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Common Fixed Point Results for Four Mappings on Ordered Vector Metric Spaces

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Abstract. A vector metric space is a generalization of a metric space, where the metric is Riesz space valued. We prove some common fixed point theorems for four mappings in ordered vector metric spaces. Obtained results extend and generalize well-known comparable results in the literature.

1. Introduction and Preliminaries

Consistent with Altun and Cevik [6, 11], the following definitions and results will be needed in the sequel.

A relation $\leq$ on E is called: (*i*) *reflexive* if $\mathbf{x} \leq \mathbf{x}$ for all $\mathbf{x} \in E$ (*ii*) *transitive* if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{z}$ imply $\mathbf{x} \leq \mathbf{z}$ (*iii*) *antisymmetric* if $\mathbf{x} \leq y$ and $\mathbf{y} \leq \mathbf{x}$ imply $\mathbf{x} = \mathbf{y}$ (*iv*) *preorder* if it is reflexive and transitive. (*v*) *translation invariant* if $\mathbf{x} \leq \mathbf{y}$ implies $(\mathbf{x} + \mathbf{z}) \leq (\mathbf{y} + \mathbf{z})$ for any $\mathbf{z} \in E$ (*vi*) *scale invariant* if $\mathbf{x} \leq \mathbf{y}$ implies $(\lambda\mathbf{x}) \leq (\lambda\mathbf{y})$ for any $\lambda > 0$. A preorder $\leq$ is called *partial order or an order relation* if it is antisymmetric.

Given a partially ordered set $(E, \leq)$, that is, the set E equipped with a partial order $\leq$, the notation $x < y$ stands for $x \leq y$ and $x \neq y$. An order interval $[x, y]$ in E is the set $\{z \in E : x \leq z \leq y\}$.

A real linear space E equipped with an order relation $\leq$ on E which is compatible with the algebraic structure of E is called an ordered linear space or ordered vector space. The ordered vector space $(E, \leq)$ is called a *vector lattice* (or a Riesz space or linear lattice) if for every $\mathbf{x}, \mathbf{y} \in E$, there exist $\mathbf{x} \wedge \mathbf{y} = \inf\{\mathbf{x}, \mathbf{y}\}$ and $\mathbf{x} \vee \mathbf{y} = \sup\{\mathbf{x}, \mathbf{y}\}$. If we denote $\mathbf{x}^+ = \mathbf{0} \vee \mathbf{x}$, $\mathbf{x}^- = \mathbf{0} \vee (-\mathbf{x})$ and $|\mathbf{x}| = \mathbf{x} \vee (-\mathbf{x})$, then $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ and $|\mathbf{x}| = \mathbf{x}^+ + \mathbf{x}^-$. The cone $\{x \in E : x \geq 0\}$ of nonnegative elements in a Riesz space E is denoted by E_+. A sequence of vectors $\{x_n\}$ in E is said to decrease to an element $x \in E$ if $x_{n+1} \leq x_n$ for every n in $\mathbb{N}$ and $x = \inf\{x_n : n \in \mathbb{N}\} = \wedge_{n\in\mathbb{N}} x_n$. We denote it by $x_n \downarrow x$. A sequence of vectors $\{x_n\}$ in E is said to increase to an element $x \in E$ if $x_n \leq x_{n+1}$ for every n in $\mathbb{N}$ and $x = \sup\{x_n : n \in \mathbb{N}\} = \vee_{n\in\mathbb{N}} x_n$. We denote it by $x_n \uparrow x$. E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds for every $a \in E_+$. A sequence (b_n) is said to be order convergent or o-convergent to b if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n. We denote this by $b_n \to_o b$. Moreover, (b_n) is said to be o−Cauchy if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$ holds for all n

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and p. E is said to be $o-$ Cauchy complete if every $o-$Cauchy sequence is $o-$convergent. For notations and other facts regarding Riesz spaces we refer to [5].

We begin with some important definitions.

Definition 1.1. (*See [6, 11]*). *Let X be a nonempty set and E a Riesz space. A mapping $d : X \times X \to E$ is said to be a vector metric or E-metric if it satisfies the following conditions:*
(E_1) *$d(x, y) = 0$ if and only if $x = y$;*
(E_2) *$d(x, y) \leq d(x, z) + d(y, z)$;*
for all $x, y, z \in X$. We call (X, d, E) a vector metric space.

For arbitrary elements x, y, z and w of a vector metric space, the following holds true:

(Em_1) $0 \leq d(x, y)$;

(Em_2) $d(x, y) = d(y, x)$;

(Em_3) $|d(x, z) - d(y, z)| \leq d(x, y)$;

(Em_4) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$.

Example 1.2. (*See [6, 11]*). *A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is called to be absolute valued metric on E.*

Definition 1.3. (*See [6, 11]*).
(*i*) *A sequence (x_n) in a vector metric space (X, d, E) vectorial converges or E-converges to some $x \in E$ (we write $x_n \to^{d,E} x$), if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n;*
(*ii*) *A sequence (x_n) is called E-Cauchy sequence if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ holds for all n and p;*
(*iii*) *A vector metric space X is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.*

Lemma 1.4. (*See [6, 11]*). *We have following properties in vector metric space X:*
(*a*) *The limit x is unique;*
(*b*) *Every subsequence of (x_n) E-converges to x;*
(*c*) *If $x_n \to^{d,E} x$ and $y_n \to^{d,E} y$, then $d(x_n, y_n) \to_o d(x, y)$.*

Lemma 1.5. (*See [6]*). *If E is a Riesz space and $a \leq ka$ where $a \in E_+$ and $k \in [0, 1)$, then $a = 0$.*

Remark 1.6. (*See [6, 11]*).
(*i*) *The difference between vector metric and Zabrejko's metric defined in [38] is that the Riesz space has also a lattice structure;*
(*ii*) *One of the differences between vector metric and Huang-Zhang's metric given in [17] is that there exists a cone due to the natural existence of ordering on Riesz space. The other difference is that vector metric omits the requirement for the vector space to be a Banach space;*
(*iii*) *Set $E = \mathbb{R}$, the concepts of vectorial convergence and convergence in metric coincide. If $X = E$ and d is absolute valued vector metric, then vectorial convergence and convergence in order are same. In the case set $E = \mathbb{R}$, the concepts of E-Cauchy sequence and Cauchy sequence are the same.*

For more details on fixed point theorems in cone metric spaces, we refer to [17, 19, 30, 32, 34] and references contained therein.

Definition 1.7. (*See [24]*). *Let $f, g : X \to X$ be mappings of a set X. If $fw = gw = z$ for some $z \in X$, then w is called a coincidence point of f and g, and z is called a point of coincidence of f and g.*

Sessa [36] defined the concept of weakly commuting to obtain common fixed point for a pairs of maps. Jungck generalized the idea of commuting mappings, first to compatible mappings [22] and then to weakly compatible mappings [23]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition.

Definition 1.8. (*See [24]*). *Let $f, g : X \to X$ be mappings of a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.*

Lemma 1.9. (*See [2]*). *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $z = fw = gw$, then z is the unique common fixed point of f and g.*

The study of unique common fixed points of mappings satisfying strict contractive conditions has been at the center of vigorous research activity. For work in this direction, we refer to [2, 4, 8, 14, 24, 30, 31] and references contained therein.

Existence of fixed points in ordered metric spaces has been introduced and applied by Ran and Reurings [33]. Recently, many researchers have obtained fixed point and common fixed point results in partially ordered metric space (see, e.g., [3], [9], [13], [16], [25], [28] and [35]). The aim of this paper is to initiate study of common fixed point of four mappings in the frame work of ordered vector metric spaces.

2. Main Results

Let X be any nonempty set and $f, g, S, T : X \to X$ four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$.

Let x_0 be an arbitrary point of X. Choose $x_1 \in X$ such that $fx_0 = Tx_1$, $x_2 \in X$ such that $gx_1 = Sx_2$. This can be done as $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$.

Continuing this way, construct a sequence $\{y_n\}$ defined by: $y_{2n-1} = Tx_{2n-1} = fx_{2n-2}$, and $y_{2n} = Sx_{2n} = gx_{2n-1}$, for all $n \geq 0$. The sequence $\{y_n\}$ in X is said to be a Jungck type iterative sequence with initial guess x_0.

Definition 2.1. *(See [3]). Let $(X, \preceq)$ be a partially ordered set. A pair (f, g) of self-maps of X is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.*

Now we give a definition of partially weakly increasing pair of mappings.

Definition 2.2. *Let $(X, \preceq)$ be a partially ordered set and f and g be two self-maps on X. An ordered pair (f, g) is said to be partially weakly increasing if $fx \preceq gfx$ for all $x \in X$.*

Note that a pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing.

For an example of an ordered pair (f, g) of self-maps f and g which is partially weakly increasing but not weakly increasing, we refer to [3].

Definition 2.3. *Let $(X, \preceq)$ be a partially ordered set. A mapping f is called a weak annihilator of g if $fgx \preceq x$ for all $x \in X$.*

Example 2.4. *Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be defined by $fx = x^2$, $gx = x^3$. Obviously, $fgx = x^6 \preceq x$ for all $x \in X$. Thus f is a weak annihilator of g.*

Definition 2.5. *Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.*

Example 2.6. *Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^{\frac{1}{3}}$. Since $x \preceq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore f is a dominating map.*

Example 2.7. *Let $X = [0, \infty)$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = \sqrt[n]{x}$ for $x \in [0, 1)$ and $fx = x^n$ for $x \in [1, \infty)$, for any $n \in \mathbb{N}$. Clearly, for every x in X we have $x \preceq fx$.*

Note that it is not hard to find four mappings to satisfy all of above definitions (see [1, 12, 29, 37]).

The following theorem is ordered vector metric version of Theorem 2.1 of [21] and Theorem 2.2 of [4].

Theorem 2.8. *Let $(X, \preceq)$ be a partially ordered set such that there exists a E-metric on X with E be an Archimedean. Let f, g, S and T be self maps on X, (T, f) and (S, g) partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, dominating maps f and g are weak annihilators of T and S, respectively. Suppose the mappings $f, g, S, T : X \to X$ satisfy the following condition:*

$$d(fx, gy) \preceq ku_{x,y}(f, g, S, T) \tag{1}$$

for all comparable elements $x, y \in X$, where $k \in [0, 1)$ is a constant and

$$u_{x,y}(f, g, S, T) \in \Big\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\Big\}.$$

If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to^{d,E} u$ implies that $x_n \preceq u$ and the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a common fixed point provided that one of $f(X)$, $g(X)$, $S(X)$, or $T(X)$ is a E-complete subspace of X. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S and T have one and only one common fixed point.

Proof. Suppose x_0 is an arbitrary point of X. Construct Jungck type iterative sequence $\{y_n\}$ in X with initial guess x_0. This can be done because $f(X) \subseteq T(X)$, and $g(X) \subseteq S(X)$. By given assumptions $x_{2n-2} \preceq fx_{2n-2} = Tx_{2n-1} \preceq fTx_{2n-1} \preceq x_{2n-1}$, and $x_{2n-1} \preceq gx_{2n-1} = Sx_{2n} \preceq gSx_{2n} \preceq x_{2n}$. Thus, for all n we have $x_n \preceq x_{n+1}$. We first show that

$$d(y_{2n+1}, y_{2n+2}) \preceq kd(y_{2n}, y_{2n+1}) \tag{2}$$

for all n. From (1), we have

$$d(y_{2n+1}, y_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \preceq ku_{x_{2n},x_{2n+1}}(f, g, S, T)$$

for $n = 1, 2, \cdots$, where

$$u_{x_{2n},x_{2n+1}}(f, g, S, T) \in \Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}\Big\}.$$

If $u_{x_{2n},x_{2n+1}}(f, g, S, T) = d(y_{2n}, y_{2n+1})$, then clearly (2) holds. If $u_{x_{2n},x_{2n+1}}(f, g, S, T) = d(y_{2n+1}, y_{2n+2})$, then according to Lemma 1.5, $d(y_{2n+1}, y_{2n+2}) = 0$ and clearly (2) holds. Finally, suppose that

$$u_{x_{2n},x_{2n+1}}(f, g, S, T) = \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}.$$

Then

$$d(y_{2n+1}, y_{2n+2}) \preceq \frac{k}{2}d(y_{2n}, y_{2n+1}) + \frac{1}{2}d(y_{2n+1}, y_{2n+2})$$

holds. Similarly, we have

$$d(y_{2n+2}, y_{2n+3}) \preceq kd(y_{2n+1}, y_{2n+2}). \tag{3}$$

Therefore, from (2) and (3), we get

$$d(y_n, y_{n+1}) \preceq k^n d(y_0, y_1).$$

Now, for all n and p, we have

$$\begin{aligned} d(y_n, y_{n+p}) &\preceq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ &\preceq (k^n + k^{n+1} + \cdots + k^{n+p-1})d(y_0, y_1) \\ &\preceq \frac{k^n}{1-k}d(y_0, y_1). \end{aligned}$$

Since E is Archimedean then $\{y_n\}$ is an E-Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists a v in $S(X)$, such that $Sx_{2n} = y_{2n} \to^{d,E} v$. Hence there exists a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(Sx_{2n}, v) \leq a_n$. On the other hand, we can find a w in X such that $Sw = v$. Now, we show that $fw = v$. Since $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \to^{d,E} v$ implies that $x_{2n+1} \preceq v$ and $v \preceq gv = gSw \preceq w$ implies that $x_{2n+1} \preceq w$. Consider

$$d(fw, v) \leq d(fw, gx_{2n+1}) + d(gx_{2n+1}, v) \leq ku_{w,x_{2n+1}}(f, g, S, T) + a_{n+1},$$

where

$$u_{w,x_{2n+1}}(f, g, S, T) \in \left\{d(Sw, Tx_{2n+1}), d(fw, Sw), d(gx_{2n+1}, Tx_{2n+1}), \frac{d(fw, Tx_{2n+1}) + d(gx_{2n+1}, Sw)}{2}\right\}$$

for all n. There are four possibilities.

Case 1.

$$d(fw, v) \leq d(Sw, Tx_{2n+1}) + a_{n+1} \leq a_{n+1} + a_{n+1} \leq 2a_n.$$

Case 2.

$$d(fw, v) \leq kd(fw, Sw) + a_{n+1} \leq kd(fw, v) + a_n.$$

Thus $d(fw, v) \leq \frac{1}{1-k} a_n$.

Case 3.

$$d(fw, v) \leq d(gx_{2n+1}, Tx_{2n+1}) + a_{n+1} \leq 2a_{n+1} + a_{n+1} \leq 3a_n.$$

Case 4.

$$\begin{aligned} d(fw, v) &\leq \frac{d(fw, Tx_{2n+1}) + d(gx_{2n+1}, Sw)}{2} + a_{n+1} \\ &\leq \frac{1}{2} d(fw, v) + 2a_n. \end{aligned}$$

Thus $d(fw, v) \leq 4a_n$.

Since the infimum of sequences on the right side of last inequality are zero, then $d(fw, v) = 0$, that is, $fw = v$. Therefore, $fw = Sw = v$.

Since $v \in f(X) \subseteq T(X)$, there exists a $z \in X$ such that $Tz = v$. Now, we show that $gz = v$. As $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to^{d,E} v$ implies that $x_{2n} \preceq v$ and $v \preceq fv = fTz \preceq z$ implies that $x_{2n} \preceq z$. Consider

$$d(v, gz) \leq d(v, fx_{2n}) + d(fx_{2n}, gz) \leq a_n + ku_{x_{2n},z}(f, g, S, T),$$

where

$$u_{x_{2n},z}(f, g, S, T) \in \left\{d(Sx_{2n}, Tz), d(fx_{2n}, Sx_{2n}), d(gz, Tz), \frac{d(fx_{2n}, Tz) + d(gz, Sx_{2n})}{2}\right\}$$

for all n. There are four possibilities.

Case 1.

$$d(v, gz) \leq a_n + d(Sx_{2n}, Tz) \leq a_n + a_{n+1} \leq 2a_n.$$

Case 2.

$$d(v, gz) \leq a_n + d(fx_{2n}, Sx_{2n}) \leq a_n + 2a_n \leq 3a_n.$$

Case 3.

$$d(v, gz) \leq a_n + kd(gz, Tz) \leq a_n + kd(gz, v).$$

Thus $d(v, gz) \leq \frac{1}{1-k} a_n$.

Case 4.

$$\begin{aligned} d(v, gz) &\leq a_n + \frac{d(fx_{2n}, Tz) + d(gz, Sx_{2n})}{2} \\ &\leq 2a_n + \frac{1}{2} d(v, gz). \end{aligned}$$

Thus $d(v, gz) \leq 4a_n$.

Since the infimum of sequences on the right side of last inequality are zero, then $d(v, gz) = 0$, that is, $gz = v$. Therefore, $gz = Tz = v$. Thus $\{f, S\}$ and $\{g, T\}$ have a common point of coincidence in X. Now, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, $fv = fSw = Sfw = Sv = v_1$ (say) and $gv = gTz = Tgz = Tv = v_2$ (say). Now

$$d(v_1, v_2) = d(fv, gv) \leq ku_{v,v}(f, g, S, T),$$

where

$$\begin{aligned} u_{v,v}(f, g, S, T) \in &\Big\{d(Sv, Tv), d(fv, Sv), d(gv, Tv), \frac{d(fv, Tv) + d(gv, Sv)}{2}\Big\} \\ &\Big\{0, d(v_1, v_2)\Big\}. \end{aligned}$$

Hence, $d(v_1, v_2) = 0$, that is, $v_1 = v_2$. If $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then v is a unique fixed point of f, g, S and T by Lemma 1.9. The proofs for the cases in which $g(X)$, $S(X)$ or $T(X)$ is complete are similar. Conversely, if f, g, S and T have a unique common fixed point, then the set of common fixed point of f, g, S and T being singleton is well ordered. □

Corollary 2.9. *Let $(X, \preceq)$ be a partially ordered set such that there exists a E-metric on X with E be an Archimedean. Let f and g be dominating self maps on X satisfy the following condition:*

$$d(fx, gy) \leq ku_{x,y}(f, g) \tag{4}$$

for all comparable elements $x, y \in X$, where $k \in [0, 1)$ is a constant and

$$u_{x,y}(f, g) \in \Big\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}[d(fx, y) + d(y, gx)]\Big\}.$$

If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to^{d,E} u$ implies that $x_n \preceq u$, then f and g have a common fixed point provided that one of $f(X)$ or $g(X)$ is a E-complete subspace of X. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Theorem 2.10. *If we replace the condition of weak compatibility of pairs $\{f, S\}$ and $\{g, T\}$ in Theorem 2.8 by the following condition: either*
(i) $\{f, S\}$ are compatible, f or S is continuous and $\{g, T\}$ are weakly compatible; or
(ii) $\{g, T\}$ are compatible, g or T is continuous and $\{f, S\}$ are weakly compatible,
then the conclusions of Theorem 2.8 remain valid.

Proof. Following the similar arguments to those in the proof of Theorem 2.8, $\{y_n\}$ is an E-Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists a v in $S(X)$, such that $Sx_{2n} = y_{2n} \to^{d,E} v$. Hence there exists a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(Sx_{2n}, v) \leq a_n$.

Assume that S is continuous. As $\{f, S\}$ are compatible, so we have $\lim_{n\to\infty} fSx_{2n} = \lim_{n\to\infty} Sfx_{2n} = Sv$. Also, $x_{2n+1} \preceq gx_{2n+1} = Sx_{2n+2}$. Now from (1), we have

$$d(fSx_{2n+2}, gx_{2n+1}) \leq ku_{Sx_{2n+2}, x_{2n+1}}(f, g, S, T),$$

where

$$u_{Sx_{2n+2},x_{2n+1}}(f,g,S,T) \in \{d(SSx_{2n+2},Tx_{2n+1}), d(fSx_{2n+2},SSx_{2n+2}), d(gx_{2n+1},Tx_{2n+1}), \frac{d(fSx_{2n+2},Tx_{2n+1})+d(gx_{2n+1},SSx_{2n+2})}{2}\}.$$

On taking limit as $n \to \infty$, we obtain $d(Sv,v) \leq ku_{v,x_{2n+1}}(f,g,S,T)$, where

$$u_{v,x_{2n+1}}(f,g,S,T) \in \{d(Sv,v), d(Sv,Sv), d(v,v), \frac{d(Sv,v)+d(v,Sv)}{2}\} = \{d(Sv,v), 0\}.$$

Now if $u_{v,x_{2n+1}}(f,g,S,T) = d(Sv,v)$, then $d(Sv,v) \leq kd(Sv,v)$, and $Sv = v$. Also if $u_{v,x_{2n+1}}(f,g,S,T) = 0$ then we obtain $d(Sv,v) \leq k(0)$ and $Sv = v$.

Now, since $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \to^{d,E} v$ as $n \to \infty$, $x_{2n+1} \preceq v$ and (1) becomes $d(fv, gx_{2n+1}) \leq ku_{v,x_{2n+1}}(f,g,S,T)$, where

$$u_{v,x_{2n+1}}(f,g,S,T) \in \{d(Sv,Tx_{2n+1}), d(fv,Sv), d(gx_{2n+1},Tx_{2n+1}), \frac{d(fv,Tx_{2n+1})+d(gx_{2n+1},Sv)}{2}\}.$$

On taking limit as $n \to \infty$, we have $d(fv,v) \leq ku_{v,x_{2n+1}}(f,g,S,T)$, where

$$u_{v,x_{2n+1}}(f,g,S,T) \in \{d(Sv,v), d(fv,Sv), d(v,v), \frac{d(fv,v)+d(v,Sv)}{2}\} = \{0, d(fv,v), \frac{d(fv,v)}{2}\}.$$

Now if $u_{v,x_{2n+1}}(f,g,S,T) = 0$, then $d(fv,v) \leq k(0)$ implies that $fv = v$. Also for $u_{v,x_{2n+1}}(f,g,S,T) = d(fv,v)$, we have $d(fv,v) \leq kd(fv,v)$ which implies that $fv = v$. Finally, If $u_{v,x_{2n+1}}(f,g,S,T) = d(fv,v)/2$, then $d(fv,v) \leq (k/2)d(fv,v)$ gives that $fv = v$. Hence we have $fv = Sv = v$.

As $v \in f(X) \subseteq T(X)$, there exists a $z \in X$ such that $Tz = v$. Now, we show that $gz = v$. As $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to^{d,E} v$ implies that $x_{2n} \preceq v$ and $v = fv = fTz \preceq z$ implies that $x_{2n} \preceq z$. Consider

$$d(v,gz) \leq d(v,fx_{2n}) + d(fx_{2n},gz) \leq a_n + ku_{x_{2n},z}(f,g,S,T),$$

where

$$u_{x_{2n},z}(f,g,S,T) \in \left\{d(Sx_{2n},Tz), d(fx_{2n},Sx_{2n}), d(gz,Tz), \frac{d(fx_{2n},Tz)+d(gz,Sx_{2n})}{2}\right\}$$

for all n. There are four possibilities.

Case 1.

$$d(v,gz) \leq a_n + d(Sx_{2n},Tz) \leq a_n + a_{n+1} \leq 2a_n.$$

Case 2.

$$d(v,gz) \leq a_n + d(fx_{2n},Sx_{2n}) \leq a_n + 2a_n \leq 3a_n.$$

Case 3.

$$d(v,gz) \leq a_n + kd(gz,Tz) \leq a_n + kd(gz,v).$$

Thus $d(v,gz) \leq \frac{1}{1-k}a_n$.

Case 4.

$$\begin{aligned} d(v, gz) &\leq a_n + \frac{d(fx_{2n}, Tz) + d(gz, Sx_{2n})}{2} \\ &\leq 2a_n + \frac{1}{2}d(v, gz). \end{aligned}$$

Thus $d(v, gz) \leq 4a_n$.
Since the infimum of sequences on the right side of last inequality are zero, $d(v, gz) = 0$, that is, $gz = v$. Therefore, $gz = Tz = v$. Thus v is the coincidence point of pair $\{g, T\}$ in X. Since $\{g, T\}$ are weakly compatible, therefore $gv = gTz = Tgz = Tv$. Now

$$d(v, gv) = d(fv, gv) \leq ku_{v,v}(f, g, S, T),$$

where

$$\begin{aligned} u_{v,v}(f, g, S, T) &\in \{d(Sv, Tv), d(fv, Sv), d(gv, Tv), \frac{d(fv, Tv) + d(gv, Sv)}{2}\} \\ &= \{d(v, gv), 0, d(gv, gv), \frac{d(v, gv) + d(gv, v)}{2}\} \\ &= \{d(v, gv), 0\}. \end{aligned}$$

If $u_{v,v}(f, g, S, T) = d(v, gv)$, then $d(v, gv) \leq kd(v, gv)$ implies $v = gv$. Also if $u_{v,v}(f, g, S, T) = 0$, then $d(v, gv) = 0$ implies that $v = gv$. Therefore v is the common fixed point of f, g, S and T.

The proofs for the other cases are similar. □

Example 2.11. *Let* $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $X = [0, \infty)$ *and* $d : X \times X \to E$ *defined by* $d(x, y)(t) = \left(|x - y|\right) e^t$, *where* $e^t \in E$. *Then* (X, d) *is a E-metric space. We consider a ordering* $\preceq$ *on* X *defined as*

$$x \preceq y \Longleftrightarrow y \leq x \text{ for all } x, y \in X.$$

Consider four mappings $f, g, T, S : X \to X$ *defined by*

$$fx = \frac{3x}{5}, \quad gx = \frac{2x}{5}, \quad Tx = \frac{5x}{3}, \quad Sx = \frac{5x}{2}, \text{ for all } x \in X.$$

Clearly, $f(X) \subseteq T(X)$ *and* $g(X) \subseteq S(X)$. *Also, the pairs* (T, f) *and* (S, g) *partially weakly increasing, that is,* $Tx = \frac{5x}{3} \geq x = fTx$, *which gives* $Tx \preceq fTx$ *and* $Sx = \frac{5x}{2} \geq x = gSx$, *which gives* $Sx \preceq gSx$. *Also,* f *and* g *are dominating maps, that is,* $fx = \frac{3x}{5} \leq x$ *and* $gx = \frac{2x}{5} \leq x$ *for all* $x \in X$ *implies that* $x \preceq fx$ *and* $x \preceq gx$ *for all* $x \in X$. *Furthermore,* f *and* g *are weak annihilators of* T *and* S, *respectively, that is,* $fTx \preceq x$ *and* $gSx \preceq x$ *for all* $x \in X$.

Now, for all $x, y \in X$,

$$d(fx, gy)(t) = \left(\left|\frac{3x}{5} - \frac{2y}{5}\right|\right) e^t = \frac{1}{5}\left(|3x - 2y|\right) e^t,$$

$$d(Sx, Ty)(t) = \left(\left|\frac{5x}{2} - \frac{5y}{3}\right|\right) e^t,$$

$$d(fx, Sx)(t) = \left(\left|\frac{3x}{5} - \frac{5x}{2}\right| e^t\right) = (\frac{19x}{10})e^t,$$

$$d(gy,Ty)(t) = \left(\left|\frac{2y}{5} - \frac{5y}{3}\right| e^t\right) = (\frac{19y}{15})e^t,$$

$$d(fx,Ty) + d(gy,Sx)(t) = \left(\left|\frac{3x}{5} - \frac{5y}{3}\right| + \left|\frac{2y}{5} - \frac{5x}{2}\right|\right) e^t.$$

If $x \geq y$, then

$$\begin{aligned} d(fx,gy)(t) &= \frac{1}{5}\left(\left|3x - 2y\right|\right) e^t \\ &\leq (\frac{3}{5}x)e^t \\ &\leq k(\frac{19x}{10})e^t \\ &= kd(fx,Sx)(t) \\ &= ku_{x,y}(f,g,S,T). \end{aligned}$$

And if $x \leq y$, then

$$\begin{aligned} d(fx,gy)(t) &= \frac{1}{5}\left(\left|3x - 2y\right|\right) e^t \\ &\leq \frac{2}{5}ye^t \\ &\leq k(\frac{19y}{15})e^t \\ &= kd(gy,Ty)(t) \\ &= ku_{x,y}(f,g,S,T). \end{aligned}$$

Thus all the conditions of Theorem 2.8 *are satisfied with $k = \dfrac{3}{4} \in [0,1)$. Note that* 0 *is the unique common fixed point of the mappings f, g, S and T.*

The following corollary extends well known Fisher's result [15] to ordered vector metric spaces with E is Archimedean.

Corollary 2.12. *Let $(X,\leq)$ be a partially ordered set such that there exists a E-metric on X with E be an Archimedean. Let f, g, S and T be self-maps on X, (T,f) and (S,g) partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, dominating maps f and g are weak annihilators of T and S, respectively. Suppose the mappings $f,g,S,T : X \to X$ satisfy*

$$d(fx,gy) \leq kd(Sx,Ty)$$

for all comparable elements $x, y \in X$. If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to^{d,E} u$ implies that $x_n \preceq u$ and the pairs of mappings $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then f, g, S and T have a common fixed point provided that one of $f(X)$, $g(X)$, $S(X)$, or $T(X)$ is a E-complete subspace of X. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S and T have one and only one common fixed point.

The following result is obtained from Theorem 2.8.

Corollary 2.13. *Let $(X,\leq)$ be a partially ordered set such that there exists a E-metric on X with E be an Archimedean. Let f, g, S and T be self-maps on X, (T,f) and (S,g) partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, dominating maps f and g are weak annihilators of T and S, respectively. Suppose the mappings $f,g,S,T : X \to X$*

satisfy the following condition

$$d(f^m x, g^n y) \leq k u_{x,y}(f^m, g^n, S^m, T^n)$$

for all comparable elements $x, y \in X$, *and some* $m, n \in \mathbb{N}$, *where* $k \in [0, 1)$ *is a constant and*

$$u_{x,y}(f^m, g^n, S^m, T^n) \in \{d(S^m x, T^n y), d(f^m x, S^m x), d(g^n y, T^n y), \frac{1}{2}[d(f^m x, T^n y) + d(g^n y, S^m x)]\}.$$

If for a nondecreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to^{d,E} u$ *implies that* $x_n \preceq u$ *and the pairs of mappings* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible, then* f, g, S *and* T *have a common fixed point provided that one of* $f(X)$, $g(X)$, $S(X)$, *or* $T(X)$ *is a E-complete subspace of X. Moreover, the set of common fixed points of* f, g, S *and* T *is well ordered if and only if* f, g, S *and* T *have one and only one common fixed point.*

The following theorem is ordered vector metric version of Theorem 2.8 of [4].

Theorem 2.14. *Let* $(X, \preceq)$ *be a partially ordered set such that there exists a E-metric on X with E be an Archimedean. Let* f, g, S *and* T *be self maps on X,* (T, f) *and* (S, g) *partially weakly increasing with* $f(X) \subseteq T(X)$ *and* $g(X) \subseteq S(X)$, *dominating maps* f *and* g *are weak annihilators of* T *and* S, *respectively. Suppose the mappings* $f, g, S, T : X \to X$ *satisfy the following condition*

$$d(fx, gy) \leq k_1 d(Sx, Ty) + k_2 d(fx, Sx) + k_3 d(gy, Ty) + k_4 d(fx, Ty) + k_5 d(gy, Sx) \tag{5}$$

for all comparable elements $x, y \in X$, *where* k_i *for* $i = 1, 2, \cdots, 5$ *are nonnegative constants with*

$$k_1 + k_2 + k_3 + 2\max\{k_4, k_5\} < 1.$$

If for a nondecreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to^{d,E} u$ *implies that* $x_n \preceq u$ *and the pairs of mappings* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible, then* f, g, S *and* T *have a common fixed point provided that one of* $f(X)$, $g(X)$, $S(X)$, *or* $T(X)$ *is a E-complete subspace of X. Moreover, the set of common fixed points of* f, g, S *and* T *is well ordered if and only if* f, g, S *and* T *have one and only one common fixed point.*

Proof. We define sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.8. From (5), we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq k_1 d(y_{2n}, y_{2n+1}) + k_2 d(y_{2n+1}, y_{2n}) + k_3 d(y_{2n+2}, y_{2n+1}) \\ &\quad + k_4 d(y_{2n+1}, y_{2n+1}) + k_5 d(y_{2n+2}, y_{2n}). \end{aligned}$$

Consequently,

$$d(y_{2n+1}, y_{2n+2}) \leq \alpha d(y_{2n}, y_{2n+1}) \tag{6}$$

where $\alpha = \frac{k_1+k_2+k_5}{1-k_3-k_5} < 1$. Similarly,

$$\begin{aligned} d(y_{2n+3}, y_{2n+2}) &= d(fx_{2n+2}, gx_{2n+1}) \\ &\leq k_1 d(y_{2n+2}, y_{2n+1}) + k_2 d(y_{2n+3}, y_{2n+2}) + k_3 d(y_{2n+2}, y_{2n+1}) \\ &\quad + k_4 d(y_{2n+3}, y_{2n+1}) + k_5 d(y_{2n+2}, y_{2n+2}). \end{aligned}$$

Consequently,

$$d(y_{2n+3}, y_{2n+2}) \leq \alpha d(y_{2n+2}, y_{2n+1}) \tag{7}$$

where $\alpha = \frac{k_1+k_3+k_4}{1-k_2-k_4} < 1$. From (6) and (7), we have

$$d(y_n, y_{n+1}) \leq \alpha^n d(y_0, y_1).$$

By the same arguments as in Theorem 2.8 we conclude that $\{y_n\}$ is a E-Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists a v in $S(X)$, such that $Sx_{2n} = y_{2n} \to^{d,E} v$. Hence there exists a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(Sx_{2n}, v) \preceq a_n$. On the other hand, we can find a w in X such that $Sw = v$. Now, we show that $fw = v$. As $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \to^{d,E} v$ implies that $x_{2n+1} \preceq v$ and $v \preceq gv = gSw \preceq w$ implies that $x_{2n+1} \preceq w$. Using (5), we get

$$\begin{aligned} d(fw, v) &\preceq d(fw, gx_{2n+1}) + d(gx_{2n+1}, v) \\ &\preceq k_1 d(Sw, Tx_{2n+1}) + k_2 d(fw, Sw) + k_3 d(gx_{2n+1}, Tx_{2n+1}) + k_4 d(fw, Tx_{2n+1}) \\ &\quad + k_5 d(gx_{2n+1}, Sw) + d(gx_{2n+1}, v) \\ &\preceq (k_1 + k_3 + k_4) d(v, Tx_{2n+1}) + (k_2 + k_4) d(fw, v) + (k_3 + k_5 + 1) d(gx_{2n+1}, v). \end{aligned}$$

Consequently,

$$d(fw, v) \preceq \frac{k_1 + 2k_3 + k_4 + k_5 + 1}{1 - k_2 - k_4} a_n,$$

for all n. Thus $d(fw, v) = 0$, i.e. $fw = v$. Since $v \in f(X) \subseteq T(X)$, there exists a $z \in X$ such that $Tz = v$. Now, we show that $gz = v$. As $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to v$ implies that $x_{2n} \preceq v$ and $v \preceq fv = fTz \preceq z$ implies that $x_{2n} \preceq z$. Now

$$\begin{aligned} d(v, gz) &\preceq d(v, fx_{2n}) + d(fx_{2n}, gz) \\ &\preceq d(v, fx_{2n}) + k_1 d(Sx_{2n}, Tz) + k_2 d(fx_{2n}, Sx_{2n}) + k_3 d(gz, Tz) \\ &\quad + k_4 d(fx_{2n}, Tz) + k_5 d(gz, Sx_{2n}) \\ &\preceq (k_1 + k_2 + k_5) d(v, Sx_{2n}) + (k_3 + k_5) d(gz, v) + (k_2 + k_4 + 1) d(fx_{2n}, v). \end{aligned}$$

Consequently,

$$d(gz, v) \preceq \frac{k_1 + 2k_2 + k_4 + k_5 + 1}{1 - k_3 - k_5} a_n,$$

for all n. Thus $d(gz, v) = 0$, i.e. $gz = v$. Now, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, $fv = fSw = Sfw = Sv = v_1$ (say) and $gv = gTz = Tgz = Tv = v_2$ (say). From (5), we get

$$\begin{aligned} d(v_1, v_2) &= d(fv, gv) \\ &\preceq k_1 d(Sv, Tv) + k_2 d(fv, Sv) + k_3 d(gv, Tv) + k_4 d(fv, Tv) + k_5 d(gv, Sv) \\ &= (k_1 + k_4 + k_5) d(v_1, v_2). \end{aligned}$$

which implies that $d(v_1, v_2) = 0$ by Lemma 1.5. Thus $v_1 = v_2$. If $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then v is a unique fixed point of f, g, S and T by Lemma 1.9. The proofs for the cases in which $g(X)$, $S(X)$ or $T(X)$ is complete are similar. Conversely, if f, g, S and T have a unique common fixed point, then the set of common fixed point of f, g, S and T being singleton is well ordered. □

Example 2.15. *Let $E = \mathbb{R}$, $X = [0, \infty)$ equipped with absolute valued metric $d(x, y) = |x - y|$ for $x, y \in X$ and $\leq$ be usual ordering on $E = \mathbb{R}$. Now, consider a new ordering $\preceq$ on X as follows:*

$$x \preceq y \Longleftrightarrow y \leq x, \qquad \forall x, y \in X.$$

Let $f, g, T, S : X \to X$ be define by $f(x) = g(x) = \ln(x + 1)$ and $T(x) = S(x) = e^x - 1$. Since we have $1 + x \leq e^x$ for each $x \in X$ so $f(x) = g(x) = \ln(x + 1) \leq x$, which implies that $x \preceq f(x)$ and $x \preceq g(x)$. Thus f and g are dominating maps. Also, we have $fT(x) = gS(x) = \ln(e^x) = x \geq x$ for each $x \in X$, which implies that $fTx \preceq x$ and $gSx \preceq (x)$. Thus f and g are weak annihilators of T and S. Moreover, we have $S(x) = T(x) = e^x - 1 \geq x = fT(x) = gS(x)$, which follows that $T(x) \preceq fT(x)$ and $S(x) \preceq gS(x)$. Thus, (T, f) and (S, g) are partially weakly increasing with $f(X) \subseteq T(X)$

and $g(X) \subseteq S(X)$*. Also, the pairs of mappings* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible and the range of all of mappings are a closed subset of* X*. Moreover, using mean value theorem we have*

$$d(fx, gy) = |\ln(x+1) - \ln(y+1)| \leq k|x - y| \leq k|e^x - e^y| = kd(Tx, Sy)$$

for all $x, y \in X$*, where* $k = \frac{1}{1+c} \in [0, 1)$ *with* c *between* x *and* y*. Thus* f*,* g *,* S *and* T *satisfy all the condition given in Theorem* 2.14*. Moreover,* 0 *is a unique common fixed point of* f*,* g*,* S *and* T*.*

Theorem 2.16. *If we replace the condition of weak compatibility of pairs* $\{f, S\}$ *and* $\{g, T\}$ *in Theorem* 2.14 *by the following condition: either*
(i) $\{f, S\}$ *are compatible,* f *or* S *is continuous and* $\{g, T\}$ *are weakly compatible; or*
(ii) $\{g, T\}$ *are compatible,* g *or* T *is continuous and* $\{f, S\}$ *are weakly compatible,*
then the conclusions of Theorem 2.14 *remain valid.*

Proof. Following the similar arguments to those given in proof of Theorem 2.10, the result follows. □

3. Application

Let $X = L^2(\Omega)$ be the set of comparable functions on Ω whose square is integrable on Ω where $\Omega = [0, 1]$. The set X is endowed with the partial order $\preceq$ given by: $x, y \in X$, $x \preceq y \Leftrightarrow x(t) \leq y(t)$, for all $t \in \Omega$. Consider the integral equations

$$\begin{aligned} x(t) &= \int_{\Omega} q_1(t, s, x(s))ds + v(t), \\ y(t) &= \int_{\Omega} q_2(t, s, x(s))ds + v(t), \end{aligned} \tag{8}$$

where $q_1, q_2 : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}_+$ are given continuous mappings. Altun and Simsek [7] obtained the common solution of integral equations (8) as an application of their result in ordered metric spaces. We shall study sufficient condition for the existence of common solution of integral equations in framework of E-metric spaces. For $E = \mathbb{R}^2$, we define $d : X \times X \to E$ by

$$d(x, y) = (\sup_{t \in \Omega} \left|x(t) - y(t)\right|, c \sup_{t \in \Omega} \left|x(t) - y(t)\right|),$$

where $c > 0$ and the coordinate wise ordering defined by $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Then d is a E-metric on X. Suppose that the following conditions holds:

(i) For each $s, t \in \Omega$, we have

$$u_1(t) \leq \int_{\Omega} q_1(t, s, u_1(s))ds$$

and

$$u_2(t) \leq \int_{\Omega} q_2(t, s, u_2(s))ds.$$

(ii) There exists $p : \Omega \to \Omega$ satisfying

$$\int_{\Omega} \left|q_1(t, s, u(s)) - q_2(t, s, v(s))\right| ds \leq p(t) \left|u(t) - v(t)\right|$$

for each $s, t \in \Omega$ with $\sup_{t \in \Omega} p(t) \leq k$ where $k \in [0, 1)$.

Then the integral equations (8) have a common solution in $L^2(\Omega)$.

Proof. Define $(fx)(t) = \int_\Omega q_1(t,s,x(s))ds + v(t)$ and $(gx)(t) = \int_\Omega q_2(t,s,x(s))ds + v(t)$. From (i), we have

$$\begin{aligned}(fx)(t) &= \int_\Omega q_1(t,s,x(s))ds + v(t)\\ &\geq x(t) + v(t)\\ &\geq x(t)\end{aligned}$$

and

$$\begin{aligned}(gx)(t) &= \int_\Omega q_2(t,s,x(s))ds + v(t)\\ &\geq x(t) + v(t)\\ &\geq x(t).\end{aligned}$$

Thus f and g are dominating maps on X. Now, for all comparable $x, y \in X$, *we have*

$$\begin{aligned}d(fx,gy) &= (\sup_{t\in\Omega}\left|(fx)(t)-(gy)(t)\right|, c\sup_{t\in\Omega}\left|(fx)(t)-(gy)(t)\right|)\\ &= \left(\sup_{t\in\Omega}\left|\int_\Omega q_1(t,s,x(s))ds - \int_\Omega q_2(t,s,y(s))ds\right|, c\sup_{t\in\Omega}\left|\int_\Omega q_1(t,s,x(s))ds - \int_\Omega q_2(t,s,y(s))ds\right|\right)\\ &\leq \left(\sup_{t\in\Omega}\int_\Omega \left|q_1(t,s,x(s)) - q_2(t,s,y(s))\right|ds, c\sup_{t\in\Omega}\int_\Omega \left|q_1(t,s,x(s)) - q_2(t,s,y(s))\right|ds\right)\\ &\leq (\sup_{t\in\Omega}p(t)\left|x(t)-y(t)\right|, c\sup_{t\in\Omega}p(t)\left|x(t)-y(t)\right|)\\ &\leq k(\sup_{t\in\Omega}\left|x(t)-y(t)\right|, c\sup_{t\in\Omega}\left|x(t)-y(t)\right|)\\ &= kd(x,y)\\ &= ku_{x,y}(f,g),\end{aligned}$$

where

$$u_{x,y}(f,g) = d(x,y) \in \left\{d(x,y), d(fx,x), d(gy,y), \frac{1}{2}[d(fx,y)+d(y,gx)]\right\}.$$

Thus (4) is satisfied. Now we can apply Corollary 2.9 to obtain the common solutions of integral equations (8) in $L^2(\Omega)$. □

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