Research Article

Common Fixed Points of Generalized Rational Type Cocyclic Mappings in Multiplicative Metric Spaces

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The aim of this paper is to present fixed point result of mappings satisfying a generalized rational contractive condition in the setup of multiplicative metric spaces. As an application, we obtain a common fixed point of a pair of weakly compatible mappings. Some common fixed point results of pair of rational contractive types mappings involved in cocyclic representation of a nonempty subset of a multiplicative metric space are also obtained. Some examples are presented to support the results proved herein. Our results generalize and extend various results in the existing literature.

1. Introduction and Preliminaries

Fixed point theory provides the most important and traditional tools for proving the existence of solutions of many problems in both pure and applied mathematics. In metric fixed point theory, the interplay between contractive condition and the existence and uniqueness of a fixed point has been very strong and fruitful. The study of fixed points of mappings which satisfy certain contractive conditions has primary applications in the solution of differential and integral equations (see, e.g., [1–5]). Theorems dealing with fixed point of certain mappings inspired and motivated the development of many other important kinds of points like coincidence points, intersection points, sectional points, and so forth.

One of the basic and most widely applied fixed point theorems in mathematical analysis is "Banach Contraction Mapping Principle" (or Banach's Fixed Point Theorem). Kirk et al. [4] obtained some fixed point results for mappings satisfying cyclical contractive conditions. The Banach contraction mappings are continuous mappings, while cyclic contraction mappings need not be continuous. Păcurar and Rus [6] studied some fixed point results for cyclic weak

contractions. Piątek [7] obtained some results on cyclic Meir-Keeler contractions in metric spaces. Using fixed point result of weakly contractive map, Karapinar [3] established some interesting fixed point results for cyclic ϕ-weak contraction mappings. Recently, Derafshpour et al. [8] obtained results on the existence of best proximity points of cyclic contractions. For more results in this direction, we refer to [9–15].

Banach contraction principle has been generalized either by extending the domain of the mapping or by considering a more general contractive condition on the mappings. Ozavsar and Cevikel [16] proved an analogous result to Banach contraction principle in the framework of multiplicative metric spaces. They also studied some topological properties of the relevant multiplicative metric space. Bashirov et al. [17] studied the concept of multiplicative calculus and proved a fundamental theorem of multiplicative calculus. They also illustrated the usefulness of multiplicative calculus with some interesting applications. Multiplicative calculus provides natural and straightforward way to compute the derivative of product and quotient of two functions [18]. It was shown that the multiplicative differential equations are more suitable than the ordinary differential equations in investigating some problems in economics and finance.

Due to its operational simplicity and support to Newtonian calculus, it has attracted the attention of several researchers in the recent years. Furthermore, based on the definition of multiplicative absolute value function, they defined the multiplicative distance between two nonnegative real numbers and between two positive square matrices. This provided the basis for multiplicative metric spaces. Florack and van Assen [19] gave applications of multiplicative calculus in biomedical image analysis. He et al. [20] studied common fixed points for weak commutative mappings on a multiplicative metric space (see also [21]). Recently, Yamaod and Sintunavarat [22] obtained some fixed point results for generalized contraction mappings with cyclic (α, β)-admissible mapping in multiplicative metric spaces.

In this paper, we obtain fixed point result of a generalized rational contractive mapping in the framework of multiplicative metric spaces. Employing this result, a common fixed point of a pair of weakly compatible mappings is obtained. We study the sufficient conditions for the existence of common fixed points of pair of rational contractive types mappings involved in cocyclic representation of a nonempty subset of a multiplicative metric space. Our results generalize and extend comparable results in [3].

By $\mathbb{R}, \mathbb{R}^+, \mathbb{R}_n^+$, and $\mathbb{N}$ we denote the set of all real numbers, the set of all nonnegative real numbers, the set of all n-tuples of positive real numbers, and the set of all natural numbers, respectively.

Consistent with [16, 17], the following definitions and results will be needed in the sequel.

Definition 1 (see [17]). Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$, the following conditions hold:

(i) $d(x, y) \geq 1$ and $d(x, y) = 1$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$.

The pair (X, d) is called a multiplicative metric space.

We define absolute valued function that includes the negative real numbers in its domain.

Definition 2. A multiplicative absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}^+$ is defined as

$$
|x| = \begin{cases} x & \text{if } x \geq 1, \\ \dfrac{1}{x} & \text{if } x \in (0, 1), \\ 1 & \text{if } x = 0, \\ -\dfrac{1}{x} & \text{if } x \in (-1, 0), \\ -x & \text{if } x \leq -1. \end{cases} \tag{1}
$$

Using the definition of multiplicative absolute value function, we can prove the following proposition.

Proposition 3. *For arbitrary $x, y \in \mathbb{R}^+$, the multiplicative absolute value function $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following:*

(1) $|x| \geq 1$.

(2) $x \leq |x|$.

(3) $1/|x| \leq x$ if $x > 0$ and $x \leq 1/|x|$ if $x \leq 0$.

(4) $|x \cdot y| \leq |x||y|$.

Example 4 (see [16]). Let $X = \mathbb{R}_n^+$. Then

$$
\begin{aligned}
d_1(x, y) &= \left|\frac{x_1}{y_1}\right| \cdot \left|\frac{x_2}{y_2}\right| \cdots \left|\frac{x_n}{y_n}\right|, \\
d_2(x, y) &= \max\left\{\left|\frac{x_1}{y_1}\right|, \left|\frac{x_2}{y_2}\right|, \dots, \left|\frac{x_n}{y_n}\right|\right\}
\end{aligned} \tag{2}
$$

define multiplicative metrices on X, where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}_n^+$.

Definition 5 (see [16]). Let (X, d) be a multiplicative metric space, x_0 an arbitrary point in X, and $\varepsilon > 1$. A multiplicative open ball $B(x_0, \varepsilon)$ of radius ε centered at x_0 is the set $\{z \in X : d(z, x_0) < \varepsilon\}$.

A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative convergent to some point $x \in X$ if, for any given $\varepsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$ for all $n \geq n_0$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 6 (see [16]). A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is multiplicative convergent to x in X if and only if $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 7. Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and x_0 an arbitrary but fixed element of X. A mapping $f : X \to Y$ is said to be multiplicative continuous at x_0 if and only if $x_n \to x_0$ in (X, d_X) implies that $f(x_n) \to f(x_0)$ in (Y, d_Y). That is, given arbitrary $\varepsilon > 1$, there exists $\delta > 1$ which depends on x_0 and ε such that $d_Y(fx, fx_0) < \varepsilon$ for all those x in X for which $d_X(x, x_0) < \delta$.

Definition 8 (see [16]). A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is said to be multiplicative Cauchy sequence if, for any $\varepsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$.

A multiplicative metric space (X, d) is said to be complete if every multiplicative Cauchy sequence $\{x_n\}$ in X is multiplicative convergent in X.

Theorem 9 (see [16]). *A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is multiplicative Cauchy if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$.*

Example 10. Let $X = C^*[a, b]$ be the collection of all real-valued multiplicative continuous functions over $[a, b] \subseteq \mathbb{R}^+$ with the multiplicative metric d defined by

$$
d(f, g) = \sup_{x \in [a,b]} \left|\frac{f(x)}{g(x)}\right| \quad \text{for arbitrary } f, g \in X, \tag{3}
$$

where $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$ is a multiplicative absolute value function. Then $(C^*[a, b], d)$ is complete.

Recall that if f and g are two self-maps on a set X and $u = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and u is called a *point of coincidence* of f and g.

Definition 11 (see [23]). Two self-maps f and g on a nonempty set X are called weakly compatible if they commute at their coincidence point.

We will also need the following proposition from [23].

Proposition 12. *Let f and g be weakly compatible self-maps on a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 13 (see [12]). Let $\{X_i : i = 1, 2, \ldots, p\}$ be a finite collection of nonempty subsets of a set X, where p is some positive integer and $f, g : X \to X$. The set X is said to have a cocyclic representation with respect to the collection $\{X_i : i = 1, 2, \ldots, p\}$ and a pair (f, g) if

(1) $X = \cup_{i=1}^{p} X_i$;

(2) $f(X_1) \subseteq g(X_2), \ldots, f(X_{p-1}) \subseteq g(X_p)$, $f(X_p) \subseteq g(X_1)$.

Definition 14. The control functions ψ and φ are defined as follows:

(i) $\psi : [1, \infty) \to [1, \infty)$ is a continuous nondecreasing function with $\psi(t) = 1$ if and only if $t = 1$.

(ii) $\varphi : [1, \infty) \to [1, \infty)$ is a lower semicontinuous function with $\varphi(t) = 1$ if and only if $t = 1$.

2. Main Results

In this section, we obtain several fixed and common fixed point results of self-maps satisfying certain generalized contractive conditions in the framework of multiplicative metric space.

We start with the following result.

Theorem 15. *Let (X, d) be a complete multiplicative metric space and $f : X \to X$. Suppose that there exist control functions ψ and φ such that*

$$\psi(d(fx, fy)) \leq \frac{\psi(M_f(x, y))}{\varphi(M_f(x, y))}, \tag{4}$$

for any $x, y \in X$, where

$$M_f(x, y) = \max \left\{ d(x, y), d(fx, x), d(fy, y), \right. \tag{5}$$
$$\left. (d(fx, y) \cdot d(fy, x))^{1/2} \right\}.$$

Then f has a unique fixed point.

Proof. Let x_0 be a given point in X. Define a sequence $\{x_n\}$ in X as $x_n = f^n x_0$ or equivalently as $x_{n+1} = fx_n$ for $n \in \mathbb{N}$. If $x_k = x_{k+1}$ for some k, then we have $fx_k = x_{k+1} = x_k$ and the

result follows. Assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$; that is, $M_f(x_{n+1}, x_n) > 1$ for all $n \in \mathbb{N}$. From (4), we have

$$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(fx_{n+1}, fx_n))$$
$$\leq \frac{\psi(M_f(x_{n+1}, x_n))}{\varphi(M_f(x_{n+1}, x_n))}, \tag{6}$$

where

$$M_f(x_{n+1}, x_n) = \max \left\{ d(x_{n+1}, x_n), d(fx_{n+1}, x_{n+1}), \right.$$
$$\left. d(fx_n, x_n), (d(fx_{n+1}, x_n) \cdot d(fx_n, x_{n+1}))^{1/2} \right\}$$
$$= \max \left\{ d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n), \right.$$
$$\left. (d(x_{n+2}, x_n) \cdot d(x_{n+1}, x_{n+1}))^{1/2} \right\}$$
$$= \max \left\{ d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), d(x_{n+2}, x_n)^{1/2} \right\}$$
$$\leq \max \left\{ d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), \right. \tag{7}$$
$$\left. (d(x_{n+2}, x_{n+1}) \cdot d(x_{n+1}, x_n))^{1/2} \right\}$$
$$\leq \max \left\{ d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), \right.$$
$$\left. \left(\frac{d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)}{2} \right) \right\}$$
$$= \max \left\{ d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}) \right\}.$$

If $M_f(x_{k+1}, x_k) \leq d(x_{k+2}, x_{k+1})$ for some $k \in \mathbb{N}$, then $\psi(d(x_{k+2}, x_{k+1})) \leq \psi(d(x_{k+2}, x_{k+1}))/\varphi(M_f(x_{k+1}, x_k))$ implies $\varphi(M_f(x_{k+1}, x_k)) \leq 1$, a contradiction. Hence $M_f(x_{n+1}, x_n) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Also, $d(x_{n+1}, x_n) \leq M_f(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$ and hence $M_f(x_{n+1}, x_n) = d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Thus, we have

$$\psi(d(x_{n+2}, x_{n+1})) \leq \frac{\psi(d(x_{n+1}, x_n))}{\varphi(d(x_{n+1}, x_n))} \tag{8}$$
$$< \psi(d(x_{n+1}, x_n))$$

which implies that

$$d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n) \tag{9}$$

for all $n \in \mathbb{N}$. Thus $\{d(x_{n+1}, x_n)\}$ is (strictly) decreasing of positive real numbers. Consequently, there exists $c \geq 1$ such that $\{d(x_{n+1}, x_n)\}$ converges to c. Suppose that $c > 1$. Now

$$\psi(d(x_{n+2}, x_{n+1})) \leq \frac{\psi(M_f(x_{n+1}, x_n))}{\varphi(M_f(x_{n+1}, x_n))}, \tag{10}$$

and lower semicontinuity of φ gives that

$$\limsup_{n \to \infty} \psi(d(x_{n+2}, x_{n+1}))$$
$$\leq \frac{\limsup_{n \to \infty} \psi(M_f(x_{n+1}, x_n))}{\liminf_{n \to \infty} \varphi(M_f(x_{n+1}, x_n))}, \tag{11}$$

which implies that $\psi(c) \leq \psi(c)/\varphi(c) < \psi(c)$, a contradiction as $\varphi(c) > 1$. Therefore, $c = 1$; that is, $\lim_{n \to \infty} d(x_{n+1}, x_n) = 1$.

Now, we claim that $\lim_{n,m \to \infty} d(x_n, x_m) = 1$. If not, then there exist $\varepsilon > 1$ and sequences $\{n_k\}$, $\{m_k\}$ in $\mathbb{N}$, with $n_k > m_k \geq k$, such that $d(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$. Without any loss of generality, we can assume that $d(x_{n_k}, x_{m_k-1}) < \varepsilon$. Since $\{d(x_{m_k-1}, x_{m_k})\}$ is a subsequence of convergent sequence $\{d(x_{n-1}, x_n)\} \to 1$ as $n \to \infty$, then $\{d(x_{m_k-1}, x_{m_k})\} \to 1$ as $k \to \infty$. Now

$$\varepsilon \leq d\left(x_{m_k}, x_{n_k}\right) \leq d\left(x_{n_k}, x_{m_k-1}\right) \cdot d\left(x_{m_k-1}, x_{m_k}\right) \quad (12)$$

implies that

$$\lim_{k \to \infty} d\left(x_{m_k}, x_{n_k}\right) = \varepsilon. \quad (13)$$

From (13) and inequality $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) \cdot d(x_{m_k-1}, x_{n_k})$, it follows that $\varepsilon \leq \lim_{k \to \infty} d(x_{m_k-1}, x_{n_k})$. Also, the inequality $d(x_{m_k-1}, x_{n_k}) \leq d(x_{m_k-1}, x_{m_k}) \cdot d(x_{m_k}, x_{n_k})$ and (13) give that $\lim_{k \to \infty} d(x_{m_k-1}, x_{n_k}) \leq \varepsilon$, and hence we have

$$\lim_{k \to \infty} d\left(x_{m_k-1}, x_{n_k}\right) = \varepsilon. \quad (14)$$

Equation (14) and inequality $d(x_{m_k-1}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k+1}) \cdot d(x_{n_k+1}, x_{n_k})$ imply that $\varepsilon \leq \lim_{k \to \infty} d(x_{m_k-1}, x_{n_k+1})$, while inequality $d(x_{m_k-1}, x_{n_k+1}) \leq d(x_{m_k-1}, x_{n_k}) \cdot d(x_{n_k}, x_{n_k+1})$ and (14) imply that $\lim_{k \to \infty} d(x_{m_k-1}, x_{n_k+1}) \leq \varepsilon$, and hence we have

$$\lim_{k \to \infty} d\left(x_{m_k-1}, x_{n_k+1}\right) = \varepsilon. \quad (15)$$

From (13) and $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k+1}) \cdot d(x_{n_k+1}, x_{n_k})$, we have $\varepsilon \leq \lim_{k \to \infty} d(x_{m_k}, x_{n_k+1})$, and the inequality $d(x_{m_k}, x_{n_k+1}) \leq d(x_{m_k}, x_{n_k}) \cdot d(x_{n_k}, x_{n_k+1})$ and (13) give that $\lim_{k \to \infty} d(x_{m_k}, x_{n_k+1}) \leq \varepsilon$. So

$$\lim_{k \to \infty} d\left(x_{m_k}, x_{n_k+1}\right) = \varepsilon. \quad (16)$$

As

$$M_f\left(x_{n_k}, x_{m_k-1}\right) = \max\left\{d\left(x_{n_k}, x_{m_k-1}\right),\right.$$
$$d\left(fx_{n_k}, x_{n_k}\right), d\left(fx_{m_k-1}, x_{m_k-1}\right),$$
$$\left.\left(d\left(x_{n_k}, fx_{m_k-1}\right) \cdot d\left(x_{m_k-1}, fx_{n_k}\right)\right)^{1/2}\right\} \quad (17)$$
$$= \max\left\{d\left(x_{n_k}, x_{m_k-1}\right), d\left(x_{n_k+1}, x_{n_k}\right),\right.$$
$$\left.d\left(x_{m_k}, x_{m_k-1}\right), \left(d\left(x_{n_k}, x_{m_k}\right) \cdot \left(x_{m_k-1}, x_{n_k+1}\right)\right)^{1/2}\right\},$$

we have $\lim_{k \to \infty} M_f(x_{n_k}, x_{m_k-1}) = \max\{\varepsilon, 1, 1, \varepsilon\} = \varepsilon$. From (4), it follows that

$$\psi\left(d\left(x_{n_k+1}, x_{m_k}\right)\right) = \psi\left(d\left(fx_{n_k}, fx_{m_k-1}\right)\right)$$
$$\leq \frac{\psi\left(M_f\left(x_{n_k}, x_{m_k-1}\right)\right)}{\varphi\left(M_f\left(x_{n_k}, x_{m_k-1}\right)\right)}. \quad (18)$$

Taking upper limit as $k \to \infty$ implies that $\psi(\varepsilon) \leq \psi(\varepsilon)/\varphi(\varepsilon) < \psi(\varepsilon)$, a contradiction as $\varepsilon > 1$. Thus, we obtain that $\lim_{n,m \to \infty} d(x_n, x_m) = 1$, and hence $\{x_n\}$ is a multiplicative Cauchy sequence in (X, d). Next, we assume there exists a point $u \in X$ such that $\lim_{n \to \infty} d(u, x_n) = 1$; equivalently,

$$\lim_{n,m \to \infty} d\left(x_n, x_m\right) = \lim_{n \to \infty} d\left(x_n, u\right) = 1. \quad (19)$$

Note that

$$d\left(fu, u\right) \leq M_f\left(u, x_n\right) = \max\left\{d\left(u, x_n\right), d\left(fu, u\right),\right.$$
$$\left.d\left(fx_n, x_n\right), \left(d\left(u, fx_n\right) \cdot d\left(x_n, fu\right)\right)^{1/2}\right\}$$
$$= \max\left\{d\left(u, x_n\right), d\left(fu, u\right), d\left(x_{n+1}, x_n\right),\right.$$
$$\left(d\left(u, x_{n+1}\right) \cdot d\left(x_n, fu\right)\right)^{1/2}\right\} \leq \max\left\{d\left(u, x_n\right),\right. \quad (20)$$
$$d\left(fu, u\right), d\left(x_{n+1}, x_n\right),$$
$$\left.\left(d\left(u, x_{n+1}\right) \cdot d\left(x_n, u\right) \cdot d\left(u, fu\right)\right)^{1/2}\right\}.$$

Taking limit as $n \to \infty$, we conclude that $\lim_{n \to \infty} M_f(u, x_n) = d(fu, u)$. Hence

$$\psi\left(d\left(fu, fx_{n+1}\right)\right) \leq \frac{\psi\left(M_f\left(u, x_n\right)\right)}{\varphi\left(M_f\left(u, x_n\right)\right)} \quad (21)$$

on taking upper limit as $n \to \infty$ implies that

$$\psi\left(d\left(fu, u\right)\right) \leq \frac{\psi\left(d\left(fu, u\right)\right)}{\varphi\left(d\left(fu, u\right)\right)} \quad (22)$$

which further implies that $fu = u$.

To prove the uniqueness of fixed point of f, assume on the contrary that $fv = v$ and $fw = w$ with $v \neq w$. Note that

$$\psi\left(d\left(v, w\right)\right) = \psi\left(d\left(fv, fw\right)\right) \leq \frac{\psi\left(M_f\left(v, w\right)\right)}{\varphi\left(M_f\left(v, w\right)\right)}, \quad (23)$$

where

$$M_f\left(v, w\right) = \max\left\{d\left(v, w\right), d\left(fv, v\right), d\left(fw, w\right),\right.$$
$$\left.\left(d\left(fv, w\right) \cdot d\left(fw, v\right)\right)^{1/2}\right\} = \max\left\{d\left(v, w\right),\right. \quad (24)$$
$$d\left(v, v\right), d\left(w, w\right), \left(d\left(v, w\right) \cdot d\left(w, v\right)\right)^{1/2}\right\} = d\left(v, w\right).$$

From (23) it follows that

$$\psi\left(d\left(v, w\right)\right) = \psi\left(d\left(fv, fw\right)\right) \leq \frac{\psi\left(d\left(v, w\right)\right)}{\varphi\left(d\left(v, w\right)\right)} \quad (25)$$
$$< \psi\left(d\left(v, w\right)\right),$$

a contradiction as $d(v, w) > 1$. Hence $v = w$. $\qquad \square$

Corollary 16. *Let (X, d) be a complete multiplicative metric space and $f : X \rightarrow X$. Suppose that there exist control functions ψ and φ such that*

$$\psi \left(d \left(f^n x, f^n y \right) \right) \leq \frac{\psi \left(M_{f^n} \left(x, y \right) \right)}{\varphi \left(M_{f^n} \left(x, y \right) \right)}, \tag{26}$$

for any $x, y \in X$ and $n \in \mathbb{N}$ where

$$M_{f^n} \left(x, y \right) = \max \left\{ d \left(x, y \right), d \left(f^n x, x \right), d \left(f^n y, y \right), \right.$$
$$\left. \left(d \left(f^n x, y \right) \cdot d \left(f^n y, x \right) \right)^{1/2} \right\}. \tag{27}$$

Then f has a unique fixed point u and $f^n u = f u$.

Proof. Set $T = f^n$. From Theorem 15, T has a unique fixed point u. Now $f(u) = f(Tu) = f^{n+1}(u) = f^n(fu) = T(fu)$ which implies that fu is also a fixed point of T. By the uniqueness of fixed point of T, we have $fu = u$. □

Now, we recall the following lemma from [24].

Lemma 17. *Let X be a nonempty set and $f : X \rightarrow X$. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \rightarrow X$ is one-to-one.*

Theorem 18. *Let (X, d) be a multiplicative metric space and $f, g : X \rightarrow X$. Suppose that there exist control functions ψ and φ such that*

$$\psi \left(d \left(fx, fy \right) \right) \leq \frac{\psi \left(M \left(gx, gy \right) \right)}{\varphi \left(M \left(gx, gy \right) \right)} \tag{28}$$

holds for any $x, y \in X$, where

$$M \left(x, y \right) = \max \left\{ d \left(x, y \right), d \left(fx, gx \right), d \left(fy, gy \right), \right.$$
$$\left. \left(d \left(fx, gy \right) \cdot d \left(fy, gx \right) \right)^{1/2} \right\}. \tag{29}$$

If $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have at most one common fixed point.

Proof. By Lemma 17, there exists $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \rightarrow X$ is one-to-one. Define a map $h : g(E) \rightarrow g(E)$ by $h(gx) = fx$. Since g is one-to-one on E, h is well defined. Note that

$$\psi \left(d \left(h \left(gx \right), h \left(gy \right) \right) \right) \leq \frac{\psi \left(M \left(gx, gy \right) \right)}{\varphi \left(M \left(gx, gy \right) \right)}, \tag{30}$$

for all $gx, gy \in g(E)$ where

$$M \left(gx, gy \right) = \max \left\{ d \left(gx, gy \right), d \left(h \left(gx \right), gx \right), \right.$$
$$\left. d \left(h \left(gy \right), gy \right), \left(d \left(h \left(gx \right), gy \right) \cdot d \left(h \left(gy \right), gx \right) \right) \right\}. \tag{31}$$

Since $g(E) = g(X)$ is complete, by Theorem 15, there exists $x_0 \in X$ such that $h(gx_0) = gx_0 = fx_0$. Hence, f and g have a unique point of coincidence. From Proposition 12, f and g have a unique common fixed point. □

3. Cocyclic Contractions

Now we obtain common fixed point results for self-maps satisfying certain cocyclic contractions defined on a multiplicative metric space. We start with the following.

Theorem 19. *Let (X, d) be a multiplicative metric space, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, and $Y = \cup_{i=1}^{p} A_i$. Suppose that $f, g : Y \rightarrow Y$ are such that*

(a) *Y has a cocyclic representation with respect to pair (f, g) and the collection $\{A_i : i = 1, 2, \ldots, p\}$;*

(b) *there exists control functions ψ and φ such that, for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$, the following,*

$$\psi \left(d \left(fx, fy \right) \right) \leq \frac{\psi \left(M_{f,g} \left(x, y \right) \right)}{\varphi \left(M_{f,g} \left(x, y \right) \right)}, \tag{32}$$

holds where

$$M_{f,g} \left(x, y \right) = \max \left\{ d \left(gx, gy \right), d \left(fx, gx \right), \right.$$
$$\left. d \left(fy, gy \right), \left(d \left(fx, gy \right) \cdot d \left(fy, gx \right) \right)^{1/2} \right\} \tag{33}$$

with $A_{p+1} = A_1$.

If $g(A_i)$ is complete subspace of X for each $i \in \{1, 2, \ldots, p\}$, then f and g have a unique coincidence point $z \in \cap_{i=1}^{p} g(A_i) = Z$ provided that $f(Z) \subseteq g(Z) \subseteq Z$ and $g(Z)$ is closed. Moreover, if f and g are weakly compatible, then f and g have at most one common fixed point.

Proof. Let x_0 be a given point in $\cup_{i=1}^{p} A_i$. Then there exists $i_0 \in \{1, 2, \ldots, p\}$ such that $x_0 \in A_{i_0}$. Choose a point x_1 in A_{i_0+1} such that $f(x_0) = g(x_1)$. This can be done because $f(A_{i_0}) \subseteq g(A_{i_0+1})$. Continuing this process, for $n > 0$, there exists $i_n \in \{1, 2, \ldots, p\}$ such that having chosen x_n in A_{i_n}, we obtain x_{n+1} in A_{i_n+1} such that $f(x_n) = g(x_{n+1})$. If, for some $n_0 \geq 0$, we have $f(x_{n_0}) = f(x_{n_0+1})$, then $f(x_{n_0}) = g(x_{n_0})$ implies that x_{n_0} is the coincidence point of f and g. Assume that $d(f(x_n), f(x_{n+1})) > 1$ for all n. From (b), we have

$$\psi \left(d \left(gx_{n+1}, gx_{n+2} \right) \right) = \psi \left(d \left(fx_n, fx_{n+1} \right) \right)$$
$$\leq \frac{\psi \left(M_{f,g} \left(x_n, x_{n+1} \right) \right)}{\varphi \left(M_{f,g} \left(x_n, x_{n+1} \right) \right)} \tag{34}$$
$$\leq \psi \left(M_{f,g} \left(x_n, x_{n+1} \right) \right),$$

where

$$M_{f,g} \left(x_n, x_{n+1} \right) = \max \left\{ d \left(gx_n, gx_{n+1} \right), d \left(fx_n, gx_n \right), \right.$$
$$d \left(fx_{n+1}, gx_{n+1} \right),$$

$$(d\left(fx_n, gx_{n+1}\right) \cdot d\left(fx_{n+1}, gx_n\right))^{1/2}\}$$

$$= \max \{d\left(gx_n, gx_{n+1}\right), d\left(gx_{n+1}, gx_n\right),$$

$$d\left(gx_{n+2}, gx_{n+1}\right),$$

$$(d\left(gx_{n+1}, gx_{n+1}\right) \cdot d\left(gx_{n+2}, gx_n\right))^{1/2}\}$$

$$\leq \max \{d\left(gx_n, gx_{n+1}\right), d\left(gx_{n+2}, gx_{n+1}\right),$$

$$(d\left(gx_{n+2}, gx_{n+1}\right) \cdot d\left(gx_{n+1}, gx_n\right))^{1/2}\}$$

$$\leq \max \{d\left(gx_n, gx_{n+1}\right), d\left(gx_{n+2}, gx_{n+1}\right),$$

$$\frac{d\left(gx_{n+2}, gx_{n+1}\right) + d\left(gx_{n+1}, gx_n\right)}{2}\}$$

$$= \max \{d\left(gx_n, gx_{n+1}\right), d\left(gx_{n+2}, gx_{n+1}\right)\}. \tag{35}$$

If, for some n, $\max\{d(gx_n, gx_{n+1}), d(gx_{n+2}, gx_{n+1})\} = d(gx_{n+2}, gx_{n+1})$, then $M_{f,g}(x_n, x_{n+1}) = d(gx_{n+2}, gx_{n+1})$ and

$$\psi\left(d\left(gx_{n+1}, gx_{n+2}\right)\right) \leq \frac{\psi\left(M_{f,g}\left(x_n, x_{n+1}\right)\right)}{\varphi\left(M_{f,g}\left(x_n, x_{n+1}\right)\right)} \tag{36}$$

$$\leq \frac{\psi\left(d\left(gx_{n+2}, gx_{n+1}\right)\right)}{\varphi\left(d\left(gx_{n+2}, gx_{n+1}\right)\right)}$$

imply that $\varphi(d(gx_{n+2}, gx_{n+1})) = 1$, a contradiction as $d(gx_{n+2}, gx_{n+1}) > 1$. Hence $d(gx_{n+2}, gx_{n+1}) \leq d(gx_n, gx_{n+1})$ for all n. Thus $M_{f,g}(x_n, x_{n+1}) \leq d(gx_n, gx_{n+1})$. On the other hand, we have $M_{f,g}(x_n, x_{n+1}) \geq d(gx_n, gx_{n+1})$. Hence $M_{f,g}(x_n, x_{n+1}) = d(gx_n, gx_{n+1})$.

Similarly we obtain that $d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1})$. Thus the sequence $\{d(gx_{n+1}, gx_n)\}$ is nonincreasing. Consequently, there exists $\delta \geq 1$ such that $\lim_{n \to \infty} d(gx_{n+1}, gx_n) = \delta$. Suppose that $\delta > 1$. Now

$$\psi\left(d\left(gx_{n+2}, gx_{n+1}\right)\right) \leq \frac{\psi\left(M_{f,g}\left(x_{n+1}, x_n\right)\right)}{\varphi\left(M_{f,g}\left(x_{n+1}, x_n\right)\right)}, \tag{37}$$

and lower semicontinuity of φ gives that

$$\limsup_{n \to \infty} \psi\left(d\left(gx_{n+2}, gx_{n+1}\right)\right)$$

$$\leq \frac{\limsup_{n \to \infty} \psi\left(M_{f,g}\left(x_{n+1}, x_{2n}\right)\right)}{\liminf_{n \to \infty} \varphi\left(M_{f,g}\left(x_{2n+1}, x_{2n}\right)\right)}, \tag{38}$$

which implies that $\psi(\delta) \leq \psi(\delta)/\varphi(\delta)$, a contradiction. Therefore $\delta = 1$. That is,

$$\lim_{n \to \infty} d\left(gx_{n+1}, gx_n\right) = 1. \tag{39}$$

Assume that $\{gx_n\}$ is not a multiplicative Cauchy sequence. Then, there is $\varepsilon > 1$, and there are even integers n_k and m_k with $m_k > n_k > k$ such that

$$d\left(gx_{m_k}, gx_{n_k}\right) \geq \varepsilon, \tag{40}$$

and $d(gx_{m_k-1}, gx_{n_k}) < \varepsilon$. Note that

$$\varepsilon \leq d\left(gx_{m_k}, gx_{n_k}\right)$$

$$\leq d\left(gx_{n_k}, gx_{m_k-1}\right) \cdot d\left(gx_{m_k-1}, gx_{m_k}\right). \tag{41}$$

From (39) and (40), it follows that

$$\lim_{k \to \infty} d\left(gx_{m_k}, gx_{n_k}\right) = \varepsilon. \tag{42}$$

By (42) and

$$d\left(gx_{n_k+1}, gx_{m_k}\right) \leq d\left(gx_{n_k+1}, gx_{n_k}\right)$$

$$\cdot d\left(gx_{n_k}, gx_{m_k}\right), \tag{43}$$

we have $\lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k}) \leq \varepsilon$. Also, by (42) and

$$d\left(gx_{m_k}, gx_{n_k}\right) \leq d\left(gx_{n_k+1}, gx_{n_k}\right) \cdot d\left(gx_{n_k+1}, gx_{m_k}\right) \tag{44}$$

we obtain that $\varepsilon \leq \lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k})$. Hence

$$\lim_{k \to \infty} d\left(gx_{n_k+1}, gx_{m_k}\right) = \varepsilon. \tag{45}$$

From (39) and

$$d\left(gx_{n_k+1}, gx_{m_k-1}\right) \leq d\left(gx_{n_k+1}, gx_{n_k}\right)$$

$$\cdot d\left(gx_{n_k}, gx_{m_k-1}\right), \tag{46}$$

we have $\lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k-1}) \leq \varepsilon$. By (45) and the inequality

$$d\left(gx_{n_k+1}, gx_{m_k}\right) \leq d\left(gx_{n_k+1}, gx_{m_k-1}\right)$$

$$\cdot d\left(gx_{m_k-1}, gx_{m_k}\right), \tag{47}$$

we have $\varepsilon \leq \lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k-1})$. Thus

$$\lim_{k \to \infty} d\left(gx_{n_k+1}, gx_{m_k-1}\right) = \varepsilon. \tag{48}$$

Note that

$$M_{f,g}\left(x_{n_k}, x_{m_k-1}\right) = \max \{d\left(fx_{n_k}, fx_{m_k-1}\right),$$

$$d\left(fx_{n_k}, gx_{n_k}\right), d\left(fx_{m_k-1}, gx_{m_k-1}\right),$$

$$(d\left(fx_{n_k}, gx_{m_k-1}\right) \cdot d\left(fx_{m_k-1}, gx_{n_k}\right))^{1/2}\}$$

$$= \max \{d\left(gx_{n_k+1}, gx_{m_k}\right), d\left(gx_{n_k+1}, gx_{n_k}\right),$$

$$d\left(gx_{m_k}, gx_{m_k-1}\right),$$

$$(d\left(gx_{n_k+1}, gx_{m_k-1}\right) \cdot d\left(gx_{m_k}, gx_{n_k}\right))^{1/2}\}. \tag{49}$$

Consequently, $\lim_{k \to \infty} M_{f,g}(x_{n_k}, x_{m_{k-1}}) = \max\{\varepsilon, 1, 1, \varepsilon\} = \varepsilon$. Since x_{n_k} and $x_{m_{k-1}}$ lie in different but adjacent labelled sets A_i and A_{i+1} for some $1 \le i \le p$, we have

$$\psi\left(d\left(gx_{n_k+1}, gx_{m_k}\right)\right) = \psi\left(d\left(fx_{n_k}, gx_{m_{k-1}}\right)\right)$$
$$\le \frac{\psi\left(M_{f,g}\left(x_{n_k}, x_{m_{k-1}}\right)\right)}{\varphi\left(M_{f,g}\left(x_{n_k}, x_{m_{k-1}}\right)\right)}. \tag{50}$$

Taking upper limit as $k \to \infty$ implies that $\psi(\varepsilon) \le \psi(\varepsilon)/\varphi(\varepsilon)$, a contradiction as $\varepsilon > 1$, and hence $\{gx_n\}$ is a multiplicative Cauchy sequence in $\cup_{i=1}^{p} g(A_i)$. Since $\cup_{i=1}^{p} g(A_i)$ is complete, there exists $y \in \cup_{i=1}^{p} g(A_i)$ such that

$$\lim_{k \to \infty} d\left(gx_n, y\right) = 1. \tag{51}$$

Consequently, we can find a point z in Y such that $g(z) = y$.

Now we show that $y \in \cap_{i=1}^{p} g(A_i) = Z$. From condition (a) and $x_0 \in A_{i_0}$ for some $i_0 \in \{1, 2, \ldots, p\}$, we can choose a subsequence $\{x_{n_k}\}$ in A_{i_0} out of the sequence $\{x_n\}$. Obviously, $\{gx_{n_k}\} \subseteq g(A_{i_0})$. As $g(A_{i_0})$ is closed, $y \in g(A_{i_0})$. Similarly, we can choose a subsequence $\{x_{n_k+1}\}$ in A_{i_0+1} out of the sequence $\{x_n\}$. Obviously, $\{gx_{n_k+1}\} \subseteq g(A_{i_0+1})$. As $g(A_{i_0+1})$ is closed, $y \in g(A_{i_0+1})$. Continuing this way, we obtain that $y \in \cap_{i=1}^{p} g(A_i)$ and hence $\cap_{i=1}^{p} g(A_i) \ne \emptyset$.

Now we show that $f(z) = g(z)$. Since $z \in Y$, there exists some i in $\{1, 2, \ldots, p\}$ such that $z \in A_i$. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$. From (b), we have

$$\psi\left(d\left(gx_{n_k+1}, fz\right)\right) = \psi\left(d\left(fx_{n_k}, fz\right)\right)$$
$$\le \frac{\psi\left(M_{f,g}\left(x_{n_k}, z\right)\right)}{\varphi\left(M_{f,g}\left(x_{n_k}, z\right)\right)}, \tag{52}$$

where

$$M_{f,g}\left(x_{n_k}, z\right) = \max\left\{d\left(gx_{n_k}, gz\right), d\left(fx_{n_k}, gx_{n_k}\right),\right.$$
$$\left. d\left(fz, gz\right), \left(d\left(fx_{n_k}, gz\right) \cdot d\left(fz, gx_{n_k}\right)\right)^{1/2}\right\}$$
$$= \max\left\{d\left(gx_{n_k}, gz\right), d\left(gx_{n_k+1}, gx_{n_k}\right), d\left(fz, gz\right),\right.$$
$$\left. \left(d\left(gx_{n_k+1}, gz\right) \cdot d\left(fz, gx_{n_k}\right)\right)^{1/2}\right\}. \tag{53}$$

On taking upper limit as $k \to \infty$ we obtain that

$$\psi\left(d\left(gz, fz\right)\right) \le \frac{\psi\left(d\left(fz, gz\right)\right)}{\varphi\left(d\left(fz, gz\right)\right)}, \tag{54}$$

and hence $f(z) = g(z)$. Thus z is the coincidence point of f and g.

Note that Z being a finite intersection of closed sets is closed and hence complete. Consider the restrictions of f and g on Z; that is, $f|_Z, g|_Z : Z \to Z$. Obviously, $f|_Z(Z) \subseteq g|_Z(Z) \subseteq Z$. Also $g|_Z(Z)$ is closed and hence complete. From Theorem 18, it follows that $f|_Z$ and $g|_Z$ have a unique

coincidence point in Z. As f and g are weakly compatible, from Proposition 12, it follows that $f|_Z$ and $g|_Z$ have a unique common fixed point. □

Corollary 20. *Let (X, d) be a multiplicative metric space, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, and $Y = \cup_{i=1}^{p} A_i$. Suppose that $f, g : Y \to Y$ are such that*

(a) *Y has a cocyclic representation with respect to pair (f^m, g^n) and the collection $\{A_i : i = 1, 2, \ldots, p\}$ for some $m, n \in \mathbb{N}$;*

(b) *there exists two control functions ψ and φ such that, for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$,*

$$\psi\left(d\left(f^m x, f^m y\right)\right) \le \frac{\psi\left(M_{f,g}\left(x, y\right)\right)}{\varphi\left(M_{f,g}\left(x, y\right)\right)} \tag{55}$$

holds for some $m, n \in \mathbb{N}$, where

$$M_{f,g}\left(x, y\right) = \max\left\{d\left(g^n x, g^n y\right), d\left(f^m x, g^n x\right),\right.$$
$$\left. d\left(f^m y, g^n y\right), \left(d\left(f^m x, g^n y\right) \cdot d\left(f^m y, g^n x\right)\right)^{1/2}\right\} \tag{56}$$

with $A_{p+1} = A_1$.

If $g^n(A_i)$ is complete subspace of X for each $i \in \{1, 2, \ldots, p\}$, then f^m and g^n have a unique coincidence point $z \in \cap_{i=1}^{p} g^n(A_i) = Z$ provided that $f^m(Z) \subseteq g^n(Z) \subseteq Z$ and $g^n(Z)$ is closed. Moreover, if f^m and g^n are weakly compatible, then f^m and g^n have a unique common fixed point. Furthermore, f and g have a unique common fixed point provided that f and g are commuting.

Proof. Set $S = f^m$ and $T = g^n$. From Theorem 19, S and T have a unique common fixed point u. Now $S(fu) = f(Su) = fu$ and $T(fu) = f(Tu) = fu$ imply that fu is the common fixed point of S and T. Also, $T(gu) = g(Tu) = gu$ and $S(gu) = g(Su) = gu$ imply that gu is also the common fixed point of S and T. By uniqueness of common fixed point of S and T, we have $fu = gu = u$. □

Example 21. Let $X = \mathbb{R}$ and let d be a multiplicative metric on X defined by $d(x, y) = a^{|x-y|}$, where $a > 1$ is a real number. For some $c > 1$, set $A_1 = [-c, 0]$, $A_2 = [0, c]$, and $A_3 = A_1$. Define $f, g : \cup_{i=1}^{2} A_i \to \cup_{i=1}^{2} A_i$ by

$$f(x) = \frac{\alpha x}{c},$$
$$g(x) = -\frac{\beta x}{c}, \tag{57}$$

where $\alpha, \beta \ge 1$ with $6\alpha \le 5\beta$ and $\beta \le c$. Note that $f(A_1) = [-\alpha, 0] \subseteq [-\beta, 0] = g(A_2)$ and $f(A_2) = [0, \alpha] \subseteq [0, \beta] = g(A_1)$. $Y = A_1 \cup A_2$ has a cocyclic representation with respect to pair (f, g) and the collection $\{A_1, A_2\}$.

Define $\psi, \varphi : [1, \infty) \to [1, \infty)$ by

$$\psi(t) = t^3,$$

$$\varphi(t) = \begin{cases} t^{1/2}, & \text{if } t \in [1, 2\beta], \\ t^2, & \text{if } t > 2\beta. \end{cases} \quad (58)$$

Clearly ψ is continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t) = \varphi(t) = 1$ if and only if $t = 1$.

We show that condition (b) is satisfied. Now, for $x \in A_1$, $y \in A_2$ implies

$$\psi(d(fx, fy)) = \psi\left(d\left(\frac{\alpha x}{c}, \frac{\alpha y}{c}\right)\right) = \psi\left(a^{(\alpha/c)(y-x)}\right)$$

$$= a^{(3\alpha/c)(y-x)} \le a^{(5\beta/2c)(y-x)}$$

$$= \left(d\left(-\frac{\beta x}{c}, -\frac{\beta y}{c}\right)\right)^{5/2}$$

$$= d(gx, gy)^{5/2} \le \left[M_{f,g}(x, y)\right]^{5/2} \quad (59)$$

$$= \frac{\left[M_{f,g}(x, y)\right]^3}{\left[M_{f,g}(x, y)\right]^{1/2}}$$

$$= \frac{\psi(M_{f,g}(x, y))}{\varphi(M_{f,g}(x, y))}.$$

When $x \in A_2$ and $y \in A_1$,

$$\psi(d(fx, fy)) = \psi\left(d\left(\frac{\alpha x}{c}, \frac{\alpha y}{c}\right)\right) = \psi\left(a^{(\alpha/c)(x-y)}\right)$$

$$= a^{(3\alpha/c)(x-y)} \le a^{(5\beta/2c)(x-y)}$$

$$= \left(d\left(-\frac{\beta x}{c}, -\frac{\beta y}{c}\right)\right)^{5/2}$$

$$= d(gx, gy)^{5/2} \le \left[M_{f,g}(x, y)\right]^{5/2} \quad (60)$$

$$= \frac{\left[M_{f,g}(x, y)\right]^3}{\left[M_{f,g}(x, y)\right]^{1/2}}$$

$$= \frac{\psi(M_{f,g}(x, y))}{\varphi(M_{f,g}(x, y))}.$$

Thus, f and g satisfy all the conditions of Theorem 19. Moreover, f and g have at most one common fixed point.

Example 22. Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}^+$ be the multiplicative metric defined by $d(x, y) = e^{|x-y|}$. Suppose

$A_1 = [-c, 0]$, $A_2 = [-c/2, c/2]$, and $A_3 = [-c/4, c]$ and $A_4 = A_1$, where $c > 0$. Define $f, g : \cup_{i=1}^3 A_i \to \cup_{i=1}^3 A_i$ by

$$fx = \begin{cases} -\dfrac{x}{c} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (61)$$

$$gx = \frac{2x}{c}.$$

Note that $f(A_1) = \{0\} \subseteq [-1, 1] = g(A_2)$, $f(A_2) = [-1/2, 0] \subseteq [-1/2, 2] = g(A_3)$, and $f(A_3) = [-1, 0] \subseteq [-2, 0] = g(A_1)$. Hence $Y = A_1 \cup A_2$ is a cocyclic representation between f and g.

Let $\psi, \varphi : [1, \infty) \to [1, \infty)$ be as defined in Example 21. To check condition (b), we consider the following cases:

(1) Let $x \in A_1$ and $y \in A_2$. If $y \in [-c/2, 0]$, then $d(fx, fy) = e^0 = 1$ and hence (b) holds. If $y \in (0, c/2]$, then we have

$$d(fx, fy) = d\left(0, -\frac{y}{c}\right) = e^{(1/c)y} \le e^{(5/c)(y-x)}$$

$$= d\left(\frac{2x}{c}, \frac{2y}{c}\right)^{5/2} = d(gx, gy)^{5/2}$$

$$\le \left[M_{f,g}(x, y)\right]^{5/2} = \frac{\left[M_{f,g}(x, y)\right]^3}{\left[M_{f,g}(x, y)\right]^{1/2}} \quad (62)$$

$$= \frac{\psi(M_{f,g}(x, y))}{\varphi(M_{f,g}(x, y))}.$$

(2) Let $x \in A_2$ and $y \in A_3$. If $x \in [-c/2, 0]$ and $y \in [-c/4, 0]$, then $d(fx, fy) = e^0 = 1$ and hence (b) is satisfied. If $x \in [-c/2, 0]$ and $y \in (0, c]$, then we obtain that

$$d(fx, fy) = d\left(0, -\frac{y}{c}\right) = e^{(1/c)y} < e^{(15/2c)y}$$

$$= d\left(-\frac{y}{c}, \frac{2y}{c}\right)^{5/2} = d(fy, gy)^{5/2}$$

$$\le \left[M_{f,g}(x, y)\right]^{5/2} = \frac{\left[M_{f,g}(x, y)\right]^3}{\left[M_{f,g}(x, y)\right]^{1/2}} \quad (63)$$

$$= \frac{\psi(M_{f,g}(x, y))}{\varphi(M_{f,g}(x, y))}.$$

When $x \in (0, c/2]$ and $y \in (-c/4, 0]$, we have

$$d(fx, fy) = d\left(-\frac{x}{c}, 0\right) = e^{(1/c)x} < e^{(15/2c)x}$$

$$= d\left(-\frac{x}{c}, \frac{2x}{c}\right)^{5/2} = d(fx, gx)^{5/2}$$

$$\leq \left[M_{f,g}(x,y)\right]^{5/2} = \frac{\left[M_{f,g}(x,y)\right]^3}{\left[M_{f,g}(x,y)\right]^{1/2}}$$

$$= \frac{\psi\left(M_{f,g}(x,y)\right)}{\varphi\left(M_{f,g}(x,y)\right)}. \tag{64}$$

For $x \in (0, c/2]$ and $y \in (0, c]$, we obtain that

$$d(fx, fy) = d\left(-\frac{x}{c}, -\frac{y}{c}\right) = e^{(1/c)|x-y|} < e^{(5/c)|x-y|}$$

$$= d\left(\frac{2x}{c}, \frac{2y}{c}\right)^{5/2} = d(gx, gy)^{5/2}$$

$$\leq \left[M_{f,g}(x,y)\right]^{5/2} = \frac{\left[M_{f,g}(x,y)\right]^3}{\left[M_{f,g}(x,y)\right]^{1/2}} \tag{65}$$

$$= \frac{\psi\left(M_{f,g}(x,y)\right)}{\varphi\left(M_{f,g}(x,y)\right)}.$$

(3) Let $x \in A_3$ and $y \in A_1$. If $x \in [-c/4, 0]$, then $d(fx, fy) = 1$ and (b) is satisfied. If $x \in (0, c]$, then we have

$$d(fx, fy) = d\left(-\frac{x}{c}, 0\right) = e^{(1/c)x} < e^{(15/2c)x}$$

$$= d\left(-\frac{x}{c}, \frac{2x}{c}\right)^{5/2} = d(fx, gx)^{5/2}$$

$$\leq \left[M_{f,g}(x,y)\right]^{5/2} = \frac{\left[M_{f,g}(x,y)\right]^3}{\left[M_{f,g}(x,y)\right]^{1/2}} \tag{66}$$

$$= \frac{\psi\left(M_{f,g}(x,y)\right)}{\varphi\left(M_{f,g}(x,y)\right)}.$$

Hence f and g satisfy all the conditions of Theorem 19. Moreover, 0 is a unique common fixed point of f and g in $\cap_{i=1}^{3} A_i$.

Denote $\Upsilon = \{\phi : [1, \infty) \rightarrow [1, \infty); \phi$ is a Lebesgue integrable with finite integral on each compact subset of $[1, \infty)$ and for each $\varepsilon > 1$, $\int_1^\varepsilon \varphi(t)dt > 1\}$.

As a consequence of Theorem 19, we obtain following common fixed point result for mappings satisfying cocyclic contractive condition of integral type in a multiplicative metric space.

Corollary 23. *Let (X, d) be a multiplicative metric space, $A_1, A_2, \ldots, A_p$, p nonempty closed subsets of X, and $Y = \cup_{i=1}^{p} A_i$. Suppose that $f, g : Y \rightarrow Y$ are mappings such that*

(a) *Y has a cocyclic representation with respect to pair (f, g) and the collection $\{A_i : i = 1, 2, \ldots, p\}$;*

(b) *there exists control functions ψ and φ such that, for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$, the following,*

$$\int_1^{\psi(d(fx, fy))} \phi(t)\,dt \leq \frac{\int_1^{\psi(M_{f,g}(x,y))} \phi(t)\,dt}{\int_1^{\varphi(M_{f,g}(x,y))} \phi(t)\,dt}, \tag{67}$$

is satisfied, where $\phi \in \Upsilon$, and

$$M_{f,g}(x, y) = \max \Big\{ d(gx, gy), d(fx, gx),$$
$$d(fy, gy), \left(d(fx, gy) \cdot d(fy, gx)\right)^{1/2} \Big\} \tag{68}$$

with $A_{p+1} = A_1$.

If $g(A_i)$ is a complete subspace of X for each $i \in \{1, 2, \ldots, p\}$, then f and g have a unique coincidence point $z \in \cap_{i=1}^{p} g(A_i) = Z$ provided that $f(Z) \subseteq g(Z) \subseteq Z$ and $g(Z)$ is closed. Moreover, if f and g are weakly compatible, then f and g have at most one common fixed point.

Proof. Define $\Psi : [1, \infty) \rightarrow [1, \infty)$ by $\Psi(x) = \int_1^x \phi(t)dt$. From (67), it follows that

$$\Psi\left(\psi\left(d(fx, fy)\right)\right) \leq \frac{\Psi\left(\psi\left(M_{f,g}(x,y)\right)\right)}{\Psi\left(\varphi\left(M_{f,g}(x,y)\right)\right)}, \tag{69}$$

which can be written as

$$\psi_1\left(d(fx, fy)\right) \leq \frac{\psi_1\left(M_{f,g}(x,y)\right)}{\varphi_1\left(M(x,y)\right)}, \tag{70}$$

where $\psi_1 = \Psi \circ \psi$ and $\varphi_1 = \Psi \circ \varphi$. Clearly, $\psi_1, \varphi_1 : [1, \infty) \rightarrow [1, \infty)$, ψ_1 is continuous and nondecreasing, φ_1 is a lower semicontinuous, and $\psi_1(t) = \varphi_1(t) = 1$ if and only if $t = 1$. Hence by Theorem 19, f and g have at most one common fixed point. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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