A Generally Weighted Moving Average Chart for Time Between Events

N. Chakraborty[1,a], S.W. Human[1,b], N. Balakrishnan[2,c]

[1]Department of Statistics, University of Pretoria, Pretoria, 0002, Lynnwood Road, Hillcrest, South Africa

[2]Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

[a]Corresponding author: e-mail niladriorama@gmail.com

[b]Corresponding author: e-mail: schalk.human@up.ac.za

[c]Corresponding author: e-mail bala@mcmaster.ca

Abstract

Shewhart-type attribute charts are known to be inefficient for small changes in monitoring nonconformities. An alternative way is to use a time-weighted chart to monitor the time between events (TBE). We propose a one-sided Generally Weighted Moving Average control chart to monitor the time between events (TBE); regarded as the GWMA-TBE chart. To aid the implementation of the chart, the necessary design parameters are provided. An extensive performance analysis shows that the GWMA-TBE chart is better than the well-known EWMA and Shewhart charts at detecting very small to moderate changes. Finally, a summary and some conclusions are provided.

Keywords

GWMA chart; Time between events; Average Run-length; Simulation; Markov chain

1. Introduction

Since its introduction by Walter A. Shewhart in the 1920's, statistical quality control (SQC) has found a variety of applications ranging from health care monitoring to financial fraud detection. In general, an item that does not comply with the standards is labelled a nonconforming or defective item. The traditional Shewhart-type attribute charts such as the c-chart or p-chart (see Montgomery, 2009) are used to monitor a process when the number of nonconformities or the number of defective items is of interest; these charts use the number of nonconformities or the number of defective items in regular, typically equi-spaced, time intervals and are known to be ineffective in detecting small shifts. Such rare events are frequently found in today's high-performing technological environment wherein the rate of failures can be very small. An alternative and perhaps more appealing way to monitor the rate of failures (called the Poisson rate) is to use control charts based on the inter-arrival times of nonconforming items. These charts are especially useful when the occurrence of defective items is rare and the rate of occurrence is very small. Such charts are called time between events (TBE) charts and are typically based on the following two assumptions: (a) the occurrence of failures in a process follows a homogeneous Poisson process, and (b) the inter-arrival times between two consecutive events follow an exponential distribution. A number of TBE control charts can be found in the literature. The Cumulative Sum (CUSUM) chart to monitor the time between events, which is simply called the exponential CUSUM chart, was proposed by Vardeman and Ray (1985); the optimal design of the exponential CUSUM chart was given by Gan (1994) and an algorithm for computing the average run-length (ARL) of the chart was subsequently given by Gan and Choi

(1994). The design and performance of the Exponentially Weighted Moving Average (EWMA) chart for the exponential distribution was studied by Gan (1998). A Shewhart-type chart for the time between events (also assuming an exponential distribution) was studied by Xie et al. (2002). In addition, Zhang et al. (2007) studied the performance of the Shewhart-type time between events chart for the gamma distribution.

Time-weighted control charts such as the CUSUM and EWMA have been shown to be efficient in detecting small sustained upward or downward step shifts in a process – see Montgomery (2009) for more details and references on the theory, design and application of the classical EWMA and CUSUM charts for the normal distribution. Capitalising on the efficiency of the EWMA chart, Sheu and Lin (2003) proposed a Generally Weighted Moving Average (GWMA) control chart for the normal distribution; this chart is a generalisation of the EWMA chart and has been shown to be more effective than the EWMA, CUSUM and Shewhart-type charts (see Hsu et al., 2009) in detecting very small shifts. For more details on the recent developments on the GWMA charts, the interested reader is referred to the works of Sheu and Yang (2006), Sheu and Chiu (2007), Chiu and Sheu (2008), Chiu (2009), Sheu and Hsieh (2009), Sheu et al. (2012), Teh et al. (2012), Sheu et al. (2013), Huang et al. (2014), Huang (2014), Lu (2015), Aslam et al. (2015) and Chakraborty et al. (2016).

As noted, for example, by Zhang et al. (2005) and Balakrishnan et al. (2015), the quick detection of any deterioration in a process is of utmost importance from a practical point of view. In the current context, i.e., monitoring the time between events, process deterioration can occur following an increase in the failure rate. So, for a high quality process, a small reduction in process performance needs to be detected as early as possible. This does not imply that process improvement is not important. Quite to the contrary, as mentioned by Maravelakis and Castagliola (2009), an improvement in a process usually occurs after corrective action has taken place, but the time of the change is usually known and a control chart is not needed to detect the change/improvement. For this reason, a one-sided control chart to detect process deterioration is pursued here.

In this article, we construct a one-sided GWMA chart based on the gamma distribution for monitoring the time between events (often regarded as the time between failures); this chart is labelled the GWMA-TBE chart and can be used in detecting a sustained downward step shift in a process. The proposed chart includes the one-sided EWMA and Shewhart charts as special cases, and it differs in two ways from the currently available TBE charts: (i) The interdependency of the GWMA charting statistics is explicitly taken into consideration, and (ii) we specifically focus on a one-sided GWMA-TBE chart based on the gamma distribution. The proposed one-sided GWMA-TBE chart is thus run-length unbiased (by construction) which is unlike the currently available two-sided control charts (based on the exponential distribution) which are biased. Note that a control chart is "biased" when the out-of-control average run-length (denoted by ARL_1) is larger than its in-control value (denoted by ARL_0) and therefore takes longer to detect a shift in the process; this is an undesirable property and normalizing transformations often do not completely eliminate this problem.

The rest of this article is organised as follows: The general theory and background on the GWMA-TBE chart are given in Section 2. In Section 3, the run-length distribution for the scenario when the parameters are known is studied; this includes the in-control (IC) design and the out-of-control (OOC) performance. The GWMA-TBE chart for the scenario when the parameters are unknown is studied in Section 4; this includes the effects of parameter estimation on the performance as well as the design of the Phase II GWMA-TBE chart. In Section 5, we show how the proposed chart can also be used to monitor the variance of a normal distribution. An illustrative example is provided in Section 6 and we finally conclude with some remarks in Section 7.

2. The GWMA Time Between Events (GWMA-TBE) control chart

Let the random variables $Y_j \sim iid\, Exp(\theta)$, $j =$ 1,2,3,…, denote the inter-arrival times between consecutive failures in a homogeneous Poisson process with rate parameter $1/\theta$. The continuous random variable $X = \sum_{j=1}^{k} Y_j$, which is the sum of the inter-arrival times of k consecutive failures, then denotes the time until the k^{th} failure and is known to follow a $Gamma(k,\theta)$ distribution. The probability density function (p.d.f.) of X is given by

$$f(x;k,\theta) = \frac{e^{-x/\theta} x^{k-1}}{\Gamma(k)\theta^k}, \quad x > 0, \theta > 0 \text{ and } k > 0. \quad (1)$$

Note the following regarding the p.d.f. in (1):

i. The expected value and variance of X are $E(X) = k\theta$ and $var(X) = k\theta^2$;

ii. Although the parameter $k > 0$ can theoretically be any positive real number, in the developments that follow it is assumed that k is a known/specified positive integer, i.e.,

$k = 1, 2, 3, \ldots$, and selected by the practitioner. For example, if we set $k = 3$, the random variable X denotes the total time between three consecutive failures;

iii. If $k = 1$, the p.d.f. in (1) reduces to that of an exponential distribution and we would be interested in monitoring the time until one failure;

iv. It is assumed that θ_0 denotes the in-control (IC) standard/value for the parameter θ; this is known as the "standard known" case and denoted by Case K. We primarily focus on the scenario when the parameter θ is known, but we also deal with the situation when θ is unknown and estimated from an in-control (IC) Phase I reference sample before online monitoring can start in Phase II; the latter scenario is referred to as the "standard unknown" case and denoted by Case U;

v. Our objective is to construct a control chart to monitor θ for a sustained downward step shift, i.e., a decrease in the inter-arrival times, which would be indicative of deterioration in the process. The key focus is to detect small or very small shifts as quickly as possible.

The GWMA-TBE control chart is constructed by taking a weighted average of a sequence of the X_i's. To this end, let N be the discrete random variable denoting the number of samples until the next occurrence of an event since its last occurrence. Then, by summing over all values of N, we can write

$$\sum_{i=1}^{\infty} \Pr\left[N = i\right] = \sum_{i=1}^{t} Pr\left[N = i\right] + \Pr[N > t] = 1. \quad (2)$$

A generally weighted moving average (GWMA) is a weighted moving average (WMA) of a sequence of X_i statistics where the probability $\Pr\left[N = i\right]$ is regarded as the weight w_i for the i^{th}

most recent statistic X_{t-i+1} among the last t of X_i statistics. In other words, the probability $\Pr[N=1]$ is the weight w_1 for the latest observation X_t and the probability $\Pr[N=t]$ is the weight w_t for the most out-dated observation X_1. The probability $\Pr[N>t]$ is considered as the weight for the starting value, denoted by Z_0, which is typically taken as the in-control (IC) expected value of the statistic under consideration, i.e., $Z_0 = E(X_i \mid \mathrm{IC}) = k\theta_0$. Therefore, the charting statistic for the GWMA-TBE chart is defined as

$$Z_t = \sum_{i=1}^{t} \Pr[N=i] X_{t-i+1} + \Pr[N>t] Z_0 \quad \text{for} \quad t = 1,2,3,\ldots \tag{3}$$

As in Sheu and Lin (2003), the distribution of N is taken to be $\Pr[N=i] = q^{(i-1)^\alpha} - q^{i^\alpha}$, where $0 \le q < 1$ and $\alpha > 0$ are the two parameters; this is the discrete two-parameter Weibull distribution (see Nakagawa and Osaki, 1975). So, the weights are given by $w_i = q^{(i-1)^\alpha} - q^{i^\alpha}$. By substituting $Z_0 = k\theta_0$ and the probability mass function (p.m.f.) of the two-parameter discrete Weibull distribution in equation (3), the charting statistic for the GWMA-TBE chart simplifies to

$$Z_t = \sum_{i=1}^{t} \left(q^{(i-1)^\alpha} - q^{i^\alpha} \right) X_{t-i+1} + q^{t^\alpha} k\theta \quad \text{for} \quad t = 1,2,3,\ldots \tag{4}$$

Note that the GWMA-TBE chart reduces to an EWMA chart when $\alpha = 1$ and $q = 1-\lambda$, where $0 < \lambda \le 1$ is the smoothing parameter of the EWMA chart. The EWMA chart further reduces to the Shewhart chart when $q = 0$ and $\alpha = 1$. The GWMA chart can therefore be viewed as a generalization of the EWMA and Shewhart-type charts with an additional parameter α which provides more flexibility in designing the chart. We introduce the EWMA-TBE and the Shewhart-TBE charts as special cases.

A closer look at the choice of weights for the GWMA chart shows that the weights $w_i = q^{(i-1)^\alpha} - q^{i^\alpha}$, $i =$ 1,2,3,…, are decreasing in i for a fixed $0 < q < 1$ and $0 < \alpha \le 1$; this implies that more weight is given to more recent observations in the sequence of X_i's. A proof of this result is given in the Appendix.

The in-control (IC) expected value and variance of the charting statistic Z_t are given by

$$E\left(Z_t \mid \mathrm{IC}\right) = \sum_{i=1}^{t}\left(q^{(i-1)^\alpha} - q^{i^\alpha}\right) E\left(X_{t-i+1}\right) + q^{t^\alpha} k\theta_0 = k\theta_0 \quad (5)$$

and

$$var\left(Z_t \mid \mathrm{IC}\right) = \sum_{i=1}^{t}\left(q^{(i-1)^\alpha} - q^{i^\alpha}\right)^2 var\left(X_{t-i+1}\right) = Q_t k\theta_0^2 \quad (6)$$

respectively, where $Q_t = \sum_{i=1}^{t}\left(q^{(i-1)^\alpha} - q^{i^\alpha}\right)^2$.

The exact time-varying (symmetric) control limits (UCL_e &LCL_e) and centerline (CL_e) of a two-sided GWMA-TBE chart are given by

$$UCL_e / LCL_e = k\theta_0 \pm L\sqrt{Q_t k\theta_0^2} \text{ and } CL_e = k\theta_0 \quad (7)$$

where $L >$ 0 is the distance of the control limits from the centerline and the subscript "*e*" denotes the exact control limits.

The steady-state control limits are used when the process has been running for several time periods and are based on the asymptotic variance of the charting statistic (see Lucas and Saccucci, 1990). The steady-state control limits and the centerline are given by

$$UCL_s / LCL_s = k\theta_0 \pm L\sqrt{Qk\theta_0^2} \text{ and } CL_s = k\theta_0, \quad (8)$$

where the subscript "s" denotes the steady-state control limits and $Q = \lim_{t\to\infty} Q_t$.

As we are primarily interested in detecting deterioration in the process, i.e., a decrease in the waiting times between failures, only a lower control limit is used in the design and implementation of the proposed GWMA-TBE chart; this limit is given by

$$LCL_s = k\theta_0 - L\sqrt{Qk\theta_0^2}. \quad (9)$$

A GWMA-TBE chart for detecting improvement in a process can be designed in a similar manner but not pursued here any further for the sake of brevity.

The following two points are worth noting here:

i. The quantity Q_t is a monotonically increasing and convergent function of t (see the Appendix for more details). To this end, Table 1 shows the value of Q_t for different choices of the parameters (q,α) and various values of t. We observe that the change in Q_t is almost negligible for $t \geq 10$. Due to the quick convergence of Q_t, the exact control limits also quickly converge towards the steady-state limits. Hence, the steady-state lower control limit is used in order to simplify the application/implementation of the proposed chart. For the sake of notational simplicity, we will use LCL hereafter to denote the steady-state lower control limit in (9);

ii. If any Z_t plots on or below LCL, the process is declared out-of-control (OOC) and a search for assignable causes is initiated. Otherwise, the process is considered to be in-control (IC), which implies no change in θ has occurred, and the charting procedure continues on.

In the ensuing section, we discuss the design of the proposed GWMA-TBE chart in more detail.

Table 1: Q_t values for different (q,α) combinations.

t	$q=0.5,\ \alpha=0.5$	$q=0.5,\ \alpha=0.9$	$q=0.5,\ \alpha=1.3$
5	0.2751	0.3208	0.3691
10	0.2773	0.3215	0.3691
50	0.2778	0.3215	0.3691
100	0.2778	0.3215	0.3691
t	$q=0.7,\ \alpha=0.5$	$q=0.7,\ \alpha=0.9$	$q=0.7,\ \alpha=1.3$
5	0.1074	0.1555	0.2227
10	0.1107	0.1614	0.2239
50	0.1129	0.1618	0.2239
100	0.1129	0.1618	0.2239
t	$q=0.9,\ \alpha=0.5$	$q=0.9,\ \alpha=0.9$	$q=0.9,\ \alpha=1.3$
5	0.0132	0.0274	0.0667
10	0.0143	0.0360	0.0866
50	0.0160	0.0427	0.0887
100	0.0163	0.0427	0.0887

3. The design and implementation of GWMA-TBE chart

Performance measures are needed to design and compare the performance of control charts. The traditional approach of evaluating a control chart is to obtain the run-length distribution and its associated characteristics. The run-length is a discrete random variable that represents the number of samples which must be collected (or, equivalently, the number of charting statistics that must be plotted) in order for the chart to detect a shift or give a signal. An intuitively appealing and popular measure of a chart's performance is the average run-length (ARL), which is the expected number of charting statistics that must be plotted in order for the chart to signal (see Human and Graham, 2007). Clearly, for an efficient control chart, one would like to have the in-control ARL (denoted ARL_0) to be "large" and the out-of-control ARL (denoted ARL_1) to be "small". Although other measures such as the standard deviation of the run-length $(SDRL)$ and various upper and lower percentiles could be and have been used to supplement the evaluation of control charts, the ARL is the most widely used measure due to its intuitive appealing interpretation. Therefore, we use primarily the ARL to design and compare the performance of the proposed GWMA-TBE chart.

The design of a control chart typically involves solving for the combination of the chart's parameters, i.e., q, α and L, so as to obtain a pre-specified in-control average run-length denoted by ARL_0^* for a given or selected value of k. The computational aspects of the run-length distribution for the GWMA-TBE chart are discussed next; this is followed by the design of the GWMA-TBE chart.

3.1 **The run-length distribution of the GWMA-TBE chart**

There are numerous methods to calculate the run-length distribution of a time-weighted chart like the GWMA chart and we investigate here three approaches: (i) An exact approach based on closed-form expressions, (ii) A Markov chain approach, and (iii) Monte Carlo Simulation. Each method is discussed in more detail below along with their pros and cons. It should be noted that ultimately we used computer simulation and double-checked the results by using the Markov chain approach for the EWMA-TBE chart; these results are available upon request.

3.1.1 Exact approach

Suppose the run-length random variable is denoted by R and that A_i denotes the signalling event at the i^{th} sample. The non-signalling event is therefore given by $A_i^c = [Z_i > LCL]$ for $i =$ 1,2,3,.... Then, in general, the run-length distribution can be written as $\Pr\left[R=r\right]=\Pr\left[\left\{\bigcap_{i=1}^{r-1} A_i^{\mathrm{c}}\right\}\bigcap A_r\right]$, for $r =$ 1,2,3,.... For any $i \geq 1$, we can re-write the event $A_i^c = [Z_i > LCL]$, as $A_i^c = [X_i > L_i]$, where $L_1 = \dfrac{LCL - qk\theta_0}{1-q}$ and

$$L_i = \frac{LCL - \sum_{j=2}^{i}\left(q^{(j-1)^{\alpha}} - q^{j^{\alpha}}\right)X_{i-j+1} - q^{i^{\alpha}}k\theta_0}{1-q}, \; i = 2,3,4,\ldots \quad (10)$$

The run-length distribution can therefore be written as

$$\Pr\left[R=1\right]=1-\Pr\left[X_1 > L_1\right] \text{ and } \Pr\left[R=r\right]=I_{r-1}-I_r, \quad (11)$$

where $I_{\mathrm{r}} = \Pr\left[\bigcap_{i=1}^{r} A_i^c\right] = \Pr\left[\bigcap_{i=1}^{r} \{X_i > L_i\}\right]$ for $r = 2,3,4,\ldots$; see the Appendix for more details.

Since the X_i's are assumed to be independent $Gamma(k,\theta)$ random variables, we have

$$I_r = \int_{L_1}^{\infty}\int_{L_2}^{\infty}\cdots\int_{L_r}^{\infty}\left\{\prod_{i=1}^{r} f\left(x_i;k,\theta\right)dx_i\right\}, \quad (12)$$

where $f\left(x_i;k,\theta\right)$ is the p.d.f. given in (1).

The ARL can also be expressed in terms of I_{r} as (see the Appendix for more details)

$$ARL = 1+\sum_{r=1}^{\infty} I_{\mathrm{r}}. \quad (13)$$

To analytically evaluate expressions (12) and (13) is time-consuming and uneconomical for two reasons:

a. The lower bounds in the integrals of I_r, i.e., the L_i's given in equation (12), are mutually dependent and functions of the sequence of preceding statistics $X_1, X_2, \ldots, X_{i-1}$; these bounds cannot be economically recursively updated and is therefore a computationally expensive approach;

b. The number of terms in expression (12) that needs to be evaluated increases dramatically as r increases;

As mentioned before, if we set $q=0$ and $\alpha=1$, we obtain the Shewhart-TBE chart. In this special case, the lower control limit in (10) reduces to $L_i = LCL$ for $i \geq 1$ and the probability of a signal is given by $\Pr[X_i \leq LCL \mid X_i \sim Gamma\left(k,1/\delta\right)]$, where the lower control limit can be obtained by solving the expression $\Pr\left[X_i \leq LCL \mid X_i \sim Gamma\left(k,1\right)\right] = 1/ARL_0^*$.

3.1.2 Markov chain approach

For the EWMA charts, using the Markov chain approach, the probability mass function (p.m.f.), ARL and the variance ($VARL$) of the run-length random variable R can be obtained as (see Theorems 5.2 and 7.4 of Fu and Lou, 2003), $P[R=r]$ = $\boldsymbol{\xi}\mathbf{Q}^{r-1}(\mathbf{I}-\mathbf{Q})1$ for $r=1,2,3,\ldots$, $ARL=\boldsymbol{\xi}(\mathbf{I}-\mathbf{Q})^{-1}1$ and $VARL=$ $\boldsymbol{\xi}(\mathbf{I}+\mathbf{Q})(\mathbf{I}-\mathbf{Q})^{-2}1-ARL^{2}$, where the sub-matrix matrix $\mathbf{Q}=\mathbf{Q}_{\nu\times\nu}$ is called the essential transition probability sub-matrix, $\mathbf{I}_{\nu\times\nu}$ is the identity matrix, $\xi_{1\times\nu}$ is the 'initial probability vector' such that $\xi_j=1$ when the process mean i.e. $k\theta_0$, is in the j^{th} state and $\xi_j=$ 0 for all other j with $\sum_{j=1}^{\nu}\xi_j=1$, $1_{\nu\times1}=(1,1,\ldots,1)^{\mathrm{T}}$ is the unit vector and ν denotes the number of transient states in the state space Ω. The Markov chain results for the one-sided EWMA-TBE chart are available upon request from the corresponding author. The interested reader is also referred to the work by Graham et al. (2011a and 2011b) for more details.

The prime obstacle using the Markov chain approach for the GWMA chart is the fact that the GWMA chart's plotting statistic cannot be viewed as following a first-order Markov chain. In fact, obtaining Z_t requires all the X_t's from start-up, i.e., $X_t, X_{t-1},\ldots,X_0$. This complication arises due to the repeated exponentiation (so-called "bottom-up" tetration) in the weights $w_i=q^{(i-1)^{\alpha}}-q^{i^{\alpha}}$ of the charting statistics. This implies that Z_t depends on all $Z_{t-1},Z_{t-2},\ldots,Z_1$. To this end, note that the probability $\Pr[Z_t \mid Z_{t-1},\ldots,Z_1]$ can be approximated by $\Pr[Y_t^{*} \mid Y_{t-1}^{*}]$, where $Y_t^{*}=(Z_t,\ldots,Z_{t-m+1})$ and m is the threshold beyond which the weights are approximately zero (see Appendix). So, $\{Y_t^{*}, t=m,m+1,m+2,\ldots\}$ becomes a first-order Markov chain and we can use the results from Fu and Lou (2003) for $t=m,m+1,\ldots$. For $t=1,2,\ldots,m-1$, one still has to use the exact approach. The major difficulty in using the Markov Chain approach for the GWMA-TBE chart is in defining the state-space of the first-order Markov chain { $Y_t^*, t = m, m+1, m+2 \ldots\}$ as it depends on the state-space of $(Z_t, \ldots, Z_{t-m+1})$, which is the Cartesian product Ω^m, $t = m, m+1, m+2 \ldots$...

Due to the above-mentioned difficulties with the exact approach and the Markov chain approach, extensive simulation has been used to calculate the run-length distribution for the proposed GWMA-TBE chart. To this end, it is important to note that, Sheu and Lin (2003), Sheu and Yang (2006) and Lu (2015) also mentioned that it is difficult to obtain the run-length distribution of the GWMA charts by using the exact approach or the Markov chain approach. The simulation approach is discussed next.

3.1.3 Monte Carlo simulation approach

The simulation algorithm uses the stochastic representation of the GWMA-TBE chart and is done according to the following five steps:

i. Select a combination of the design parameters, i.e., (q, α, L), set $\theta_0 = 1$ and calculate the control limit according to expression (9);
ii. Generate an individual observation from a $Gamma(k,1)$ distribution and calculate the charting statistic Z_t according to expression (4) with the starting value taken as $Z_0 = k$;
iii. If $Z_t > LCL$, the process is considered to be in-control and a run-length counter is incremented;
iv. Steps (ii) and (iii) are repeated and Z_t is sequentially updated until $Z_t \le LCL$; when this event occurs, a signal is given and the process is declared to be out-of-control. The simulation stops and the run-length is recorded;

v. Steps (ii)–(iv) are repeated 10,000 times.

We have also used simulation for the EWMA-TBE chart in order to be consistent and be able to compare them with the Markov chain approach. The 10,000 simulated run-lengths can be used to empirically calculate the characteristics of the run-length distribution.

3.2 The in-control (IC) design of the GWMA-TBE chart

For a given or chosen value of k, the two parameters q and α are varied over a certain range and for each (q,α) combination, the values of the charting constant, i.e. $L>0$, are obtained so that the attained in-control ARL is close to (in this case slightly above or below due to the use of simulation) the nominal or specified value ARL_0^*. The typical industry standards for the ARL_0^* are 370 or 500 and we consider the former in our study. The typical recommendation for the smoothing parameter $0<\lambda\leq 1$ for an EWMA chart is to choose smaller values for smaller shifts (see Montgomery, 2009, page 423). Because the GWMA chart reduces to an EWMA chart when $q=1-\lambda$ and $\alpha=1$, a larger value of q, i.e. closer to 1, should be a reasonable choice for the GWMA chart to detect small shifts. To this end, Sheu and Lin (2003) noted that (q,α) combinations in the intervals $0.5 \leq q \leq 0.9$ and $0.5 \leq \alpha < 1$ enhanced the sensitivity of the GWMA-$\bar{X}$ chart and outperformed the EWMA-$\bar{X}$ chart for small shifts (i.e., less than 1 standard deviations in the location). The same range of (q,α) values were also considered by Sheu and Yang (2006), Teh et al. (2012), Sheu et al. (2013), and Lu (2015). In our simulation study, we set $k=$ 1,2,3,4,5 and considered the range $q=$ 0.5,0.6,0.7,0.8,0.9,0.95 and $\alpha=$ 0.5,0.6,0.7,0.8,0.9,1.0,1.3, respectively.

Using simulation along with a grid search algorithm, we obtained the charting constant $L > 0$ for the chosen (q, α) combination and specified value of k, so that the attained ARL_0 is approximately equal to $ARL_0^* = 370$; the values of L determined in this way are presented in Table 2 along with the attained ARL_0 values. The L values in Table 2 will be useful for the design and implementation of the GWMA-TBE chart; this includes designing an EWMA or Shewhart-type TBE chart. For example, if we choose $k = 2$ and let q =0.8, α =0.7, a value of $L =$ 1.953 leads to an attained ARL_0 equal to 370.05. To highlight the design parameters of the EWMA and Shewhart-type TBE charts, the row for $\alpha = 1.0$ have been highlighted.

From Table 2, we note in general that multiple combinations of the parameters (q, α, L) will yield the same ARL_0 for some chosen or specified value of k. This is somewhat challenging because, apart from desiring a sufficiently large ARL_0, the ARL_1 should be small for an effective GWMA-TBE chart. Therefore, the (q, α, L) combination with the minimum ARL_1 for a specified shift δ is said to be the optimal combination. The optimal design of the GWMA-TBE chart consists of specifying the desired ARL_0 and ARL_1 values as well as the magnitude of the process shift that is anticipated and then select the (q, α, L) combination that provides the desired ARL performance; typically, the (q, α, L) combination with the minimum ARL_1 is selected. The solution to this problem is an optimization problem in 3-dimensional space. Although a detailed study on the optimal design for the GWMA-TBE chart is out-of-scope for this paper, in the next section we investigate and comment on the “near optimal” design given the out-of-control (OOC) ARL values for different shift sizes.

Table 2: Values of L for the GWMA-TBE chart for different (q, α) when k = 1, 2, 3, 4, 5 and $ARL_0^* = 370$.

		q												Shewhart	
k	***α***	**0.5**		**0.6**		**0.7**		**0.8**		**0.9**		**0.95**		***q* = 0**	
1	**0.50**	1.343	369.64	1.427	370.50	1.512	370.20	1.594	371.19	1.620	371.34	1.552	370.34		
	0.60	1.384	372.64	1.482	372.21	1.585	369.94	1.687	371.27	1.728	370.96	1.615	370.27		
	0.70	1.413	372.22	1.520	370.82	1.635	370.02	1.750	370.06	1.806	370.96	1.689	370.55		
	0.80	1.432	369.18	1.544	371.04	1.665	370.62	1.792	371.86	1.868	370.79	1.764	370.59		
	0.90	1.444	368.61	1.559	372.79	1.681	371.98	1.808	371.12	1.893	370.82	1.818	370.99		
	EWMA (1.0)	1.451	373.85	1.563	369.10	1.685	372.03	1.811	370.90	1.907	371.40	1.859	370.05	0.002706	370
	1.30	1.447	368.10	1.549	372.79	1.656	372.24	1.770	371.09	1.880	370.01	1.891	370.73		
2	**0.50**	1.626	370.30	1.694	371.30	1.762	371.71	1.817	371.52	1.808	370.27	1.717	370.32		
	0.60	1.667	370.20	1.747	370.33	1.827	370.60	1.897	369.96	1.893	370.59	1.746	370.53		
	0.70	1.698	372.84	1.786	370.94	1.873	370.26	1.953	370.05	1.960	371.34	1.802	370.61		
	0.80	1.721	371.34	1.813	371.74	1.907	370.85	1.989	370.16	2.006	370.37	1.856	370.90		
	0.90	1.7	37	1.8	37	1.9	37	2.0	37	2.0	37	1.9	37		

		36	0.42	31	0.54	27	0.66	11	1.28	34	0.82	04	0.24		
	EWMA (1.0)	1.748	371.03	1.842	369.75	1.935	370.81	2.019	371.44	2.045	370.16	1.944	370.60	0.075386	370
	1.30	1.758	370.92	1.842	371.86	1.923	369.59	1.999	370.28	2.042	370.70	1.991	370.34		
3	**0.50**	1.786	371.32	1.843	369.85	1.899	371.08	1.937	369.54	1.911	369.87	1.807	370.59		
	0.60	1.823	371.60	1.889	369.82	1.956	369.85	2.009	370.30	1.978	369.85	1.813	370.09		
	0.70	1.852	372.01	1.925	369.98	1.999	369.79	2.059	370.41	2.033	370.30	1.846	369.82		
	0.80	1.872	369.63	1.950	370.59	2.028	370.83	2.091	370.74	2.069	370.16	1.892	370.23		
	0.90	1.889	370.20	1.968	370.33	2.045	369.96	2.108	370.03	2.095	370.63	1.938	370.24		
	EWMA (1.0)	1.900	370.74	1.977	369.52	2.054	370.00	2.115	370.95	2.110	370.74	1.972	370.24	0.270712	370
	1.30	1.914	370.03	1.982	369.56	2.049	370.58	2.100	369.94	2.111	370.53	2.036	370.66		
4	**0.50**	1.892	370.48	1.938	370.21	1.985	370.97	2.012	369.64	1.972	370.04	1.859	370.11		
	0.60	1.925	370.23	1.982	369.75	2.038	370.87	2.076	370.35	2.029	370.52	1.855	370.20		
	0.70	1.952	371.02	2.015	369.41	2.079	369.89	2.122	369.62	2.075	370.49	1.877	370.78		
	0.80	1.972	371.44	2.040	370.21	2.108	370.15	2.153	370.67	2.107	370.26	1.917	370.25		
	0.90	1.987	370.5	2.057	370.7	2.123	369.2	2.170	370.5	2.133	370.0	1.956	370.1		

			8		8		7		1		4		0		
	EWMA (1.0)	1.999	369.62	2.067	370.04	2.131	370.15	2.176	369.99	2.143	370.70	1.989	370.39	0.564345	370
	1.30	2.016	371.34	2.074	369.50	2.128	370.99	2.164	369.42	2.156	369.80	2.059	370.58		
5	**0.50**	1.969	369.23	2.012	370.24	2.052	370.41	2.074	370.86	2.026	370.34	1.910	370.07		
	0.60	2.001	371.55	2.051	370.19	2.100	369.76	2.128	369.52	2.071	369.82	1.891	370.51		
	0.70	2.025	371.11	2.082	371.70	2.136	369.42	2.171	370.25	2.110	370.98	1.905	370.80		
	0.80	2.044	370.06	2.105	370.80	2.163	370.13	2.200	370.39	2.141	370.71	1.941	370.52		
	0.90	2.060	371.59	2.120	370.42	2.176	369.75	2.216	371.16	2.159	370.18	1.977	370.32		
	EWMA (1.0)	2.071	370.40	2.132	370.54	2.186	371.14	2.222	370.59	2.175	370.60	2.008	370.93	0.930615	370
	1.30	2.088	369.63	2.141	371.14	2.182	369.94	2.212	370.73	2.187	369.73	2.078	369.99		

3.3 The out-of-control (OOC) performance of the GWMA-TBE chart

To study the out-of control (OOC) performance of the proposed GWMA-TBE chart, we used all the combinations of the design parameters shown in Table 2; these combinations ensure that the ARL_0's are close to 370 (when $\delta = 1$, i.e., no shift occurred) and therefore guarantee that all the charts are at an equal footing. Because we use only a lower control limit (as we are interested in a sustained downward step shift), we only focus on values for $\delta < 1$ and we specifically use $\delta =$ 0.975, 0.95, 0.925, 0.9, 0.85, 0.8, 0.7, 0.5 and 0.25. Also, we do not consider any shifts $\delta <$ 0.25, i.e., more than a 75% decrease, as the Shewhart-type charts are well-known to be more efficient than the GWMA and/or EWMA charts for large changes.

The results for the OOC performance comparisons are shown in Tables 3, 4 and 5 for $k = 1, 2 \text{ and } 3$, respectively, and for some combinations of (q, α, L). The results for $k = 4, 5$ and other combinations of (q, α, L) are not presented here for conciseness, but are available from the authors upon request. Note the following:

i. Each cell in Tables 3, 4 and 5 displays the ARL, the standard deviation of the run-length ($SDRL$) as well as the 5th, 25th, 50th, 75th and 95th percentiles;

ii. The tables include the results for the EWMA and the Shewhart-type TBE charts. The highlighted columns in the tables display the results for the EWMA chart whereas the columns on the right-hand side display the results for the Shewhart-type chart;

Although these tables are dense and contain lots of information, a quick comparison of the results reveals the following main points:

i. Both the GWMA-TBE and the EWMA-TBE charts outperform the Shewhart-type TBE chart for all values of k and for all shifts; this includes the scenario where θ decreases to a quarter of its original value, i.e., when $\delta = 0.25$;

ii. The GWMA-TBE can be designed to outperform the EWMA-TBE chart for very small to moderate shifts; this can be done using a suitably selected combination of (q, α, L). For example, consider Table 3 with the results for $k = 1$ and focus on the section where $q = 0.8$: The three GWMA charts with $\alpha = 0.5$, 0.7 and 0.9 all outperform the EWMA chart when $\alpha = 1$. The same is true for other values of k and (q, α, L). This clearly illustrates the benefit of the additional design parameter (α) in constructing a GWMA control chart;

iii. As the value of k increases, the performance for both the GWMA-TBE and EWMA-TBE charts improves. For instance, when $k =$ 1 and $\delta =$ 0.9, the ARL_1 for a GWMA-TBE chart with $q =$ 0.9, $\alpha =$ 0.7, $L =$ 1.806 is 125.37 while the ARL_1 for GWMA-TBE chart when $k =$ 2 and $\delta =$ 0.9 and $q =$ 0.9, $\alpha =$ 0.7, $L =$ 1.960 is 93.96 . This result indicates that a higher value of k improves the sensitivity of the GWMA and EWMA charts. However, caution should be applied when implementing these charts in practice since a larger value of k also implies more observations/failures has to be collected before a decision can be made about the status of the process. The specific choice of k is left to the practitioners;

iv. As mentioned earlier, the optimal design of the GWMA-TBE chart typically consists of specifying the desired ARL_0 and ARL_1 values as well as the magnitude of the process shift (δ) that is anticipated and then select the (q, α, L) combination that provides the desired ARL performance. However, Chan and Zang (2000) argued that it is also important to take

Table 3: IC and OOC ARL, SDRL as well as the 5th, 25th, 50th (or $MDRL$), 75th and 95th percentiles of the run-length for different combinations of (q, α, L) when $k = 1$.

	q = 0.8					q = 0.9					q = 0.95					q = 0
	α = 0.5	0.7	0.9	α = 1	1.3	0.5	0.7	0.9	α = 1	1.3	0.5	0.7	0.9	α = 1	1.3	α = 1
Shift	L = **1.594**	**1.75**	**1.808**	**1.811**	**1.77**	**1.62**	**1.806**	**1.893**	**1.907**	**1.88**	**1.552**	**1.689**	**1.818**	**1.859**	**1.891**	**0.002706**
1	**371.19**	**370.06**	**371.12**	**370.90**	**371.09**	**371.34**	**370.96**	**370.82**	**371.40**	**370.01**	**370.34**	**370.55**	**370.99**	**370.05**	**370.73**	**370.00**
	377.06	**353.90**	**357.28**	**359.12**	**357.79**	**405.96**	**356.76**	**353.62**	**352.78**	**359.74**	**457.50**	**391.52**	**362.97**	**356.97**	**360.36**	**369.50**
	28	28	27	26	24	28	30	29	28	26	25	30	32	31	29	19
	104	114	113	112	113	95	113	117	116	112	82	98	110	113	112	106
	251	263	261	259	261	232	261	264	266	260	201	238	255	258	260	256
	501	509	512	509	517	497	509	505	509	506	472	505	509	504	507	512
	1144	1094	1100	1109	1092	1204	1111	1094	1100	1110	1321	1158	1101	1101	1085	1107
0.975	**273.18**	**288.11**	**302.93**	**310.04**	**318.43**	**253.72**	**270.12**	**287.22**	**295.72**	**311.89**	**236.13**	**253.28**	**273.79**	**283.85**	**303.12**	**360.76**
	269.29	**281.96**	**294.95**	**303.02**	**311.12**	**260.59**	**259.63**	**279.33**	**285.96**	**305.36**	**264.28**	**253.10**	**260.43**	**271.17**	**291.30**	**360.26**
	25	25	23	22	21	25	27	26	25	23	23	27	29	28	26	18
	85	89	91	92	94	75	85	90	92	92	66	78	88	91	94	104
	188	202	214	217	225	167	189	201	206	217	146	170	192	201	212	250
	372	394	418	429	438	342	371	393	408	430	306	341	374	389	417	499
	806	851	879	910	939	778	794	836	863	919	757	765	806	825	891	1079
0.95	**207.90**	**228.4**	**250.8**	**259.7**	**279.5**	**184.3**	**202.6**	**225.5**	**238.4**	**260.3**	**167.9**	**185.3**	**207.8**	**219.0**	**244.8**	**351.53**

		5	**1**	**2**	**5**	**9**	**0**	**3**	**6**	**2**	**9**	**4**	**6**	**0**	**0**	
	196.39	**217.78**	**242.32**	**253.41**	**272.40**	**176.03**	**190.19**	**215.65**	**231.67**	**252.75**	**169.33**	**175.08**	**192.27**	**206.52**	**236.21**	**351.02**
	23	22	21	20	19	22	24	23	23	21	21	25	26	25	23	18
	69	73	78	79	84	63	69	74	76	79	55	64	71	73	78	101
	149	163	178	183	197	129	144	159	167	183	113	129	149	156	173	243
	281	312	344	357	386	247	273	305	323	359	222	248	280	296	332	487
	605	659	735	761	833	532	588	661	707	769	493	532	600	635	724	1052
0.925	**161.35**	**183.16**	**207.62**	**217.72**	**244.17**	**142.07**	**157.98**	**180.49**	**192.39**	**220.86**	**126.82**	**140.88**	**160.07**	**169.41**	**200.71**	**342.29**
	146.62	**172.63**	**199.84**	**211.37**	**239.31**	**128.26**	**140.88**	**170.02**	**182.99**	**214.08**	**117.16**	**125.09**	**143.70**	**154.03**	**194.54**	**341.79**
	21	20	19	18	18	21	22	22	21	19	19	23	24	23	20	18
	58	61	66	68	74	53	58	61	64	68	47	54	59	61	64	98
	117	131	149	155	172	103	115	129	136	156	91	102	116	122	141	237
	218	248	283	300	337	189	213	243	260	304	166	187	214	228	270	474
	451	525	604	634	725	396	439	521	554	649	359	391	445	476	591	1024
0.9	**128.63**	**148.58**	**173.00**	**183.58**	**213.68**	**111.55**	**125.37**	**144.40**	**156.72**	**187.81**	**99.91**	**111.23**	**127.30**	**136.09**	**164.27**	**333.05**
	111.80	**135.05**	**162.90**	**175.02**	**210.89**	**93.50**	**108.34**	**130.53**	**145.04**	**179.73**	**84.90**	**93.29**	**109.76**	**120.53**	**153.85**	**332.55**
	19	18	17	17	16	19	21	19	19	17	18	21	22	22	19	17
	50	52	56	58	65	46	49	51	53	59	41	46	50	51	55	96
	96	108	123	131	150	84	93	104	113	133	75	83	94	99	117	231
	173	200	237	253	295	148	167	194	214	257	131	146	170	181	224	461
	352	416	497	521	626	299	341	406	443	545	267	296	346	379	470	996
0.85	**86.84**	**100.97**	**122.68**	**132.82**	**160.59**	**75.55**	**83.87**	**96.77**	**105.44**	**133.96**	**67.82**	**74.67**	**86.01**	**89.47**	**110.70**	**314.58**

	68.90	**88.18**	**112.32**	**123.64**	**155.89**	**56.69**	**65.30**	**82.70**	**93.15**	**125.72**	**50.39**	**54.68**	**66.51**	**72.52**	**99.40**	**314.07**
	16	16	15	14	13	17	18	17	16	14	16	19	20	19	16	16
	38	39	42	44	50	35	38	38	39	44	32	36	39	38	40	90
	67	74	88	95	113	60	65	72	78	96	54	59	67	68	80	218
	116	135	168	182	222	100	110	129	142	183	89	97	113	119	150	435
	222	275	348	384	462	186	213	260	291	386	167	183	219	234	309	941
0.8	**62.68**	**70.97**	**87.16**	**95.27**	**122.38**	**55.39**	**60.14**	**68.47**	**74.44**	**96.27**	**50.15**	**54.54**	**60.23**	**63.69**	**78.05**	**296.10**
	46.11	**58.20**	**78.69**	**86.64**	**116.87**	**37.46**	**43.04**	**55.89**	**63.49**	**87.90**	**33.09**	**35.16**	**42.16**	**47.98**	**68.62**	**295.60**
	14	14	13	12	11	15	16	15	14	12	14	17	17	17	14	15
	30	30	32	34	39	29	30	30	30	34	26	30	31	30	30	85
	50	53	62	68	86	46	48	52	55	69	42	45	48	50	57	205
	83	93	117	130	166	72	78	89	98	131	64	70	77	82	103	410
	152	184	244	269	360	129	145	177	200	271	116	124	144	158	212	886
0.7	**37.00**	**39.70**	**46.70**	**51.79**	**69.10**	**33.98**	**35.55**	**38.09**	**40.30**	**51.45**	**31.51**	**33.79**	**35.60**	**36.55**	**41.68**	**259.15**
	22.59	**27.46**	**37.54**	**43.80**	**62.94**	**18.95**	**20.40**	**25.55**	**29.19**	**44.11**	**17.01**	**17.44**	**19.51**	**21.71**	**31.13**	**258.65**
	12	11	10	9	9	12	13	12	12	10	12	14	14	14	12	13
	20	20	20	21	24	20	21	20	20	20	19	21	22	21	20	74
	32	33	36	38	50	30	31	31	32	38	28	30	31	31	32	179
	47	51	61	69	94	43	45	48	51	68	40	42	44	45	53	359
	82	94	121	138	197	71	76	89	98	138	64	67	74	79	104	775
0.5	**17.26**	**17.05**	**17.65**	**18.47**	**23.36**	**16.91**	**17.29**	**16.90**	**16.84**	**18.18**	**16.23**	**17.55**	**17.92**	**17.64**	**16.87**	**185.25**
	7.14	**7.63**	**9.76**	**11.43**	**18.50**	**6.30**	**6.22**	**6.60**	**7.32**	**11.17**	**5.82**	**5.60**	**5.64**	**5.77**	**7.32**	**184.75**
	8	8	7	7	6	9	10	9	9	7	9	10	11	11	9	9
	12	12	11	11	10	12	13	12	12	10	12	13	14	13	12	53
	16	15	15	15	17	16	16	15	15	15	15	17	17	16	15	128
	21	21	22	23	30	20	21	20	20	22	19	21	21	21	20	256
	31	32	37	41	60	29	29	30	31	40	27	28	28	28	31	553

0.25	**8.89**	**8.53**	**7.97**	**7.72**	**7.45**	**9.19**	**9.55**	**9.18**	**8.85**	**7.89**	**9.05**	**10.18**	**10.58**	**10.40**	**9.23**	**92.88**
	2.04	**1.89**	**1.91**	**2.01**	**2.75**	**1.88**	**1.69**	**1.57**	**1.54**	**1.68**	**1.75**	**1.58**	**1.45**	**1.39**	**1.28**	**92.37**
	6	6	6	5	5	7	7	7	7	6	7	8	9	9	8	5
	7	7	7	6	6	8	8	8	8	7	8	9	10	9	8	27
	9	8	8	7	7	9	9	9	9	7	9	10	10	10	9	64
	10	10	9	9	8	10	10	10	10	9	10	11	11	11	10	128
	13	12	12	12	13	13	13	12	12	11	12	13	13	13	12	277

Table 4: IC and OOC ARL, $SDRL$ as well as the 5th, 25th, 50th (or $MDRL$), 75th and 95th percentiles of the run-length for different combinations of (q, α, L) when $k = 2$.

	q = 0.8					q = 0.9					q = 0.95					q = 0
	α = 0.5	0.7	0.9	α = 1	1.3	0.5	0.7	0.9	α = 1	1.3	0.5	0.7	0.9	α = 1	1.3	α = 1
Shift	L = 1.817	1.953	2.011	2.019	1.999	1.808	1.96	2.034	2.045	2.042	1.717	1.802	1.904	1.944	1.991	0.075386
1	**371.52**	**370.05**	**371.28**	**371.44**	**370.28**	**370.27**	**371.34**	**370.82**	**370.16**	**370.70**	**370.32**	**370.61**	**370.24**	**370.60**	**370.34**	**370.00**
	378.34	**360.38**	**361.18**	**360.66**	**360.61**	**396.94**	**365.67**	**356.44**	**354.33**	**359.03**	**444.35**	**382.38**	**358.36**	**353.30**	**350.58**	**369.50**
	27	27	26	25	24	26	30	29	28	27	24	29	31	31	28	19
	106	112	112	111	110	97	110	116	115	113	83	101	113	117	116	106
	251	258	259	259	258	237	255	259	259	260	210	246	259	260	262	256
	509	512	513	511	513	503	513	513	515	511	482	505	512	514	517	512
	1122	1086	1097	1114	1096	1164	1099	1084	1080	1095	1267	1141	1085	1074	1072	1107
0.975	**250.45**	**267.09**	**285.27**	**291.62**	**300.39**	**226.46**	**245.81**	**264.33**	**270.61**	**288.49**	**207.60**	**228.87**	**246.12**	**256.15**	**275.71**	**352.18**
	241.42	**257.57**	**277.99**	**283.24**	**292.09**	**224.36**	**232.48**	**249.89**	**256.43**	**277.66**	**218.13**	**223.43**	**232.86**	**240.43**	**262.95**	**351.68**
	22	22	22	21	21	22	25	24	23	22	20	25	26	26	23	18
	79	83	86	86	90	71	81	85	85	87	63	74	81	85	88	101
	176	187	200	206	213	154	174	186	190	204	135	158	175	183	194	244
	346	367	391	404	420	308	339	363	374	398	275	310	336	352	379	488
	728	783	847	864	869	667	709	774	791	855	635	671	713	742	810	1054
0.95	**177.21**	**196.5**	**220.9**	**228.9**	**248.4**	**153.3**	**170.7**	**191.8**	**201.2**	**228.0**	**136.5**	**152.0**	**170.1**	**181.3**	**205.8**	**334.80**

		6	**0**	**2**	**1**	**4**	**1**	**2**	**0**	**2**	**1**	**7**	**5**	**5**	**2**	
	162.96	**187.96**	**215.02**	**222.64**	**238.78**	**139.46**	**155.19**	**179.22**	**191.98**	**221.24**	**127.18**	**136.33**	**154.02**	**166.34**	**193.75**	**334.30**
	20	19	19	18	19	19	21	21	20	18	18	22	23	23	20	17
	62	64	69	70	75	56	61	64	65	70	50	57	62	64	67	96
	128	139	157	162	176	110	122	137	143	161	95	109	122	129	146	232
	241	269	298	312	348	208	233	263	273	313	182	204	230	246	278	463
	499	570	643	669	726	430	478	555	590	669	391	424	475	510	602	1001
0.925	**129.58**	**146.92**	**168.74**	**178.75**	**204.59**	**110.50**	**124.04**	**142.55**	**151.26**	**180.32**	**98.68**	**108.74**	**122.78**	**131.96**	**155.80**	**317.87**
	113.86	**135.45**	**162.60**	**171.70**	**197.18**	**92.16**	**108.21**	**130.78**	**141.33**	**174.36**	**83.13**	**89.77**	**105.94**	**117.34**	**147.62**	**317.37**
	17	17	16	16	16	18	19	18	18	16	16	20	21	20	18	16
	49	51	55	57	63	45	48	51	52	57	40	45	49	50	52	91
	95	106	119	126	144	83	91	103	108	126	74	82	90	96	111	220
	176	201	232	245	281	148	167	193	204	246	131	146	165	176	210	440
	358	416	492	524	599	299	338	394	433	529	266	287	336	367	447	951
0.9	**98.75**	**112.63**	**132.93**	**142.36**	**169.50**	**83.99**	**93.96**	**108.19**	**115.79**	**142.27**	**75.25**	**82.71**	**92.58**	**99.40**	**119.62**	**301.37**
	81.95	**100.70**	**124.80**	**134.47**	**163.78**	**65.47**	**76.64**	**96.16**	**104.79**	**135.32**	**58.70**	**63.12**	**74.49**	**84.07**	**110.07**	**300.87**
	15	15	14	14	14	15	17	16	15	14	15	18	19	18	15	15
	40	42	44	46	53	37	40	41	42	46	33	38	40	40	42	87
	75	82	94	101	119	66	71	79	84	101	59	65	70	74	85	209
	132	153	180	195	234	112	124	145	156	194	100	109	123	132	162	417
	261	310	383	413	495	215	245	296	323	409	191	208	240	264	343	901
0.85	**62.16**	**69.59**	**83.74**	**91.41**	**115.52**	**54.60**	**59.01**	**65.92**	**70.82**	**90.10**	**49.24**	**53.59**	**58.10**	**61.01**	**73.19**	**269.69**

	46.71	**57.74**	**75.42**	**83.50**	**109.79**	**37.84**	**42.68**	**52.66**	**59.46**	**82.06**	**33.63**	**35.61**	**41.05**	**45.34**	**63.12**	**269.19**
	13	12	11	11	11	13	14	14	13	11	12	15	16	15	13	14
	29	29	31	32	37	27	28	29	29	32	25	28	29	29	29	77
	50	53	61	65	81	45	47	51	53	65	41	44	46	48	55	187
	82	92	113	124	159	72	77	87	95	121	64	69	75	79	98	373
	155	185	234	257	332	129	144	173	190	252	116	123	139	153	197	806
0.8	**43.26**	**46.60**	**55.11**	**60.76**	**78.57**	**38.45**	**40.53**	**44.02**	**46.63**	**59.51**	**35.28**	**37.84**	**40.01**	**41.32**	**47.92**	**239.77**
	29.13	**35.07**	**46.54**	**53.34**	**73.57**	**23.93**	**25.78**	**32.28**	**36.32**	**51.58**	**21.40**	**22.00**	**24.41**	**26.90**	**38.39**	**239.27**
	11	10	10	9	9	11	12	11	11	10	11	13	14	13	11	12
	22	22	22	23	27	21	22	21	21	23	20	22	22	22	21	69
	36	37	41	44	57	33	34	35	36	44	30	32	34	34	36	166
	57	61	73	81	106	50	53	57	61	80	46	49	51	53	63	332
	102	117	151	168	224	85	91	108	118	163	77	80	88	96	123	717
0.7	**24.39**	**24.95**	**27.29**	**29.16**	**38.05**	**22.74**	**23.40**	**23.79**	**24.34**	**28.47**	**21.42**	**22.88**	**23.52**	**23.56**	**24.51**	**185.22**
	13.60	**15.28**	**19.54**	**22.02**	**33.10**	**11.57**	**11.81**	**13.39**	**14.93**	**21.22**	**10.58**	**10.47**	**10.93**	**11.53**	**15.43**	**184.72**
	8	8	7	7	6	9	9	9	9	7	9	10	11	11	9	9
	14	14	13	13	15	14	15	14	14	13	14	15	16	15	14	53
	21	21	22	23	28	20	21	20	20	22	19	21	21	21	20	128
	31	32	35	38	51	29	29	30	31	37	27	28	29	29	30	256
	51	55	66	73	105	45	46	50	54	71	42	43	44	46	55	553
0.5	**11.16**	**10.79**	**10.53**	**10.56**	**11.56**	**11.06**	**11.39**	**11.07**	**10.77**	**10.43**	**10.74**	**11.79**	**12.14**	**11.98**	**10.96**	**97.22**
	4.39	**4.24**	**4.61**	**5.03**	**7.22**	**3.93**	**3.70**	**3.63**	**3.68**	**4.60**	**3.68**	**3.42**	**3.25**	**3.19**	**3.36**	**96.72**
	6	6	5	5	5	6	7	7	6	6	6	7	8	8	7	5
	8	8	7	7	6	8	9	8	8	7	8	9	10	10	9	28
	10	10	9	9	9	10	11	10	10	9	10	11	12	11	10	67
	14	13	13	13	14	13	13	13	13	12	13	14	14	14	12	134
	19	19	19	20	26	18	18	18	18	20	17	18	18	18	17	290
0.25	**5.58**	**5.50**	**5.24**	**5.08**	**4.70**	**5.81**	**6.28**	**6.24**	**6.09**	**5.57**	**5.79**	**6.78**	**7.31**	**7.32**	**6.77**	**26.82**
	1.24	**1.0**	**0.9**	**0.9**	**0.9**	**1.1**	**1.0**	**0.8**	**0.8**	**0.7**	**1.1**	**0.9**	**0.8**	**0.8**	**0.7**	**26.3**

		8	**9**	**6**	**5**	**7**	**0**	**9**	**4**	**4**	**1**	**6**	**5**	**0**	**1**	**2**
	4	4	4	4	4	4	5	5	5	5	4	5	6	6	6	1
	5	5	5	4	4	5	6	6	6	5	5	6	7	7	6	8
	5	5	5	5	4	6	6	6	6	5	6	7	7	7	7	18
	6	6	6	6	5	6	7	7	7	6	6	7	8	8	7	36
	8	7	7	7	6	8	8	8	8	7	8	8	9	9	8	79

Table 5: IC and OOC ARL, $SDRL$ as well as the 5th, 25th, 50th (or $MDRL$), 75th and 95th percentiles of the run-length for different combinations of (q, α, L) when $k = 3$.

	q = 0.8					q = 0.9					q = 0.95					q = 0
	α = 0.5	0.7	0.9	α = 1	1.3	0.5	0.7	0.9	α = 1	1.3	0.5	0.7	0.9	α = 1	1.3	α = 1
Shift	L = 1.937	2.059	2.108	2.115	2.1	1.911	2.033	2.095	2.11	2.111	1.807	1.846	1.938	1.972	2.036	0.2707115
1	**369.54**	**370.41**	**370.03**	**370.95**	**369.94**	**369.87**	**370.30**	**370.63**	**370.74**	**370.53**	**370.59**	**369.82**	**370.24**	**370.24**	**370.66**	**370.00**
	370.91	**364.72**	**370.59**	**369.13**	**366.18**	**393.58**	**363.05**	**361.54**	**363.08**	**367.37**	**439.30**	**383.02**	**360.12**	**363.62**	**358.67**	**369.50**
	24	25	24	23	24	24	27	27	26	25	22	27	31	30	26	19
	110	114	114	114	113	96	114	115	115	115	84	99	115	115	116	106
	253	259	256	257	256	239	259	261	260	259	214	246	262	261	264	256
	509	509	502	507	505	507	510	508	510	505	488	512	510	506	507	512
	1110	1092	1110	1112	1116	1162	1093	1093	1109	1106	1273	1128	1074	1086	1092	1107
0.975	**230.85**	**249.80**	**265.66**	**269.50**	**289.32**	**206.23**	**225.89**	**243.96**	**253.03**	**270.84**	**187.62**	**204.14**	**223.72**	**231.28**	**256.17**	**344.70**
	220.86	**241.24**	**261.80**	**265.91**	**287.22**	**199.30**	**213.07**	**233.28**	**243.97**	**265.89**	**189.35**	**196.38**	**211.04**	**218.26**	**247.41**	**344.20**
	20	20	20	20	20	20	22	22	21	20	19	23	24	23	21	18
	73	78	82	83	87	65	73	77	80	84	58	65	74	75	80	99
	164	178	188	190	202	146	162	174	180	191	129	144	160	167	181	239
	317	340	362	367	396	282	307	333	347	369	250	278	305	314	352	477
	663	740	780	791	855	597	645	710	743	794	560	590	629	657	750	1031
0.95	**153.87**	**172.9**	**194.2**	**202.4**	**223.1**	**133.4**	**147.0**	**164.4**	**175.3**	**201.4**	**118.7**	**129.2**	**145.0**	**152.4**	**178.8**	**320.59**

		3	**6**	**7**	**8**	**6**	**4**	**6**	**8**	**8**	**2**	**4**	**4**	**6**	**2**	
	138.74	**161.63**	**188.05**	**195.59**	**221.78**	**118.71**	**129.44**	**150.89**	**163.07**	**195.39**	**107.66**	**113.92**	**128.19**	**135.59**	**166.97**	**320.09**
	17	17	16	16	15	17	19	18	18	16	16	19	21	20	18	16
	54	56	60	62	67	49	53	56	58	62	44	48	53	54	58	92
	114	127	139	146	157	99	110	121	128	144	86	95	108	113	130	222
	209	236	263	277	302	179	199	223	240	276	157	173	197	207	244	444
	429	494	561	585	670	371	407	465	502	585	333	358	401	425	512	959
0.925	**109.92**	**124.87**	**142.88**	**151.62**	**175.52**	**93.98**	**103.33**	**117.22**	**126.08**	**150.10**	**83.07**	**89.05**	**100.97**	**107.96**	**128.38**	**297.63**
	94.98	**113.18**	**135.27**	**144.63**	**171.80**	**77.26**	**88.10**	**105.79**	**117.23**	**144.70**	**68.20**	**71.94**	**85.22**	**94.13**	**119.30**	**297.13**
	15	15	14	14	13	15	16	16	15	14	14	17	18	18	15	15
	42	44	46	48	53	38	41	42	43	47	34	38	41	41	44	85
	83	91	102	108	124	72	77	84	90	107	63	68	75	79	92	206
	148	170	198	209	242	127	139	159	172	205	112	118	135	145	175	412
	300	351	409	440	503	247	278	328	359	435	219	232	271	295	365	890
0.9	**81.90**	**92.42**	**107.59**	**115.86**	**137.64**	**69.70**	**75.38**	**85.98**	**92.78**	**113.67**	**62.45**	**66.26**	**73.74**	**77.92**	**95.28**	**275.79**
	66.58	**81.94**	**99.55**	**109.15**	**131.68**	**53.59**	**59.95**	**74.81**	**82.89**	**107.02**	**47.41**	**49.37**	**57.96**	**64.33**	**86.64**	**275.29**
	13	13	12	12	12	13	14	14	13	12	13	15	16	16	14	14
	34	34	36	38	44	31	33	33	34	37	29	31	33	33	34	79
	63	68	77	83	98	55	58	63	67	81	49	52	57	58	68	191
	112	125	148	159	190	93	101	116	126	156	83	87	97	104	130	382
	212	253	304	330	393	175	195	234	259	328	156	163	189	205	263	825
0.85	**49.34**	**54.44**	**63.84**	**69.17**	**87.00**	**43.51**	**45.88**	**50.02**	**54.00**	**67.31**	**39.69**	**41.70**	**44.84**	**46.38**	**55.44**	**235.39**
	36.02	**43.92**	**56.23**	**62.19**	**82.43**	**29.19**	**31.84**	**38.75**	**44.50**	**60.49**	**25.72**	**26.29**	**30.26**	**32.98**	**46.95**	**234.89**

	11	11	10	10	9	11	12	11	11	10	11	12	13	13	12	12
	23	23	24	24	28	22	23	23	23	24	21	23	23	23	22	68
	40	42	46	50	61	36	37	39	41	48	33	35	37	37	41	163
	65	72	86	93	119	57	59	66	71	91	52	54	57	59	73	326
	120	142	176	194	252	100	109	125	143	189	90	94	104	113	149	704
0.8	**33.49**	**35.41**	**39.96**	**43.47**	**56.11**	**30.48**	**31.42**	**32.80**	**34.37**	**42.12**	**28.26**	**29.54**	**30.91**	**31.37**	**34.74**	**199.15**
	21.85	**25.58**	**32.80**	**37.94**	**51.64**	**18.30**	**18.99**	**22.03**	**25.04**	**35.87**	**16.48**	**16.21**	**17.50**	**18.82**	**25.84**	**198.65**
	9	9	8	8	7	9	10	10	10	8	9	11	11	11	10	10
	17	17	17	17	19	17	18	17	17	17	16	18	18	18	17	57
	28	29	30	32	40	26	27	27	27	31	25	26	27	26	27	138
	44	46	52	51	76	39	40	42	44	56	36	37	39	39	45	275
	76	87	106	118	159	66	68	77	85	114	60	62	66	68	87	595
0.7	**18.76**	**18.60**	**19.39**	**20.36**	**24.90**	**17.78**	**18.03**	**17.86**	**17.98**	**19.68**	**16.82**	**17.72**	**18.49**	**18.03**	**17.98**	**138.24**
	10.01	**10.55**	**12.60**	**14.33**	**20.44**	**8.69**	**8.46**	**8.99**	**9.78**	**13.48**	**7.97**	**7.53**	**7.57**	**7.75**	**9.80**	**137.74**
	7	7	6	6	5	7	8	8	7	7	7	8	9	9	8	7
	11	11	10	10	10	11	12	11	11	10	11	12	13	12	11	40
	17	16	16	16	19	16	16	16	15	16	15	16	16	16	15	95
	24	23	25	26	33	22	22	22	22	25	21	22	22	22	22	191
	38	39	45	48	66	35	34	36	37	47	32	32	33	33	38	413
0.5	**8.40**	**8.14**	**7.78**	**7.67**	**7.72**	**8.44**	**8.76**	**8.55**	**8.34**	**7.78**	**8.25**	**9.11**	**9.57**	**9.46**	**8.73**	**56.42**
	3.15	**2.92**	**2.93**	**3.04**	**3.89**	**2.89**	**2.62**	**2.45**	**2.42**	**2.65**	**2.70**	**2.46**	**2.27**	**2.19**	**2.11**	**55.92**
	4	5	4	4	4	5	5	5	5	5	5	6	7	7	6	3
	6	6	6	6	5	6	7	7	7	6	6	7	8	8	7	16
	8	8	7	7	7	8	8	8	8	7	8	9	9	9	8	39
	10	10	9	9	9	10	10	10	10	9	10	10	11	11	10	78
	14	14	13	14	15	14	14	13	13	13	13	14	14	14	13	168
0.25	**4.19**	**4.27**	**4.15**	**4.07**	**3.78**	**4.40**	**4.89**	**5.02**	**4.98**	**4.62**	**4.41**	**5.28**	**5.88**	**5.96**	**5.73**	**10.40**
	0.94	**0.79**	**0.67**	**0.64**	**0.64**	**0.89**	**0.76**	**0.66**	**0.62**	**0.59**	**0.85**	**0.72**	**0.66**	**0.62**	**0.57**	**9.89**
	3	3	3	3	3	3	4	4	4	4	3	4	5	5	5	1
	4	4	4	4	3	4	4	5	5	4	4	5	5	6	5	3
	4	4	4	4	4	4	5	5	5	5	4	5	6	6	6	7
	5	5	4	4	4	5	5	5	5	5	5	6	6	6	6	14

	6	6	5	5	5	6	6	6	6	5	6	7	7	7	7	30

into account the $SDRL$ relative to the ARL when designing an EWMA chart; i.e.; they added the constraint that $SDRL \leq ARL$; this was done to avoid large variation in the run-length distribution. So, using their criterion for the optimal design of the EWMA chart for the proposed GWMA-TBE chart, the optimal design would consist of specifying the desired ARL_0 and ARL_1 values as well as the magnitude of the process shift (δ) that is anticipated and then select the (q, α, L) combination that provides the desired ARL performance subject to the constraint that $SDRL \leq ARL$. We used both definitions to obtain the “near optimal design”. Note that, we specifically refer to this as the “near optimal design” because in the way the design was carried out, i.e., the two parameters q and α were varied over a certain range and for each (q, α) combination that was investigated the value of the charting constant, i.e., $L > 0$, was obtained so that the attained in-control ARL is close to the nominal or specified value ARL_0^*. This approach might exclude the “true” optimal design.

Using the results in Tables 3, 4 and 5, the “near optimal” design would be those (q, α, L) combinations that result in the smallest ARL_1 for a specified shift (δ) given the $ARL_0 = 370$; with or without the constraint $SDRL \leq ARL$.

Table 6 provides the “near optimal” combinations of the parameters (q, α, L) as well as the ARL_1 values for different δ and $k =$ 1, 2, 3, 4 and 5. From Table 6, we observe the following:

a. For smaller shifts, i.e., for values of δ closer to 1, a larger value of q (closer to 1) and a smaller value of α (closer to 0.5) works best. The converse also holds, i.e., for larger

Table 6: Near optimal (q, α, L) combinations with corresponding ARL values.

No constraint	$k=1$				$k=2$				$k=3$				$k=4$				$k=5$			
	ARL	q	α	L	ARL	q	α	L	ARL	q	α	L	ARL	q	α	L	ARL	q	α	L
0.975	236.13	0.95	0.5	1.552	207.60	0.95	0.5	1.717	187.62	0.95	0.5	1.807	170.78	0.95	0.5	1.859	163.59	0.95	0.5	1.91
0.95	167.99	0.95	0.5	1.552	136.51	0.95	0.5	1.717	118.72	0.95	0.5	1.807	103.06	0.95	0.5	1.859	95.71	0.95	0.5	1.91
0.925	126.82	0.95	0.5	1.552	98.68	0.95	0.5	1.717	83.07	0.95	0.5	1.807	71.36	0.95	0.5	1.859	65.67	0.95	0.5	1.91
0.9	99.91	0.95	0.5	1.552	75.25	0.95	0.5	1.717	62.45	0.95	0.5	1.807	53.24	0.95	0.5	1.859	48.59	0.95	0.5	1.91
0.85	67.82	0.95	0.5	1.552	49.24	0.95	0.5	1.717	39.69	0.95	0.5	1.807	34.05	0.95	0.5	1.859	30.36	0.95	0.5	1.91
0.8	50.15	0.95	0.5	1.552	35.28	0.95	0.5	1.717	28.26	0.95	0.5	1.807	24.02	0.95	0.5	1.859	21.31	0.95	0.5	1.91
0.7	31.51	0.95	0.5	1.552	21.42	0.95	0.5	1.717	16.82	0.95	0.5	1.807	14.21	0.95	0.5	1.859	12.46	0.95	0.5	1.91
0.6	21.96	0.95	0.5	1.552	14.67	0.95	0.5	1.717	11.28	0.9	1.3	2.111	9.22	0.9	1.3	2.156	7.82	0.8	1	2.222
0.5	16.23	0.95	0.5	1.552	10.43	0.9	1.3	2.042	7.67	0.8	1	2.115	6.18	0.8	1.3	2.164	5.21	0.8	1.3	2.212
0.33	9.74	0.9	1.3	1.88	5.79	0.8	1.3	1.999	4.23	0.7	1.3	2.049	3.37	0.6	1.3	2.074	2.79	0.5	1.3	2.088
0.25	7.45	0.8	1.3	1.77	4.40	0.7	1.3	1.923	3.16	0.5	1.3	1.914	2.56	0.5	1.3	2.016	2.24	0.5	1.3	2.088
$SDRL \leq ARL$	$k=1$				$k=2$				$k=3$				$k=4$				$k=5$			
	ARL	q	α	L	ARL	q	α	L	ARL	q	α	L	ARL	q	α	L	ARL	q	α	L
0.975	27	0.	0	1.	23	0.	0	1.	21	0.	0	1.	20	0.	0	1.	18	0.	0	1.

	0.12	9	.7	806	7.09	95	.8	856	4.95	95	.8	892	5.07	95	.9	956	5.12	95	.8	941
0.95	202.60	0.9	0.7	1.806	160.64	0.95	0.8	1.856	136.66	0.95	0.8	1.892	127.55	0.95	0.9	1.956	109.16	0.95	0.8	1.941
0.925	157.98	0.9	0.7	1.806	115.48	0.95	0.8	1.856	94.57	0.95	0.8	1.892	85.50	0.95	0.9	1.956	72.86	0.95	0.8	1.941
0.9	125.37	0.9	0.7	1.806	87.41	0.95	0.8	1.856	69.67	0.95	0.8	1.892	61.21	0.95	0.9	1.956	52.56	0.95	0.8	1.941
0.85	83.87	0.9	0.7	1.806	55.84	0.95	0.8	1.856	43.27	0.95	0.8	1.892	37.25	0.95	0.9	1.956	32.24	0.95	0.8	1.941
0.8	60.14	0.9	0.7	1.806	38.98	0.95	0.8	1.856	30.30	0.95	0.8	1.892	25.91	0.95	0.9	1.956	22.48	0.95	0.8	1.941
0.7	35.55	0.9	0.7	1.806	23.38	0.95	0.8	1.856	17.86	0.9	0.9	2.095	14.70	0.9	1	2.143	12.64	0.9	1	2.175
0.6	23.90	0.9	0.7	1.806	15.19	0.9	1	2.045	11.28	0.9	1.3	2.111	9.22	0.9	1.3	2.156	7.82	0.8	1	2.222
0.5	16.84	0.9	1	1.907	10.43	0.9	1.3	2.042	7.67	0.8	1	2.115	6.18	0.8	1.3	2.164	5.21	0.8	1.3	2.212
0.33	9.74	0.9	1.3	1.88	5.79	0.8	1.3	1.999	4.23	0.7	1.3	2.049	3.42	0.5	1	1.999	2.79	0.5	1.3	2.088
0.25	7.45	0.8	1.3	1.77	4.40	0.7	1.3	1.923	3.16	0.5	1.3	1.914	2.67	0.5	1	1.999	2.24	0.5	1.3	2.088

shifts, a smaller value of q (closer to 0.5) and larger value of α (greater and equal to 1) works best;

b. The EWMA-TBE chart, i.e. when $\alpha = 1$, only features as the optimal design for $\delta \leq$ 0.7. This result confirms the fact that a GWMA chart can be designed to outperform an EWMA chart for very small, small and moderate shifts.

3.4. The zero-state versus the steady-state average run-length

The out-of-control average run-lengths (ARL_1) of the previous sections are called the zero-state ARL's and are based on the assumption that the shift occurs at start-up, i.e., at time $t = 1$. However, it is also of interest to see whether a GWMA-TBE chart designed for optimal performance at start-up works well if the shift occurs later in the process, say at time $t =$ 50, 100, 150, 300, etc. This is called the steady-state performance and the ARL is referred to as the steady-state ARL; the assumption is basically that a stable process has been operating in-control for some time before the shift occurs. We calculated the steady-state ARL for some (q, α, L) combinations when $k = 1, 2$ and compared it to the zero-state ARL. The results are presented in Table 7 and from this table we observe the following:

i. The zero-state and steady-state ARL are the same for all practical purposes. The minor differences that are observed are due to the inherent simulation variability;

ii. The GWMA-TBE chart generally performs similar or, in many cases, better than the EWMA-TBE chart when $\delta \geq$ 0.7 (i.e., for very small, small or moderate shifts). For $\delta <$ 0.7 (i.e., for larger shifts), the EWMA-TBE chart generally performs better than the GWMA-TBE. These results hold irrespective of when the shift has occurred;

Table 7: The zero-state and steady-state ARL values for the GWMA-TBE chart.

					Shift										
k	**Time of shift**	**q**	**α**	**L**	**0.975**	**0.95**	**0.925**	**0.9**	**0.85**	**0.8**	**0.7**	**0.6**	**0.5**	**0.33**	**0.25**
$k = 1$	**$t = 1$**	**0.90**	**0.70**	**1.806**	270.1	202.6	158.0	125.4	83.9	60.1	35.6	23.9	17.3	11.3	9.6
		0.90	**1.00**	**1.907**	295.7	238.5	192.4	156.7	105.4	74.4	40.3	25.0	16.8	10.5	8.9
	$t = 50$	**0.90**	**0.70**	**1.806**	263.4	199.2	153.5	123.4	82.1	59.3	36.0	24.3	17.9	11.9	10.2
		0.90	**1.00**	**1.907**	287.6	231.3	186.5	152.4	101.7	72.0	39.5	24.3	16.7	10.5	9.0
	$t = 100$	**0.90**	**0.70**	**1.806**	266.4	202.4	157.9	126.5	85.1	61.6	36.6	24.9	18.4	12.4	10.6
		0.90	**1.00**	**1.907**	290.2	233.9	188.3	153.2	103.4	73.2	39.5	24.3	16.8	10.7	9.2
	$t = 150$	**0.90**	**0.70**	**1.806**	266.4	201.8	157.0	126.3	86.0	61.4	37.1	25.2	18.7	12.9	11.4
		0.90	**1.00**	**1.907**	289.0	233.9	189.6	153.5	103.0	72.9	39.4	24.3	16.9	11.1	9.7
	$t = 300$	**0.90**	**0.70**	**1.806**	272.7	203.8	159.2	128.1	86.9	62.7	37.9	26.2	19.4	13.5	11.9
		0.90	**1.00**	**1.907**	287.1	235.4	191.0	152.6	103.0	72.9	39.7	25.0	17.4	11.8	10.7
	Shewhart	**0.00**	**1.00**	**0.003**	360.8	351.5	342.3	333.1	314.6	296.1	259.2	222.2	185.3	122.4	92.9
$k = 2$	**$t = 1$**	**0.90**	**0.70**	**1.960**	245.8	170.7	124.0	94.0	59.0	40.5	23.4	15.6	11.4	7.5	6.3
		0.90	**1.00**	**2.045**	270.6	201.2	151.3	115.8	70.8	46.6	24.3	15.2	10.8	7.1	6.1
	$t = 50$	**0.90**	**0.70**	**1.960**	240.0	167.4	122.4	92.6	58.1	40.5	23.8	16.2	12.0	8.2	7.3
		0.90	**1.00**	**2.045**	263.8	195.9	146.0	112.1	68.6	45.1	23.6	14.9	10.7	7.3	6.6
	$t = 100$	**0.90**	**0.70**	**1.960**	242.4	169.9	124.6	94.4	59.8	42.1	24.7	16.9	12.7	9.1	8.3
		0.90	**1.00**	**2.045**	261.3	193.7	145.1	112.2	68.9	45.6	24.0	15.4	11.2	8.0	7.3
	$t = 150$	**0.90**	**0.70**	**1.960**	242.6	171.1	124.9	95.0	60.2	42.2	24.9	17.2	13.1	9.6	8.8

		0.90	**1.00**	**2.045**	260.9	194.0	145.5	111.5	68.1	45.5	24.3	15.6	11.7	8.4	8.0
	***t* = 300**	**0.90**	**0.70**	**1.960**	245.7	173.2	125.6	95.5	60.4	43.0	26.2	18.2	14.2	10.9	10.8
		0.90	**1.00**	**2.045**	262.5	196.3	148.9	113.3	69.7	45.9	24.9	16.3	12.4	9.6	9.2
	Shewhart	**0.00**	**1.00**	**0.075**	352.2	334.8	317.9	301.4	269.7	239.8	185.2	137.7	97.2	44.6	26.8

iii. Both the GWMA-TBE and EWMA-TBE charts outperform the Shewhart-TBE; this is a result that was also observed when studying the results of Tables 3, 4 and 5 with the zero-state ARL's.

Note that the charting statistics for the Shewhart-TBE chart are mutually independent, irrespective of the time the shift occurs, and therefore the time of shift can always be taken as $t=1$ when considering the OOC run-length.

4. Phase II GWMA-TBE chart

When the in-control (IC) value of the parameter θ is unknown, it is typically estimated from an in-control Phase I sample (the so-called reference or calibration sample) before prospective (i.e., online) monitoring starts in Phase II; this scenario is referred to as the "standard unknown" case and denoted by Case U. Note that the Phase I sample is taken when the process was thought to operate in-control and without any special causes of concern (see Montgomery (2009), page 198). The point estimate of θ is denoted by $\hat{\theta}$ and is used to estimate the starting value Z_0 and to estimate the Phase II lower control limit of the GWMA-TBE chart; in both cases, $\hat{\theta}$ is substituted for the known parameter value θ_0. The charting statistic and estimated lower control limit of the GWMA-TBE chart in Case U are defined as follows:

$$Z_t = \sum_{i=1}^{t}\left(q^{(i-1)^{\alpha}} - q^{i^{\alpha}}\right)X_{t-i+1} + q^{t^{\alpha}}k\hat{\theta} \text{ for } t=1,2,3,\ldots \quad (14)$$

and

$$L\hat{C}L = k\hat{\theta} - L\sqrt{Qk\hat{\theta}^2}. \quad (15)$$

The Phase II GWMA-TBE chart operates in the same manner as the GWMA-TBE chart of Case K, i.e., the charting statistic in (14) is plotted on a control chart with the $L\hat{C}L$ given in (15) and if a point plots on or below the lower control limit the chart signals and the process is declared OOC so that a search for assignable causes is started.

The following points should be noted:

i. Obtaining an in-control Phase I reference sample is an important problem in its own right, but we do not study this problem here. We assume that we have an in-control Phase I reference sample. The interested reader is referred to Chakraborti et al. (2008) for an in-depth overview of Phase I control charting procedures as well as the operation and implementation of these charts;

ii. To estimate the unknown value of θ, a suitable point estimator is required. We use the Maximum Likelihood Estimator (MLE) which is also the Uniform Minimum Variance Unbiased Estimator (UMVUE). To this end, if we let $X_1, X_2, \ldots, X_m \sim iidGamma(k,\theta)$ denote the IC Phase I reference sample of size $m \geq 1$, the MLE is given by $\hat{\theta} = \sum_{i=1}^{m} X_i / km$ and follows a $Gamma(km, \theta / km)$ distribution;

iii. Because a point estimate is substituted for the unknown parameter value in Case U, the starting value and lower control limits are both random variables (as indicated by the ^ -notation). Therefore, it is of interest to examine the effects of estimation on the Phase II run-length distribution and hence the performance and robustness of the GWMA-TBE chart using the design parameters of Case K. Stated differently, we want to know if the design parameters for the Case K GWMA-TBE chart may be used for the Phase II GWMA-TBE chart in Case U. To this end, it is important to stress that, given a point estimate $\hat{\theta}$ (calculated using an in-control Phase I sample), we obtain the conditional Phase II run-length distribution and we observe the conditional performance of the GWMA-TBE chart. That is, the observed performance of the chart is based on that specific in-control Phase I sample. Thus, the conditional performance and the conditional run-length distribution will be different for each practitioner based on his/her own in-control Phase I sample. Therefore, the conditional performance does not give us a complete insight into the overall performance of the chart. In order to get an overall picture and a more general idea about the effects of parameter estimation, we study the unconditional run-length distribution; this unconditional distribution can be thought of as the run-length distribution averaged over all possible values of the parameter estimates. Focusing on the average run-length (ARL), we can write the aforementioned statements mathematically as follows:

$$UARL = E_{\hat{\theta}}\left(E\left(R|\hat{\theta}\right)\right) = \int_0^\infty E\left(R|\hat{\theta}\right) f\left(\hat{\theta}\right) d\hat{\theta}, \quad (16)$$

where $UARL$ denotes the unconditional ARL, $E\left(R|\hat{\theta}\right)$ denotes the conditional ARL, i.e., the expectation of the run-length $\left(R\right)$ conditional on a point estimate $\hat{\theta}$, and $f\left(\hat{\theta}\right)$ denotes the p.d.f. of $\hat{\theta}$. Similar expressions can be obtained for other characteristics of the run-length distribution as well.

4.1 Effects of parameter estimation on the performance of the Phase II GWMA-TBE chart

To assess if the design parameters for the Case K GWMA-TBE chart may be used for the Phase II GWMA-TBE chart in Case U, a simulation study was performed to calculate the in-control and out-of-control average run-length of the Phase II GWMA-TBE chart. Without loss of generality, we simulated the in-control Phase I sample from a $Gamma(k,1)$ distribution. We used $m=$ 50, 100, 500 and 1000 with four sets of design parameters:

i. ($k=1, q=0.95, \alpha=0.5, L=1.552$), ($k=1, q=0.95, \alpha=1, L=1.859$),

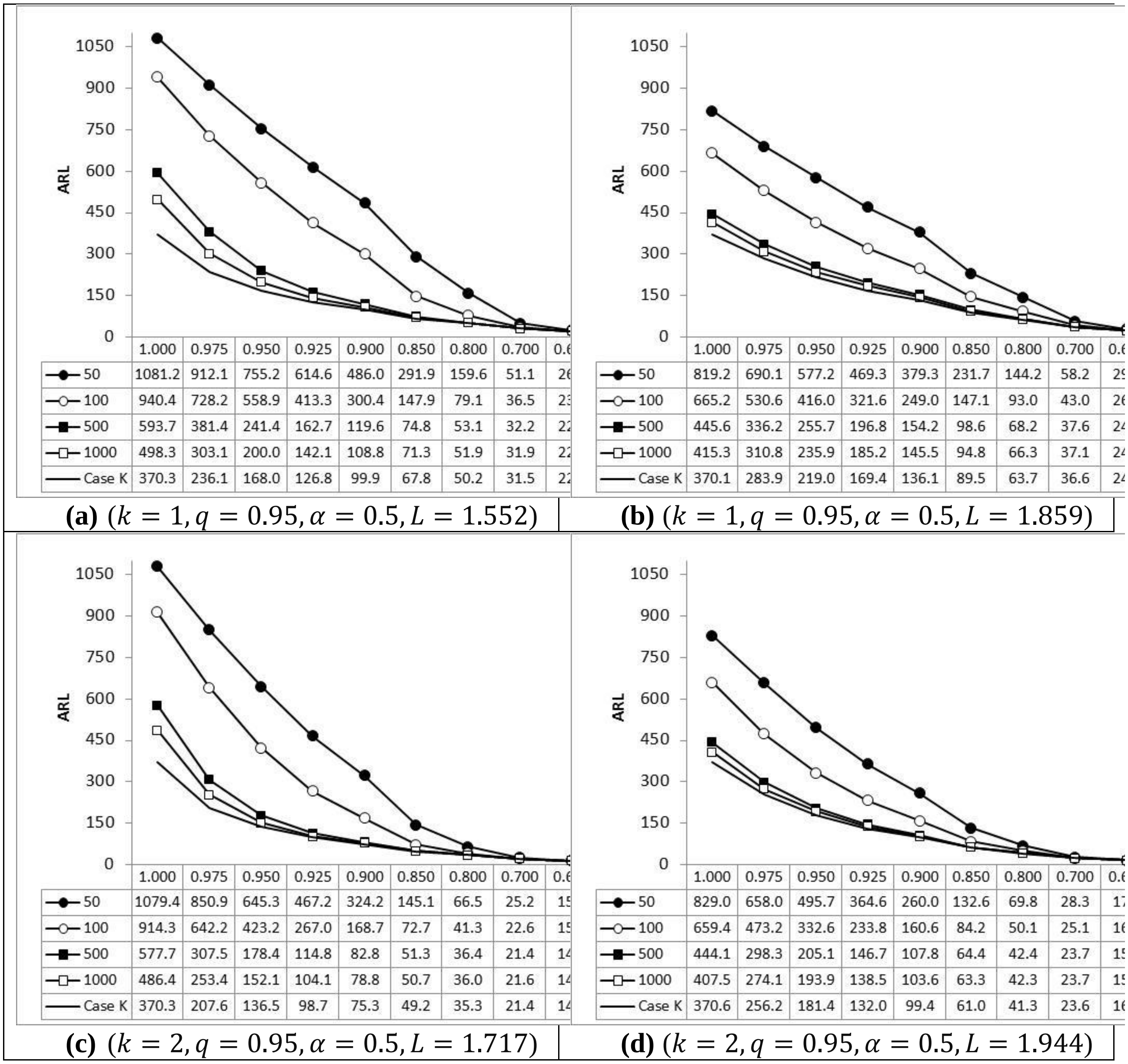
ii. ($k=2, q=0.95, \alpha=0.5, L=1.717$), ($k=2, q=0.95, \alpha=1, L=1.944$).

Note that the first and second sets use $k=1$ whereas the third and fourth sets use $k=2$. Also, the first and third sets result in GWMA-TBE charts whereas the second and fourth sets result in EWMA-TBE charts.

The results are displayed in panels (a), (b), (c) and (d) of Figure 1. These graphs display the ARL curves, i.e., ARL on the vertical axis versus the size of the shift (δ) on the horizontal axis. The graphs also display the corresponding ARL-curve of Case K and is used as the reference to which we compare the ARL-curves of Case U. The ARL values are shown in the data tables below the graphs for ease of reference. From these graphs, we observe the following points:

i. The in-control ARL in Case U is substantially larger than the corresponding in-control ARL in Case K; in some cases even greater than 1,000. The out-of-control ARL in Case U is also larger than the corresponding out-of-control ARL of Case K. Only if $\delta \leq 0.5$ do we see that the charts perform relatively similar;

ii. The smaller the in-control Phase I reference sample is, i.e., the smaller the value of m, the larger the difference between the Case U and Case K and ARL is. For the Case U chart (based on the design parameters of the Case K chart), to perform anything like the Case K chart requires more than 1,000 Phase I observations;



(a) $(k = 1, q = 0.95, \alpha = 0.5, L = 1.552)$

	1.000	0.975	0.950	0.925	0.900	0.850	0.800	0.700	0.6
50	1081.2	912.1	755.2	614.6	486.0	291.9	159.6	51.1	26
100	940.4	728.2	558.9	413.3	300.4	147.9	79.1	36.5	23
500	593.7	381.4	241.4	162.7	119.6	74.8	53.1	32.2	22
1000	498.3	303.1	200.0	142.1	108.8	71.3	51.9	31.9	22
Case K	370.3	236.1	168.0	126.8	99.9	67.8	50.2	31.5	22

(b) $(k = 1, q = 0.95, \alpha = 0.5, L = 1.859)$

	1.000	0.975	0.950	0.925	0.900	0.850	0.800	0.700	0.6
50	819.2	690.1	577.2	469.3	379.3	231.7	144.2	58.2	29
100	665.2	530.6	416.0	321.6	249.0	147.1	93.0	43.0	26
500	445.6	336.2	255.7	196.8	154.2	98.6	68.2	37.6	24
1000	415.3	310.8	235.9	185.2	145.5	94.8	66.3	37.1	24
Case K	370.1	283.9	219.0	169.4	136.1	89.5	63.7	36.6	24

(c) $(k = 2, q = 0.95, \alpha = 0.5, L = 1.717)$

	1.000	0.975	0.950	0.925	0.900	0.850	0.800	0.700	0.6
50	1079.4	850.9	645.3	467.2	324.2	145.1	66.5	25.2	15
100	914.3	642.2	423.2	267.0	168.7	72.7	41.3	22.6	15
500	577.7	307.5	178.4	114.8	82.8	51.3	36.4	21.4	14
1000	486.4	253.4	152.1	104.1	78.8	50.7	36.0	21.6	14
Case K	370.3	207.6	136.5	98.7	75.3	49.2	35.3	21.4	14

(d) $(k = 2, q = 0.95, \alpha = 0.5, L = 1.944)$

	1.000	0.975	0.950	0.925	0.900	0.850	0.800	0.700	0.6
50	829.0	658.0	495.7	364.6	260.0	132.6	69.8	28.3	17
100	659.4	473.2	332.6	233.8	160.6	84.2	50.1	25.1	16
500	444.1	298.3	205.1	146.7	107.8	64.4	42.4	23.7	15
1000	407.5	274.1	193.9	138.5	103.6	63.3	42.3	23.7	15
Case K	370.6	256.2	181.4	132.0	99.4	61.0	41.3	23.6	16

Figure 1: ARL values for the GWMA-TBE chart for unknown θ when $k = 1$ and 2.

iii. The EWMA-TBE chart is less impacted (compared to the GWMA-TBE chart) by the estimation from Phase I which is seen by looking at the ARL values on the vertical axis.

The above observations demonstrate that in those scenarios where it is unrealistic to wait a long time to gather the necessary Phase I observations, an alternative and more realistic approach is needed since we cannot use the Case K design parameters. To this end, one should adjust the control limit, i.e., contract the control limit. This is investigated next.

4.2 The in-control (IC) design of the Phase II GWMA-TBE chart

The design of the Phase II GWMA-TBE chart requires one to use expression (16) and adjust the width of the lower control limit (L) so that the in-control unconditional ARL is equal to a pre-specified value such as 370. This means we want to solve for the value of L that satisfies the following equation:

$$UARL_0 = E_{\hat{\theta}}\left(E\left(R|\hat{\theta}, IC\right)\right) = \int_0^{\infty} E\left(R|\hat{\theta}, IC\right) f\left(\hat{\theta}\right) d\hat{\theta} = 370. \quad (17)$$

The integral in equation (17) can be approximated by using computer simulation and calculating the average of a sufficiently large number of in-control conditional average run-lengths, i.e., calculating $\frac{1}{N}\sum_{j=1}^{N} E\left(R|\hat{\theta}, IC\right)$, N denotes the number of simulations and $E\left(R|\hat{\theta}, IC\right)$ denotes the conditional in-control average run-length. Using a search algorithm, we have found the value of L that satisfies $\frac{1}{N}\sum_{j=1}^{N} E\left(R|\hat{\theta}, IC\right) \approx 370$ for some (q, α) combinations. These values are displayed in Table 8 along with the value of L for the Case K chart; the latter correspond to the values listed in Table 2. From Table 8, we observe that the value of L converges to the Case K

Table 8: Design parameters for the Phase II GWMA-TBE chart in Case U.

			m				
k	***q***	***α***	**50**	**100**	**500**	**1000**	**Case K**
1	**0.95**	0.50	0.927	1.099	1.387	1.463	1.552
		1.00	1.383	1.586	1.789	1.819	1.859
2		0.50	1.020	1.240	1.551	1.625	1.717
		1.00	1.422	1.649	1.868	1.901	1.944

value as the number of observations increases. So, the smaller the value of m, the narrower the control limits should be to compensate for the large ARL we observed in Figure 1.

In the next two sections, we discuss a generalization of the proposed GWMA chart and an illustrative example is provided in order to demonstrate the application of the proposed chart.

5. GWMA-TBE for monitoring the variance

The proposed GWMA chart based on the $Gamma(k,\theta)$ distribution can also be used to monitor the known in-control (IC) variance of a normal distribution for a sustained downward step shift. To this end, if X_{ij} denotes the j^{th} observation in the i^{th} sample of size $n \geq 1$, where $X_{ij} \sim iidN\left(\mu, \sigma_0^2\right)$ for $i = 1,2,3,\ldots$, $j = 1,2,\ldots,n$, and σ_0^2 is the known in-control (IC) standard/value for the variance, the statistic $\dfrac{(n-1)S_i^2}{2\sigma_0^2}$ is distributed as $Gamma\left(k = \dfrac{n-1}{2}, \theta_0 = 1\right)$, where $S_i^2 = \dfrac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2$ denotes the sample variance. If the mean of the normal distribution is known beforehand and equal to μ_0, the sample variance will be calculated as $S_i^2 = \dfrac{1}{n}\sum_{j=1}^{n}\left(X_{ij} - \mu_0\right)^2$ and the statistic $\dfrac{nS_i^2}{2\sigma_0^2}$ is distributed $Gamma\left(k = \dfrac{n}{2}, \theta_0 = 1\right)$. So, the design parameters given in Table 2 can be used to construct a GWMA chart to monitor the known variance. For example, if samples of size $n = 5$ is taken from a normal distribution with known variance and unknown mean, it follows that $\dfrac{2S_i^2}{\sigma_0^2} \sim Gamma(2,1)$. So, using Table 2 and setting ($q = 0.95, \alpha = 1.0, L = 1.944$) will result in an

EWMA chart with an in-control ARL of 370.60. The corresponding out-of-control ARL performance of the EWMA chart, for monitoring the variance, can be found from Table 4 with $\delta = \sigma_1^2 / \sigma_0^2$. Note that, in this case, when $\delta < 1$, there would have been a decrease in the variance which would be equivalent to an improvement in the process.

Note that if the variance is unknown and has to be estimated from an in-control Phase I reference sample using the pooled variance estimator S_p^2 with v_1 degrees of freedom, for example the statistic $\dfrac{v_1 S_i^2}{v_2 S_p^2}$ follows an F_{v_2, v_1} distribution, where v_2 denotes the degrees of freedom for a future or Phase II sample variance S_i^2. The proposed GWMA-TBE chart is, however, not suited for this scenario.

6. Illustrative example

To illustrate the application of the proposed GWMA-TBE chart, a simulated dataset is used. Because the proposed GWMA-TBE chart is for monitoring a high-performance process, for which failure rate is assumed to be very small, the known value of the in-control (IC) process failure rate is taken as $1/\theta_0 = 1/5000 = 0.0002$; this implies the time until the k^{th} failure, X, follows a $Gamma(k, 5000)$ distribution.

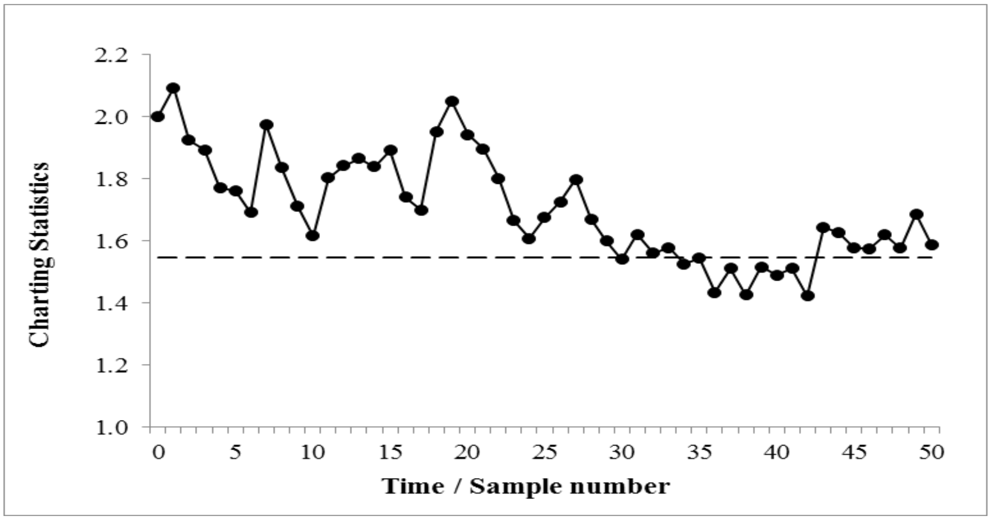
The simulated dataset consists of 50 random observations from a $Gamma(k = 2,\ 4000)$ distribution. If we assume that the known value of θ is 5000, the simulated dataset may be viewed as observations from an out-of-control (OOC) process following a shift of $\delta =$ 4000/5000 = 0.8; this is a deterioration in the process. Note that, because $k = 2$, we will be monitoring the total time between two consecutive failures. Also, because the preceding developments assumed that $\theta = 1$, the simulated data has to be scaled by dividing by $\theta_0 = 5000$.

Two sets of design parameters are used: ($q =$ 0.9, $\alpha = 0.7$, $L =$ 1.960) and ($q = 0.9$, $\alpha = 1.0$, $L =$ 2.045). The first set results in a GWMA-TBE chart whereas the second set results in an EWMA-TBE chart (because $\alpha = 1.0$) with smoothing parameter $\lambda = 1 - 0.9 =$ 0.1. These two sets of parameters are chosen for the illustration purposes only of the proposed chart and any other combination can be chosen in this regard. However, while choosing a parameter combination in practice, it is important to note that, a larger value of q and a smaller value of α usually works well for smaller shifts, i.e., for values of δ closer to 1.

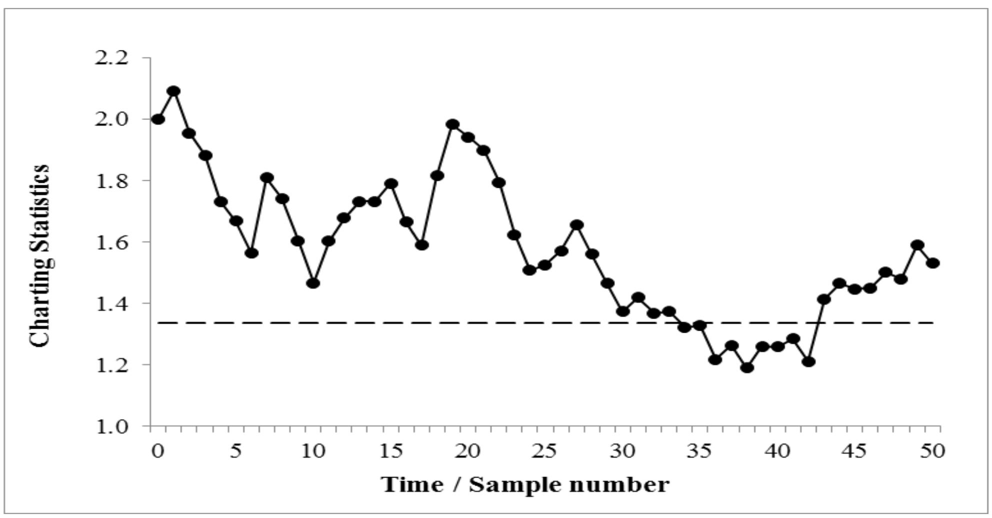
From Table 2, we can see that both charts are designed so that their $ARL_0 \approx 370$. From Table 4, we observe that the GWMA-TBE chart has an OOC ARL of 40.53 while the EWMA-TBE chart has an OOC ARL of 46.63. So, we would expect the GWMA-TBE chart to signal before the EWMA-TBE chart.

The lower control limits are calculated using equation (9) and are equal to 1.546 and 1.336, respectively. The charting statistics are calculated using equation (4) with $\theta_0 = 1$; this means the starting value is taken as 2.

Figure 2 displays the GWMA-TBE and EWMA-TBE charts. From this figure, we observe that the GWMA-TBE chart signals at time 30 whereas the EWMA-TBE chart signals at time 34. Note that it is expected that the GWMA-TBE chart outperforms the EWAM chart for these choices of the design parameters.



(a) GWMA-TBE chart



(b) EWMA-TBE chart

Figure 2: GWMA-TBE chart and EWMA-TBE chart for the example.

7. Concluding remarks

Efficient control charts are crucial to improve the quality of a process. To this end, an efficient control chart should detect any change as quickly as possible since the faster a change is detected the quicker a corrective action can be taken. However, when monitoring nonconformities or defects, the traditional Shewhart-type attribute charts (such as the c-chart and the p-chart) are known to be inefficient at detecting small changes quickly and poor to detect changes when the failure rate is very small. For this reason, we have proposed a time-weighted chart, which sequentially accumulates all the information over time, and monitor the time between events (TBE), i.e., the time between consecutive defects. Accumulating all historical information provides greater sensitivity to detect small changes effectively. We have specifically proposed a one-sided Generally Weighted Moving Average (GWMA) control chart based on the gamma distribution to monitor the TBE; this chart has been called the GWMA-TBE chart. It has been shown that the proposed GWMA chart includes the one-sided Exponentially Weighted Moving Average (EWMA) and Shewhart-type charts as special cases. We have investigated the scenarios when the scale parameter of the gamma distribution is known (referred to as the "standard known") as well as when it is unknown (referred to as the "standard unknown"). It has been shown that when one estimates the unknown parameter from an in-control Phase I reference sample, the run-length (and in particular the ARL) is adversely affected. Three methods for calculating the run-length distribution and the associated characteristics of the run-length distribution have been investigated; this includes (i) Exact closed-form expressions, (ii) A Markov chain approach, and (iii) Monte Carlo Simulation. Due to the difficulties of numerically evaluating the closed-form expression and using the Markov chain approach for the GWMA-

TBE chart, we have used computer simulation. To aid the implementation of the chart, the necessary design parameters for Case K and Case U have been provided, which guarantees that the in-control ARL is equal to a specified nominal value. An extensive performance analysis has been carried-out and a set of near optimal design parameters have been provided. The performance analysis has shown that the GWMA-TBE chart is better than the well-known EWMA and Shewhart charts at detecting very small to moderate changes. We have also shown how the GWMA-TBE chart can be used to monitor the variance of a normal distribution.

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Appendix

A1. Decreasing weights

We consider two scenarios:

i. For $\alpha = 1$ and $0 < q < 1$, we have $w_i = q^{(i-1)} - q^i = q^{i-1}(1-q)$ which is decreasing function of i because $0 < q < 1$;

ii. For $0 < \alpha < 1$ and $0 < q < 1$, we have $q^{(i-1)^\alpha} > q^{i^\alpha}$ for all i. We can then re-write the weights $w_i = q^{(i-1)^\alpha} - q^{i^\alpha} = q^{(i-1)^\alpha}\left(1 - q^{i^\alpha - (i-1)^\alpha}\right)$. Now, because $q^{(i-1)^\alpha}$ is a decreasing function for all i, w_i will be decreasing if the remaining part of the product, i.e., $\left(1 - q^{i^\alpha - (i-1)^\alpha}\right)$, is decreasing. To show this, we need to show that the exponent of q viz., $w_i^* = \left(i^\alpha - (i-1)^\alpha\right)$ is decreasing for $i = 1, 2, 3, \ldots$ and $0 < \alpha < 1$.

It can be easily shown that for $n \geq 1$ non-decreasing pairs of positive real numbers $(a_i, b_i), i =$ 1(1) n, such that $a_i \leq b_i \; \forall i = 1(1)n$,

$$\frac{\left(\prod_{i=1}^{n} a_i\right) + \left(\prod_{i=1}^{n} b_i\right)}{2} \geq \prod_{i=1}^{n} \frac{a_i + b_i}{2}. \quad \text{(i)}$$

If $a_i = a$ and $b_i = b$ for all i such that $a \leq b$, then we can re-write the inequality in (i) as

$$\frac{a^n + b^n}{2} \geq \left(\frac{a+b}{2}\right)^n. \quad \text{(ii)}$$

Taking $b = (i+1)^{\frac{1}{n}}$, $a = (i-1)^{\frac{1}{n}}$ in (ii) for some integer $i \geq 1$, we get

$$i \geq \left(\frac{(i+1)^{\frac{1}{n}} + (i-1)^{\frac{1}{n}}}{2} \right)^n . \text{ (iii)}$$

Since $\alpha < 1$, we can write $\alpha = \frac{1}{n}$ for some positive integer $n > 1$ and replacing $\alpha = \frac{1}{n}$ in (iii), we get, $i^\alpha \geq \frac{(i+1)^\alpha + (i-1)^\alpha}{2} \Rightarrow i^\alpha - (i-1)^\alpha \geq (i+1)^\alpha - i^\alpha \Rightarrow w_i^* \geq w_{i+1}^*$. Therefore, $w_i^* = i^\alpha - (i-1)^\alpha$ is decreasing for $i = 1, 2, 3, \ldots$ and $0 < \alpha < 1$. This implies that $1 - q^{i^\alpha - (i-1)^\alpha}$ is decreasing since $0 < q < 1$. Thus, we have $w_i = q^{(i-1)^\alpha}\left(1 - q^{i^\alpha - (i-1)^\alpha}\right)$ to be decreasing for $i = 1, 2, 3, \ldots$ and $0 < \alpha < 1$, which completes the proof.

A2. $\boldsymbol{Q_t = \sum_{i=1}^{\infty}\left(q^{(i-1)^\alpha} - q^{i^\alpha}\right)^2}$ **is convergent as** $\boldsymbol{t \to \infty}$ **, for** $0 \leq \boldsymbol{q} < 1$**,** $\boldsymbol{\alpha} > 0$

We have $Q_t = \sum_{i=1}^{t}\left(q^{(i-1)^\alpha} - q^{i^\alpha}\right)^2 = \sum_{i=1}^{t} q^{2(i-1)^\alpha} + \sum_{i=1}^{t} q^{2i^\alpha} - 2\sum_{i=1}^{t} q^{(i-1)^\alpha} q^{i^\alpha}$. We consider two scenarios:

i. For $q = 0$, $\lim_{t \to \infty} Q_t = \sum_{i=1}^{\infty}\left(q^{(i-1)^\alpha} - q^{i^\alpha}\right)^2 = 1$ which is trivial.

ii. For $0 < q < 1$, we have $q^{(i-1)^\alpha} > q^{i^\alpha} \Rightarrow q^{(i-1)^\alpha} q^{i^\alpha} > q^{2i^\alpha}$. Substituting $q^{(i-1)^\alpha} q^{i^\alpha}$ by q^{2i^α}, it follows that $Q_t < \left(\sum_{i=1}^{t} q^{2(i-1)^\alpha} + \sum_{i=1}^{t} q^{2i^\alpha} - 2\sum_{i=1}^{t} q^{2i^\alpha}\right) = \sum_{i=1}^{t} q^{2(i-1)^\alpha} - \sum_{i=1}^{t} q^{2i^\alpha}$. It then follows that $Q_t < \left(1 - q^{2t^\alpha}\right) < 1$.

So, for $0<q<1$, the monotonically increasing sequence $\{Q_t; t=1,2,3,\ldots\}$ is bounded above and is therefore convergent. Further, $0<Q_t<1$ for all values of t, which implies that $0<\lim_{t\to\infty} Q_t<1$ and so there exists a number (denoted by Q) in the interval (0,1) such that $\lim_{t\to\infty} Q_t = Q$ for $0<q<1$. This completes the proof.

A3. $\mathbf{Pr}\left[\boldsymbol{R}=\boldsymbol{r}\right]=\boldsymbol{I}_{r-1}-\boldsymbol{I}_r$ **, where** $\boldsymbol{I}_r=\mathbf{Pr}\left[\bigcap_{i=1}^{r} \boldsymbol{A}_i^c\right]$ **for** $\boldsymbol{r}=2,3,4,\ldots$

$$\Pr\left[R=r\right]=\Pr\left[\left\{\bigcap_{i=1}^{r-1} A_i^c\right\}\bigcap A_r\right]=\Pr\left[\bigcap_{i=1}^{r-1} A_i^c\right]-\Pr\left[\left\{\bigcap_{i=1}^{r-1} A_i^c\right\}\bigcap A_i^c\right]=\Pr\left[\bigcap_{i=1}^{r-1} A_i^c\right]-\Pr\left[\bigcap_{i=1}^{r} A_i^c\right]$$

. Therefore, $\Pr\left[R=r\right]=I_{r-1}-I_r$, where $I_r=\Pr\left[\bigcap_{i=1}^{r} A_i^c\right]$.

A4. $\boldsymbol{ARL}=\mathbf{1}+\sum_{r=1}^{\infty}\boldsymbol{I}_{\mathbf{r}}$

Because $\Pr\left[R=r\right]=I_{\mathrm{r}-1}-I_{\mathrm{r}}$, we have $ARL=\sum_{r=1}^{\infty} r\Pr\left[R=r\right]$. Upon expanding and re-arranging some of the terms, we obtain

$$ARL=\Pr\left[R=1\right]+\sum_{r=2}^{\infty} r\Pr\left[R=r\right] = 1-\Pr\left[A_1^c\right]+\sum_{r=2}^{\infty} r\left(I_{\mathrm{r}-1}-I_{\mathrm{r}}\right)=1-\Pr\left[A_1^c\right]+I_1+\sum_{r=1}^{\infty} I_{\mathrm{r}} = 1+\sum_{r=1}^{\infty} I_{\mathrm{r}},$$

as required.