t} + \mathcal{O}(\Delta_t)\cdot$$

The central difference

$$\frac{\partial C}{\partial S} = \frac{C_{m+1,n} - C_{n,m-1}}{2\Delta S} + \mathcal{O}(\Delta S),\ \frac{\partial C}{\partial t} = \frac{C_{m,n+1} - C_{n-1,m}}{2\Delta_t} + \mathcal{O}(\Delta_t)\cdot$$

The second-order difference

$$\frac{\partial^2 C}{\partial S^2} = \frac{C_{m+1,n} - 2C_{m,n} + C_{m-1,n}}{(\Delta S)^2} + \mathcal{O}(\Delta S^2)\cdot$$

Depending on the combinations of the schemes, we have either the explicit or implicit method, see [AP05]. We define an array of $N+1$ equally spaced grid points, i.e. $t_0, t_1, \ldots, t_N$ to discretise time. Similarly, we define an array of $M+1$ equally spaced grid points, i.e. $S_0, S_1, \ldots, S_M$ to discretise the underlying asset. So we have:

$$\Delta_t = t_{n+1} - t_n \equiv \frac{T}{N},\ \ \Delta S = S_{m+1} - S_m \equiv \frac{S_{\max} - S_{\min}}{M}\cdot$$

This leads us to a rectangular region on the (t, S_t) plane, with boundaries $(0, S_{\max})$ and $(0, T)$.

The Crank-Nicolson method applies central approximation for $\frac{\partial C}{\partial t}, \frac{\partial C}{\partial S}$ and $\frac{\partial^2 C}{\partial S^2}$. Hence we have the following approximation for the differential equations in (6.18):

$$\begin{aligned}
\frac{\partial C}{\partial t} &= \frac{C_m^n - C_m^{n-1}}{\Delta_t} + \mathcal{O}(\Delta_t)\cdot \\
\frac{\partial C}{\partial S} &= \frac{1}{2}\left(\frac{C_{m+1}^{n-1} - C_{m-1}^{n-1}}{2\Delta S} + \frac{C_{m+1}^{n} - C_{m-1}^{n}}{2\Delta S}\right) + \mathcal{O}(\Delta S)\cdot \\
\frac{\partial^2 C}{\partial S^2} &= \frac{1}{2}\left(\frac{C_{m+1}^{n-1} - 2C_m^{n-1} + C_{m-1}^{n-1}}{(\Delta S)^2} + \frac{C_{m+1}^{n} - 2C_m^{n} + C_{m-1}^{n}}{(\Delta S)^2}\right) + \mathcal{O}(\Delta S^2)\cdot
\end{aligned}$$

Discretisation of the PDE

$$\begin{aligned}
&\frac{C_m^n - C_m^{n-1}}{\Delta_t} + \left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right)\cdot\frac{1}{2}\left(\frac{C_{m+1}^{n-1} - C_{m-1}^{n-1}}{2\Delta S} + \frac{C_{m+1}^{n} - C_{m-1}^{n}}{2\Delta S}\right) \\
&+ \frac{1}{2}(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S)\cdot\frac{1}{2}\left(\frac{C_{m+1}^{n-1} - 2C_m^{n-1} + C_{m-1}^{n-1}}{(\Delta S)^2} + \frac{C_{m+1}^{n} - 2C_m^{n} + C_{m-1}^{n}}{(\Delta S)^2}\right) \\
&= \frac{r}{2}\left(C_m^n + C_m^{n-1}\right) + \mathcal{O}(\Delta_t, \Delta S^2) \\
&\therefore C_m^n - C_m^{n-1} + \frac{\Delta_t}{4\Delta S}\cdot\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right)\cdot\left(C_{m+1}^{n-1} - C_{m-1}^{n-1} + C_{m+1}^{n} - C_{m-1}^{n}\right) \\
&+ \frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right)\cdot\left(C_{m+1}^{n-1} - 2C_m^{n-1} + C_{m-1}^{n-1} + C_{m+1}^{n} - 2C_m^{n} + C_{m-1}^{n}\right) \\
&= \frac{r\Delta_t}{2}\left(C_m^n + C_m^{n-1}\right) + \underbrace{\mathcal{O}(\Delta_t^2, \Delta_t\Delta S^2)}_{\text{local truncation error}}\text{ . Ignoring the error term, we have}
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right) - \frac{\Delta_t}{4\Delta S}\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right)\right)C_{m-1}^{n-1} \\
&+ \left(-\frac{r\Delta_t}{2} - \frac{\Delta_t}{2\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right) - 1\right)C_m^{n-1} \\
&+ \left(\frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right) + \frac{\Delta_t}{4\Delta S}\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right)\right)C_{m+1}^{n-1} \\
&= \left(-\frac{\Delta_t}{4\Delta S}\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right) - \frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right)\right)C_{m-1}^{n} \\
&+ \left(\frac{\Delta_t}{2\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right) + \frac{r\Delta_t}{2} - 1\right)C_m^{n} + \\
&\left(-\frac{\Delta_t}{4\Delta S}\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right) - \frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right)\right)C_{m+1}^{n}
\end{aligned} \tag{6.19}$$

We simplify equation (6.19) as

$$-a_m C_{m-1}^{n-1} + (1-b_m)C_m^{n-1} - c_m C_{m+1}^{n-1} = a_m C_{m-1}^n + (1+b_m)C_m^n + c_m C_{m+1}^n, \text{ where} \tag{6.20}$$

$$a_m = \left(\frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2 \Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right) - \frac{\Delta_t}{4\Delta S}\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right)\right).$$

$$b_m = -\frac{r\Delta_t}{2} - \frac{\Delta_t}{2\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right).$$

$$c_m = \left(\frac{\Delta_t}{4\Delta S}\left((\mu_1 - m\mu_2\Delta S) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S}\right) + \frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2\Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right)\right).$$

A Matrix System

$$AC_{n-1} = BC_n + D_{n-1} + D_n, \quad \text{where } n = N, \dots, 1.$$

$$C_n = \begin{bmatrix} C_1^n \\ C_2^n \\ \vdots \\ C_{M-1}^n \end{bmatrix}, \quad D_n = \begin{bmatrix} a_1 C_0^n \\ 0 \\ \vdots \\ 0 \\ c_{M-1}C_M^n \end{bmatrix}$$

$$A = \begin{bmatrix} 1-b_1 & -c_1 & 0 & \cdots & 0 & 0 \\ -a_2 & 1-b_2 & -c_2 & \cdots & 0 & 0 \\ & -a_3 & 1-b_3 & \cdots & 0 & 0 \\ & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & \cdots & -a_{M-1} & 1-b_{M-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 1+b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & 1+b_2 & c_2 & \cdots & 0 & 0 \\ & a_3 & 1+b_3 & \cdots & 0 & 0 \\ & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & \cdots & a_{M-1} & 1+b_{M-1} \end{bmatrix}$$

In a complete case (6.9) we follow similar steps as above to obtain:

$$a_m = \frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2 \Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right) - \frac{rm\Delta_t}{2}.$$

$$b_m = -\frac{r\Delta_t}{2} - \frac{\Delta_t}{2\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2 \Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right).$$

$$c_m = \frac{rm\Delta_t}{2} + \frac{\Delta_t}{4\Delta S^2}\left(\sigma_1^2 + \sigma_2^2 m^2 \Delta S^2 + 2\rho\sigma_1\sigma_2 m\Delta S\right).$$

Discretisation of the PDE introduces some error in the Crank-Nicolson scheme, see [FN13, AP05]. There are three main factors that characterise a numerical scheme, that is consistency, stability, and convergence, see [Str04]. We study these factors in the next section.

6.1.2 Convergence, Consistency and Stability

Consistency: an FD scheme is said to be consistent if the scheme converges to the PDE, as the time and space tend to zero. Stability: an FD scheme is said to be stable if the distance between the exact and approximation solution remains bounded, as the number of steps gets large. Convergence: an FD scheme is said to converge, if the distance between the exact and approximation solution, tends to zero as time and space tend to zero, see [MM98]. We state the following mathematical conditions for convergence and consistency from [MM98, Str04].

Definition 12 ***Convergence*** *and* ***Consistency***
For a given PDE, $\Lambda u = f$ and a finite difference (FD) scheme $\Lambda_{k,h} v = f$, a FD scheme is said to be consistent with a PDE if for any smooth function $\phi(t,x)$,
$\Lambda\phi - \Lambda_{k,h}\phi \to 0$ *as* $k,h \to 0$,
the convergence is pointwise-convergence at each point (t,x).

We show that the CN finite difference scheme is consistent with the PDE (6.9, 6.18). We have Λ from the PDE in (6.18)

$$\Lambda\phi = \phi_t + \left((\mu_1 - \mu_2 S_t) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\right)\phi_S + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\phi_{SS} - rC.$$

Substituting

$$\phi_t = \frac{\phi_m^n - \phi_m^{n-1}}{k} + \mathcal{O}(\Delta_t)\cdot$$
$$\phi_S = \frac{1}{2}\left(\frac{\phi_{m+1}^{n-1} - \phi_{m-1}^{n-1}}{2h} + \frac{\phi_{m+1}^{n} - \phi_{m-1}^{n}}{2h}\right) + \mathcal{O}(\Delta S)\cdot$$
$$\phi_{SS} = \frac{1}{2}\left(\frac{\phi_{m+1}^{n-1} - \phi_{m}^{n-1} + \phi_{m-1}^{n-1}}{h^2} + \frac{\phi_{m+1}^{n} - 2\phi_{m}^{n} + \phi_{m-1}^{n}}{h^2}\right) + \mathcal{O}(\Delta S^2)\cdot$$

We compute the Taylor's series expansion of the function ϕ in t and S about (t_n, S_m),

$$\phi_m^{n-1} = \phi_m^n - k\phi_t + \frac{1}{2}k^2\phi_{tt} + \mathcal{O}(k^3)$$
$$\phi_{m+1}^{n-1} = \phi_m^n - k\phi_t + \frac{1}{2}k^2\phi_{tt} + h\phi_S + \frac{1}{2}h^2\phi_{SS} - kh\phi_{tS} + \mathcal{O}(k^3) + \mathcal{O}(h^3)$$
$$\phi_{m-1}^{n-1} = \phi_m^n - k\phi_t + \frac{1}{2}k^2\phi_{tt} - h\phi_S + \frac{1}{2}h^2\phi_{SS} + kh\phi_{tS} + \mathcal{O}(k^3) + \mathcal{O}(h^3)$$
$$\phi_{m+1}^{n} = \phi_m^n + h\phi_S + \frac{1}{2}h^2\phi_{SS} + \mathcal{O}(h^3)$$
$$\phi_{m-1}^{n} = \phi_m^n - h\phi_S + \frac{1}{2}h^2\phi_{SS} + \mathcal{O}(h^3)$$

Therefore,

$$\Lambda_{k,h}\phi = \phi_t - \frac{1}{2}k\phi_{tt} + \left((\mu_1 - \mu_2 S_t) - \lambda\sqrt{\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t}\right)(\phi_S - \frac{1}{2}k\phi_{tS})$$
$$+ \frac{1}{2}(\sigma_1^2 + \sigma_2^2 S_t^2 + 2\rho\sigma_1\sigma_2 S_t)\phi_{SS} - rC$$

Thus,

$\Lambda\phi - \Lambda_{k,h}\phi \to 0$ as $(k,h) \to 0$. Therefore, this scheme is consistent.

Theorem 2 ***The stability condition***

A one-step finite difference scheme is stable in a stability region Λ if and only if there is a constant K (independent of θ, k and h) such that

$$|g(\theta, k, h)| \leq 1 + Kk$$

with $(k,h) \in \Lambda$. If $g(\theta, k, h)$ is independent of k and h, then the stability condition is

$$|g(\theta)| \leq 1$$

See [Str04] for proof.

We show that the CN finite difference scheme is stable with the PDE (6.9, 6.18) using Von Neumann Stability analysis.

Definition 13 ***Von Neumann Stability Analysis***

A finite difference scheme that approximate C_m^n is said to be stable if the substitution $C_m^n = g^n e^{im\theta}$ satisfies the condition that $|g(\theta)| \leq 1$ or $|g(\theta)|^2 \leq 1$ where, $g(\theta)$ is the amplification factor and θ is the phase angle, see [Str04].

From equation (6.20) we make the substitution $C_m^n = g^n e^{im\theta}$,

$$-a_m C_{m-1}^{n-1} + (1-b_m)C_m^{n-1} - c_m C_{m+1}^{n-1} = a_m C_{m-1}^n + (1+b_m)C_m^n + c_m C_{m+1}^n$$

$$-a_m g^{n-1} e^{i(m-1)\theta} + (1-b_m)g^{n-1}e^{im\theta} - c_m g^{n-1} e^{i(m+1)\theta} = a_m g^n e^{i(m-1)\theta} + (1+b_m)g^n e^{im\theta} + c_m g^n e^{i(m+1)\theta},$$

by the identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Thus

$$g(\theta) = \frac{(1-b_m) - \cos(\theta)(a_m + c_m) + i\sin(\theta)(a_m - c_m)}{(1+b_m) + \cos(\theta)(a_m + c_m) - i\sin(\theta)(a_m - c_m)}.$$

Thus, $|g(\theta)|^2 \leq 1$, hence the scheme is stable.

6.2 Model Calibration

We approximate σ_1 and σ_2 since these parameters are not obtained directly from the market. For a model to be realistic it needs to return the current prices or at least approximately, that implies that we need to fit these parameters to our model. The process of fitting parameters to the model is called calibration. Calibration is based on the assumption that there are sufficiently many liquidly traded contingent claims, see [KW12]. The standard calibration procedure minimises the distance between model prices (F^{model}) and market prices (P^{market}). There are various types of error measure, in this dissertation, we use the sum of squared errors (SSE) i.e.

$$\text{SSE}(x) = \sum_{i=1}^{N} \left| P_i^{\text{market}}(x) - F_i^{\text{model}}(x) \right|^2, \tag{6.21}$$

where N is the total number of market prices and $x = (x_1, \ldots\ldots, x_n)^T$ is a vector of model parameters. We want to find a parameter vector x^* that best fits the model prices to market prices, i.e. $P_i^{\text{market}}(x) \approx F_i^{\text{model}}(x), i = 1, \ldots, N$. Then the calibration procedure can be viewed as an optimization problem of the form

$$\min_{x \varepsilon \chi} f(x), \quad \xi \subseteq \mathbb{R}^n,$$

where ξ is the admissible domain of the model parameters, $x_1, \ldots, x_n$, see [KW12, Miy11].

Let $\{C^{\text{model}}(K_i, T_i),\ i = 1, 2, \ldots, N\}$ denote N model prices of European call options based on the hybrid model, where K is the strike price and T is the maturity. Also denote N market prices of European call options by $\{C^{\text{market}}(K_i, T_i),\ i = 1, 2, \ldots, N\}$. For each i, we define a function of σ_1 and σ_2 by

$$f_i(\sigma_1, \sigma_2) = C^{\text{market}}(K_i, T_i) - C^{\text{model}}(K_i, T_i)\,. \tag{6.22}$$

Then we estimate the parameters by minimizing the function

$$\min_{(\sigma_1, \sigma_2)} \sum_{i=1}^{N} |f_i(\sigma_1, \sigma_2)|^2\,. \tag{6.23}$$

We use the Black-Scholes prices as the market prices and the hybrid prices as model prices.

S_{min}	0
S_{max}	170
K	145
r	0.075
T	82/252
λ	0.11193

Table 6.1: European option inputs.

We use the *optimx* function in R to minimize equation (6.23), see the code in the Appendix.

Figure 6.1: Calibration results.

Parameter	Estimate
$\hat{\sigma_1}$	0.076
$\hat{\sigma_2}$	1.014

Table 6.2: Fitted parameters for the hybrid model.

6.3 Results

In Figure (6.2) we plot the surface of the European call option value using the CN-method for the PDE in (6.18). In Figure (6.3a) we compare market and model prices for both PDE's in (6.9 and 6.18). In Figure (6.3a) we plot the absolute error and the Black-Scholes European call option prices. In Figure (6.3) we plot the delta[7] of a call option vs a share price. We use inputs in Table (6.1) and (6.2).

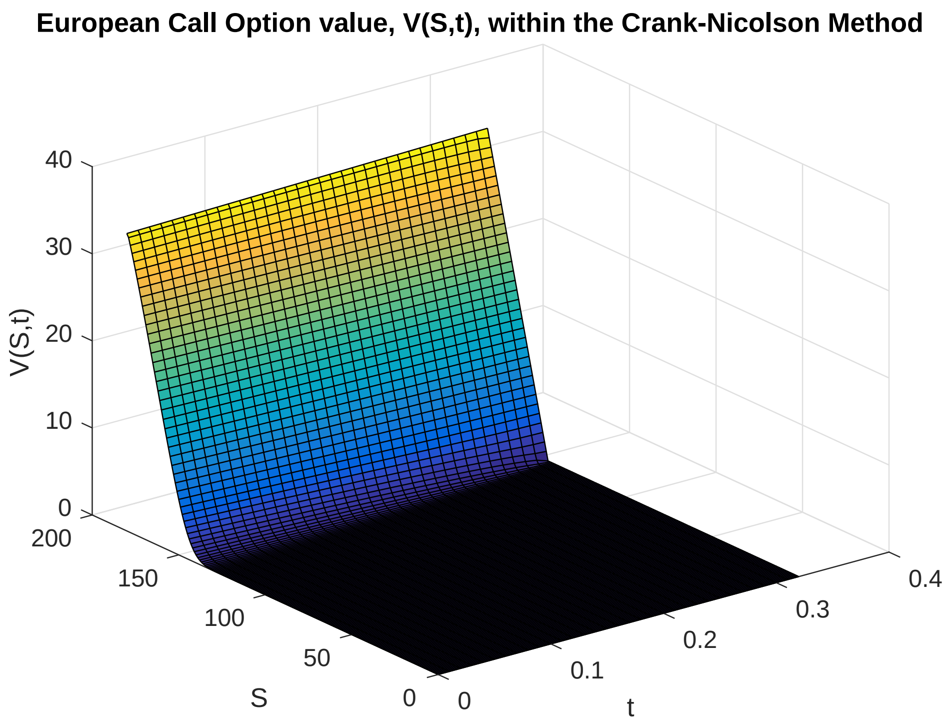
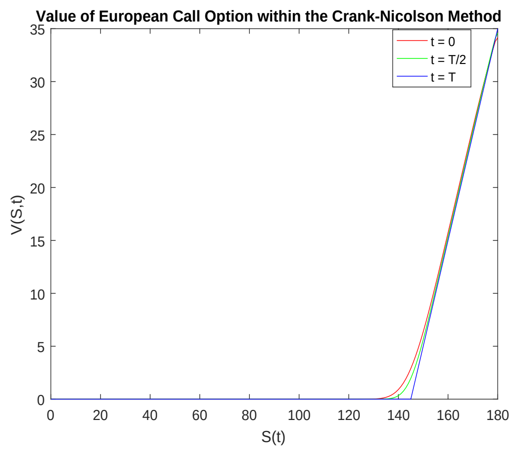
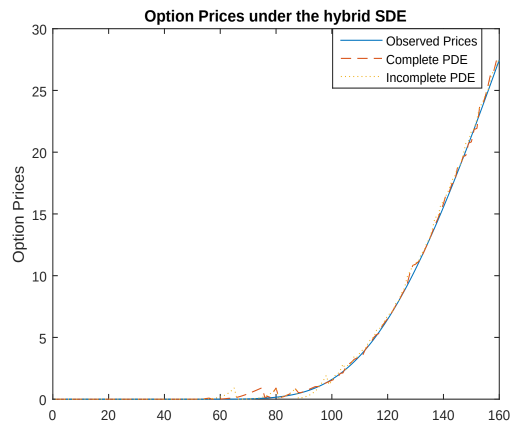


Figure 6.2: European Call Option for the incomplete case.

[7] We apply the FD method to estimate the delta (Δ) i.e. $\Delta = \frac{f_{i+1}-f_i}{2\delta}$ where f_i is a price of a European call option (PDE in 6.18), and δ is a small number. Other *Greeks* can be computed using FD method.

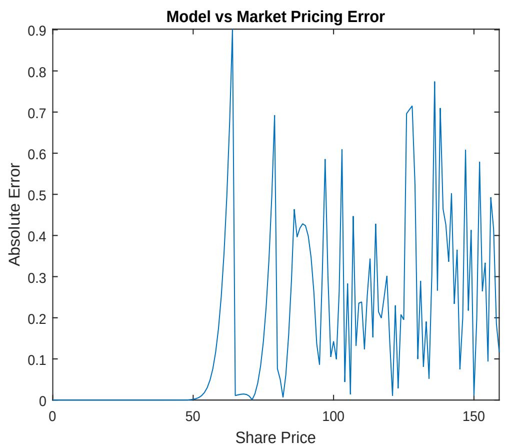


(a) European Call option value at times: $t = 0, T/2$ and T (maturity).

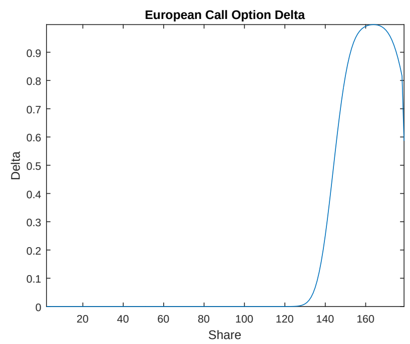


(b) Market vs the hybrid Model call option prices.

Figure 6.3: European Call option results.



(a) Absolute errors between market and model prices.



(b) European Call Option delta using the hybrid model.

Figure 6.4: Absolute errors and the delta of the hybrid model.

6.4 Chapter Summary

In this chapter, we have applied the hybrid SDE in Chapter 2 to price European style call options, in a complete and incomplete market set up. The hybrid SDE has two sources of noise and one risky asset, which makes it incomplete by the Meta-theorem. In a special case where the two sources of noise are perfectly correlated, the underlying model is complete. The main difficulty of pricing contingent claims in an incomplete market is the uniqueness of a martingale measure. In an incomplete market there are multiple measures, so to price we need to fix a measure. We followed the techniques by [Bra13, Bjö09, Hes93] to derive the PDE in terms of market price of risk.

In an incomplete market, the market price of risk acts as a martingale measure, see [Bra13, Bjö09]. The market price of risk $\lambda = \frac{\mu - r}{\sigma}$ then $\mu = r + \lambda\sigma$. The drift (μ) is equal to the risk-free rate plus the market price of risk, dependent on volatility (σ). If $\lambda = 0$ then $\mu = r$, which leads us to a risk-neutral world, see [Bra13, Bjö09]. We solved the PDE for European style options using the Crank-Nicolson method and calibrated our model to the market prices using the least squares method. Our results are consistent with results of [IL13]. An alternative approach to pricing and hedging in an incomplete market, is to minimise the expected squared hedging errors, this method is called variance-optimal hedging, see [BC10, SS08].

Chapter 7

Conclusion

In this dissertation, we studied [SS15] model and applied it in computations of risk measures such as Value-at-Risk and Expected Shortfall. Finally, we applied the model to price European style options in two settings, namely pricing in a complete and in an incomplete market set up.

Chapter 2 was devoted to the derivation of the arithmetic-geometric hybrid Brownian Motion model. In this model, we consider two types of market traders namely, the fundamental and technical traders. Both these types of traders are seeking information about the underlying asset S_t, and trade based on that information. However, their method of seeking such information differs. The underlying (S_t) price will fluctuate up and down based on the demand and supply of S_t. The net effect on the market prices over a short period of time is described by the log-returns (X_t), which we defined in equation (2.1). We assume that (X_t) is sufficient in describing the current state of the market. In this model, fundamental traders trade independently of the current value of (X_t), while technical traders trade based on the historical performance of (X_t) i.e. $\{X_s \,|0 \leq s \leq t\}$.

The current market state (X_t) is determined by the buy and sell orders to the market. We derived the mean and variance of the counting process of the number of trades, from a characteristic function of the inter-arrival distribution following the work of [SS15]. After a lengthy argument, we derived the mean and standard deviation of the fundamental and technical traders. The buying and selling of (S_t) cause price fluctuations, which we described the linear return impact function[1] in equations (2.35 - 2.36). Lastly, we approximate the trade arrival model by two independent Brownian motions. We naturally obtained a discrete-time stochastic equation in (2.38). A hybrid SDE is obtained by taking limits in equation (2.42). It is clear from the SDE that, the Arithmetic Brownian motion is related to the fundamental trades and the Geometric Brownian motion is related to the technical trades. We introduced the correlation (ρ) between the two Brownian motions and re-wrote the SDE in equation (2.42) with a single noise in equation (2.43). The hybrid SDE has five parameters, namely $\sigma_1, \mu_1, \sigma_2, \mu_2$ and ρ.

We then showed the strong solution of the hybrid SDE exists using Theorem 1. In section 2.2 we provided the explicit solution of the hybrid SDE and showed how to simulate sample paths in Figure (2.1). In Section 2.3 we calculated the mean of the hybrid SDE and a variance for the special case when ($\rho = 0 = \mu_1$) in equation (2.59). In Table (2.1) we calculated the probability of sigma-events (extreme movements) under the Gaussian and Student-t (4 d.o.f) distribution. Extreme events are greatly probable in the Student-t distribution compared to the Gaussian distribution. The reason is that the Student-t distribution has fat-tails compared to the Gaussian distribution, see [DCHW08].

In Chapter 3 we derived the quantilised Fokker-Planck equation (QFPE 3.1) associated with the general SDE in equation (3.1). We then applied the QFPE in equation (3.10) to derive equilibrium fat-tail distributions, following the work of [SS08, SS15]. In Section 3.2 we derived the Pearson type IV distribution, and the Student-t distribution in Section 3.3. Both these distributions have fat-tail property, however, the Pearson type IV distribution is much more flexible than the Student-t distribution. Both these distributions were derived under equilibrium conditions in the QFPE. However, the market turns to be unstable in times of financial panics. We, therefore, applied the full Fokker-Planck equation[2] on the hybrid SDE under non-equilibrium conditions. Under special cases where $\nu = \{0, 2, -2, -4\}$ we explicitly derived the PDF's, captured by Figures (3.3, 3.6, 3.5, 3.7). For large values of ν (d.o.f), it becomes impossible to explicitly derive the PDF's. All the non-equilibrium densities display the variance explosion phenomenon.

[1]The return impact function is simply a description of the order book, see [Sch15].

[2]The Fokker-Planck equation is a PDE that describes the time evolution of the probability density function of a stochastic process, see [Bjö09, Chapter 5].

In Chapter 4 we introduced and analysed our data, we downloaded four stocks of the top four financial institutions in South Africa (Barclays Africa, Nedbank, Old Mutual and Standard Bank), and the FTSE/JSE Top 40 index. We computed the first four moments of our data, and the results are captured in the Table (4.1). All shares display moderate skewness and excess in kurtosis (heavy tails). In Section 4.2 we fitted the Gaussian distribution in our data and applied the QQ-plot to test for normality. Figures (4.6a - 4.10a) capture the normality test results. It is clear from our results that the Normal distribution greatly underestimates the tails. We then fitted the Pearson type IV and the Student-t distributions in our data, using the method of maximum likelihood. The Pearson type IV and the Student-t distributions fitted our data much better, we further verified our results by applying the QQ-plot. The results for distribution fitting of the Pearson type IV and the Student-t distributions are captured in Tables (4.2 - 4.3) and Figures (4.11 - 4.20a).

In the next chapter, we analysed Value-at-Risk (VaR) and the Expected Shortfall (ES) using the Historical method and the Variance-Covariance (VC) method. We estimated one-day VaR and ES at 99% confidence level for all the stocks and the index. We use the Pearson type IV and Student-t distributions. Under the VC method, we adopted the GARCH(1,1) model, to deal with the volatility clustering following the work of [BM+12, BCY08, BMK09, SMNZ12, SZ13]. Based on the results obtained, the Pearson type IV and the Student-t distributions produced good results, whereas the Normal distribution produced mediocre results, as we expected. This is not surprising since the Normal distribution has thin tails. We applied the Success-failure ratio (LR) test and the Kupiec likelihood ratio test, to backtest our VaR estimates. Backtesting results indicate that the Pearson type IV and the Student-t distributions produced consistent VaR estimates, compared to the Normal distribution. Our results are consistent with those of [MLR14, BMK09, SMNZ12]. All the VaR, ES and backtesting results are in Section 5.5.1. One major drawback of VaR is that it is not a coherent risk measure while volatility and ES are coherent, see [ADEH99].

Furthermore we computed VaR estimates using non-equilibrium density functions in equations (5.7 - 5.10). We fitted the distributions to our data using the method of maximum likelihood and applied numerical integration to compute VaR. The non-equilibrium distributions produced good and consistent results, but no better than the Pearson type IV and the Student-t distributions. The reason for that might be because both equilibrium (the Pearson type IV and the Student-t distributions) and non-equilibrium densities have heavy tails. Adopting the GARCH(1,1) model to the non-equilibrium distributions requires further work. Finally, in Chapter 6, we applied the hybrid SDE to price European style options. The hybrid SDE (2.42) has two sources of noise and one risky asset, then by the Meta-theorem, the hybrid SDE is considered to be incomplete. However, the hybrid SDE can be reduced to an SDE with a single noise (2.43), by taking the correlation between the two sources of noise. We, therefore, priced options in a complete and incomplete market set up. We then applied the Crank-Nicolson method to price options and calibrated our model using the method of least squares. Further research can be done by implementing minimum variance hedging techniques under the hybrid model.

Appendix A

Stochastic Calculus And The Black-Scholes Theory

In this chapter, we present important results from stochastic calculus which we use in this dissertation and the theory of the Black-Scholes model. We omit proofs in this chapter, detailed proofs will be referenced. Throughout this chapter, we model an asset in continuous time stochastic processes, in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The main sources for this chapter are [Øks03, Fam65, Bjö09].

Definition 14 ***Stochastic Process***
A stochastic process is a collection of random variables $\{X(t)\}$*. For any* t*,* $t = 0, 1, \ldots T$*,* $X(t)$ *is a random variable on* $(\Omega, \mathcal{F})$*.*

Definition 15 ***Random variable***
A random variable (X) *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a function* $X : \Omega \to \mathbb{R}$ *which assigns a number* $X(\omega)$ *to every outcome* $\omega \in \Omega$*.*

A Markov process is a particular type of stochastic process, where only the current value of a random variable is relevant for predicting the future. It is also known as the *memoryless* process.

Definition 16 ***The Markov Property***

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with a bounded Borel function $f : \mathbb{R}^n \to \mathbb{R}$. Then a stochastic process (X_t) is called Markovian if

$$\mathbb{E}[f(X_{t+h})|\mathcal{F}_t] = \mathbb{E}[f(X_h)]$$

for $t, h \geq 0$. See proof in [Øks03, Chapter 7].

Definition 17 ***A Renewal Process***

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with $\mathbb{P}[X_k > 0] = 1$. We define the partial sums as follows

$$\begin{aligned} X_1 &= Z_1 \\ X_n &= Z_n - Z_{n-1}, \quad n > 0, \quad Z_0 = 0 \\ &\vdots \\ Z_n &= \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}. \end{aligned}$$

The sequence $Z_1, Z_2, \cdots$ is called the renewal times and is increasing, and the sequence $X_1, X_2, \cdots$ is called the inter-renewal times. Then the process $\{N(t) : t \geq 0\}$ given by

$$N(t) = \sum_{n=1}^{\infty} \mathbb{1}_{\{Z_n \leq t\}} = \begin{cases} 1 & \text{if } Z_n \leq t \\ 0 & \text{otherwise.} \end{cases}$$

is called the renewal process.

A.1 The Black-Scholes Model

Under the Black-Scholes model we consider two assets, one is the risk-free asset (money account or bond) with price B_t at time t, described by the following dynamics:

$$\frac{dB_t}{B_t} = r dt \tag{A.1}$$

where r is the non-negative interest rate.

The other asset is the stock (risky asset) with the price X_t at time t, described by the following dynamics:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{A.2}$$

where $\mu \in \mathbb{R}$ is a constant mean rate of return, $\sigma > 0$ is a constant volatility, and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

A.2 Wiener Process

Definition 18 *A stochastic process W_t is called a Wiener process if the following hold.*

1. $W(0) = 0$ *a.s.*
2. *The process W_t has independent increment, i.e. if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.*
3. *For $s < t$ the stochastic variable $W(t) - W(s)$ has Gaussian distribution i.e. $N(0, \sqrt{t-s})$ with mean 0 and variance $\sqrt{t-s}$.*
4. *W_t has continuous trajectories.*

A.3 Itô Formula

Suppose the process (X_t) has a stochastic differential given by

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where $\mu(X_t, t)$ and $\sigma(X_t, t)$ are adapted processes, and let f be a $C^{1,2}$-function. Define the process Z by $Z(t) = f(t, X(t))$. Then Z has a stochastic differential given by

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$

	dt	dW_t^1	dW_t^2
dt	0	0	0
dW_t^1	0	dt	0
dW_t^2	0	0	dt

Table A.1: Itô's multiplication table, where W_t^1 and W_t^2 are two independent Weiner processes.

A.4 The Black-Scholes PDE

Let $V(t, X_t) = f(t, X_t)$ be a European style contingent claim, whose value depends on both the underlying (X_t) and time. The function f solves the Black-Scholes PDE

$$\frac{\partial f}{\partial t}(t,x) + (r-q)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) - rf(t,x) = 0 \tag{A.3}$$

where q is the dividend yield, and r is the continuously compounded risk-free rate. The Black-Scholes PDE is derived via delta-hedging strategy, which involves continuous trading in the underlying asset. The Black-Scholes imperfections are well captured by [Hul06, Bjö09].

A.5 Hedging Basics

In this section we discuss the basic hedging strategy for European style contingent claims. Let us assume we have some option with payoff $h(X_T)$, where the stock price is modelled by the GBM or by the hybrid SDE. We want to replicate the payoff $h(X_T)$ by using a portfolio of underlying asset and the risk-free asset. We consider a vector of a hedging strategy $\bar{\phi}_t = (\phi_t^1, \phi_t^2)$, where ϕ_t^1 represents the number of the units of the underlying asset held at time t, and ϕ_t^2 represents the number of units of the riskless bond held at time t. We require $\mathbb{E}[\int_0^T \phi_t^1]dt < \infty$ and $\mathbb{E}[\int_0^T \phi_t^2]dt < \infty$. The solution of the ODE in equation (A.1) is $B_t = e^{rt}$ with $B_0 = 1$. Then the value of this portfolio (Φ_t) at time t is $\Phi_t = \phi_t^1 X_t + \phi_t^2 e^{rt}$ In order for this portfolio to replicate the payoff at time T, we require

$$\phi_T^1 X_T + \phi_T^2 e^{rT} = h(X_T) \tag{A.4}$$

This portfolio is assumed to be self-financing, in differential form,it is written as

$$d(\phi_t^1 X_t + \phi_t^2 e^{rt}) = \phi_t^1 dX_t + r\phi_t^2 e^{rt} dt. \tag{A.5}$$

In a complete market equation (A.4) is easily achieved, but in an incomplete market, the equality does not always hold, see [KW12, BC12, BC10].

Appendix B

GARCH and Conditional Volatility

The volatility forecast models in time series of financial data has been popular since the 1980's. In recent years volatility forecast models have been used in estimating risk measures, see [BM+12, BCY08]. The first such model was developed in 1982 by Engel, the Autoregressive Conditional Heteroscedasticity (ARCH) model, which was further generalised by Bollerslev (1986) to Generalised Autoregressive Conditional Heteroscedasticity (GARCH) model. The GARCH model is used to estimate conditional mean and variance of financial data. The conditional variance depends on the all the past innovations of order p, but all previous conditional variances of order q.

B.1 ARCH and GARCH (p,q) Model

The ARCH Model

$$\sigma_t^2 = \omega + \sum_{i=1}^{h} \alpha_i R_{t-i}^2 \tag{B.1}$$

where h is the number of lags. The parameters must satisfy the following conditions, see [Dan11]:

1. $\forall i = 1, \dots, h,\ \alpha_i, \omega > 0$ To ensure positive volatility forecasts
2. $\sum_{i=1}^{h} \alpha_i < 1$ To ensure covariance stationarity so that unconditional volatility is defined.

One major drawback with the ARCH model concerns the long lag (h) lengths required to capture the impact of historical returns on current volatility. So by including lagged volatility during ARCH model creation we have the potential to incorporate the impact of historical returns, see [Dan11].

A general GARCH (p, q) model is described as follows:

$$R_t|\Omega_{t-1} \sim \Psi(\mu_t, \sigma_t^2) \tag{B.2}$$

$$\mu_t = c + \rho R_{t-1} \tag{B.3}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_i e_{t-j}^2 \quad \text{and} \tag{B.4}$$

$$e_t = R_t - \mu_t \tag{B.5}$$

where R_t is the return, μ_t is the conditional mean, σ_t^2 is the conditional variance, Ω_{t-1} is the set of all information available at time t, and $\Psi(\mu_t, \sigma_t^2)$ is the conditional distribution of R_t. The parameters satisfy the conditions $p > 0$, $q \geq 0$, $\omega > 0$ and $\alpha_i, \beta_j \geq 0$ for $i = 1, \ldots, p$, $j = 1, \ldots, p$.

The most common GARCH type model is the GARCH (1,1) model. Substituting $p = q = 1$ in the above equations we have

$$\mu_t = \rho R_{t-1} \tag{B.6}$$

$$\sigma_t^2 = \omega + \alpha(R_{t-1} - \mu_{t-1})^2 + \beta\sigma_{t-1}^2 \tag{B.7}$$

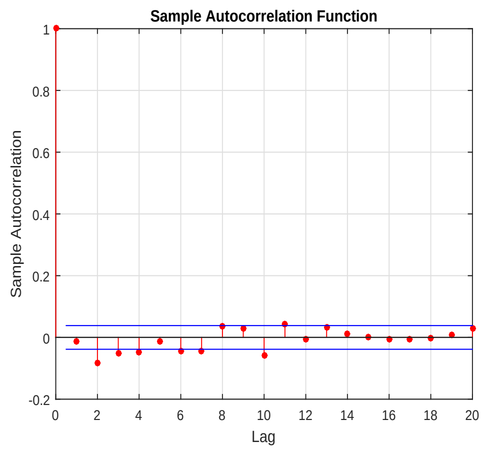
where $|\rho| < 1$, $\alpha, \beta > 0$ and $\alpha + \beta < 1$.

The unconditional volatility is given by
$\mathbb{E}(\omega + \alpha R_{t-1}^2 + \beta\sigma_{t-1}^2) = \omega + \alpha\sigma^2 + \beta\sigma^2$, where $\sigma^2 = \omega + \alpha\sigma^2 + \beta\sigma^2$.
$\Rightarrow \sigma^2 = \dfrac{\omega}{1 - \alpha - \beta}$.

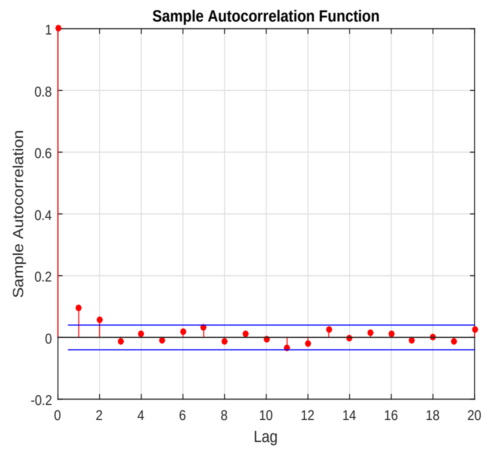
Definition 19 ***Autocorrelation function***
A autocorrelation function (ACF) $\rho(h)$ of a co-variance process $(X_t)_{t\in\mathbb{Z}}$ is

$$\rho(h) = \rho(X_h, X_0), \ \forall h \in \mathbb{Z} \tag{B.8}$$

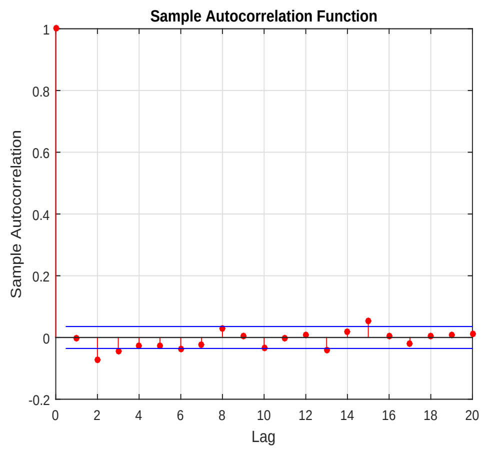
where h is a lag.



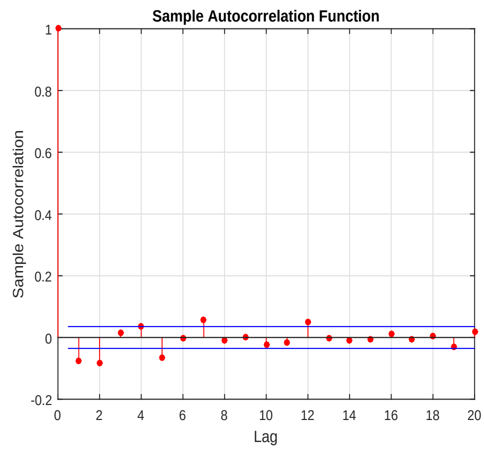
(a) Barclays Africa Autocorrelations of the entire sample data.



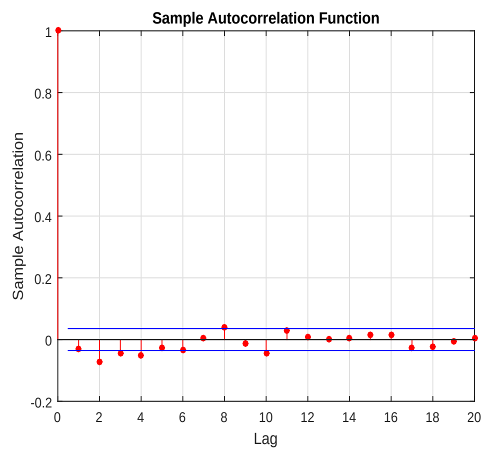
(b) JSE Autocorrelations of the entire sample data.



(c) Nedbank Autocorrelations of the entire sample data.



(d) Old Mutual Autocorrelations of the entire sample data.



2.8in

Figure B.1: Standard Bank Autocorrelations of the entire sample data.

Company	ω	α	β	σ^2_{t+1}
Barclays Africa	1.15396×10^{-5}	0.09448	0.87418	0.00024
FTSE/JSE Top 40	9.70991×10^{-6}	0.10738	0.86522	0.00038
Nedbank	1.00783×10^{-5}	0.08011	0.89097	0.00039
Old Mutual	5.4403×10^{-6}	0.00784	0.90992	0.00037
Standard Bank	7.38973×10^{-6}	0.08351	0.89510	0.00029

Table B.1: GARCH(1,1) parameter estimates under the Normal distribution.

Company	ω	α	β	σ^2_{t+1}
Barclays Africa	8.8329×10^{-6}	0.08742	0.88938	0.00024
FTSE/JSE Top 40	8.4975×10^{-6}	0.11516	0.87145	0.00037
Nedbank	9.897312×10^{-6}	0.08151	0.890313	0.00039
Old Mutual	4.92554×10^{-6}	0.08752	0.904423	0.00038
Standard Bank	7.53362×10^{-6}	0.08474	0.893843	0.00027

Table B.2: GARCH(1,1) parameter estimates under the Student-t distribution.

Company	ω	α	β	σ^2_{t+1}
Barclays Africa	1.37901×10^{-5}	0.09549	0.87932	0.00029
FTSE/JSE Top 40	3.92088×10^{-5}	0.02771	0.71392	0.00039
Nedbank	4.89233×10^{-6}	0.02794	0.95752	0.00040
Old Mutual	2.06528×10^{-5}	0.09572	0.81371	0.00035
Standard Bank	2.68447×10^{-5}	0.09272	0.82644	0.00031

Table B.3: GARCH(1,1) parameter estimates under the PearsonIV distribution.

Appendix C

Codes

C.1 R Calibration

```
Prices1 <- read.csv2("C:/Users/INNOCENTMBONA/Desktop/University Work/R work
/Pretoria/MSc/Prices1.csv", header=FALSE)
> dat=data.frame((Market_Prices=Prices1[,1]),Model_Prices=Prices1[,2])
> min.RSS<-function(data,const){
+ with(data,sum((const[1]+const[2]*Market_Prices-Model_Prices)^2))
+ }
> result<-optim(const=c(0,1),min.RSS,data=dat)
> plot ( Model_Prices~Market_Prices ,data=dat , main="Least square regressi
> abline (a = result $const[ 1 ], b = result $const[ 2 ] , col = "red")
```

C.2 The Crank Nicolson FD Matlab

```
% Matlab Code: Evaluates an European Call / Put option by the
% Crank-Nicolson Scheme method under the hybrid model. The code was
    adopted from "http://www.goddardconsulting.ca/matlab-finite-
    diff-crank-nicolson.html"

function [V,Cvector]= eHybrid(K,S0,r,Sharevec,timevec,sigma1,sigma2
    ,rho,lambda,OptionType)
```

```
% Inputs: K - Strike price
%       : S0 - Share price
%       : r - Risk free interest rate
%       : Sharevec - Vector of stock prices (i.e. grid points)
%       : tvec - Vector of times (i.e. grid points)
%       : oType - must be 'PUT' or 'CALL'.
%
% Output: V - the option price.
%       : Cvector - the option price vector.

%% The Crank-Nicolson Method

% Number of grid points
M = length(Sharevec)-1;
N = length(timevec)-1;

% Number of grid sizes
 dt=(timevec(end)-timevec(1))/M; % Time step
 ds=(Sharevec(end)-Sharevec(1))/N; % Share step

% Matrix coeficients for the Crank-Nicolson method
m =  0:M;

SIGMA=sigma1^2+sigma2^2*m.^2*ds+2*rho*sigma1*sigma2*m*ds;

% Incomplete Case

mu1=0.01;mu2=0.02;
Am=(dt/(4*ds^2))*(SIGMA)-(dt/(4*ds))*((mu1-m.*mu2*ds)-lambda.*sqrt(
    SIGMA));
Bm=-(r*dt)/2-(dt/(2*ds^2))*(SIGMA);
```

```
Cm=(dt/(4*ds))*((mu1-m.*mu2*ds)-lambda.*sqrt(SIGMA))+(dt/(4*ds^2))
    *(SIGMA);

% Complete Case
% Am=(dt/(4*ds^2))*(SIGMA)-((dt*m*r)/(2));
% Bm=-(r*dt)/2-(dt/(2*ds^2))*(SIGMA);
% Cm=((dt*m*r)/(2))+(dt/(4*ds^2))*(SIGMA);

% Pre-allocate the output
OptionPrice(1:M+1,1:N+1) = nan;

% Boundary conditions
switch OptionType
    case 'CALL'
        % Specify the expiry time boundary condition
        OptionPrice(:,end) = max(Sharevec-K,0);
        % Put in the minimum and maximum price boundary conditions
        % assuming that the largest value in the Svec is
        % chosen so that the following is true for all time
        OptionPrice(1,:) = 0;
        OptionPrice(end,:) = (Sharevec(end)-K)*exp(-r*timevec(end
            :-1:1));
    case 'PUT'
        % Specify the expiry time boundary condition
        OptionPrice(:,end) = max(K-Sharevec,0);
        % Put in the minimum and maximum price boundary conditions
        % assuming that the largest value in the Svec is
        % chosen so that the following is true for all time
        OptionPrice(1,:) = (K-Sharevec(1))*exp(-r*timevec(end:-1:1)
            );
        OptionPrice(end,:) = 0;
end

```

```
% Tridiagonal matrixs
C = -diag(Am(3:M),-1) + diag(1-Bm(2:M)) - diag(Cm(2:M-1),1);
[L,U] = lu(C);
D = diag(Am(3:M),-1) + diag(1+Bm(2:M)) + diag(Cm(2:M-1),1);

% Solve at each node
offset = zeros(size(D,2),1);
for idx = N:-1:1
    if length(offset)==1
        offset = Am(2)*(OptionPrice(1,idx)+OptionPrice(1,idx+1)) + ...
            Cm(end)*(OptionPrice(end,idx)+OptionPrice(end,idx+1));
    else
        offset(1) = Am(2)*(OptionPrice(1,idx)+OptionPrice(1,idx+1));
        offset(end) = Cm(end)*(OptionPrice(end,idx)+OptionPrice(end,idx+1));
    end
    OptionPrice(2:M,idx) = U\(L\(D*OptionPrice(2:M,idx+1) + offset));
end

% Calculate the option price
V= interp1(Sharevec,OptionPrice(:,1),S0);
Cvector=OptionPrice(:,1);
% Plot the value of the option value, V(S,t), as a function of S
% at times:t=0, T/2 and T (maturity).
figure(1);
plot(Sharevec,OptionPrice(:,1)','r-',Sharevec,OptionPrice(:,round((N+1)/2))','g-',Sharevec,OptionPrice(:,N+1)','b-');
xlabel('S(t)');
ylabel('V(S,t)');
legend('t = 0','t = T/2','t = T')
```

```
title('Value of European Call Option within the Crank-Nicolson
    Method');

%% Figure of the Value of the option, V(S,t)
figure(2);
surf(timevec,Sharevec,OptionPrice(1:M+1,1:N+1));
title('European Call Option value, V(S,t), within the Crank-
    Nicolson Method')
xlabel('t')
ylabel('S')
zlabel('V(S,t)')

%% Black-Scholes
% S0 = 0:1:180;
% K = 145;
% r = 0.076;
% sigma = 0.076;
% for T=82/252:-0.05:0
% plot (S0 ,blsprice(S0 ,K,r ,T, sigma)) ;
% hold on;
% end
% grid on
% figure(3);
% title('The Black-Scholes Prices')
% xlabel('Share Price')
% ylabel('Option Prices')
end
```

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