SOLUTION OF INVERSE PROBLEM – REGULARIZATION VIA THERMODYNAMICAL CRITERION

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ABSTRACT

In engineering practice, measuring temperature on both sides of a wall (of, for example, turbine casing or combustion chamber) is not always possible. On the other hand, measurement of both temperature and heat flux on the outer surface of the wall is possible. For transient heat conduction equation, measurements of temperature and heat flux supplemented by the initial condition state the Cauchy problem, which is ill-conditioned In this paper, the stable solution is obtained for the Cauchy problem using the Laplace transformation and the minimisation of continuity in the process of integration of convolution. Test examples confirm proposed algorithm for the inverse problem solution.

INTRODUCTION

In thermal problems, the coefficients of governing equations such as the thermal conductivity, density and specific heat, as well as the intensity and location of internal heat sources, if they exist, and appropriate boundary and initial conditions should be specified. Such problems are referred to as 'direct thermal problems' and may be accurately solved using standard numerical methods since they are well posed. However, in many practical applications which arise in engineering, a part of the boundary is not accessible for heat flux or temperature measurements. For example, the temperature or the heat flux may be seriously affected by the presence of a sensor which results in a loss of accuracy in measurement or, more simply, the surface of the body may be unsuitable for attaching a sensor to measure the temperature or the heat flux. Inner surface of turbine casing or combustion chamber can be indicated as examples. The situation when neither the heat flux nor temperature can be determined on a part of the boundary, while both of them are prescribed on the remaining part, leads to an ill-posed problem called the 'Cauchy problem'[11, 12] which is more difficult to solve both analytically and numerically.

The Cauchy problem is not new in literature [1-7]. Due to its ill-posed character many approximation methods are used. In paper [1] the problem is reduced to a linear integral Volterra equation of second type which admits a unique solution. Method of fundamental solution was used in paper [2] for solving the steady Cauchy problem. In papers [3-4] method of finite difference with the Fourier transform techniques was used. Legandre polynomials were used in paper [5] for solving 1-D Cauchy problem. Wavelet-Galerkin method with the Fourier transform was used in paper [6]. The unique solution of Cauchy problem was considered in paper [7].

The purpose of this paper is to propose a stable solution obtained for the Cauchy problem using the Laplace transformation and the linear approximation of function in convolution product and thermodynamical regularization. Test examples confirm proposed algorithm for the solution of inverse problem.

NOMENCLATURE

c	[J/K]	heat capacity
E_{total}	[J]	total energy
F	[K]	sought temperature at the point x=δ
H	[K]	temperature distribution at point x=δ
J	[-]	functional
k	[W/mK]	thermal conductivity
L	[-]	differential operator
p	[-]	variable
Q	[W/m^2]	heat flux
q	[-]	dimensionless heat flux
T	[K]	temperature
t	[s]	time
w	[-]	regularization matrix
x	[m]	Cartesian coordinates

Special characters

α	[-]	regularization parameter
$\Delta\tau$	[s]	dimensionless time step
δ	[m]	thickness
η	[-]	Heaviside function
μ_n	[-]	eigenvalue
ξ	[-]	dimensionless coordinate
ρ	[kg/m^3]	density
σ_s	[-]	intensity of entropy production
τ	[-]	dimensionless time
χ	[-]	unknown function
ψ	[-]	integral kernel
ψ_s	[-]	local dissipation function
Ω	[m^2]	calculated area
ϑ	[-]	dimensionless temperature

FUNDAMENTAL EQUATION

For region shown in Fig. 1, the governing equation and conditions describing heat flow are as follows:

- heat conduction equation

$$\rho c \cdot \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right), \quad x \in (0, \delta), \; t > 0 \qquad (1)$$

- initial condition

$$T(x,0) = T_0(x) \qquad (2)$$

- boundary conditions

$$T(x = \delta, t) = H(t) \qquad (3)$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=\delta} = Q(t) \qquad (4)$$

$$T(x = 0, t) = F(t) = ? \qquad (5)$$

The distribution of the temperature $F(\tau)$ (5) is sought.

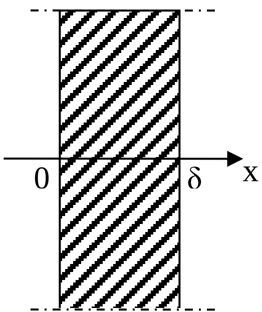

Figure 1 Calculation area

In consideration of the boundary conditions (3) and (4), the problem formulated by (1-4) is the Cauchy problem. For the next considerations the following non-dimensional variables are introduced:

$$T_{max} = \max_{\substack{x\in\langle 0,1\rangle \\ t\geq 0}} \left(T(x,t)\right), \quad \vartheta = \frac{T}{T_{max}}, \quad \xi = \frac{x}{\delta}, \quad \tau = \frac{k}{\rho c}\cdot\frac{t}{\delta^2} \qquad (6)$$

and now non-dimensional formulation of the linear problem is as follows:

- heat conduction equation

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \xi^2}, \quad \xi \in (0,1), \; \tau > 0 \qquad (7)$$

- initial condition

$$\vartheta(\xi,0) = \vartheta_0(\xi), \; \xi \in \langle 0,1\rangle \qquad (8)$$

- boundary condition at surface $\xi = 1$

$$\vartheta(1,\tau) = h(\tau) = H/T_{max}, \; \tau > 0 \qquad (9)$$

$$-\frac{\partial \vartheta(1,\tau)}{\partial \xi} = q(\tau), \quad q = \frac{\delta}{\lambda \cdot T_{max}} \cdot Q, \; \tau > 0 \qquad (10)$$

- unknown boundary condition at surface $\xi = 0$

$$\vartheta(0,\tau) = \chi(\tau), \;\; \chi(\tau) = F(\tau)/T_{max} \;\; \tau > 0 \qquad (11)$$

In consideration of equations (6)-(9) linearity, the Laplace transformation will be used for their solution. Let

$$L\vartheta(\xi,\tau) = \bar{\vartheta}(\xi,s) = \int_0^\infty \vartheta(\xi,\tau)\cdot e^{-s\tau} d\tau \qquad (12)$$

then the system of equations (6)-(9) is transformed to following form:

- heat conduction equation with the initial condition

$$s \cdot \bar{\vartheta}(\xi,s) - \vartheta(\xi,0) = \frac{d^2\bar{\vartheta}}{d\xi^2}, \;\; \vartheta(\xi,0) = \vartheta_0(\xi) \qquad (13)$$

- boundary conditions at surface $\xi = 1$

$$\bar{\vartheta}(1,s) = \bar{h}(s) \qquad (14)$$

$$\frac{-d\bar{\vartheta}(1,s)}{d\xi} = \bar{q}(s) \qquad (15)$$

- unknown boundary condition at surface $\xi = 0$

$$\bar{\vartheta}(0,s) = \bar{\chi}(s) \qquad (16)$$

Idea of determining the unknown distribution (5) is based on determination of direct problem, namely the solution to the equation (7) with conditions (8)-(11), and next on the determination of the relation between functions $\chi(t)$ and $h(t)$.

For simplicity, it is assumed $\vartheta_0(\xi) = \vartheta_0 = \text{const}$, then the solution to the direct problem has a form

$$\bar{\vartheta}(\xi,s) = \bar{\chi}(s)\cdot\frac{\cosh\sqrt{s}(1-\xi)}{\cosh\sqrt{s}} + \bar{q}(s)\cdot\frac{\sinh\sqrt{s}\xi}{\sqrt{s}\cdot\cosh\sqrt{s}} + \frac{\vartheta_0}{s}\cdot\left(1 - \frac{\cosh\sqrt{s}(1-\xi)}{\cosh\sqrt{s}}\right) \qquad (17)$$

For $\xi = 1$

$$\bar{\vartheta}(1,s) = \bar{\chi}(s)\cdot\frac{1}{\cosh\sqrt{s}} + \bar{q}(s)\frac{\mathrm{tgh}\sqrt{s}}{\sqrt{s}} + \frac{\vartheta_0}{s}\left(1 - \frac{1}{\cosh\sqrt{s}}\right) \qquad (18)$$

Unknown function $\bar{\chi}(s)$ we will search on the base of the known distribution (9), namely

$$\bar{\vartheta}(1,s) = s\cdot\bar{\chi}(s)\cdot\frac{1}{s\cdot\cosh\sqrt{s}} + \bar{q}(s)\frac{\mathrm{tgh}\sqrt{s}}{\sqrt{s}} + \frac{\vartheta_0}{s}\left(1 - \frac{1}{\cosh\sqrt{s}}\right) = \bar{h}(s)$$

In this way we have the Volterra integral equation of second kind for determining the function $\chi(t)$ in a form

$$L^{-1}[s\cdot\bar{\chi}(s)] * L^{-1}\left[\frac{1}{s\cosh\sqrt{s}}\right] + L^{-1}[s\bar{q}(s)] * L^{-1}\left[\frac{\mathrm{tgh}\sqrt{s}}{s\sqrt{s}}\right] + \vartheta_0\left\{\eta(t) - L^{-1}\left[\frac{1}{s\cosh\sqrt{s}}\right]\right\} = h(t) \qquad (19)$$

Finally, for $q(\tau) = 0$ and $\vartheta_0 = 0$, the form of the solution (19) can be written as follows

$$\vartheta(\xi,\tau) = \int_0^\tau \chi(p) \cdot 2\sum_{n=1}^{\infty} \mu_n \cdot \sin \mu_n \xi e^{-\mu_n^2(\tau-p)} \cdot dp =$$

$$= \int_0^\tau \chi(p) \cdot \psi(\xi,\tau,p) dp, \quad \xi \in \langle 0,1 \rangle \tag{20}$$

where

$$\psi(\xi,\tau,p) = 2\sum_{n=1}^{\infty} \mu_n \cdot \sin \mu_n \cdot e^{-\mu_n^2(\tau-p)}$$

From condition (9) on the base of (20) we obtained the equation for the determination function $\chi(\tau)$

$$\vartheta(1,\tau) = \int_0^\tau \chi(p) \cdot \psi(1,\tau,p) \cdot dp = h(\tau) \tag{21}$$

or

$$\int_0^\tau \chi(p) \cdot \psi(1,\tau,p) \cdot dp = h(\tau) \tag{22}$$

The equation is an integral equation of Volterra kind.

NUMERICAL CALCULATIONS

In order to test solution to integral equation (22) we will compare the numerical solution with the analytical one.
The discretization of eq. (22) was done on the equidistance net.
Analytical solution to equation (7) with the initial condition $\vartheta(\xi,0) = 0$ and the following boundary conditions

$$\vartheta(\xi=0,\tau) = T_b \cdot \left(1 - e^{-\beta\tau}\right), \quad -\frac{\partial \vartheta(\xi=1,\tau)}{\partial \xi} = Bi \cdot \vartheta(\xi=1,\tau) \tag{23}$$

has the form [10]

$$\vartheta(\xi,\tau) =$$

$$= T_b \cdot \left(1 - \frac{Bi}{Bi+1} \cdot \xi\right)\left(1 - e^{-\beta\tau}\right) + 2T_b \cdot \beta \cdot e^{-\beta\tau} \cdot \sum_{n=1}^{\infty} w_n(\xi) \cdot \frac{1}{p_n^2 - \beta} -$$

$$- 2T_b \cdot \beta \cdot \sum_{n=1}^{\infty} w_n(\xi) - \frac{1}{p_n^2 - \beta} \cdot e^{-p_n^2\tau},$$

$$w_n(\xi) = -\frac{\sin p_n \xi}{p_n} \cdot \left(1 - \frac{Bi}{Bi^2 + Bi + p_n^2}\right) \tag{24}$$

where numbers p_n are the following roots of equation

$$tgp_n = -\frac{p_n}{Bi}, \quad n = 1,2,\ldots, \quad \text{and for} \quad Bi \to 0, \quad p_n = \frac{\pi}{2}(2n-1)$$

and

$$\lim_{\tau \to \infty} \vartheta(\xi,\tau) = T_b \cdot \left(1 - \frac{Bi}{Bi+1} \cdot \xi\right)$$

Solution (24) is used for determining functions h(t) and q(t) given by formulas (9) and (10).

INVERSE PROBLEM

At each segment $<\tau_{j-1}, \tau_j>$, $j = 1,\ldots, M$ function $\chi(\tau)$ in (22) is approximated by $\chi = \chi_{j-1} \cdot \Theta + \chi_j \cdot (1 - \Theta)$, $0 < \Theta < 1$, therefore, between segments the function $\chi(\tau)$ is not differentiated and a jump of the first derivative appears there

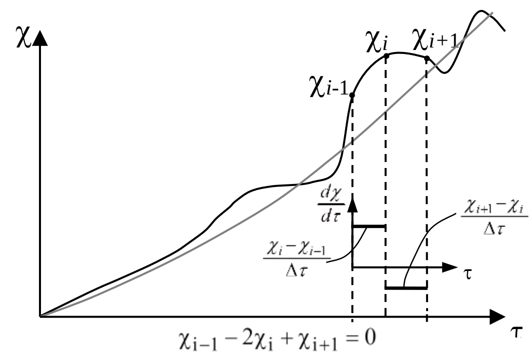

Figure 2 Idea of linear regularization of the solution $\chi(\tau)$

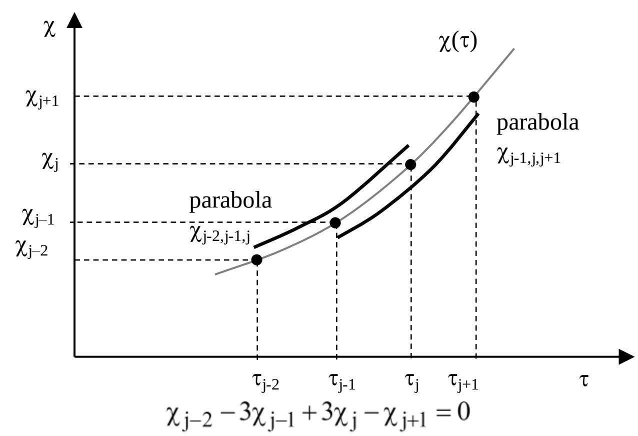

Figure 3 Idea of parabolic regularization of the solution $\chi(\tau)$

Leading the first parabola $\chi_{j-2,j-1,j}$ through the following points (τ_{j-2}, χ_{j-2}), (τ_{j-1}, χ_{j-1}), (τ_j, χ_j) and the second parabola $\chi_{j-1,j,j+1}$ through points (τ_{j-1}, χ_{j-1}), (τ_j, χ_j), (τ_{j+1}, χ_{j+1}) we require that the difference between the first derivative in common points of both parabolas should be equal zero, what in case of uniform mesh τ_{j-1} and τ_j, $\tau_j - \tau_{j-1} = h$, $k = 1,2,\ldots, M$ leads to one equation on region $<\tau_{j-1}, \tau_{j+1}>$

$$\chi_{j-2} - 3\chi_{j-1} + 3\chi_j - \chi_{j+1} = 0, \; j = 2,\ldots, M-2 \tag{25}$$

In case of the linear regularization, as shown in Fig. 2, the conditions that lead to reduction of jumps of the first derivation are

$$\chi_{j-1} - 2\chi_j + \chi_{j+1} = 0, \; j = 1,2,\ldots, M-1 \tag{26}$$

Discretization of the equation (22) leads to the following matrix form

$$[\psi]\{\chi\}=\{h\} \tag{27}$$

Dimension of matrix $[\psi]$ is equal $\dim[\psi]=(M+1)\times(M+1)$. Because the inverse problem is ill posed, then the condition (25) or (26) is added to the system (27) to reduce the oscillations in solution to the inverse problem

$$\begin{bmatrix}[\psi]\\ \alpha\cdot[w]\end{bmatrix}\{\chi\}=\begin{Bmatrix}\{h\}\\ \{0\}\end{Bmatrix},\ \dim[\psi]=(M+1)\times(M+1) \tag{28}$$

where matrix $[w]$ related with the condition (25) has the following form, and α denotes the regularization parameter

$$[w]=\begin{bmatrix}1 & -3 & 3 & -1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 1 & -3 & 3 & -1\end{bmatrix}$$

$$\dim[w]=(M-3)\times(M+1) \tag{29}$$

and for the condition (26) we have

$$[w]=\begin{bmatrix}1 & -2 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 1 & -2 & 1\end{bmatrix},\ \dim[w]=(M-1)\times(M+1) \tag{30}$$

Solution of over-determined system of equations (27) can be considered as a minimization of functional

$$J(\{\chi\})=\|[\psi]\{\chi\}-\{h\}\|^2+\|[\alpha\cdot w]\{\chi\}\|^2,\ \alpha>0 \tag{31}$$

or in standard mathematical form (11), (12)

$$J(X)=\|AX-Y\|^2+\|[\alpha\cdot w]X\|^2$$
$$A=[\psi],\quad X=\{\chi\},\quad Y=\{h\}$$

OPTIMAL CHOICE OF REGULARIZATION PARAMETER VIA THREMODYNAMICAL CRITERION

Processes of heat flow are accompanied by entropy production and energy dissipation. The intensity of the source of entropy production is given by formula [9]

$$\sigma_s=\frac{1}{T}\left(\rho c\frac{\partial T}{\partial t}-\operatorname{div}(k\nabla T)-q\right)+\frac{k(\nabla T)^2}{T^2}= \\ =\frac{1}{T}LT+\frac{k(\nabla T)^2}{T^2} \tag{32}$$

and the function of energy dissipation by formula [9]

$$\psi_s=T\sigma_s=\rho c\frac{\partial T}{\partial t}-\operatorname{div}(h\nabla T)-q+\frac{h(\nabla T)^2}{T}= \\ =LT+\frac{h(\nabla T)^2}{T} \tag{33}$$

Function of temperature satisfies the heat conduction equation for LT = 0, thus total intensity of the source of entropy σ_{total} and relatively, of energy dissipation ψ_{total} in the time interval <0,t> and the region Ω is equal to

$$\sigma_{total}=\int_0^t\int_\Omega \sigma_s\cdot d\Omega dt=\int_0^t\int_\Omega \frac{k(\nabla T)^2}{T^2}d\Omega dt$$
$$\psi_{total}=\int_0^t\int_\Omega \psi_s\cdot d\Omega dt=\int_0^t\int_\Omega \frac{k(\nabla T)^2}{T}d\Omega dt \tag{34}$$

The total energy is expressed by the integral

$$E_{total}=\int_0^t\int_\Omega k(\nabla T)^2 d\Omega dt \tag{35}$$

Physical processes proceed with minimal increment of entropy σ_{total} and minimal dissipation of energy ψ_{total}. Therefore, the regularization parameter will correspond to the minimal value σ_{total} or ψ_{total} and will also depend on the form of the regularized functional (31).

NUMERICAL CALCULATION

For given temperature distribution at the edge of x = 0 (Fig. 3) with β=10.0 and q(x= δ)=0, the temperature distribution at the edge x= δ was calculated. The course of temperature at x= δ was disturbed with random error δ = 5%, the value of disturbed temperature was treated as measured value. The solution of the inverse problem obtained for different regularization parameters and in different coordinate systems is shown in figures from 4 to 9.

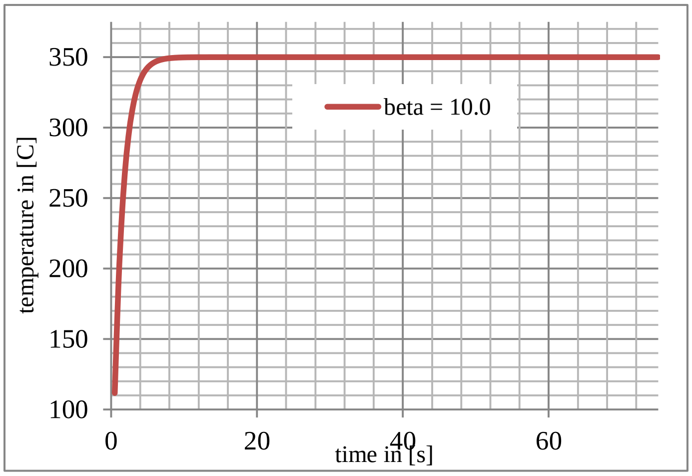

Figure 3 Analytical temperature distribution at surface x=0.0 : T=T0*(1-exp(-beta*tau))

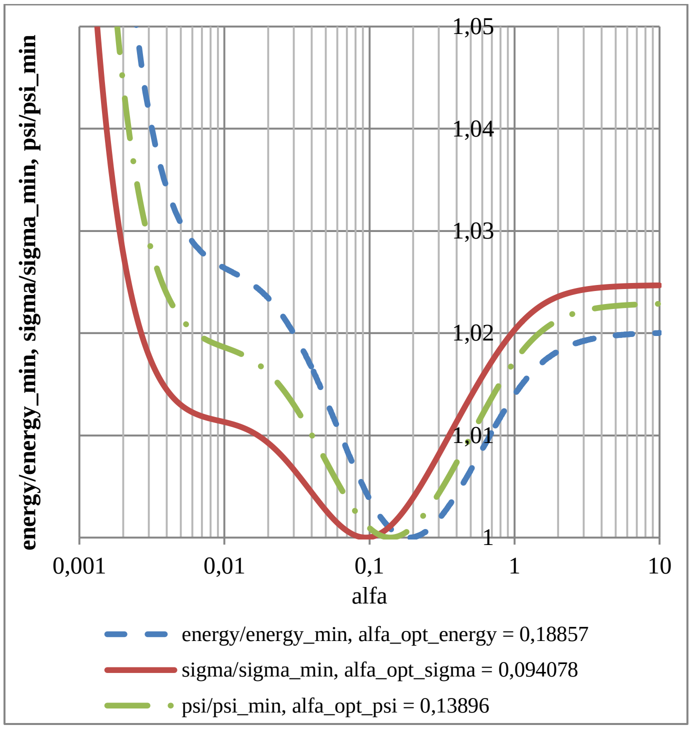

Figure 4 Distribution of: energy/energy_min, sigma/sigma_min, psi/psi_min as the function of regularization parameter alfa, beta = 10, |delta_random| = 5.0%

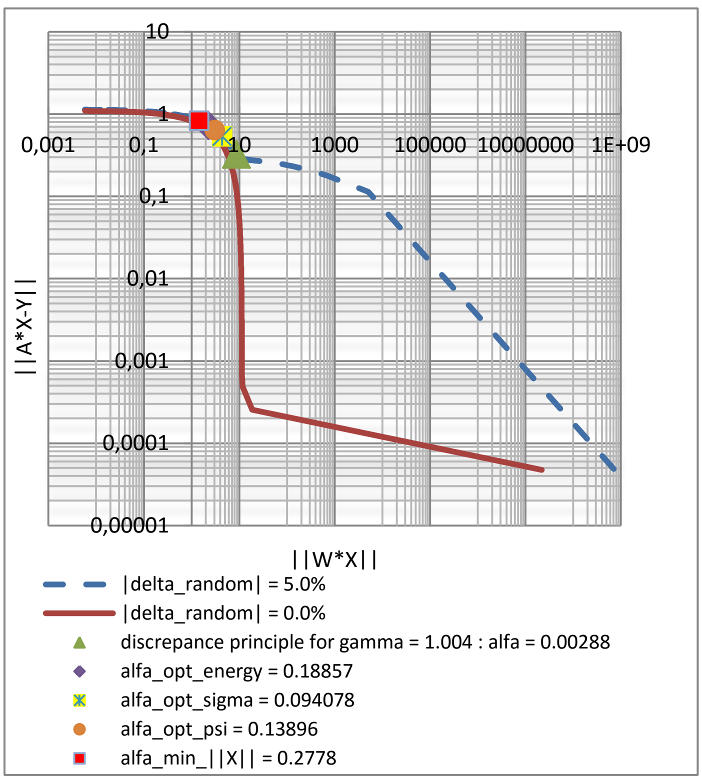

Figure 5 Distribution of ||A*X-Y|| = f(||w*X||) for Tikhonow-Phillips functional J(alfa) = ||A*X-Y||**2+ ||alfa*W*X||**2, beta = 10

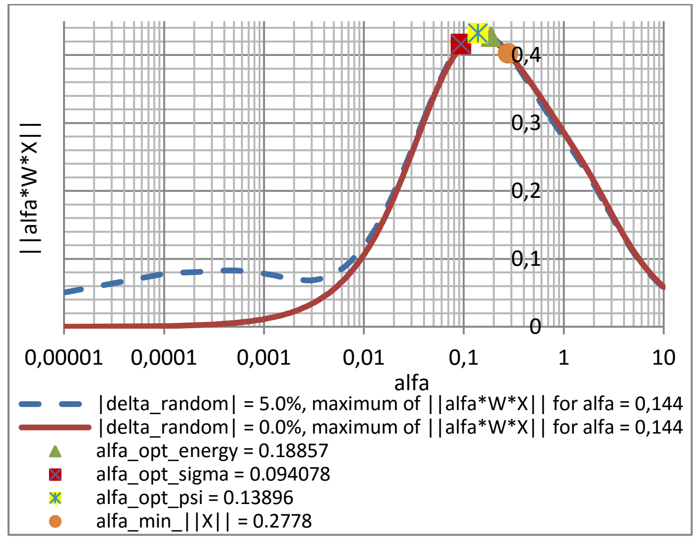

Figure 6 Tikhonow-Phillips functional J(alfa) = ||A*X-Y||**2+ ||alfa*W*X||**2, beta = 10

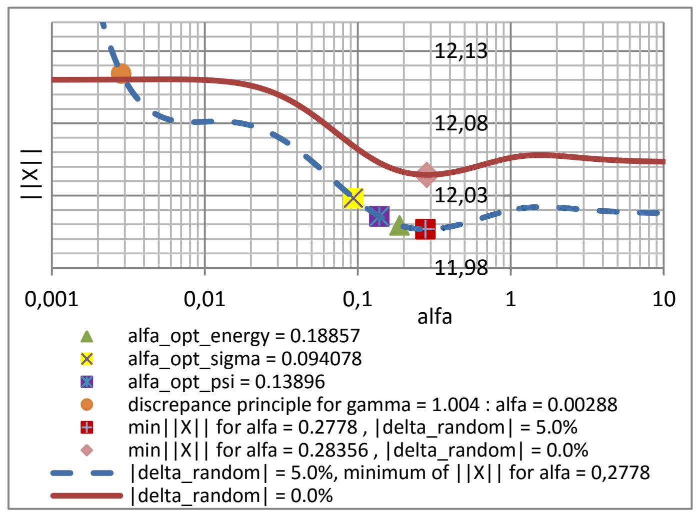

Figure 7 Distribution of ||X|| = f(alfa), beta = 10

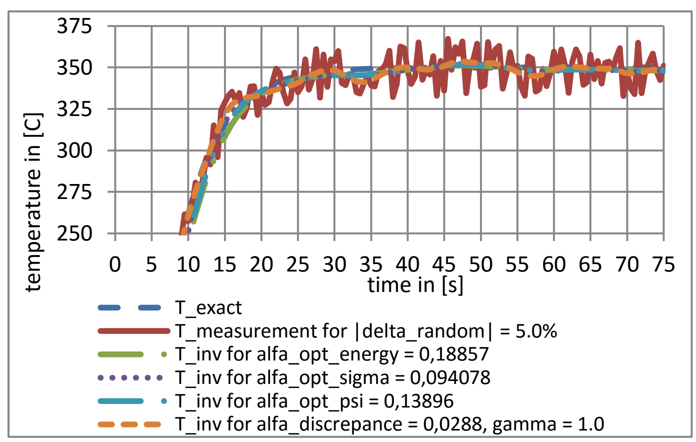

Figure 8 Temperature distribution at surface x=1.0 , beta = 10 , |delta_random| = 5.0% for different regularization parameters

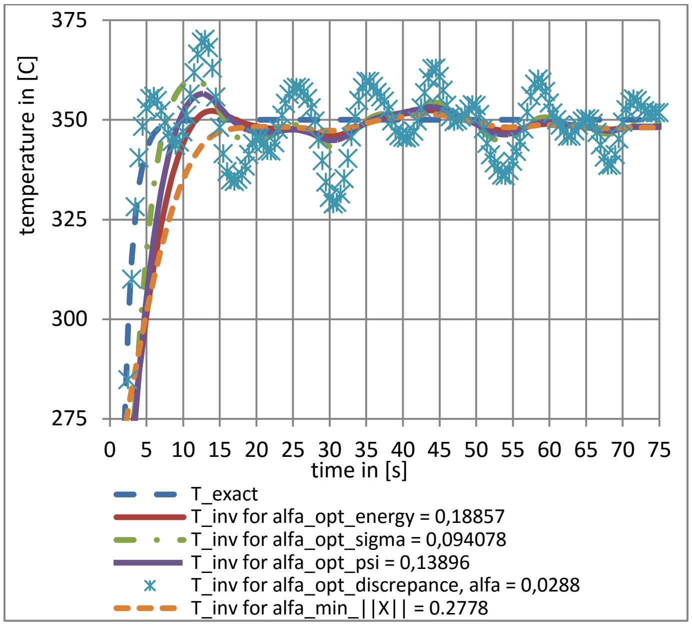

Figure 9 Temperature distribution at surface x=0.0 , beta = 10, |delta_random| = 5.0% for different regularization parameters

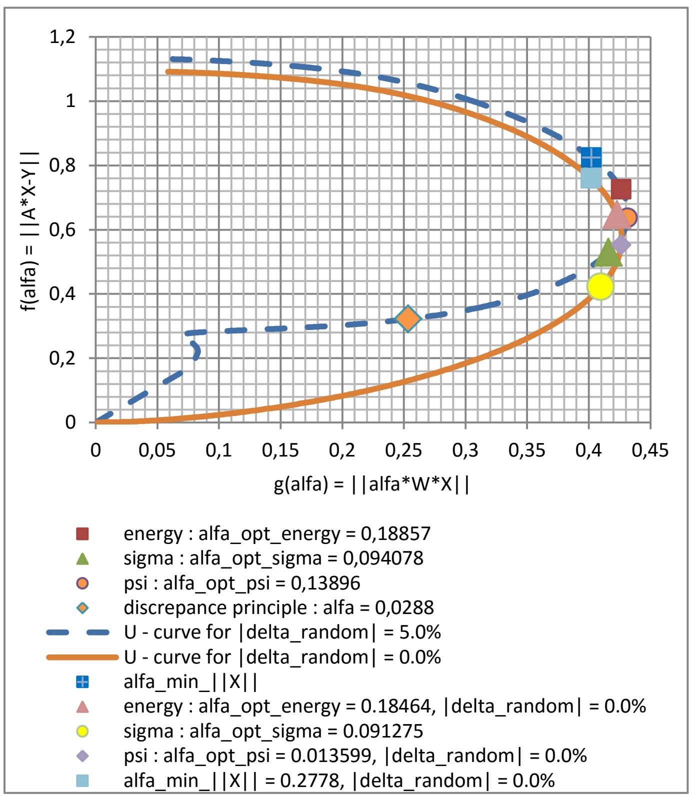

Figure 10 U - curve for Tikhonow-Phillips functional J(alfa) = ||A*X-Y||**2+ ||alfa*W*X||**2, beta = 10

CONCLUSIONS

The U-curve (Fig. 10) expresses the dependence between terms of functional for the regularized solution. It may indicate where the solutions to the inverse problem, obtained using various methods (such as the Morozov principle, the minimum intensity of entropy production, minimum of dissipation energy, L-curve, etc.), are placed on it. Analysis of the location of the solutions, obtained using various methods, on the U-curve indicates the area (slow heating: low beta-parameter), where those solutions are within relatively short distance from the result obtained using the Morozov method. Extrema points of the functional of the intensity of entropy production and of the dissipation are in the vicinity of the minimum of ||X|| norm of the regularized solution. Numerical experiment shows that the curvature of the L-curve corresponds to the parameter gamma = 1 in the Morozov method (discrepancy principle). Future work: investigation of inverse Cauchy–type problem for steady-state thermal loading of the turbine blade. Intensity of entropy production will be investigated in case of higher rank of regularization matrices and 2D problem [13].

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