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A New ϕ-Generalized Quasi Metric Space with Some Fixed Point Results and Applications

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Abstract. In this paper, the concept of a new ϕ-generalized quasi metric space is introduced. A number of well-known quasi metric spaces are retrieved from ϕ-generalized quasi metric space. Some general fixed point theorems in a ϕ-generalized quasi metric spaces are proved, which generalize, modify and unify some existing fixed point theorems in the literature. We also give applications of our results to obtain fixed points for contraction mappings in the domain of words and to prove the existence of periodic solutions of delay differential equations.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

Banach contraction principle is one of the most celebrated result in metric fixed theory. Many authors have given various generalizations of this principle in ambient different spaces (see e.g. [1]-[14]).

In the past few years, quasi metric spaces have been one of the interesting topics for the researchers in the field of fixed point theory due to two reasons. The first reason is that the assumptions of quasi metric are weaker than the more general metric. Consequently, the obtained fixed point results in this space are more general and hence the corresponding results in metric space are covered. The second reason is the fact that fixed point problems in metric space can be reduced to related fixed point problems in the context of quasi metric space (see [4]). Very recently, Karapinar [18] introduced (α-ψ)-contraction mappings on generalized quasi metric spaces without the Hausdorffness assumption. Zhu et al. [19] introduced a new concept of quasi-b-metric-like spaces as a generalization of b-metric-like spaces and quasi metric-like spaces and some fixed point theorems were investigated in quasi-b-metric-like spaces.

In this article, we prove the existence of fixed point of (α-ψ)-contraction mappings in the context of our newly introduced ϕ-generalized quasi metric space without the Hausdorffness assumption. Consequently, our results generalize, modify and unify several results in the literature.

In this paper, we arrange our work as follows.

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In section 2, we define the ϕ-generalized quasi metric spaces and discuss some of their topological concepts.

In section 3, we recover some well-known quasi metric spaces like standard quasi metric spaces, dislocated quasi metric spaces, b-quasi metric spaces and generalized quasi metric spaces from ϕ-generalized quasi metric spaces.

In section 4, we define the (α-ψ)-contraction mapping, prove the main theorem of uniqueness of fixed point of this mapping on ϕ-generalized quasi metric space and consider immediate consequences of our main theorem on the existence and uniqueness of fixed point of k-contraction and ψ-contraction mappings.

In section 5, we present some applications of our results.

Before we proceed to our actual work, let us recall the following.

Let X be a nonempty set and $T : X \longrightarrow X$ a self map. We say that $x \in X$ is a fixed point of T if $Tx = x$ and denote by $F(T)$ or $Fix(T)$ the set of all fixed points of T. For any given $x \in X$ we define $T^n x$ inductively by $T^0 x = x_0$ and $T^{n+1}x = T(T^n x)$, we call $T^n x$, the n^{th} iterate of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subseteq X$ given by

$$x_n = T(x_{n-1}) = T^n x_0, \;\; n = 1, 2, \cdots$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the *Picard iteration* starting at x_0.

A mapping $T : X \longrightarrow X$ where (X, d) is a metric space, is said to be a *contraction* if there exists $k \in [0, 1)$ such that for all $x, y \in X$

$$d(T(x), T(y)) \leq kd(x, y). \tag{1}$$

If the metric space (X, d) is complete, the mapping satisfying (1) has a unique fixed point (Banach contraction principle).

Inequality (1) implies continuity of T. We have some contraction condition which will imply existence of fixed point in a complete metric space but do not imply continuity (see [3]).

1.1. *ϕ-generalized quasi metric space*

Real life is full of instances where symmetry with respect to distance is not necessary. Such a distance is referred to as a quasi metric in contrast with a metric needing symmetry. For example, given a set X of mountain villages, the typical walking times between elements of X form a quasi metric because traveling up hill takes longer than traveling down hill. Another example is a geometry topology having one-way streets, where a path from point A to point B comprises a different set of streets than a path from B to A.There is an abundant literature devoted to *distances* where the requirement of symmetry is omitted. Quasi metrics have some interest in topology, but they are also used in applied mathematics in the calculus of variation.

Since quasi metric spaces form a generalization of metric spaces, any sound completion theory for such spaces should strictly generalize the usual completion theory for metric spaces. Traditionally this is done by generalizing the concept of Cauchy sequence and/or that of the convergence of a sequence.

We now define our ϕ-generalized quasi metric. Let $\phi : [0, +\infty] \longrightarrow [0, +\infty)$ be a continuous and nondecreasing for all $t > 0$ and $\phi(0) = 0$. Let X be a nonempty set and $D_\phi : X \times X \longrightarrow [0, +\infty]$ a given mapping. For every $x \in X$, let us define the set

$$(D_\phi, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0 \right\}.$$

Definition 1.1. *Let X be a nonempty set. A function $D_\phi : X \times X \longrightarrow [0, +\infty]$ is called a ϕ-generalized quasi metric on X if the following two conditions are met:*

$(D_\phi)_1$ for every $x, y \in X$

$$D_\phi(x, y) = 0 \implies x = y.$$

$(D_\phi)_2$ there exists ϕ such that if $(x, y) \in X \times X$, $\{x_n\} \in (D_\phi, X, x)$, then

$$D_\phi(x, y) \leq \phi\left(\limsup_{n\to\infty} D_\phi(x_n, y)\right).$$

The pair (X, D_ϕ) is called a ϕ-generalized quasi metric space.

Remark 1.2.

(i) The notion of a ϕ-generalized quasi metric space is a generalization of the notion of a ϕ-generalized metric space.

(ii) In Definition 1.1, if $(D_\phi)_1$ is replaced by $D_\phi(x, x) = 0$ for all $x \in X$, then D_ϕ is called a ϕ−generalized quasi-pseudometric on X.

(iii) For a ϕ-generalized quasi metric D_ϕ on X, the conjugate ϕ-generalized quasi metric D_ϕ^{-1} on X of D_ϕ is defined by $D_\phi^{-1}(x, y) = D_\phi(y, x)$.

(iv) If D_ϕ is a generalized T_0-quasi-pseudometric on X, then the function D_ϕ^u defined by $D_\phi^u = D_\phi^{-1} \bigvee D_\phi$ that is, $D_\phi^u(x, y) = \max\left\{D_\phi(x, y), D_\phi(y, x)\right\}$, defines a ϕ-generalized metric on X.

(v) The general quasi metric spaces retrieve standard quasi metric spaces, quasi-partial metric spaces, quasi-b-metric spaces, dislocated quasi metric spaces and so on.

The lack of symmetry in the definition of quasi metric spaces may cause a lot of troubles, mainly concerning completeness, compactness and total boundedness in such spaces. However, there are a lot of completeness notions in quasi metric spaces, all agreeing with the usual notion of completeness in the case of metric spaces, each of them having its advantages and weaknesses.

We describe briefly some of these notions along with some of their properties.

Definition 1.3. *A sequence $\{x_n\}$ in a ϕ-generalized quasi metric space (X, D_ϕ) converges to x, called*

(a) D_ϕ-convergent or *left convergence* if

$$x_n \xrightarrow{D_\phi} x \iff D_\phi(x, x_n) \longrightarrow 0.$$

(b) D_ϕ^{-1}-convergent or *right convergence* if

$$x_n \xrightarrow{D_\phi^{-1}} x \iff D_\phi(x_n, x) \longrightarrow 0.$$

(c) D_ϕ^u-convergent if

$$x_n \xrightarrow{D_\phi^u} x \iff x_n \xrightarrow{D_\phi} x \text{ and } x_n \xrightarrow{D_\phi^{-1}} x$$

Definition 1.4. *A sequence $\{x_n\}$ in a ϕ-generalized quasi metric space (X, D_ϕ) is called*

(d) left (right) K-Cauchy if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $D_\phi(x_k, x_n) < \epsilon$ for all $n, k \in \mathbb{N}$ with $n_\epsilon \leq k \leq n$ ($n_\epsilon \leq n \leq k$).

(e) D_ϕ^u-Cauchy if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $D_\phi(x_n, x_k) < \epsilon$ for all $n, k \in \mathbb{N}$ with $n, k \geq n_\epsilon$.

Remark 1.5. *From the above definitions we conclude that:*

(i) a sequence is left K-Cauchy with respect to D_ϕ if and only if it is right K-Cauchy with respect to D_ϕ^{-1}.

(ii) a sequence is D^u_ϕ-Cauchy if and only if it is both left and right K-Cauchy.

We now give an example of a ϕ-generalized quasi metric space and also show that left K-Cauchy does not necessarily imply D^u_ϕ-Cauchy.

Example 1.6. *Let $X = (0, 1)$ and define D_ϕ on X by*

$$D_\phi(x, y) := \begin{cases} x - y, & \text{if } x \geq y, \\ 1, & \text{if } y > x. \end{cases} \tag{2}$$

and define $\phi(t) := 2t$ for all $t > 0$.

We now show that (X, D_ϕ) is a ϕ-generalized quasi metric space. Clearly, from the definition $D_\phi(x, y) = 0$ implies $x = y$. So the property $(D_\phi)_1$ is satisfied. For $(D_\phi)_2$, we assume $\{x_n\} \in (D_\phi, X, x)$ and consider the following cases.

Case 1. Let $x_n \geq y$ $\forall n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0$. Then it follows that $x \geq y$.

$$\phi\left(\limsup_{n\to\infty} D_\phi(x_n, y)\right) = 2\left(\limsup_{n\to\infty}(x_n - y)\right) \geq (x - y) = D_\phi(x, y).$$

Case 2. Let $y > x_n$ $\forall n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0$. Then it follows that $y \geq x$. If $x = y$, then $(D_\phi)_2$ is satisfied. Otherwise

$$\phi\left(\limsup_{n\to\infty} D_\phi(x_n, y)\right) = 2\left(\lim_{n\longrightarrow\infty} 1\right) \geq 1 = D_\phi(x, y).$$

Case 3. Let $x_n \geq y$ for infinitely many $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0$. Then the same result holds as in Case (1).

Case 4. Let $y > x_n$ for infinitely many $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0$. Then the same result holds as in Case (2).

Case 5. Let $x_n \geq y$ for finitely many $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0$. Then either x_n is ultimately constant sequence and the same result holds as in Case 1 or there exists $n_0 \in \mathbb{N}$ such that $y > x_n$ for all $n \geq n_0$ and the same result holds as in Case 2.

Hence property $(D_\phi)_2$ is satisfied in all the cases. Therefore, (X, D_ϕ) is a ϕ-generalized quasi metric space.

Let $\{x_n\}$ be the sequence in X defined as follows:

$$x_n = \frac{1}{4} + \frac{1}{2^n}$$

for all $n \in \mathbb{N}$.

Then for all $k < n$, $D_\phi(x_k, x_n) \to 0$ as $k, n \to \infty$, so that $\{x_n\}$ is left K-Cauchy. However, $\{x_n\}$ is not right K-Cauchy because for all $n < k$, $D_\phi(x_k, x_n) = 1$ and $\{x_n\}$ is not D^u_ϕ -Cauchy sequence. Then left K-Cauchy does not necessarily imply D^u_ϕ-Cauchy.

Definition 1.7. *A ϕ-generalized quasi metric space (X, D_ϕ) is called*

(i) left-K-complete if every left K-Cauchy sequence in X is D_ϕ-convergent.

(ii) Smyth-complete if every left K-Cauchy sequence in X is D^u_ϕ-convergent.

(iii) bicomplete if (X, D^u_ϕ) is a complete general metric space.

The definition implies that every Smyth-complete quasi metric space is left K-complete. Dually, right-completeness notions are derived from the above definition.

1.2. Retrieving other quasi metrics from ϕ-generalized quasi metric spaces

In this section, we see that the ϕ-generalized quasi metric spaces retrieve a number of various other metrics. Hence it ends with a true generalization of a quasi metric spaces. We recall the following concepts.

(1) Let X be a nonempty set. A function $d : X \times X \longrightarrow [0, +\infty)$ is called a standard quasi metric on X provided that, for all $x, y, z \in X$ if it satisfies the following conditions:

$$\begin{aligned}(d_1)\ d(x,y) &= 0 \Longleftrightarrow x = y;\\ (d_2)\ d(x,y) &\leq d(x,z) + d(z,y).\end{aligned}$$

The pair (X, d) is said to be standard quasi metric space.

(2) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0, +\infty)$ is called a b-quasi metric on X provided that, for all $x, y, z \in X$ if it satisfies the following conditions:

$$\begin{aligned}(b_1)\ d(x,y) &= 0 \Longleftrightarrow x = y;\\ (b_2)\ d(x,y) &\leq s[d(x,z) + d(z,y)].\end{aligned}$$

The pair (X, d) is said to be b-quasi metric space. It is clear that definition of b-quasi metric space is a extension of standard quasi metric space. The topological concept of convergence in such spaces is similar to that of standard quasi metric spaces.

(3) Let X be a nonempty set. A function $d : X \times X \longrightarrow [0, +\infty)$ is called a dislocated quasi metric on X provided that, for all $x, y, z \in X$ if it satisfies the following conditions:

$$\begin{aligned}(HS_1)\ d(x,y) &= 0 \Longrightarrow x = y;\\ (HS_2)\ d(x,y) &\leq d(x,z) + d(z,y).\end{aligned}$$

The pair (X, d) is said to be a dislocated quasi metric space. The topological concept of convergence in such spaces is similar to that of standard quasi metric spaces.

(4) Let X be a nonempty set and a mapping $D : X \times X \longrightarrow [0, +\infty]$. For every $x \in X$, define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \longrightarrow \infty} D(x_n, x) = 0 \right\}. \tag{3}$$

Then D is said to be a generalized quasi metric on X if it satisfies the following conditions:

($\mathbf{D_1}$) for every $(x, y) \in X \times X$, we have

$$D(x,y) = 0 \Longrightarrow x = y.$$

($\mathbf{D_2}$) there exists $C > 0$ such that if $(x, y) \in X \times X$, $\{x_n\} \in C(D, X, x)$, then

$$D(x,y) \leq C\left(\limsup_{n\to\infty} D(x_n, y)\right).$$

The pair (X, D) is called a generalized quasi metric space. The quasi metric spaces $(1) - (3)$ are special cases of the generalized quasi metric (4) above.

We now give the following important proposition which shows that our ϕ-generalized quasi metric is more general than the generalized quasi metric space as in (4) above.

Proposition 1.8. *Any generalized quasi metric on X is a ϕ-generalized quasi metric on X.*

Proof. *We have just to prove that D satisfies the property $(D_\phi)_2$. Let D be a generalized quasi metric on X. Let $x \in X$ and $x_n \in C(D, X, x)$. For every $y \in X$, by the property (D_2), we have*

$$D(x, y) \le C\left(\limsup_{n\to\infty} D(x_n, y)\right),$$

for every natural number n. Then the property $(D_\phi)_2$ is satisfied with $\phi(t) = Ct$ for all $t \in [0, +\infty)$. □

Remark 1.9. *Although all the quasi metrics given in* (1) − (3) *are now special cases of our ϕ-generalized quasi metric by the above proposition, yet we can retrieve them independently. Note that in all the cases only $(D_\phi)_2$ needs to be satisfied. Now* (1) *and* (3) *follow by taking $\phi(t) = t$ for all $t \in [0, +\infty)$.* (2) *follows by taking $\phi(t) = st$ for all $t \in [0, +\infty)$ and $s \ge 1$.*

1.3. Generalized (α-ψ)-contractions in a ϕ-generalized quasi metric space

In this section, we characterize (α-ψ)-contraction mappings in the setting of ϕ-generalized quasi metric spaces and investigate the existence and uniqueness of a fixed point of such mappings.

Definition 1.10. *Let Ψ be the family of functions $\psi : [0, +\infty] \longrightarrow [0, +\infty)$ which is monotone nondecreasing and $\lim_{n\longrightarrow\infty} \psi^n(t) = 0$, for a given $t > 0$ and $\psi(0) = 0$.*

Lemma 1.11. *[16] Let $t > 0$, $\psi(t) < t$ if and only if $\lim_{n\longrightarrow\infty} \psi^n(t) = 0$, where ψ^n is the n-times repeated composition of ψ with itself.*

Definition 1.12. *Let (X, D_ϕ) be a ϕ-generalized quasi metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is an (α-ψ)-contraction mapping if there exist two functions $\alpha : X \times X \longrightarrow [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)D_\phi(Tx, Ty) \le \psi\left(D_\phi(x, y)\right), \ \forall x, y \in X. \tag{4}$$

Remark 1.13. *From Definition 1.12, we see that a k-contraction mapping is an (α-ψ)-contraction mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in [0, 1)$, $t \ge 0$ and a ψ-contraction mapping is an (α-ψ)-contraction mapping with $\alpha(x, y) = 1$ for all $x, y \in X$.*

Definition 1.14. *[17] Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, +\infty)$.We say that T is α-admissible if for all $x, y \in X$, we have*

$$\alpha(x, y) \ge 1 \Longrightarrow \alpha(Tx, Ty) \ge 1.$$

Definition 1.15. *[20] Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, +\infty)$.We say that T is a triangular α-admissible if*

(T_1) $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$, $x, y \in X$,

(T_2) $\alpha(x, z) \ge 1$, and $\alpha(z, y) \ge 1$, imply $\alpha(x, y) \ge 1$, $x, y, z \in X$.

Lemma 1.16. *[20] Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n) \ge 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

For every $x \in X$, let

$$\delta(D_\phi, T, x) = \sup\left\{D_\phi(T^i(x), T^j(x)) : i \ne j \in \mathbb{N} \cup \{0\}\right\}.$$

Theorem 1.17. *Let (X, D_ϕ) be a generalized Smyth-complete quasi metric space and $T : X \longrightarrow X$ is a (α-ψ)-contraction map which satisfies*

i) *T is a triangular α-admissible;*

ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$;*

iii) *$q, q' \in F(T)$ implies $D_\phi(q, q') < \infty$, $\alpha(q, q') \geq 1$ and $\alpha(x_0, q) \geq 1$.*

If there exists $x_0 \in X$ such that $\delta(D_\phi, T, x_0) < \infty$, then $\{T^n x_0\}$ D_ϕ-converges to a unique fixed point q of T.

Proof. By ii), there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. Let us define a sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then x_{n_0} is a fixed point of T. For the rest of the proof, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Regarding the assumption i), we derive

$$\alpha(x_1, x_0) = \alpha(Tx_0, x_0) \geq 1 \Longrightarrow \alpha(Tx_1, Tx_0) = \alpha(x_2, x_1) \geq 1. \tag{5}$$

Recursively, we get

$$\alpha(x_{n+1}, x_n) = \alpha(T^n x_0, T^{n-1} x_0) \geq 1 \;\; \forall n \in \mathbb{N}. \tag{6}$$

For $i, j \in \mathbb{N} \cup \{0\}$, assume that $i \neq j$. Taking (4) into account and applying Lemma 1.16, we find that

$$\begin{aligned} D_\phi(T^{n+i}x_0, T^{n+j}x_0) &\leq \alpha(T^{n-1+i}x_0, T^{n-1+j}x_0) D_\phi(T^{n+i}x_0, T^{n+j}x_0) \\ &\leq \psi\left(D_\phi(T^{n-1+i}x_0, T^{n-1+j}x_0)\right), \end{aligned}$$

which implies that

$$\delta(D_\phi, T, T^n x_0) \leq \psi\left(\delta(D_\phi, T, T^{n-1}x_0)\right).$$

Then for every $n \in \mathbb{N}$, we have

$$\delta(D_\phi, T, T^n x_0) \leq \psi^n\left(\delta(D_\phi, T, x_0)\right).$$

Thus for every $n, m \in \mathbb{N}$, we have

$$D_\phi(T^n x_0, T^{n+m}x_0) \leq \delta(D_\phi, T, T^n x_0) \leq \psi^n\left(\delta(D_\phi, T, x_0)\right).$$

Using the definition of ψ in (1.10) and the fact $\delta(D_\phi, T, x_0) < \infty$, we obtain

$$\lim_{n,m \longrightarrow \infty} D_\phi(T^n x_0, T^{n+m}x_0) = 0,$$

and this amounts to saying that $\{T^n x_0\}$ is a left K-Cauchy sequence in (X, D_ϕ). Since the space (X, D_ϕ) is generalized Smyth-complete quasi metric space, there exits a $q \in X$ such that

$$\lim_{n \longrightarrow \infty} D^u_\phi(T^n x_0, q) = 0.$$

Thus we have

$$\lim_{n \longrightarrow \infty} D_\phi(T^n x_0, q) = 0 \text{ and } \lim_{n \longrightarrow \infty} D_\phi(q, T^n x_0) = 0.$$

On the other hand, we have, since T is a $(\alpha$-$\psi)$-contractive and by $(D_\phi)_2$, for all $n \in \mathbb{N}$,

$$\begin{aligned} D_\phi(q, Tq) &\leq \phi\left(\limsup_{n \to \infty} D_\phi(T^n x_0, Tq)\right) \\ &\leq \phi\left(\limsup_{n \to \infty} \alpha(T^{n-1}x_0, q) D_\phi(T^n x_0, Tq)\right) \\ &\leq \phi\left(\psi\left(\limsup_{n \to \infty} D_\phi(T^{n-1}x_0, q)\right)\right) = 0. \end{aligned}$$

This yields $D_\phi(q,Tq)=0$ and by $(D_\phi)_1$, we get $Tq=q$, that is, q is a fixed point of T. Now, suppose that $q'\in X$ is another fixed point of T such that $D_\phi(q,q')<\infty$. Since T is an $(\alpha$-$\psi)$-contraction, we have

$$D_\phi(q,q')=D_\phi(Tq,Tq')\leq\alpha(q,q')D_\phi(Tq,Tq')\leq\psi\left(D_\phi(q,q')\right),$$

If $D_\phi(q,q')>0$, then by Lemma 1.11 we have

$$D_\phi(q,q')=D_\phi(Tq,Tq')\leq\psi\left(D_\phi(q,q')\right)<D_\phi(q,q'),$$

which is a contradiction. Thus we have $D_\phi(q,q')=0$. From the property $(D_\phi)_1$ it follows that $q=q'$. □

Remark 1.18. *Theorem 1.17 also holds if $\delta(D_\phi,T,x_0)<\infty$ is replaced by*

$$\sup\left\{D_\phi(x_0,T^rx_0):r\in\mathbb{N}\right\}<\infty.$$

Indeed, there exists $x_0\in X$ such that

$$\sup\left\{D_\phi(x_0,T^rx_0):r\in\mathbb{N}\right\}<\infty.$$

Since T is an $(\alpha$-$\psi)$-contraction mapping, we have

$$\delta(D_\phi,T,x_0)\leq\sup\left\{D_\phi(x_0,T^rx_0):r\in\mathbb{N}\right\}.$$

The result now follows as above.

Corollary 1.19. *Let (X,D) be a complete generalized quasi metric space and $T:X\longrightarrow X$ be a mapping. Suppose that, we have*

$$D(Tx,Ty)\leq\psi(D(x,y))$$

for all $x,y\in X$. If there exists $x_0\in X$ such that

$$\delta(D_\phi,T,x)=\sup\left\{D_\phi(T^i(x),T^j(x)):i\neq j\in\mathbb{N}\cup\{0\}\right\}<\infty.$$

then the sequence $\{T^nx_0\}$ left D-converges to a fixed point of T. Moreover, T has one and only one fixed point.

Now we give an example in order to validate our Theorem 1.17.

Example 1.20. *Let $X=[0,\infty)$ and define*

$$D_\phi(x,y):=\begin{cases}2(x-y), \text{ if } x>y,\\ y-x, \text{ if } y\geq x.\end{cases}$$

$\phi(t)=2t$ for all $t\in[0,\infty)$.With a similar procedure as in Example 1.23 given below, one can show that (X,D_ϕ) is a Smyth-complete quasi metric space. Let $T:X\longrightarrow X$ be a mapping defined as

$$T(x):=\begin{cases}\dfrac{x}{8}, \text{ if } x\in[0,1],\\ \dfrac{x}{4}, \text{ if } x\in(1,\infty).\end{cases}$$

Let

$$\alpha(x,y):=\begin{cases}2, \text{ if } (x,y)\in[0,1]\times[0,1],\\ 0, \text{ if } (x,y)\in[0,1]\times(1,\infty),\\ 0, \text{ if } (x,y)\in(1,\infty)\times[0,1],\\ 0, \text{ if } (x,y)\in(1,\infty)\times(1,\infty).\end{cases}$$

and $\psi(t)=\frac{t}{2}$. Obviously, one can show that T is $(\alpha$-$\psi)$-contraction map. We now show that T is a triangular α-admissible mapping. Let $x,y\in X$, if $\alpha(x,y)\geq1$, then $x,y\in[0,1]$. On the other hand, for all $x,y\in[0,1]$ we have $T(x)\leq1$ and $T(y)\leq1$. It follows that $\alpha(Tx,Ty)\geq1$. Also, if $\alpha(x,z)\geq1$ and $\alpha(z,y)\geq1$, then $x,y,z\in[0,1]$ and hence $\alpha(x,y)\geq1$. Thus, the assertion holds by the same arguments as above. Notice that $\alpha(0,T0)\geq1$.

Hence, all the conditions of Theorem 1.17 hold and 0 is a unique fixed point of T.

We now present an extension of k-contraction and ψ-contraction type mappings [15] to the setting of ϕ-generalized quasi metric spaces.

Definition 1.21. *A mapping T is a*

(i) k-contraction if there exists $k \in [0,1)$ such that

$$D_\phi(Tx, Ty) \leq kD_\phi(x,y),$$

for all $x, y \in X$.

(ii) ψ-contraction if

$$D_\phi(Tx, Ty) \leq \psi\left(D_\phi(x,y)\right),$$

for every $(x,y) \in X \times X$.

Note that if $\psi(t) = kt$ for all $t \geq 0$ and $k \in [0,1)$, then k-contraction is ψ-contraction.

Corollary 1.22. *Let (X, D_ϕ) be a generalized Smyth-complete quasi metric space and $T : X \longrightarrow X$ is a k-contraction. Suppose there exists $x_0 \in X$ such that $\delta\left(D_\phi, T, x_0\right) < \infty$. Then $\{T^n x_0\}$ D_ϕ-converges to $q \in X$, a fixed point of T. Moreover, if $q' \in X$ is another fixed point of T such that $D_\phi(q,q') < \infty$, then $q = q'$.*

Proof. In Theorem 1.17, we take $\alpha(x,y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and $k \in [0,1)$. □

Here is an example in support of the above corollary.

Example 1.23. *Let $X = \left\{\frac{1}{2^n} : n = 0, 1, 2, \cdots\right\} \cup \{0\}$ and define*

$$D_\phi(x,y) := \begin{cases} 2(x-y), & \text{if } x > y, \\ y - x, & \text{if } y \geq x. \end{cases}$$

for all $x, y \in X$ and define $\phi(t) = 2t$ for all $t \in [0, \infty)$. We now show that (X, D_ϕ) is a Smyth-complete quasi metric space. Clearly D_ϕ satisfies property $(D_\phi)_1$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that either $x_n = \frac{1}{2^n}$ or $y_n = c$, where c is constant for all $n \in \mathbb{N} \cup \{0\}$. Let $\{x_n\} \in (D_\phi, X, x)$, that is, $\lim_{n\longrightarrow\infty} D_\phi(x_n, x) = 0$.

Case 1. *Let $x_n > y$ for all $n \in \mathbb{N}$. It follows that $x \geq y$. If $x = y$, then $(D_\phi)_2$ is satisfied. Otherwise*

$$\begin{aligned} D_\phi(x,y) &= 2(x-y) = \lim_{n\longrightarrow\infty} 2(x_n - y) \leq 2(\limsup_{n\to\infty} 2(x_n - y)) \\ &= \phi\left(\limsup_{n\to\infty} 2(x_n - y)\right) = \phi\left(\limsup_{n\to\infty} D_\phi(x_n, y)\right). \end{aligned}$$

Case 2. *Let $y \geq x_n$ for all $n \in \mathbb{N}$. It follows that $y \geq x$.*

$$\begin{aligned} D_\phi(x,y) &= y - x = \lim_{n\longrightarrow\infty} (y - x_n) \leq 2(\limsup_{n\to\infty}(y - x_n)) \\ &= \phi\left(\limsup_{n\to\infty}(y - x_n)\right) = \phi\left(\limsup_{n\to\infty} D_\phi(x_n, y)\right). \end{aligned}$$

Case 3. *Let $x_n > y$ for infinitely many $n \in \mathbb{N}$. Then the same result holds as in Case 1.*

Case 4. *Let $y \geq x_n$ for infinitely many $n \in \mathbb{N}$. Then the same result holds as in Case 2.*

Case 5. *Let $x_n > y$ for finitely many $n \in \mathbb{N}$. Then either x_n is ultimately constant sequence and the same result holds as in case 1 or there exists $n_0 \in \mathbb{N}$ such that $y \geq x_n$ for all $n \geq n_0$ and the same result holds as in Case 2.*

Hence D_ϕ satisfies property $(D_\phi)_2$. Therefore (X, D_ϕ) is a ϕ-generalized quasi metric space.

Note that $D_\phi(x_k, x_n) = 2(\frac{1}{2^k} - \frac{1}{2^n}) < \frac{1}{2^{k-1}}$ for all $n > k$ and $D_\phi(y_k, y_n) = D_\phi(c, c) = 0$ for all $n > k$, so that $\{x_n\}$ and $\{y_n\}$ are left K-Cauchy in X. Since $D_\phi(0, \frac{1}{2^n}) = D_\phi(\frac{1}{2^n}, 0) \longrightarrow 0$ and $D_\phi(c, c) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, hence sequences $\{x_n\}$ and $\{y_n\}$ are D^u_ϕ-convergent. Therefore (X, D_ϕ) is a generalized Smyth-complete quasi metric space.

Let $T : X \longrightarrow X$ be a mapping defined as follows:

$$T(x) := \begin{cases} \dfrac{1}{2^{n+1}}, \text{ if } x = \dfrac{1}{2^n}, n = 0, 1, 2, \ldots, \\ 0, \text{ if } x = 0. \end{cases}$$

Then T is k-contraction for $k \in [0, 1)$ as shown below.

Case 1. *Let $x = y$.*

$$D_\phi(Tx, Ty) = 0 \leq kD_\phi(x, y).$$

Case 2. *Let $x = 0$ and $y = \frac{1}{2^n}$ $(n = 0, 1, 2, \cdots)$.*

$$D_\phi(Tx, Ty) = D_\phi\left(0, \frac{1}{2^{n+1}}\right) = \frac{1}{2^{n+1}} \leq \frac{1}{2}\left(\frac{1}{2^n}\right) = kD_\phi(x, y).$$

Case 3. *Let $x = \frac{1}{2^n}$ $(n = 0, 1, 2, \cdots)$ and $y = 0$.*

$$D_\phi(Tx, Ty) = D_\phi\left(\frac{1}{2^{n+1}}, 0\right) = 2\left(\frac{1}{2^{n+1}} - 0\right) \leq \frac{1}{2}\left(2\left(\frac{1}{2^n} - 0\right)\right) = kD_\phi(x, y).$$

Case 4. *Let $x = \frac{1}{2^n}$ and $y = \frac{1}{2^m}$ $(m > n)$.*

$$\begin{aligned} D_\phi(Tx, Ty) &= D_\phi\left(\frac{1}{2^{n+1}}, \frac{1}{2^{m+1}}\right) = 2\left(\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}\right) = \frac{1}{2}\left(2\left(\frac{1}{2^n} - \frac{1}{2^m}\right)\right) \\ &= \frac{1}{2}D_\phi(x, y) = kD_\phi(x, y). \end{aligned}$$

Case 5. *Let $x = \frac{1}{2^m}$ and $y = \frac{1}{2^n}$ $(m > n)$.*

$$\begin{aligned} D_\phi(Tx, Ty) &= D_\phi\left(\frac{1}{2^{m+1}}, \frac{1}{2^{n+1}}\right) = \frac{1}{2^{n+1}} - \frac{1}{2^{m+1}} = \frac{1}{2}\left(\frac{1}{2^n} - \frac{1}{2^m}\right) \\ &= \frac{1}{2}D_\phi(x, y) = kD_\phi(x, y). \end{aligned}$$

Hence, all the conditions of Corollary 1.22 hold. Thus T has a unique fixed point. Note that for any $x_0 \in X$, we have $\delta\left(D_\phi, T, x_0\right) < \infty$. Then $\{T^n x_0\}$ D_ϕ-converges to a fixed point of T.

Corollary 1.24. *Suppose that (X, D_ϕ) is a generalized Smyth-complete quasi metric space and T is a ψ-contraction for some ψ as in Definition 1.10. If there exists $x_0 \in X$ such that $\delta\left(D_\phi, T, x_0\right) < \infty$, then $\{T^n x_0\}$ D_ϕ-converges to a fixed point q of T. Moreover, if $q' \in X$ is another fixed point of T such that $D_\phi(q, q') < \infty$, then $q = q'$.*

Proof. In Theorem 1.17, we put $\alpha(x, y) = 1$ for all $x, y \in X$. □

Example 1.25. *Define $\psi(t) := \frac{t}{2}$ for all $t \geq 0$. In Example 1.23, it is easy to show that T is ψ-contraction by the same argument as we showed that T is k-contraction. Hence, all conditions of Corollary 1.24 hold. Thus T has a unique fixed point. For any $x_0 \in X$, we have $\delta\left(D_\phi, T, x_0\right) < \infty$. Then $\{T^n x_0\}$ D_ϕ-converges to a fixed point of T.*

1.4. Applications

1.4.1. An application to the domain of words

Let $\sum$ be a nonempty set of alphabets and the set of all (finite or infinite) sequences ("words") over $\sum$ is denoted by $\sum^\infty$. Conventionally, we denote the empty sequence (word) by $\emptyset$ and suppose that $\emptyset \in \sum^\infty$. Let the prefix order $\sqsubseteq$ on $\sum^\infty$ be defined as:

$$x \sqsubseteq y \Longleftrightarrow x \text{ is a prefix of } y.$$

Now, for each sequence (word) $x \neq \emptyset$ in $\sum^\infty$, let $\ell(x) \in [1, \infty]$ be the length of x and assume that $\ell(\emptyset) = 0$. Also, if $x \in \sum^\infty$ has finite length, then

$$x := x_1 x_2 \cdots x_{\ell(x)},$$

otherwise (i.e., in the case of infinite sequence) we write

$$x := x_1 x_2 \dots .$$

Now, if $x, y \in \sum^\infty$, then $x \sqcap y$ denotes the common prefix of x and y. Note that $x = y$ if and only if $x \sqsubseteq y$ and $y \sqsubseteq x$ and $\ell(x) = \ell(y)$. Define a mapping $D_\phi : \sum^\infty \times \sum^\infty \longrightarrow [0, \infty]$ (with convention $2^{-\infty} = 0$) by

$$D_\phi(x, y) := \begin{cases} 0, \text{ iff } x = y, \\ 2^{-\ell(x)}, \text{ iff } \ell(x) > \ell(y), \\ 2^{-\ell(y)}, \text{ iff } \ell(y) > \ell(x), \\ 2^{-\ell(x \sqcap y)}, \text{ otherwise.} \end{cases}$$

and $\phi(x) = 2t$ for all $t \in [0, \infty)$. We now show that $(\sum^\infty, D_\phi)$ is a ϕ-generalized quasi metric space. Since $D_\phi(x, y) = 0$ implies $x = y$, hence property $(D_\phi)_1$ is satisfied.

Let $\{x_n\} \in D_\phi(\sum^\infty, x)$, that is, $\lim_{n \longrightarrow \infty} D_\phi(x_n, x) = 0$.

Case 1. Let $\ell(x_n) > \ell(y)$ for all $n \in \mathbb{N}$. Then it follows that $\ell(x) > \ell(y)$.

$$\begin{aligned} D_\phi(x, y) &\leq 2D_\phi(x, y) = 2 \times 2^{-\ell(x)} = 2\left(\lim_{n \longrightarrow \infty} 2^{-\ell(x_n)}\right) \\ &\leq 2\left(\limsup_{n \to \infty} 2^{-\ell(x_n)}\right) = \phi\left(\limsup_{n \to \infty} 2^{-\ell(x_n)}\right) \\ &= \phi\left(\limsup_{n \to \infty} D_\phi(x_n, y)\right). \end{aligned}$$

Case 2. Let $\ell(y) > \ell(x_n)$ for all $n \in \mathbb{N}$. Then it follows that $\ell(y) > \ell(x)$.

$$\begin{aligned} D_\phi(x, y) &\leq 2D_\phi(x, y) = 2 \times 2^{-\ell(y)} = 2\left(\lim_{n \longrightarrow \infty} 2^{-\ell(y)}\right) \\ &= 2\left(\limsup_{n \to \infty} 2^{-\ell(y)}\right) = \phi\left(\limsup_{n \to \infty} 2^{-\ell(y)}\right) \\ &= \phi\left(\limsup_{n \to \infty} D_\phi(x_n, y)\right). \end{aligned}$$

Case 3. Let $x_n \neq y$, $\ell(y) \geq \ell(x_n)$ or $\ell(x_n) \geq \ell(y)$ for all $n \in \mathbb{N}$. Then it follows that $x \neq y$, $\ell(y) \geq \ell(x)$ or $\ell(x) \geq \ell(y)$.

$$\begin{aligned} D_\phi(x, y) &\leq 2D_\phi(x, y) = 2 \times 2^{-\ell(x \sqcap y)} = 2\left(\lim_{n \longrightarrow \infty} 2^{-\ell(x_n \sqcap y)}\right) \\ &= 2\left(\limsup_{n \to \infty} 2^{-\ell(y)}\right) = \phi\left(\limsup_{n \to \infty} 2^{-\ell(x \sqcap y)}\right) \\ &= \phi\left(\limsup_{n \to \infty} D_\phi(x_n, y)\right). \end{aligned}$$

In all the cases, we have $D_\phi(x,y) \leq \phi\left(\limsup_{n\to\infty} D_\phi(x_n,y)\right)$ for all $n \in \mathbb{N}$ and for all $x,y \in \sum^\infty$. Hence D_ϕ satisfies property $(D_\phi)_2$. Therefore $(\sum^\infty, D_\phi)$ is a ϕ-generalized quasi metric space.

Next, we consider the average case time complexity analysis of sorting algorithm called Quicksort, according to (see [21], [22]). Precisely, it yields the following recurrence relation

$$T(n) = \begin{cases} 0, & \text{if } n = 1 \\ \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1), & \text{if } n \geq 2. \end{cases} \tag{7}$$

Consider $\sum^\infty = [0,\infty)$, say the set of nonnegative real numbers. We associate to T the functional $\Theta : \sum^\infty \longrightarrow \sum^\infty$ that corresponds $\Theta(x) := (\Theta(x))_1(\Theta(x))_2 \cdots$ to $x := x_1x_2\cdots$ and is defined as follows:

$$\begin{cases} (\Theta(x))_1 = T1 = 0, \\ (\Theta(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}x_{n-1}, \text{ for all } n \geq 2. \end{cases}$$

Lemma 1.26. *[23] $\ell(\Theta(x)) = \ell(x) + 1$ for all $x \in \sum^\infty$ (in particular, $\ell(\Theta(x)) = \infty$ whenever $\ell(x) = \infty$) and $\ell(\Theta(x \sqcap y)) \leq \ell(\Theta(x) \sqcap \Theta(y)) = \infty$ for all $x,y \in \sum^\infty$.*

Next we show the existence and uniqueness of solution for the recurrence equation (7). We claim that Θ is a k-contraction on the Smyth-complete quasi metric $(\sum^\infty, D_\phi)$, with contraction constant $\frac{1}{2}$.

Case 1. Let $\Theta(x) = \Theta(y)$.

$$D_\phi(\Theta(x),\Theta(y)) = 0 \leq \frac{1}{2}D_\phi(x,y) = kD_\phi(x,y).$$

Case 2. Let $\ell(\Theta(y)) > \ell(\Theta(x))$. Then it follows that $\ell(x) > \ell(y)$.

$$\begin{aligned} D_\phi(\Theta(x),\Theta(y)) &\leq 2^{-\ell(\Theta(y))} = 2^{-\{\ell(y)+1\}} \\ &= \frac{1}{2} \times 2^{-\ell(y)} = kD_\phi(x,y). \end{aligned}$$

Case 3. Let $\ell(\Theta(x)) > \ell(\Theta(y))$. Then it follows that $\ell(x) > \ell(y)$.

$$\begin{aligned} D_\phi(\Theta(x),\Theta(y)) &\leq 2^{-\ell(\Theta(x))} = 2^{-\{\ell(x)+1\}} \\ &= \frac{1}{2} \times 2^{-\ell(x)} = kD_\phi(x,y). \end{aligned}$$

Case 4. Let $\Theta(x) \neq \Theta(y)$, $\ell(\Theta(y)) \geq \ell(\Theta(x))$ or $\ell(\Theta(x)) \geq \ell(\Theta(y))$.

$$\begin{aligned} D_\phi(\Theta(x),\Theta(y)) &= 2^{-\ell(\Theta(x)\sqcap\Theta(y))} \leq 2^{-\ell(\Theta(x\sqcap y))} \\ &= 2^{-\{\ell(x\sqcap y)+1\}} \\ &= \frac{1}{2} \times 2^{-\ell(x\sqcap y)} = kD_\phi(x,y). \end{aligned}$$

Therefore Θ is a k-contraction on $(\sum^\infty, D_\phi)$ with contraction constant $\frac{1}{2}$. So, by corollary 1.22, Θ has a unique fixed point $z = z_1z_2\cdots$, which is obviously the unique solution to the recurrence equation T, that is, $z_1 = 0$ and $z_n = \frac{2(n-1)}{n} + \frac{n+1}{n}z_{n-1}$ for all $n \geq 2$.

1.4.2. An application in delay differential equations

In this section, we prove the existence of periodic solution to delay differential equation as an application of obtained results. We consider the following problem:

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t, x_t),\, t \in I, \\ x(0) &= \eta(0) = x(\eta)(a), \\ x(\theta) &= \eta(\theta),\, \theta \in [-\tau, 0], \end{aligned} \tag{8}$$

where $I = [0, a]$, $a > 0$ and $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function, $x(t) \in \mathbb{R}^n$ and $x_t(\theta) = x(t + \theta)$, $\eta : [-\tau, 0] \longrightarrow \mathbb{R}^n$, that is, $\eta \in C([-\tau, 0], \mathbb{R}^n)$ (the space of all continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}^n$), where $-\tau \leq \theta \leq 0$, $\tau > 0$. It is well known that a delay differential equation from its theory admits a unique solution $x(t) = x(t; \eta)$ of (8) for every Lipschitz continuous function f and every initial condition

$$x(0) = \eta(0),\, x(\theta) = \eta(\theta),\, -\tau \leq \theta \leq 0,$$

and the solution $x(., \eta)$ continuously depends on η and f. For more details on delay differential equations we refer the reader to [24] and the references therein.

Lemma 1.27. *[24] Problem (8) is equivalent to the integral equation:*

$$x_t(\eta)(\theta) = \begin{cases} e^{-\lambda\theta} \int\limits_0^a G(t, s)(f(s, x_s) + \lambda x(s))ds, \text{ if } t + \theta \geq 0, \\ \eta(t + \theta), \text{ if } t + \theta \leq 0, \end{cases} \tag{9}$$

where

$$G(t, s) = \begin{cases} \dfrac{e^{-\lambda(t-s)}}{1 - e^{-\lambda a}}, \text{ if } 0 \leq s \leq t + \theta, \\ \dfrac{e^{-\lambda(t+a-s)}}{1 - e^{-\lambda a}}, \text{ if } t + \theta \leq s \leq a. \end{cases} \tag{10}$$

Define $\|x\| = \sup\limits_{t \in I} |x(t)|$ *for an arbitrary* $x \in C([0, a], \mathbb{R}^n)$ *where*

$$|x(t)| = |x_1(t), x_2(t), ..., x_n(t)| = \sqrt{x_1^2(t) + x_2^2(t) + ... + x_n^2(t)}$$

Then $(C([0, a], \mathbb{R}^n), \|\cdot\|)$ *is a Banach space endowed with the* ϕ*-generalized quasi metric* D_ϕ *defined by*

$$D_\phi(x, y) = \begin{cases} 2 \sup_{t \in I} \left| x(t) - y(t) \right|, \text{ if } |x(t)| > |y(t)|, \\ \sup_{t \in I} \left| x(t) - y(t) \right|, \text{ if } |y(t)| \geq |x(t)|, \end{cases} \tag{11}$$

where $x, y \in C([0, a], \mathbb{R}^n)$*. Note that* $(C([0, a], \mathbb{R}^n), D_\phi)$ *is a generalized Smyth-complete quasi metric space. Define*

$$S = \{x \in C([0, a], \mathbb{R}^n) : x(0) = \eta(0) = x(a), \eta \in C([-\tau, 0], \mathbb{R}^n)\}$$

and

$$x(t) = \begin{cases} e^{-\lambda\theta} \int\limits_0^a G(t - \theta, s)(f(s, x_s) + \lambda x(s))ds, \end{cases} \tag{12}$$

where $G(t - \theta, s)$ *is defined in (10). For each* $x \in S$*, set* $f(x)(t) = f(t, x_t)$*, where*

$$x_t(\theta) = \begin{cases} x(t + \theta), \text{ if } 0 \leq t + \theta \leq a, \\ \eta(t + \theta), \text{ if } -\tau \leq t + \theta \leq 0. \end{cases} \tag{13}$$

Let us consider the operator for each $x \in S$ as follows:

$$T(x)(t) = e^{-\lambda\theta} \int_0^a G(t-\theta, s)(f(x)(s) + \lambda x(s))ds. \tag{14}$$

Now make the following assumption that are necessary to prove the existence and uniqueness of solution of (8). There exists $\lambda > 0$, $\mu > 0$ and $\mu < \lambda$ such that

$$\begin{cases} \left\|(f(t,x) + \lambda x(0)) - (f(t,y) + \lambda y(0))\right\| \leq \frac{\mu}{2} D_\phi(x,y), \text{ if } |x(t)| > |y(t)|, \\ \left\|(f(t,x) + \lambda x(0)) - (f(t,y) + \lambda y(0))\right\| \leq \mu D_\phi(x,y), \text{ if } |y(t)| \geq |x(t)|, \end{cases} \tag{15}$$

for all $x, y \in S$.

Theorem 1.28. *If assumption (15) is satisfied, then there exists a unique solution to (8).*

Proof. For each $x, y \in S$, we get

$$\begin{aligned} \left\|T(x)(t) - T(y)(t)\right\| &= \left\| e^{-\lambda\theta} \int_0^a G(t-\theta, s)\left(f(x)(s) + \lambda x(s) - (f(y)(s) + \lambda y(s))\right)ds \right\| \\ &\leq e^{-\lambda\theta} \int_0^a \|G(t-\theta, s)\| \left\|f(x)(s) + \lambda x(s) - (f(y)(s) + \lambda y(s))\right\| ds \\ &\leq e^{-\lambda\theta} \int_0^a G(t-\theta, s) \left\|f(x)(s) + \lambda x(s) - (f(y)(s) + \lambda y(s))\right\| ds. \end{aligned}$$

where
$G(t-\theta, s) = \dfrac{e^{-\lambda(t-\theta-s)}}{1-e^{-\lambda a}}$, if $0 \leq s \leq t$, and $G(t-\theta, s) = \dfrac{e^{-\lambda(t-\theta+a-s)}}{1-e^{-\lambda a}}$, if $t \leq s \leq a$.

Case 1. Let $|x(t)| > |y(t)|$ and using assumption 15.

$$\begin{aligned} \left\|T(x)(t) - T(y)(t)\right\| &= D_\phi(T(x), T(y)) \\ &\leq \frac{\mu}{2} D_\phi(x,y) e^{-\lambda\theta} \int_0^a G(t-\theta, s)ds \\ &= \frac{\mu}{2} D_\phi(x,y) \left[e^{-\lambda\theta} \int_0^t \frac{e^{-\lambda(t-\theta-s)}}{1-e^{-\lambda a}} ds + e^{-\lambda\theta} \int_t^a \frac{e^{-\lambda(t-\theta+a-s)}}{1-e^{-\lambda a}} ds \right] \\ &= \frac{\mu}{2} D_\phi(x,y) \left[\int_0^t \frac{e^{\lambda(s-t)}}{1-e^{-\lambda a}} ds + \int_t^a \frac{e^{\lambda(s-t-a)}}{1-e^{-\lambda a}} ds \right] \\ &= \frac{\mu}{2\lambda} D_\phi(x,y) = k D_\phi(x,y), \text{ where } k = \frac{\mu}{2\lambda} < 1. \end{aligned}$$

Case 2. Let $|y(t)| \geq |x(t)|$ and using assumption 15.

$$\begin{aligned}
\left\|T(x)(t) - T(y)(t)\right\| &= D_{\phi}(T(x), T(y)) \\
&\leq \mu D_{\phi}(x, y) e^{-\lambda\theta} \int_{0}^{a} G(t-\theta, s) ds \\
&= \mu D_{\phi}(x, y) \left[e^{-\lambda\theta} \int_{0}^{t} \frac{e^{-\lambda(t-\theta-s)}}{1-e^{-\lambda a}} ds + e^{-\lambda\theta} \int_{t}^{a} \frac{e^{-\lambda(t-\theta+a-s)}}{1-e^{-\lambda a}} ds \right] \\
&= \mu D_{\phi}(x, y) \left[\int_{0}^{t} \frac{e^{\lambda(s-t)}}{1-e^{-\lambda a}} ds + \int_{t}^{a} \frac{e^{\lambda(s-t-a)}}{1-e^{-\lambda a}} ds \right] \\
&= \frac{\mu}{\lambda} D_{\phi}(x, y) = k D_{\phi}(x, y), \text{ where } k = \frac{\mu}{\lambda} < 1.
\end{aligned}$$

In both cases, we have $D_{\phi}(T(x), T(y)) \leq kD_{\phi}(x, y)$. All the conditions of Corollary 1.22 are satisfied. Consequently problem (8) has a unique solution.

□

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