University of Pretoria

Department of Mathematics and Applied Mathematics

Master's Thesis
July 13, 2017

Existence results for a class of semi-linear initial value problems

Author:
Gray Manicom
10147722

Supervisor:
Dr. Janharm van der Walt

Existence results for a class of semi-linear initial value problems

by
Gray T. Manicom
10147722

Submitted in partial fulfilment of the requirements for the degree
Magister Scientiae
in Applied Mathematics

in the Faculty of the Natural and Agricultural Sciences
University of Pretoria
Pretoria
July 13, 2017

Supervised by

Dr. Janharm van der Walt
Department of Mathematics and Applied Mathematics

Declaration

I, declare that the thesis, which I hereby submit for the degree Magister Scientiae in Applied Mathematics at the University of Pretoria, is my own work and has not been previously submitted by me for a degree at this or any other tertiary institution.

Signature: ....

Date: ....

Summary

The main result of this thesis is an existence result for parabolic semi-linear problems. This is done by reformulating the semi-linear problem as an abstract Cauchy problem

$$\begin{aligned} u_t(t) &= Au(t) + f(t, u(t)),\ t > 0 \\ u(0) &= u_0 \end{aligned} \tag{1}$$

for $u_0 \in X$, where X is a Banach space. We then develop and use the theory of compact semigroups to prove an existence result.

In order to make this result applicable, we give a characterization of compact semigroups in terms of its resolvent operator and continuity in the uniform operator topology. Thus, using the theory of analytic semigroups, we are able to determine under what conditions on A a solution to (1) exists.

Furthermore, we consider the asymptotic behaviour and regularity of such solutions. By developing perturbation theory, we are easily able to apply our existence result to a larger class of problems. We then demonstrate these results with an example.

This work is significant in providing a novel approach to a group of previously established results. The content can be considered pure mathematics, but it is of significant importance in real world situations. The structure of the thesis, and the choice of certain definitions, lends itself to be easily understood and interpreted in the light of these real world situations and is intended to be easily followed by an applied mathematician. An important part of this process is to develop the problem in a real Hilbert space and then to consider the complexification of the problem in order to reset it in a complex Hilbert space, in which we can apply the theory of analytic semigroups. A large number of real world problems fall into the class of problems discussed here, not only in biology as demonstrated, but also in physics, chemistry, and elsewhere.

I would like to thank:

My supervisor, Dr. Jan Harm van der Walt, for his patience and guidance;

My family, friends and church, for their support and encouragement;

The University of Pretoria, for affording me this opportunity;

The SARChI chair at the University of Pretoria, for providing me with funding.

Dedicated to the late Brian Manicom.

Contents

Chapter 1

Introduction

1.1 Aims of the Thesis

The main aim of this thesis is to prove an existence result for a class of semi-linear abstract Cauchy problems. We do this by first reformulating the problem as an infinite-dimensional dynamical system,

$$
\begin{aligned}
u_t(t) &= Au(t) + f(t, u(t)),\ t > 0 \\
u(0) &= u_0,
\end{aligned}
\tag{1.1}
$$

and then developing and using semigroup theory to determine conditions under which a solution to (1.1) will exist. Furthermore, we aim to use semigroup theory to prove a regularity result for the solution to our problem and determine the asymptotic behaviour of the solution.

We shall then demonstrate the applicability of these results to parabolic semi-linear PDEs with an example.

1.2 Structure of the Thesis

The first chapter introduces the topic of the thesis and, in Section 1.3, discusses why the theory developed in the thesis is important. Basic results from the theory of C_0-semigroups are given in Section 1.4. In Section 1.5 we state the problem, that is, a semi-linear PDE, and reformulate it as an abstract Cauchy problem. We then discuss two types of solutions, classical and mild, to the abstract Cauchy problem.

In Chapter 2 we begin by investigating compact semigroups and giving a characterization of such semigroups in terms of the compactness of the resolvent operator and the continuity of the semigroup in the uniform operator topology. This provides motivation for Section 2.2 in which we show that elliptic operators acting on the Hilbert space $\mathcal{L}^2(\Omega)$ have compact resolvent operators, and Section 2.3 in which we discuss sectorial operators and define analytic semigroups in terms of these operators. We prove basic properties of analytic semigroups. In Section 2.4 we prove perturbation results concerning strongly continuous and analytic semigroups. We proceed to define the complexification of a Hilbert space and the complexification of an operator on that Hilbert space in Section 2.5, and as a result we determine which characteristics of a semigroup or infinitesimal generator are preserved when moving, in the way defined, from a real Hilbert space to its complexification.

Chapter 3 is dedicated to the existence, asymptotic behaviour and regularity of mild solutions to the abstract Cauchy problem (1.1). We start, in Section 3.1, by showing that classical solutions are mild solutions. We then prove an existence result for a local mild solution to the abstract Cauchy problem (1.1) in Section 3.2, using the theory of compact semigroups developed in Section 2.1. In Section 3.3 we determine conditions on f under which the local mild solution to (1.1) is a global mild solution. Furthermore, in Section 3.4 we prove a regularity result to show under what conditions a mild solution will be a classical solution. This result is based on the theory of analytic semigroups developed in Section 2.3.

In Chapter 4 we demonstrate the applicability of these results with a biological example. We state the problem and then reformulate it into the form of the abstract Cauchy problem in Section 4.1. In Section 4.2 we apply our existence, asymptotic behaviour and regularity results from Chapter 3 to the example problem. We then discuss how the perturbation theory developed in Section 2.4 allows us to do the same with a broader class of problems in Section 4.3.

We conclude the thesis in Chapter 5 by asking whether or not the aims of the thesis were met.

We give three appendices. Appendix A contains some basic results and definitions regarding integration, in particular, the Bochner integral for the integration of vector-valued functions and Cauchy's Integral Formula. In Appendix B we consider Hölder continuous functions, relevant to our regularity results. Our final appendix, Appendix C, contains miscellaneous results needed throughout the thesis, but which do not contribute to the main aims or flow of the thesis.

1.3 Why Semigroups?

Jean-Baptiste Alphonse Karr is famous for saying "plus ça change, plus c'est la même chose", usually translated as "the more things change, the more they stay the same."From a mathematical perspective we assume that the way in which things change stays the same, allowing us to model such change as dynamical systems.

In the physical realm things are almost always changing, yet, for most of history, geometric and algebraic methods were used to deal with problems that are essentially static. Indeed, prior to the development of Calculus by Newton and Leibniz in the latter half of the 17th century, the only types of motion that could be described in a mathematically precise manner were those of a particle moving at uniform velocity along a straight line and a particle moving at constant angular momentum along a circular path. However, differential and integral Calculus provided the tools for the exact mathematical formulation of dynamic problems.

Since then mathematics has developed as a tool to help us understand the phenomenon of change over time by building models, most often differential equations, to try and understand the nature of that change and to predict how the object in question would change under certain conditions. In the context of differential equations such conditions are usually formulated as initial and boundary conditions. Mathematical analysis is done on the model to test whether or not a reasonably realistic solution of the formulated model exists. The insights from our analysis are used either to solve our problem exactly or to compute an approximate solution. These models and their solutions must be tested against empirical data in order to validate them. If there is some sense of validity, then further analysis or refining of the model can be done.

One of the main insights in the development of the mathematical theory of differential equations is that such equations can be expressed in operator theoretic terms; that is, in terms of functions acting on functions. This allows complicated problems in finite dimensions to be transformed into formally simpler problems in higher, or infinitely many, dimensions. Peano [19] was the first to do this explicitly, by expressing a system of first order linear ODEs in matrix form as

$$\begin{aligned} \bar{x}'(t) &= A\bar{x}(t) + f(t),\ t > 0 \\ \bar{x}(0) &= \bar{x}_0, \end{aligned} \tag{1.2}$$

and solving it using the explicit formula

$$\bar{x}(t) = e^{tA}\bar{x}_0 + \int_0^t e^{(t-s)A} f(s)ds \tag{1.3}$$

where

$$e^{tA} = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!}. \tag{1.4}$$

In the case of PDEs, the situation is much more complex. Consider an evolution equation, such as the one-dimensional heat equation

$$u_t(x,t) = \gamma u_{xx}(x,t),\ 0 < x < 1,\ t > 0 \tag{1.5}$$

with initial and boundary conditions given, for instance, by

$$u(x,0) = u_0(x),\ 0 < x < 1 \tag{1.6}$$

and

$$u(0,t) = u(1,t) = 0,\ t > 0. \tag{1.7}$$

The initial-boundary value problem (1.5)-(1.7) describes the relationship between the change in heat over time and the distribution of heat over space. In the case of an ODE such as (1.2) the state of the system at any given time $t > 0$ is given by $\bar{x}(t) \in \mathbb{R}^n$. On the other hand, for the problem (1.5)-(1.7), the state of the system at time $t > 0$ is given by a function

$$\bar{u}(t) : [0,1] \ni x \mapsto \bar{u}(t)[x] := u(x,t) \in \mathbb{R}.$$

Therefore, when expressing the initial-boundary value problem (1.5)-(1.7) as a dynamical system

$$\begin{aligned} \bar{u}'(t) &= A\bar{u}(t),\ t > 0 \\ \bar{u}(0) &= \bar{u}_0, \end{aligned} \tag{1.8}$$

the operator A acts on some infinite dimensional function space X, with $u_0 \in X$. The boundary condition (1.7) is typically incorporated into the definition of $\mathcal{D}(A) \subseteq X$.

For the ODE (1.2), the solution (1.3) can be expressed in terms of the operator (1.4). Two difficulties prevent us from adopting the same approach for the dynamical system (1.8). Firstly, the operator A is typically unbounded. Secondly, in most cases $\mathcal{D}(A)$ is not the whole of X. For these reasons the series (1.4) need not converge.

Semigroups were introduced to deal with this dilemma. The first time the term *semigroup* was formerly used was in 1904 [10, page vi], but only from the 1930s-1950s was the basic theory developed, with major contributions by Einar Hille [11], Ralph Phillips [12], Kôsaku Yosida [27] and Dunford and Schwartz [8]. However, it was not until the work of McIntosh in the 1980s that the functional calculus for sectorial operators was introduced, [3, page 101].

The basic idea underlying semigroup theory is the following: If A is an operator acting on X, we define a function

$$[0, \infty) \ni t \mapsto e^{tA} \in B(X),$$

where $B(X)$ is the set of all bounded linear operators acting on X. This function has all the main properties of e^{tA} when A is a bounded linear operator. In particular, $e^{0A} = I$, if $t, s > 0$ then $e^{(t+s)A} = e^{tA}e^{sA}$, and the right-hand time derivative of e^{tA} is Ae^{tA}. We call the family of operators e^{tA}, $t \geq 0$, a semigroup, and the operator A the infinitesimal generator of that semigroup. If we use $u(t)$ in place of $\bar{x}(t)$, u_0 in the place of $\bar{x}_0$, and the semigroup e^{tA} in place of (1.4), then (1.3) defines a solution to (1.5)-(1.7), given that $\lim_{t\to 0^+} e^{tA}u_0 = u_0$.

This raises some questions: How does the semigroup e^{tA} behave? Is e^{tA} bounded, and in what sense? What can we deduce about e^{tA} as a function of time based on our knowledge of A? To help answer these questions, consider a general initial value problem

$$\begin{aligned} u_t(t) &= Au(t) + f, \, t > 0 \\ u(0) &= u_0, \end{aligned} \tag{1.9}$$

for a function $f : [0, \infty) \mapsto X$, with X some Banach space, $A : X \supset \mathcal{D}(A) \mapsto X$ a linear operator, and $u_0 \in X$.

If the problem is linear, that is, if $f = f(t)$ does not depend on $u(t)$, then the initial-value problem is *well-posed* if and only if A generates a C_0-semigroup, see Section 1.4. However, semigroups provide the added benefit that the solution to problem (1.9) is continuously dependent on A [10, Theorem IV]. Furthermore, perturbation theorems concerning semigroups allow you to easily characterize and find solutions to (1.9) when A is changed in appropriate ways. For example if A generates a C_0-semigroup and B is a bounded operator then $A + B$ generates a C_0-semigroup.

Even with the aforementioned benefits of semigroup theory when applied to linear problems, semigroup theory is particularly useful when applied to non-linear problems. This is because, to a large degree, the "linearity is irrelevant"-[10, Page 9]. Goldstein justifies this remarkable claim further, for example in [10, Theorem I'] and the Crandall-Liggett Theorem [10, Theorem II']. These theorems allow us to define a C_0-semigroup dependant only on properties of A and thus prove that a well-posed solution exists for any f, linear or non-linear, in the domain of A. Indeed, most of the theory for linear problems can be adopted for non-linear problems with little effort, given that the space you are working in is reflexive, see for example, [18, Chapter 6]. Even if the space is not reflexive, some sense of well-posedness of the solution can still be obtained, but with a little more work, and a little more compromise.

Semigroup theory has a wide range of applications to both linear and non-linear Partial Differential Equations. In physics it has proven particularly useful in quantum mechanics. As an example, consider Schroedinger's Equation. The linear problem is dealt with in [18, Section 7.5], and the non-linear one in [18, Section 8.1] and [7, Chapter 14]. Scattering Theory is discussed in [10, Section 2.14] and nuclear stability in [5, Section 6.3]. It has also been proven useful in continuum mechanics, such as the Navier-Stokes equation [10, Sections 15.19 and 15.25], the Korteweg-de Vries equation [18, Section 8.5] and vibration models [23]. In probability theory it is helpful in dealing with Markov processes, see for example [10, Sections 15.8-15.12] and [7, Chapter 12]. In other fields semigroup theory may be applied to Fischer's model for invasive species and models for spatial patterns in biology and ecology as in [15, Part III]. Indeed, sectorial operators and the analytic semigroups they generate have become central to our understanding of parabolic problems. In this thesis we consider the human mechanism of perspiration. It is worthwhile noting that, although this work focusses on the contribution of semigroup theory to Partial Differential Equations, semigroup theory has also made a large contribution to other areas of functional analysis.

For an extensive list of references on either theory or application of semigroup theory and its historical development, we refer the reader to the bibliographies and corresponding notes and historical remarks contained in both Pazy [18] and Goldstein [10].

1.4 C_0-Semigroups

In this section, for the convenience of the reader, we gather together the basic notions and results relating to C_0-semigroups. These notions and results are typically treated in any first course on the topic of semigroups, and is therefore considered as pre-knowledge. All the given results can be found in [18, Sections 1.1-1.5], unless otherwise indicated, and we refer the reader to this work for a more comprehensive discussion on C_0-semigroups. We develop our semigroup theory for operators acting on Banach spaces.

Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. For any $\bar{z} = x + iy \in \mathbb{C}$ we denote the real part of $\bar{z}$ by $\Re\bar{z}$, that is, $\Re\bar{z} = x$, and the imaginary part of $\bar{z}$ by $\Im\bar{z}$, that is, $\Im\bar{z} = y$.

Definition 1.4.1. (**Semigroup**) A family of bounded linear operators $T(t)$ on X, defined for $t \geq 0$, is called a *semigroup* on X if

$$\begin{aligned} &T(t+s) = T(t)T(s),\ t, s \geq 0 \\ &T(0) = I. \end{aligned}$$

Definition 1.4.2. (**Infinitesimal Generator**) Consider a semigroup $T(t)$, $t \geq 0$, on X. For $h > 0$, define the linear operator A_h by

$$A_h x = h^{-1}(T(h) - I)x,\ x \in X.$$

Let

$$\mathcal{D}(A) = \left\{ x \in X \,\middle|\, \lim_{h \to 0^+} A_h x \text{ exists} \right\}$$

and define the operator A with domain $\mathcal{D}(A)$ by

$$Ax = \lim_{h \to 0^+} A_h x,\ x \in \mathcal{D}(A).$$

The operator A is called the *infinitesimal generator* of $T(t)$.

Definition 1.4.3 (**Strongly Continuous Semigroup**)**.** A semigroup $T(t)$, $t \geq 0$, is called strongly continuous if

$$\lim_{h \to 0^+} T(h)x = x \text{ for each } x \in X.$$

A strongly continuous semigroup is also called a C_0-semigroup.

Definition 1.4.4. (**Uniformly Continuous Semigroup**) A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t \to 0^+} \|T(t) - I\| = 0.$$

Uniformly continuous semigroups are clearly C_0-semigroups, and if X is a finite dimensional Banach space, then strongly continuous semigroups are uniformly continuous. A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if it is bounded, [18, Theorem 1.2]. However, in applications to PDE's, the infinitesimal generator is rarely bounded, thus we will deal with strongly continuous semigroups, which may be generated by unbounded operators.

Theorem 1.4.1. *If $T(t)$, $t \geq 0$, is a C_0-semigroup, then there exist real numbers $M \geq 1$ and $\beta > 0$ such that*

$$\|T(t)\| \leq Me^{\beta t}$$

for all $t \geq 0$.

Corollary 1.4.2. *If $T(t)$, $t \geq 0$, is a C_0-semigroup, then for each $x \in X$ the function $\mathbb{R}^+ \ni t \mapsto T(t)x \in X$ is continuous.*

Theorem 1.4.3. *Let $T(t)$, $t \geq 0$, be a C_0-semigroup and let A be its infinitesimal generator. Then the following results hold:*

(i) *For $x \in X$ and $t \geq 0$,*

$$\lim_{h \to 0^+} h^{-1} \int_t^{t+h} T(s)x ds = T(t)x.$$

(ii) *For $x \in X$ and $t > 0$, $\int_0^t T(s)x ds \in \mathcal{D}(A)$ and*

$$A \int_0^t T(s)x ds = T(t)x - x.$$

(iii) *For $x \in \mathcal{D}(A)$ and $t \geq 0$, $T(t)x \in \mathcal{D}(A)$ and*

$$\frac{d}{dt}[T(t)x] = AT(t)x = T(t)Ax.$$

(iv) *For $x \in \mathcal{D}(A)$ and $t, s \geq 0$,*

$$T(s)x - T(t)x = \int_t^s AT(\tau)x d\tau = \int_t^s T(\tau)Ax d\tau.$$

Here it is worthwhile recalling the meaning and intuition underlying the definition of a semigroup as mentioned in the introduction, and some of the properties listed in Theorem 1.4.3. First note that the properties of a C_0-semigroup, particularly those from Definition 1.4.1 and (iii) of Theorem 1.4.3, correspond to the properties of the well-known operator-valued function $S(t) = e^{tA}$, defined when A is bounded. This reminds us of the use of semigroups in providing the existence of solutions to PDEs of the form $u_t = Au$. Indeed, for any $u_0 \in \mathcal{D}(A)$, if $T(t)$, $t \geq 0$, is a C_0-semigroup then the function $u(t) = T(t)u_0$ for $t \geq 0$ is a classical solution of the initial value problem $u_t = Au$ with initial condition $u(0) = u_0$.

Since the abstract Cauchy problem (1.13) can be formulated in terms of the weak time derivative of $u(t)$, one might ask what happens if we take the weak derivative in (iii). Pazy shows in [18, Section 2.1] that if we define the A_w as the weak right-hand derivative of a C_0-semigroup $T(t)$, $t \geq 0$, with infinitesimal generator A, that is, A is the strong right-hand derivative of $T(t)$, then $A = A_w$.

Theorem 1.4.4 (Uniqueness of Infinitesimal Generators). *Let $T(t)$, $t \geq 0$, and $S(t)$, $t \geq 0$ be semigroups of bounded linear operators with infinitesimal generators A and B respectively. Then $A = B$ if and only if $T(t) = S(t)$ for all $t \geq 0$.*

Theorem 1.4.5. *If A is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, then $\mathcal{D}(A)$ is dense in X and A is a closed linear operator.*

Since in practice we need to determine the character of the semigroup from the properties of its infinitesimal generator, we are more interested in whether the converse of Theorem 1.4.5 is true. We shall see from the Hille-Yosida Theorem to come that the denseness of $\mathcal{D}(A)$ in X will always be required for C_0-semigroups, and A will always need to be closed.

However, since A is unbounded, it is difficult to use properties of A directly to determine characteristics of the semigroup of bounded operators $T(t)$, $t \geq 0$, that it generates, if indeed it is the infinitesimal generator of a semigroup. The most basic way to relate an unbounded operator A with a bounded operator is to consider the inverse A^{-1}. However, this may not always be defined, and it may not always help. In our case we consider the resolvent operator, $(\lambda I - A)^{-1}$, for $\lambda \in \mathbb{K}$ such that the operator exists. The additional requirements for an operator to generate a C_0-semigroup depend on the resolvent operator, defined below.

Definition 1.4.5 (Resolvent Set). For an operator A on X, the resolvent set $\rho(A)$ is defined as

$$\rho(A) = \{\lambda \in \mathbb{K} \mid (\lambda I - A)^{-1} \in B(X)\}.$$

That is, the resolvent set is the set of all $\lambda \in \mathbb{K}$ such that $(\lambda I - A)^{-1}$ exists and is a bounded linear operator on X.

Definition 1.4.6 (Resolvent Operator). Let A be an operator on X. The resolvent operator $R(\lambda, A)$ is defined for $\lambda \in \rho(A) \subset \mathbb{K}$ as

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

Theorem 1.4.6 (The Resolvent Identity). *[10, Chapter 1 1.3] If $\lambda, \mu \in \rho(A)$ then*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

Theorem 1.4.7. *[10, Chapter 1 1.2] Suppose that $A : X \supset \mathcal{D}(A) \mapsto X$. If the resolvent set $\rho(A)$ of A is non-empty then A is closed. Furthermore, if A is closed then*

$$\rho(A) = \{\lambda \in \mathbb{K} \mid (\lambda I - A) \textit{ is a bijection}\}.$$

Theorem 1.4.8 (Hille-Yosida). *Suppose that A is a linear operator on X. There exists an $M \geq 1$ and an $\omega \in \mathbb{R}$ such that A is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, satisfying $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$, if and only if*

(i) A is closed and $\overline{\mathcal{D}(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains the ray (ω, ∞) and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|} \text{ for } \lambda > \omega.$$

It is worthwhile noting that to prove the Hille-Yosida Theorem we form a sequence (e^{tA_n}), where A_n is a bounded operator for each $n \in \mathbb{N}$, which converges to the semigroup $T(t)$ generated by the unbounded operator A.

Traditionally the Hille-Yosida theorem has $M = 1$ and provides necessary and sufficient conditions for generating a contraction C_0-semigroup, that is, where $\|T(t)\| \leq 1$ for $t \geq 0$. This is sufficient for most purposes, for if we have a semigroup $T(t)$, $t \geq 0$ and $\|T(t)\| \leq M$ where $M \geq 1$, then we can find an equivalent norm $\|\cdot\|_1$ on $B(X)$ such that $\|T(t)\|_1 \leq 1$ for all $t \geq 0$. That is, $T(t)$ is a contraction semigroup with respect to $\|\cdot\|_1$, see [10, Exercise 2.19]. Here, however, we deal with exponentially bounded semigroups.

1.5 Problems and Solutions

1.5.1 Problems

Semi-linear Parabolic Problems

Let Ω be an open and bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. For some fixed $T > 0$ let $\Omega_T = \Omega \times (0, T)$. The initial-value problem is given by

$$\begin{aligned} u_t(x,t) &= Au(x,t) + f(x,t,u(x,t)), && (x,t) \in \Omega_T \\ u(x,0) &= u_0(x), && x \in \Omega \\ u(x,t) &= g(x), && x \in \partial\Omega,\ 0 < t < T. \end{aligned} \tag{1.10}$$

The function $f : \Omega_T \times \mathbb{R} \mapsto \mathbb{R}$ is taken to be continuous and second-order Lebesgue integrable over its domain. The boundary $\partial\Omega$ is taken to be smooth. The linear operator A is taken to be a second-order partial differential operator in divergence form

$$Au(x,t) = \sum_{i,j=1}^{n} (\alpha^{ij}(x)u(x,t)_{x_i})_{x_j} + \sum_{i=1}^{n} \beta^i(x)u(x,t)_{x_i} + \gamma(x)u(x,t) \tag{1.11}$$

for coefficient functions $\alpha^{ij} \in \mathcal{C}^1(\bar{\Omega})$, $\beta^i \in \mathcal{L}^\infty(\Omega)$ and $\gamma \in \mathcal{L}^\infty(\Omega)$ for $i, j = 1, ..., n$. We further assume that the operator $\dfrac{\partial}{\partial t} - A$ is parabolic, that is, the operator A is uniformly elliptic. Thus there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} \alpha^{ij}(x)\zeta_i\zeta_j \geq \theta\|\zeta\|^2 \tag{1.12}$$

for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$. In this case A is a generalization of the Laplacian ∇^2.

In dealing with problems of this sort it is worth keeping a physical interpretation in mind. Parabolic problems of this type are often used to model diffusion processes. Here $u(\cdot, t)$ represents the concentration or density profile of a substance within a particular medium at a time $t \in (0, T)$. The shape of the medium determines the spatial domain Ω of u, the boundary conditions are given by g and the initial condition u_0 describes the concentration profile at time $t = 0$.

The second-order term $\sum_{i,j=1}^{n}(\alpha^{ij}u_{x_i})_{x_j}$ of our operator A determines how the concentration u *diffuses* or *coalesces* in the medium through space and with time. The ellipticity assumption forces diffusion from regions with a high concentration to regions with a low concentration. This agrees with most physical examples like energy or chemical flow, but might not be appropriate for cash flow, as, no doubt, most people understand full well. Some mediums would allow rapid diffusion, others not, and some may allow more rapid diffusion in particular directions. How the medium affects diffusion is described by the coefficient functions α^{ij}. The first-order term $\sum_{i=1}^{n}\beta^i u_{x_i}$ determines the *transport* of the concentration u through the spatial domain Ω. This is not to be confused with the flow due to diffusion through the medium, but is due to some forces that act on the substance, either due to the nature of its interaction with the medium or some external input. The zeroth-order term γu describes the direct *creation* or *depletion* of the substance within the medium.

The function f represents some external factor which affects the distribution profile of the substance, and can itself be dependent on the concentration profile.

As an example, consider squirting oil into the bottom of a glass of water with a syringe. The body of water contained in the glass is the spatial domain Ω. The function $u(x, t)$ represents the concentration of oil at a point $x \in \Omega$ and at time $t \geq 0$. The initial concentration profile is u_0. The oil will float to the top due to the fact that it has a lower density than the water. This is the transport of the oil to the surface. The rate at which the oil flows to the surface is captured by the functions β^i. However, as the oil rises in the water, the oil stream spreads out in different directions. This is the diffusion of the oil in the water. The behaviour of the diffusion of oil through the water is determined by the functions α^{ij}. If the water were replaced with something more dense, the diffusion would likely be slower, and different functions α^{ij} would be needed. If we kept squirting oil from the syringe into the water for $t > 0$, perhaps at different places, then the impact of that on the overall concentration of oil in the cup would be described by the function γ. If we stirred the water then the oil would also be transported by the current, and this would likely affect both the diffusivity and transport terms. If we were to add some substance that chemically reacts with the oil, so that the amount of reaction depends on the concentration of the oil, then that contribution would be determined by the function f.

Although this is a very simple example of a semi-linear parabolic problem, to truly be able to understand or predict the behaviour of oil squirted into a glass of water is no mean feat, and a large body of sophisticated mathematics is needed in order to analyse these types of problems.

The Abstract Cauchy Problem

One way to formulate problems of this type is as an infinite-dimensional dynamical system, or abstract Cauchy problem. Let X be a Banach space of functions defined on Ω, and let U be an open subset of X. Define $u(t) : [0, T) \mapsto X$ as

$$u(t) : x \in \Omega \mapsto u(t)[x] := u(x, t)$$

and $f(t, u) : [0, T) \times U \mapsto X$ as

$$f(t) : x \in \Omega \mapsto f(t, u)[x] := f(x, t, u(x, t)).$$

The function $u(t)$ is called the *state* of the system at time t. We can now write our problem in the form

$$\begin{aligned} &u_t(t) = Au(t) + f(t, u(t)),\ t \in (0, T) \\ &u(0) = u_0 \end{aligned} \tag{1.13}$$

where $u_0 \in X$ and $A : X \supset \mathcal{D}(A) \mapsto X$. The main aim of this thesis is to prove an existence result for this more general form of problem using semigroup theory and to apply this result to a concrete example of a semi-linear parabolic problem.

1.5.2 Solutions

We now consider two different types of solutions to the abstract Cauchy problem (1.13).

Definition 1.5.1. (**Classical Solution**) We say that $u : [0, T) \mapsto X$ is a *classical solution* to (1.13) if $u(t) \in \mathcal{D}(A)$ for $0 < t < T$, $u(t)$ is continuous on $[0, T)$, and $u(t)$ is continuously differentiable on $(0, T)$. Furthermore, $u_t(t) = Au(t) + f(t, u(t))$ for $0 < t < T$ and $u(0) = u_0$.

In practice a classical solution, while very desirable, might not exist. In these cases, we are obliged to consider weaker notions of solution. In this thesis our existence theorem provides sufficient conditions for the existence of a local mild solution.

Definition 1.5.2. (**Mild Solution**) Suppose the linear operator A from problem (1.13) is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$. We say that $u : [0, T) \mapsto X$ is a *mild solution* to problem (1.13) if

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds \tag{1.14}$$

for $0 \leq t < T$.

Note that if u is a mild solution to (1.13) and f is continuous then $u(t) \in \mathcal{D}(A)$ for $0 < t < T$, u is continuous on $[0, T)$ and $u(0) = u_0 \in \overline{\mathcal{D}(A)}$. However, u need not be differentiable. Thus a mild solution is not a classical solution, even though the converse is true, as shown in Section 3.1. Indeed a mild solution is a classical solution if and only if it is continuously differentiable, see [2, Propositions 3.1.2 and 3.1.9].

Chapter 2

Compact and Analytic Semigroups

Many different existence theorems for the abstract Cauchy problem (1.13) exist, each with their own conditions, and we refer the reader to the introductions of [17], or, more recently, [6], for an extensive list of references. Most of these theorems add restrictions to the function f, requiring it to be locally Lipschitz, monotone or have some sort of relationship to the fractional powers of the operator A. However, in this thesis we consider a restriction on the semigroup $T(t)$, requiring it to be compact, and the only demand on f is that it is continuous.

There is no useful characterization of a compact semigroups in terms of its infinitesimal generator. However we do have a characterization in terms of another property of the semigroup. A semigroup is compact if and only if the semigroup is continuous in the uniform operator topology for $t > 0$ and the resolvent operator is compact for every $\lambda \in \rho(A)$, as we show in Theorem 2.1.6.

In practice this compactness theorem is of limited use, since it requires $T(t)$ to be continuous in the uniform operator topology for $t > 0$. Therefore, using the compactness theorem requires us to know something about the semigroup, when in applications we only have access to information regarding its infinitesimal generator. Thus we need to find a way to determine whether a semigroup is uniformly continuous with respect to the uniform operator norm in terms of the infinitesimal generator A.

The idea then is to determine the characteristics of A needed in order for it to be the infinitesimal generator of an *analytic semigroup*. The theory of analytic semigroups has proven to be particularly useful when applied to semi-linear parabolic problems, see [17], [18, Chapter 6], or [6] for examples. Analytic semigroups act on complex Banach spaces and are functions of a complex variable. Importantly for us, are continuous with respect to the uniform operator norm for $t > 0$. They are also useful in proving regularity results of the solution, as we show in Section 3.4.

However, in many applications, including those we deal with in this thesis, the dynamical system is defined on a real Banach space, often a Hilbert space. A standard approach in such cases is to study the extension of the operator A, acting on a real space X, to the complexification $\tilde{X}$ of X. The general idea of this approach is shown in the Figure 2.1.

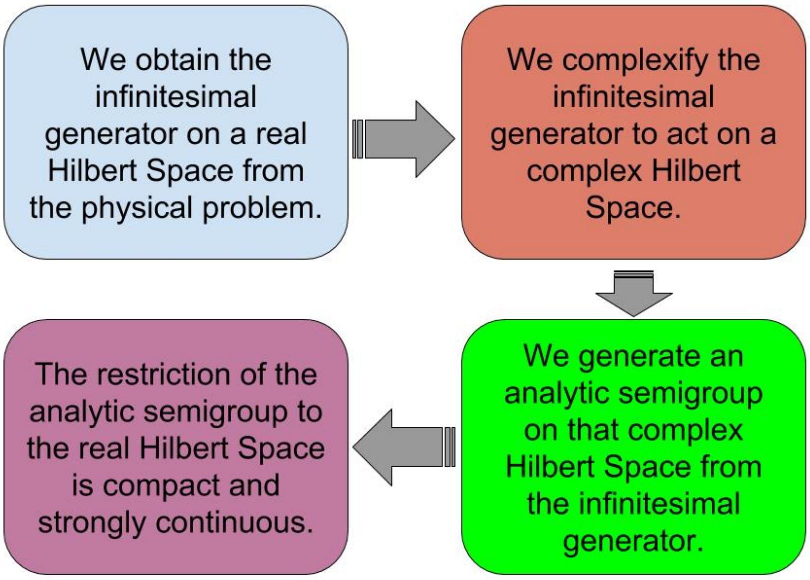

Figure 2.1: General Idea to Prove Existence.

In this chapter we develop the semigroup theory we shall use to prove the existence result for mild solutions of the abstract Cauchy problem (1.13). We start by considering compact semigroups and give the aforementioned characterization of such semigroups. As preparation for applications to semi-linear parabolic problems, we prove compactness results for the resolvent of elliptic operators acting on Hilbert space $\mathcal{L}^2(\Omega)$ for suitable open sets $\Omega \subseteq \mathbb{R}^n$. To make this characterization useful to us, we then investigate analytic semigroups. We then develop perturbation theory regarding analytic semigroups, which enables us to apply our existence result to a broader class of problems. Finally, we consider the complexification of operators acting on Hilbert spaces.

2.1 Compact Semigroups

In this section we define and characterize compact semigroups acting on a Banach space X.

Definition 2.1.1 (**Compact Operator**)**.** An linear operator L from a Banach Space X to another Banach Space Y is called compact if the image under L of any bounded subset of X is a precompact subset of Y.

We will make use of the following standard results concerning compact operators.

Theorem 2.1.1. *[8, VI.5.3] Suppose (T_n) is a sequence of compact operators from one Banach space to another, and suppose that (T_n) converges to a operator T with respect to the operator norm. Then T is also compact.*

Theorem 2.1.2. *[8, VI.5.4] Linear combinations of compact operators are compact operators, and the composition of a compact operator and a bounded linear operator is a compact operator.*

Definition 2.1.2 (**Compact Semigroup**)**.** A C_0-semigroup $T(t)$, $t \geq 0$, is called compact for $t > t_0 > 0$ if, for each $t > t_0$, $T(t)$ is a compact linear operator. A C_0-semigroup $T(t)$, $t \geq 0$, is called compact if it is compact for $t > 0$.

Theorem 2.1.3. *[7, Theorem 7.1.4] If $T(t)$, $t \geq 0$, is a C_0-semigroup, and $T(a)$ is compact for some $a > 0$, then $T(t)$ is compact and continuous in the uniform operator topology for all $t \geq a$.*

Proof. Suppose $T(a)$ is compact and let $t \geq a$. Since $T(t)x = T(a)T(t-a)x$ for all $x \in X$, $T(t)$ is compact by Theorem 2.1.2. Now suppose $\mathcal{B}$ is the unit ball in X and $\bar{X}$ is the compact closure of the image of $\mathcal{B}$ under $T(a)$. Because $\overline{X}$ is compact and $T(t)$, $t \geq 0$, is a C_0-semigroup, then for any $\epsilon > 0$ we can find a $\delta > 0$ such that if $0 \leq t < \delta$ then $\|[T(t) - I]x\|_X < \epsilon$ for all $x \in \bar{X}$. Thus, if we let $a \leq b \leq t < b + \delta$ and $x \in \mathcal{B}$, then $0 \leq t - b \leq \delta$, $T(a)x \in \bar{X}$, and

$$\begin{aligned} \|T(t)x - T(b)x\|_X &= \|T(b-a)[T(t-b) - I]T(a)x\|_X \\ &\leq \|T(b-a)\|\epsilon. \end{aligned}$$

Thus $\|T(t) - T(b)\| \leq M\epsilon$ where $M = \|T(b-a)\|$. Hence $T(t)$ is continuous with respect to the uniform operator norm from the right on $[a, \infty)$. Similarly it can be shown that $T(t)$ is continuous with respect to the uniform operator norm from the left on $[a, \infty)$. □

Theorem 2.1.4. *Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is a linear operator and $R(\lambda, A)$ is compact for some $\lambda \in \rho(A)$. Then $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$.*

Proof. Suppose $\lambda \in \rho(A)$ such that $R(\lambda, A)$ is compact. For any $\mu \in \rho(A)$, by the resolvent identity

$$R(\mu, A) = R(\lambda, A) + (\lambda - \mu)R(\mu, A)R(\lambda, A).$$

Thus, by Theorem 2.1.2, $R(\mu, A)$ is compact for all $\mu \in \rho(A)$. □

To proceed we need the following Lemma.

Lemma 2.1.5. *Let $A : X \supseteq \mathcal{D}(A) \mapsto X$ be the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and for some $M \geq 1$ and $\omega \in \mathbb{R}$. For $\lambda \in \rho(A)$, if $\Re\lambda > \omega$ then*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt.$$

Proof. Consider $\lambda \in \rho(A)$ with $\Re\lambda > \omega$. Let $R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t) x dt$ for all $x \in X$. Since $T(t)$ is a C_0-semigroup, the mapping $\mathbb{R}^+ \ni t \mapsto T(t)x$ is continuous for each $x \in X$, and by assumption $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ so that the integral exists. Thus for any $x \in X$

$$\begin{aligned}
\|R(\lambda)x\|_X &= \left\| \int_0^\infty e^{-\lambda t} T(t) x dt \right\|_X \\
&\leq \int_0^\infty e^{-\Re\lambda t} \|T(t)\| \|x\|_X dt \\
&\leq \int_0^\infty e^{-\Re\lambda t} M e^{\omega t} \|x\|_X dt \\
&= M\|x\|_X \int_0^\infty e^{-(\Re\lambda - \omega)t} dt \\
&= \frac{M}{\Re\lambda - \omega} \|x\|_X
\end{aligned} \tag{2.1}$$

Thus $R(\lambda)$ is a bounded linear operator. For $h > 0$

$$\begin{aligned}
\frac{T(h)-I}{h}R(\lambda)x &= \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t+h)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt \\
&= \frac{1}{h}\int_h^\infty e^{-\lambda(s-h)}T(s)xds - \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt \\
&= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}T(t)xdt \\
&= \frac{e^{\lambda h}-1}{h}R(\lambda)x - e^{\lambda h}h^{-1}\int_0^h e^{-\lambda t}T(t)xdt. \qquad (2.2)
\end{aligned}$$

Applying L'Hôpital's rule to the first term on the right-hand side of (2.2) and Theorem 1.4.3 (i) to the second, equation (2.2) gives

$$\lim_{h\to 0^+}\frac{T(h)-I}{h}R(\lambda)x = \lambda R(\lambda)x - x.$$

Thus for each $x \in X$, $\lim_{h\to 0^+}\frac{T(h)-I}{h}R(\lambda)x$ exists so that $R(\lambda)x \in \mathcal{D}(A)$, and $AR(\lambda)x = \lambda R(\lambda)x - x$. Thus $(\lambda I - A)R(\lambda)x = x$ for every $x \in X$ so that

$$(\lambda I - A)R(\lambda) = I. \qquad (2.3)$$

Furthermore, by Theorem 1.4.5, A is closed, so by Theorem 1.4.3 (iii) and Proposition A.1.3, for all $x \in \mathcal{D}(A)$ we have

$$\begin{aligned}
R(\lambda)Ax &= \int_0^\infty e^{-\lambda t}T(t)Axdt \\
&= \int_0^\infty e^{-\lambda t}AT(t)xdt \\
&= A\left[\int_0^\infty e^{-\lambda t}T(t)xdt\right] \\
&= AR(\lambda)x. \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4) it follows that $R(\lambda)(\lambda I - A)x = x$ for all $x \in \mathcal{D}(A)$. Thus $R(\lambda) = (\lambda I - A)^{-1} = R(\lambda, A)$. □

The following is an important characterization of compact semigroups.

Theorem 2.1.6 (Characterization of Compact Semigroups). *A C_0-semigroup $T(t)$, $t \geq 0$, is compact if and only if it is continuous in the uniform operator topology for $t > 0$ and the resolvent $R(\lambda, A)$ of its infinitesimal generator A is a compact linear operator for every λ in the resolvent set $\rho(A)$ of A.*

Proof. First suppose that $T(t)$ is a C_0-semigroup which is compact for $t > 0$. Then by Theorem 1.4.1 we can find an $M \geq 1$ and an $\omega \in \mathbb{R}$ such that the resolvent set contains $\{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$, and $\|T(t)\| \leq Me^{\omega t}$. Thus, by Lemma 2.1.5, the resolvent operator is given by

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) ds \tag{2.5}$$

for all $\lambda \in \rho(A)$, $\Re\lambda > \omega$. Furthermore, by Theorem 2.1.3 the semigroup $T(t)$ is continuous in the uniform operator topology for $t > 0$ so that the integral (2.5) is convergent with respect to the operator norm. Fix $\epsilon > 0$ and $\lambda \in \rho(A)$, and define

$$R_\epsilon(\lambda) = \int_\epsilon^\infty e^{-\lambda s} T(s) ds.$$

By Theorems 2.1.1, 2.1.2 and the definition of the integral, for any $\alpha > \epsilon$ we have that $\displaystyle\int_\epsilon^\alpha e^{-\lambda s} T(s) ds$ will be compact since $T(s)$ is compact for all $s > 0$. Thus since the integral converges with respect to the operator norm $\displaystyle\lim_{\alpha\to\infty} \int_\epsilon^\alpha e^{-\lambda s} T(s) ds$ is compact, by Theorem 2.1.1. Thus $R_\epsilon(\lambda)$ is compact. Furthermore

$$\begin{aligned} \|R(\lambda, A) - R_\epsilon(\lambda)\| &= \left\| \int_0^\epsilon e^{-\lambda s} T(s) ds \right\| \\ &\leq M \int_0^\epsilon |e^{(\omega-\lambda)s}| ds \\ &\leq M\epsilon. \end{aligned}$$

Therefore $R_\epsilon(\lambda)$ converges to $R(\lambda, A)$ with respect to the operator norm, so by Theorem 2.1.1, $R(\lambda, A)$ is compact. By Theorem 2.1.4 then $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$.

Conversely suppose that $R(\lambda, A)$ is compact for $\lambda \in \rho(A)$ and that $T(t)$ is continuous with respect to the operator norm for $t > 0$. For all $\lambda \in \rho(A)$, $\Re\lambda > \omega$

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) ds$$

exists and converges with respect to the operator norm since

$$\|e^{-\lambda s} T(s)\| \leq |e^{-\lambda s}| \|T(s)\| \leq Me^{(\omega - \Re\lambda)s}$$

for all $s > 0$. Let $\gamma \in \mathbb{R}^+$ such that $\gamma > \omega$. Then $\gamma \in \rho(A)$. Note that

$$\gamma \int_0^\infty e^{-\gamma s} T(t) ds = \gamma T(t) \int_0^\infty e^{-\gamma s} ds = T(t),$$

and that if $\delta > 0$ then

$$\int_0^\delta \gamma e^{-\gamma s} ds \leq 1.$$

Thus for any fixed $\delta > 0$ and $t > 0$

$$\begin{aligned}
\|\gamma R(\gamma, A)T(t) - T(t)\| &= \left\| \gamma \int_0^\infty e^{-\gamma s}[T(t+s) - T(t)]ds \right\| \\
&\leq \int_0^\delta \gamma e^{-\gamma s} \|T(t+s) - T(t)\| ds \\
&\quad + \int_\delta^\infty \gamma e^{-\gamma s} \|T(t+s) - T(t)\| ds \\
&\leq \int_0^\delta \gamma e^{-\gamma s} \|T(t+s) - T(t)\| ds \\
&\quad + \int_\delta^\infty \gamma e^{-\gamma s} \|T(t)\| \|T(s) - I\| ds \\
&\leq \sup_{0 \leq s \leq \delta} \|T(t+s) - T(t)\| \\
&\quad + \int_\delta^\infty M\gamma e^{-\gamma s} e^{\omega t}(Me^{\omega s} + 1) ds \\
&\leq \sup_{0 \leq s \leq \delta} \|T(t+s) - T(t)\| \\
&\quad + M^2 \gamma (\gamma - \omega)^{-1} e^{\omega(t+\delta)} e^{-\gamma\delta} + M\gamma e^{\omega t} e^{-\gamma\delta}. \qquad (2.6)
\end{aligned}$$

Fix $\epsilon > 0$. Since δ is arbitrary we can take δ small enough so that

$$\sup_{0 \leq s \leq \delta} \|T(t+s) - T(t)\| < \frac{\epsilon}{3}. \qquad (2.7)$$

For that fixed δ we can then take γ large enough so that both

$$M^2 \gamma (\gamma - \omega)^{-1} e^{\omega(t+\delta)} e^{-\gamma\delta} < \frac{\epsilon}{3} \qquad (2.8)$$

and

$$M\gamma e^{\omega t} e^{-\gamma\delta} < \frac{\epsilon}{3}. \qquad (2.9)$$

It follows from (2.7), (2.8), (2.9) and (2.6) that

$$\lim_{\gamma \to \infty} \|\gamma R(\gamma, A)T(t) - T(t)\| = 0.$$

Now since $R(\lambda, A)$ is compact for every $\lambda \in \rho(A)$ and $\rho(A)$ contains all real numbers $\gamma > \omega$ then $\gamma R(\gamma, A)T(t)$ is compact for all $\gamma > \omega$ by Theorem 2.1.2. Thus, since $\gamma R(\gamma, A)T(t)$ converges to $T(t)$ with respect to the uniform operator norm, then by Theorem 2.1.1, $T(t)$ is compact for every $t > 0$. $\square$

Recall that the C_0-semigroup $T(t)$, $t \geq 0$, is continuous in the uniform operator topology for $t \geq 0$ if and only if it has a bounded infinitesimal generator. Since we are interested in the case where A is an unbounded operator, we have that $T(t)$ is not continuous with respect to the uniform operator norm for $t \geq 0$. Thus

$$\lim_{t \to 0^+} \|T(t) - I\| > 0$$

or does not exist, while, for each $u_0 \in X$,

$$\lim_{t \to 0^+} \|T(t)u_0 - u_0\| = 0.$$

The characterization of compact semigroups given above requires that the resolvent operator of the infinitesimal generator be compact and the semigroup be continuous in the uniform operator topology for $t > 0$. Thus

$$\lim_{t \to a^+} \|T(t) - T(a)\| = 0$$

for all $a > 0$. In the following section, Section 2.2, we show that if A is an elliptic operator acting on $\mathcal{L}^2(\Omega)$ then A has a compact resolvent. In Section 2.3 we show that analytic semigroups are semigroups whose infinitesimal generator is a sectorial operator, and that these semigroups are strongly continuous for $t \geq 0$ and continuous in the uniform operator topology for $t > 0$.

2.2 Elliptic Operators on a Hilbert Space

In this section we show that if A is an elliptic operator on $\mathcal{L}^2(\Omega)$ for suitable open sets $\Omega \subseteq \mathbb{R}^n$ then $\rho(A)$ contains $\{\lambda \in \mathbb{R} \mid \lambda > \omega\}$ for some $\omega \geq 0$ and the resolvent operator $R(\lambda, A)$ is compact for some $\lambda \in \rho(A)$. This result is extremely useful in application to elliptic and parabolic problems, as we demonstrate in Chapter 4.

We first recall from (1.11) and (1.12) what it means for A to be an elliptic operator. Suppose Ω is an open and bounded subset of $\mathbb{R}^n$ and $\partial\Omega$ is smooth. Then $A : \mathcal{L}^2(\Omega) \supset \mathcal{D}(A) \mapsto \mathcal{L}^2(\Omega)$ is called a uniformly elliptic operator in divergence form if we can write A in the form

$$Au = \sum_{i,j=1}^{n} (\alpha^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} \beta^i(x)u_{x_i} + \gamma(x)u \tag{2.10}$$

and there exists a constant $\theta > 0$ such that for all $\zeta \in \mathbb{R}^n$ then

$$\sum_{i,j=1}^{n} \alpha^{ij}(x)\zeta_i\zeta_j \geq \theta\|\zeta\|^2 \tag{2.11}$$

where $\alpha^{ij} \in \mathcal{C}^1(\bar{\Omega})$, $\beta^i \in \mathcal{L}^\infty(\Omega)$ and $\gamma \in \mathcal{L}^\infty(\Omega)$ for $i, j = 1, ..., n$.

To proceed we need the following theorem from Evans [9]:

Theorem 2.2.1 (Energy Estimates). *([9], Section 6.2.2, Theorem 2) Suppose $B : H_0^1(\Omega) \times H_0^1(\Omega) \mapsto \mathbb{R}$ given by*

$$B[u, v] = \int_\Omega \left[\sum_{i,j=1}^{n} \alpha^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} \beta^i(x)u_{x_i}v + \gamma(x)uv\right] dx$$

is the bilinear form corresponding to an elliptic operator $A : \mathcal{D}(A) \mapsto \mathcal{L}^2(\Omega)$ in divergence form (2.10)*.*

Then there exist constants $\delta, \epsilon > 0$ and $\omega \geq 0$ such that

(i) $\quad |B[u, v]| \leq \delta\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}$

and

(ii) $\quad \epsilon\|u\|^2_{H_0^1(\Omega)} \leq B[u, u] + \omega\|u\|^2_{\mathcal{L}^2(\Omega)}$

for all $u, v \in H_0^1(\Omega)$.

Theorem 2.2.2. *Suppose Ω is an open and bounded subset of $\mathbb{R}^n$, $\partial\Omega$ is smooth, $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$, and $A : \mathcal{D}(A) \mapsto \mathcal{L}^2(\Omega)$ is a uniformly elliptic operator in divergence form. Then there exists a constant $\omega \geq 0$ such that the resolvent set $\rho(A)$ of A contains the ray $\{\lambda \in \mathbb{R} \mid \lambda > \omega\}$. Furthermore, $R(\lambda, A)$ is compact for each $\lambda > \omega$.*

Proof. We begin by showing that there exists a $\lambda \in \mathbb{R}$ such that $(\lambda I - A)^{-1}$ exists, that is, for all $g \in \mathcal{L}^2(\Omega)$ there exists a unique $u \in \mathcal{D}(A)$ such that $(\lambda I - A)u = g$.

Let $\lambda \in \mathbb{R}$ be arbitrary, and let $u \in \mathcal{D}(A)$. Then there exists an $f \in \mathcal{L}^2(\Omega)$ such that

$$(\lambda I - A)u = f. \tag{2.12}$$

Multiplying both sides of (2.12) with $v \in H_0^1(\Omega)$ and integrating over Ω, we get

$$\lambda(u, v)_{\mathcal{L}^2(\Omega)} - (Au, v)_{\mathcal{L}^2(\Omega)} = (f, v)_{\mathcal{L}^2(\Omega)}. \tag{2.13}$$

Equations (2.13) and (2.10) give us

$$\lambda(u,v)_{\mathcal{L}^2(\Omega)} - \int_\Omega \left[\sum_{i,j=1}^{n} (\alpha^{ij} u_{x_i})_{x_j} v + \sum_{i=1}^{n} \beta^i u_{x_i} v + \gamma uv \right] dx = (f,v)_{\mathcal{L}^2(\Omega)}.$$

Applying integration by parts to $\int_\Omega \sum_{i,j=1}^{n} (\alpha^{ij} u_{x_i})_{x_j} v dx$ with $v \in H_0^1(\Omega)$ gives

$$\lambda(u,v)_{\mathcal{L}^2(\Omega)} + \int_\Omega \left[\sum_{i,j=1}^{n} \alpha^{ij} u_{x_i} v_{x_j} - \sum_{i=1}^{n} \beta^i u_{x_i} v - \gamma uv \right] dx = (f,v)_{\mathcal{L}^2(\Omega)}.$$

We define bilinear forms $B[u,v]$ and $B_\lambda[u,v]$ associated with $-A$ and $(\lambda I - A)$ respectively, for each $v \in H_0^1(\Omega)$, by

$$B[u,v] = \int_\Omega \left[\sum_{i,j=1}^{n} \alpha^{ij} u_{x_i} v_{x_j} - \sum_{i=1}^{n} \beta^i u_{x_i} v - \gamma uv \right] dx$$

and

$$B_\lambda[u,v] = \lambda(u,v)_{\mathcal{L}^2(\Omega)} + B[u,v]. \tag{2.14}$$

Then

$$B_\lambda[u,v] = (f,v)_{\mathcal{L}^2(\Omega)} \tag{2.15}$$

for each $v \in H_0^1(\Omega)$. Since A is elliptic, by Theorem 2.2.1, we can find constants $\epsilon > 0$ and $\omega \geq 0$ such that

$$\epsilon \|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \omega \|u\|^2_{\mathcal{L}^2(\Omega)}.$$

Let $\lambda \geq \omega \geq 0$. Then

$$\begin{aligned} \epsilon \|u\|^2_{H_0^1(\Omega)} &\leq B[u,u] + \lambda \|u\|^2_{\mathcal{L}^2(\Omega)} \\ &= B_\lambda[u,u]. \end{aligned} \tag{2.16}$$

It is clear that $(\lambda I - A)$ is elliptic. Thus by Theorem 2.2.1 we can find a constant $\delta > 0$ such that

$$|B_\lambda[u,v]| \leq \delta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \tag{2.17}$$

for all $u, v \in H_0^1(\Omega)$. From (2.16) and (2.17) the requirements for the Lax-Milgram Theorem, Theorem C.0.6 are met. Furthermore, the function h_f :

$\mathcal{L}^2(\Omega) \mapsto \mathbb{R}$ defined by $h_f(v) = (f, v)_{\mathcal{L}^2(\Omega)}$ is a bounded linear functional on $\mathcal{L}^2(\Omega)$ by the Reisz Representation Theorem, Theorem C.0.7. Thus, for every $g \in \mathcal{L}^2(\Omega)$ we can find a unique $u \in \mathcal{D}(A)$ so that $(g, v)_{\mathcal{L}^2(\Omega)} = B_\lambda[u, v]$ for all $v \in H_0^1(\Omega)$. This holds whenever $g = (\lambda I - A)u$. Thus $(\lambda I - A)^{-1}$ exists and $u = (\lambda I - A)^{-1}g$.

We now show that $(\lambda I - A)^{-1}$ is a bounded linear operator. Substituting (2.15) into (2.16) and applying Cauchy-Schwartz inequality we have that

$$\begin{aligned}\epsilon\|u\|^2_{H_0^1(\Omega)} &\leq (g, u)_{\mathcal{L}^2(\Omega)} \\ &\leq \|g\|_{\mathcal{L}^2(\Omega)}\|u\|_{\mathcal{L}^2(\Omega)} \\ &\leq \|g\|_{\mathcal{L}^2(\Omega)}\|u\|_{H_0^1(\Omega)}.\end{aligned}$$

Thus for every $g \in \mathcal{L}^2(\Omega)$

$$\epsilon\|(\lambda I - A)^{-1}g\|_{H_0^1(\Omega)} \leq \|g\|_{\mathcal{L}^2(\Omega)}. \tag{2.18}$$

It follows from (2.18) that $(\lambda I - A)^{-1}$ is a bounded linear operator, and thus $\lambda \in \rho(A)$. Since this is true for all $\lambda \geq \omega$ then $\rho(A)$ contains the ray $\{\lambda \in \mathbb{R} \mid \lambda > \omega\}$.

Furthermore, by the Rellich-Kondrachov Compactness Theorem, Theorem C.0.8, $H_0^1(\Omega)$ is compactly embedded in $\mathcal{L}^2(\Omega)$. Thus (2.18) implies that $R(\lambda, A)$ is compact. □

2.3 Analytic Semigroups

We now consider sectorial operators, and define analytic semigroups in terms of these operators. In this section X denotes a complex Banach space, and A is a linear operator on X.

The definition of analytic function is given by A.2.1. A function from $\mathbb{C}$ into X is called holomorphic at point $a \in \mathbb{C}$ if it is infinitely differentiable within some open disk containing a. A function is holomorphic if and only if it is complex analytic, see [4, Page 46]. Thus, due to the equivalent definitions, the words holomorphic and analytic are used interchangeably, even though one can also have *real* analytic functions.

Before we proceed we show that for a linear operator $A : X \supset \mathcal{D}(A) \mapsto X$ with a non-empty resolvent set, the mapping $\rho(A) \ni \lambda \mapsto R(\lambda, A) \in B(X)$ is analytic on $\rho(A)$.

Theorem 2.3.1. *Consider a linear operator $A : X \supset \mathcal{D}(A) \mapsto X$. Let $\lambda_0 \in \rho(A)$. Then the open ball*

$$D = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}\}$$

is contained in $\rho(A)$, and if $\lambda \in D$, then

$$\begin{aligned} R(\lambda, A) &= \sum_{0}^{\infty}(-1)^n(\lambda-\lambda_0)^n R^{n+1}(\lambda_0, A) \\ &= R(\lambda_0, A)[I+(\lambda-\lambda_0)R(\lambda_0, A)]^{-1}. \end{aligned}$$

Therefore the resolvent set $\rho(A)$ is open in $\mathbb{C}$ and $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$.

Proof. Consider $\lambda \in D = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}\}$ and $y \in X$. To show that $R(\lambda, A)$ exists we show that there exists a unique $x \in \mathcal{D}(A)$ such that $\lambda x - Ax = y$. We then show that $R(\lambda, A)$ is a bounded linear operator. The equation $\lambda x - Ax = y$ is equivalent to $(\lambda - \lambda_0)x + (\lambda_0 I - A)x = y$. Thus letting $z = (\lambda_0 I - A)x$ so that $x = R(\lambda_0, A)z$ we get the equivalent equation

$$z + (\lambda - \lambda_0)R(\lambda_0, A)z = [I + (\lambda - \lambda_0)R(\lambda_0, A)]z = y.$$

Consider the series

$$S(\lambda) = \sum_{n=0}^{\infty}(-1)^n(\lambda-\lambda_0)^n R^n(\lambda_0, A).$$

Since $|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}$ implies that

$$\|(\lambda - \lambda_0)R(\lambda_0, A)\| < 1$$

the series $S(\lambda)$ converges absolutely with respect to the operator norm, thus defining a bounded linear operator on X. Thus, if $\lambda \in D$ then $S(\lambda)$ is a bounded linear operator and

$$\begin{aligned} & S(\lambda)[I+(\lambda-\lambda_0)R(\lambda_0, A)] \\ &\quad = \sum_{n=0}^{\infty}(-1)^n(\lambda-\lambda_0)^n R^n(\lambda_0, A)[I+(\lambda-\lambda_0)R(\lambda_0, A)] \\ &\quad = \sum_{n=0}^{\infty}(-1)^n(\lambda-\lambda_0)^n R^n(\lambda_0, A) + \sum_{n=0}^{\infty}(-1)^n(\lambda-\lambda_0)^{n+1} R^{n+1}(\lambda_0, A) \\ &\quad = I + (\lambda-\lambda_0)R(\lambda_0, A) - (\lambda-\lambda_0)R(\lambda_0, A) - (\lambda-\lambda_0)^2 R^2(\lambda_0, A) + \ldots \\ &\quad = I. \end{aligned}$$

Similarly, $[I+(\lambda-\lambda_0)R(\lambda_0, A)]S(\lambda) = I$. Thus for $\lambda \in D$, $S(\lambda)$ is the inverse of $[I + (\lambda - \lambda_0)R(\lambda_0, A)]$, so that the equation

$$z + (\lambda - \lambda_0)R(\lambda_0, A)z = y$$

has a unique solution $z \in X$ for every $y \in X$. Thus for every $y \in X$, $(\lambda I - A)x = y$ if and only if

$$x = R(\lambda_0, A)z = R(\lambda_0, A)S(\lambda)y.$$

Thus, since $R(\lambda_0, A)S(\lambda)$ a bounded linear operator then $\lambda \in \rho(A)$. Hence D is contained in $\rho(A)$. Furthermore,

$$R(\lambda, A) = R(\lambda_0, A)S(\lambda) = \sum_{n=0}^{\infty}(-1)^n(\lambda - \lambda_0)^n R^{n+1}(\lambda_0, A),$$

so that the mapping $\lambda \mapsto R(\lambda, A)$ is analytic. $\square$

Sectorial Operators

Definition 2.3.1 (Sector). For $\omega \in \mathbb{R}$ and $\theta > 0$ we define the sector $S_{\theta,\omega}$ as

$$S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}.$$

If $\omega = 0$ then we write $S_{\theta,0}$ as S_θ. See Figure 2.2 for a graphical representation of the sector $S_{\theta,\omega}$.

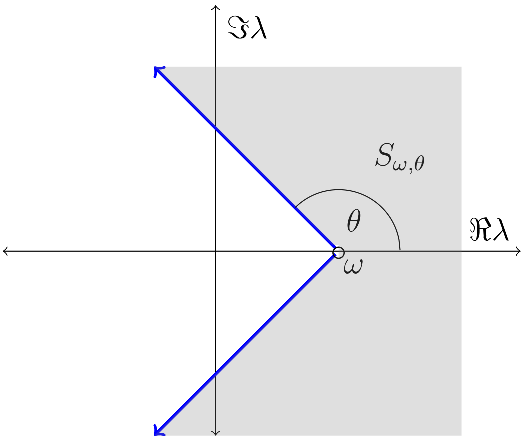

Figure 2.2: The sector $S_{\theta,\omega}$.

Definition 2.3.2 (Sectorial Operator). An operator $A : X \supset \mathcal{D}(A) \mapsto X$ is said to be sectorial if there exist constants $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $M > 0$ such that

(i) $S_{\theta,\omega} \subset \rho(A)$

(ii) $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - \omega|}$ for all $\lambda \in S_{\theta,\omega}$.

The following is a useful sufficient condition for an operator to be sectorial.

Theorem 2.3.2. *[14, Theorem 2.1.11] Let $A : X \supset \mathcal{D}(A) \mapsto X$ be a linear operator such that $\rho(A)$ contains a half-plane $\Pi = \{\lambda \in \mathbb{C} \mid \Re\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\| \leq M \text{ when } \Re\lambda \geq \omega,$$

with $\omega \in \mathbb{R}$ and $M > 0$. Then A is sectorial.

Proof. We start by showing that there exists a $\theta \in (\frac{\pi}{2}, \pi)$ such that $S_{\theta,\omega} \subset \rho(A)$. If $\omega < 0$ then we can choose any $\omega^* \geq 0$ and use it to define a sector S_{θ,ω^*} on which the conditions for A being sectorial hold true. Thus we assume that $\omega \geq 0$. For any $r \in \mathbb{R}$, $\omega + ir \in \Pi \subset \rho(A)$. Since $\|\lambda R(\lambda, A)\| \leq M$ for $\Re\lambda \geq \omega$, then $\|R(\omega + ir, A)\|^{-1} \geq \dfrac{|\omega + ir|}{M}$ for all $r \in \mathbb{R}$. Thus, from Theorem 2.3.1, for all $r \in \mathbb{R}$ the open ball $B_{M^{-1}|\omega+ir|}(\omega + ir)$ is contained within the resolvent set $\rho(A)$. Let

$$S = \{\lambda \in \mathbb{C} \mid \lambda \neq \omega, |arg(\lambda - \omega)| \leq \pi - \arctan 2M\}.$$

We show that S is contained in $\bigcup_{r\in\mathbb{R}} B_{M^{-1}|\omega+ir|}(\omega + ir) \cup \Pi$ so that $S \subset \rho(A)$.

To do this assume $r > 0$ and let $x + iy \in S$, with $y > 0$. If $x \geq \omega$ then $x + iy \in \Pi$ and we are done. If $x < \omega$, let $\theta = \arg(x + iy - \omega)$ so that

$$\begin{aligned} \frac{y}{x - \omega} &= \tan(\theta) \\ &\leq \tan(\pi - \arctan 2M) \\ &= -\tan(\arctan 2M) \\ &= -2M. \end{aligned}$$

Thus $\left(\dfrac{y}{x - \omega}\right)^2 \geq 4M^2 > M^2$ and

$$|x + iy - (\omega + ir)|^2 = y^2 \left(\frac{x - \omega}{y}\right)^2 + (y - r)^2 < \frac{y^2}{M^2} + (y - r)^2.$$

Setting $r = y$ gives $|x + iy - (\omega + ir)|^2 < \dfrac{y^2}{M^2} \leq \dfrac{\omega^2 + r^2}{M^2}$. Thus $x + iy$ is contained in the open ball $B_{M^{-1}|\omega+ir|}(\omega + ir)$. The case for $y < 0$ follows similarly with

$$\frac{y}{x - \omega} \geq \tan(\pi - \arctan(2M)) = 2M.$$

Thus $S \subset \bigcup_{r \in \mathbb{R}} B_{M^{-1}|\omega+ir|}(\omega + ir) \cup \Pi$.

We now show that there exists an $M^* > 0$ such that $\|R(\lambda, A)\| \leq \dfrac{M^*}{|\lambda - \omega|}$ for all $\lambda \in S$. Suppose $\lambda \in S$. If $0 \leq \omega \leq \Re\lambda$ then

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \leq \frac{M}{|\lambda - \omega|}. \tag{2.19}$$

Suppose $\Im\lambda > 0$. If $\Re\lambda < \omega$ then $\dfrac{\pi}{2} < \arg\lambda \leq \pi - \arctan 2M$, so that $\tan(\arg\lambda) \leq -2M$. Thus $\lambda = w + ir - \dfrac{\alpha r}{M}$ for some $r \neq 0$ and $\alpha \in \left(0, \dfrac{1}{2}\right]$. By Theorem 2.3.1, since $\Re\lambda < \omega$ we have that

$$\begin{aligned}
\|R(\lambda, A)\| &= \left\| \sum_{n=0}^{\infty} (-1)^n \left[\lambda - (\omega + ir)\right]^n \left[R(\omega + ir, A)\right]^{n+1} \right\| \\
&\leq \sum_{n=0}^{\infty} \left[-\frac{\alpha r}{M}\right]^n \left[\frac{M}{|\omega + ir|}\right]^{n+1} \\
&\leq \frac{M}{\sqrt{\omega^2 + r^2}} \sum_{n=0}^{\infty} \left[-\frac{\alpha r}{\sqrt{\omega^2 + r^2}}\right]^n \\
&= \frac{M}{\sqrt{\omega^2 + r^2}} \left[1 - \left(\frac{\alpha r}{\sqrt{\omega^2 + r^2}}\right) + \left(\frac{\alpha r}{\sqrt{\omega^2 + r^2}}\right)^2 - \ldots\right] \\
&\leq \frac{2M}{|r|},
\end{aligned} \tag{2.20}$$

since $\left|\dfrac{\alpha r}{\sqrt{\omega^2 + r^2}}\right| \leq \dfrac{1}{2}$. However, with $\lambda = w + ir - \dfrac{\alpha r}{M}$ we have

$$|r| = |\lambda - \omega| \left(1 + \frac{\alpha^2}{M^2}\right)^{-\frac{1}{2}} \geq |\lambda - \omega| \left(1 + \frac{1}{4M^2}\right)^{-\frac{1}{2}}$$

since $0 < \alpha \leq \frac{1}{2}$. Substituting this back into (2.20) gives

$$\|R(\lambda, A)\| \leq 2M \left(1 + \frac{1}{4M^2}\right)^{\frac{1}{2}} |\lambda - \omega|^{-1}. \tag{2.21}$$

Setting $M^* = 2M \left(1 + \dfrac{1}{4M^2}\right)^{\frac{1}{2}} > M$, (2.19) and (2.21) gives our result, and thus A is sectorial. $\square$

Analytic Semigroups

We begin this section by associating with the sectorial operator A a family of bounded linear operators e^{tA}, $t > 0$, on X. We then show that e^{tA}, $t > 0$, with $e^{0A} = I$, is a semigroup. Lunardi [14] calls this family of bounded operators the analytic semigroup generated by A.

Unlike with C_0-semigroups, when dealing with sectorial operators it is no longer necessary that $\mathcal{D}(A)$ is dense in X, see [14, Page X]. This is demonstrated by Lunardi in [14, Proposition 2.1.4], where he shows that analytic semigroups have all the properties of C_0-semigroups listed in Theorem 1.4.3 on $\overline{\mathcal{D}(A)}$, where $\overline{\mathcal{D}(A)}$ is not necessarily the whole of X. However, some results are true when $\mathcal{D}(A)$ is dense in X and other results not, see [3, Page 102]. Furthermore, suppose $\mathcal{D}(A)$ was not dense in X. Let $X_0 = \overline{\mathcal{D}(A)}$ and A_0 be the part of A in X_0. Then X_0 a Banach space with respect to $\|\cdot\|_X$ and all the results proved in this section hold true for A_0 acting on X_0. In particular, if A is sectorial on X then A_0 is sectorial on X_0, A_0 is the infinitesimal generator of an analytic semigroup e^{tA_0}, $t \geq 0$, in X_0, and this semigroup is a C_0-semigroup in X_0, see [14, Remark 2.1.5].

Therefore we assume that $\mathcal{D}(A)$ is dense in X. Under this assumption we shall show in Theorem 2.3.6 that analytic semigroups are C_0-semigroups on X.

Proposition 2.3.3. *Let $A : X \supseteq \mathcal{D}(A) \mapsto X$ be a sectorial operator such that $S_{\theta,\omega} \subset \rho(A)$ for some $\omega \in \mathbb{R}$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$. The operator e^{tA} defined as the integral*

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, \; t > 0,$$

where $r > 0$, $\eta \in \left(\frac{\pi}{2}, \theta\right)$, and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} \mid |\arg \lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \eta, |\lambda| = r\}$$

oriented counter-clockwise, is a bounded linear operator on X for all $t > 0$, and the integral is absolutely uniformly convergent.

Proof. The curve $\omega + \gamma_{r,\eta}$ is shown in Figure 2.3. Since A is sectorial there exists a $M > 0$ such that $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - \omega|}$ for all $\lambda \in (\omega + \gamma_{r,\eta})$. For any $n \in \mathbb{N}$, $n > \max\left\{r, \dfrac{-\omega}{\cos(\eta)}\right\}$, let $\Gamma_n = \overline{B}_n(\omega) \cap (\omega + \gamma_{r,\eta})$, where $\overline{B}_n(\omega)$ is the closed ball in $\mathbb{C}$ with radius n centred at $\omega + 0i$. Note that for all $n > \max\left\{r, \dfrac{-\omega}{\cos(\eta)}\right\}$, if $\lambda \in (\omega + \gamma_{r,\eta}) \setminus \Gamma_n$ then $\Re\lambda = |\lambda| \cos(\eta) + \omega < 0$.

If $t > 0$ and $\lambda \in (\omega + \gamma_{r,\eta}) \setminus \Gamma_n$ then

$$\begin{aligned}\|e^{t\lambda}R(\lambda, A)\| &\leq |e^{t\lambda}|\|R(\lambda, A)\| \\ &\leq e^{t\Re\lambda}\frac{M}{|\lambda|} \\ &\leq e^{t(n\cos(\eta)+\omega)}\frac{M}{n},\end{aligned}$$

with $\cos(\eta) < 0$. Thus, since Γ_n is a compact piecewise differentiable curve, for each $t > 0$ we have that $\int_{\Gamma_n} \|e^{t\lambda}R(\lambda, A)x\|_X d\lambda < \infty$ for every $x \in X$. Furthermore, the function $\rho(A) \ni \lambda \mapsto e^{t\lambda}R(\lambda, A) \in B(X)$ is measurable by Theorem 2.3.1. Thus, by Theorem A.1.2, $e^{t\lambda}R(\lambda, A) \in \mathcal{L}^1(\Gamma_n; B(X))$.

Fix $\alpha > 0$. We now show that $\frac{1}{2\pi i}\int_{\Gamma_n} e^{t\lambda}R(\lambda, A)d\lambda$ is absolutely and uniformly convergent on $[\alpha, \infty)$. Fix $\epsilon > 0$ and $t \in [\alpha, \infty)$. Consider the curve $(\omega + \gamma_{r,\eta}) \setminus \Gamma_n$ with positive imaginary part, say $(\omega + \gamma^+_{r,\eta}) \setminus \Gamma_n$. For each $\lambda \in (\omega+\gamma^+_{r,\eta})\setminus\Gamma_n$, there exists a real number $\delta \geq n$ such that $\lambda = \omega+\delta e^{i\eta}$. Thus

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{(\omega+\gamma^+_{r,\eta})\setminus\Gamma_n} \|e^{t\lambda}R(\lambda, A)\|d\lambda\right| &\leq \left|\frac{1}{2\pi i}\right|\left|\int_{(\omega+\gamma^+_{r,\eta})\setminus\Gamma_n} |e^{t\lambda}|\|R(\lambda, A)\|d\lambda\right| \\ &= \frac{1}{2\pi}\left|\int_{(\omega+\gamma^+_{r,\eta})\setminus\Gamma_n} e^{t\Re\lambda}\|R(\lambda, A)\|d\lambda\right| \\ &= \frac{1}{2\pi}\left|\int_n^\infty e^{t(\omega+\delta\cos(\eta))}\|R(\omega+\delta e^{i\eta}, A)\|e^{i\eta}d\delta\right| \\ &\leq \frac{1}{2\pi}\int_n^\infty e^{\alpha(\omega+\delta\cos(\eta))}\frac{M}{|\delta|}|e^{i\eta}|d\delta \\ &\leq \frac{M}{2\pi n}\int_n^\infty e^{\alpha(\omega+\delta\cos(\eta))}d\delta \\ &= \frac{Me^{\alpha\omega}}{2\pi\alpha|\cos(\eta)|n}e^{\alpha\cos(\eta)n} \\ &\leq \frac{Me^{\alpha\omega}}{2\pi\alpha|\cos(\eta)|n} \\ &= \frac{K_\alpha}{n}\end{aligned}$$

where $K_\alpha = \frac{Me^{\alpha\omega}}{2\pi\alpha|\cos(\eta)|} > 0$. It follows that if $n > \max\left\{r, \frac{-\omega}{\cos(\eta)}, \frac{K_\alpha}{\epsilon}\right\}$

then

$$\left| \frac{1}{2\pi i} \int_{(\omega+\gamma_{r,\eta}^{+})\setminus\Gamma_n} \|e^{t\lambda} R(\lambda, A)\| d\lambda \right| < \epsilon. \tag{2.22}$$

Note that for each $\lambda^+ \in (\omega + \gamma_{r,\eta}^+) \setminus \Gamma_n$, $\lambda^+ = \omega + \delta e^{i\eta}$, there exists a corresponding $\lambda^- \in (\omega + \gamma_{r,\eta}) \setminus \Gamma_n$, $\lambda^- = \omega + \delta e^{-i\eta}$ with negative imaginary part. However, since $\cos(\eta) = \cos(-\eta)$, then $\Re\lambda^- = \Re\lambda^+$. Thus if we integrate over the curve $(\omega + \gamma_{r,\eta}) \setminus \Gamma_n$ with negative imaginary part (2.22) follows in the same way.

Thus for $t \in [\alpha, \infty)$ we have that $\frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \|e^{t\lambda} R(\lambda, A)\| d\lambda < \infty$ so that $e^{t\lambda} R(\lambda, A) \in \mathcal{L}^1(\omega + \gamma_{r,\eta}; B(X))$. By Theorem A.2.1

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma_n} e^{t\lambda} R(\lambda, A) d\lambda = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda$$

and the integral is absolutely and uniformly convergent on $[\alpha, \infty)$. Taking $\alpha \to 0$ gives that the convergence is absolute and uniform on $(0, \infty)$. Thus for any sectorial operator A, we can define a bounded linear operator

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda$$

for $t > 0$, and the integral is absolutely uniformly convergent.

$\square$

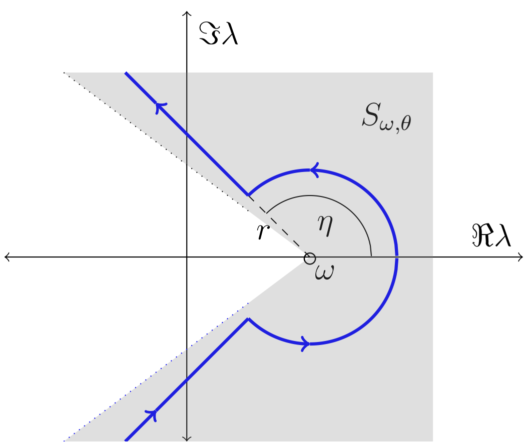

Figure 2.3: The curve $\omega + \gamma_{r,\eta}$.

The following basic proposition shows that, in general, we can take $\omega = 0$.

Proposition 2.3.4 (**Exponential Shift**). *Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is a sectorial operator such that there exist constants $\omega \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that $S_{\theta,\omega}$ is contained in $\rho(A)$, and there exists an $M > 0$ such that if $\lambda \in S_{\theta,\omega}$ then $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - \omega|}$. Then the operator*

$$B : X \supset \mathcal{D}(A) \ni x \mapsto Ax - \omega x \in X$$

is sectorial and satisfies the following properties:

(i) If $\lambda + \omega \in \rho(A)$ then $\lambda \in \rho(B)$ and $R(\lambda, B) = R(\lambda + \omega, A)$;

(ii) $S_\theta \subset \rho(B)$;

(iii) If $\lambda \in S_\theta$ then $\|\lambda R(\lambda, B)\| \leq M$.

(iv) The family e^{tB}, $t \geq 0$, is a family of bounded linear operators, and $e^{tB} = e^{-\omega t} e^{tA}$, $t \geq 0$.

Proof. (i): Suppose $\lambda + \omega \in \rho(A)$. Then $(A - (\lambda + \omega)I)^{-1} = (B - \lambda I)^{-1}$ is a bounded linear operator. Thus $\lambda \in \rho(B)$ and $R(\lambda, B) = R(\lambda + \omega, A)$.

(ii): Let $\lambda \in S_\theta$. Then $\lambda + \omega \in S_{\theta,\omega} \subset \rho(A)$. Thus, by (i), $\lambda \in \rho(B)$. Hence $S_\theta \subset \rho(B)$.

(iii): Let $\lambda \in S_\theta \subset \rho(B)$. Then, by (i),

$$\|R(\lambda, B)\| = \|R(\lambda + \omega, A)\| \leq \frac{M}{|\lambda|}.$$

(iv): For $t > 0$ we have by Proposition (2.3.3) that

$$\begin{aligned} e^{tB} &= \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{\lambda t} R(\lambda, B) d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{(\lambda - \omega)t} R(\lambda - \omega, B) d\lambda \\ &= \frac{1}{2\pi i} e^{-\omega t} \int_{\omega + \gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda \\ &= e^{-\omega t} e^{tA}, \end{aligned}$$

is a bounded linear operator. In the case of $t = 0$ then we just have the identity.

$\square$

Theorem 2.3.5. *Suppose A is a sectorial operator with $\rho(A)$ containing $S_{\theta,\omega}$, for constants $\theta \in (\frac{\pi}{2}, \pi)$ and $\omega \in \mathbb{R}$, and that if $\lambda \in S_{\theta,\omega}$ then $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - \omega|}$ for some $M > 0$. Then the family of bounded linear operators*

$$\begin{aligned} e^{tA} &= \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda,\ t > 0, \\ e^{0A}x &= x,\ x \in X, \end{aligned}$$

has the following properties:

(i) $e^{tA}x \in \mathcal{D}(A^k)$ *for each* $t > 0$, $x \in X$, $k \in \mathbb{N}$. *If* $x \in \mathcal{D}(A^k)$ *for some* $k \in \mathbb{N}$, *then*
$$A^k e^{tA} x = e^{tA} A^k x, \text{ for } t > 0.$$

(ii) $e^{tA} e^{sA} = e^{(t+s)A}$ *for all* $t, s \geq 0$.

(iii) *(a)* $\|e^{tA}\| \leq M_0 e^{\omega t}$, $t > 0$;

(b) For every $k \in \mathbb{N}$ *there exists a constant* $M_k > 0$ *such that*
$$\|t^k (A - \omega I)^k e^{tA}\| \leq M_k e^{\omega t},\ t > 0.$$

(c) For every $\epsilon > 0$ *and* $k \in \mathbb{N}$ *there exists a* $C_{k,\epsilon} > 0$ *such that*
$$\|t^k A^k e^{tA}\| \leq C_{k,\epsilon} e^{(\omega+\epsilon)t},\ t > 0.$$

(iv) The function $t \mapsto e^{tA}$ *belongs to* $\mathcal{C}^\infty((0,\infty); B(X))$, *and for all* $k \in \mathbb{N}$
$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA},\ t > 0.$$

Moreover this function has an analytic extension to the sector
$$S_{\theta - \frac{\pi}{2}} = \{t \in \mathbb{C} \mid t \neq 0,\ |\arg t| < \theta - \frac{\pi}{2}\}.$$

Proof. The following equations are used in the subsequent proofs. Let $t \geq 0$. For each $\lambda \in \rho(A)$, since $(\lambda I - A)R(\lambda, A)x = x$ for $x \in X$ and $R(\lambda, A)(\lambda I - A)x = x$ for $x \in \mathcal{D}(A)$ we have that

$$R(\lambda, A)Ax = AR(\lambda, A)x, \text{ for } x \in \mathcal{D}(A), \tag{2.23}$$
$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x, \text{ for } x \in X. \tag{2.24}$$

(i): Fix $t > 0$. Note that

$$\begin{aligned}\int_{\omega+\gamma_{r,\eta}} e^{\lambda t} d\lambda = &\int_{\infty}^{r} e^{\omega+\delta e^{-i\eta}} e^{-i\eta} d\delta + \int_{-\eta}^{0} e^{\omega+re^{i\theta}} re^{i\theta} d\theta \\ &+ \int_{0}^{\eta} e^{\omega+re^{i\theta}} re^{i\theta} d\theta + \int_{r}^{\infty} e^{\omega+\delta e^{i\eta}} e^{i\eta} d\delta \\ =&0. \end{aligned} \tag{2.25}$$

From (2.24) we have

$$\begin{aligned}\|e^{\lambda t} AR(\lambda, A)x\|_X &= e^{t\Re\lambda} \|\lambda R(\lambda, A)x - x\|_X \\ &\leq e^{t\Re\lambda} \left(\frac{|\lambda| M}{|\lambda - \omega|} + 1 \right) \|x\|_X \end{aligned} \tag{2.26}$$

for all $\lambda \in S_{\theta,\omega} \supset (\omega + \gamma_{r,\eta})$ and $x \in X$. For all $x \in X$, since

$$\lim_{\Re\lambda \to -\infty} e^{t\Re\lambda} \left(\frac{M|\lambda|}{|\lambda - \omega|} + 1 \right) \|x\|_X = 0$$

then

$$\frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} AR(\lambda, A)x d\lambda$$

exists. By Theorem 1.4.7 the sectorial operator A is closed. Thus by Propositions 2.3.3 and A.1.3 we have, for all $x \in X$, that

$$e^{tA}x = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} R(\lambda, A)x d\lambda \in \mathcal{D}(A)$$

and

$$\frac{1}{2\pi i} A \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} R(\lambda, A)x d\lambda = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} AR(\lambda, A)x d\lambda. \tag{2.27}$$

From (2.27), (2.24) and (2.25), we have

$$\begin{aligned} Ae^{tA}x &= \frac{1}{2\pi i} A \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} R(\lambda, A)x d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} AR(\lambda, A)x d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{\lambda t} [\lambda R(\lambda, A)x - x] d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \lambda e^{\lambda t} R(\lambda, A)x d\lambda. \end{aligned} \tag{2.28}$$

In a similar manner to (2.26) we get that

$$\lim_{\Re\lambda \to -\infty} \|\lambda^{k-1} e^{\lambda t} AR(\lambda, A)x\| \leq \lim_{\Re\lambda \to -\infty} e^{t\Re\lambda} |\lambda|^{k-1} \left(\frac{M|\lambda|}{|\lambda - \omega|} + 1 \right) \|x\|_X$$
$$= 0 \tag{2.29}$$

for all $k \in \mathbb{N}$ and $x \in X$. Thus, from (2.28) and (2.29), it follows in a similar manner to before that for all $k \in \mathbb{N}$ and $x \in X$, $e^{tA}x \in \mathcal{D}(A^k)$ and

$$A^k e^{tA} x = \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} \lambda^k e^{\lambda t} R(\lambda, A) x d\lambda. \tag{2.30}$$

From (2.23) and (2.27), if $x \in \mathcal{D}(A)$ then

$$\begin{aligned} Ae^{tA}x &= \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{\lambda t} AR(\lambda, A) x d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{\lambda t} R(\lambda, A) Ax d\lambda \\ &= e^{tA} Ax. \end{aligned}$$

Similarly, for any $k \in \mathbb{N}$, if $x \in \mathcal{D}(A^k)$ then $A^k e^{tA} x = e^{tA} A^k x$.

(ii): Let $t, s > 0$. From Proposition 2.3.4 we can define a sectorial operator $B : X \supset \mathcal{D}(A) \mapsto X$ such that $\rho(B)$ contains S_θ, $\|R(\lambda, B)\| = \|R(\lambda + \omega, A)\| \leq \dfrac{M}{|\lambda|}$ for $\lambda \in S_\theta$, and $e^{tB} = e^{-\omega t} e^{tA}$ for $t \geq 0$. If $e^{(t+s)B} = e^{tB} e^{sB}$ then

$$e^{(t+s)A} = e^{\omega(t+s)} e^{(t+s)B} = e^{\omega t} e^{tB} e^{\omega s} e^{sB} = e^{tA} e^{sA}.$$

Thus it is sufficient to prove that $e^{(t+s)B} = e^{tB} e^{sB}$. Let $0 < r_1 < r_2$ and $\frac{\pi}{2} < \eta_2 < \eta_1$. It is clear that

$$\int_{\gamma_{r_1,\eta_1}} \int_{\gamma_{r_2,\eta_2}} e^{t\lambda + s\mu} R(\lambda, B) R(\mu, B) d\lambda d\mu$$

exists due to the fact that $\|R(\lambda, B)\| \leq \dfrac{M}{|\lambda|}$ for $\lambda \in S_\theta$. Thus by Fubini's Theorem, Theorem A.1.4, and the Resolvent Identity, Theorem 1.4.6, we have

$$\begin{aligned}
e^{tB}e^{sB} &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r_1,\eta_1}} e^{t\lambda}R(\lambda,B)d\lambda \int_{\gamma_{r_2,\eta_2}} e^{s\mu}R(\mu,B)d\mu \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r_1,\eta_1}} \int_{\gamma_{r_2,\eta_2}} e^{t\lambda+s\mu}R(\lambda,B)R(\mu,B)d\lambda d\mu \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r_1,\eta_1}} \int_{\gamma_{r_2,\eta_2}} e^{t\lambda+s\mu}\frac{R(\lambda,B)-R(\mu,B)}{\mu-\lambda}d\lambda d\mu \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r_1,\eta_1}} e^{t\lambda}R(\lambda,B)\int_{\gamma_{r_2,\eta_2}} e^{s\mu}(\mu-\lambda)^{-1}d\mu d\lambda \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r_2,\eta_2}} e^{s\mu}R(\mu,B)\int_{\gamma_{r_1,\eta_1}} e^{t\lambda}(\mu-\lambda)^{-1}d\lambda d\mu.
\end{aligned}$$

By Lemma A.2.3 $(1)(a)$, since γ_{r_1,η_1} lies to the left of γ_{r_2,η_2} in the complex plane and $e^{s\mu}$ is analytic and bounded on γ_{r_2,η_2} and everything to the left of it, we have that

$$\begin{aligned}
e^{tB}e^{sB} &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r_1,\eta_1}} e^{t\lambda}R(\lambda,B)(2\pi i e^{s\lambda})d\lambda \\
&= \left(\frac{1}{2\pi i}\right) \int_{\gamma_{r_1,\eta_1}} e^{(t+s)\lambda}R(\lambda,B)d\lambda \\
&= e^{(t+s)B}.
\end{aligned}$$

$(iii)(a)$: To prove $(iii)(a)$ we again use the operator B as described in the proof of (ii), so that $\|R(\lambda,B)\| \leq \dfrac{M}{|\lambda|}$ for $\lambda \in S_\theta$. Then

$$\|e^{tA}\| = \|e^{\omega t}e^{tB}\| = e^{\omega t}\|e^{tB}\|$$

for all $t > 0$. Thus, for $M_0 > 0$, $e^{tA} \leq M_0 e^{\omega t}$ if and only if $\|e^{tB}\| \leq M_0$ for all $t > 0$. Hence it is sufficient to show that there exists an $M_0 > 0$

such that $\|e^{tB}\| \leq M_0$ for all $t > 0$. For any fixed $t > 0$, we have

$$\begin{aligned}
\|e^{tB}\| =& \left\| \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, B) d\lambda \right\| \\
=& \frac{1}{2\pi} \left\| \int_r^\infty e^{t\delta e^{i\eta}} R(\delta e^{i\eta}, B) e^{i\eta} d\delta + \int_\infty^r e^{t\delta e^{-i\eta}} R(\delta e^{-i\eta}, B) e^{-i\eta} d\delta \right. \\
& \left. + \int_{-\eta}^{\eta} e^{tre^{i\theta}} R(re^{i\theta}, B) ire^{i\theta} d\theta \right\| \\
\leq& \frac{1}{2\pi} \left(2 \int_r^\infty e^{t\delta\cos(\eta)} \frac{M}{\delta} d\delta + \int_{-\eta}^{\eta} e^{tr\cos(\theta)} \frac{M}{r} r d\theta \right) \\
\leq& \frac{M}{2\pi} \left(2 \int_r^\infty \delta^{-1} e^{t\delta\cos(\eta)} d\delta + \int_{-\eta}^{\eta} e^{tr\cos(\theta)} d\theta \right).
\end{aligned}$$

Note that $\cos(\eta) < 0$ so that $\int_r^\infty \delta^{-1} e^{t\delta\cos(\eta)} d\delta$ exists. Then

$$\|e^{tB}\| \leq \frac{MN}{2\pi} := M_0$$

when $N > 0$ is greater than $\left(2 \int_r^\infty \delta^{-1} e^{t\delta\cos(\eta)} d\delta + \int_{-\eta}^{\eta} e^{tr\cos(\theta)} d\theta \right)$.

$(iii)(b)$: Fix $k \in \mathbb{N}$. Note that if there exists a positive constant M_k such that $\|t^k B^k e^{tB}\| \leq M_k$ for all $t > 0$ then

$$\|t^k (A - \omega I)^k e^{tA}\| = \|t^k B^k e^{\omega t} e^{tB}\| \leq M_k e^{\omega t}$$

for all $t > 0$ and we have our result. Thus we show that there exists a positive constant M_k such that $\|t^k B^k e^{tB}\| \leq M_k$ for all $t > 0$. Fix $t > 0$. From (i), for any $x \in X$ we have that $e^{tB}x \in \mathcal{D}(B)$. Thus, by (2.27), (2.24) and (2.25), and using the same integral manipulation as in $(iii)(a)$, we have

$$\begin{aligned}
\|Be^{tB}\| &= \left\| \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} BR(\lambda, B) d\lambda \right\| \\
&= \left\| \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, B) d\lambda - \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} d\lambda \right\| \\
&= \frac{1}{2\pi} \left\| \int_{\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, B) d\lambda \right\| \\
&\leq \frac{M}{2\pi} \left(2 \int_r^\infty e^{t\delta\cos(\eta)} d\delta + \int_{-\eta}^{\eta} e^{tr\cos(\theta)} r d\theta \right).
\end{aligned}$$

Letting $\zeta = t\delta$ and $\zeta_0 = tr$ gives

$$\|Be^{tB}\| \leq \frac{M}{2\pi t}\left(2\int_{\zeta_0}^{\infty} e^{\zeta\cos(\eta)}d\zeta + \int_{-\eta}^{\eta} e^{\zeta_0\cos(\theta)}\zeta_0 d\theta\right) = \frac{M_1}{t}$$

for $0 < M_1 = \dfrac{M}{2\pi}\left(2\displaystyle\int_{\zeta_0}^{\infty} e^{\zeta\cos(\eta)}d\zeta + \int_{-\eta}^{\eta} e^{\zeta_0\cos(\theta)}\zeta_0 d\theta\right) < \infty$. Fix $k \in \mathbb{N}$. Since $e^{tB}x \in \mathcal{D}(B)$ for all $x \in X$, $t > 0$, and $Be^{tB} = e^{tB}B$ on $\mathcal{D}(B)$ by (i), then $B^k e^{tB} = (Be^{\frac{t}{k}B})^k$. Thus letting $M_k = (M_1 k)^k > 0$ we have

$$\|B^k e^{tB}\| = \|(Be^{\frac{t}{k}B})^k\| \leq \left(\frac{M_1 k}{t}\right)^k = \frac{M_k}{t^k}. \tag{2.31}$$

$(iii)(c)$: Fix $\epsilon > 0$ and $k \in \mathbb{N}$. Note that $k^k \leq e^k k!$ so that (2.31) gives $\|B^k e^{tB}\| \leq \dfrac{M_k}{t^k} \leq (M_1 e)^k k! t^{-k}$. Thus, by the binomial formula, we get

$$\begin{aligned}
\|A^k e^{tA}\| &= \|(B+\omega I)^k e^{\omega t} e^{tB}\| \\
&= e^{\omega t}\left\|\sum_{j=0}^{k}\binom{k}{j}\omega^{k-j}B^j e^{tB}\right\| \\
&\leq e^{\omega t}\left[\sum_{j=0}^{k}\binom{k}{j}\omega^{k-j}\|B^j e^{tB}\|\right] \\
&\leq e^{\omega t}\left[\sum_{j=0}^{k}\binom{k}{j}\omega^{k-j}(M_1 e)^j j! t^{-j}\right] \\
&= e^{\omega t}t^{-k}\left[\sum_{j=0}^{k}\frac{k!}{(k-j)!}\omega^{k-j}(M_1 e)^j t^{k-j}\right] \\
&= e^{\omega t}t^{-k}k!\left[\sum_{i=0}^{k}\frac{t^i}{i!}\omega^i (M_1 e)^{k-i}\right] \\
&\leq e^{\omega t}t^{-k}k!\left[\sum_{i=0}^{k}\frac{(t\epsilon)^i}{i!}\left(\frac{|\omega|}{\epsilon}\right)^i (M_1 e)^{k-i}\right].
\end{aligned}$$

Let $C_{k,\epsilon} = k! \max\left\{(M_1 e)^k, \left(\frac{|\omega|}{\epsilon}\right)(M_1 e)^{k-1}, ..., \left(\frac{|\omega|}{\epsilon}\right)^k\right\} > 0$. Then

$$\begin{aligned}\|A^k e^{tA}\| &\le e^{\omega t} t^{-k} C_{k,\epsilon} \left(\sum_{i=0}^{k} \frac{(t\epsilon)^i}{i!}\right) \\ &\le e^{\omega t} t^{-k} C_{k,\epsilon} e^{t\epsilon} \\ &= C_{k,\epsilon} e^{(\omega+\epsilon)t} t^{-k}.\end{aligned}$$

(iv): Let $t > 0$ and $x \in X$. Then by (i), $e^{tA}x \in \mathcal{D}(A)$. Since the integral defining e^{tA} is uniformly convergent in t, then by (2.28)

$$\begin{aligned}\frac{d}{dt} e^{tA} &= \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \frac{d}{dt} e^{t\lambda} R(\lambda, A) d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, A) d\lambda \\ &= A e^{tA}.\end{aligned}$$

Similarly, for $k \in \mathbb{N}$, we have that $e^{tA}x \in \mathcal{D}(A^k)$ and by (2.30), $\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}$. Thus the function $t \mapsto e^{tA}$ belongs to $\mathcal{C}((0,\infty); B(X))$. Now let $\epsilon > 0$ such that $2\epsilon < \theta - \frac{\pi}{2}$, and let $\eta = \theta - \epsilon$, so that $\frac{\pi}{2} < \eta < \theta$. Then, following an argument similar to that of Proposition 2.3.3, the function

$$t \mapsto e^{tA} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda$$

is well-defined if $\Re(t\lambda) < 0$ as $\Re\lambda \to -\infty$. Note that this is the case if $\frac{\pi}{2} < |\arg(t\lambda)| = |\arg t + \arg \lambda|$. But if

$$t \in S_\epsilon = \left\{t \in \mathbb{C} \mid t \neq 0, |\arg t| < \theta - 2\epsilon - \frac{\pi}{2}\right\}$$

then $|\arg(t\lambda)| \ge \frac{\pi}{2} + \epsilon$. Hence the map $t \mapsto e^{tA}$ extends to S_ϵ for all $\epsilon > 0$ such that $2\epsilon < \theta - \frac{\pi}{2}$. Thus we may take the derivative with respect to t and we see that the mapping $t \mapsto e^{tA}$ is analytic on S_ϵ with $\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}$ for all $k \in \mathbb{N}$ as before. Since this is true for each ϵ such that $2\epsilon \in (0, \theta - \frac{\pi}{2})$, $t \mapsto e^{tA}$ is analytic on

$$S_{\theta-\frac{\pi}{2}} = \left\{t \in \mathbb{C} \mid t \neq 0, |\arg t| < \theta - \frac{\pi}{2}\right\}.$$

□

Most commonly, the properties of above theorem are used to define an analytic semigroup, as is done in Pazy [18], Goldstein [10] and Showalter [22]. For our purposes, however, it is more useful to define the semigroup in terms of its infinitesimal generator A, since in applications to PDEs A is directly linked to the physical problem that we are dealing with. Thus we follow the method of Lundardi [14]. This definition is given below.

Definition 2.3.3 (**Analytic Semigroup**). Let $A : X \supseteq \mathcal{D}(A) \mapsto X$ be a sectorial operator. The family $\{e^{tA} \mid t \geq 0\}$ defined as

$$\begin{aligned} e^{tA} &= \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)d\lambda,\ t > 0, \\ e^{0A}x &= x,\ x \in X, \end{aligned}$$

where $r > 0$, $\eta \in (\frac{\pi}{2}, \theta)$, and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} \mid |\arg\lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \eta, |\lambda| = r\}$$

oriented counter-clockwise, is the analytic semigroup generated by A on X.

It follows, by Theorem 2.3.5 (ii), that the family of bounded linear operators e^{tA}, $t \geq 0$, as defined in Definition 2.3.3, is indeed a semigroup. We now show that e^{tA}, $t \geq 0$, is a C_0-semigroup.

Theorem 2.3.6. *If A is a sectorial operator, then the analytic semigroup generated by A is a C_0-semigroup with infinitesimal generator A.*

Proof. Since A is sectorial there exist constants $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $M > 0$ such that $S_{\omega,\theta} \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - \omega|}$ for $\lambda \in S_{\omega,\theta}$. Furthermore, the operator A generates an analytic semigroup e^{tA}, $t \geq 0$, defined as $e^{0A} = I$ and $e^{tA} = \dfrac{1}{2\pi i}\displaystyle\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)d\lambda$ for all $t > 0$ and for some $r > 0$. Suppose $x \in \mathcal{D}(A)$. Consider $\zeta \in \mathbb{R}$ such that $\zeta > \omega + r$. Then $\zeta \in \rho(A)$. Let $y = (\zeta I - A)x$, then $x = R(\zeta, A)y$.

Fix $t > 0$. By the resolvent identity, Theorem 1.4.6, we have that

$$\begin{aligned}
e^{tA}x =& e^{tA}R(\zeta, A)y \\
=& \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)R(\zeta, A)y d\lambda \\
=& \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}\frac{R(\lambda, A) - R(\zeta, A)}{\zeta - \lambda} y d\lambda \\
=& \frac{-1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}\frac{R(\lambda, A)}{\lambda - \zeta} y d\lambda + \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}\frac{R(\zeta, A)}{\lambda - \zeta} y d\lambda. \qquad (2.32)
\end{aligned}$$

If $\Re\lambda < \omega$, then

$$\|e^{t\lambda}R(\zeta, A)y\|_X \leq e^{t\Re\lambda}\|R(\zeta, A)\|\|y\|_X \leq e^{t\omega}\|R(\zeta, A)\|\|y\|_X$$

and $e^{t\lambda}R(\zeta, A)$ is bounded. Thus, since ζ lies to the right of $\omega+\gamma_{r,\eta}$, it follows from Lemma A.2.3 (1)(b) that

$$\int_{\omega+\gamma_{r,\eta}} \frac{e^{t\lambda}R(\zeta, A)}{\lambda - \zeta} y d\lambda = 0. \qquad (2.33)$$

For $\lambda \in (\omega + \gamma_{r,\eta})$ with $\Re\lambda < 0$,

$$\begin{aligned}
\left\| e^{t\lambda}\frac{R(\lambda, A)}{\lambda - \zeta} y\right\|_X &\leq e^{t\Re\lambda}\frac{M\|y\|_X}{|\lambda - \omega||\lambda - \zeta|} \\
&\leq \frac{M\|y\|_X}{|\lambda - \omega||\lambda - \zeta|}.
\end{aligned}$$

Hence

$$\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}\frac{R(\lambda, A)}{\lambda - \zeta} y d\lambda \qquad (2.34)$$

is uniformly convergent in $t \geq 0$. Therefore, by (2.32), (2.33) and (2.34),

$$\begin{aligned}
\lim_{t\to 0^+} e^{tA}x &= \lim_{t\to 0^+} \frac{-1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}\frac{R(\lambda, A)}{\lambda - \zeta} y d\lambda \\
&= \frac{-1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} \lim_{t\to 0^+} e^{t\lambda}\frac{R(\lambda, A)}{\lambda - \zeta} y d\lambda \\
&= \frac{-1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} \frac{R(\lambda, A)}{\lambda - \zeta} y d\lambda. \qquad (2.35)
\end{aligned}$$

We have that $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - \omega|}$ for all $\lambda \in S_{\omega,\theta}$. Thus, for any $\rho \in (0, r)$, $R(\lambda, A)$ will be bounded on $\mathcal{D} = S_{\omega,\theta} \setminus B_\rho(\omega)$, where $B_\rho(\omega)$ is the open ball with radius ρ centred at $\omega + 0i$. Furthermore, $\mathcal{D}$ contains the half-plane $\{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$ and the curve $\omega + \gamma_{r,\eta}$. By Theorem 2.3.1, the mapping $\rho(A) \ni \lambda \mapsto R(\lambda, A) \in B(X)$ is analytic. It follows from Lemma A.2.3 (2)(a) with ζ to the right of $\omega + \gamma_{r,\eta}$ that

$$\int_{\omega+\gamma_{r,\eta}} \frac{R(\lambda, A)}{\lambda - \zeta} y d\lambda = -2\pi i R(\zeta, A)y. \tag{2.36}$$

Substituting (2.36) into (2.35) gives

$$\lim_{t\to 0^+} e^{tA}x = R(\zeta, A)y = x \tag{2.37}$$

for every $x \in \mathcal{D}(A)$. We now prove this result for $x \in X$. Fix $x \in X$. Since $\overline{\mathcal{D}(A)} = X$, there exists a sequence $(x_n) \in \mathcal{D}(A)$ such that (x_n) converges to x in X. By Theorem 2.3.5 $(iii)(a)$ there exists a $M_0 > 0$ such that $\|e^{tA}\| \leq M_0 e^{\omega t}$. Thus

$$\begin{aligned} \|e^{tA}x - x\|_X &\leq \|e^{tA}x - e^{tA}x_n\|_X + \|e^{tA}x_n - x_n\|_X + \|x_n - x\|_X \\ &\leq (Me^{\omega t} + 1)\|x - x_n\|_X + \|e^{tA}x_n - x_n\|_X \end{aligned} \tag{2.38}$$

for all $n \in \mathbb{N}$. Fix $\epsilon > 0$. Since $\mathbb{R} \ni t \mapsto M_0 e^{\omega t} \in \mathbb{R}$ is continuous and (x_n) converges to x there exists a $\delta_1 > 0$ and a $N \in \mathbb{N}$ such that if $t \in (0, \delta_1)$ then

$$(Me^{\omega t} + 1)\|x - x_N\|_X \leq \frac{\epsilon}{2}. \tag{2.39}$$

Since $x_N \in \mathcal{D}(A)$, (2.37) gives us that $\lim\limits_{t\to 0^+} e^{tA}x_N = x_N$. Thus there exists a $\delta \in (0, \delta_1]$ such that if $t \in (0, \delta)$ then

$$\|e^{tA}x_N - x_N\|_X < \frac{\epsilon}{2}. \tag{2.40}$$

Substituting (2.39) and (2.40) into (2.38) gives that if $t \in (0, \delta)$ then

$$\|e^{tA}x - x\|_X < \epsilon.$$

Hence

$$\lim_{t\to 0^+} e^{tA}x = x \tag{2.41}$$

for all $x \in X$ so that e^{tA}, $t \geq 0$, is a C_0-semigroup on X.

Next we show that A is the infinitesimal generator of e^{tA}, $t \geq 0$. That is, we show that, for all $x \in \mathcal{D}(A)$,

$$\lim_{t\to 0^+} \frac{1}{t}(e^{tA}x - x) = Ax.$$

Fix $x \in \mathcal{D}(A)$. Then $Ae^{tA}x = e^{tA}Ax$ for $t > 0$ by Theorem 2.3.5 (i). We show that the function $t \mapsto Ae^{tA}x$ is continuous on $[0, \infty)$. Since the mapping $t \mapsto e^{tA}x$ is an belongs to $\mathcal{C}^\infty((0,\infty); B(X))$ by Theorem 2.3.5 (iv) then $t \mapsto Ae^{tA}x$ is continuous on $(0, \infty)$. Thus we only need to show continuity at $t = 0$. Let $s > 0$, then

$$\begin{aligned}\|Ae^{sA}x - Ae^{0A}x\|_X &= \|A(e^{sA} - I)x\|_X \\ &= \|(e^{sA} - I)Ax\|_X.\end{aligned}$$

By (2.41), $\lim\limits_{s\to 0^+}(e^{sA} - I)Ax = 0$. Thus

$$\lim_{s\to 0^+} \|Ae^{sA}x - Ae^{0A}x\|_X = 0,$$

and the function $t \mapsto Ae^{tA}x$ is continuous on $[0, \infty)$. Thus $t \mapsto Ae^{tA}x$ is Bochner integrable.

By Proposition A.1.5 (a), the function

$$F(t) = \int_0^t Ae^{sA}x ds,\ t \geq 0$$

is differentiable on $(0, \infty)$ and, by Theorem 2.3.5 (iv),

$$F'(t) = Ae^{tA}x = \frac{d}{dt}e^{tA}x$$

for $t > 0$. Thus

$$F(t) = e^{tA}x + c_x$$

for $t > 0$ and for some $c_x \in X$. However,

$$0 = F(0) = x + c_x$$

so that $c_x = -x$. Hence

$$\int_0^t Ae^{sA}x ds = F(t) = e^{tA}x - x$$

for $t > 0$. Thus, for every $t > 0$,

$$\frac{1}{t}(e^{tA}x - x) = \frac{1}{t}\int_0^t Ae^{sA}x ds. \tag{2.42}$$

By Proposition A.1.5 (b) we have that for $t > 0$

$$\begin{aligned}\lim_{t\to 0^+}\left\|\frac{1}{t}\int_0^t Ae^{sA}xds - Ax\right\|_X &= \lim_{t\to 0^+}\left\|\frac{1}{t}\int_0^t (Ae^{sA}x - Ax)ds\right\|_X \\ &\leq \lim_{t\to 0^+}\frac{1}{t}\int_0^t \|Ae^{sA}x - Ae^{0t}x\|_X ds \\ &= 0.\end{aligned}$$

Thus for $t > 0$

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t Ae^{sA}xds = Ax. \tag{2.43}$$

From (2.42) and (2.43) it follows that for all $t > 0$

$$\lim_{t\to 0^+}\frac{1}{t}(e^{tA}x - x) = Ax.$$

Thus A is the infinitesimal generator of the C_0-semigroup e^{tA}. □

It follows that all the properties of C_0-semigroups stated in Theorem 1.4.3 hold for analytic semigroups. However, Theorem 2.3.5 (i) is a stronger result than Theorem 1.4.3 (iii).

Theorem 2.3.7. *If e^{tA}, $t \geq 0$, is an analytic semigroup then e^{tA} is uniformly continuous for $t > 0$.*

Proof. Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of e^{tA}, $t \geq 0$. Let $b > 0$ be arbitrary and let M_b be an upper bound for $\{\|e^{tA}\| \mid t \in (0, b)\}$. By Theorem 2.3.5 (i) and (iv), $e^{tA}x \in \mathcal{D}(A)$ and $\frac{d}{dt}e^{tA}x = Ae^{tA}x$ for any $x \in X$ and $t > 0$. Thus the domain of Ae^{tA}, $t > 0$, is all of X. Furthermore, since A is a closed linear operator and since e^{tA} is bounded on $(0, b)$, then Ae^{tA} is a closed operator for each $t \in (0, b)$. Thus, by the Closed Graph Theorem, Theorem C.0.10, Ae^{tA} is a bounded linear operator on X for each $t \in (0, b)$. To show that e^{tA} is uniform continuous for $t > 0$, consider $t_1 \in (0, b)$ and $t_2 \in \mathbb{R}$ such that $0 < t_1 \leq t_2 \leq t_1 + b$. Since e^{tA}, $t \geq 0$, is a C_0-semigroup, by Theorem 2.3.6, then by Theorem 1.4.3 (iv), for any $x \in X$ we have

$$\begin{aligned}\|e^{t_2A}x - e^{t_1A}x\|_X &= \left\|\int_{t_1}^{t_2} Ae^{sA}xds\right\|_X \\ &= \left\|\int_{t_1}^{t_2} e^{(s-t_1)A}Ae^{t_1A}xds\right\|_X \\ &\leq (t_2 - t_1)M_b\|Ae^{t_1A}\|\|x\|_X.\end{aligned}$$

Therefore $\|e^{t_2A} - e^{t_1A}\| \leq (t_2 - t_1)M_b\|Ae^{t_1A}\|$ so that

$$\lim_{t_2 \to t_1^+} \|e^{t_2A} - e^{t_1A}\| = 0.$$

In the same way

$$\lim_{t_2 \to t_1^-} \|e^{t_1A} - e^{t_2A}\| = 0.$$

Since b is arbitrary e^{tA} is uniformly continuous for $t > 0$. $\square$

2.4 Perturbation Theory

As mentioned in the introduction, one of the main benefits of semigroup theory is the relatively easy way it lends itself to perturbation theory. This allows one to generalise results that hold for a given operator A to operators of type $A + B$. In particular, if A is the infinitesimal generator of a C_0-semigroup then perturbation theory determines specific conditions on the operator B for $A + B$ to be the infinitesimal generator of a C_0-semigroup. The simplest cases are when B is bounded. We consider only a few basic results related to strongly continuous and analytic semigroups, since this enables us to extend our existence theorem to a broader class of problems, as discussed in Remark 2.4.3. These results can be found in [18, Chapter 3]. For a more complete discussion on the perturbation of semigroups we refer the reader to [7, Chapter 11].

Let X be a Banach space.

In what follows we make use of the following technical Lemma.

Lemma 2.4.1. *[18, Lemma 1.5.1] Let A be a linear operator on a Banach space X for which $\rho(A)$ contains $(0, \infty)$. If there exists an $M > 0$ such that*

$$\|\lambda^n R(\lambda, A)^n\| \leq M \text{ for } n = 1, 2, ..., \ \lambda > 0$$

then there exists a norm $\|\cdot\|_X^*$ on X which is equivalent to the original norm $\|\cdot\|_X$ on X in the sense that

$$\|x\|_X \leq \|x\|_X^* \leq M\|x\|_X \text{ for every } x \in X$$

and

$$\|\lambda R(\lambda, A)x\|_X^* \leq \|x\|_X^* \text{ for all } x \in X, \ \lambda > 0.$$

Remark 2.4.1. It is clear that we can extend this theorem to the case where $\rho(A)$ contains (ω, ∞) for some $\omega \in \mathbb{R}$. In this case,

$$\|(\lambda - \omega)R(\lambda, A)x\|_X^* \leq \|x\|_X^* \text{ for every } x \in X, \ \lambda > \omega.$$

Furthermore, Pazy shows in the proof of [18, Theorem 1.5.2] that if A is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, such that $\|T(t)\| \leq M$ for some $M \geq 1$ and for all $t \geq 0$, then $\|T(t)x\|_X^* \leq \|x\|_X^*$ for each $x \in X$ and $t \geq 0$. This estimate can be generalized to the case where $\rho(A) \supset (\omega, \infty)$ in the same way as before, and, as Pazy suggests in [18, Section 1.5, Equation (5.17)], if $\|T(t)\| \leq Me^{\omega t}$ then

$$\|T(t)x\|_X^* \leq \|x\|_X^* e^{\omega t}$$

for each $x \in X$.

Theorem 2.4.2. *Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, on X, satisfying $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$, for some $M \geq 1$ and $\omega > 0$. If B is a bounded linear operator on X then $A + B$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, on X satisfying $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$ for $t \geq 0$.*

Proof. From the Hille-Yosida Theorem, Theorem 1.4.8, we have that A is closed, $\mathcal{D}(A)$ is dense in X, $\rho(A)$ contains (ω, ∞) and

$$\|R(\lambda, A)\| \leq \frac{M}{(\lambda - \omega)} \text{ for } \lambda > \omega,\ n \in \mathbb{N}.$$

Since B is bounded, $A+B$ is closed and $\mathcal{D}(A+B)$ is dense in X. Furthermore, from Lemma 2.4.1 and Remark 2.4.1 there exists a norm $\|\cdot\|_X^*$ on X such that, for every $x \in X$, the following hold:

(i) $\|x\|_X \leq \|x\|_X^* \leq M\|x\|_X$,

(ii) $\|R(\lambda, A)x\|_X^* \leq (\lambda - \omega)^{-1}\|x\|_X^*$ for $\lambda > \omega$

(iii) $\|T(t)x\|_X^* \leq \|x\|_X^* e^{\omega t}$.

Denote by $\|\cdot\|^*$ the operator norm on the set of bounded linear operators on X with respect to the norm $\|\cdot\|_X^*$. That is,

$$\|T\|^* = \sup\{\|Tx\|_X^* \mid x \in X,\ \|x\|_X^* = 1\}.$$

It follows from (i) that $\|\cdot\|^*$ is equivalent to the standard operator norm $\|\cdot\|$. Furthermore,

$$\|R(\lambda, A)\|^* \leq (\lambda - \omega)^{-1} \tag{2.44}$$

for $\lambda > \omega$ by (ii) and $\|T(t)\|^* \leq e^{\omega t}$, $t \geq 0$, by (iii).

We now show that there exist $\lambda \in \rho(A)$ such that the operator $I - BR(\lambda, A)$ is invertible. Fix $\lambda \in \rho(A)$ such that $\lambda > \omega + \|B\|^*$. Then (2.44) gives

$$\|BR(\lambda, A)\|^* \leq \|B\|^*(\lambda - \omega)^{-1} < 1. \tag{2.45}$$

Suppose

$$[I - BR(\lambda, A)]x = [I - BR(\lambda, A)]y,$$

for some $x, y \in X$. Then, by the reverse triangle inequality and (2.45),

$$\begin{aligned} 0 &= \|[I - BR(\lambda, A)](x - y)\|_X^* \\ &\geq |\|x - y\|_X^* - \|BR(\lambda, A)(x - y)\|_X^*| \\ &= \|x - y\|_X^* - \|BR(\lambda, A)(x - y)\|_X^* \\ &\geq \|x - y\|_X^*(1 - \|BR(\lambda, A)\|^*). \end{aligned} \tag{2.46}$$

From (2.45) and (2.46) we have that $\|x - y\|_X^* = 0$ and so $x = y$. Thus the operator $[I - BR(\lambda, A)]$ is injective. Furthermore, for all $x \in X$

$$\begin{aligned} [I - BR(\lambda, A)] \sum_{n=0}^{\infty} [BR(\lambda, A)]^n x &= I - BR(\lambda, A)x + BR(\lambda, A)x - ... \\ &= x. \end{aligned} \tag{2.47}$$

Thus $[I - BR(\lambda, A)]$ is surjective. In a similar manner to (2.47) we can show that $\sum_{n=0}^{\infty} [BR(\lambda, A)]^n [I - BR(\lambda, A)]x = x$ for all $x \in X$. Thus $[I - BR(\lambda, A)]$ is invertible and

$$[I - BR(\lambda, A)]^{-1} = \sum_{n=0}^{\infty} [BR(\lambda, A)]^n.$$

Set

$$R = R(\lambda, A)[I - BR(\lambda, A)]^{-1} = \sum_{n=0}^{\infty} R(\lambda, A)[BR(\lambda, A)]^n. \tag{2.48}$$

Now

$$\begin{aligned} [\lambda I - (A + B)]R &= [(\lambda I - A) - B]R(\lambda, A)[I - BR(\lambda, A)]^{-1} \\ &= [I - BR(\lambda, A)]^{-1} - BR(\lambda, A)[I - BR(\lambda, A)]^{-1} \\ &= [I - BR(\lambda, A)][I - BR(\lambda, A)]^{-1} \\ &= I. \end{aligned}$$

We also have from (2.48) that for each $x \in \mathcal{D}(A)$

$$
\begin{aligned}
R[\lambda I-(A+B)]x =&\sum_{n=0}^{\infty} R(\lambda,A)[BR(\lambda,A)]^n[\lambda I-(A+B)]x\\
=&R(\lambda,A)[(\lambda I-A)-B]x\\
&+\sum_{n=1}^{\infty} R(\lambda,A)[BR(\lambda,A)]^n[\lambda I-(A+B)]x\\
=&x-R(\lambda,A)Bx+R(\lambda,A)BR(\lambda,A)[(\lambda I-A)-B]x\\
&+R(\lambda,A)[BR(\lambda,A)]^2[(\lambda I-A)-B]x+\ldots\\
=&x-R(\lambda,A)Bx+R(\lambda,A)Bx-R(\lambda,A)BR(\lambda,A)Bx\\
&+R(\lambda,A)BR(\lambda,A)Bx-R(\lambda,A)[BR(\lambda,A)]^2Bx+\ldots\\
=&x.
\end{aligned}
$$

Thus $\lambda \in \rho(A+B)$ and

$$R(\lambda,A+B)=R=R(\lambda,A)[I-BR(\lambda,A)]^{-1}. \tag{2.49}$$

Moreover, from (2.48), (2.44) and (2.45) we have

$$
\begin{aligned}
\|R(\lambda,A+B)\|^* &= \left\|\sum_{n=0}^{\infty} R(\lambda,A)[BR(\lambda,A)]^n\right\|^*\\
&\le (\lambda-\omega)^{-1}\sum_{n=0}^{\infty}(\|BR(\lambda,A)\|^*)^n\\
&= (\lambda-\omega)^{-1}(1-\|BR(\lambda,A)\|^*)^{-1}.
\end{aligned} \tag{2.50}
$$

Now, from (2.44) we have that

$$(\lambda-\omega)\|BR(\lambda,A)\|^* \le (\lambda-\omega)\|B\|^*\|R(\lambda,A)\|^* \le \|B\|^*.$$

Thus

$$1 \le \frac{(\lambda-\omega)-(\lambda-\omega)\|BR(\lambda,A)\|^*}{(\lambda-\omega)-\|B\|^*}$$

so that

$$(\lambda-\omega)^{-1} \le \frac{1-\|BR(\lambda,A)\|^*}{\lambda-\omega-\|B\|^*},$$

and we conclude that

$$(\lambda-\omega)^{-1}(1-\|BR(\lambda,A)\|^*)^{-1} \le (\lambda-\omega-\|B\|^*)^{-1}. \tag{2.51}$$

Substituting (2.51) into (2.50) gives that

$$\|R(\lambda, A+B)\|^* \leq (\lambda - \omega - \|B\|^*)^{-1}. \tag{2.52}$$

Thus $\rho(A+B)$ contains $Q = \{\lambda \in \rho(A) \mid \lambda > \omega + \|B\|^*\}$ and (2.52) holds for $\lambda \in Q$. By the Hille-Yosida Theorem we then have that $A+B$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, satisfying

$$\|S(t)\|^* \leq e^{(\omega + \|B\|^*)t}$$

for $t \geq 0$. Returning to the original, equivalent norm $\|\cdot\|$ on $B(X)$ gives

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$$

for $t \geq 0$ as desired. □

In is natural to wonder how the C_0-semigroup $T(t)$, $t \geq 0$, generated by A, and the C_0-semigroup $S(t)$, $t \geq 0$, generated by $A+B$, where B is a bounded linear operator, are related. For more on this topic we refer the reader to [18, Proposition 3.1.2, Corollary 3.1.3].

We now consider perturbations of the infinitesimal generators of analytic semigroups. The first result in this regard is the following.

Theorem 2.4.3. *Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is a sectorial operator and $B : X \supset \mathcal{D}(B) \mapsto X$ is a closed linear operators such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. Then there exists a $\delta \in \left(0, \frac{1}{2}\right)$ such that if there exist constants $a \in [0, \delta]$ and $b \geq 0$ such that*

$$\|Bx\|_X \leq a\|Ax\|_X + b\|x\|_X \text{ for all } x \in \mathcal{D}(A), \tag{2.53}$$

then $A+B$ is a sectorial operator.

Proof. Since A is sectorial, there exist constants $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $M > 0$ such that

$$S_{\theta,\omega} \subset \rho(A)$$

and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|} \text{ for all } \lambda \in S_{\theta,\omega}. \tag{2.54}$$

Let $\delta = \frac{1}{2}(1+M)^{-1} \in \left(0, \frac{1}{2}\right)$. Assume that (3.44) holds for some $a \in [0, \delta]$ and $b \geq 0$. From (2.53), (2.54) and (2.24) we have that for every $x \in X$ and

$\lambda \in S_{\theta,\omega}$

$$\begin{aligned}
\|BR(\lambda, A)x\|_X &\leq a\|AR(\lambda, A)x\|_X + b\|R(\lambda, A)x\|_X \\
&= a\|\lambda R(\lambda, A)x - x\|_X + b\|R(\lambda, A)x\|_X \\
&= a\|(\lambda - \omega)R(\lambda, A)x + \omega R(\lambda, A)x - x\|_X + b\|R(\lambda, A)x\|_X \\
&\leq a(M+1)\|x\|_X + \frac{M(b + a|\omega|)}{|\lambda - \omega|}\|x\|_X \\
&\leq a(M+1)\|x\|_X + \frac{M(b + |\omega|)}{|\lambda - \omega|}\|x\|_X. \qquad (2.55)
\end{aligned}$$

Thus $BR(\lambda, A)$ is bounded. In particular, if $\Re\lambda > \omega + 2M(b + |\omega|)$ then (2.55) implies that $\|BR(\lambda, A)\| < 1$. Then, in the same way as the proof of Theorem 2.4.2, in particular (2.46) and (2.47), the operator $[I - BR(\lambda, A)]$ is invertible with a bounded inverse.

Continuing with the same reasoning as in the proof of Theorem 2.4.2, in particular (2.48) and (2.49), we have that if $\Re\lambda > \omega + 2M(b + |\omega|)$ then $\lambda \in \rho(A + B)$ and

$$[\lambda I - (A + B)]^{-1} = R(\lambda, A)[I - BR(\lambda, A)]^{-1}. \qquad (2.56)$$

Let $\omega' > \omega + 2M(b + |\omega|)$. Then $S_{\theta,\omega'} \subset S_{\theta,\omega}$ and, from (2.56), $S_{\theta,\omega'} \subset \rho(A+B)$. Furthermore, for $\lambda \in S_{\theta,\omega'}$ we have

$$\begin{aligned}
\|R(\lambda, A + B)\| &= \|R(\lambda, A)[I - BR(\lambda, A)]^{-1}\| \\
&\leq \frac{MN}{|\lambda - \omega|}
\end{aligned}$$

where $N = \|[I - BR(\lambda, A)]^{-1}\|$. Thus $A + B$ is a sectorial operator. $\square$

Remark 2.4.2. If A and B satisfy the relationship (2.53) then they are called *relatively bounded*, see [7, Section 11.1].

Corollary 2.4.4. *Let $A : X \supset \mathcal{D}(A) \mapsto X$ be the infinitesimal generator of an analytic semigroup. If B is a bounded linear operator on X then $A + B$ is the infinitesimal generator of an analytic semigroup.*

Proof. Since B is bounded we have

$$\|Bx\|_X \leq a\|Ax\|_X + \|B\|\|x\|_X$$

for every $a \geq 0$ and $x \in X$. The result follows from Theorem 2.4.3. $\square$

Remark 2.4.3. Pazy shows in [18, Section 3.1, Proposition 1.4] that if X is a Banach space, $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$, and B is a bounded linear operator on X, then $A+B$ is the infinitesimal generator of a compact semigroup $S(t)$, $t \geq 0$. However, we do not need this result since our aim is to apply our existence theory to parabolic problems. Indeed, in such cases we obtain compactness of the semigroup via analyticity of the semigroup and compactness of the resolvent operator of its infinitesimal generator, see Theorems 2.1.6 and 2.3.7. Note that, in the case of parabolic problems, A is an elliptic operator acting on $\mathcal{L}^2(\Omega)$ for some open set $\Omega \subset \mathbb{R}^n$. A perturbation of A by an operator of the form

$$Bu = \sum_{i=1}^{n} \beta^i(x) u_{x_i} + \gamma(x) u, \tag{2.57}$$

is also elliptic, and therefore has a compact resolvent operator $R(\lambda, A+B)$ for some $\lambda \in \rho(A+B)$, by Theorem 2.2.2.

To make Remark 2.4.3 precise we consider the complexification of A acting on the complexification of Hilbert space $\mathcal{L}^2(\Omega)$.

2.5 Complexification of Operators

In order to apply the existence, regularity and perturbation results for the abstract Cauchy problem from sections 3.2, 3.4 and 2.4 respectively, we need to work with analytic semigroups defined on a complex Banach space. However, the problems we are considering are set in a real Hilbert space, and it is intuitive that the physical solutions should take values in a real Hilbert space. In this section we show how to associate with a real Hilbert space H a complex space $\tilde{H}$. For each operator A on H there is a corresponding operator $\tilde{A}$ on $\tilde{H}$, and for each semigroup $T(t)$, $t \geq 0$, on H there is a corresponding semigroup $\tilde{T}(t)$, $t \geq 0$, on $\tilde{H}$. Many of the properties of a semigroup on H can be deduced from the semigroup induced on $\tilde{H}$. For a more generous discussion on the complexification of Banach spaces we refer the reader to [16].

Operators on the Complexification of a Real Hilbert Space

Definition 2.5.1 (**Complexification of a real Hilbert Space**)**.** Let H be a real Hilbert space. The complexification of H is the complex vector space $\tilde{H}$ defined as

$$\tilde{H} := \{x + iy \mid x \in H, y \in H\}$$

with addition and scalar multiplication given by

$$\begin{aligned}(x+iy)+(u+iv) &= (x+u)+i(y+v)\\ (\alpha+i\beta)(x+iy) &= (\alpha x-\beta y)+i(\alpha y+\beta x)\end{aligned}$$

for all $(x+iy),(u+iv)\in\tilde{H}$ and $(\alpha+i\beta)\in\mathbb{C}$.

In the following two theorems we show that $\tilde{H}$ is a complex Hilbert space.

Theorem 2.5.1. *The function $(\cdot,\cdot)_{\tilde{H}}:\tilde{H}\times\tilde{H}\mapsto\mathbb{C}$ defined as*

$$(x+iy,u+iv)_{\tilde{H}} := (x,u)_H+(y,v)_H+i(y,u)_H-i(x,v)_H$$

is an inner product on $\tilde{H}$.

Proof. The function $(\cdot,\cdot)_{\tilde{H}}$ is an inner product on $\tilde{H}$ if, for any $a,b\in\mathbb{C}$ and $\mathbf{x},\mathbf{u},\mathbf{q}\in\tilde{H}$, we have

(i) $(\mathbf{x},\mathbf{u})_{\tilde{H}}=\overline{(\mathbf{u},\mathbf{x})_{\tilde{H}}}$.

(ii) $(\mathbf{x}+\mathbf{u},\mathbf{p})_{\tilde{H}}=(\mathbf{x},\mathbf{p})_{\tilde{H}}+(\mathbf{u},\mathbf{p})_{\tilde{H}}$.

(iii) $(a\mathbf{x},\mathbf{p})_{\tilde{H}}=a(\mathbf{x},\mathbf{p})_{\tilde{H}}$

(iv) $(\mathbf{x},\mathbf{x})_{\tilde{H}}\geq 0$.

(v) $(\mathbf{x},\mathbf{x})_{\tilde{H}}=0 \implies \mathbf{x}=0$.

Properties (i) to (v) above all follow easily from the properties of $(\cdot,\cdot)_H$ and the definition of $(\cdot,\cdot)_{\tilde{H}}$. We demonstrate the general idea by proving (iii). Let $\mathbf{x}=x+iy\in\tilde{H}$, $\mathbf{p}=p+iq\in\tilde{H}$ and $a=\alpha+i\beta\in\mathbb{C}$. Then

$$\begin{aligned}(a[p+iq],x+iy)_{\tilde{H}} &= ([\alpha p-\beta q]+i[\beta p+\alpha q],x+iy)_{\tilde{H}}\\ &= (\alpha p-\beta q,x)_H+(\beta p+\alpha q,y)_H-i(\alpha p-\beta q,y)_H\\ &\quad +i(\beta p+\alpha q,x)_H\\ &= (\alpha+i\beta)[(p,x)_H+(q,y)_H]+(\beta-i\alpha)[(p,y)_H-(q,x)_H]\\ &= (\alpha+i\beta)\{[(p,x)_H+(q,y)_H]-i[(p,y)_H-(q,x)_H]\}\\ &= a(p+iq,x+iy)_{\tilde{H}}.\end{aligned}$$

□

Remark 2.5.1. Note that if $x+iy\in\tilde{H}$ then

$$\|x+iy\|^2_{\tilde{H}}=(x+iy,x+iy)_{\tilde{H}}=\|x\|^2_H+\|y\|^2_H.$$

Furthermore, for each $x\in H$, $x+0i\in\tilde{H}$ and $\|x\|_H=\|x+0i\|_{\tilde{H}}$.

Theorem 2.5.2. *The vector space $\tilde{H}$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\tilde{H}}$.*

Proof. We have already shown in Theorem 2.5.1 that $\tilde{H}$ is an inner product space with inner product $(\cdot,\cdot)_{\tilde{H}}$. We need to show that $\tilde{H}$ is complete with respect to the norm $\|\cdot\|_{\tilde{H}}$. Suppose $(\mathbf{z}_n)$ is a Cauchy sequence in $\tilde{H}$ with respect to $\|\cdot\|_{\tilde{H}}$. For $n \in \mathbb{N}$, $\mathbf{z}_n = x_n + iy_n$ with $x_n, y_n \in H$. If $n, m \in \mathbb{N}$ we have

$$\|\mathbf{z}_n - \mathbf{z}_m\|_{\tilde{H}}^2 = \|x_n - x_m\|_H^2 + \|y_n - y_m\|_H^2, \tag{2.58}$$

so that

$$\|x_n - x_m\|_H \leq \|\mathbf{z}_n - \mathbf{z}_m\|_{\tilde{H}}$$

and

$$\|y_n - y_m\|_H \leq \|\mathbf{z}_n - \mathbf{z}_m\|_{\tilde{H}}.$$

Thus (x_n) and (y_n) are Cauchy sequences in H, and since H is a Hilbert space they converge to limits in H, say x and y respectively. Thus the sequence $(\mathbf{z}_n)$ converges to $\mathbf{z} = x + iy \in \tilde{H}$, since $\|\mathbf{z}_n - \mathbf{z}\|_{\tilde{H}}^2 = \|x_n - x\|_H^2 + \|y_n - y\|_H^2$. □

Definition 2.5.2 (Complexification of a Linear Operator). Let H be a real Hilbert space and A a linear operator on H with domain $\mathcal{D}(A)$. The complexification of A, denoted $\tilde{A}$, acts on the complex Hilbert space $\tilde{H}$ and is given by

$$\mathcal{D}(\tilde{A}) := \{x + iy \mid x \in \mathcal{D}(A), y \in \mathcal{D}(A)\} \subset \tilde{H},$$
$$\tilde{A}(x + iy) := Ax + iAy,\ x + iy \in \mathcal{D}(\tilde{A}).$$

Theorem 2.5.3. *An operator $A : H \supseteq \mathcal{D}(A) \mapsto H$ is closed if and only if its complexification $\tilde{A} : \tilde{H} \supseteq \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$ is closed.*

Proof. Suppose that the operator $A : H \supseteq \mathcal{D}(A) \mapsto H$ is closed. That is, if $(x_n) \subset \mathcal{D}(A)$ is a sequence converging to $x \in H$ and (Ax_n) converges to $y \in H$, both with respect to $\|\cdot\|_H$, then $x \in \mathcal{D}(A)$ and $Ax = y$. Now suppose $\tilde{A}$ is the complexification of A. We consider a sequence $(\mathbf{z}_n) = (x_n + iy_n) \subset \mathcal{D}(\tilde{A})$ such that $(\mathbf{z}_n)$ converges to $\mathbf{z} = x + iy \in \tilde{H}$ and $(\tilde{A}\mathbf{z}_n)$ converges to $\mathbf{w} = u + iv \in \tilde{H}$, both with respect to $\|\cdot\|_{\tilde{H}}$. That is,

$$\lim_{n\to\infty} \|\mathbf{z}_n - \mathbf{z}\|_{\tilde{H}} = 0 \text{ and } \lim_{n\to\infty} \|\tilde{A}\mathbf{z}_n - \mathbf{w}\|_{\tilde{H}} = 0.$$

By Definition 2.5.2, $x_n, y_n \in \mathcal{D}(A)$ for each $n \in \mathbb{N}$. By Remark 2.5.1 we have that $\lim_{n\to\infty} \|x_n - x\|_H = 0$, $\lim_{n\to\infty} \|y_n - y\|_H = 0$, $\lim_{n\to\infty} \|Ax_n - u\|_H = 0$ and

$\lim_{n\to\infty} \|Ay_n - v\|_H = 0$. Since A is closed, $x, y \in \mathcal{D}(A)$ and $u = Ax$, $v = Ay$. Hence $\mathbf{z} \in \mathcal{D}(\tilde{A})$ and $\mathbf{w} = \tilde{A}\mathbf{z}$. Thus $\tilde{A}$ is closed.

Conversely suppose $\tilde{A}$ is closed, and that $(x_n) \subset \mathcal{D}(A)$ is a sequence converging to $x \in H$, such that (Ax_n) converges to $y \in H$, both with respect to $\|\cdot\|_H$. It is clear that $\mathbf{x}_n = x_n + 0i \in \mathcal{D}(\tilde{A})$ and that $\tilde{A}\mathbf{x}_n = Ax_n + 0i$ for each $n \in \mathbb{N}$. Since (x_n) converges to x and (Ax_n) converges to y, it follows that $(\mathbf{x}_n)$ converges to $\mathbf{x} = x + 0i$ and $(\tilde{A}\mathbf{x}_n)$ converges to $\mathbf{y} = y + 0i$, both with respect to $\|\cdot\|_{\tilde{H}}$. Thus since $\tilde{A}$ is closed, $\mathbf{x} \in \mathcal{D}(\tilde{A})$ and $\mathbf{y} = \tilde{A}x$. By Definition 2.5.2 it follows that $x \in \mathcal{D}(A)$ and $y = Ax$. Thus A is closed. $\square$

Proposition 2.5.4. *Consider a linear operator A on H. Then $\overline{\mathcal{D}(A)} = H$ if and only if $\overline{\mathcal{D}(\tilde{A})} = \tilde{H}$.*

Proof. Suppose that $\overline{\mathcal{D}(A)} = H$ and let $\mathbf{z} = x + iy \in \tilde{H}$. Then by the definition of $\tilde{H}$, $x, y \in H$ and we can find sequences (x_n) and (y_n) contained in $\mathcal{D}(A)$ such that $\lim_{n\to\infty} \|x_n - x\|_H = 0$ and $\lim_{n\to\infty} \|y_n - y\|_H = 0$. Define $\mathbf{z}_n := x_n + iy_n \in \mathcal{D}(\tilde{A})$ for each $n \in \mathbb{N}$. Then by Remark 2.5.1

$$\lim_{n\to\infty} \|\mathbf{z}_n - \mathbf{z}\|^2_{\tilde{H}} = \lim_{n\to\infty} \|x_n - x\|^2_H + \lim_{n\to\infty} \|y_n - y\|^2_H = 0.$$

Thus $\overline{\mathcal{D}(\tilde{A})} = \tilde{H}$.

Conversely suppose that $\overline{\mathcal{D}(\tilde{A})} = \tilde{H}$. Let $x \in H$. Then $\mathbf{x} = x + 0i \in \tilde{H}$. Now since $\overline{\mathcal{D}(\tilde{A})} = \tilde{H}$ we can find a sequence $(\mathbf{z}_n) = (x_n + iy_n)$ in $\mathcal{D}(\tilde{A})$ converging to $\mathbf{x}$. Since

$$\|x - x_n\|^2_H \leq \|x - x_n\|^2_H + \|y_n\|^2_H = \|\mathbf{x} - \mathbf{z}_n\|^2_{\tilde{H}}$$

for every $n \in \mathbb{N}$ it follows that (x_n) converges to x. Furthermore, $\mathbf{z}_n = x_n + iy_n \in \mathcal{D}(\tilde{A})$ for every $n \in \mathbb{N}$, so $x_n \in \mathcal{D}(A)$ for every $n \in \mathbb{N}$ by definition of $\tilde{A}$. Hence $\overline{\mathcal{D}(A)} = H$. $\square$

Theorem 2.5.5. *A linear operator $T : H \mapsto H$ is bounded if and only if its complexification $\tilde{T} : \tilde{H} \mapsto \tilde{H}$ is bounded. Furthermore, in the case that they are bounded, $\|T\| = \|\tilde{T}\|$.*

Proof. Suppose $T : H \mapsto H$ is bounded. Then for all $\mathbf{z} = x + iy \in \tilde{H}$ we have that

$$\begin{aligned}
\|\tilde{T}\mathbf{z}\|^2_{\tilde{H}} &= \|Tx + iTy\|^2_{\tilde{H}} \\
&= \|Tx\|^2_H + \|Ty\|^2_H \\
&\leq \|T\|^2(\|x\|^2_H + \|y\|^2_H) \\
&= \|T\|^2\|\mathbf{z}\|^2_{\tilde{H}}.
\end{aligned}$$

Thus $\|\tilde{T}\mathbf{z}\|_{\tilde{H}} \leq \|T\|\|\mathbf{z}\|_{\tilde{H}}$ for all $\mathbf{z} \in \tilde{H}$, so $\tilde{T}$ is bounded and $\|\tilde{T}\| \leq \|T\|$.

Conversely, suppose that $\tilde{T} : \tilde{H} \mapsto \tilde{H}$ is bounded. Then for all $x \in H$ we have

$$\begin{aligned}\|Tx\|_H^2 &= \|\tilde{T}(x + 0i)\|_{\tilde{H}}^2 \\ &\leq \|\tilde{T}\|^2\|x + 0i\|_{\tilde{H}}^2 \\ &= \|\tilde{T}\|^2\|x\|_H^2.\end{aligned}$$

Thus $\|Tx\|_H \leq \|\tilde{T}\|\|x\|_H$ for all $x \in H$, so T is bounded and $\|T\| \leq \|\tilde{T}\|$. $\square$

Theorem 2.5.6. *Suppose that $A : H \supseteq \mathcal{D}(A) \mapsto H$ is the infinitesimal generator of a C_0-semigroup $T(t)$ for $t \geq 0$. Then the following hold:*

(i) $\tilde{A} : \tilde{H} \supseteq \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$ is the infinitesimal generator of the C_0-semigroup $\tilde{T}(t)$ for $t \geq 0$;

(ii) If $T(t)$ satisfies $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$ and for some $M \geq 1$ and $\omega \in \mathbb{R}$, then $\|\tilde{T}(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

Proof. We first prove (i). Suppose $A : H \supset \mathcal{D}(A) \mapsto H$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$. Then the complexification of A is $\tilde{A} : \tilde{H} \supset \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$ where $\tilde{A}\mathbf{z} = Ax + iAy$ for $\mathbf{z} = x + iy \in \mathcal{D}(\tilde{A})$. We have that $\tilde{T}(t) : \tilde{H} \mapsto \tilde{H}$ is defined as $\tilde{T}(t)\mathbf{z} = T(t)x + iT(t)y$ for $t \geq 0$ and $\mathbf{z} = x + iy \in \tilde{H}$. We now show that $\tilde{T}(t)$ is a C_0-semigroup generated by $\tilde{A}$. For each $t \geq 0$ the operator $\tilde{T}(t)$ is bounded by Theorem 2.5.5. For $t, s > 0$ and $\mathbf{z} = x + iy \in \tilde{H}$ we have that

$$\begin{aligned}\tilde{T}(t + s)\mathbf{z} &= T(t + s)[x] + iT(t + s)[y] \\ &= T(t)[T(s)x] + iT(t)[T(s)y] \\ &= \tilde{T}(t)[T(s)x + iT(s)y] \\ &= \tilde{T}(t)\tilde{T}(s)\mathbf{z}.\end{aligned}$$

Furthermore, $\tilde{T}(0)\mathbf{z} = T(0)x + iT(0)y = \mathbf{z}$ for all $\mathbf{z} = x + iy \in \tilde{H}$. Thus $\tilde{T}(t)$, $t \geq 0$ is a semigroup. Furthermore

$$\lim_{t\to 0^+} \tilde{T}(t)\mathbf{z} = \lim_{t\to 0^+} T(t)x + i \lim_{t\to 0^+} T(t)y = x + iy = \mathbf{z}.$$

Thus $\tilde{T}(t)$, $t \geq 0$, is a C_0-semigroup. Let A^* be the infinitesimal generator of $\tilde{T}(t)$, $t \geq 0$. Choose $\mathbf{z} = x + iy \in \mathcal{D}(A^*)$. Then

$$\begin{aligned}A^*\mathbf{z} &= \lim_{h\to 0^+} h^{-1}(\tilde{T}(h) - I)[x + iy] \\ &= \lim_{h\to 0^+} h^{-1}(T(h) - I)x + i \lim_{h\to 0^+} h^{-1}(T(h) - I)y.\end{aligned}$$

Now since $\mathbf{z} = x+iy \in \mathcal{D}(A^*)$ then $A^*\mathbf{z}$ exists. Thus $\lim_{h\to 0^+} h^{-1}(T(h)-I)x$ and $\lim_{h\to 0^+} h^{-1}(T(h)-I)y$ exist so that $x, y \in \mathcal{D}(A)$, and $\lim_{h\to 0^+} h^{-1}(T(h)-I)x = Ax$ and $\lim_{h\to 0^+} h^{-1}(T(h)-I)y = Ay$. It follows that

$$\begin{aligned} A^*\mathbf{z} &= Ax + iAy \\ &= \tilde{A}\mathbf{z}. \end{aligned}$$

Thus $\mathcal{D}(A^*) \subseteq \mathcal{D}(\tilde{A})$ and $\tilde{A}$ is an extension of A^*.

We now show that $\mathcal{D}(\tilde{A}) \subseteq \mathcal{D}(A^*)$, that is, if $x+iy \in \mathcal{D}(\tilde{A})$ then $x+iy \in \mathcal{D}(A^*)$. Let $x+iy \in \mathcal{D}(\tilde{A})$ so that $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(A)$. Thus

$$\lim_{h\to 0^+} h^{-1}(T(h)-I)x \text{ and } \lim_{h\to 0^+} h^{-1}(T(h)-I)y \text{ exist.} \tag{2.59}$$

Now consider

$$h^{-1}(\tilde{T}(h)-I)(x+iy) = h^{-1}(T(h)-I)x + ih^{-1}(T(h)-I)y. \tag{2.60}$$

From (2.59) and (2.60) we have that $\lim_{h\to 0^+} h^{-1}(\tilde{T}(h)-I)(x+iy)$ exists. Thus $x+iy \in \mathcal{D}(A^*)$ so that $\mathcal{D}(\tilde{A}) \subseteq \mathcal{D}(A^*)$. Hence $A^* = \tilde{A}$.

The statement (ii) follows immediately from Theorem 2.5.5. $\square$

Theorem 2.5.7. *Suppose $T > 0$ and U is an open bounded subset of H. Let $\tilde{U}$ be the complexification of U. Further suppose $A : H \supset \mathcal{D}(A) \mapsto H$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, and $f : (0,T)\times U \mapsto H$ is continuous. Consider the function $\tilde{f} : (0,T) \times \tilde{U} \mapsto \tilde{H}$ given by*

$$\tilde{f}(t, u+iv) = f(t,u) + 0i.$$

If $u : [0,T) \mapsto H$ is a mild solution of (1.13) with initial condition $u_0 \in H$, then the function $\tilde{u} : [0,T) \mapsto \tilde{H}$ given by $\tilde{u}(t) = u(t) + 0i$ is a mild solution of

$$\begin{aligned} \frac{d}{dt}\tilde{u}(t) &= \tilde{A}\tilde{u}(t) + \tilde{f}(t, \tilde{u}(t)),\ t \in (0,T) \\ \tilde{u}(0) &= u_0 + 0i. \end{aligned} \tag{2.61}$$

Proof. Clearly $\tilde{U}$ is an open bounded subset of $\tilde{H}$. Fix $t \in (0,T)$ and $u_0 \in H$. Since u is a mild solution of (1.13) with initial condition u_0 then u satisfies the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds,\ t \geq 0.$$

By Theorem 2.5.6, $\tilde{A}$ is the infinitesimal generator of a C_0-semigroup $\tilde{T}(t)$, $t \geq 0$. For $t \geq 0$, consider the integral equation

$$\begin{aligned}
\tilde{T}(t)(u_0 + 0i) + \int_0^t \tilde{T}(t-s)\tilde{f}(s, \tilde{u}(s))ds =& T(t)u_0 + 0i \\
&+ \int_0^t \tilde{T}(t-s)[f(s, u(s)) + 0i]ds \\
=& T(t)u_0 + 0i \\
&+ \int_0^t [T(t-s)f(s, u(s)) + 0i]ds \\
=& T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds \\
&+ 0i \\
=& u(t) + 0i \\
=& \tilde{u}(t).
\end{aligned}$$

Thus $\tilde{u}(t)$ is a mild solution of (2.61). $\square$

Theorem 2.5.8. *A linear operator $T : H \mapsto H$ is compact if and only if its complexification $\tilde{T} : \tilde{H} \mapsto \tilde{H}$ is compact.*

Proof. Suppose that T is a compact operator on a real Hilbert space H. Then T maps any bounded subset U of H onto a precompact subset of H by definition. By Theorem C.0.9 this is equivalent to saying that if $(x_n) \subset H$ is a bounded sequence then $(Tx_n) \subset H$ has a convergent subsequence. Now consider the complexification $\tilde{T}$ of T acting on the complex Hilbert space $\tilde{H}$, and suppose $(\mathbf{z}_n) \subset \tilde{H}$ is a bounded sequence. We want to show that the sequence $(\tilde{T}\mathbf{z}_n)$ has a convergent subsequence.

By the definition of $\tilde{H}$, for every $n \in \mathbb{N}$ we can find $x_n, y_n \in H$ such that $\mathbf{z}_n = x_n + iy_n$. Since $\|x_n\|_H \leq \|\mathbf{z}_n\|_{\tilde{H}}$ and $\|y_n\|_H \leq \|\mathbf{z}_n\|_{\tilde{H}}$ for every $n \in \mathbb{N}$, it follows that (x_n) and (y_n) are bounded sequences in H. Thus, since T is compact, (Tx_n) has a convergent subsequences, say, (Tx_{n_i}) with limit $x \in H$. Since (y_{n_i}) is a subsequence of (y_n), it is bounded. By the compactness of T, (Ty_{n_i}) has a subsequence $(Ty_{n_{i_j}})$ that converges to some $y \in H$. Since (Tx_{n_i}) converges to x, then so does its subsequence $(Tx_{n_{i_j}})$. Hence $(\tilde{T}\mathbf{z}_{n_{i_j}}) = (Tx_{n_{i_j}} + iTy_{n_{i_j}})$ converges to $\mathbf{z} = x + iy$ so that $\tilde{T}$ is compact.

Conversely suppose that $\tilde{T}$ is a compact operator on $\tilde{H}$. Consider a bounded sequence $(x_n) \subset H$. We want to show that the sequence (Tx_n) has a convergent subsequence. We define $\mathbf{z}_n = x_n + 0i \in \tilde{H}$ for every $n \in \mathbb{N}$. It follows that $(\mathbf{z}_n) \subset \tilde{H}$ is a bounded sequence, since $\|\mathbf{z}_n\|_{\tilde{H}} = \|x_n\|_H$ for all $n \in \mathbb{N}$. Thus, by the compactness of $\tilde{T}$ and by Theorem C.0.9, $(\tilde{T}\mathbf{z}_n)$ has a convergent subsequence, say $(\tilde{T}\mathbf{z}_{n_k})$, converging to $\mathbf{z} = x + iy \in \tilde{H}$. Furthermore, for every $k \in \mathbb{N}$ we have that $\tilde{T}\mathbf{z}_{n_k} = \tilde{T}(x_{n_k} + 0i)$. Hence $\mathbf{z} = x + 0i$ and

$$0 = \lim_{k\to\infty} \|\tilde{T}\mathbf{z}_{n_k} - \tilde{T}\mathbf{z}\|_{\tilde{H}} = \lim_{k\to\infty} \|Tx_{n_k} - Tx\|_H.$$

Thus (Tx_{n_k}) is a convergent subsequence of (Tx_n), converging to Tx, so that T is compact. □

The Resolvent and its Complexification

In order to prove the main results of this section we make use of the following proposition and theorem.

Proposition 2.5.9. *[14, Proposition A.0.2] Let $\Omega \subset \mathbb{C}$ be an open set and let $\{\tilde{F}(\lambda) \mid \lambda \in \Omega\}$ be a family of bounded linear operators satisfying the resolvent identity*

$$\tilde{F}(\lambda) - \tilde{F}(\mu) = (\mu - \lambda)\tilde{F}(\lambda)\tilde{F}(\mu), \textit{ for every } \lambda, \mu \in \Omega. \tag{2.62}$$

Assume that for some $\lambda_0 \in \Omega$, the operator $\tilde{F}(\lambda_0)$ is injective. Then there exists a unique linear operator $\tilde{A} : \tilde{H} \ni \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$ such that $\Omega \subset \rho(\tilde{A})$, and $R(\lambda, \tilde{A}) = \tilde{F}(\lambda)$ for $\lambda \in \Omega$.

Proof. Since $\tilde{F}(\lambda_0)$ is injective for some $\lambda_0 \in \Omega$, we can define an operator $\tilde{A}$ such that

$$\mathcal{D}(\tilde{A}) = \text{Range } \tilde{F}(\lambda_0), \ \ \tilde{A}x = \lambda_0 x - \tilde{F}(\lambda_0)^{-1}x \text{ for all } x \in \mathcal{D}(\tilde{A}).$$

Fix $y \in \tilde{H}$. By the resolvent identity, equation (2.62), for every $\lambda \in \Omega$

$$\begin{aligned}\tilde{F}(\lambda)y &= \tilde{F}(\lambda_0)y - (\lambda - \lambda_0)\tilde{F}(\lambda_0)\tilde{F}(\lambda)y \\ &= \tilde{F}(\lambda_0)[y - (\lambda - \lambda_0)\tilde{F}(\lambda)y]\end{aligned} \tag{2.63}$$

so that $\tilde{F}(\lambda)y \in \mathcal{D}(\tilde{A})$. Thus $F(\lambda)(H) = \mathcal{D}(\tilde{A})$ for every $\lambda \in \Omega$.

For any $x \in \mathcal{D}(\tilde{A})$, $y \in \tilde{H}$ and $\lambda \in \Omega$

$$(\lambda I - \tilde{A})x = y \tag{2.64}$$

if and only if

$$(\lambda - \lambda_0)x + \tilde{F}(\lambda_0)^{-1}x = y. \tag{2.65}$$

If we now apply $\tilde{F}(\lambda)$ to both sides of the equation then

$$(\lambda - \lambda_0)\tilde{F}(\lambda)x + \tilde{F}(\lambda)\tilde{F}(\lambda_0)^{-1}x = \tilde{F}(\lambda)y. \tag{2.66}$$

Now by applying $\tilde{F}(\lambda_0)^{-1}$ to both sides of the resolvent identity $\tilde{F}(\lambda) - \tilde{F}(\lambda_0) = (\lambda_0 - \lambda)\tilde{F}(\lambda)\tilde{F}(\lambda_0)$ we get

$$\tilde{F}(\lambda)\tilde{F}(\lambda_0)^{-1} - I = -(\lambda - \lambda_0)\tilde{F}(\lambda). \tag{2.67}$$

Substituting (2.67) into (2.66) gives us

$$\tilde{F}(\lambda)y = (\lambda - \lambda_0)\tilde{F}(\lambda)x - (\lambda - \lambda_0)\tilde{F}(\lambda)x + x = x.$$

Thus if $(\lambda I - \tilde{A})x = y$, then $x = \tilde{F}(\lambda)y$. Conversely, if for some $y \in \tilde{H}$ and $\lambda \in \Omega$ we set $x = \tilde{F}(\lambda)y$, then by (2.63)

$$\begin{aligned}(\lambda - \lambda_0)x + \tilde{F}(\lambda_0)^{-1}x &= (\lambda - \lambda_0)\tilde{F}(\lambda)y + \tilde{F}(\lambda_0)^{-1}\tilde{F}(\lambda)y \\ &= (\lambda - \lambda_0)\tilde{F}(\lambda)y + \tilde{F}(\lambda_0)^{-1}\tilde{F}(\lambda_0)[y - (\lambda - \lambda_0)\tilde{F}(\lambda)y] \\ &= y.\end{aligned}$$

Thus for any $x \in \mathcal{D}(\tilde{A})$, $y \in \tilde{H}$ and $\lambda \in \Omega$,

$$(\lambda I - \tilde{A})x = y \text{ iff } x = \tilde{F}(\lambda)y.$$

Thus $(\lambda I - \tilde{A})$ has a bounded inverse $R(\lambda, \tilde{A}) = \tilde{F}(\lambda)$ for all $\lambda \in \Omega$.

To show uniqueness, suppose that there was an operator $\tilde{B}$ such that $\tilde{F}(\lambda_0) = R(\lambda_0, \tilde{B})$. Then $\mathcal{D}(\tilde{B}) = \tilde{F}(\lambda_0)(H) = \mathcal{D}(\tilde{A})$. Thus, for all $x \in \mathcal{D}(A)$

$$\tilde{F}(\lambda_0)(\lambda_0 I - \tilde{B})x = x = \tilde{F}(\lambda_0)(\lambda_0 I - \tilde{A})x,$$

which implies that

$$\tilde{F}(\lambda_0)^{-1}\tilde{F}(\lambda_0)(\lambda_0 I - \tilde{B})x = \tilde{F}(\lambda_0)^{-1}\tilde{F}(\lambda_0)(\lambda_0 I - \tilde{A})x,$$

from which it follows that

$$(\lambda_0 I - \tilde{B})x = (\lambda_0 I - \tilde{A})x.$$

Thus $\tilde{B} = \tilde{A}$. □

Remark 2.5.2. Operators $\tilde{F}(\lambda)$ which satisfy the resolvent identity, as in Proposition 2.5.9, are called *pseudo resolvents*, see[18, Chapter 1.9].

Theorem 2.5.10. *[14, Proposition 2.1.9 (b)] Let $\{\tilde{T}(t) \mid t > 0\}$ be a family of bounded linear operators on $\tilde{H}$ such that the following hold:*

(i) $\tilde{T}(t)\tilde{T}(s) = \tilde{T}(t+s)$, *for every* $t, s > 0$;

(ii) *There exist constants* $\omega \in \mathbb{R}$ *and* $M > 0$ *such that* $\|\tilde{T}(t)\| \leq Me^{\omega t}$ *for each* $t > 0$;

(iii) *Either*

(a) *there is a* $t > 0$ *such that* $\tilde{T}(t)$ *is one-to-one, or*

(b) *for every* $x \in \tilde{H}$, $\lim_{h\to 0^+} \tilde{T}(h)x = x$.

Then there exists an unique linear operator $\tilde{A} : \tilde{H} \supset \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$ *such that*

$$R(\lambda, \tilde{A}) = \int_0^\infty e^{-\lambda t}\tilde{T}(t)dt,$$

for all $\lambda \in \Pi = \{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$.

Proof. By (ii), for all $t > 0$

$$\|e^{-\lambda t}\tilde{T}(t)\| \leq Me^{(\omega - \Re\lambda)t}.$$

Thus the integral $\tilde{F}(\lambda) = \int_0^\infty e^{-\lambda t}\tilde{T}(t)dt$ exists for all $\lambda \in \Pi$. Furthermore, for any $x \in \tilde{H}$,

$$\begin{aligned}
\|\tilde{F}(\lambda)x\|_{\tilde{H}} &= \left\|\int_0^\infty e^{-\lambda t}\tilde{T}(t)xdt\right\|_{\tilde{H}} \\
&\leq \int_0^\infty \|e^{-\lambda t}\tilde{T}(t)x\|_{\tilde{H}}dt \\
&\leq \int_0^\infty Me^{(\omega-\Re\lambda)t}\|x\|_{\tilde{H}}dt \\
&= \frac{M}{\Re\lambda - \omega}\|x\|_{\tilde{H}}.
\end{aligned}$$

Hence $\tilde{F}(\lambda)$ is a bounded linear operator for each $\lambda \in \Pi$. We now show that the family $\{\tilde{F}(\lambda) \mid \lambda \in \Pi\}$ satisfies the conditions of Proposition 2.5.9. To do this let $\lambda, \mu \in \Pi$. Then by Fubini's Theorem, Theorem A.1.4,

$$\begin{aligned}
\tilde{F}(\lambda)\tilde{F}(\mu) &= \int_0^\infty e^{-\lambda t}\tilde{T}(t)dt \int_0^\infty e^{-\mu s}\tilde{T}(s)ds \\
&= \int_0^\infty \int_0^\infty e^{-\lambda t}\tilde{T}(t)e^{-\mu s}\tilde{T}(s)dtds \\
&= \int_0^\infty \left[\int_0^\infty e^{-\mu(t+s)}e^{-(\lambda-\mu)t}\tilde{T}(t+s)dt\right] ds.
\end{aligned}$$

Let $\sigma = t + s$. Then

$$\begin{aligned}
\tilde{F}(\lambda)\tilde{F}(\mu) &= \int_0^\infty e^{-\mu\sigma}\tilde{T}(\sigma)\int_0^\sigma e^{-(\lambda-\mu)t}dtd\sigma \\
&= \int_0^\infty e^{-\mu\sigma}\tilde{T}(\sigma)\frac{e^{(\mu-\lambda)\sigma}-1}{\mu-\lambda}d\sigma \\
&= \frac{1}{\mu-\lambda}\left[\int_0^\infty e^{-\lambda\sigma}\tilde{T}(\sigma)d\sigma - \int_0^\infty e^{-\mu\sigma}\tilde{T}(\sigma)d\sigma\right] \\
&= \frac{1}{\mu-\lambda}[\tilde{F}(\lambda)-\tilde{F}(\mu)].
\end{aligned}$$

Thus F satisfies the resolvent identity on the half-plane Π.

We now show that $\tilde{F}(\lambda)$ is one-to-one for $\lambda \in \Pi$. For the sake of contradiction assume that for some $\lambda \in \Pi$, $\tilde{F}(\lambda)$ is not one-to-one. That is, for some $x \neq 0$ we can find a $\lambda_0 \in \Pi$ such that $\tilde{F}(\lambda_0)x = 0$. Then for any $\lambda \in \Pi$, by the resolvent identity,

$$\tilde{F}(\lambda)x = \tilde{F}(\lambda_0)x + (\lambda_0 - \lambda)\tilde{F}(\lambda)\tilde{F}(\lambda_0)x = 0.$$

Therefore, for every $x^* \in \tilde{H}$ and $\lambda \in \Pi$,

$$\begin{aligned}
0 &= (\tilde{F}(\lambda)x, x^*)_{\tilde{H}} \\
&= \left(\int_0^\infty e^{-\lambda t}\tilde{T}(t)xdt, x^*\right)_{\tilde{H}} \\
&= \int_0^\infty e^{-\lambda t}(\tilde{T}(t)x, x^*)_{\tilde{H}}dt
\end{aligned} \tag{2.68}$$

since the inner product is continuous. Note that $L(x) = \int_0^\infty e^{-\lambda t}(\tilde{T}(t)x, x^*)_{\tilde{H}}dt$ is the Laplace transform of $(\tilde{T}(t)x, x^*)_{\tilde{H}}$. Furthermore, Laplace transforms are injective. Thus (2.68) implies that $0 = (\tilde{T}(t)x, x^*)_{\tilde{H}}$ for every $t > 0$. Hence $\tilde{T}(t)x = 0$ for $t > 0$ since x^* is arbitrary.

However, if $(iii)(a)$ holds, then there exists a $t^* > 0$ such that $\tilde{T}(t^*)$ is one-to-one. Then

$$\tilde{T}(t^*)x = 0 = \tilde{T}(t^*)0$$

so that $x = 0$, a contradiction.

Alternatively, if $(iii)(b)$ holds, then $x^* = \lim_{t\to 0^+} \tilde{T}(t)x^* = 0$, also a contradiction.

Hence $\tilde{F}(\lambda)$ is injective. Thus by Proposition 2.5.9 there exists a unique operator $\tilde{A} : \tilde{H} \supset \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$, where $\mathcal{D}(\tilde{A}) = \text{Range } \tilde{F}(\lambda)$, such that $R(\lambda, \tilde{A}) = \tilde{F}(\lambda)$ for $\lambda \in \Pi$. $\square$

Theorem 2.5.11. *Suppose $A : H \supset \mathcal{D}(A) \mapsto H$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, satisfying $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$, for some $M \geq 1$ and $\omega \in \mathbb{R}$. Then the resolvent set of the complexification of A, $\rho(\tilde{A})$, contains the half-plane $\Pi = \{\lambda \in \mathbb{C} \mid Re\lambda > \omega\}$. Furthermore, for $\lambda \in \Pi$, $\|R(\lambda, \tilde{A})\| \leq \dfrac{M}{|\lambda - \omega|}$.*

Proof. By Theorem 2.5.6 we have that $\tilde{A} : \tilde{H} \supset \mathcal{D}(\tilde{A}) \mapsto \tilde{H}$ is the infinitesimal generator of a C_0-semigroup $\tilde{T}(t)$, $t \geq 0$, satisfying $\|\tilde{T}(t)\| \leq Me^{\omega t}$ for $t \geq 0$. By Lemma 2.1.5 we have that

$$R(\lambda, \tilde{A}) = \int_0^\infty e^{-\lambda t}\tilde{T}(t)dt$$

for all $\lambda \in \rho(\tilde{A})$. Now $\tilde{T}(t)$ satisfies conditions (i), (ii) and $(iii)(b)$ of Theorem 2.5.10, and thus we can find an unique operator $\tilde{B} : \tilde{H} \supset \mathcal{D}(\tilde{B}) \mapsto \tilde{H}$ such that

$$R(\lambda, \tilde{B}) = \int_0^\infty e^{-\lambda t}\tilde{T}(t)dt$$

for all $\lambda \in \Pi$. Fix $\lambda \in \mathbb{R}$, $\lambda > \omega$. By the Hille-Yosida Theorem, Theorem 1.4.8, $\lambda \in \rho(A)$. Thus $\lambda \in \Pi \cap \rho(\tilde{A})$ and $R(\lambda, \tilde{A}) = R(\lambda, \tilde{B})$. It follows that

$$\mathcal{D}(\tilde{A}) = R(\lambda, \tilde{A})(H) = R(\lambda, \tilde{B})(H) = \mathcal{D}(\tilde{B})$$

and for $x \in \mathcal{D}(\tilde{A}) = \mathcal{D}(\tilde{B})$

$$R(\lambda, \tilde{A})(\lambda I - \tilde{A})x = x = R(\lambda, \tilde{A})(\lambda I - \tilde{B})x.$$

By the injectivity of $R(\lambda, \tilde{A})$ it follows that

$$(\lambda I - \tilde{A})x = (\lambda I - \tilde{B})x$$

so that $\tilde{A}x = \tilde{B}x$. Thus $\tilde{A} = \tilde{B}$ so that $\Pi \subset \rho(\tilde{A})$.

Furthermore, if $\lambda \in \Pi$ and $\mathbf{z} \in \tilde{H}$ then

$$\begin{aligned}
\|R(\lambda, \tilde{A})\mathbf{z}\|_{\tilde{H}} &= \left\| \int_0^\infty e^{-\lambda t} \tilde{T}(t)\mathbf{z} dt \right\|_{\tilde{H}} \\
&\leq \int_0^\infty e^{-\Re\lambda t} \|\tilde{T}(t)\| \|\mathbf{z}\|_{\tilde{H}} dt \\
&\leq \int_0^\infty e^{-\Re\lambda t} M e^{\omega t} \|\mathbf{z}\|_{\tilde{H}} dt \\
&= M\|\mathbf{z}\|_{\tilde{H}} \int_0^\infty e^{-(\Re\lambda - \omega)t} dt \\
&= \frac{M}{\Re\lambda - \omega} \|\mathbf{z}\|_{\tilde{H}} \\
&\leq \frac{M}{|\lambda - \omega|} \|\mathbf{z}\|_{\tilde{H}}.
\end{aligned}$$

Thus, for $\lambda \in \Pi$, $\|R(\lambda, \tilde{A})\| \leq \dfrac{M}{|\lambda - \omega|}$. □

Theorem 2.5.12. *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, satisfying $\|\widetilde{T(t)}\| \leq Me^{\omega t}$ for some $M \geq 1$ and some $\omega \in \mathbb{R}$. If $\lambda \in \mathbb{R}$ and $\lambda > \omega$, then $\widetilde{R(\lambda, A)} = R(\lambda, \tilde{A})$.*

Proof. Consider $\lambda \in \mathbb{R}$ such that $\lambda > \omega$. It follows directly from the Hille-Yosida Theorem, Theorem 1.4.8 that $\lambda \in \rho(A)$. Thus $\lambda \in \rho(\tilde{A})$. By Lemma 2.1.5 we have, for each $\mathbf{z} = x + iy \in \tilde{H}$, that

$$\begin{aligned}
\widetilde{R(\lambda, A)}\mathbf{z} &= R(\lambda, A)x + iR(\lambda, A)y \\
&= \int_0^\infty e^{-\lambda t}(T(t)x + iT(t)y) dt \\
&= \int_0^\infty e^{-\lambda t} \tilde{T}(t)\mathbf{z} dt \\
&= R(\lambda, \tilde{A})\mathbf{z}.
\end{aligned}$$

□

Note that the results presented in this section hold if the Hilbert space H is replaced with a Banach space X, see for example [14, Proposition 2.1.9]. However,

$$\|x + iy\|_{\tilde{X}} = \left(\|x\|_X^2 + \|y\|_X^2\right)^{\frac{1}{2}}$$

does not define a norm on $\tilde{X}$, so that the norm on X must be defined in some other way. Furthermore, since we apply the results in this section to parabolic problems, the Hilbert space setting considered here is sufficient.

Chapter 3

Mild Solutions

In this chapter we investigate mild solutions to the abstract Cauchy problem (1.13). We start by showing that classical solutions to (1.13) are indeed mild solutions. Then we prove an existence result showing under which conditions local mild solutions exist. Furthermore, we investigate conditions under which these local mild solutions are global mild solutions. Finally, we prove a regularity result for the mild solutions of (1.13), giving conditions on A and f so that all mild solutions of (1.13) are classical solutions.

Throughout this chapter, X denotes a Banach space.

3.1 Classical vs Mild Solutions

We now show that a classical solution of (1.13), in the sense of Definition 1.5.1, is a mild solution of (1.13). To do this we prove the following two propositions.

Proposition 3.1.1. *Suppose $T(t)$ is a C_0-semigroup for $t \geq 0$, $I \subseteq \mathbb{R}$ is an open interval, and $f : I \mapsto X$ is a continuous function. Then for $t > 0$ and $s \in I$ we have*

$$\lim_{h \to 0^+} T(t+h)f(s+h) = T(t)f(s).$$

Proof. For any fixed $s \in I$, $t > 0$ and real number $h \geq 0$, we have

$$\begin{aligned}\|T(t+h)f(s+h) - T(t)f(s)\|_X \leq & \|T(t+h)f(s+h) - T(t+h)f(s)\|_X \\ & + \|T(t+h)f(s) - T(t)f(s)\|_X \\ \leq & \|T(t+h)\| \|f(s+h) - f(s)\|_X \\ & + \|T(t)\| \|T(h)f(s) - f(s)\|_X. \end{aligned} \tag{3.1}$$

Since $T(t)$, $t \geq 0$, is a C_0-semigroup, then by Theorem 1.4.1 there exists an $M \geq 1$ and an $\omega \in \mathbb{R}$ such that $\|T(t+h)\| \leq Me^{\omega(t+h)}$. Furthermore, since f is continuous, then

$$\lim_{h \to 0} \|T(t+h)\| \|f(s+h) - f(s)\|_X = 0. \tag{3.2}$$

Furthermore, since $\|T(t)\| \leq Me^{\omega t}$ and $T(t)$, $t \geq 0$, is a C_0-semigroup, then

$$\lim_{h \to 0^+} \|T(t)\| \|T(h)f(s) - f(s)\|_X = 0. \tag{3.3}$$

Thus substituting (3.2) and (3.3) into (3.1), and taking the limit as $h \to 0^+$, we have our result. $\square$

Proposition 3.1.2. *Fix $\beta > 0$ and $0 < T \leq \infty$. Suppose $T(t)$, $t \geq 0$, is a C_0-semigroup with infinitesimal generator A, and the function $f : (0, T) \mapsto \mathcal{D}(A)$ is differentiable. Consider the functions $g : (0, T) \mapsto X$ given by $g(\theta) = T(\theta+\beta)f(\theta)$, and $h : [\beta, T) \mapsto X$ given by $h(\theta) = T(\theta-\beta)f(\theta)$. Then g and h are differentiable from the right on $(0, T)$ and (β, T) respectively. Furthermore*

$$g'(\theta) = AT(\theta+\beta)f(\theta) + T(\theta+\beta)f'(\theta),$$

and

$$h'(\theta) = -AT(\theta-\beta)f(\theta) + T(\theta-\beta)f'(\theta).$$

Proof. Fix $\theta \in (0, T)$ and $h \geq 0$. We have

$$\begin{aligned}
\frac{g(\theta+h) - g(\theta)}{h} =& \frac{T(\theta+\beta+h)f(\theta+h) - T(\theta+\beta)f(\theta)}{h} \\
=& \frac{T(\theta+\beta)T(h)f(\theta+h) - T(\theta+\beta)f(\theta)}{h} \\
=& T(\theta+\beta)\frac{T(h)f(\theta+h) - f(\theta)}{h} \\
=& T(\theta+\beta)h^{-1}\bigg(T(h)[f(\theta+h) - f(\theta)] \\
& + [T(h) - I]f(\theta)\bigg).
\end{aligned} \tag{3.4}$$

We now define a function

$$j(x) = \begin{cases} \frac{f(\theta+x)-f(\theta)}{x} & \text{for } x \in (-\theta, \infty),\ x \neq 0 \\ f'(\theta) & \text{for } x = 0, \end{cases}$$

which is continuous on $(-\theta, \infty)$ since f is differentiable. Thus by Proposition 3.1.1 we have that

$$\begin{aligned}\lim_{h\to 0^+} \frac{T(h)[f(\theta+h)-f(\theta)]}{h} &= \lim_{h\to 0^+}[T(h)j(h)] \\ &= j(0) \\ &= f'(\theta). \end{aligned} \tag{3.5}$$

By Definition 1.4.2 we have, since $f(\theta) \in \mathcal{D}(A)$ for all $\theta \in (0, T)$, that

$$\lim_{h\to 0^+} \frac{(T(h)-I)}{h} f(\theta) = Af(\theta). \tag{3.6}$$

Thus, considering (3.5) and (3.6), taking the limit as $h \to 0^+$ in (3.4) we see that g is differentiable from the right at θ and that

$$g'(\theta) = AT(\theta+\beta)f(\theta) + T(\theta+\beta)f'(\theta).$$

That h is differentiable from the right at any $\theta > \beta$, and that

$$h'(\theta) = -AT(\theta-\beta)f(\theta) + T(\theta-\beta)f'(\theta),$$

follows in the same way. □

Theorem 3.1.3. *Let $0 < T \leq \infty$, and U be an open subset of X. If $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of a C_0-semigroup, then a classical solution of (1.13) is a mild solution of (1.13).*

Proof. Since u is a classical solution, $u(t) \in \mathcal{D}(A)$ for $0 < t < T$ and u is differentiable on $(0, T)$. Thus, from Proposition 3.1.2, we have that

$$\begin{aligned} u(t) - T(t)u_0 &= T(0)u(t) - T(t)u(0) \\ &= \int_0^t \left(\frac{d}{ds}[T(t-s)u(s)]\right) ds \\ &= \int_0^t \left(-T(t-s)Au(s) + T(t-s)\frac{d}{ds}u(s)\right) ds \\ &= \int_0^t T(t-s)\left(\frac{d}{ds}u(s) - Au(s)\right) ds \\ &= \int_0^t T(t-s)f(s, u(s))ds, \end{aligned}$$

taking each derivative from the right and the final identity following from the fact that u is a classical solution of (1.13). □

3.2 Existence

In this section we consider an existence theorem from Pazy [17] for mild solutions of the abstract Cauchy problem (1.13) in the case where A is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$.

This existence theorem follows from two classical results, namely, the Arzelà-Ascoli Theorem [8, IV.6.7] and the Schauder Fixed Point Theorem [8, V.10.6]. For the convenience of the reader the results are given below.

Definition 3.2.1 (**Equicontinuous Functions**)**.** Suppose S is a compact metric space. A subset F of $C(S;X)$ is equicontinuous if, for every $x \in S$ and $\epsilon > 0$, there exists a neighbourhood U_x of x so that $\|f(y) - f(x)\|_X < \epsilon$ whenever $f \in F$ and $y \in U_x$.

Theorem 3.2.1 (**Arzelà-Ascoli Theorem**)**.** *Let S be a compact metric space. Then a subset F of $C(S;X)$ is precompact with respect to the uniform norm if and only if it is bounded and equicontinuous.*

Theorem 3.2.2 (**Schauder Fixed Point Theorem**)**.** *Suppose K is a non-empty convex subset of X. If T is a continuous mapping from K to itself such that $T(K)$ is contained in a compact subset of K, then T has a fixed point.*

The previous two theorems hold under more general assumptions. Namely, the Arzelà-Ascoli Theorem holds for any subset F of the set continuous functions acting on a compact Hausdorff space S, and the Schauder Fixed Point Theorem holds for any locally convex linear topological space X.

Theorem 3.2.3 (**Local Existence**)**.** *Let U be an open subset of X and let $0 < T \leq \infty$. If $f : [0,T) \times U \mapsto X$ is continuous and A is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$, then for every $u_0 \in U$ there exists a $t^* = t^*(u_0)$, $0 < t^* < T$, and a continuous mapping u from $[0,t^*]$ to U which is a mild solution of* (1.13) *on* $[0,t^*]$.

Proof. Fix $u_0 \in U$, $t_0 \in (0,T)$ and let $M > 0$ be an upper bound for the non-empty set $\{\|T(t)\| \mid t \in [0,t_0]\}$.

We now show that there exists a $\rho > 0$ such that $\overline{B_\rho(u_0)} \subset U$ and the set $\{f(t,v) \mid t \in [0,t_0],\ v \in \overline{B_\rho(u_0)}\}$ is bounded in X. Fix $\epsilon > 0$ and $t \in [0,t_0]$, and let $f_t = f(t,u_0)$. By the continuity of f we can find a $\delta_t > 0$ and a $\rho_t > 0$ such that if $s \in [0,t_0] \cap (t-\delta_t, t+\delta_t)$ and $v \in \overline{B_{\rho_t}(u_0)} \subset U$ then $f(s,v) \in B_\epsilon(f_t)$. Let $I_t = (t-\delta_t, t+\delta_t)$ for each $t \in [0,t_0]$. Then $\{I_t \mid t \in [0,t_0]\}$ is an open cover for $[0,t_0]$. Thus there exists $t_1, t_2, ...t_n \in [0,t_0]$ such that

$[0, t_0] \subset \bigcup_{i=1}^{n} I_{t_i}$. Let $\rho = \min\{\rho_{t_1}, ..., \rho_{t_n}\}$, then $\overline{B_\rho(u_0)} = \bigcap_{i=1}^{n} \overline{B_{\rho_{t_i}}(u_0)}$. Thus for all $t \in [0, t_0]$, if $v \in \overline{B_\rho(u_0)}$ then $f(t, v) \in B_\epsilon(f_{t_i})$ for some $i = 1, ..., n$. Thus we can find an $N > 0$ such that $\|f(t, v)\|_X \leq N$ for all $t \in [0, t_0]$ and $v \in \overline{B_\rho(u_0)}$.

Since $T(t)$ is a C_0-semigroup for $t \geq 0$, we can find a constant $b > 0$ such that

$$\|T(t)u_0 - u_0\|_X \leq \frac{\rho}{2} \text{ for } t \in [0, b].$$

Let

$$t^* = \min\left\{t_0, b, \frac{\rho}{2MN}\right\} > 0.$$

Set $Y = C([0, t^*]; X)$, and

$$Y_0 = \{u \in Y \mid u(0) = u_0,\, u(t) \in \overline{B_\rho(u_0)} \text{ for } t \in [0, t^*]\}.$$

It is clear that Y_0 is a closed, bounded, convex and non-empty subset of Y. We define a mapping F on Y_0 by setting

$$(Fu)(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds,\ 0 \leq t \leq t^*.$$

Using the Schauder Fixed Point Theorem we show that F has a fixed point in Y_0. We begin by showing that if $u \in Y_0$ then $Fu \in Y_0$. It is clear that $(Fu)(0) = u_0$. We also have that Fu is continuous on $[0, t^*]$, since f is continuous on $[0, t^*] \times U$ and $T(t)$, $t \geq 0$, is a C_0-semigroup. Furthermore, for each $t \in [0, t^*]$,

$$\begin{aligned}
\|(Fu)(t) - u_0\|_X &= \left\|T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds - u_0\right\|_X \\
&\leq \|T(t)u_0 - u_0\|_X + \left\|\int_0^t T(t-s)f(s, u(s))ds\right\|_X \\
&\leq \frac{\rho}{2} + t^*MN \\
&\leq \rho.
\end{aligned}$$

Thus F maps Y_0 into Y_0. We now show that F is a continuous mapping of Y_0 into Y_0. Fix $u \in Y_0$ and $\epsilon > 0$, and consider $v \in Y_0$ and $t \in [0, t^*]$. Then

$$\begin{aligned}
\|(Fu)(t) - (Fv)(t)\|_X &= \left\|\int_0^t T(t-s)[f(s, u(s)) - f(s, v(s))]ds\right\|_X \\
&\leq M \int_0^{t^*} \|f(s, u(s)) - f(s, v(s))\|_X ds. \qquad (3.7)
\end{aligned}$$

By the continuity of f and u, for every $t \in [0, t^*]$ there exists a $\delta_t^{'} > 0$ and a $\rho_t^{'} > 0$ such that if $s \in [0, t^*] \cap (t - \delta_t^{'}, t + \delta_t^{'})$ and $\|u(t) - w\|_X < \rho_t^{'}$ then

$$\|f(t, u(t)) - f(s, w)\|_X < \frac{\epsilon}{2Mt^*} \tag{3.8}$$

and

$$\|u(t) - u(s)\|_X < \frac{\rho_t^{'}}{2}.$$

Let $I_t^{'} = (t - \delta_t^{'}, t + \delta_t^{'})$ for each $t \in [0, t^*]$. Then $\{I_t^{'} \mid t \in [0, t^*]\}$ is an open cover for $[0, t^*]$. Therefore there exists $t_1^{'}, t_2^{'}, ..., t_n^{'} \in [0, t^*]$ such that $[0, t^*] \subset \bigcup_{i=1}^{n} I_{t_i^{'}}^{'}$. Let $\rho^{'} = \frac{1}{2}\min\{\rho_{t_1^{'}}^{'}, ..., \rho_{t_n^{'}}^{'}\}$ and assume that $\sup_{s \in [0,t^*]} \|u(s) - v(s)\|_X < \rho^{'}$. It follows that for all $s \in [0, t^*]$, there exists an $i = 1, ..., n$ such that $\|u(t_i) - u(s)\|_X < \frac{\rho_{t_i}^{'}}{2}$ and

$$\begin{aligned}
\|u(t_i) - v(s)\|_X &\leq \|u(t_i) - u(s)\|_X + \|u(s) - v(s)\|_X \\
&< \frac{\rho_{t_i}^{'}}{2} + \rho^{'} \\
&< \rho_{t_i}^{'}.
\end{aligned}$$

Hence from (3.8) we have that

$$\begin{aligned}
\|f(s, u(s)) - f(s, v(s))\|_X \leq &\|f(s, u(s)) - f(t_i, u(t_i))\|_X \\
&+ \|f(t_i, u(t_i)) - f(s, v(s))\|_X \\
< &\frac{\epsilon}{Mt^*}.
\end{aligned}$$

Thus, considering (3.7), we have that there exists a $\rho^{'} > 0$ such that if $\sup_{s \in [0,t^*]} \|u(s) - v(s)\|_X < \rho^{'}$ then $\|(Fu)(t) - (Fv)(t)\|_X < \epsilon$ for all $t \in [0, t^*]$. It follows that F is a continuous mapping from Y_0 to Y_0.

We now show that F maps Y_0 onto a precompact subset of Y_0 using the Arzelà-Ascoli Theorem. Consider the set

$$Z^* = \{Fu \mid u \in Y_0\}.$$

We need to show that Z^* is an equicontinuous family of functions. Fix $s_1, s_2 \in [0, t^*]$ and $u \in Y_0$. Without loss of generality, assume that $s_1 > s_2$.

Then

$$\begin{aligned}\|(Fu)(s_1)-(Fu)(s_2)\|_X \leq & \|(T(s_1)-T(s_2))u_0\|_X \\ & + \left\| \int_0^{s_1} T(s_1-s)f(s,u(s))ds - \int_0^{s_2} T(s_2-s)f(s,u(s))ds \right\|_X \\ \leq & \|(T(s_1)-T(s_2))u_0\|_X + \left\| \int_{s_2}^{s_1} T(s_1-s)f(s,u(s))ds \right\|_X \\ & + \left\| \int_0^{s_2} [T(s_1-s)-T(s_2-s)]f(s,u(s))ds \right\|_X \\ \leq & \|(T(s_1)-T(s_2))u_0\|_X + (s_1-s_2)MN \\ & + N\int_0^{s_2} \|T(s_1-s)-T(s_2-s)\|ds. \end{aligned} \tag{3.9}$$

Note that right-hand side of (3.9) is independent of $u \in Y_0$. Fix $\epsilon > 0$. Since $T(t)$, $t \geq 0$, is a C_0-semigroup there exists a δ_1 such that if $(s_1 - s_2) < \delta_1$ then

$$\|(T(s_1)-T(s_2))u_0\|_X < \frac{\epsilon}{3}. \tag{3.10}$$

If $\epsilon \geq 12MNs_2$ then

$$\begin{aligned} N\int_0^{s_2} \|T(s_1-s)-T(s_2-s)\|ds & \leq 2NMs_2 \\ & < \frac{\epsilon}{3}. \end{aligned} \tag{3.11}$$

Now suppose $\epsilon < 12MNs_2$ so that $\gamma = \dfrac{\epsilon}{12MN} \in (0, s_2)$. Then

$$\begin{aligned}\int_0^{s_2} \|T(s_1-s)-T(s_2-s)\|ds = & \int_0^{s_2-\gamma} \|T(s_1-s)-T(s_2-s)\|ds \\ & + \int_{s_2-\gamma}^{s_2} \|T(s_1-s)-T(s_2-s)\|ds \\ \leq & \int_0^{s_2-\gamma} \|T(s_1-s)-T(s_2-s)\|ds + 2M\gamma \\ \leq & \int_0^{s_2-\gamma} \|T(s_1-s)-T(s_2-s)\|ds + \frac{\epsilon}{6N}. \end{aligned} \tag{3.12}$$

By Theorem 2.1.6, since $T(t)$, $t \geq 0$, is compact for $t > 0$, $T(t)$ is continuous in the uniform operator topology for $t > 0$. Since $[\gamma, t^*]$ is compact, $T(t)$ is uniformly continuous with respect to the uniform norm on $[\gamma, t^*]$. Thus there exists a $\delta_2 > 0$ such that, for all $r_1, r_2 \in [\gamma, t^*]$, if $|r_1 - r_2| < \delta_2$ then

$$\|T(r_1) - T(r_2)\| < \frac{\epsilon}{6Nt^*}. \tag{3.13}$$

If $0 \le s \le s_2 - \gamma$ then $s_1 - s_2 + \gamma \le s_1 - s \le s_1$. Since $s_1 > s_2$ then $\gamma < s_1 - s_2 + \gamma \le s_1 - s \le s_1 \le t^*$. Thus $(s_1 - s) \in [\gamma, t^*]$. Similarly, if $0 \le s \le s_2 - \gamma$ then $(s_2 - s) \in [\gamma, t^*]$. Thus, by (3.13), if $|s_1 - s_2| = |(s_1 - s) - (s_2 - s)| < \delta_2$ then

$$\int_0^{s_2 - \gamma} \|T(s_1 - s) - T(s_2 - s)\| ds < \frac{\epsilon}{6N}. \tag{3.14}$$

Substituting (3.14) into (3.12) we get that if $(s_1 - s_2) < \delta_2$ then

$$\int_0^{s_2} \|T(s_1 - s) - T(s_2 - s)\| ds < \frac{\epsilon}{3N}. \tag{3.15}$$

Let $\delta = \min\left\{\delta_1, \delta_2, \frac{\epsilon}{3MN}\right\}$. Then, from (3.9), (3.10), (3.11) and (3.15), we have that if $(s_1 - s_2) < \delta$ then

$$\|(Fu)(s_1) - (Fu)(s_2)\|_X < \epsilon.$$

Thus Z^* is equicontinuous.

We have already shown that $Z^* = F(Y_0) \subseteq Y_0$, so that Z^* is bounded. Thus by the Arzelà-Ascoli Theorem, Theorem 3.2.1, Z^* is a precompact set. Since Y_0 is closed, $Z^* = F(Y_0)$ is contained in a compact subset of Y_0.

Thus F is a continuous mapping from Y_0 to itself, and $F(Y_0)$ is contained in a compact subset of Y_0. Therefore, by Schauder's Fixed Point Theorem, Theorem 3.2.2, F has a fixed point, say u, in Y_0. Thus $u \in C([0, t^*], X)$, $u(0) = u_0$, $u(t) \in U$ for all $t \in [0, t^*]$ and

$$u(t) = T(t)u_0 + \int_0^t T(t - s) f(s, u(s)) ds$$

for $t \in [0, t^*]$. $\square$

Note that the proof of Theorem 3.2.3 shows that under the conditions of the theorem, we can find a solution to the integral equation

$$u(t) = g(t) + \int_0^t T(t - s) f(s, u(s)) ds,$$

for any continuous function $g : [0, b) \mapsto X$.

We mention two weaknesses of this existence theorem. Firstly, the theorem assumes that A generates a compact semigroup and yet, as previously mentioned, it is difficult to prove that A generates a compact semigroup.

Secondly, existence theorems using semigroup theory are numerically weak, as demonstrated by [7, Theorem 8.3.10]. This shows that if an estimate of $\|R(\lambda, A)\|$ differs from the required bound, that is, $\|R(\lambda, A)\| \leq \dfrac{M}{|\Re\lambda - \omega|}$ when $\Re\lambda > \omega$, by even an infinitesimally small amount, then a corresponding semigroup does not necessarily exist. Thus it is easy to err when using a numerical method to approximate $R(\lambda, A)x$ for $x \in X$.

3.3 Asymptotic Behaviour

In this section we show that under certain conditions a local mild solution of (1.13) can be extended to a global solution.

Note that if E is a subspace of X, $f : E \mapsto \mathbb{R}$ and a is a limit point of E, then the limit superior of $f(x)$ as $x \to a$, denoted $\overline{\lim\limits_{x\to a}} f(x)$, is defined by

$$\overline{\lim_{x\to a}} f(x) = \lim_{\epsilon\to 0^+} (\sup\{f(x) \mid x \in E \cap B_\epsilon(a) \setminus \{a\}\}).$$

Theorem 3.3.1. *Suppose A is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$, and $f : [0, \infty) \times X \mapsto X$ is continuous. If f maps bounded sets in $[0, \infty) \times X$ into bounded sets in X, then for every $u_0 \in X$ a mild solution (1.14) of (1.13) can be extended to maximal interval of existence $[0, t_{\max})$. If $t_{\max} < \infty$ then*

$$\overline{\lim_{t\to t_{\max}}} \|u(t)\| = \infty.$$

Proof. Suppose u is a mild solution of (1.13) on $[0, t_1]$ and consider the problem

$$\begin{aligned} w_t(t) &= Aw(t) + f(t + t_1, w(t)),\ t > 0 \\ w(0) &= u(t_1) \end{aligned} \tag{3.16}$$

where $w(0) = u(t_1) \in X$. Then, by Theorem 3.2.3, there exists a $t_2 = t_2(u(t_1)) > 0$ and a continuous function $w : [0, t_2] \mapsto X$ such that

$$w(t) = T(t)u(t_1) + \int_0^t T(t-s)f(s + t_1, w(s))ds$$

is a mild solution to (3.16) on $[0, t_2]$. We now show that we can extend the mild solution u of (1.13) to $[0, t_1 + t_2]$.

For $t = t_1 + \gamma$, $0 \leq \gamma \leq t_2$, let $u(t) = w(\gamma)$. For $t_1 < t \leq t_1 + t_2$, with $t = t_1 + \gamma$, we have

$$\begin{aligned}
T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds =& T(t_1+\gamma)u_0 \\
&+ \int_0^{t_1+\gamma} T(t_1+\gamma-s)f(s,u(s))ds \\
=& T(\gamma)T(t_1)u_0 \\
&+ \int_0^{t_1} T(t_1+\gamma-s)f(s,u(s))ds \\
&+ \int_{t_1}^{t_1+\gamma} T(t_1+\gamma-s)f(s,w(s-t_1))ds \\
=& T(\gamma)\left[T(t_1)u_0 + \int_0^{t_1} T(t_1-s)f(s,u(s))ds\right] \\
&+ \int_{t_1}^{t_1+\gamma} T(t_1+\gamma-s)f(s,w(s-t_1))ds \\
=& T(\gamma)u(t_1) \\
&+ \int_{t_1}^{t_1+\gamma} T(t_1+\gamma-s)f(s,w(s-t_1))ds.
\end{aligned} \tag{3.17}$$

Letting $\alpha = s - t_1$, (3.17) gives

$$\begin{aligned}
T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds =& T(\gamma)u(t_1) \\
&+ \int_0^{\gamma} T(\gamma-\alpha)f(\alpha+t_1,w(\alpha))d\alpha \\
=& w(\gamma) \\
=& u(t).
\end{aligned}$$

Thus u is a mild solution of (1.13) on $[0, t_1 + t_2]$.

Suppose $[0, t_{\max})$, $t_{\max} < \infty$, is the maximal interval to which the solution can be extended. For the sake of attaining a contradiction, suppose $\overline{\lim\limits_{t \to t_{\max}}} \|u(t)\| < \infty$. Then there exist constants $M, K > 0$ such that $\|T(t)\| \leq M$ and $\|u(t)\| \leq K$ for all $t \in [0, t_{\max})$. Furthermore, by our assumption that f maps bounded sets in $[0, \infty) \times X$ into bounded sets in X, there exists an $N > 0$ such that $\|f(t, u(t))\| \leq N$ for all $t \in [0, t_{\max}]$. Fix $\epsilon > 0$ and $0 < t < t' < t_{\max}$. Define $\rho \in (0, t)$ by

$$\rho = \min\left\{\frac{\epsilon}{8MN}, \frac{t}{2}\right\}. \tag{3.18}$$

Then

$$\begin{aligned}
\|u(t') - u(t)\|_X = & \left\| T(t')u_0 + \int_0^{t'} T(t'-s)f(s,u(s))ds \right. \\
& \left. - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds \right\|_X \\
\leq & \|T(t')u_0 - T(t)u_0\|_X + \left\| \int_t^{t'} T(t'-s)f(s,u(s)) \right\|_X \\
& + \left\| \int_0^{t-\rho} [T(t'-s) - T(t-s)]f(s,u(s))ds \right\|_X \\
& + \left\| \int_{t-\rho}^{t} [T(t'-s) - T(t-s)]f(s,u(s))ds \right\|_X \\
\leq & \|T(t')u_0 - T(t)u_0\|_X + (t'-t)MN \\
& + N \int_0^{t-\rho} \|T(t'-s) - T(t-s)\| ds \\
& + N \int_{t-\rho}^{t} \|T(t'-s)\| + \|T(t-s)\| ds \\
\leq & \|T(t')u_0 - T(t)u_0\|_X + (t'-t)MN + 2MN\rho \\
& + N \int_0^{t-\rho} \|T(t'-s) - T(t-s)\| ds. \qquad (3.19)
\end{aligned}$$

By the definition of a compact semigroup, $T(t)$, $t \geq 0$, is a C_0-semigroup. Thus there exists a $\delta_1 > 0$ such that if $(t'-t) < \delta_1$ then

$$\|T(t')u_0 - T(t)u_0\|_X < \frac{\epsilon}{4}. \qquad (3.20)$$

Since $T(t)$, $t \geq 0$, is a compact semigroup, it is continuous in the uniform operator topology for $t > 0$ by Theorem 2.1.6. Hence it is uniformly continuous in the uniform operator topology on the compact set $[\rho, t']$. If $0 < s < t-\rho$ then $(t'-s) \in [\rho, t']$ and $(t-s) \in [\rho, t']$. Thus there exists a $\delta_2 > 0$ such that if $[(t'-s) - (t-s)] = (t'-t) < \delta_2$ then

$$\|T(t'-s) - T(t-s)\| < \frac{\epsilon}{4Nt_{\max}} \qquad (3.21)$$

for all $s \in (0, t-\rho)$. Let $\delta^* = \min\left\{\delta_1, \delta_2, \dfrac{\epsilon}{4MN}\right\}$. Then (3.18), (3.20), (3.21) and (3.19) imply that if $(t'-t) < (t_{\max} - t) < \delta^*$ then

$$\|u(t') - u(t)\|_X < \epsilon.$$

Therefore $\lim\limits_{t \to t_{\max}} u(t) = u(t_{\max})$ exists. But then we can apply the same procedure as at the beginning of the proof to extend the solution $u(t)$ to $[0, b]$ where $b > t_{\max}$, contradicting the fact that $[0, t_{\max})$ is the maximal interval of existence. Thus $\overline{\lim\limits_{t \to t_{\max}}} \|u(t)\| = \infty$. $\square$

Corollary 3.3.2. *Suppose A is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$, and $f : [0, \infty) \times X \mapsto X$ is continuous. If f maps bounded sets in $[0, \infty) \times X$ into bounded sets in X, and there exist two real-valued locally Lebesgue-integrable functions k_1 and k_2 on $[0, \infty)$ such that*

$$\|f(t, u)\|_X \leq k_1(t)\|u\|_X + k_2(t)$$

for all $t \in [0, \infty)$ and $u \in X$, then, for every $u_0 \in X$, any local mild solution (1.14) *of* (1.13) *can be extended to a global mild solution.*

Proof. Fix $u_0 \in X$. By Theorem 3.2.3 there exists a $t^* = t^*(u_0)$ such that a local mild solution $u(t)$ of (1.13) exists on $[0, t^*]$. Assume that $[0, t_{\max})$ is the maximal interval of existence of $u(t)$ with $t_{\max} < \infty$. There exists a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \in [0, t_{\max}]$. Let

$$\alpha_{u_0}(t) = M\|u_0\|_X + M \int_0^t k_2(s)ds$$

for $t \in [0, \infty)$. Then for $t \in [0, t_{\max})$

$$\begin{aligned}
\|u(t)\|_X &= \left\| T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds \right\|_X \\
&\leq \|T(t)u_0\|_X + \int_0^t \|T(t-s)f(s, u(s))\|_X ds \\
&\leq M\|u_0\|_X + M \int_0^t \|f(s, u(s))\|_X ds \\
&\leq M\|u_0\|_X + M \int_0^t k_2(s)ds + \int_0^t Mk_1(s)\|u(s)\|_X ds \\
&= \alpha_{u_0}(t) + \int_0^t Mk_1(s)\|u(s)\|_X ds. \qquad (3.22)
\end{aligned}$$

By Grönwall's Inequality, Theorem C.0.11, (3.22) gives

$$\|u(t)\|_X \leq \alpha_{u_0}(t) + \int_0^t \alpha_{u_0}(s)Mk_1(s)e^{\int_s^t Mk_1(r)dr}ds \qquad (3.23)$$

for all $t \in [0, t_{\max})$. Define $\Phi_{u_0}(t) : [0, \infty) \mapsto [0, \infty)$ by

$$\Phi_{u_0}(t) = \begin{cases} \alpha_{u_0}(t) + \int_0^t \alpha_{u_0}(s) M k_1(s) e^{\int_s^t M k_1(r) dr} ds, & t \in [0, t_{\max}) \\ \alpha_{u_0}(t_{\max}) + \int_0^{t_{\max}} \alpha_{u_0}(s) M k_1(s) e^{\int_s^{t_{\max}} M k_1(r) dr} ds, & t \in [t_{\max}, \infty). \end{cases}$$

Clearly $\Phi_{u_0}(t)$ is continuous on $[0, \infty)$ and from (3.23)

$$\|u(t)\|_X \leq \Phi_{u_0}(t)$$

for $t \in [0, t_{\max})$. It follows that $\overline{\lim_{t \to t_{\max}}} \|u(t)\|_X < \infty$ contradicting Theorem 3.3.1. Hence any mild solution u of (1.13) can be extended to a global solution. $\square$

3.4 Regularity

Recall from Section 1.5.2 that a mild solution of (1.13) need not be a classical solution of (1.13) since it need not be differentiable. Indeed, a mild solution is classical if and only if it is continuously differentiable, see [2, Propositions 3.1.2 and 3.1.9].

In this section we prove a regularity result for the mild solutions of (1.13). This result gives sufficient conditions on f and A for the mild solutions of (1.13) to be classical solutions. In particular, f is required to be Hölder continuous. See Appendix B for the relevant definitions and results regarding Hölder continuous functions.

Before proceeding to the main result for the regularity of mild solutions to (1.13), we consider useful regularity results for the homogeneous abstract Cauchy problem and the linear non-homogeneous abstract Cauchy problem. We do not prove the results concerning the homogeneous problem for the sake of presentation, but we do for the linear non-homogeneous problem, as the result for the semi-linear case relies heavily on it and it lends intuition to the relationship between the mild and classical solutions of (1.13). This section is based on work done by Pazy in [17] and [18, Sections 3.1-3.2], and Arendt *et al.* in [2].

In this section X is a complex Banach space.

The Homogeneous Problem

The homogeneous case of the abstract Cauchy problem is given by

$$\begin{aligned} u_t(t) &= Au(t), \, t > 0 \\ u(0) &= u_0 \end{aligned} \tag{3.24}$$

with $u_0 \in X$.

Remark 3.4.1. If $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, then $u(t) = T(t)u_0$ is a mild solution to (3.24). If $u_0 \in \mathcal{D}(A)$ then $u(t) = T(t)u_0$ is a classical solution to (3.24), see [2, Proposition 3.1.9 (h)]. Continuously differentiable semigroups give us a stronger result. If A is the infinitesimal generator of a continuously differentiable semigroup $T(t)$, $t \geq 0$, then for all $u_0 \in X$ the function $u(t) = T(t)u_0$ is a classical solution to (3.24). It follows that if A is the infinitesimal generator of an analytic semigroup e^{tA}, $t \geq 0$, then for all $u_0 \in X$ the function $u(t) = e^{tA}u_0$ is a classical solution to (3.24), see [2, Corollary 3.7.21].

The Linear Problem

We now consider the linear non-homogeneous abstract Cauchy problem

$$\begin{aligned} u_t(t) &= Au(t) + g(t),\ t > 0 \\ u(0) &= u_0 \end{aligned} \tag{3.25}$$

where $u_0 \in X$ and $g : [0, \infty) \mapsto \mathbb{R}$ is independent of u. Assume $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of an analytic semigroup e^{tA}, $t \geq 0$. Then

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} g(s) ds \tag{3.26}$$

is a mild solution to (3.25).

The idea is to first prove that if g is Hölder continuous, then a mild solution (3.26) of (3.25) is a classical solution of (3.25). We then show that a mild solution (3.26) is Hölder continuous, which enables us to prove a similar result for the semi-linear case where g acts on u.

Theorem 3.4.1. *Let A be the infinitesimal generator of an analytic semigroup e^{tA}, $t \geq 0$. If, for some $T > 0$, the function $g \in \mathcal{L}^1(0, T; X)$ is Hölder continuous on $(0, T)$, then for every $u_0 \in X$ the mild solution (3.26) to problem (3.25) is a classical solution.*

Proof. Fix $u_0 \in X$. We split this proof into two parts:

1) We show that if

$$v(t) = \int_0^t e^{(t-s)A} g(s) ds$$

is an element of $\mathcal{D}(A)$ for $t \in (0, T)$, and $Av(t)$ is continuous on $(0, T)$, then every mild solution

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} g(s) ds = e^{tA}u_0 + v(t) \tag{3.27}$$

of (3.25) is a classical solution of (3.25).

2) We show that if g satisfies the conditions stated in the theorem then $v(t) \in \mathcal{D}(A)$ for $t \in (0, T)$ and $Av(\cdot)$ is continuous on $(0, T)$.

We begin with the proof of part 1. Suppose that $u(t)$ is a mild solution to the linear abstract Cauchy problem (3.25), that is, $u(t)$ satisfies (3.27). Furthermore, assume that $v(t) \in \mathcal{D}(A)$ for all $t \in (0, T)$ and $Av(\cdot)$ is continuous on $(0, T)$. Fix $t \in (0, T)$ and $h > 0$ is such that $t + h \in (0, T)$. Then

$$\begin{aligned}
&h^{-1}[v(t+h) - v(t)] - h^{-1}\int_t^{t+h} e^{(t+h-s)A} g(s)ds \\
=&h^{-1}\int_0^{t+h} e^{(t+h-s)A} g(s)ds - h^{-1}\int_0^{t} e^{(t-s)A} g(s)ds - h^{-1}\int_t^{t+h} e^{(t+h-s)A} g(s)ds \\
=&h^{-1}\int_0^{t} e^{(t+h-s)A} g(s)ds - h^{-1}\int_0^{t} e^{(t-s)A} g(s)ds \\
=&h^{-1}(e^{hA} - I)v(t).
\end{aligned} \tag{3.28}$$

Since $v(t) \in \mathcal{D}(A)$ for $t \in (0, T)$, we have

$$\lim_{h \to 0^+} h^{-1}(e^{hA} - I)v(t) = Av(t). \tag{3.29}$$

Furthermore we have

$$\begin{aligned}
&\left\| h^{-1}\int_t^{t+h} e^{(t+h-s)A} g(s)ds - g(t) \right\|_X \\
&\qquad = \left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A} g(s) - g(t)]ds \right\|_X \\
&\qquad \leq \left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A} g(s) - e^{(t+h-s)A} g(t)]ds \right\|_X \\
&\qquad\quad + \left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A} g(t) - g(t)]ds \right\|_X .
\end{aligned} \tag{3.30}$$

Fix $\epsilon > 0$ and let M_T be an upper bound for the set $\{\|e^{sA}\| \mid 0 \leq s < T\}$. Then

$$\begin{aligned}
&\left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A} g(s) - e^{(t+h-s)A} g(t)]ds \right\|_X \\
&\qquad \leq h^{-1}\int_t^{t+h} \|e^{(t+h-s)A} g(s) - e^{(t+h-s)A} g(t)\|_X ds \\
&\qquad \leq \frac{M_T}{h}\int_t^{t+h} \|g(s) - g(t)\|_X ds.
\end{aligned}$$

Since g is continuous, by Proposition A.1.5, $\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \|g(s)-g(t)\|_X ds = 0$, thus there exists a $\delta_1 > 0$ such that if $h < \delta_1$ then

$$\left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A}g(s) - e^{(t+h-s)A}g(t)]ds \right\|_X \le \frac{\epsilon}{2}. \tag{3.31}$$

Furthermore, by letting $v = t+h-s$ we get

$$\begin{aligned} \left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A}g(t) - g(t)]ds \right\|_X &= \left\| h^{-1}\int_0^{h} [e^{vA}g(t) - g(t)]dv \right\|_X \\ &\le h^{-1}\int_0^h \|e^{vA}g(t) - g(t)\|_X dv. \end{aligned}$$

Since e^{vA}, $v \ge 0$, is a C_0-semigroup then $v \mapsto e^{vA}g(t)$ is continuous, and thus Bochner integrable. Hence, by Proposition A.1.5, $\lim_{h\to 0^+} h^{-1}\int_0^h \|e^{vA}g(t) - g(t)\|_X dv = 0$. Therefore there exists a $\delta_2 > 0$ such that if $h < \delta_2$ then

$$\left\| h^{-1}\int_t^{t+h} [e^{(t+h-s)A}g(t) - g(t)]ds \right\|_X \le \frac{\epsilon}{2}. \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.30) gives that if $h < \min\{\delta_1, \delta_2\}$ then

$$\left\| h^{-1}\int_t^{t+h} e^{(t+h-s)A}g(s)ds - g(t) \right\|_X < \epsilon.$$

Thus

$$\lim_{h\to 0^+} h^{-1}\int_t^{t+h} e^{(t+h-s)A}g(s)ds = g(t). \tag{3.33}$$

Substituting (3.29) and (3.33) into (3.28) and taking $h \to 0^+$ gives that $v(t)$ is differentiable from the right on $(0,T)$, and

$$\frac{d}{dt}v(t) = \lim_{h\to 0^+} h^{-1}[v(t+h) - v(t)] = Av(t) + g(t). \tag{3.34}$$

Furthermore, since $Av(t)$ and $g(t)$ are continuous on $(0,T)$, it follows that $v(t)$ is continuously differentiable on this interval, and

$$v(0) = 0. \tag{3.35}$$

From Remark 3.4.1, since A is the infinitesimal generator of an analytic semigroup e^{tA}, $t \ge 0$, then $e^{tA}u_0$ is a classical solution to the homogeneous abstract Cauchy problem (3.24) for all $u_0 \in X$. Therefore $e^{tA}u_0$ is continuously differentiable on $(0,T)$ and

$$e^{0A}u_0 = u_0. \tag{3.36}$$

Thus, from (3.27), (3.35) and (3.36), we have that $u(0) = u_0$, and $u(t)$ is continuously differentiable on $(0,T)$ for each $u_0 \in X$. Furthermore, from (3.27), Theorem 2.3.5 (iv) and (3.34), we have

$$\begin{aligned}\frac{d}{dt}u(t) &= \frac{d}{dt}[e^{tA}u_0 + v(t)]\\ &= Ae^{tA}u_0 + Av(t) + g(t)\\ &= Au(t) + g(t)\end{aligned}$$

for all $u_0 \in X$ and $t \in (0,T)$. Thus $u(t)$ is a classical solution of (3.25).

We now prove part 2. For $t \in (0,T)$ we have

$$v(t) = \int_0^t e^{(t-s)A}g(s)ds = v_1(t) + v_2(t)$$

where

$$v_1(t) = \int_0^t e^{(t-s)A}[g(s) - g(t)]ds$$

and

$$v_2(t) = \int_0^t e^{(t-s)A}g(t)ds.$$

From Theorem 1.4.3 (ii) we have that $v_2(t) \in \mathcal{D}(A)$ for each $t \in (0,T)$ and that

$$Av_2(t) = (e^{tA} - I)g(t).$$

Since $g(t)$ is continuous on $(0,T)$, it follows that $Av_2(t)$ is continuous on this interval.

We now show that $v_1(t) \in \mathcal{D}(A)$ for each $t \in (0,T)$ using the fact that A is closed. Fix $t \in (0,T)$ and let $\epsilon \in (0,t)$. Define the function $v_{1,\epsilon}(t)$ by

$$v_{1,\epsilon}(t) = \int_0^{t-\epsilon} e^{(t-s)A}[g(s) - g(t)]ds.$$

Clearly

$$\lim_{\epsilon\to 0^+} v_{1,\epsilon}(t) = \int_0^t e^{(t-s)A}[g(s) - g(t)]ds = v_1(t). \tag{3.37}$$

We now show that $v_{1,\epsilon}(t) \in \mathcal{D}(A)$. Let $f(s) = e^{(t-s)A}[g(s)-g(t)]$, $s \in [0, t-\epsilon]$. By Theorem 2.3.5 (i) we have that $f(s) \in \mathcal{D}(A)$ for each $s \in [0, t-\epsilon]$.

Furthermore, for any $s_1, s_2 \in [0, t-\epsilon]$, assuming without loss of generality that $s_1 > s_2$, we have

$$\begin{aligned}\|f(s_1) - f(s_2)\|_X =& \|e^{(t-s_1)A}[g(s_1) - g(t)] - e^{(t-s_2)A}[g(s_2) - g(t)]\|_X \\ \leq& \|e^{(t-s_1)A}\| \Big(\|g(s_1) - e^{(s_1-s_2)A} g(s_1)\|_X \\ &+ \|e^{(s_1-s_2)A}[g(s_1) - g(s_2)]\|_X + \|g(t) - e^{(s_1-s_2)A} g(t)]\|_X \Big).\end{aligned}$$

It follows that f is continuous on $[0, t-\epsilon]$ since $e^{(t-s)A}$ is bounded on $[0, t-\epsilon]$ by Theorem 2.3.5 $(iii)(a)$, e^{tA}, $t \geq 0$, is a C_0-semigroup by Theorem 2.3.6, g is continuous, and from Proposition 3.1.1. Thus f is Bochner integrable.

For $s \in [0, t-\epsilon]$ Theorem 2.3.5 (iv) gives that

$$\begin{aligned}Af(s) &= Ae^{(t-s)A}[g(s) - g(t)] \\ &= \left[\frac{d}{d(t-s)} e^{(t-s)A}\right][g(s) - g(t)].\end{aligned}$$

Since $s \in [0, t-\epsilon]$ then $(t-s) \in [\epsilon, t]$ and thus, by Theorem 2.3.5 (iv), $A \circ f$ is continuous on $[0, t-\epsilon]$. Thus $A \circ f$ is Bochner integrable. Hence, by Theorem A.1.3, $v_{1,\epsilon}(t) \in \mathcal{D}(A)$ and

$$Av_{1,\epsilon}(t) = \int_0^{t-\epsilon} Ae^{(t-s)A}[g(s) - g(t)]ds. \tag{3.38}$$

We now show that $\int_0^t Ae^{(t-s)A}[g(s) - g(t)]ds$ exists and that

$$\lim_{\epsilon \to 0^+} Av_{1,\epsilon}(t) = \int_0^t Ae^{(t-s)A}[g(s) - g(t)]ds.$$

By Theorem 2.3.5 (iv),

$$Ae^{(t-s)A}[g(s) - g(t)] = \left[\frac{d}{d(t-s)} e^{(t-s)A}\right][g(s) - g(t)]$$

is continuous on $[0, t)$. Furthermore, by Theorem 2.3.5 $(iii)(c)$, there exists constants $C_{1,1} > 0$ and $\omega \in \mathbb{R}$ such that

$$\|Ae^{(t-s)A}[g(s) - g(t)]\|_X \leq C_{1,1} e^{(\omega+1)(t-s)} (t-s)^{-1} \|g(s) - g(t)\|_X \tag{3.39}$$

on $[0, t)$. Since $g(t)$ is Hölder continuous on $(0, T)$ there exist constants $C > 0$ and $\alpha \in (0, 1]$ such that

$$\|g(s) - g(t)\|_X \leq C|s - t|^\alpha \tag{3.40}$$

for all $s \in (0,t]$. Thus, by Proposition B.0.5, we can take $\alpha < 1$. From (3.39) and (3.40) we have that

$$\|Ae^{(t-s)A}[g(s)-g(t)]\|_X \leq Le^{(\omega+1)(t-s)}(t-s)^{\alpha-1} \tag{3.41}$$

for $s \in [0,t)$ with $\alpha < 1$ where $L = C_{1,1}C$. Thus $\displaystyle\int_0^t Ae^{(t-s)A}[g(s)-g(t)]ds$ exists for every $s \in [0,t)$. Now from (3.38) and (3.41) we have

$$\begin{aligned}
\left\| \int_0^t Ae^{(t-s)A}[g(s)-g(t)]ds - Av_{1,\epsilon}(t) \right\|_X &= \left\| \int_{t-\epsilon}^t Ae^{(t-s)A}[g(s)-g(t)]ds \right\|_X \\
&\leq \int_{t-\epsilon}^t \|Ae^{(t-s)A}[g(s)-g(t)]\|_X ds \\
&\leq \int_{t-\epsilon}^t Le^{(\omega+1)(t-s)}(t-s)^{\alpha-1}ds
\end{aligned}$$

for $s \in [0,t)$ with $\alpha < 1$. Let $K = \max\{L, e^{(\omega+1)T}L\}$. Then K is independent of ϵ and

$$\begin{aligned}
\left\| \int_0^t Ae^{(t-s)A}[g(s)-g(t)]ds - Av_{1,\epsilon}(t) \right\|_X &\leq K \int_{t-\epsilon}^t (t-s)^{\alpha-1}ds \\
&= \frac{K}{\alpha}\epsilon^\alpha. \qquad (3.42)
\end{aligned}$$

Thus

$$\lim_{\epsilon\to 0^+} Av_{1,\epsilon}(t) = \int_0^t Ae^{(t-s)A}[g(s)-g(t)]ds. \tag{3.43}$$

Since A is closed, (3.37) and (3.43) give that $v_1(t) \in \mathcal{D}(A)$ and

$$Av_1(t) = \int_0^t Ae^{(t-s)A}[g(s)-g(t)]ds.$$

We now show that Av_1 is continuous on $(0,T)$. From (3.42) it follows that, for each $\delta \in (0,T)$, $Av_{1,\epsilon} \to Av_1$ uniformly on $[\delta, T)$ as $\epsilon \to 0^+$. Since $Av_{1,\epsilon}(t)$ is continuous on $(0,T)$ for all $\epsilon \in (0,T)$, it follows that, taking $\delta \to 0$, $Av_1(t)$ is continuous on $(0,T)$.

Thus $v(t) \in \mathcal{D}(A)$ for $t \in (0,T)$ and $Av(\cdot)$ is continuous on $(0,T)$.

□

The following lemma will be useful to us in the proof of a regularity result, similar to the one above, for the semi-linear problem.

Lemma 3.4.2. *Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of an analytic semigroup e^{tA}, $t \geq 0$, and $g \in \mathcal{L}^2(0,T;X)$. If $u(t)$ is a mild solution to the linear Cauchy problem* (3.25)*, that is, $u(t)$ satisfies* (3.26) *with $u_0 \in X$, then, for every $\epsilon \in (0,T)$, $u(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[\epsilon,T)$. That is, for every $\epsilon > 0$ there exists a constant $K_\epsilon > 0$ such that*

$$\|u(t) - u(s)\|_X \leq K_\epsilon |t-s|^{\frac{1}{2}}$$

for all $t, s \in [\epsilon, T)$. Furthermore, if $u_0 \in \mathcal{D}(A)$ then $u(t)$ is Hölder continuous on $[0,T)$ with exponent $\frac{1}{2}$.

Proof. Fix $\epsilon \in (0,T)$. Let $\|e^{tA}\| \leq M$ on $[0,T)$. From Theorem 2.3.5 $(iii)(c)$ there exist constants $\omega \in \mathbb{R}$ and $C_{1,1} > 0$ such that

$$\|Ae^{tA}\| \leq C_{1,1}e^{(\omega+1)t}t^{-1} \leq Ct^{-1} \tag{3.44}$$

on $(0,T)$ where $C = \max\{C_{1,1}, C_{1,1}e^{(\omega+1)T}\}$. If $x \in X$ then from (3.44) the mapping $[\epsilon,T) \ni t \mapsto Ae^{tA}x$ belongs to $\mathcal{L}^1([\epsilon,T);X)$. Thus, from Theorems 2.3.6, 1.4.3 (ii) and 2.3.5 (i), and using (3.44), if $t, t+h \in (0,T)$ then

$$\begin{aligned}
\|e^{(t+h)A}x - e^{tA}x\|_X &= \|e^{tA}(e^{hA}x - x)\|_X \\
&= \left\| e^{tA}\int_0^h Ae^{sA}xds \right\|_X \\
&= \left\| \int_0^h Ae^{(t+s)A}xds \right\|_X \\
&= \left\| \int_t^{t+h} Ae^{rA}xdr \right\|_X \\
&\leq \int_t^{t+h} \|Ae^{rA}\|\|x\|_X dr \\
&\leq \int_t^{t+h} Cr^{-1}\|x\|_X dr \\
&\leq C\|x\|_X t^{-1}\int_t^{t+h} dr \\
&\leq Ct^{-1}h\|x\|_X,
\end{aligned}$$

so that

$$\|e^{(t+h)A} - e^{tA}\| \leq hCt^{-1}. \tag{3.45}$$

In particular, if $t, t+h \in [\epsilon, T)$ then

$$\|e^{(t+h)A} - e^{tA}\| \leq \frac{hC}{\epsilon}.$$

Thus $t \mapsto e^{tA}x$ is Lipschitz continuous on $[\epsilon, T)$ for all $x \in X$.

However, if $x \in \mathcal{D}(A)$ and $t, t+h \in [0,T)$, then by Theorem 2.3.5 (i) and $(iii)(a)$ there exists an $M^* > 0$ such that

$$\begin{aligned}
\left\| \int_t^{t+h} Ae^{rA}xdr \right\|_X &= \left\| \int_t^{t+h} e^{rA}Axdr \right\|_X \\
&\leq \int_t^{t+h} \|e^{rA}A\| \|x\|_X dr \\
&\leq \int_t^{t+h} M^* \|x\|_X dr \\
&= hM^* \|x\|_X,
\end{aligned}$$

where M^* is an upper bound for the bounded operator $\|e^{tA}A\|$ on $(0,T)$. Thus $t \mapsto e^{tA}x$ is Lipschitz continuous on $[0,T)$ for all $x \in \mathcal{D}(A)$.

Hence, Proposition B.0.5 gives that the first term on the right-hand side of

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s)ds$$

will be Hölder continuous with exponent $\frac{1}{2}$ on $[\epsilon, T)$ if $u_0 \in X$, and on $[0,T)$ if $u_0 \in \mathcal{D}(A)$. Hence, for $u(t)$ to be Hölder continuous with exponent $\frac{1}{2}$ on $[\epsilon, T)$ and $[0,T)$ for $u_0 \in X$ and $u_0 \in \mathcal{D}(A)$ respectively, it is sufficient to show that $v(t) = \displaystyle\int_0^t e^{(t-s)A}g(s)ds$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0,T)$.

To do this fix $t \in [0,T)$ and $h > 0$ such that $t + h < T$. Then

$$\begin{aligned}
v(t+h) - v(t) &= \int_0^{t+h} e^{(t+h-s)A}g(s)ds - \int_0^t e^{(t-s)A}g(s)ds \\
&= \int_t^{t+h} e^{(t+h-s)A}g(s)ds \\
&\quad + \int_0^t [e^{(t+h-s)A} - e^{(t-s)A}]g(s)ds. \qquad (3.46)
\end{aligned}$$

We deal with each integral in (3.46) independently. Consider the norm of the first integral. By the Cauchy-Schwartz inequality

$$\begin{aligned}
\left\| \int_t^{t+h} e^{(t+h-s)A} g(s)ds \right\|_X &\leq M \int_t^{t+h} \|g(s)\|_X ds \\
&\leq M \left(\int_t^{t+h} ds \right)^{\frac{1}{2}} \left(\int_t^{t+h} \|g(s)\|_X^2 ds \right)^{\frac{1}{2}} \\
&\leq M h^{\frac{1}{2}} \|g\|_{\mathcal{L}^2(0,T;X)} \\
&= K_1 h^{\frac{1}{2}} \qquad (3.47)
\end{aligned}$$

where $K_1 = M\|g\|_{\mathcal{L}^2(0,T;X)}$.

Before we proceed with the second integral, note that

$$\|e^{(t+h)A} - e^{tA}\| \leq 2M \qquad (3.48)$$

on $[0,T)$. Thus, from (3.45) and (3.48), letting $C^* = \max\{C, 2M\}$ and $\mu(h,t) = \min\left\{1, \frac{h}{t}\right\}$ we get that

$$\|e^{(t+h)A} - e^{tA}\| \leq C^* \mu(h,t). \qquad (3.49)$$

Now, by the Cauchy-Schwartz inequality and (3.49) we have

$$\begin{aligned}
\left\| \int_0^t [e^{(t+h-s)A} - e^{(t-s)A}] g(s)ds \right\|_X &\leq \int_0^t \|e^{(t+h-s)A} - e^{(t-s)A}\| \|g(s)\|_X ds \\
&\leq C^* \int_0^t |\mu(h,t-s)| \|g(s)\|_X ds \\
&\leq C^* \left(\int_0^t \mu(h,t-s)^2 ds \right)^{\frac{1}{2}} \\
&\quad \left(\int_0^t \|g(s)\|_X^2 ds \right)^{\frac{1}{2}} \\
&\leq C^* \|g\|_{\mathcal{L}^2(0,T;X)} \left(\int_0^t \mu(h,t-s)^2 ds \right)^{\frac{1}{2}}. \qquad (3.50)
\end{aligned}$$

Letting $\tau = t - s$ we have

$$\begin{aligned}
\int_0^t \mu(h,t-s)^2 ds &= \int_0^t \mu(h,\tau)^2 d\tau \\
&\leq \int_0^h d\tau + \int_h^\infty \frac{h^2}{\tau^2} d\tau \\
&= 2h. \qquad (3.51)
\end{aligned}$$

Substituting (3.51) into (3.50) gives

$$\left\| \int_0^t [e^{(t+h-s)A} - e^{(t-s)A}] g(s) ds \right\|_X \leq K_2 h^{\frac{1}{2}} \tag{3.52}$$

where $K_2 = \sqrt{2} C^* \|g\|_{\mathcal{L}^2(0,T;X)}$. Finally, substituting (3.47) and (3.52) into (3.46) gives that

$$\|v(t+h) - v(t)\|_X \leq K^* h^{\frac{1}{2}}$$

where $K^* = K_1 + K_2$. Hence $v(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0, T)$. Therefore

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g(s)ds$$

is Hölder continuous with exponent $\frac{1}{2}$ on $[\epsilon, T)$ for $u_0 \in X$, and on $[0, T)$ for $u_0 \in \mathcal{D}(A)$. □

The Semi-Linear Problem

We now proceed to the semi-linear problem (1.13), and show that if f is Hölder continuous then a mild solution (1.14) of (1.13) is a classical solution of (1.13).

Theorem 3.4.3. *Suppose $T > 0$, U is an open subset of X and $A : X \supset \mathcal{D}(A) \mapsto X$ is the infinitesimal generator of an analytic semigroup e^{tA}, $t \geq 0$. If $f : [0, T) \times U \mapsto X$ is Hölder continuous in both of its variables, that is, there exist constants $K > 0$ and $\alpha \in (0, 1]$ such that*

$$\|f(t_1, u_1) - f(t_2, u_2)\|_X \leq K(|t_1 - t_2|^\alpha + \|u_1 - u_2\|_X^\alpha) \tag{3.53}$$

for every $t_1, t_2 \in [0, T)$ and $u_1, u_2 \in U$, then every mild solution $u(t)$ of (1.13) is a classical solution of (1.13).

Proof. Suppose $u(t)$ is a mild solution to (1.13), that is

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s, u(s))ds \tag{3.54}$$

for $t \in [0, T)$. If we let $g(t) = f(t, u(t))$ for each $t \in [0, T)$, then

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} g(s)ds \tag{3.55}$$

is a mild solution of problem (3.25).

We now show that $u(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0,T)$. By Lemma 3.4.2, $u(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[\frac{T}{5},T)$. Thus there exists a constant $C_1>0$ such that, if $t_1,t_2\in[\frac{T}{5},T)$ then

$$\|u(t_1)-u(t_2)\|_X \leq C_1|t_1-t_2|^{\frac{1}{2}}. \tag{3.56}$$

For $t\in[0,\frac{2T}{3})$, let $v(t)=u\left(t+\frac{T}{3}\right)$ and $h(t,v(t))=f\left(t+\frac{T}{3},v(t)\right)$. Then, for $t\in[0,\frac{2T}{3})$, we have

$$\begin{aligned}
v(t) =& u\left(t+\frac{T}{3}\right)\\
=& e^{\left(t+\frac{T}{3}\right)A}u_0+\int_0^{t+\frac{T}{3}} e^{\left(t+\frac{T}{3}-s\right)A}f(s,u(s))ds\\
=& e^{tA}e^{\frac{T}{3}A}u_0+e^{tA}\int_0^{\frac{T}{3}} e^{\left(\frac{T}{3}-s\right)A}f(s,u(s))ds\\
&+\int_{\frac{T}{3}}^{t+\frac{T}{3}} e^{\left(t+\frac{T}{3}-s\right)A}f(s,u(s))ds\\
=& e^{tA}u\left(\frac{T}{3}\right)+\int_{\frac{T}{3}}^{t+\frac{T}{3}} e^{\left(t+\frac{T}{3}-s\right)A}f(s,u(s))ds.
\end{aligned}\tag{3.57}$$

Letting $\alpha=s-\frac{T}{3}$, (3.57) gives

$$\begin{aligned}
v(t) =& e^{tA}u\left(\frac{T}{3}\right)+\int_0^t e^{(t-\alpha)A}f\left(\alpha+\frac{T}{3},u\left(\alpha+\frac{T}{3}\right)\right)d\alpha\\
=& e^{tA}u\left(\frac{T}{3}\right)+\int_0^t e^{(t-\alpha)A}h(\alpha,v(\alpha))d\alpha.
\end{aligned}$$

Thus $v(t)$ is a mild solution to the problem

$$\begin{aligned}
&v'(t)=Av(t)+h(t,v(t)),\ t\in\left(0,\frac{2T}{3}\right)\\
&v(0)=u\left(\frac{T}{3}\right).
\end{aligned}$$

Let $g_h(t)=h(t,v(t))$ for $t\in(0,\frac{2T}{3})$, then $v(t)$ is a mild solution to the problem

$$\begin{aligned}
&v'(t)=Av(t)+g_h(t),\ t\in\left(0,\frac{2T}{3}\right)\\
&v(0)=u\left(\frac{T}{3}\right).
\end{aligned}$$

Thus, by Lemma 3.4.2, $v(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[\frac{T}{3}, \frac{2T}{3})$. Since $u(t) = v\left(t - \dfrac{T}{3}\right)$, then $u(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0, \frac{T}{3})$. Thus there exists a constant $C_2 > 0$ such that, if $t_1, t_2 \in [0, \frac{T}{3})$ then

$$\|u(t_1) - u(t_2)\|_X \leq C_2|t_1 - t_2|^{\frac{1}{2}}. \tag{3.58}$$

Now we have that u is Hölder continuous with exponent $\frac{1}{2}$ on both $[0, \frac{T}{3})$ and $[\frac{T}{5}, T)$. Suppose $t_1 \in [0, \frac{T}{4})$ and $t_2 \in [\frac{T}{4}, T)$. Then (3.56) and (3.58) give that

$$\begin{aligned}
\|u(t_1) - u(t_2)\|_X &\leq \left\|u(t_1) - u\left(\frac{T}{4}\right)\right\|_X + \left\|u\left(\frac{T}{4}\right) - u(t_2)\right\|_X \\
&\leq C_2\left|t_1 - \frac{T}{4}\right|^{\frac{1}{2}} + C_1\left|t_2 - \frac{T}{4}\right|^{\frac{1}{2}} \\
&\leq \max\{C_1, C_2\}\left(\left|t_1 - \frac{T}{4}\right|^{\frac{1}{2}} + \left|t_2 - \frac{T}{4}\right|^{\frac{1}{2}}\right) \\
&\leq 2\max\{C_1, C_2\}|t_1 - t_2|^{\frac{1}{2}}.
\end{aligned}$$

Thus $u(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0, T)$.

By Proposition B.0.4, f and u are continuous on $[0, T]$, and thus $g \in \mathcal{L}^1(0, T; X)$. Furthermore, from (3.53) and the fact that $u(t)$ is Hölder continuous with exponent $\frac{1}{2}$ on $[0, T)$, there exist constants $D > 0$, $K > 0$ and $\alpha \in (0, 1]$ such that if $t_1, t_2 \in [0, T)$ then

$$\begin{aligned}
\|g(t_1) - g(t_2)\|_X &= \|f(t_1, u(t_1)) - f(t_2, u(t_2))\|_X \\
&\leq K(|t_1 - t_2|^{\alpha} + \|u(t_1) - u(t_2)\|_X^{\alpha}) \\
&\leq K(|t_1 - t_2|^{\alpha} + D^{\alpha}|t_1 - t_2|^{\frac{1}{2}\alpha}) \\
&= K(|t_1 - t_2|^{\frac{1}{2}\alpha} + D^{\alpha})|t_1 - t_2|^{\frac{1}{2}\alpha} \\
&\leq C|t_1 - t_2|^{\frac{1}{2}\alpha}
\end{aligned}$$

where $C = K(T^{\frac{1}{2}\alpha} + D^{\alpha})$. Thus g is Hölder continuous on $[0, T)$. Hence, by Theorem 3.4.1, (3.55) is a classical solution to the linear Cauchy problem (3.25) for each $u_0 \in X$. It follows that (3.54) is a classical solution to the abstract Cauchy problem (1.13). □

Chapter 4

Example

4.1 Biological Model

Many biological organisms are heavily reliant on the stability of their core body temperature to survive. Being able to maintain a constant body temperature is therefore imperative. Warm-blooded animals like elephants cool themselves off by flapping their ears, splashing themselves with water, and covering themselves in a layer of mud or dust. Cold-blooded animals like snakes lie on rocks in the sun to warm themselves up.

Apart from regulating their body temperatures by making use of their environments, these organisms may also have an internal thermoregulatory mechanism. In humans these mechanisms work to maintain a core body temperature averaging $36.2°C$, "with most of the internal temperatures controlled within the range of $35 - 39°C$", [26, Introduction]. An example of this is the way humans constrict their blood vessels near the skin and shiver if they get too cold, or dilate their blood vessels near the skin or sweat if they get too hot. Often these mechanisms will only operate when their body temperature has exceeded, or dropped below, some threshold limit.

Tests done by Wyndham and Atkins [25] show that in humans "sweat rate does not increase until rectal temperature rises above a threshold value of $36.5°C$; thereafter the increase in sweat rate depends upon the level of mean skin temperature, being greater the higher the mean skin temperature is". Furthermore, they claim that "sweat rate does not increase markedly until mean skin temperature rises above $33°C$". There has been "considerable disagreement as to whether the peripheral or core temperature is the controlling factor for the sweat mechanism", [24, Introduction]. Therefore further investigations and refinement of the mathematical models for perspiration as a thermoregulatory mechanism may be necessary.

We consider a simple mathematical model for the distribution of heat on the skin of humans who are at rest, with particular consideration of perspiration as a thermoregulatory mechanism. Suppose $\Omega \in \mathbb{R}^2$ is an open and bounded set representing the spatial domain of the skin surface in question. We suppose that the boundary $\partial\Omega$ of Ω is smooth, that is, $\partial\Omega \in C^\infty$. For some fixed $T > 0$, let $\Omega_T = \Omega \times (0, T)$. Let $u(x, t)$ represent the skin temperature at point $x \in \Omega$ and time $t \in (0, T)$, and $L : \Omega \mapsto \mathbb{R}$ the threshold limit. That is, perspiration starts at $x \in \Omega$ and $t \in (0, T)$ whenever $u(x, t) > L(x)$. The function u_0 is the distribution of heat on the surface of the skin at time $t = 0$. We assume that we have Dirichlet boundary conditions, that is, $u(x, t) = 0$ for $x \in \partial\Omega$ for each $t \in [0, T)$. A biological interpretation of this assumption is that we take the average heat at the surface of the skin, say $H \in \mathbb{R}$, to correspond with $u(x, t) = 0$ and assume that the skin surrounding the area in question remains at that temperature. In the case of Wyndam and Atkins [25], $L(x) = 33°C - H$ for all $x \in \Omega$. We consider only the diffusion across the skin due to the laws of thermodynamics, as described by the heat equation

$$\frac{\partial}{\partial t} u(x, t) = \nabla^2 u(x, t),$$

and the contribution of perspiration, given by $p(x, t)$, when the skin temperature exceeds the threshold $L(x)$.

This provides us with a simple semi-linear parabolic problem (1.10) given by

$$\begin{aligned}
&u_t(x, t) = \nabla^2 u(x, t) + p(x, t) \max\{u(x, t) - L(x), 0\},\ (x, t) \in \Omega_T \\
&u(x, 0) = u_0,\ x \in \Omega \\
&u(x, t) = 0,\ x \in \partial\Omega
\end{aligned} \tag{4.1}$$

where $p(x, t) : \overline{\Omega} \times [0, \infty) \mapsto \mathbb{R}^-$ is assumed to be continuous and uniformly bounded on its domain, and $L \in \mathcal{L}^2(\Omega)$. It is clear that the operator ∇^2 is in divergence form (1.11) taking $\alpha_{i,j}(x, t) = 1$, $\beta^i(x, t) = 0$ and $\gamma(x, t) = 0$ on Ω_T, with $i, j = 1, 2$. Furthermore, it is clear that the operator $\dfrac{\partial}{\partial t} - \nabla^2$ is parabolic on Ω_T.

We reformulate this problem as an infinite-dimensional dynamical system, in the form of an abstract Cauchy problem (1.13). Our Banach space X, as referred to throughout this thesis, shall be taken to be the Hilbert space $\mathcal{L}^2(\Omega)$. Due to the Dirichlet boundary conditions, we take $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Let $u(t) : [0, T) \mapsto \mathcal{L}^2(\Omega)$ be the state of u at time t, that is,

$$u(t) : x \in \Omega \mapsto u(t)[x] := u(x, t).$$

Furthermore, define $p : [0, T) \mapsto \mathcal{L}^2(\Omega)$ by

$$p(t) : x \in \Omega \mapsto p(t)[x] := p(x, t) \in \mathbb{R}$$

for each $t \in [0, T)$ and $f : [0, T) \times \mathcal{L}^2(\Omega) \mapsto \mathcal{L}^2(\Omega)$ by

$$f(t, u) : x \in \Omega \mapsto f(t, u)[x] := p(x, t) \max\{u(x, t) - L(x), 0\}.$$

Then

$$f(t, u(t)) = p(t) \sup\{u(t) - L, 0\}$$

for each $t \in [0, T)$ and our problem is given by

$$\begin{aligned} u_t(t) &= \nabla^2 u(t) + f(t, u(t)),\ t \in (0, T) \\ u(0) &= u_0. \end{aligned} \tag{4.2}$$

Remark 4.1.1. The fact that f maps $[0, T) \times \mathcal{L}^2(\Omega)$ into $\mathcal{L}^2(\Omega)$ and is continuous is non-trivial

Firstly, for $t \in [0, T)$ and $u \in \mathcal{L}^2(\Omega)$, f need not belong to $\mathcal{L}^2(\Omega)$ since the product of two functions p and u belonging to $\mathcal{L}^2(\Omega)$ is itself not necessarily contained in $\mathcal{L}^2(\Omega)$. However, in our case it is true. For all $t \in [0, T)$, since p is uniformly bounded on $\Omega \times [0, T)$, there exists a $K > 0$ such that

$$|p(x, t)| \leq K$$

for all $x \in \Omega$ and $t \in [0, T)$. Thus for any $t \in [0, T)$ and $u \in \mathcal{L}^2(\Omega)$,

$$\begin{aligned} \int_\Omega f(t, u)^2 dx &= \int_\Omega p(x, t)^2 \max\{u(x) - L(x), 0\}^2 dx \\ &\leq K^2 \int_\Omega \max\{u(x) - L(x), 0\}^2 dx. \end{aligned}$$

Since $L \in \mathcal{L}^2(\Omega)$ and $u \in \mathcal{L}^2(\Omega)$, the integral exists and is finite so that $f(t, u) \in \mathcal{L}^2(\Omega)$ for all $t \in [0, T)$.

Secondly, f is continuous. It is easily verified that for every $u_1, u_2 \in \mathcal{L}^2(\Omega)$ and $x \in \Omega$

$$\begin{aligned} &|\max\{u_1(x) - L(x), 0\} - \max\{u_2(x) - L(x), 0\}| \\ &\qquad \leq |u_1(x) - L(x) - [u_2(x) - L(x)]| \\ &\qquad = |u_1(x) - u_2(x)|. \end{aligned} \tag{4.3}$$

Fix $\epsilon > 0$, $u_2 \in \mathcal{L}^2(\Omega)$ and $t_2 \in [0,T)$. Then, for every $t_1 \in [0,T)$ and $u_1 \in \mathcal{L}^2(\Omega)$ we have

$$\begin{aligned}
&\|f(t_1,u_1) - f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} \\
\leq &\|f(t_1,u_1) - f(t_1,u_2)\|_{\mathcal{L}^2(\Omega)} + \|f(t_1,u_2) - f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} \\
= &\left(\int_\Omega p(x,t_1)^2[\max\{u_1(x) - L(x), 0\} - \max\{u_2(x) - L(x), 0\}]^2 dx\right)^{\frac{1}{2}} \\
&+ \left(\int_\Omega [p(x,t_1) - p(x,t_2)]^2 \max\{u_2(x) - L(x), 0\}^2 dx\right)^{\frac{1}{2}} \\
\leq &\left(K^2 \int_\Omega |u_1(x) - u_2(x)|^2 dx\right)^{\frac{1}{2}} \\
&+ (\|u_2\|^2_{\mathcal{L}^2(\Omega)} + \|L\|^2_{\mathcal{L}^2(\Omega)})^{\frac{1}{2}} \left(\int_\Omega [p(x,t_1) - p(x,t_2)]^2 dx\right)^{\frac{1}{2}} \\
= &K\|u_1 - u_2\|_{\mathcal{L}^2(\Omega)} + (\|u_2\|^2_{\mathcal{L}^2(\Omega)} + \|L\|^2_{\mathcal{L}^2(\Omega)})^{\frac{1}{2}} \|p(t_1) - p(t_2)\|_{\mathcal{L}^2(\Omega)}.
\end{aligned} \tag{4.4}$$

Since p is continuous on $\overline{\Omega} \times [0,T)$ there exists a $\delta_1 > 0$ such that if $|t_1 - t_2| < \delta_1$ then

$$\|p(t_1) - p(t_2)\|_{\mathcal{L}^2(\Omega)} < \frac{\epsilon}{2(\|u_2\|^2_{\mathcal{L}^2(\Omega)} + \|L\|^2_{\mathcal{L}^2(\Omega)})^{\frac{1}{2}}}. \tag{4.5}$$

Let $\delta_2 = \dfrac{\epsilon}{2K}$. Then from (4.5) and (4.4) it follows that if $|t_1 - t_2| < \delta_1$ and $\|u_1 - u_2\|_{\mathcal{L}^2(\Omega)} < \delta_2$ then

$$\|f(t_1,u_1) - f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} < \epsilon.$$

Thus f is continuous on $[0,T) \times \mathcal{L}^2(\Omega)$.

4.2 Mild Solutions of the Biological Model

We begin this section by showing that a local mild solution to the semi-linear problem (4.2) exists. We then consider the asymptotic behaviour and regularity of this solution. Finally, we consider whether solutions to more complex models of the mechanism of sweating exist.

Let $\widetilde{\mathcal{L}^2(\Omega)}$ be the complexification of $\mathcal{L}^2(\Omega)$ as defined in Definition 2.5.1. By Theorem 2.5.2 the vector space $\widetilde{\mathcal{L}^2(\Omega)}$ is a complex Hilbert space with inner product $(\cdot,\cdot)_{\widetilde{\mathcal{L}^2(\Omega)}}$ as defined in Theorem 2.5.1. Let $\tilde{A}$ be the complexification of $A = \nabla^2$ acting on $\widetilde{\mathcal{L}^2(\Omega)}$, as in Definition 2.5.2.

4.2.1 Existence

We begin this subsection by showing that the operator A satisfies the conditions of the Hille-Yosida theorem, so that a mild solution to the homogeneous problem exists. We then use results from Section 2.5 on the complexification of operators to show that $\tilde{A}$ is the infinitesimal generator of an analytic semigroup and that a local mild solution to the semi-linear problem (4.2) exists.

Theorem 4.2.1. *The closure of the domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ of $A = \nabla^2$ on $\mathcal{L}^2(\Omega)$ is $\mathcal{L}^2(\Omega)$, and A is closed.*

Proof. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ and $C_0^\infty(\Omega) \subset \mathcal{D}(A)$ then $\mathcal{L}^2(\Omega) = \overline{\mathcal{D}(A)}$.

Furthermore, since A is an elliptic operator in divergence form, then by Theorem 2.2.2 the resolvent set $\rho(A)$ of A is non-empty. Thus by Theorem 1.4.7 the operator A is closed.

□

Theorem 4.2.2. *The resolvent set $\rho(A)$ of $A : \mathcal{D}(A) \mapsto \mathcal{L}^2(\Omega)$ contains $\mathbb{R}^+$, and there exists an $M > 0$ such that*

$$\|\lambda R(\lambda, A)\| \leq M,\ \lambda > 0.$$

Proof. Suppose $B[u, v] : H_0^1(\Omega) \times H_0^1(\Omega) \mapsto \mathbb{R}$ given by

$$B[u, v] = \int_\Omega \sum_{i,j=1}^{2} u_{x_i} v_{x_j} dx$$

is the bilinear form corresponding to the uniform elliptic operator $-A = -\nabla^2$ in divergence form (2.10). Then it follows directly from the ellipticity condition (2.11) on A that there exists a $\theta > 0$ such that

$$\theta \|u\|_{H_0^1(\Omega)} \leq B[u, u] \tag{4.6}$$

for $u \in H_0^1(\Omega)$. If we compare (4.6) with Theorem 2.2.1 and the proof of Theorem 2.2.2, in particular (2.16), then we can choose the ω in Theorem 2.2.2 to be 0, and the resolvent set $\rho(A)$ of A contains the ray $\{\lambda \in \mathbb{R} \mid \lambda > 0\}$.

We now show that there exists a $M > 0$ such that for all $\lambda > 0$ and $f \in \mathcal{L}^2(\Omega)$

$$\|u\|_{\mathcal{L}^2(\Omega)} = \|(\lambda I - A)^{-1} f\|_{\mathcal{L}^2(\Omega)} \leq \frac{M}{\lambda} \|f\|_{\mathcal{L}^2(\Omega)}.$$

Let $\lambda > 0$ and $u \in \mathcal{D}(A)$ so that

$$(\lambda I - \nabla^2) u = f \tag{4.7}$$

where $f \in \mathcal{L}^2(\Omega)$. Multiplying both sides of (4.7) by u and integrating, we get

$$\int_\Omega \lambda u^2 dx - \int_\Omega (\nabla^2 u) u dx = \int_\Omega f u dx.$$

Applying Integration by Parts gives

$$\lambda \|u\|^2_{\mathcal{L}^2(\Omega)} + \|\nabla u\|^2_{\mathcal{L}^2(\Omega)} = (f, u)_{\mathcal{L}^2(\Omega)}.$$

Thus by the Cauchy-Swartz inequality

$$\lambda \|u\|^2_{\mathcal{L}^2(\Omega)} \leq (f, u)_{\mathcal{L}^2(\Omega)} \leq \|f\|_{\mathcal{L}^2(\Omega)} \|u\|_{\mathcal{L}^2(\Omega)}$$

and we have that

$$\|u\|_{\mathcal{L}^2(\Omega)} \leq \frac{1}{\lambda} \|f\|_{\mathcal{L}^2(\Omega)}. \tag{4.8}$$

The result follows upon setting $M = 1$. □

Theorem 4.2.3. *The operator A is the infinitesimal generator of a contraction C_0-semigroup $T(t)$, $t \geq 0$. That is, the semigroup satisfies $\|T(t)\| \leq 1$ for $t \geq 0$.*

Proof. From Theorems 4.2.1 and 4.2.2, the conditions of the Hille-Yosida Theorem, Theorem 1.4.8, are satisfied, with $M = 1$ and $\omega = 0$. □

It follows that, for all $u_0 \in \mathcal{L}^2(\Omega)$, $u(t) = T(t)u_0$, $t \geq 0$, is a mild solution to (4.2) in the case where $f = 0$.

Recall Figure 2.1. We now have all the required knowledge to show how the flow-chart works out in practice. Consider the complexification, $\tilde{A}$ of A, acting on $\widetilde{\mathcal{L}^2(\Omega)}$.

Theorem 4.2.4. *The complexification $\tilde{A}$ of A is a sectorial operator.*

Proof. By Theorems 4.2.3 and 2.5.11 the resolvent set of $\tilde{A}$ contains the half-plane $\Pi = \{\lambda \in \mathbb{C} \ : \ \Re\lambda > 0\}$ and $\|R(\lambda, \tilde{A})\| \leq \dfrac{M}{|\lambda|}$ for $\lambda \in \Pi$. Thus, for any $\epsilon > 0$

$$\{\lambda \in \mathbb{C} \ : \ \Re\lambda \geq \epsilon\} \subset \rho(\tilde{A}). \tag{4.9}$$

Furthermore,

$$\|R(\lambda, \tilde{A})\| \leq \frac{1}{|\lambda|} \tag{4.10}$$

for all $\lambda \in \mathbb{C}$, $\Re\lambda \geq \epsilon$. From (4.9) and (4.10), Theorem 2.3.2 gives that $\tilde{A}$ is a sectorial operator. □

Corollary 4.2.5. *The complexification $\tilde{A}$ of A is the infinitesimal generator of an analytic semigroup $e^{t\tilde{A}}$, $t \geq 0$.*

Proof. This follows directly from Theorem 4.2.4. □

Theorem 4.2.6. *For every $u_0 \in \mathcal{L}^2(\Omega)$ there exists a $t^* = t^*(u_0)$, $0 < t^* < T$, and a continuous mapping u from $[0, t^*]$ to $\mathcal{L}^2(\Omega)$ that is a mild solution on $[0, t^*]$ to the abstract Cauchy problem* (4.2)*.*

Proof. We will use Theorem 3.2.3 to prove the result.

From Corollary 4.2.5 we have that $\tilde{A}$ is the infinitesimal generator of an analytic semigroup $e^{t\tilde{A}}$, $t \geq 0$, on $\widetilde{\mathcal{L}^2(\Omega)}$. Thus $e^{t\tilde{A}}$, $t \geq 0$, is a C_0-semigroup, and, by Theorem 2.3.7, $e^{t\tilde{A}}$, $t \geq 0$, is continuous in the uniform operator topology for $t > 0$.

Furthermore, since A is uniformly elliptic, by Theorem 2.2.2 the resolvent operator $R(\lambda, A)$ is compact for all $\lambda > 0$. Thus, by Theorems 2.5.12, 2.5.8 and 2.1.4, the resolvent operator $R(\lambda, \tilde{A})$ will be compact for all $\lambda \in \rho(\tilde{A})$. Therefore, by Theorem 2.1.6, the semigroup $e^{t\tilde{A}}$, $t \geq 0$, is compact.

By Theorem 2.5.6, $\tilde{A}$ is the infinitesimal generator of the C_0-semigroup $\tilde{T}(t)$, $t \geq 0$. By Theorem 1.4.4 the semigroup generated by $\tilde{A}$ is unique, so that $\tilde{T}(t) = e^{t\tilde{A}}$ for each $t \geq 0$. Hence $\tilde{T}(t)$, $t \geq 0$, is compact.

Thus, by Theorem 2.5.8, the C_0-semigroup $T(t)$, $t \geq 0$, with infinitesimal generator A, is compact. Furthermore, f is continuous on $[0, T) \times \mathcal{L}^2(\Omega)$ as shown in Remark 4.1.1.

Thus by Theorem 3.2.3 the result holds. □

We now investigate the asymptotic behaviour and regularity of the local mild solution u of (4.2).

4.2.2 Asymptotic Behaviour

Theorem 4.2.7. *For each $u_0 \in \mathcal{L}^2(\Omega)$ the local mild solution $u(t)$ of* (4.2) *is a global solution.*

Proof. Recall our assumption that p is continuous and uniformly bounded on $\overline{\Omega} \times [0, \infty)$. Thus f is defined on $[0, \infty) \times \mathcal{L}^2(\Omega)$ with values in $\mathcal{L}^2(\Omega)$. We begin by showing that f maps bounded sets in $[0, \infty) \times \mathcal{L}^2(\Omega)$ to bounded sets in $\mathcal{L}^2(\Omega)$. Consider bounded sets $[a, b] \subset [0, \infty)$ and $U \subset \mathcal{L}^2(\Omega)$. Let $K > 0$ be an upper bound for the set $\{|p(x, t)| \mid t \in [a, b],\ x \in \overline{\Omega}\}$ and $M > 0$ be an upper bound for the set $\{\|u\|_{\mathcal{L}^2(\Omega)} \mid u \in U\}$. Then for $t \in [a, b]$ and

$u \in U$

$$\begin{aligned} \|f(t,u)\|^2_{\mathcal{L}^2(\Omega)} &= \int_\Omega p(x,t)^2 \max\{u(x) - L(x), 0\}^2 dx \\ &\leq K^2(\|u\|^2_{\mathcal{L}^2(\Omega)} + \|L(x)\|^2_{\mathcal{L}^2(\Omega)}) \\ &\leq K^2(M^2 + \|L(x)\|^2_{\mathcal{L}^2(\Omega)}). \end{aligned} \tag{4.11}$$

Thus f maps $[a,b] \times U$ onto a bounded set.

Note that (4.11) holds for all $t \in [0,\infty)$ and $u \in \mathcal{L}^2(\Omega)$. Thus

$$\|f(t,u)\|_{\mathcal{L}^2(\Omega)} \leq K^2\|u\|_{\mathcal{L}^2(\Omega)} + K^2\|L(x)\|^2_{\mathcal{L}^2(\Omega)}$$

for all $t \in [0,\infty)$ and $u \in \mathcal{L}^2(\Omega)$. Thus, by Corollary 3.3.2, the local mild solution u of (4.2) can be extended to a global solution. □

4.2.3 Regularity

In this subsection we show that a mild solution of (4.2) is a classical solution, provided that $p : \Omega \times [0,T) \mapsto \mathbb{R}$ is Hölder continuous in t. In order to prove this regularity result, we first establish the Hölder continuity of f.

Theorem 4.2.8. *Suppose the function $p : \overline{\Omega} \times [0,T) \mapsto \mathbb{R}^-$ is Hölder continuous in t, such that there exist constants $C > 0$ and $\alpha \in (0,1]$ such that*

$$|p(x,t_1) - p(x,t_2)| \leq C|t_1 - t_2|^\alpha$$

for all $x \in \Omega$, $t_1, t_2 \in [0,T)$. Then f is Hölder continuous on $[0,T) \times B_\rho(0)$ for all $\rho > 0$, where $B_\rho(0)$ is the open ball in $\mathcal{L}^2(\Omega)$ centred at 0 with radius ρ.

Proof. Fix $\rho > 0$, $t_1, t_2 \in [0,T)$, and $u_1, u_2 \in B_\rho(0)$. Then

$$\begin{aligned} \|f(t_1,u_1) - f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} \leq & \|f(t_1,u_1) - f(t_2,u_1)\|_{\mathcal{L}^2(\Omega)} \\ & + \|f(t_2,u_1) - f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)}. \end{aligned} \tag{4.12}$$

We have that

$$\begin{aligned} \|f(t_1,u_1) - f(t_2,u_1)\|^2_{\mathcal{L}^2(\Omega)} &= \int_\Omega [p(x,t_1) - p(x,t_2)]^2 \max\{u_1(x) - L(x), 0\}^2 dx \\ &\leq C^2|t_1 - t_2|^{2\alpha} \int_\Omega [u_1(x) - L(x)]^2 dx \\ &\leq C^2|t_1 - t_2|^{2\alpha}[\|u_1\|^2_{\mathcal{L}^2(\Omega)} + \|L\|^2_{\mathcal{L}^2(\Omega)}] \\ &< C^2[\rho^2 + \|L\|^2_{\mathcal{L}^2(\Omega)}]|t_1 - t_2|^{2\alpha}. \end{aligned}$$

Thus

$$\|f(t_1,u_1)-f(t_2,u_1)\|_{\mathcal{L}^2(\Omega)} \leq M|t_1-t_2|^\alpha. \tag{4.13}$$

where $M = C[\rho^2 + \|L\|^2_{\mathcal{L}^2(\Omega)}]^{\frac{1}{2}} > 0$. Since $p(x,t)$ is uniformly bounded there exists a $K > 0$ such that $|p(x,t)| \leq K$ for all $(x,t) \in \overline{\Omega} \times [0,T)$. Thus from (4.3) we have

$$\begin{aligned}\|f(t_2,u_1)-f(t_2,u_2)\|^2_{\mathcal{L}^2(\Omega)} =& \int_\Omega p(x,t_2)^2[\max\{u_1(x)-L(x),0\} \\ &- \max\{u_2(x)-L(x),0\}]^2 dx \\ \leq& K^2\int_\Omega [u_1(x)-u_2(x)]^2dx \\ =& K^2\|u_1-u_2\|^2_{\mathcal{L}^2(\Omega)}.\end{aligned}$$

Thus

$$\|f(t_2,u_1)-f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} \leq K\|u_1-u_2\|_{\mathcal{L}^2(\Omega)}. \tag{4.14}$$

Proposition B.0.5 and (4.14) imply that there exists a $N > 0$ such that

$$\|f(t_2,u_1)-f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} \leq N\|u_1-u_2\|^\alpha_{\mathcal{L}^2(\Omega)} \tag{4.15}$$

for all $u_1, u_2 \in B_\rho(0)$. Substituting (4.13) and (4.15) into (4.12) gives that

$$\begin{aligned}\|f(t_1,u_1)-f(t_2,u_2)\|_{\mathcal{L}^2(\Omega)} &\leq M|t_1-t_2|^\alpha + N\|u_1-u_2\|^\alpha_{\mathcal{L}^2(\Omega)} \\ &\leq \max\{M,N\}(|t_1-t_2|^\alpha + \|u_1-u_2\|^\alpha_{\mathcal{L}^2(\Omega)}),\end{aligned}$$

and f is Hölder continuous with exponent α. □

Theorem 4.2.9. *Suppose the function $p : \overline{\Omega} \times [0,T) \mapsto \mathbb{R}^-$ is Hölder continuous in t, such that there exist constants $C > 0$ and $\alpha \in (0,1]$ such that*

$$|p(x,t_1)-p(x,t_2)| \leq C|t_1-t_2|^\alpha$$

for all $x \in \Omega$, $t_1,t_2 \in [0,T)$. Then every mild solution of (4.2) is a classical solution of (4.2).

Proof. Suppose $u(t)$ is a mild solution of (4.2). Consider the functions $\tilde{f} : [0,T) \times \widetilde{\mathcal{L}^2(\Omega)} \mapsto \widetilde{\mathcal{L}^2(\Omega)}$ given by $\tilde{f}(t,u+iv) = f(t,u)+0i$ and $\tilde{u} : [0,T) \mapsto \widetilde{\mathcal{L}^2(\Omega)}$ given by $\tilde{u}(t) = u(t)+0i$. By Theorem 4.2.8 the function f is Hölder continuous on $[0,T) \times B_\rho(0)$ for all $\rho > 0$. It is clear that taking $\rho \to \infty$, $\tilde{f}$ will be Hölder continuous on $[0,T) \times \mathcal{L}^2(\Omega)$. Furthermore, by Theorem 2.5.7, $\tilde{u}(t)$ is a mild solution of

$$\begin{aligned}&\frac{d}{dt}\tilde{u}(t) = \tilde{A}\tilde{u}(t) + \tilde{f}(t,\tilde{u}(t)),\ t \in (0,T) \\ &\tilde{u}(0) = u_0 + 0i,\end{aligned} \tag{4.16}$$

where $\tilde{A}$ is the infinitesimal generator of an analytic semigroup. Thus, by Theorem 3.4.3, $\tilde{u}(t)$ is a classical solution of (4.16). Hence $\tilde{u}(t) \in \mathcal{D}(\tilde{A})$ for $0 < t < T$, $\tilde{u}(t)$ is continuous on $[0, T)$, and $\tilde{u}(t)$ is continuously differentiable on $(0, T)$. If follows that $u(t) \in \mathcal{D}(A)$ for $0 < t < T$, $u(t)$ is continuous on $[0, T)$ and $u(t)$ is continuously differentiable on $(0, T)$. Furthermore, $u(0) + 0i = \tilde{u}(0) = u_0 + 0i$ and

$$\begin{aligned}
\frac{d}{dt}u(t) + 0i &= \frac{d}{dt}(u(t) + 0i) \\
&= \frac{d}{dt}\tilde{u}(t) \\
&= \tilde{A}\tilde{u}(t) + \tilde{f}(t, \tilde{u}(t)) \\
&= \tilde{A}(u(t) + 0i) + \tilde{f}(t, u(t) + 0i) \\
&= Au(t) + 0i + f(t, u(t)) + 0i \\
&= Au(t) + f(t, u(t)) + 0i
\end{aligned}$$

for all $t \in (0, T)$. It follows that $u(0) = u_0$ and $\frac{d}{dt}u(t) = Au(t) + f(t, u(t))$ for all $t \in (0, T)$. Thus $u(t)$ is a classical solution to (4.2). $\square$

4.3 Comments on the Model

Although this is an overly simplistic model, we use it for two reasons: First, it demonstrates the applicability of the main results of this thesis in showing the existence of a solution to a parabolic semi-linear equation.

Second, most more accurate models that consider perspiration in research include many factors other than perspiration and diffusion, since the core body temperature, not the skin temperature, is what the researchers are mostly interested in. Accordingly they add more terms to the operator A or adjust $f(t, u(t))$ in appropriate ways. Indeed, many terms such as the zeroth-order term $\rho u(t)$, where $\rho \in \mathbb{R}$, can be included as an additional term in either A or in $f(t, u(t))$. These additional terms are generally either zeroth-order or first-order terms. Models such as [28, Chapter 9, Equation (46)] add a zeroth-order term, and models such as [24, (2.1)] and [24, (2.14)] add both. Thus we can follow the ideas mentioned in Remark 2.4.3 to show existence for this broader class of problem. Therefore the existence and regularity results will still apply to many of these more complex models.

To demonstrate this, suppose the additional terms are zeroth-order and are given by linear operator $B : \mathcal{L}^2(\Omega) \supset \mathcal{D}(B) \mapsto \mathcal{L}^2(\Omega)$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. Since A is elliptic the operator $A + B$ will be elliptic, since the ellipticity only depends on the second-order term, that is, the diffusion A. Thus by Theorem 2.2.2 $R(\lambda, A+B)$ will be compact for some $\lambda \in \rho(A+B)$. Furthermore, B will be bounded when $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Thus since $\tilde{A}$ is the infinitesimal generator of an analytic semigroup, by Corollary 2.4.4 the operator $\widetilde{A+B} = \tilde{A} + \tilde{B}$ is the infinitesimal generator of an analytic semigroup.

It follows that a local mild solution to the problem

$$\begin{aligned} u_t(t) &= (A+B)u(t) + f(t, u(t)),\ t > 0 \\ u(0) &= u_0 \end{aligned} \tag{4.17}$$

exists.

Note that the regularity result only depends on the behaviour of f, thus it will also hold for problem (4.17). In particular, Theorem 4.2.9 shows that the regularity of the solution depends on conditions for the function p, which represents the contribution of perspiration. Whether or not these conditions for p are realistic assumptions falls outside the scope of this thesis.

For a discussion on some of the better known models, we refer the reader to [24, Review of Thermoregulatory Modelling] and [26, Literature Survey].

Chapter 5

Conclusion

At the beginning of this thesis we stipulated the main aims of this thesis. Here we discuss whether those goals have been met.

We considered a parabolic semi-linear PDE and reformulated it as an infinite-dimensional system. We then developed theory concerning compact and analytic semigroups, and used this knowledge to find conditions under which a local mild solution to the infinite-dimensional dynamical system exists. This required proving some results on integration over a complex plane. To make this theory applicable we discussed the complexification of operators acting on a real Hilbert space to operators acting on a complex Hilbert space.

We then investigated the asymptotic behaviour of the solution, showing when the local mild solution was global. Furthermore, we developed a regularity result, showing under which conditions a mild solution of the infinite-dimensional system will be a classical solution. This required some basic knowledge of Hölder continuous functions.

Finally we demonstrated that these results are applicable in "real-life" with a biologically motivated example, making use of perturbation theory developed earlier to demonstrate the applicability of the presented research to more complex problems.

Thus we believe that the main aims of the thesis were met.

As demonstrated, the results of this thesis are applicable to certain biological problems. Indeed, a large number of other physical models fall into the class of problems discussed throughout this thesis. Thus the work is of real world significance. The key benefit of this thesis is to develop the main results from the foundation of the real world problem, as opposed to developing the theory independently. This makes it easy to see the relevance of the work to real world problems.

This thesis could be expanded to include a broader group of problems, most obviously by considering fully non-linear parabolic PDEs or by developing wider-reaching perturbation theory. Furthermore, numerical solutions could be developed to approximate the mild solutions shown to exist. However, as mentioned previously, this theory does not easily lend itself to the development of corresponding numerical methods, so this would be a task largely independent of the work discussed in this thesis.

Appendix A

Integration

In this appendix we introduce some results on integration of vector-valued functions and integration along curves in $\mathbb{C}$.

A.1 The Bochner Integral

We begin by defining an integral for vector-valued functions and giving a few basic results. In this case, we use the Bochner integral. We refer the reader to [8, III.2] for an introduction to measure theory. Unless otherwise indicated, the results and definitions of this section can be found in [2, Section 1.1], along with a more detailed analysis of the Bochner Integral.

In this section X denotes a Banach space and I denotes an interval in $\mathbb{R}$ or rectangle in $\mathbb{R}^2$.

Definition A.1.1 (Step Function). A function $f : I \mapsto X$ is a step function if it can be written in the form $f(t) = \sum_{i=1}^{n} x_i \chi_{I_i}(t)$ for some $n \in \mathbb{N}$. Here $x_i \in X$ and $I_i \subset I$ is measurable for each $i = 1, ..., n$ and χ_{I_i} denotes the characteristic (indicator) function of I_i.

Definition A.1.2 (Measurable Function). A function $f : I \mapsto X$ is measurable if there is a sequence of step functions g_n such that $f(t) = \lim_{n\to\infty} g_n(t)$ for almost all $t \in I$.

Proposition A.1.1. *A function $f : I \mapsto X$ is measurable if it is continuous, or if there exists a sequence (f_n) of measurable functions $f_n : I \mapsto X$ such that (f_n) converges pointwise to f almost everywhere.*

For a step function $g : I \mapsto X$ given by $g(t) = \sum_{i=1}^{n} x_i \chi_{I_i}(t)$, $t \in I$, we define

$$\int_I g(t)dt := \sum_{i=1}^{n} x_i m(I_i)$$

where $m(I_i)$ is the Lebesgue measure of I_i.

Definition A.1.3 (**Bochner Integral**)**.** A function $f : I \mapsto X$ is called *Bochner integrable* if there exist step functions $g_n : I \mapsto X$ such that (g_n) converges to f pointwise almost everywhere, and

$$\lim_{n\to\infty} \int_I \|f(t) - g_n(t)\|_X dt = 0.$$

If f is Bochner Integrable then the *Bochner Integral of f on I* is

$$\int_I f(t)dt := \lim_{n\to\infty} \int_I g_n(t)dt.$$

Remark A.1.1. Another integral we could use for functions with values in a Banach space is the Pettis integral. On a Banach space the Pettis integral is more general than the Bochner integral. However, we choose to use the Bochner integral for the following three reasons: Firstly, the Bochner integral is more intuitive as a natural extension of the Lebesgue integral to vector-valued functions. Secondly, we are interested in integrating functions that are continuous on I, and these are Bochner integrable. Thirdly, the class of Bochner integrable functions is easily characterized, as shown in the following theorem.

Theorem A.1.2 (**Bochner**)**.** *A function $f : I \mapsto X$ is Bochner integrable if and only if f is measurable and $\|f\|_X$ is Lebesgue integrable. If f is Bochner integrable then*

$$\left\|\int_I f(t)dt\right\|_X \leq \int_I \|f(t)\|_X dt.$$

It is clear by definition that if $f : I \mapsto X$ is Bochner integrable and T is a bounded linear operator from X to a Banach space Y then $T \circ f : I \ni t \mapsto T(f(t))$ is Bochner integrable and $T \int_I f(t)dt = \int_I Tf(t)dt$. We shall need a result similar to this for a closed operator A.

Proposition A.1.3. *Suppose $A : X \supset \mathcal{D}(A) \mapsto X$ is a closed linear operator on X and $f : I \mapsto \mathcal{D}(A)$ is Bochner integrable. If $A \circ f : I \mapsto X$ is Bochner integrable, then $\int_I f(t)dt \in \mathcal{D}(A)$ and*

$$A \int_I f(t)dt = \int_I Af(t)dt.$$

The following Theorem, introduced by Guido Fubini in 1907 for scalar-valued functions, is a result that gives conditions under which it is possible to compute an integral over a two-dimensional area using iterated integrals. A direct result of this theorem, which is useful for us, is that the order of integration can be changed for iterated integrals.

Theorem A.1.4 (Fubini's Theorem). *Suppose $I = I_1 \times I_2$ is a rectangle in $\mathbb{R}^2$, $f : I \mapsto X$ is measurable and*

$$\int_{I_1} \int_{I_2} \|f(s,t)\|_X dtds < \infty.$$

Then f is Bochner integrable and the iterated integrals

$$\int_{I_1} \int_{I_2} f(s,t)dtds \text{ and } \int_{I_2} \int_{I_1} f(s,t)dsdt$$

exist and are equal, and they coincide with the integral $\int_I f(s,t)d(s,t)$.

Proposition A.1.5. *Let $f : [a,b] \mapsto X$ be Bochner integrable and $F(t) := \int_a^t f(s)ds$ for $t \in [a,b]$. Then*

a) F is differentiable almost everywhere in (a,b) and $F' = f$ almost everywhere.

b) $\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|_X ds = 0$ for t almost everywhere in (a,b).

We can now define $\mathcal{L}^p$-spaces for X-valued functions. The definition and properties are analogous to the case of scalar-valued functions.

Definition A.1.4. The set $\mathcal{L}^1(I;X)$ denotes the set of Bochner integrable functions equipped with norm

$$\|f\|_{\mathcal{L}^1(I;X)} := \int_I \|f(t)\|_X dt.$$

For $1 < p < \infty$ the space $\mathcal{L}^p(I;X)$ consists of all measurable functions $f : I \mapsto X$ such that

$$\int_I \|f(t)\|_X^p dt < \infty$$

equipped with norm

$$\|f\|_{\mathcal{L}^p(I;X)} := \left(\int_I \|f(t)\|_X^p dt\right)^{\frac{1}{p}}.$$

The set $\mathcal{L}^\infty(I;X)$ consists of all measurable functions $f : I \mapsto X$ such that

$$\operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X < \infty$$

equipped with norm

$$\|f\|_{\mathcal{L}^\infty(I;X)} := \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X.$$

In the usual way, we identify functions which differ only on sets of measure zero, that is, two functions are considered equal if they are equal pointwise almost everywhere. For $1 \leq p \leq \infty$ the space $\mathcal{L}^p(I;X)$ is a Banach space equipped with its respective norm $\|\cdot\|_{\mathcal{L}^p(I;X)}$, see [2, Theorem 1.1.10] and [13, Chapter 6, Theorem 6.28]. Furthermore, it can be shown that for $1 \leq p < \infty$ both the collection of all step functions $g : I \mapsto X$ and the collection of functions of the form $g(t) = \sum_{i=1}^{n} c_i\phi_i(t)$ for $t \in I$, where $c_i \in X$ and $\phi_i \in \mathcal{C}_0^\infty(I)$ for all $i = 1, ..., n$, are dense in $\mathcal{L}^p(I;X)$, see [13, Chapter 6, Proposition 6.29].

Definition A.1.5. Suppose $I = [a, \infty)$ for some $a \in \mathbb{R}$. If $f \in \mathcal{L}^1([a,\tau];X)$ for all $\tau \in [a,\infty)$ then we say that $\int_a^\infty f(t)dt$ converges as an *improper integral* and we define

$$\int_a^\infty f(t)dt := \lim_{\tau\to\infty} \int_a^\tau f(t)dt.$$

If $f \in \mathcal{L}^1(I;X)$, that is, $\int_a^\infty \|f(t)\|_X dt < \infty$, then we say that the integral is *absolutely convergent.*

If f is a scalar-valued function then the above definition holds in a similar manner and is taken as pre-knowledge, see [20, Chapter 6, Exercise 8].

A.2 Cauchy's Integral Formula

We now consider some results on the integration of both complex-valued and vector-valued functions on the complex plane. In particular, we consider *analytic* functions. We refer the reader to [1] for more on the topic of integration on the complex plane.

In this section X denotes a complex Banach space.

Definition A.2.1 (**Analytic Function**). A function f on $\mathbb{C}$ with values either in $\mathbb{C}$ or in X is called (complex) analytic if we can write $f(x)$ as a power series around x_0 for x near x_0, that is

$$f(x) = \sum_{i=0}^{\infty} a_n (x - x_0)^n$$

where a_n is either in $\mathbb{C}$ or X.

Definition A.2.2 (**Integral Along a Curve**). Suppose $\Omega \subseteq \mathbb{C}$ is open and $\Gamma \subset \Omega$ is a smooth curve with parametrization $z : \mathbb{R} \supseteq I \mapsto \mathbb{C}$. Furthermore, suppose that f is a function on Ω with values in either $\mathbb{C}$ or X such that $f(z(t))$ is Lebesgue or Bochner integrable on I respectively. Then

$$\int_\Gamma f(z)dz := \int_I f(z(t))\frac{dz}{dt}dt.$$

If Γ is piecewise smooth such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_n$ with Γ_i smooth for each $i = 1, ..., n$ then

$$\int_\Gamma f(z)dz := \sum_{i=1}^{n} \int_{\Gamma_i} f(z)dz.$$

The following theorem follows straight from Definitions A.1.5 and A.2.2.

Theorem A.2.1. *Suppose $\Omega \subseteq \mathbb{C}$ is open and $\Gamma \subset \Omega$ is a piecewise smooth curve of infinite length. Let $B_n(0)$ be the disk in $\mathbb{C}$ with radius $n \in \mathbb{R}$ centred at the origin and $\Gamma_n = \Gamma \cap B_n(0)$. If $f \in \mathcal{L}^1(\Gamma_n; X)$ for every $n \in \mathbb{R}$, then $\int_\Gamma f(t)dt$ converges as an improper integral and*

$$\int_\Gamma f(t)dt := \lim_{n\to\infty} \int_{\Gamma_n} f(t)dt.$$

If $f \in \mathcal{L}^1(I; X)$, that is, $\int_\Gamma \|f(t)\|_X dt < \infty$, then the integral is absolutely convergent.

If f is a scalar-valued function then the above theorem holds in a similar manner, and is taken as pre-knowledge.

Theorem A.2.2 (Cauchy's Integral Formula). *[1, Theorem 6, page 119] Suppose that f is an analytic on an open disk $\Omega \subset \mathbb{C}$, with values in either $\mathbb{C}$ or X. Let Γ be a piecewise differentiable, closed and positively oriented curve in Ω. For any point a not on Γ*

$$n(\Gamma, a) \cdot f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz,$$

where $n(\Gamma, a)$ is the index of a with respect to Γ.

A bit of further explanation here is useful. The index $n(\Gamma, a)$ is also called the *winding number* of Γ with respect to a. In essence it is the number of times the curve Γ winds around, or encloses, the point a, while moving in the counter-clockwise direction. Thus if a lies outside the curve Γ then $n(\Gamma, a) = 0$. Supposing Γ does not form a loop within itself, that is, Γ does not loop around any point more than once, and that Γ is positively oriented, then if a lies within curve Γ then $n(\Gamma, a) = 1$. If Γ were negatively oriented and were to loop around a then $n(\Gamma, a) = -1$. Indeed, $n(-\Gamma, a) = -n(\Gamma, a)$.

Although the theorem stated above from Ahlfors is for an open disk Ω, the theorem can be extended to any open, simply connected region Ω, see [1, Sections 4.2-4.3] and , in particular, [1, Theorem 16]. The intuition is that a simply connected region is a region without any holes, and thus its complement is a single connected piece.

We can thus conclude from Theorem A.2.2 that if Ω is any open simply connected region containing a piecewise differentiable, closed, and positively oriented curve Γ, which does not form a loop within itself, and $f(z)$ is an analytic function on Ω, then

$$\int_\Gamma \frac{f(x)}{z-a} dx = 0 \text{ for any point } a \text{ outside of } \Gamma,$$
$$\int_\Gamma \frac{f(x)}{z-a} dx = 2\pi i f(a) \text{ for any point } a \text{ inside of } \Gamma.$$

This is what we shall refer to as Cauchy's Integral Formula. We now provide a version of Cauchy's Integral Formula for the case where Γ is a curve that is not closed and has infinite length, in particular, the curve $\omega + \gamma_{r,\eta}$ as shown in Figure 2.3. This curve splits the complex plane into two parts.

Lemma A.2.3. *Suppose $\gamma_{r,\eta}$ is the curve*

$$\{\lambda \in \mathbb{C} \mid |arg\lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} \mid |arg\lambda| \leq \eta, |\lambda| = r\}$$

oriented counter-clockwise, where $r > 0$ and $\eta \in (\frac{\pi}{2}, \pi)$. Let $\omega \in \mathbb{R}$. Suppose f is a bounded analytic function on an open set $\mathcal{D}$ with values either in $\mathbb{C}$ or X. Let

$$S_L = \{x + iy \in \mathbb{C} \mid x < u, u + iy \in (\omega + \gamma_{r,\eta})\},$$

that is, everything to the left of $(\omega + \gamma_{r,\eta})$, and

$$S_R = \{x + iy \in \mathbb{C} \mid x > v, v + iy \in (\omega + \gamma_{r,\eta})\},$$

that is, everything to the right of $(\omega + \gamma_{r,\eta})$. Then

(1) If $\mathcal{D}$ contains $S_L \cup (\omega + \gamma_{r,\eta})$, then for $\mu \in \mathcal{D}$

(a) $\displaystyle\int_{\omega+\gamma_{r,\eta}} \frac{f(\lambda)}{\lambda - \mu} d\lambda = 2\pi i f(\mu)$ *whenever $\mu \in S_L$.*

(b) $\displaystyle\int_{\omega+\gamma_{r,\eta}} \frac{f(\lambda)}{\lambda - \mu} d\lambda = 0$ *whenever $\mu \in S_R$.*

(2) If $\mathcal{D}$ contains $S_R \cup (\omega + \gamma_{r,\eta})$, then for $\mu \in \mathcal{D}$

(a) $\displaystyle\int_{\omega+\gamma_{r,\eta}} \frac{f(\lambda)}{\lambda - \mu} d\lambda = -2\pi i f(\mu)$ *whenever $\mu \in S_R$.*

(b) $\displaystyle\int_{\omega+\gamma_{r,\eta}} \frac{f(\lambda)}{\lambda - \mu} d\lambda = 0$ *whenever $\mu \in S_L$.*

Proof. We start with case (1) where $\mathcal{D}$ contains the curve $(\omega + \gamma_{r,\eta})$ and everything to the left of it. Without loss of generality take $\omega = 0$. Fix $R > r > 0$ and $\mu \in \mathcal{D}$. Consider the curve

$$C_R = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \geq \eta, |\lambda| = R\}$$

and the closed curve

$$\begin{aligned}\Gamma_R = &\{\lambda \in \mathbb{C} \mid |\arg \lambda| = \eta, r \leq |\lambda| \leq R\} \cup \\ &\{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \eta, |\lambda| = r\} \cup C_R,\end{aligned}$$

both oriented counter-clockwise. By Cauchy's Integral Theorem, since Γ_R is closed for every R, and $\dfrac{f(\lambda)}{\lambda - \mu}$ is analytic on $\mathcal{D}$ which contains Γ_R, then

$$\int_{\Gamma_R} \frac{f(\lambda)}{\lambda - \mu} d\lambda = 2\pi i f(\mu) \text{ if } \mu \text{ inside } \Gamma_R$$
$$\int_{\Gamma_R} \frac{f(\lambda)}{\lambda - \mu} d\lambda = 0 \text{ if } \mu \text{ outside } \Gamma_R.$$

Note that anything the right of $\gamma_{r,\eta}$ would lie outside Γ_R for all $R > r$, while anything to the left of $\gamma_{r,\eta}$ would lie inside Γ_R for R sufficiently large. Thus

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{f(\lambda)}{\lambda-\mu}d\lambda = 2\pi i f(\mu) \text{ if } \mu \text{ to the left of } \gamma_{r,\eta} \tag{A.1}$$

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{f(\lambda)}{\lambda-\mu}d\lambda = 0 \text{ if } \mu \text{ to the right of } \gamma_{r,\eta}. \tag{A.2}$$

Since f is bounded on $\mathcal{D}$, we can find a number $M > 0$ such that $\|f(\lambda)\| \leq M$ for all $\lambda \in \mathcal{D}$. We parametrise C_R as $\lambda(s) = Re^{i(1-s)\eta+is(2\pi-\eta)}$ for $0 \leq s \leq 1$, and use the reverse triangle inequality to get

$$\begin{aligned}
\int_{C_R}\left\|\frac{f(\lambda)}{\lambda-\mu}\right\|d\lambda &\leq \int_{C_R}\frac{M}{|\lambda-\mu|}d\lambda \\
&= 2\pi M\int_0^1 \left|Re^{i(1-s)\eta+is(2\pi-\eta)}-\mu\right|^{-1}ds \\
&\leq 2\pi M\int_0^1 \left||Re^{i(1-s)\eta+is(2\pi-\eta)}|-|\mu|\right|^{-1}ds \\
&= 2\pi M|R-|\mu||^{-1}.
\end{aligned}$$

Thus

$$\lim_{R\to\infty}\int_{C_R}\left\|\frac{f(\lambda)}{\lambda-\mu}\right\|d\lambda = 0. \tag{A.3}$$

Thus by Theorem $(A.2.1)$

$$\int_{\gamma_{r,\eta}}\frac{f(\lambda)}{\lambda-\mu}d\lambda = \lim_{R\to\infty}\left[\int_{\Gamma_R}\frac{f(\lambda)}{\lambda-\mu}d\lambda - \int_{C_R}\frac{f(\lambda)}{\lambda-\mu}d\lambda\right], \tag{A.4}$$

with Γ_R positively oriented and the integral converges absolutely. Substituting equations (A.1), (A.2) and (A.3) into (A.4) gives our result.

The proof for case (2) follows in the same manner. Here we define C_R by

$$C_R = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \eta, |\lambda| = R\},$$

and define Γ_R as before. However, for Γ_R to be positively oriented we reverse the orientation of $\gamma_{r,\eta}$, thus

$$\int_{\gamma_{r,\eta}}\frac{f(\lambda)}{\lambda-\mu}d\lambda = -\lim_{R\to\infty}\left[\int_{\Gamma_R}\frac{f(\lambda)}{\lambda-\mu}d\lambda - \int_{C_R}\frac{f(\lambda)}{\lambda-\mu}d\lambda\right].$$

$\square$

Appendix B

Hölder Continuous Functions

In this section I is an interval in $\mathbb{R}$.

Definition B.0.3 (Hölder Continuous Functions)**.** A function $f : I \mapsto X$ is called *Hölder continuous* if there exist constants $C > 0$ and $\alpha \in (0, 1]$ such that

$$\|f(t) - f(s)\|_X \leq C|t - s|^\alpha$$

for all $s, t \in I$. The number α is called the *exponent* of the Hölder condition. If the inequality holds true for $\alpha = 1$ then the function is called *Lipschitz continuous*.

Note that continuously differentiable functions are Hölder continuous. In particular, they are Lipschitz continuous. Furthermore Hölder continuous functions are continuous.

We will require the following two basic propositions.

Proposition B.0.4. *Suppose that, for some open interval $I = (a, b)$ and $1 \leq p \leq \infty$, the function $f \in \mathcal{L}^p(I; X)$ is Hölder continuous on I. Then f is continuous on the closed interval $\overline{I} = [a, b]$.*

Proof. We start by showing that f is continuous at a. Consider a sequence (t_n) such that $t_n \in (a, b)$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} |t_n - a| = 0. \tag{B.1}$$

Since f is Hölder continuous on (a, b) there exist constants $\alpha \in (0, 1]$ and $C > 0$ such that

$$\|f(t) - f(s)\|_X \leq C|t - s|^\alpha$$

for all $t, s \in (a, b)$. Thus for every $n, m \in \mathbb{N}$

$$\|f(t_n) - f(t_m)\|_X \leq C|t_n - t_m|^\alpha. \tag{B.2}$$

Since (t_n) is convergent it is a Cauchy sequence. It follows from $(B.2)$ that $(f(t_n))$ is a Cauchy sequence with respect to $\|\cdot\|_X$. However, X is a Banach space with respect to $\|\cdot\|_X$. Thus there exists a $f^* \in X$ such that

$$\lim_{n\to\infty} \|f(t_n) - f^*\|_X = 0. \tag{B.3}$$

By the continuity of f on (a, b), $(B.1)$ and $(B.3)$ imply that $f^* = f(a)$. Thus f is continuous at a. In the same way we can show that f is continuous at b. $\square$

Proposition B.0.5. *Suppose $f : I \mapsto X$ is a Hölder continuous function on I with exponent $\beta \in (0, 1]$. Then f is Hölder continuous with exponent α for every $\alpha \in (0, \beta)$.*

Proof. For any $I \subset \mathbb{R}$, by Proposition B.0.4, f is continuous on $\overline{I}$. Thus f is bounded on $\overline{I}$ and there exists a constant K such that $\|f(t)\| \le K$ for all $t \in \overline{I}$. Furthermore, since f is Hölder continuous with exponent $\beta \in (0, 1]$, there exists a constant $C \ge 0$ such that

$$\|f(t) - f(s)\|_X \le C|t - s|^\beta$$

for all $t, s \in I$. Let $\alpha \in (0, \beta)$. If $|t - s| \le 1$ then $|t - s|^{\beta-\alpha} \le 1$ and

$$\begin{aligned} \|f(t) - f(s)\|_X &\le C|t - s|^\beta \\ &= C|t - s|^\alpha |t - s|^{\beta-\alpha} \\ &\le C|t - s|^\alpha. \end{aligned}$$

If $|t - s| > 1$ then $|t - s|^\alpha \ge 1$ and

$$\begin{aligned} \|f(t) - f(s)\|_X &\le \|f(t)\|_X + \|f(s)\|_X \\ &\le 2K \\ &\le 2K|t - s|^\alpha. \end{aligned}$$

Therefore

$$\|f(t) - f(s)\|_X \le \max\{C, 2K\}|t - s|^\alpha$$

for all $t, s \in I$. Thus f is Hölder continuous on I with exponent α. $\square$

Appendix C

Miscellaneous

Here we collect together some useful results which are used in the thesis but which do not belong to the scope of the thesis.

Theorem C.0.6 (**Lax-Milgram Theorem**). *([9], Section 6.2.1, Theorem 1) Suppose H is a Hilbert space. Assume that $B : H \times H \mapsto \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that*

(i) $\quad |B[u,v]| \leq \alpha \|u\|_H \|v\|_H \ (u, v \in H)$

and

(ii) $\quad \beta \|u\|_H^2 \leq B[u,u] \ (u \in H)$.

Finally, let $f : H \mapsto \mathbb{R}$ be a bounded linear functional, that is, f belongs to the dual space of H.

Then there exists a unique element $u \in H$ such that

$$B[u,v] = f(v)$$

for all $v \in H$.

Theorem C.0.7 (**Reisz Representation Theorem**). *[9, Appendix D.3. Theorem 2] Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$, and let H^* denote its dual space, consisting of all bounded linear functionals from H into the field $\mathbb{R}$ or $\mathbb{C}$. If x is an element of H, then the function φ_x, for all y in H defined by*

$$\varphi_x(y) = (y, x)_H$$

is an element of H^*. Furthermore, every element of H^* can be written uniquely in this form.

Theorem C.0.8 (**Rellich-Kondrachov Compactness Theorem**). *[9, Theorem 1, Section 5.7] Assume U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1 \leq p < n$ and $p^* = \dfrac{pn}{n-p}$. Then*

$$W^{1,p}(U) \subset\subset \mathcal{L}^q(U)$$

for each $1 \leq q < p^*$.

The statement $W^{1,p}(U) \subset\subset \mathcal{L}^q(U)$ is that $W^{1,p}(U)$ is *compactly embedded* in $\mathcal{L}^q(U)$. That is,

$$\|x\|_{\mathcal{L}^q(U)} \leq C\|x\|_{W^{1,p}(U)}$$

for some constant C, and each bounded sequence in $W^{1,p}(U)$ is precompact in $\mathcal{L}^q(U)$.

Theorem C.0.9. *([8], I.6.15) If K is a set in a metric space X then the following are equivalent:*

(i) K is sequentially compact.

(ii) K is precompact.

(iii) K is totally bounded and $\overline{K}$ is complete.

Furthermore, a compact metric space is complete and separable.

Theorem C.0.10 (**Closed Graph Theorem**). *[9, Appendix D.3. Theorem 1] If X and Y are Banach spaces, and $T : X \mapsto Y$ is a linear operator, then T is continuous if and only if its graph is closed in $X \times Y$.*

Theorem C.0.11 (**Grönwall's Inequality in Integral Form**). *[9, Appendix B.2. (k)] Let I be an interval on the real line with a greatest lower bound a, and let α, β be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I. If β is non-negative and if u satisfies the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds.$$

Theorem C.0.12 (Uniform Boundedness Principle). *[21, Theorem 2.5] Let X be a Banach space and Y a normed vector space. If a sequence of bounded operators (T_n) from X onto Y converges pointwise, that is, the limit of $(T_n x)$ exists for all $x \in X$, then these pointwise limits define a bounded operator T. Furthermore, (T_n) converges to T uniformly on compact subsets of X.*

Bibliography

[1] L. Ahlfors. *Complex Analysis: An Introduction to the Theory of Analytic Functions for One Complex Variable*. McGraw-Hill Book Company, 2nd edition, 1966.

[2] W. Arendt, C. Batty, M. Heiber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*. Birkhauser, Basel, 2001.

[3] C. Batty. Unbounded operators: Functional calculus, generation, perturbations. *Extracta Mathematicae*, 24(2):99–133, 2009.

[4] B. Churchill and Verhey. *Complex Varibles and Applications*. McGraw-Hill, 1948.

[5] C. Corduneanu. *Integral Equations and Applications*. Cambridge University Press, 2008.

[6] S. Cui. Local and global existence of solutions to semilinear parabolic initial value problems. *Pergamon- Nonlinear Analysis*, 43:293–323, 2001.

[7] E. B. Davies. *Linear Operators and Their Spectra*. Number 106 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2007.

[8] N. Dunford and J. Schwartz. *Linear Operators Part 1: General Theory*, volume VII. Interscience Publishers, Inc., New York a division of John Wiley and Sons, Inc.

[9] L. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.

[10] J. A. Goldstein. *Semigroups of Linear Operators and Applications*. Oxford University Press, 1985.

[11] E. Hille. *Functional Analysis and Semi-groups*, volume 31. American Mathematical Society, 1948.

[12] E. Hille and R. Phillips. *Functional Analysis and Semi-groups*, volume 31. American Mathematical Society Collected Publishings, 1957.

[13] J. K. Hunter and B. Nachtergaele. *Applied Analysis*. World Scientific, 2005.

[14] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, volume 16 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhauser, 1995.

[15] E. Meron. *Nonlinear Physics of Ecosystems*. CRC Press, Taylor and Francis Group, 2015.

[16] G. Munoz, Y. Sarantopolous, and A. Tonge. Complexification of real banach spaces, polynomials and multilinear maps. *Studia Mathematica*, 134, 1999.

[17] A. Pazy. A class of semi-linear equations of evolution. *Israel Journal of Mathematics*, 20(1), 1975.

[18] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, 1983.

[19] G. Peano. Integration par series des equations diffirentielles lineaires. *Atti Accad. Sci. Torino*, 1887.

[20] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, Inc., 1976.

[21] W. Rudin. *Functional Analysis*. McGraw-Hill, 2nd edition, 1991.

[22] R. Showalter. Hilbert space methods for partial differential equations. *Electronic Journal of Differential Equations*, 1994. Monograph 1.

[23] N. van Rensburg and A. van der Merwe. Analysis of the solvability of linear vibration models. *Applicable Analysis: An International Journal*, 81:1143–1159, 2002.

[24] J. J. Walker. Development of an empirical model of human sweating and a semi-empirical model of human thermoregulation. *Retrospective Theses and Dissertations*, (Paper 12180), 1999.

[25] C. H. Wyndham and A. R. Atkins. A physiological scheme and mathematical model of temperature regulation in man. *Pflugers Archive (European Journal of Physiology)*, March 1968.

[26] E. D. Yildirim. A mathematical model of the human thermal system. Master's thesis, Graduate School of Engineering and Sciences of Izmir Institute of Technology, 2005.

[27] K. Yosida. On the differentiability and the representation of one-parameter semi-group of linear operators. *Journal of the Mathematical Society of Japan*, 1(1), 1948.

[28] A. Zolfaghari and M. Maerefat. *Developments in Heat Transfer*. InTech, 2011. Chapter 9 - Bioheat Transfer.