BÖDEWADT FLOW OF A FLUID-PARTICLE SUSPENSION WITH STRONG SUCTION

Muhammad Rahman* and Helge I. Andersson
*Author for correspondence
Department of Energy and Process Engineering,
Norwegian University of Science and Technology-NTNU
Trondheim, 7491,
Norway,
E-mail: muhammad.rahman@ntnu.no

ABSTRACT

The three-dimensional revolving flow of a particle-fluid suspension above a plane surface has been considered. The flow represents an extension of the classical Bödewadt flow to a two-fluid problem. The governing equations for the two phases are coupled through an interaction force with the particle relaxation time τ as a free parameter. By means of a similarity transformation, the coupled set of non-linear ODEs becomes a two-point boundary value problem. The numerical results showed that the radial inward particle velocity increased whereas the circumferential velocity decreased by shortening τ, thereby strengthening the spiralling particle motion. On the contrary, the fluid motion was reduced as a result of the particle-fluid interactions.

NOMENCLATURE

A	[-]	Suction parameter
a	[m]	Particle radius
F	[-]	Radial velocity
G	[-]	Circumferential velocity
H	[-]	Axial velocity
m	[kg]	Particle mass
P	[-]	Pressure
p	$[N/m^2]$	Pressure
Q	[-]	Ratio between the densities
r	[m]	Radial coordinate
u,v,w	[m/s]	Cylindrical polar velocity components for fluid phase
u_p,v_p,w_p	[m/s]	Cylindrical polar velocity components for particle phase
z	[m]	Axial coordinate

Special characters

β	[-]	Reciprocal Stokes number
η	[-]	Dimensionless similarity variable
θ	[rad]	Circumferential coordinate
μ	$[N.s/m^2]$	Dynamic viscosity
ν	$[m^2/s]$	Kinematic viscosity
ρ	$[kg/m^3]$	Density of the fluid
τ	[s]	Relaxation time
Ω	[1/s]	Angular velocity

Subscripts

p	Particle
r	Radial direction
z	Axial direction
θ	Circumferential direction

INTRODUCTION

The steadily revolving flow of a viscous fluid above a planar surface is commonly known as Bödewadt flow; see Bödewadt [1]. The fluid motion well above the surface is characterised by a uniform angular velocity, which is reduced through a viscous boundary layer in order for the fluid to adhere to the no-slip condition at the solid surface. The reduction of the circumferential velocity component in the vicinity of the surface reduces the radially directed centripetal acceleration (or centrifugal force) such that the prevailing radial pressure gradient induces an inward fluid motion. In order to assure mass conservation, this inward fluid motion gives in turn rise to an axial upward flow. Such a spiralling flow exists near the planar surface, although more complex variations of the velocity field have been reported further away, but yet before the uniformly rotating flow conditions are reached. The oscillatory nature of the three velocity components reported by Bödewadt [1] has been subject to criticism, but this criticism was deem unjustified by Zandbergen and Dijkstra [2]. It is interesting to notice that these oscillations are damped and even suppressed in presence of a magnetic field (King and Lewellen [3]), partial slip (Sahoo, Abbasbandy and Poncet [4]) or suction (Nath and Venkatachala [5]). With a sufficiently high suction velocity through the planar surface, the axial flow is directed in the downward direction rather than upward, as is the case in the classical Bödewadt flow. In view of its fundamental importance as a prototype swirling flow the Bödewadt flow has received renewed focus in recent years. The inviscid instability of the Bödewadt boundary layer was examined by MacKerrell [6] whereas Sahoo [7] and Sahoo and Poncet [8] demonstrated that also such revolving flows of a non-Newtonian Reiner-Rivlin fluid admit exact similarity solutions.

The swirling flow induced by a steadily rotating disk was first described by von Kármán [9]. The von Kármán flow is essentially a reversed Bödewadt flow, albeit without the oscillatory features that characterize the latter. Zung [10] studied a von Kármán flow of a fluid-particle suspension and his analysis was subsequently extended by Sankara and Sarma [11] to include surface suction and further explored by Allaham and Peddieson [12].

The aim of the present study is to adopt a similar approach as that advocated by Zung [10] to Bödewadt flow of a fluid-particle suspension. After first having shown that the governing two-phase flow equations admit similarity solutions, numerical solutions of the coupled set of non-linear ordinary differential equations will show how the particle phase is revolving along with the fluid and also how the presence of particles will affect the swirling motion of the fluid phase.

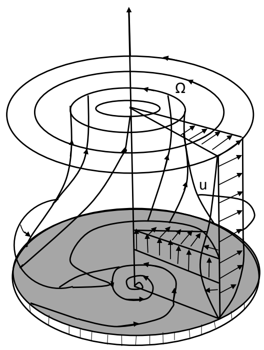

Figure 1. Sketch of the three-dimensional flow field set up by a steadily revolving flow high above the shaded surface. The variations of the radial (u) and circumferential (v) velocity components are indicated. The axial velocity component is upward, as in the classical B*ö*dewadt problem [1]. In presence of strong suction, as considered herein, the axial velocity is downward.

MATHEMATICAL MODEL EQUATIONS

Consider the steadily revolving flow of a fluid-particle suspension above a planar surface. In cylindrical polar coordinates (r,θ,z) the governing mass conservation and momentum equations for the fluid and particle phases become:

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw)=0, \tag{1}$$

$$\frac{\partial}{\partial r}(r\rho_p u_p)+\frac{\partial}{\partial z}(r\rho_p w_p)=0, \tag{2}$$

$$u\frac{\partial u}{\partial r}-\frac{v^2}{r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right]+\frac{F_r}{\rho}, \tag{3}$$

$$u\frac{\partial v}{\partial r}+\frac{uv}{r}+w\frac{\partial v}{\partial z}=\nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}\right]+\frac{F_\theta}{\rho}, \tag{4}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right]+\frac{F_z}{\rho}, \tag{5}$$

$$u_p\frac{\partial u_p}{\partial r}-\frac{v_p^2}{r}+w_p\frac{\partial u_p}{\partial z}=-\frac{1}{\rho_p}\frac{\partial p}{\partial r}-\frac{F_r}{\rho_p}, \tag{6}$$

$$u_p\frac{\partial v_p}{\partial r}+\frac{u_p v_p}{r}+w_p\frac{\partial v_p}{\partial z}=-\frac{F_\theta}{\rho_p}, \tag{7}$$

$$u_p\frac{\partial w_p}{\partial r}+w_p\frac{\partial w_p}{\partial z}=-\frac{1}{\rho_p}\frac{\partial p}{\partial z}-\frac{F_z}{\rho_p}, \tag{8}$$

where (u, v, w) and (u_p, v_p, w_p) are the velocity components of the fluid and particle phases in the radial, circumferential and axial directions, respectively. Here, we have assumed rotational symmetry about the vertical z-axis, $i.e. \partial/\partial\theta=0$. The kinematic viscosity of the fluid is ν and the densities of the fluid and particle phases are ρ and ρ_p. The above set of governing equations is the same as that considered by Zung [10] and Sankara and Sarma [11] for swirling von K*á*rm*á*n flow of a fluid-particle suspension above a steadily rotating disk, except that the a priori unknown pressure p was assigned only to the fluid phase. In the present study, however, the pressure gradients are shared between the two phases in proportion to their density ratio. This alternative formulation was suggested by Allaham and Peddeison [12] but not adopted in their subsequent calculations. In the present problem, however, it is essential to include pressure gradient terms also in the particle-phase equations of motion. Indeed, a radial pressure gradient is required to balance the centripetal acceleration in the far field in equation (6).

Particle-fluid interactions are accounted for by means of the following force components

$$F_r=\rho_p(u_p-u)/\tau, \tag{9}$$

$$F_\theta=\rho_p(v_p-v)/\tau, \tag{10}$$

$$F_z=\rho_p(w_p-w)/\tau, \tag{11}$$

included in the fluid phase equations and their negative counterparts in the particle phase equations. These expressions represent force per volume and are based on the assumption of a linear drag law, i.e. Stokes drag, where τ

$$\tau=m/6\pi\mu a, \tag{12}$$

is the relaxation time of a single spherical particle with mass m and radius a immersed in a fluid with dynamic viscosity $\mu=\rho\nu$.

The Stokes force defined in equations (9)-(11) represents a pragmatic way to account for fluid-particle interactions. The linear drag law assumes that the fluid is viscous and the particles are small. The relative velocity between fluid and particles, i.e. the slip velocity, should be sufficiently small to make the Reynolds number based on the slip velocity smaller than unity. These assumptions can be realistic for instance for dust particles in air and sand particles in water.

The viscous fluid phase sticks to the permeable planar surface at $z = 0$ and attains a state of solid body rotation with angular velocity Ω high above the surface:

$$\left.\begin{array}{llll} u=0, & v=0, & w=\sqrt{\nu\Omega}A & at \quad z=0 \\ u=0, & v=r\Omega, & p=\frac{1}{2}\rho r^2\Omega^2 & as \quad z\to\infty \end{array}\right\}, \tag{13}$$

where $A \leq 0$ is a dimensionless suction velocity. The inviscid particle phase can be assumed to follow the motion of the fluid phase far above the solid surface, i.e.

$$\left.\begin{array}{ll} u_p=u=0, & v_p=v=r\Omega, \\ w_p=w, & \rho_p=\rho, \end{array}\right\} \; for\; z\to\infty. \tag{14}$$

It is noteworthy that the boundary conditions (14) for the particle phase are imposed only far above the solid surface. Allaham and Peddieson [12] considered the possibility of including viscous effects also in the particle equations. If so, the particle equations would become of second-order and no-slip conditions should be imposed at the solid surface. In the present study, however, we follow the way paved by Zung [10] and Sankara and Sarma [11] and viscous terms have been ignored in the particle equations. The equations (6)-(8) are first-order PDEs and the boundary conditions (14) suffice and make the problem well-posed.

SIMILARITY TRANSFORMATION AND RESULTING ODEs

The two-phase flow that arises above the planar surface can be characterized by the length scale $\sqrt{\nu/\Omega}$, the time scale Ω^{-1} and the velocity scale $\sqrt{\nu\Omega}$. One can therefore introduce the same dimensionless similarity variables as used already by von K*á*rm*á*n [9] and B*ö*dewadt [1]:

$$\eta = z\sqrt{\Omega/\nu}. \tag{15}$$

The same similarity transformations are used for the fluid phase velocities and pressure:

$$\begin{array}{c} u(r,z) = r\Omega F(\eta), \\ v(r,z) = r\Omega G(\eta), \\ w(r,z) = \sqrt{\nu\Omega}H(\eta), \\ p(r,z) = \rho(-\nu\Omega P(\eta)+\frac{1}{2}r^2\Omega^2). \end{array} \tag{16}$$

The latter pressure transformation was not required in B*ö*dewadt's original approach which was based on the boundary layer approximations. That assumption implies that the pressure is constant all across the boundary layer.

The variables characterizing the particle phase can be recast into dimensionless forms by means of the transformation used by Sankara and Sarma [11]:

$$\begin{array}{c} u_p(r,z) = r\Omega F_p(\eta), \\ v_p(r,z) = r\Omega G_p(\eta), \\ w_p(r,z) = \sqrt{\nu\Omega}H_p(\eta), \\ \rho_p(r,z) = \rho Q(\eta), \end{array} \tag{17}$$

where Q is the ratio between the densities of the two phases. The governing set of partial differential equations (1)-(8) transforms into a set of ordinary differential equations:

$$2F + H' = 0, \tag{18}$$

$$Q'H_p + QH_p{}' + 2QF_p = 0, \tag{19}$$

$$F'' - HF' - F^2 + G^2 + \beta Q(F_p - F) = 1, \tag{20}$$

$$G'' - HG' - 2FG + \beta Q(G_p - G) = 0, \tag{21}$$

$$P' = -2FH + 2F' - \beta Q(H_p - H), \tag{22}$$

$$F_p{}'H_p + F_p^2 - G_p^2 + \beta(F_p - F) = -1/Q, \tag{23}$$

$$G_p{}'H_p + 2F_pG_p + \beta(G_p - G) = 0, \tag{24}$$

$$QH_p{}'H_p + \beta Q(H_p - H) = P', \tag{25}$$

where the prime denotes differentiation with respect to the similarity variable η. The accompanying boundary conditions (13)-(14) transform into:

$$F(\eta) = 0,\; G(\eta) = 0,\; H(\eta) = A, \quad at \;\; \eta = 0,$$

$$\left.\begin{array}{lll} F(\eta)=0, & G(\eta)=1, & P(\eta)=1, \\ F_p(\eta)=0, & G_p(\eta)=1, & H_p(\eta)=H(\eta), \\ Q(\eta)=1. & & \end{array}\right\} as \quad \eta\to\infty. \tag{26}$$

The resulting two-fluid flow problem now depends only on two dimensionless parameters, namely the suction parameter A and the interaction parameter $\beta = 1/\Omega\tau$ where τ was introduced in equation (12). The ratio between a particle time scale and a fluid time scale, e.g. $\tau/\Omega^{-1} = \beta^{-1}$, is often referred to as a Stokes number.

NUMERICAL INTEGRATION TECHNIQUE

The primary interest is in the flow field. The pressure gradient P' can therefore be eliminated from the axial components of the fluid and particle equations to give

$$H_p{}' = \left(2F' - 2\beta Q(H_p - H) - 2FH\right)/H_pQ. \qquad (27)$$

The bvp4c MATLAB solver, which gives very good results for non-linear ODEs with multipoint BVPs, has been used. This finite-difference code utilizes the 3-stage Lobatto IIIa formula, that is a collocation formula and the collocation polynomial provides a C1-continuous solution that is fourth-order accurate uniformly in [a,b]. For multipoint BVPs, the solution is C1-continuous within each region, but continuity is not automatically imposed at the interfaces. Mesh selection and error control are based on the residual of the continuous solution. Analytical condensation is used when the system of algebraic equations is formed; see Shampine et al.[13].

The coupled set of non-linear ODEs are integrated for $A = -3.0$ and some different values of β. Suction [5], as well as the presence either of surface slip [4] or a magnetic force field [3], tends to stabilize the revolving flow with respect to the oscillating behaviour of the three velocity components reported already by Bödewadt [1]. This is the likely reason why it turned out to be difficult to perform the present computations without suction, i.e. for $A = 0$.

RESULT AND DISCUSSIONS

The primarily interest is in particle-fluid interactions in the three-dimensional flow field. Four different values of the particle fluid interaction parameter β in the range from 0.3 to 4.0 have therefore been considered. The suction parameter A was kept constant and equal to -3.0. This particular parameter value is 50% higher than the strongest suction considered by Nath and Venkatachala [5] albeit without a particle phase.

One can see from Figure 2 that the radial inward flow is reduced in the presence of particles and this reduction increases with β . This is caused by the interaction force F_r which is positive since $u_p > u$ everywhere except in the near vicinity of the surface. It is noteworthy that also the particle phase flows towards the symmetry axis. The particle velocity u_p in Figure 3 is radially inward and increases in magnitude all the way towards the surface. This inward motion strengthens monotonically with increasing interaction parameter β. Contrary to u, however, u_p does not obey no-slip at the surface. This gives rise to a change-of-sign of the slip velocity $u_p - u$ next to the surface and thereby a reversal of the radial force F_r defined in equation (9). The slip velocity presented in Fig 4 shows that the thickness of the near-wall layer with $F_r < 0$ increases from about 0.03 to $0.2\sqrt{\nu/\Omega}$ as β increases from 0.3 to 4.0.

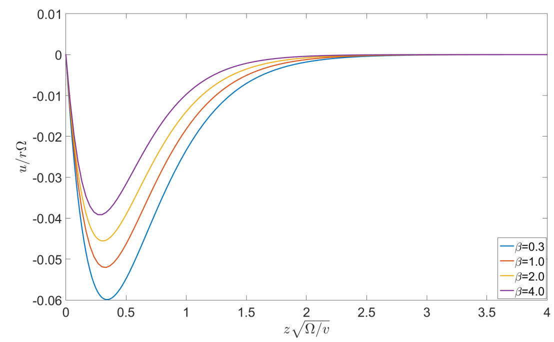

Figure 2. Radial fluid velocity component $F = u/r\Omega$ for some different values of the interaction parameter β.

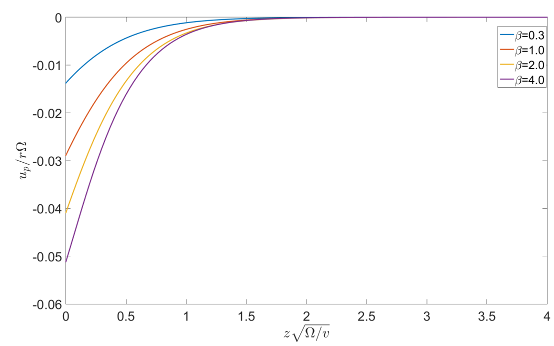

Figure 3. Radial particle velocity component $F_p = u_p/r\Omega$ for some different values of the interaction parameter β.

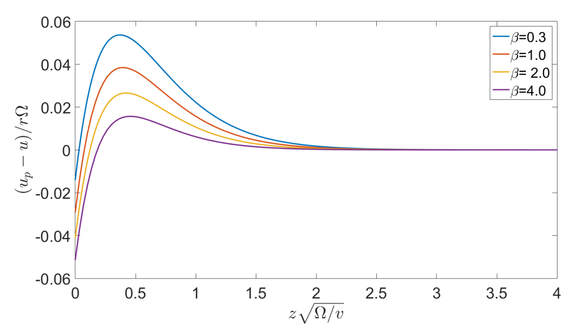

Figure 4. Difference between the particle and fluid velocities $F_p - F$ for some different values of the interaction parameter β.

The interaction parameter β has an almost negligible effect on the circumferential fluid velocity v in Figure 5 which reduces monotonically from that of solid body rotation $v = r\Omega$ to no-slip $v = 0$ at the surface. The circumferential slip velocity $v_p - v$ becomes inevitably positive in the viscous boundary layer since the particle velocity v_p is neither affected by viscous forces nor obeys no-slip. The circumferential interaction force F_θ on the fluid phase is positive, whereas the reaction force on the particle phase $-F_\theta < 0$ and therefore tends to enhance the deceleration of the particle motion with increasing β, as one can observe in Figure 6.

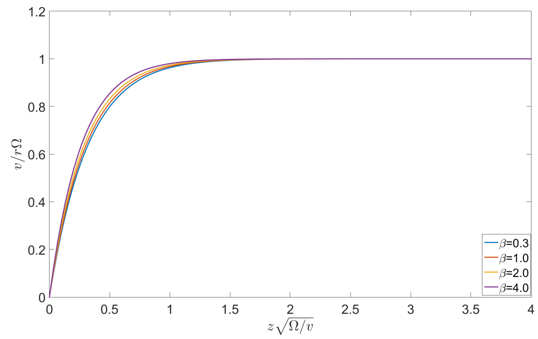

Figure 5. Circumferential fluid velocity component $G = v/r\Omega$ for some different values of the interaction parameter β.

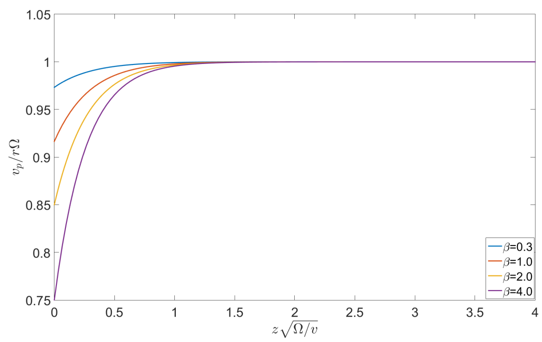

Figure 6. Circumferential particle velocity component $G_p = v_p/r\Omega$ for some different values of the interaction parameter β.

The fluid phase flows axially towards the surface, i.e. $w < 0$, as can be seen in Figure 7. This opposite flow direction compared to that in classical Bödewadt flow $(w > 0)$ is due the strong surface suction. The magnitude of the downward flow can be seen to increase from that beyond $\eta = z\sqrt{\Omega/\nu} \approx 2$ where solid-body rotation prevails through the viscous boundary layer until $w/\sqrt{\nu\Omega} = -3.0$ at the surface $z = 0$. The interaction force F_z is generally positive since the slip velocity $w_p - w > 0$ (compare Figs 7 and 8). The axial convection is $w_p \partial w_p/\partial z = 1/2 \partial w_p^2/\partial z$ is partially balanced by the negative reaction force $-F_z$ in the particle equation (8). However, the dimensionless pressure P decreases monotonically upwards, i.e. $P' < 0$. This variation gives rise to a positive pressure gradient in the axial direction, i.e. a pressure force that acts towards the surface and thus tends to support the axial motion towards the surface.

The relative density Q, i.e. the ratio between the particle and fluid densities ρ_p/ρ, is a variable quantity that has been obtained as an integral part of the numerical solution of the flow problem. The Q-profiles in Figure 9 show that the particle density ρ_p is only a few percent lower than the fluid density near the surface but increases to match ρ outside of the viscous boundary layer.

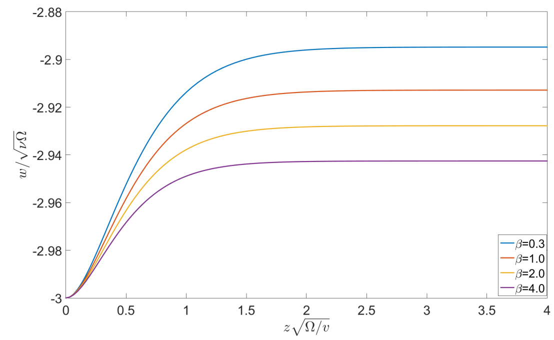

Figure 7. Axial fluid velocity component $H = w/\sqrt{\nu\Omega}$ for some different values of the interaction parameter β.

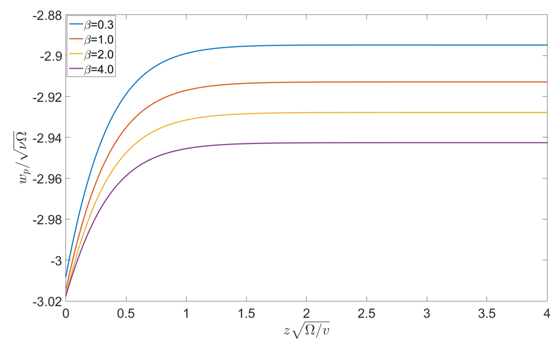

Figure 8. Axial particle velocity component $H_p = w_p/\sqrt{\nu\Omega}$ for some different values of the interaction parameter β.

Let us finally look at the motion of the particle phase in the limit as $\beta \to 0$ if $\rho_p = \rho$. The horizontal velocity components become $u_p = 0$ and $v_p = r\Omega$, i.e. the radial component vanishes and the linearly increasing circumferential velocity makes the centripetal acceleration exactly balance the radial pressure force.

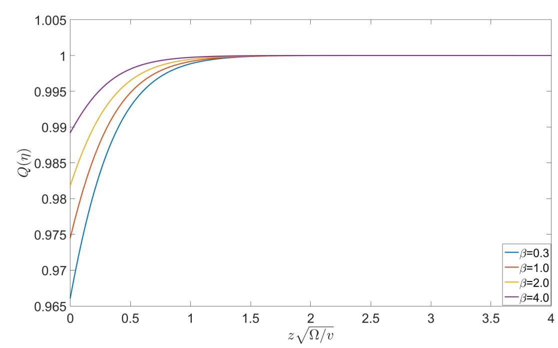

Figure 9. Ratio of particle and fluid densities $Q = \rho_p/\rho$ for some different values of the interaction parameter β.

Some primary boundary layer characteristics obtained from the numerical solutions are given in Table 1. For the fluid phase:$-F'(0)$, $G'(0)$, $-H(\infty)$. For the particle phase: $-F_p(0)$, $G_p(0)$, $-H_p(0)$. The following observations can be made from the tabulated results. The magnitude of the slope of the radial velocity $F'(0)$ decreases with increasing fluid-particle interactions. The slope of the circumferential fluid velocity $G'(0)$ increases with increasing β (not clearly visible in Figure 5). The magnitude of the downward fluid velocity $-H(\infty)$ is slightly increased from 2.8948 to 2.9426 with increasing β . The circumferential particle velocity $G_p(0)$ decreases from 0.9732 to 0.7501 as β increases from 0.3 to 4.0. The inward radial particle velocity $-F_p(0)$ increases from about 0.0138 to 0.0512 as β is increased.

Table 1. Flow characteristics for suction parameter $A = -3.0$.

β	$-F'(0)$	$-F_p(0)$	$G_p(0)$	$G'(0)$	$-H_p(0)$	$-H(\infty)$
0.3	0.4547	0.0138	0.9732	3.1441	3.0008	2.8948
1.0	0.4156	0.0289	0.9165	3.3142	3.0145	2.9129
2.0	0.3853	0.0409	0.8501	3.5125	3.0173	2.9278
4.0	0.3607	0.0512	0.7501	3.8098	3.0174	2.9426

CONCLUDING REMARKS

The present study adopted a two-fluid approach to model the presence of solid particles in a viscous carrier fluid. The particles have been treated as a continuous solid phase modelled by the Eulerian partial differential equations (6)-(8). Alternatively, the particles could have been treated as individual point-particles and modelled by Lagrangian ordinary differential equations (one set of ODEs for each and every particle), as described for instance by Zhao et al. [14].

The governing equations of the three-dimensional two-fluid problem have been transformed into a coupled set of ordinary differential equations by means of an exact similarity transformation. The resulting set of ODEs is a two parameter problem in terms of the dimensionless suction velocity A and the fluid-particle interaction parameter β. Here, β is the ratio between the rotation time scale Ω^{-1} and the particle relaxation time τ. β is therefore a reciprocal Stokes number.

The particles are spiralling inwards in the vicinity of the surface. The radial inward velocity $|u_p|$ increases and the circumferential velocity v_p decreases with increasing interaction parameter β, i.e. the spiralling increases with β. This phenomenon is commonly known as the tea cup effect: when the tea is stirred, tea leaves near the bottom move towards the centre of the cup and heap up. Contrary to the particle phase, the inward spiralling of the fluid phase is gradually reduced as the fluid-particle interaction parameter is increased. The circumferential flow is only modestly affected, but the surface shear stress is nevertheless increased.

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