Analytical Solution of 1-D Multiple Layer Dual Phase Lag Heat Conduction Problem with Generalized Time Dependent Boundary Condition

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ABSTRACT

The Fourier law of heat conduction is unable to describe the phenomena such as self-heating of micro-electronics, situations involving very low temperature near to absolute zero, heat transport in living tissues and heat sources like laser heating. The dual phase lag (DPL) heat conduction model is found to be more useful to describe these phenomena. Several analytical solutions of 1D single layer to multiple layer (composite material) DPL heat conduction problems for mixed boundary conditions (BCs) are obtained by representing the BCs with Newton's law of cooling combined with Fourier law of heat conduction. However, this is contradictory to the assumption of non-applicability of Fourier law of heat conduction in the DPL heat conduction model. In the present work it is shown that several approaches such as separation of variables (SOV), finite integral transform (FIT) and orthogonal eigenfunction expansion method (OEEM) are not applicable if BCs are consistent with DPL heat conduction model assumptions. Moreover, such methods are also not applicable in cases of heat conduction in multiple layer composite material. Since, only in the case of Dirichlet type BCs and single layer material such discrepancy is not present hence, the above mentioned approaches can be applied to obtain analytical solutions. It is also shown that the Laplace transform (LT) can be successfully used to obtain analytical solutions of single as well as multiple layer DPL heat conduction problems with generalized BCs when Taylor series expansion of the phase lag operator is taken into consideration.

NOMENCLATURE

$A(s)$	[-]	Coefficient matrix in Eqn. (41)
A, B, C	[-]	Confidents to describe boundary conditions
$C(s)$	[-]	Coefficient matrix of the solution of Eqn. (39)
C_P	[$Jkg^{-1}K^{-1}$]	Thermal capacity or specific heat
$D(s)$	[-]	Inhomogeneous component matrix in Eqn. (41)
$\bar{f}(s)$	[-]	Laplace transform of a function in s-space
$f(t)$	[-]	Function of time
$f(0)$	[-]	Function of time at $t = 0$
g	[Jm^{-3}]	Volumetric heat source
I_0	[-]	Modified Bessel function of first kind (zeroth order)
k	[$Wm^{-1}K^{-1}$]	Thermal conductivity
K_0	[-]	Modified Bessel function of 2nd kind (zeroth order)
L	[-]	Symbol for Laplace transform
M	[-]	Number of layers
p	[0, 1, 2]	Index for coordinate system
q''	[Wm^{-3}]	Heat flux
r	[m]	Distance or radius
t	[s]	Time
T	[K]	Temperature
T_0	[K]	Initial temperature

Special characters

α	[m^2s^{-1}]	Thermal diffusivity
$\nabla\cdot$	[-]	Gradient
∂	[-]	Partial derivative
ρ	[kgm^{-3}]	Mass density
τ_q	[s]	Relaxation time for heat flux
τ_T	[s]	Relaxation time for temperature gradient
ω	[-]	Frequency defined in Eqn. (39)

Subscripts and superscripts

i	Index for layer
in	Inner surface of 1st layer
n	Term in a series or derivative
out	Outer surface of Mth layer
s	Laplace transform variable

INTRODUCTION

The application of dual phase lag (DPL) heat conduction is of great importance due to various applications where Fourier heat conduction is not valid [1]–[7]. The generalized non-Fourier heat model (specially DPL) is given by Tzou [8] and it is capable of describing different types of heat conduction phenomena. But the mathematical formulation is so complicated that it becomes difficult to get solution even by using numerical schemes. A few works have been so far carried out to obtain analytical solution for such problems. Even less attempts have been made for such solutions in context of inhomogeneous mixed boundary conditions (BCs), which are mainly obtained by combining Fourier heat conduction with Newton's law of cooling. This contradicts the basic postulate of mathematical model. The incorporation of DPL model in these BCs makes the problem even more difficult because of complex relation between heat flux and temperature gradient. The analytical approaches like separation of variables (SOV), finite integral transform (FIT) and orthogonal eigenfunction expansion method (OEEM) are inapplicable into time and space components for such single as well as multiple layer heat conduction problems.

The multiple layer 1D DPL heat conduction problems are solved numerically [9], [10] as well as analytically[11], [12] only in the Cartesian coordinate systems. But both these techniques are used to solve problems approximating relation between heat flux and gradient only up to 1st term in the Taylor series expansion. Moreover, the BCs are time independent for both of these methods. The similar concept was used by Chou *et. al.*[13] to study laser effect on 2D slabs and the DPL heat conduction problem and solved numerically. The single as well as multiple layer DPL heat conduction problems can be solved analytically by using SOV[14], [15], OEEM and FIT if BCs and interface conditions are obtained using Fourier model. However, as noted earlier, this contradicts DPL heat conduction. The requirement is to get rid of inseparable terms in BCs as obtained using DPL model.

In the current work, Laplace Transform (LT) method is used for this purpose. This approach provides equations in *s*-space for which analytical solutions are possible. The formulations are provided for general DPL model as well as for 1st order approximation of Taylor's series expansion. For the later formulation, an inhomogeneous wave like equation is obtained for each layer in space. The analytical solutions of the wave equation are provided for Cartesian, cylindrical and spherical coordinates. A system of equations is formulated by replacing analytical solutions in inhomogeneous BCs and interface conditions. The matrix inversion method is applied to find out the coefficients of those analytical solutions. The inverse LT can provide the final solutions in *t*-space.

MATHEMATICAL DESCRIPTION OF DPL HEAT CONDUCTION PROBLEM FOR MULTIPLE LAYER

Taylor's series expansion:
The expansion of any function $f(t)$ around τ, assuming $\tau << 1$, is given as follows

$$f(t+\tau) = f(t) + \tau\frac{\partial f(t)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 f(t)}{\partial t^2} + \frac{\tau^3}{3!}\frac{\partial^3 f(t)}{\partial t^3} + \cdots + \frac{\tau^n}{n!}\frac{\partial^n f(t)}{\partial t^n} + \cdots + \infty \quad (1)$$

or

$$f(t+\tau) = e^{\tau\frac{\partial}{\partial t}} f(t) \quad (2)$$

where, $e^{\tau\frac{\partial}{\partial t}} = 1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2} + \frac{\tau^3}{3!}\frac{\partial^3}{\partial t^3} + \cdots + \frac{\tau^n}{n!}\frac{\partial^n}{\partial t^n} + \cdots + \infty.$

Relation between heat flux and temperature gradient
DPL model: The relation between heat flux and temperature gradient for i^{th} layer of the composite is,

$$q_i''(r, t+\tau_q) = -k_i \nabla T_i(r, t+\tau_T) \quad (3)$$

So, the relation between heat flux and temperature gradient for DPL heat conduction model is as follows considering the Taylor's series expansion [16],

$$e^{\tau_{q,i}\frac{\partial}{\partial t}} q_i''(r,t) = -k_i e^{\tau_{T,i}\frac{\partial}{\partial t}} \nabla T_i(r,t) \quad (4)$$

1st law of thermodynamics for multiple layer: Energy conservation for each i^{th} layer of the composite is given as follows[17],

$$\rho_i C_{p,i}\frac{\partial T_i(r,t)}{\partial t} = -\nabla . q_i''(r,t) + g_i(r,t) \quad (5)$$

Transport equation:
DPL heat conduction model: Combining relation between heat flux and temperature gradient with 1st law of thermodynamics, 1D model for multiple layer DPL heat conduction can be written as

$$\frac{1}{\alpha_i} e^{\tau_{q,i}\frac{\partial}{\partial t}} \frac{\partial T_i(r,t)}{\partial t} = e^{\tau_{T,i}\frac{\partial}{\partial t}} \frac{1}{r^p}\frac{\partial}{\partial r}\left\{r^p \frac{\partial T_i(r,t)}{\partial r}\right\} + \frac{1}{k_i} e^{\tau_{q,i}\frac{\partial}{\partial t}} g_i(r,t) \quad (6)$$

here $\alpha_i = \frac{k_i}{\rho_i c_{p,i}}$ is thermal diffusivity of each layer and

$$p = \begin{cases} 0 & \text{for Cartesian coordinates} \\ 1 & \text{for cylindrical coordinates.} \\ 2 & \text{for spherical coordinates} \end{cases}$$

Initial conditions (ICs):

$$T_i(r,t)|_{t=0} = T_{0,i}(r) \quad (7)$$

$$\left.\frac{\partial^n T_i(r,t)}{\partial t^n}\right|_{t=0} = 0 \quad n = 1,2,\ldots,\ldots\infty \quad (8)$$

Boundary conditions (BCs):
The BCs are obtained by combining DPL heat conduction model with Newton's law of cooling, i.e. considering non-Fourier heat conduction. The consideration does not contradict postulates of DPL heat conduction.

DPL model and Newton's law of cooling:
At the inner surface, $r = r_0$

$$\mathrm{A}_{in} e^{\tau_{T,1}\frac{\partial}{\partial t}} \left.\frac{\partial T_1(r,t)}{\partial r}\right|_{r=r_0} + \mathrm{B}_{in} e^{\tau_{q,1}\frac{\partial}{\partial t}} T_1(r,t)|_{r=r_0} = e^{\tau_{q,1}\frac{\partial}{\partial t}} C_{in}(t) \quad (9)$$

At the interface between i^{th} and $(i+1)^{th}$ layer, $r = r_i$ and $i = 1, \ldots,$ M-1

$$T_i(r,t)|_{r=r_i} = T_{i+1}(r,t)|_{r=r_i} \quad (10)$$

$$k_i e^{\tau_{T,i}\frac{\partial}{\partial t}} \left.\frac{\partial T_i(r,t)}{\partial r}\right|_{r=r_i} = k_{i+1} e^{\tau_{T,i+1}\frac{\partial}{\partial t}} \left.\frac{\partial T_{i+1}(r,t)}{\partial r}\right|_{r=r_i} \quad (11)$$

At the outer surface, $r = r_{\mathrm{M}}$

$$\mathrm{A}_{out} e^{\tau_{T,\mathrm{M}}\frac{\partial}{\partial t}} \left.\frac{\partial T_{\mathrm{M}}(r,t)}{\partial r}\right|_{r=r_{\mathrm{M}}} + \mathrm{B}_{out} e^{\tau_{q,\mathrm{M}}\frac{\partial}{\partial t}} T_{\mathrm{M}}(r,t)|_{r=r_{\mathrm{M}}} = e^{\tau_{q,\mathrm{M}}\frac{\partial}{\partial t}} C_{out}(t) \quad (12)$$

1st order approximation:

$$\frac{1}{\alpha_i}\left[1 + \tau_{q,i}\frac{\partial}{\partial t}\right]\frac{\partial T_i(r,t)}{\partial t} = \left[1 + \tau_{T,i}\frac{\partial}{\partial t}\right]\frac{1}{r^p}\frac{\partial}{\partial r}\left\{r^p \frac{\partial T_i(r,t)}{\partial r}\right\} + \frac{1}{k_i}\left[1 + \tau_{q,i}\frac{\partial}{\partial t}\right] g_i(r,t) \quad (13)$$

BCs:

At the inner surface, $r = r_0$

$$\mathrm{A}_{in}\left[1+\tau_{T,1}\frac{\partial}{\partial t}\right]\frac{\partial T_1(r,t)}{\partial r}\bigg|_{r=r_0}+\mathrm{B}_{in}\left[1+\tau_{q,1}\frac{\partial}{\partial t}\right]T_1(r,t)\big|_{r=r_0}=\left[1+\tau_{q,1}\frac{\partial}{\partial t}\right]C_{in}(t) \tag{14}$$

At the interface between i^{th} and $(i+1)^{\text{th}}$ layer, $r = r_i$ and $i = 1, \ldots,$ M-1

$$T_i(r,t)\big|_{r=r_i}=\ T_{i+1}(r,t)\big|_{r=r_i} \tag{15}$$

$$k_i\left[1+\tau_{T,i}\frac{\partial}{\partial t}\right]\frac{\partial T_i(r,t)}{\partial r}\bigg|_{r=r_i}=\ k_{i+1}\left[1+\tau_{T,i+1}\frac{\partial}{\partial t}\right]\frac{\partial T_{i+1}(r,t)}{\partial r}\bigg|_{r=r_i} \tag{16}$$

At the outer surface, $r = r_{\mathrm{M}}$

$$\mathrm{A}_{out}\left[1+\tau_{T,\mathrm{M}}\frac{\partial}{\partial t}\right]\frac{\partial T_{\mathrm{M}}(r,t)}{\partial r}\bigg|_{r=r_{\mathrm{M}}}+\mathrm{B}_{out}\left[1+\tau_{q,\mathrm{M}}\frac{\partial}{\partial t}\right]T_{\mathrm{M}}(r,t)\big|_{r=r_{\mathrm{M}}}=\left[1+\tau_{q,\mathrm{M}}\frac{\partial}{\partial t}\right]C_{out}(t) \tag{17}$$

The DPL heat conduction problems with Neumann and mixed type BCs for single layer as well as multiple layer cannot be solved by eigenfunction based methods such as SOV, FIT and OEEM. This is due to the fact that the homogeneous BCs and interface conditions are inseparable in time and space components.

SOLUTION PROCEDURE: APPLICATION OF LAPLACE TRANSFOR

Laplace transform: The LT of different functions are shown as follows,

$$\bar{f}(s)=L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt \tag{18}$$

$$L\{e^{at}f(t)\}=\int_0^\infty e^{(a-s)t}f(t)dt=\bar{f}(s-a) \tag{19}$$

$$L\left\{\frac{df(t)}{dt}\right\}=\int_0^\infty e^{-st}\frac{df(t)}{dt}dt=s\bar{f}(s)-f(0) \tag{20}$$

$$L\left\{\frac{d^2f(t)}{dt^2}\right\}=s^2\bar{f}(s)-sf(0)-f'(0) \tag{21}$$

$$L\left\{\frac{d^3f(t)}{dt^3}\right\}=s^3\bar{f}(s)-s^2f(0)-sf'(0)-f''(0) \tag{22}$$

$\vdots$

$$L\left\{\frac{d^nf(t)}{dt^n}\right\}=s^n\bar{f}(s)-s^{n-1}f(0)-\cdots-sf^{n-2}(0)-f^{n-1}(0) \tag{23}$$

$\vdots$.

Now, taking LT of Eqn. (1) and replacing terms described in Eqn. (18), Eqns. (20-23) and so on, the following expression can be obtained.

$$L\left\{e^{\tau\frac{\partial}{\partial t}}f(t)\right\}=\left[1+\tau s+\frac{\tau^2s^2}{2!}+\cdots+\frac{\tau^ns^n}{n!}+\cdots\infty\right]\bar{f}(s)-\left[\tau+\frac{\tau^2}{2!}s+\cdots+\frac{\tau^n}{n!}s^{n-1}+\cdots\infty\right]f(0)-\left[\frac{\tau^2}{2!}+\frac{\tau^3}{3!}s+\cdots+\frac{\tau^n}{n!}s^{n-2}+\cdots\infty\right]f'(0)-\left[\frac{\tau^3}{3!}+\frac{\tau^4}{4!}s+\cdots+\frac{\tau^n}{n!}s^{n-3}+\cdots\infty\right]f''(0)-\cdots-\left[\frac{\tau^n}{n!}+\frac{\tau^{n+1}}{n+1!}s+\cdots\infty\right]f^{n-1}(0)-\left[\frac{\tau^{n+1}}{n+1!}+\frac{\tau^{n+2}}{n+2!}s+\cdots\infty\right]f^n(0)-\cdots\infty \tag{24}$$

The Eqn. (24) can be expressed as follows,

$$L\left\{e^{\tau\frac{\partial}{\partial t}}f(t)\right\}=\left[1+\tau s+\frac{\tau^2s^2}{2!}+\cdots+\frac{\tau^ns^n}{n!}+\cdots\infty\right]\bar{f}(s)-\frac{1}{s}\left[\tau s+\frac{\tau^2s^2}{2!}+\cdots+\frac{\tau^ns^n}{n!}+\cdots\infty\right]f(0)-\frac{1}{s^2}\left[\frac{\tau^2s^2}{2!}+\frac{\tau^3s^3}{3!}+\cdots+\frac{\tau^ns^n}{n!}+\cdots\infty\right]f'(0)-\frac{1}{s^3}\left[\frac{\tau^3s^3}{3!}+\frac{\tau^4s^4}{4!}+\cdots+\frac{\tau^ns^n}{n!}+\cdots\infty\right]f''(0)-\cdots-\frac{1}{s^n}\left[\frac{\tau^ns^n}{n!}+\frac{\tau^{n+1}s^{n+1}}{n+1!}+\cdots\infty\right]f^{n-1}(0)-\frac{1}{s^n}\left[\frac{\tau^{n+1}s^{n+1}}{n+1!}+\frac{\tau^{n+2}s^{n+2}}{n+2!}+\cdots\infty\right]f^n(0)-\cdots\infty \tag{25}$$

The expression in Eqn. (25) can be rewritten as follows,

$$L\left\{e^{\tau\frac{\partial}{\partial t}}f(t)\right\}=e^{\tau s}\bar{f}(s)-\frac{1}{s}[e^{\tau s}-1]f(0)-\frac{1}{s^2}[e^{\tau s}-1-\tau s]f'(0)-\frac{1}{s^3}\left[e^{\tau s}-1-\tau s-\frac{\tau^2s^2}{2!}\right]f''(0)-\cdots-\frac{1}{s^n}\left[e^{\tau s}-1-\tau s-\frac{\tau^2s^2}{2!}-\cdots-\frac{\tau^{n-2}s^{n-2}}{n-2!}-\frac{\tau^{n-1}s^{n-1}}{n-1!}\right]f^{n-1}(0)-s^n\left[e^{\tau}-1-\tau-\frac{\tau^2s^2}{2!}-\cdots-\frac{\tau^{n-1}s^{n-1}}{n-1!}-\frac{\tau^ns^n}{n!}\right]f^n(0)-\cdots\infty \tag{26}$$

This can be expressed as

$$L\left\{e^{\tau\frac{\partial}{\partial t}}f(t)\right\}=e^{\tau s}\bar{f}(s)-\left[\left(\frac{e^{\tau s}-e^{\tau\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)f(t)\right]_{t=0} \tag{27}$$

where $\frac{1}{s-\frac{\partial}{\partial t}}=\frac{1}{s}+\frac{1}{s^2}\frac{\partial}{\partial t}+\frac{1}{s^3}\frac{\partial^2}{\partial t^2}+\cdots+\frac{1}{s^{n+1}}\frac{\partial^n}{\partial t^n}+\cdots\infty$.

Similarly,

$$L\left\{e^{\tau\frac{\partial}{\partial t}}\frac{\partial f(t)}{\partial t}\right\}=se^{\tau s}\bar{f}(s)-\left[\left(\frac{se^{\tau s}-\frac{\partial}{\partial t}e^{\tau\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)f(t)\right]_{t=0} \tag{28}$$

The modified governing equation for DPL heat conduction can be written, by taking the LT of Eqn. (6) combining with Eqn. (27) and Eqn. (28), as follows

Transport equation:

$$\frac{1}{\alpha_i}\left\{se^{\tau_{q,i}s}\bar{T}_i(r,s)-\left[\left(\frac{se^{\tau_{q,i}s}-\frac{\partial}{\partial t}e^{\tau_{q,i}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)T_i(r,t)\right]_{t=0}\right\}=e^{\tau_{T,i}s}\frac{1}{r^p}\frac{\partial}{\partial r}\left\{r^p\frac{\partial\bar{T}_i(r,s)}{\partial r}\right\}-\left[\left(\frac{e^{\tau_{T,i}s}-e^{\tau_{T,i}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)\frac{1}{r^p}\frac{\partial}{\partial r}\left\{r^p\frac{\partial T_i(r,t)}{\partial r}\right\}\right]_{t=0}+\frac{1}{k_i}\left\{e^{\tau_{q,i}s}\bar{g}_i(r,s)-\left[\left(\frac{e^{\tau_{q,i}s}-e^{\tau_{q,i}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)g_i(r,t)\right]_{t=0}\right\} \tag{29}$$

BCs:

At the inner surface, $r = r_0$

$$\mathrm{A}_{in}\left\{e^{\tau_{T,1}s}\frac{\partial\bar{T}_1(r,s)}{\partial r}\bigg|_{r=r_0}-\left[\left(\frac{e^{\tau_{T,1}s}-e^{\tau_{T,1}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)\frac{\partial T_1(r,t)}{\partial r}\bigg|_{r=r_0}\right]_{t=0}\right\}+ \mathrm{B}_{in}\left\{e^{\tau_{q,1}s}\bar{T}_1(r,s)|_{r=r_0}-\left[\left(\frac{e^{\tau_{q,1}s}-e^{\tau_{q,1}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)T_1(r,t)|_{r=r_0}\right]_{t=0}\right\}= e^{\tau_{q,1}s}\bar{C}_{in}(s)-\left[\left(\frac{e^{\tau_{q,1}s}-e^{\tau_{q,1}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)C_{in}(t)\right]_{t=0} \quad (30)$$

At the interface between i^{th} and $(i+1)^{\text{th}}$ layer, $r = r_i$ and $i = 1, \ldots,$ M-1

$$\bar{T}_i(r,s)|_{r=r_i} = \bar{T}_{i+1}(r,s)|_{r=r_i} \quad (31)$$

$$k_i\left\{e^{\tau_{T,i}s}\frac{\partial\bar{T}_i(r,s)}{\partial r}\bigg|_{r=r_i}-\left[\left(\frac{e^{\tau_{T,i}s}-e^{\tau_{T,i}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)\frac{\partial T_i(r,t)}{\partial r}\bigg|_{r=r_i}\right]_{t=0}\right\}= k_{i+1}\left\{e^{\tau_{T,i+1}s}\frac{\partial\bar{T}_{i+1}(r,s)}{\partial r}\bigg|_{r=r_i}- \left[\left(\frac{e^{\tau_{T,i+1}s}-e^{\tau_{T,i+1}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)\frac{\partial T_{i+1}(r,t)}{\partial r}\bigg|_{r=r_i}\right]_{t=0}\right\} \quad (32)$$

At the outer surface, $r = r_{\mathrm{M}}$

$$\mathrm{A}_{out}\left\{e^{\tau_{T,\mathrm{M}}s}\frac{\partial\bar{T}_{\mathrm{M}}(r,s)}{\partial r}\bigg|_{r=r_{\mathrm{M}}}- \left[\left(\frac{e^{\tau_{T,\mathrm{M}}s}-e^{\tau_{T,\mathrm{M}}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)\frac{\partial T_{\mathrm{M}}(r,t)}{\partial r}\bigg|_{r=r_{\mathrm{M}}}\right]_{t=0}\right\}+ \mathrm{B}_{out}\left\{e^{\tau_{q,\mathrm{M}}s}\bar{T}_{\mathrm{M}}(r,s)|_{r=r_{\mathrm{M}}}- \left[\left(\frac{e^{\tau_{q,\mathrm{M}}s}-e^{\tau_{q,\mathrm{M}}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)T_1(r,t)|_{r=r_{\mathrm{M}}}\right]_{t=0}\right\}= e^{\tau_{q,\mathrm{M}}}\bar{C}_{out}(s)- \left[\left(\frac{e^{\tau_{q,\mathrm{M}}s}-e^{\tau_{q,\mathrm{M}}\frac{\partial}{\partial t}}}{s-\frac{\partial}{\partial t}}\right)C_{out}(t)\right]_{t=0} \quad (33)$$

1st order approximation:

The 1st order approximation of Taylor series expansion for the set of above equations can be expressed in the following form,

Transport equation:

$$\frac{1}{\alpha_i}\left\{s(1+s\tau_{q,i})\bar{T}_i(r,s)-(1+s\tau_{q,i})T_{0,i}(r)- \tau_{q,i}\overbrace{\left[\frac{\partial T_i(r,t)}{\partial t}\right]_{t=0}}^{0}\right\}= (1+s\tau_{T,i})\frac{1}{r^p}\frac{\partial}{\partial r}\left\{r^p\frac{\partial\bar{T}_i(r,s)}{\partial r}\right\}- \tau_{T,i}\frac{1}{r^p}\frac{d}{dr}\left\{r^p\frac{dT_{0,i}(r)}{dr}\right\}+\frac{1}{k_i}\left\{(1+s\tau_{q,i})\bar{g}_i(r,s)-\tau_{q,i}g_{0,i}(r)\right\} \quad (34)$$

BCs:

At the inner surface, $r = r_0$

$$\mathrm{A}_{in}\left\{(1+s\tau_{T,1})\frac{\partial\bar{T}_1(r,s)}{\partial r}\bigg|_{r=r_0}-\frac{dT_{0,1}(r)}{dr}\bigg|_{r=r_0}\right\}+\mathrm{B}_{in}\left\{(1+ s\tau_{q,1})\bar{T}_1(r,s)|_{r=r_0}-T_{0,1}(r)\big|_{r=r_0}\right\}=(1+s\tau_{q,1})\bar{C}_{in}(s)- C_{in}(0) \quad (35)$$

At the interface between i^{th} and $(i+1)^{\text{th}}$ layer, $r = r_i$ and $i = 1, \ldots,$ M-1

$$\bar{T}_i(r,s)|_{r=r_i} = \bar{T}_{i+1}(r,s)|_{r=r_i} \quad (36)$$

$$k_i\left\{(1+s\tau_{T,i})\frac{\partial\bar{T}_i(r,s)}{\partial r}\bigg|_{r=r_i}-\frac{\partial T_{0,i}(r)}{\partial r}\bigg|_{r=r_i}\right\}= k_{i+1}\left\{(1+ s\tau_{T,i+1})\frac{\partial\bar{T}_{i+1}(r,s)}{\partial r}\bigg|_{r=r_i}-\frac{\partial T_{0,i+1}(r)}{\partial r}\bigg|_{r=r_i}\right\} \quad (37)$$

At the outer surface, $r = r_{\mathrm{M}}$

$$\mathrm{A}_{out}\left\{(1+s\tau_{T,\mathrm{M}})\frac{\partial\bar{T}_{\mathrm{M}}(r,s)}{\partial r}\bigg|_{r=r_0}-\frac{\partial T_{0,\mathrm{M}}(r)}{\partial r}\bigg|_{r=r_0}\right\}+\mathrm{B}_{out}\left\{(1+ s\tau_{q,\mathrm{M}})\bar{T}_{\mathrm{M}}(r,s)|_{r=r_0}-T_{0,\mathrm{M}}(r)\big|_{r=r_0}\right\}=(1+s\tau_{q,\mathrm{M}})\bar{C}_{out}(s)- C_{out}(0) \quad (38)$$

Here, the intention is to solve the problem described in Eqn. (34) with BCs in Eqns. (35-38), analytically. However, the above equations need to be expressed in a standard form before solving the problem and these are as follows,

$$\frac{1}{r^p}\frac{\partial}{\partial r}\left\{r^p\frac{\partial\bar{T}_i(r,s)}{\partial r}\right\}-\omega_i^2(s)\bar{T}_i(r,s)=\bar{F}_i(r,s) \quad (39)$$

where $\omega_i^2(s)=\frac{s(1+s\tau_{q,i})}{\alpha_i(1+s\tau_{T,i})}$ and

$$\bar{F}_i(r,s)=\frac{1}{1+s\tau_{T,i}}\left[\tau_{T,i}\frac{1}{r^p}\frac{d}{dr}\left\{r^p\frac{dT_{0,i}(r)}{dr}\right\}+\frac{\tau_{q,i}}{k_i}g_{0,i}(r)-(1+ s\tau_{q,i})\left\{\frac{1}{\alpha_i}T_{0,i}(r)+\frac{1}{k_i}\bar{g}_i(r,s)\right\}\right].$$

The solution of Eqn. (39) is well known and given as follows,

$$\bar{T}_i(r,s)=\begin{cases}\left[C_{2i-1}(s)-\frac{1}{2w_i(s)}\int_{r_{i-1}}^{r_i}e^{rw_i(s)}\bar{F}_i(r,s)dr\right]e^{-rw_i(s)}+\left[C_{2i}(s)+\frac{1}{2w_i(s)}\int_{r_{i-1}}^{r_i}e^{-rw_i(s)}\bar{F}_i(r,s)dr\right]e^{rw_i(s)} & \text{if } p=0\\ \left[C_{2i-1}(s)-\frac{1}{2\pi}\int_{r_{i-1}}^{r_i}rK_0(rw_i(s))\bar{F}_i(r,s)dr\right]I_0(rw_i(s))+\left[C_{2i}(s)+\frac{1}{2\pi}\int_{r_{i-1}}^{r_i}rI_0(rw_i(s))\bar{F}_i(r,s)dr\right]K_0(rw_i(s)) & \text{if } p=1\\ \frac{1}{r}\left[C_{2i-1}(s)-\frac{1}{2w_i(s)}\int_{r_{i-1}}^{r_i}re^{rw_i(s)}\bar{F}_i(r,s)dr\right]e^{-rw_i(s)}+\frac{1}{2rw_i(s)}\left[C_{2i}(s)+\int_{r_{i-1}}^{r_i}re^{-rw_i(s)}\bar{F}_i(r,s)dr\right]e^{rw_i(s)} & \text{if } p=2\end{cases} \quad (40)$$

In the above equation, only two unknowns are present $C_{2i-1}(s)$ and $C_{2i}(s)$ for each i^{th} solution.

A system of 2M number of equations can be formulated from the BCs in Eqns. (35-38) and can be written as follows,

$$[A(s)]_{2M\times 2M}[C(s)]_{2M\times 1} = [D(s)]_{2M\times 1} \quad (41)$$

The matrix inversion method can be applied in Eqn. (41) to get 2M number of coefficients which are,

$$[C(s)]_{2M\times 1} = [A(s)]^{-1}_{2M\times 2M}[D(s)]_{2M\times 1} \quad (42)$$

The solution of the PDE in *s*-space needs inverse LT to get solution in *t*-space. Several inversion methods are available in literature and are not described here for the sake of brevity. Anyone of them can be used to get the final solution.

CONCLUSION

A detailed methodology for the solution of 1D multiple layer DPL heat conduction using Laplace Transform is developed in the present work. The solution is provided for first order Taylor series approximation of the governing equations. This approach can be used for all three coordinate systems (Cartesian, cylindrical and spherical). Moreover, mixed and Neumann BCs are considered using DPL approximation which are consistent with the governing equation. The interface flux continuity condition is also defined in a similar consistent fashion.

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