THERMAL NETWORKS CONSIDERING GRAPH THEORY AND THERMODYNAMICS

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ABSTRACT

Heat transfer in solids may be dealt with the heat equation, which is a partial differential equation, from which different analytical solutions for the study of heat transfer throughout solids and at their surfaces may be found. This implies the resolution of a distributed parameter model. On the other hand, the possibility of considering the thermal-electrical analogy is usually assumed, this being based mainly on the similarity between Ohm's and Fourier's laws under the assumption that the different variables used in electrical networks may be regarded as analogues to the thermal network variables. This implies the use of a lumped parameter model, which may be represented as a system of differential and algebraic equations (DAE) linked to the graphical representation of the thermal network. In this latter case the limitations of such analogy for describing heat flow should be taken into account. Therefore, it would be important to consider thermal networks independently of the thermal-electrical analogy. For this, thermal networks may be built as particular cases of directed graphs, within graph theory, since thermal networks may have physical meaning without the electrical analogy. The interpretation of a graph as a thermal network may directly use physical principles of heat and thermodynamics. This enables us to propose an alternative to the use of the electrical analogy, since electrical networks are only a particular application of graph theory consistent with electromagnetic laws which are not analogous to thermodynamic laws. Furthermore, the construction and the use of thermal networks for analysing heat transfer problems may be simplified from this perspective.

1. INTRODUCTION

Heat transfer in solids is usually approached by using calculus. The textbook by Carslaw and Jaeger is an early reference which proposes analytical solutions for the study of heat transfer throughout solids and at their surfaces [1]. These authors present the heat equation as a partial differential equation for solids, and they notice the thermal-electrical analogy. This analogy is based mainly on the similarity between Ohm's and Fourier's laws, implying that the different variables used in electrical and thermal networks are usually presented as analogues [2]. At solid surfaces in contact with a fluid (air) the analogy with Ohm's law can be extended by considering Newton's cooling law, which may represent heat losses between the solid's surface and its surroundings. Heat losses at a solid's surface may be due to convective (air-solid) and radiative (surface-surface) phenomena. Today, Newton's cooling law dealing with convective losses and radiative losses may be handled by linearizing the radiative losses given by Stefan-Boltzmann's law. The assumption of linear energy flow in form of heat (heat flow) within solids and at their surfaces, independently of the physical linear law considered, enables the use of the heat equation beyond heat conduction within solids.

On the other hand, due to the limitations of thermal-electrical analogy for describing heat flow, thermal networks should be seen as a particular case within graph theory, which is a wide field of mathematics [3]. Moreover, thermal networks may be studied by using linear algebra. The interpretation of a graph as a thermal network needs to utilize physical principles of heat, i.e. thermodynamic laws. This allows us to propose an alternative to the use of the electrical analogy, since electrical networks constitute another particular application of graph theory consistent with electromagnetic laws which are not analogous to thermodynamic laws [4].

Firstly, we will offer insight into the classical foundations of the heat conduction theory in solids before providing all the information necessary for building a thermal network to be employed in heat transfer modelling of a dynamic system. In addition, we show that thermal networks may be seen as the graphical representation of a linear time invariant system of differential algebraic equations (DAE). Thermal networks are formed by nodes (vertices in graph theory) and branches (edges in graph theory) weighted by a thermal resistance. It is important to highlight that thermal networks or the system of DAE will be linear in parameters but temperature or heat sources, which are the physical variables, may be non-linear functions. Moreover, linear in parameters does not mean that the thermal network represents a homogeneous system in space [5]. The density, specific heat capacity or conductivity may be a function of space as well as time in general. An inhomogeneous system would simply require a thermal network of higher order, i.e. a greater number of nodes and branches. Each node of the thermal network has a single value for each variable at a given time. The parameters of the thermal networks will be assumed to be time invariant; however, the temperature and heat sources provided may be time variant. In short, linearity will be related to the parameters of the thermal network or the system of DAE, not to the physical variables

and thermal networks to be used in transient further steady-state problems.

Finally, thermal networks are usually associated with the use of lumped parameter models. As lumped parameter models are historically presented as opposite to distributed parameter systems [6], a brief discussion is included on the concepts of lumped and distributed parameter models, since these are considered important to appreciate the subtle difference between them.

2. HEAT FLUX (HEAT FLOW RATE)

Heat flux or heat flow rate is the amount of energy in form of heat which goes from one system at a higher temperature to another system at a lower temperature per time unit.

2.1. LINEAR HEAT FLUX

The mathematical theory of heat conduction may be said to be founded upon a hypothesis suggested by experimental evidence [1] and it states that the energy which flows, *Q*, in one direction across a solid per time and surface unit is given by:

$$\frac{Q}{St} = \frac{\kappa(\theta_{s1} - \theta_{s2})}{d} \tag{1}$$

where κ is the thermal conductivity, which may be considered a constant property dependent on the solid; $(\theta_{s1} - \theta_{s2})$ is the difference of temperatures between the two surfaces of the solid; and *S* and *d* are the surface and the thickness of the solid, respectively.

Eq. **Erreur ! Source du renvoi introuvable.**(1) reaches the range of physical law when carried to the limits of zero surface, thickness, and time. Assuming heat flux density in three dimensions, we get:

$$q_S = -\kappa \boldsymbol{\nabla} \theta \tag{2}$$

where $\boldsymbol{\nabla}$ is the gradient operator and the density heat flux is noted as q_S ($\mathrm{W/m^2}$). The surface, *S*, is usually considered constant and it can be considered explicitly in Eq. (2). Then, we will get the heat flux, q, instead of the density heat flux.

$$q = q_S S \tag{3}$$

The importance of Eq. (2) resides in its linearity, and it is usually known as Fourier's law [7].

An important detail needs to be highlighted—that is, time is not considered in thermodynamics since system changes are studied from one equilibrium state to another. For this reason, heat or heat transfer are considered energy in transit with units of energy. Hereafter, once time is considered, we will refer to heat "rate" transfer or heat flux as energy per time unit but density heat flux as energy per time and space unit. The differences between heat transfer and heat flux need to be noted to avoid confusion.

In engineering, time is relevant and we consider heat flux density as the energy transferred across a surface per unit area and per unit time as expressed by Fourier's law. Heat flux occurs only as a consequence of a gradient of temperatures between systems at different temperatures which are connected thermally. Classically, this is interpreted while assuming the existence of isothermal surfaces and assuming that temperature is a potential function. The exchange of energy will be a function only of the difference in temperatures between isothermals in such a context. In this study, thermal networks will use the concept of nodes (vertices in graph theory) instead of isothermal surfaces. In fact, in the case of isothermal surfaces the energy is assumed to be transferred between points of different isothermal surfaces. An isothermal surface has a single temperature by definition; and a node of a thermal network will also have a single temperature. Nodes may be seen as a generalization of isothermal surfaces since they may be seen as isothermal surfaces or volumes, or they can even be considered a single point. A volume may refer to absolute or relative space; nodes (points, surfaces, and volumes) will be considered fixed in space.

Next, it should be noted that the origin of the heat conduction theory is not the differential expression, Eq. (2), but the algebraic expression, Eq. (1), which should be seen as suggested more than verified by experience [1]. The origin is crucial since the definition of thermal conductivity arises from such an algebraic equation, Eq. (1):

$$\kappa = \frac{Qd}{(\theta_{s1} - \theta_{s2})St} \tag{4}$$

Therefore, thermal conductivity is not a measurable quantity but rather needs to be estimated from measurements of the physical variables. From Eq. (4), the theoretical assumption of a time invariant and constant thermal conductivity between two surfaces is usually made, which refers, upon a classical interpretation, to the resistance of heat spreading throughout the volume between the surfaces [1].

2.2. ENERGY CONSERVATION

The next important point is the assumption of the principle of energy conservation, which is a first principle of physics. From a classical standpoint, the energy, E, entering or leaving a system (in or out) is equal to the change of energy within the system. The energy balance equation of a system is:

$$\Delta E = E_{out} - E_{in} \tag{5}$$

In thermodynamics, it is assumed that energy may be transferred in form of heat from a system in equilibrium to other system/s and vice versa up to reaching overall equilibrium. The energy balance is reached between a system and its surroundings (other system/s) and, in this context; a macroscopic system will be a node. The concept of a node will be used for building thermal networks based on graph theory, where a node is called a vertex. The aim is to enable the study of the dynamic system for solving heat transfer problems using linear algebra; since thermodynamics do not consider the study of dynamic systems. We will say that the resistance occurs between two nodes instead of between two surfaces. In this way, as explained above, the nodes may be surfaces but they can also be volumes or points. The resistance will be the resistance of two nodes for exchanging energy in form of heat due to their different temperatures. Furthermore, the use of nodes, instead of isothermal surfaces only, allows us to

consider the algebraic linear law, Eq. (1), in general not limited only to heat conduction within solids.

On the other hand, it is important to note that physics is a science of measurements, and measurements are comparisons. Therefore, it is always necessary to state a baseline for energy measurements, i.e. in the case of energy measurements a zero energy value. In the case of energy transferred in form of heat, the temperature will be the measurement for quantifying the energy of a system (a node). For this purpose, the increment of enthalpy of a system, ΔH, is usually defined as [7]:

$$\Delta H = mc\Delta\theta \tag{6}$$

where m(kg) is the mass of the system given by its density times its volume, c(J/kgK) is the specific heat capacity of the system, and θ(K) is the temperature of the system. The zero enthalpy corresponds to 0ºC and enthalpy, *H*, can be written as:

$$H = mc\theta \tag{7}$$

Eq. (4) and (5) assumes that the change of energy in the system (node) is due only to the temperature change of the system (node). Such expressions are usually considered from a thermodynamic viewpoint. In a dynamic context, this expression of the energy of the system may be derived in time assuming that only the temperature is time variant:

$$\frac{dH}{dt} = mc\frac{d\theta}{dt} \tag{8}$$

This approach will be considered in this dissertation, where no mass transfers of phase changes are considered. The use of thermal networks will not be limited by these assumptions, but such subjects would require further development, which is beyond the scope of this study.

Next, the relation between the enthalpy, *H*, and the internal energy, *U*, of a system (node) should be noted. By definition, the enthalpy is the addition of the internal energy of the system and the mechanical work done over/by the system [8]:

$$\Delta H = \Delta U + \Delta PV + P\Delta V \tag{9}$$

One of the main hypotheses in the present dissertation is that the processes to be treated will be at constant pressure and volume. Thus, the changes in enthalpy and in internal energy will be equal. That is, the exchange of heat will be the only cause of all the energy change in the system and no work is done by/over the system (node). In this way, the energy transfer is due to heat flux between the system and its surroundings, and heat flux is due only to temperature differences between the system and its surroundings. This is the initial hypothesis of textbooks which deal with heat transfer in solids [1, 7]. In thermal networks, a node (system) may or may not be the abstract representation of a solid.

A key point is that the mass of the system (node), given by its density times its volume, and the specific heat capacity will be assumed to be constant and time invariant. The product of the mass by the specific heat capacity is equal to the heat or thermal capacity, *C*(J/K):

$$C = mc \tag{10}$$

The heat capacity is a measurable physical quantity given by definition as the amount of heat which produces a change of temperature in a system (node) of 1ºC:

$$C = \frac{Q}{\Delta\theta} \tag{11}$$

From Eq. (10) and (11) we get:

$$Q = mc\Delta\theta \tag{12}$$

In the present study, we recall again that heat, *Q*(J), will be the only cause of the change of temperature of a system and it will correspond exactly with the change of the internal energy that will be given by the change of enthalpy.

$$Q = \Delta H \tag{13}$$

2.2. THE HEAT EQUATION

The heat equation can be derived from Eq. (5). In the present case the energy change of a system (node) is assumed to be due only to the temperature change—that is, heat is the only cause of the change of internal energy without mass transfer or phase change. It may be given by the time variation of the enthalpy, Eq. (8):

$$\Delta E = \frac{dH}{dt} \tag{14}$$

or:

$$\Delta E = \rho Vc\frac{d\theta}{dt} \tag{15}$$

where the density, ρ(kg/m^3), the volume, V(m^3), and the specific heat capacity, c(J/kgK), are considered time invariant for the system (node).

The heat flux entering or leaving the system (node) is given by:

$$E_{out} - E_{in} = (q_{out} - q_{in})S \tag{16}$$

where *S* is the surface of the system (node).

When we use Eq. (15) and Eq. (16), assuming a cubic volume and considering the first heat flux in the *x*-direction, Eq. (5) becomes:

$$\rho cS\delta x\frac{\partial\theta}{\partial t} = \delta q_x S \tag{17}$$

where the surface cancels, and taking the limit of $\delta x \to 0$, we may get:

$$\rho c\frac{\partial\theta}{\partial t} = \frac{\partial q_x}{\partial x} \tag{18}$$

It is possible to follow an analogous procedure in *y* and *z* directions considering heat flux as a vector, $\mathbf{q} = (q_x, q_y, q_z)$. Then, we get the known as the heat equation when Fourier's law is considered and heat sources, p, are added to the system:

$$\rho c\frac{\partial\theta}{\partial t} = -\boldsymbol{\nabla}\cdot(-\kappa\boldsymbol{\nabla}\theta) + p \tag{19}$$

It is important to note that we will work with linear algebra not with calculus. In any case, this simplified way of formulating the heat equation is useful to bear in mind that calculus comes later than linear algebra.

3. GRAPH THEORY AND THERMAL NETWORKS

The basis of networks is the graph theory, which is considered to have been started by Leonard Euler and the problem of the seven bridges of Konigsberg [3]. A formal definition given by Wallis [9] states that a graph consists of a finite set of objects called vertices together with a set of unordered pairs of vertices called edges. Graphs are usually represented by diagrams in which the vertices are points, and edges between two points (x, y) are shown as lines from x to y. Graphs have multiple applications and the general graph theory is broad. In particular, we will examine oriented or directed graphs (digraphs). A digraph is like a graph except that each edge is given a direction and one vertex is designated as a start and the other as a finish; a digraph can be restricted to allow not more than one edge to connect two points.

An important application of digraphs involves electrical networks, which, in particular, consider algebraic graph theory based on Kirchhoff's voltage and current laws as well as Ohm's law [10]. This early application explains the origin of the thermal-electrical analogy, but Fourier's law inspired Ohm's law [4], and Kirchhoff's voltage law does not apply for temperature sources since temperature sources in a series cannot be added algebraically; that is, the sum of the drops in temperature in a closed thermal network cannot be zero. This latter restriction is mandatory for avoiding the violation of thermodynamic laws. This crucial difference, among others [4], weakens the analogy and makes it recommendable to state thermal networks from graph theory directly despite using the thermal-electrical analogy. Another option usually often proposed is the use of thermodynamic bond graphs [11, 12]. Nonetheless, bond graphs also play with analogies at a level higher than graph theory; the idea of bond graphs is to extend the use of analogies to mechanical or hydraulic systems, further electrical or thermal systems. The use of the same mathematical expressions is proposed but changing the meaning of vertices (potentials) and edges (forces) [11].

Thermal networks may be seen as directed graphs and this will be taken as the prime idea in this dissertation instead of the usual analogy with electrical networks [1, 7, 13], which are also directed graphs. The reason for not using the analogy resides in its limitation for steady-state problems and for the differences between heat and electricity [4], as for example the different rules that apply to the addition of temperature sources and electrical voltage sources. The addition of temperature sources must obey thermodynamic laws; this means that two temperature sources connected in series cannot be added together to get an equivalent temperature source which is the algebraic sum of them, while it is well known that electrical voltage sources in a series can be added together to give an equivalent electrical voltage source which is the algebraic sum of them. In short, Kirchhoff's voltage law has no analogy in thermal systems. Moreover, the difference regarding the order of the time derivative between the elliptic partial differential equation representing electromagnetic fields and the parabolic partial differential equation representing temperature fields is well known. Consequently, there is no analogous term for the inductance, so that the existence of such a term, called inertance, would go against the second law of thermodynamics [12].

Gustav Kirchhoff used the incipient graph theory for his advances in the study of electrical circuits [10], as James J. Sylvester made use of graph theory in his paper "Chemistry and Algebra" [3]. Electrical and chemical circuits are only applications of the graph theory. Currently, the power of graphs has extended their use from chemistry and physics to all disciplines, including social or information sciences. The advantage of graph theory is that it offers simplicity to the representation of a problem, and this simplicity allows a new insight which facilitates the analysis of the problem itself. In the case of thermal networks, they will be expressed as directed graphs and they will be represented by matrices considering linear algebra in their study. From mathematics to physics, a thermal network will be expressed as a directed graph instead of adapting *ad hoc*, by an incomplete analogy, an electrical network to be a thermal network.

Then, the problem becomes to put physics on digraphs to establish thermal networks. Firstly, the vertices will be the thermal nodes and the edges will be thermal branches. The nodes will be characterized by a unique temperature and will be connected in the sense given by Fourier's law, while thermal resistances will be placed at branches and thermal capacitances at nodes. The absent of mass transfer makes capacitances time invariant since it is related to the constant mass of the element and its specific heat. Instead of mass, the coefficient mass per volume is used, i.e. density. Mass transfer is a notable issue and it should be coupled with the solution of the problem in future works.

We may conclude that, basically, a network is a graph where vertices and edges become entities in the physical sense. Once a graph is applied to a particular problem, we have a network, implying a non-abstract sense to vertices and edges. In the case of thermal networks, vertices and edges have a physical sense, nodes represent temperatures, and they are weighted by thermal capacities; the edges represent physical interactions between nodes and they are weighted by thermal resistances from which heat flows from node to node. Thermal networks will imply, by construction, the linearity of heat transfer between nodes, which is further verified by experience [14].

2.3. THERMAL NETWORKS AND DAE

From a thermal network, it is possible to design a system of DAE. A thermal network may be arranged in (Figure 1):

- branches, containing thermal resistances, R_k, and temperatures sources, b_k, –which are crossed by heat flow rates, q_k, and
- nodes to which thermal capacities and flow sources are connected.

This formalism allows a unique representation for all branches of the model studied [5, 14, 15].

The temperature difference over a resistance is R_k is:

$$\mathbf{e} = -\mathbf{A}\boldsymbol{\theta} + \mathbf{b} \tag{20}$$

where:
$\mathbf{e} = [e_1, \dots, e_k, \dots e_m]^T$ is the vector of temperature drops over thermal resistances, with m the number of branches in the model, which is equal to the number of resistances in the model;
$\boldsymbol{\theta} = [\theta_1, \dots, \theta_l, \dots, \theta_n]^T$ is the vector of temperature values in the nodes, with n the number of nodes;
$\mathbf{b} = [b_1, \dots, b_k, \dots, b_m]^T$ is the vector of temperature sources on the branches;
and $\mathbf{A}$ is the incidence matrix of the thermal network, with dimension $m \times n$; $\mathbf{A}$ is a matrix operator which makes the difference of temperatures. The m rows of the incidence matrix $\mathbf{A}$ correspond to the branches containing heat rate flows, q_k, crossing the resistances, R_k, and the n columns correspond to the nodes representing the temperatures θ_l.

The elements of the incidence matrix $\mathbf{A}$ are given by:

$$a_{kl} = \begin{cases} -1 & \text{if flow } q_k \text{ exits from node } \theta_l \\ 0 & \text{if } q_k \text{ is not connected to node } \theta_l \\ +1 & \text{if flow } q_k \text{ enters in node } \theta_l \end{cases}$$

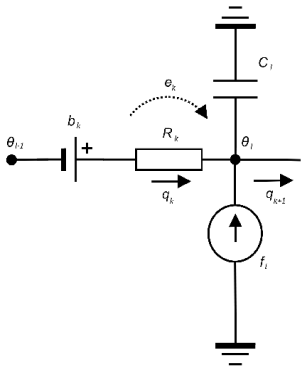

Figure 1. Typical branch and nodes in thermal networks

The heat flow rate in all branches of a thermal network is:

$$\mathbf{q} = \mathbf{G}\mathbf{e} \tag{21}$$

where:
$\mathbf{q} = [q_1, \dots, q_k, \dots, q_m]^T$ is the vector of heat rates in the branches
$\mathbf{G} = \begin{bmatrix} R_1^{-1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & R_m^{-1} \end{bmatrix}$ is a diagonal matrix of thermal conductances.

The balance of heat rates in a node θ_l states that the variation in time of the energy accumulated in the thermal capacity, $C_l\dot{\theta}_l$, is equal to the algebraic sum of heat rates entering or exiting the node θ_l and the heat rate source f_l connected to the node. The balance equation of heat rates for all nodes of a thermal network can be written as:

$$\mathbf{C}\dot{\boldsymbol{\theta}} = \mathbf{A}^T\mathbf{q} + \mathbf{f} \tag{22}$$

where
$\mathbf{C} = \begin{bmatrix} C_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_n \end{bmatrix}$ is a diagonal matrix of thermal capacities.
$\mathbf{A}^T$ is the transpose of the incidence matrix; it is a matrix operator which makes the algebraic sum of heat transfer rates $\mathbf{q}$ in nodes.
$\mathbf{f} = [f_1, \dots, f_l, \dots, f_n]^T$ is the vector of heat rate sources connected to the temperature nodes.
Substituting Eq. (20) and Eq. (21) in Eq. (22), we can write the set of heat balance equations as:

$$\mathbf{C}\dot{\boldsymbol{\theta}} = -\mathbf{A}^T\mathbf{G}\mathbf{A}\boldsymbol{\theta} + \mathbf{A}^T\mathbf{G}\mathbf{b} + \mathbf{f} \tag{23}$$

or, by noting $\mathbf{K} \equiv -\mathbf{A}^T\mathbf{G}\mathbf{A}$ and $\mathbf{K}_b \equiv \mathbf{A}^T\mathbf{G}$, as:

$$\mathbf{C}\dot{\boldsymbol{\theta}} = \mathbf{K}\boldsymbol{\theta} + \mathbf{K}_b\mathbf{b} + \mathbf{f} \tag{24}$$

The Eq. (23) gives a set of differential algebraic equations for the nodes with negligible capacity (the elements of matrix $\mathbf{C}$ corresponding to negligible capacities are assumed to be zero).

Eqs. (20)-(22) are present in one expression in the set of differential algebraic equations, Eq. (23). Eqs. (20)-(22) may be also arranged using a block matrix:

$$\begin{bmatrix} \mathbf{G}^{-1} & \mathbf{A} \\ -\mathbf{A}^T & \mathbf{C}\dfrac{d}{dt} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix} \tag{25}$$

Eq. (25) is an adaptation of the equilibrium equations to dynamic heat transfer problems where the term related to the dynamics of the system, $\mathbf{C}\frac{d}{dt}$, is placed separately from the conductance matrix, $\mathbf{G}$, instead of changing it by an impedance matrix, as is done for electrical networks [16].

2.4. DISTRIBUTED AND LUMPED PARAMETER MODELS

Heat equation and thermal networks are two different ways of dealing with heat flow. The difference resides in considering finite networks or finite systems of differential and algebraic equations (DAE) instead of a partial differential equation (PDE), which may become an infinite system of DAE [5], for dealing with the space distribution of a physical system.

It is usually considered that thermal networks are lumped parameter models in which the variables are not functions of the space, while the heat equation is considered a distributed parameter model in which the variables are dependent on space. As has been stated above, lumped parameter models may be described by a finite system of differential algebraic equations (DAE), and distributed parameter models may be described by partial differential equations (PDE). We could consider that lumped and distributed parameter models may be described, in general, by a system of DAE, allowing us to consider the same mathematical formalism for describing both kinds of models. In mathematics, operators are also introduced, which may be differentials or not, for using the same formalism and defining both distributed and lumped parameter models in the same way [17]. Once the same mathematical formalism is used to describe a system regardless of considering it distributed or lumped, the difference remains in the assumption that a lumped

parameter model is an approximation of a distributed parameter model, which is supposed to be able to describe the physical system perfectly [6]. The dependence on space of a lumped parameter model is given by its order, which is not infinite.

This is a subtle difference between distributed and lumped parameters models since, for practical applications, solving a distributed parameter problem usually requires its reduction to a lumped parameter problem, i.e. a finite order model. The reduction of the distributed system to a lumped system implies a connection between the two representations. This may be done before or after solving the time integration [18]. Such reduction is usually done by space integration of the distributed system, and for this reason it should be noted that a lumped system designed in such way could be considered a distributed system where the spatial dependence is present implicitly. In this way, a lumped model should not be considered the opposite of a distributed model since it may be used to determine the variables at every space placement. The question would be to estimate the order of the lumped model needed to accurately describe a thermal system distributed in space [5].

2.5. CONCLUSIONS

It has been shown that the heat flow between systems at different temperature may be studied, further using the classical heat equation and Fourier's law, by using thermal networks built considering thermodynamics and graph theory. In such a case, thermodynamic systems are represented by nodes, and the most important principle to be kept is that of energy conservation. Furthermore, the interactions between the nodes of thermal networks may be considered to be governed by a linear law, as is usual in classical physics. Thermal networks introduced in this way do not have the limitations of the analogy with electrical networks; the main problem of thermal-electrical analogy is that Kirchhoff's voltage law does not apply for thermal networks, since temperatures cannot be added algebraically as can be done for electrical voltage.

Finally, it should to be highlighted that lumped parameter models (thermal networks) may consider the distribution of the variables in space implicitly. The only theoretical difference regarding a distributed parameter model (heat equation) is based on the assumption that the distributed parameter model uses the exact value of the variables at every point in space, while the lumped parameter model will use only the exact value of the variables at a finite number of space nodes.

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