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Unification of several distance functions and a common fixed point result

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Abstract

In this work, we initiate the notions of dislocated-A_b-quasi-metric and A_b-quasi-metric-like spaces. Then we establish the existence of a common fixed point of weakly compatible mappings satisfying a contractive condition on a closed neighborhood of dislocated A_b-quasi-metric spaces. Some examples are given to show that these spaces are more general than various known comparable metric spaces. Our result unify, complement, and generalize various known results in the literature.

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1 Introduction and preliminaries

In software engineering, algorithms are designed by means of recursive denotational specifications. The running time and the memory space of computing such algorithms are two important factors that determine the efficiency of the software.

Scott [1] used a T_0 model for lambda calculus to construct a system of logic. He then employed fixed point techniques as a suitable mathematical tool for program verifications in denotational semantics of programming languages.

Scott results were extended by Matthews [2] by defining a partial ordering on T_2 models. He introduced the notion of partial metric spaces and studied their essential topological properties. Matthews successfully reinforced Scott's fixed point techniques with a metric. His approach turned out to be very productive and attracted the attention of several researchers, who studied fixed point results on partial metric spaces.

Arshad *et al.* [3] proved the existence of some fixed point results for mappings satisfying a contractive condition in a closed neighborhood of a certain point in an ordered dislocated metric space.

The aim of this paper is to introduce a notion of generalized partial metric spaces called dislocated A_b-quasi-metric spaces. We study basic topological properties of dislocated A_b-quasi-metric spaces and provide some examples to support the concepts defined herein. We also obtain common fixed point results of weakly compatible mappings satisfying local contractive condition in such spaces. Our results unify, improve, and generalize several comparable results in [4] and [3].

In the sequel, the letters $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the sets of positive integers, nonnegative integers, real numbers, and positive real numbers, respectively.

Definition 1.1 Let X be a nonempty set, and $s \geq 1$ a given real number. A function $A_b : X^n \to [0, \infty)$ is called a dislocated A_b-quasi-metric on X if for any $a, x_i \in X$, $i = 1, 2, \ldots, n$, $n \geq 3$, the following conditions hold:

(A_{b_1}) $A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = A_b(x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1) = 0$ implies that $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$;

(A_{b_2})

$$A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n)$$
$$\leq s\big[A_b\big(x_1, x_1, x_1, \ldots, (x_1)_{(n-1)}, a\big)$$
$$+ A_b\big(x_2, x_2, x_2, \ldots, (x_2)_{(n-1)}, a\big)$$
$$+ A_b\big(x_3, x_3, x_3, \ldots, (x_3)_{(n-1)}, a\big)$$
$$\vdots$$
$$+ A_b\big(x_{(n-1)}, x_{(n-1)}, x_{(n-1)}, \ldots, (x_{(n-1)})_{(n-1)}, a\big)$$
$$+ A_b\big(x_n, x_n, x_n, \ldots, (x_n)_{(n-1)}, a\big)\big].$$

The pair (X, A_b) is called a dislocated A_b-quasi-metric space with coefficient s.

Lemma 1.2 *Let (X, A_b) be a dislocated A_b-quasi-metric space with coefficient $s \geq 1$. Then $A_b(x, x, x, \ldots, x, y) = A_b(y, y, y, \ldots, y, x)$ for all $x, y \in X$.*

Example 1.3 Let $X = \mathbb{R}$. Define the function $A_b : X^n \to [0, \infty)$ by

$$A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = |2x_1 - x_2|^2 + |2x_1 - x_3|^2 + \cdots + |2x_1 - x_n|^2$$
$$+ |2x_2 - x_3|^2 + |2x_2 - x_4|^2 + \cdots + |2x_2 - x_n|^2$$
$$\vdots$$
$$+ |2x_{n-2} - x_{n-1}|^2 + |2x_{n-2} - x_n|^2$$
$$+ |2x_{n-1} - x_n|^2.$$

Indeed, if $A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = A_b(x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1) = 0$, then $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$. Note that

$$A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = |2x_1 - x_2|^2 + |2x_1 - x_3|^2 + \cdots + |2x_1 - x_n|^2$$
$$+ |2x_2 - x_3|^2 + |2x_2 - x_4|^2 + \cdots + |2x_2 - x_n|^2$$
$$\vdots$$
$$+ |2x_{n-2} - x_{n-1}|^2 + |2x_{n-2} - x_n|^2$$
$$+ |2x_{n-1} - x_n|^2$$
$$\leq \big(|2x_1 - a| + |x_2 - a|\big)^2$$

$$+ \left(|2x_1 - a| + |x_3 - a|\right)^2$$

$$+ \cdots + \left(|2x_1 - a| + |x_n - a|\right)^2$$

$$+ \left(|2x_2 - a| + |x_3 - a|\right)^2$$

$$+ \left(|2x_2 - a| + |x_4 - a|\right)^2$$

$$+ \cdots + \left(|2x_2 - a| + |x_n - a|\right)^2$$

$$\vdots$$

$$+ \left(|2x_{n-2} - a| + |x_{n-1} - a|\right)^2$$

$$+ \left(|2x_{n-2} - a| + |x_n - a|\right)^2$$

$$+ \left(|2x_{n-1} - a| + |x_n - a|\right)^2$$

$$\leq 2\big[A_b\big(x_1, x_1, x_1, \ldots, (x_1)_{(n-1)}, a\big)$$

$$+ A_b\big(x_2, x_2, x_2, \ldots, (x_2)_{(n-1)}, a\big)$$

$$+ A_b\big(x_3, x_3, x_3, \ldots, (x_3)_{(n-1)}, a\big)$$

$$\vdots$$

$$+ A_b\big(x_{(n-1)}, x_{(n-1)}, x_{(n-1)}, \ldots, (x_{(n-1)})_{(n-1)}, a\big)$$

$$+ A_b\big(x_n, x_n, x_n, \ldots, (x_n)_{(n-1)}, a\big)\big].$$

Hence, (X, A_b) is a dislocated A_b-quasi-metric space with coefficient $s = 2$.

Example 1.4 Let $X = \mathbb{R}$. Consider the dislocated A_b-quasi-metric on X^3 given by

$$A_b(x_1, x_2, x_3) = |2x_1 - x_2|^2 + |2x_1 - x_3|^2 + |2x_2 - x_3|^2.$$

If $x_1 = 4$, $x_2 = 6$, and $x_3 = 10$, then $A_b(4, 6, 10) \neq A_b(10, 6, 4)$ implies that (X, A_b) is not a G_{p_b}-metric space [5].

Definition 1.5 Let (X, A_b) be a dislocated A_b-quasi-metric space with $s \geq 1$, $x_0 \in X$, and r a positive real number r. The set

$$\overline{B(x_0, r)} = \big\{ y \in X : A_b(y, y, y, \ldots, y, x_0) \leq r \big\}$$

is called a closed ball centered at x_0 with radius r.

Definition 1.6 A subset G of a dislocated A_b-quasi-metric space (X, A_b) is said to be an open set if for each $x \in G$, there exists a positive real number r such that $B(x, r) \subseteq G$. A set $F \subseteq X$ is called closed if $X \setminus F$ is open. The collection of open sets in a dislocated A_b-quasi-metric space (X, A_b) is called the topology induced by the metric A_b.

Lemma 1.7 *Let (X, A_b) be a dislocated A_b-quasi-metric space with $s \geq 1$. Then for all $x, y \in X$, we have*

$$A_b(x, x, x, \ldots, x, z) \leq s\big[(n-1)A_b(x, x, x, \ldots, x, y) + A_b(y, y, y, \ldots, y, z)\big].$$

Definition 1.8 Let (X, A_b) be a dislocated A_b-quasi-metric space with $s \geq 1$ and $n \geq 3$. A sequence $\{x_k\}$ in X is said to be convergent in X if there exists a point $x \in X$ such that $\lim_{k \to \infty} A_b(x_k, x_k, x_k, \ldots, x_k, x) = 0$. In this case, we write $\lim_{k \to \infty} x_k = x$.

Lemma 1.9 *Let (X, A_b) be a dislocated A_b-quasi-metric space with $s \geq 1$ and $n \geq 3$. The limit of a convergent sequence $\{x_k\}$ in X is unique.*

Proof Suppose that $\{x_k\}$ converges to x and y. Then given $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$A_b(x_k, x_k, x_k, \ldots, x_k, x) < \frac{\epsilon}{2(n-1)s}$$

for every $k \geq N_1$ and

$$A_b(x_k, x_k, x_k, \ldots, x_k, y) < \frac{\epsilon}{2s}$$

for every $k \geq N_2$. Choose $N = \max\{N_1, N_2\}$. Note that

$$
\begin{aligned}
A_b(x, x, x, \ldots, x, y) \\
\leq s\big[(n-1)A_b(x, x, x, \ldots, x, x_k) + A_b(x_k, x_k, x_k, \ldots, x_k, y)\big] \\
= s\big[(n-1)A_b(x_k, x_k, x_k, \ldots, x_k, x) + A_b(x_k, x_k, x_k, \ldots, x_k, y)\big] \\
< s\left[(n-1) \times \frac{\epsilon}{2(n-1)s} + \frac{\epsilon}{2s}\right] = \epsilon
\end{aligned}
$$

for every $k \geq N$. Since ϵ is arbitrary, $A_b(x, x, x, \ldots, x, y) = A_b(y, y, y, \ldots, y, x) = 0$, and hence $x = y$. □

Definition 1.10 Let (X, A_b) be a dislocated A_b-quasi-metric space with $s \geq 1$ and $n \geq 3$. A sequence $\{x_k\}$ in X is called Cauchy if $\lim_{k,m \to \infty} A_b(x_k, x_k, x_k, \ldots, x_k, x_m) = 0$, that is, if for each $\epsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k, m \geq N$, we have $A_b(x_k, x_k, x_k, \ldots, x_k, x_m) \leq \epsilon$.

Lemma 1.11 *Every convergent sequence in a dislocated A_b-quasi-metric space is a Cauchy sequence.*

Remark 1.12 The converse of Lemma 1.11 does not hold in general. Let $X = \mathbb{Q}$ (the set of rational numbers), and let A_b be a dislocated A_b-quasi-metric defined in Example 1.18. Let $\{x_k\}$ be a sequence defined by $x_k = (1 + \frac{1}{k})^k$. Note that

$$
\begin{aligned}
A_b(x_k, x_k, x_k, \ldots, x_k, x_m) &= s(n-1)|x_k - x_m|^2 \\
&= s(n-1)\left|\left(1 + \frac{1}{k}\right)^k - \left(1 + \frac{1}{m}\right)^m\right|^2 \longrightarrow 0
\end{aligned}
$$

as $k, m \longrightarrow \infty$. Thus, $\{x_k\}$ is a Cauchy sequence. But $x_k \longrightarrow e$ as $k \longrightarrow \infty$. Hence, $\{x_k\}$ is not convergent in $\mathbb{Q}$.

Definition 1.13 A dislocated A_b-quasi-metric space X is said to be complete if every Cauchy sequence in (X, A_b) is convergent.

Definition 1.14 Let (X, A_{b_X}) and (Y, A_{b_Y}) be dislocated A_b-quasi-metric spaces with $s \geq 1$ and $n \geq 3$. A function $f : X \to Y$ is called continuous if, given $x \in X$ and $\epsilon > 0$, we can find a $\delta_{(x,\epsilon)} > 0$ such that $A_{b_Y}(f(x), f(x), \ldots, f(x), f(y)) < \epsilon$ whenever $A_{b_X}(x, x, \ldots, x, y) < \delta_{(x,\epsilon)}$.

Theorem 1.15 *Let (X, A_{b_X}) and (Y, A_{b_Y}) be dislocated A_b-quasi-metric spaces with $s \geq 1$ and $n \geq 3$. A function $f : X \longrightarrow Y$ is continuous at a point $x_0 \in X$ if and only if it is sequentially continuous at x_0.*

Lemma 1.16 *Let (X, A_b) be a dislocated A_b-quasi-metric space. Then the function $A_b(x, x, x, \ldots, x, y)$ is continuous in all of its arguments. In other words, if there exist sequences $\{x_k\}$ and $\{y_k\}$ such that $\lim_{k \to \infty} x_k = x$ and $\lim_{k \to \infty} y_k = y$, then $\lim_{k \to \infty} A_b(x_k, x_k, x_k, \ldots, x_k, y_k) = A_b(x, x, x, \ldots, x, y)$ and $\lim_{k \to \infty} A_b(y_k, y_k, y_k, \ldots, y_k, x_k) = A_b(y, y, y, \ldots, y, x)$.*

Definition 1.17 Let X be a nonempty set, and $s \geq 1$ a given real number. A function $A_b : X^n \to [0, \infty)$ is called A_b-quasi-metric-like on X if for any $a, x_i \in X$, $i = 1, 2, \ldots, n$, $n \geq 3$, the following conditions hold:

(A_{b_1}) $A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = 0$ implies $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$;

(A_{b_2})

$$
A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n)
$$
$$
\leq s \Big[A_b\big(x_1, x_1, x_1, \ldots, (x_1)_{(n-1)}, a\big)
$$
$$
+ A_b\big(x_2, x_2, x_2, \ldots, (x_2)_{(n-1)}, a\big)
$$
$$
+ A_b\big(x_3, x_3, x_3, \ldots, (x_3)_{(n-1)}, a\big)
$$
$$
\vdots
$$
$$
+ A_b\big(x_{(n-1)}, x_{(n-1)}, x_{(n-1)}, \ldots, (x_{(n-1)})_{(n-1)}, a\big)
$$
$$
+ A_b\big(x_n, x_n, x_n, \ldots, (x_n)_{(n-1)}, a\big) \Big].
$$

The pair (X, A_b) is called an A_b-quasi-metric-like space.

Example 1.18 Let $X = \mathbb{R}$. Define the function $A_b : X^n \to [0, \infty)$ by

$$
A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = |x_1 - x_2|^2 + |x_1 - x_3|^2 + \cdots + |x_1 - x_n|^2
$$
$$
+ |x_2 - x_3|^2 + |x_2 - x_4|^2 + \cdots + |x_2 - x_n|^2
$$
$$
\vdots
$$
$$
+ |x_{n-2} - x_{n-1}|^2 + |x_{n-2} - x_n|^2
$$
$$
+ |x_{n-1} - x_n|^2.
$$

Then (X, A_b) is an A_b-quasi-metric-like space with coefficient $s = 2$. Indeed, if $n = 3$, then we have

$$
A_b(x_1, x_2, x_3) = |x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2.
$$

Note that $A_b(x_1, x_2, x_3) = 0$ implies that $x_1 = x_2 = x_3$. Also,

$$
\begin{aligned}
A_b(x_1, x_2, x_3) &= |x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2 \\
&= |x_1 - a + a - x_2|^2 + |x_1 - a + a - x_3|^2 \\
&\quad + |x_2 - a + a - x_3|^2 \\
&\leq \big(|x_1 - a| + |x_2 - a|\big)^2 + \big(|x_1 - a| + |x_3 - a|\big)^2 \\
&\quad + \big(|x_2 - a| + |x_3 - a|\big)^2 \\
&= |x_1 - a|^2 + 2|x_1 - a||x_2 - a| + |x_2 - a|^2 \\
&\quad + |x_1 - a|^2 + 2|x_1 - a||x_3 - a| + |x_3 - a|^2 \\
&\quad + |x_2 - a|^2 + 2|x_2 - a||x_3 - a| + |x_3 - a|^2 \\
&\leq 2\big[2|x_1 - a|^2 + 2|x_2 - a|^2 + 2|x_3 - a|^2\big] \\
&= 2\big[A_b(x_1, x_1, a) + A_b(x_2, x_2, a) + A_b(x_3, x_3, a)\big].
\end{aligned}
$$

Thus, (X, A_b) is an A_b-quasi-metric-like space with coefficient $s = 2$.

Proposition 1.19 *If (X, A_b) is a quasi-metric-like space, then it is a dislocated A_b-quasi-metric space, but the converse does not hold in general.*

Proof Let (X, A_b) be an A_b-quasi-metric-like space with coefficient $s \geq 1$. Then $A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = 0$ implies that $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$, which in turn gives that $A_b(x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1) = 0$.

Let (X, A_b) be a dislocated A_b-quasi-metric space considered in Example 1.3. Since $A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = 0$ does not imply $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$, (X, A_b) is not an A_b-quasi-metric-like space. $\square$

Example 1.20 Let $X = \mathbb{R}$. Define the function $A_b : X^2 \to [0, \infty)$ by

$$
A_b(x_1, x_2) = |2x_1 - x_2|^2.
$$

Then (X, A_b) is a dislocated quasi-b-metric space.

Definition 1.21 Let (X, d) be a metric space, and $f, g : X \to X$. The pair (f, g) is said to be weakly compatible on X if f and g commute at their coincidence points, that is, if $fx = gx$ for some $x \in X$, then $gfx = fgx$.

2 Main results

In this section, we obtain a common fixed point result for mappings satisfying generalized local contractive condition in the setup of dislocated A_b-quasi-metric space X. We start with the following result.

Theorem 2.1 *Let f, g, T, and S be self mappings on a dislocated A_b-quasi-metric space X with $s \geq 1$, $x_0, x_1 \in X$, $y_0 = Sx_0 = gx_1$, and $r > 1$. Suppose that $S(Y) \subseteq g(Y)$, $T(Y) \subseteq f(Y)$, and one of $S(Y)$, $g(Y)$, $T(Y)$, or $f(Y)$ is a complete subspace of Y, where $Y = \overline{B(y_0, r)}$. Suppose*

that there exists $\alpha \in [0, \frac{1}{2})$ such that $0 \le \gamma = (n-1)s\alpha < 1$ with $0 < \frac{\gamma}{1-\gamma} < 1$ and for any $x, y \in Y$, we have

$$A_b(Sx, Sx, \ldots, Sx, Ty) \le \alpha M(x, y) \quad and \quad A_b(y_0, y_0, \ldots, y_0, Tx_1) \le r(1-h), \tag{2.1}$$

where

$$M(x, y) = \max\{A_b(fx, fx, \ldots, fx, gy), A_b(fx, fx, \ldots, fx, Sx), A_b(gy, gy, \ldots, gy, Ty),$$
$$A_b(Sx, Sx, \ldots, Sx, gy), A_b(fx, fx, \ldots, fx, Ty)\}.$$

Then (f, S) and (g, T) have a unique point of coincidence in Y. Furthermore, if (f, S) and (g, T) are weakly compatible, then there exists a unique common fixed point of f, T, S, and g in Y.

Proof Since $T(Y) \subseteq f(Y)$, there exists a point x_2 in Y such that $y_1 = T(x_1) = f(x_2), \ldots$, so there exist $\{x_k\}$ and $\{y_k\}$ in Y such that $y_{2k} = Sx_{2k} = gx_{2k+1}$ and $y_{2k+1} = Tx_{2k+1} = fx_{2k+2}$ for all k in $\mathbb{N}_0$.

Now we show that $y_k \in Y$ for all $k \in \mathbb{N}$.

Since $A_b(y_0, y_0, \ldots, y_0, Tx_1) = A_b(y_0, y_0, \ldots, y_0, y_1) \le r(1-h) < r$, it follows that $y_1 \in Y$. Suppose that $\{y_2, y_3, \ldots, y_j\} \subseteq Y$ for some $j \in \mathbb{N}$. If $j = 2t$, then by (2.1) we have

$$A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1})$$
$$= A_b(Sx_{2t}, Sx_{2t}, \ldots, Sx_{2t}, Tx_{2t+1})$$
$$\le \alpha\big(\max\{A_b(fx_{2t}, fx_{2t}, \ldots, fx_{2t}, gx_{2t+1}), A_b(fx_{2t}, fx_{2t}, \ldots, fx_{2t}, Sx_{2t}),$$
$$A_b(gx_{2t+1}, gx_{2t+1}, \ldots, gx_{2t+1}, Tx_{2t+1}), A_b(Sx_{2t}, Sx_{2t}, \ldots, Sx_{2t}, gx_{2t+1}),$$
$$A_b(fx_{2t}, fx_{2t}, \ldots, fx_{2t}, Tx_{2t+1})\}\big)$$
$$= \alpha\big(\max\{A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}), A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}),$$
$$A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1}), A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t}),$$
$$A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t+1})\}\big)$$
$$= \alpha\big(\max\{A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}), A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1}),$$
$$A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t+1})\}\big)$$
$$\le \alpha\big(\max\{A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}), A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1}),$$
$$s\big[(n-1)A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}) + A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1})\big]\}\big)$$
$$= \alpha s\big[(n-1)A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}) + A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1})\big]$$
$$\le \alpha s(n-1)\big[A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}) + A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1})\big].$$

Thus, we obtain that $A_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1}) \le h(A_b(y_{2t-1}, y_{2t-1}, \ldots, y_{2t-1}, y_{2t}))$, where $h = \frac{\gamma}{1-\gamma}$. Similarly, for $j = 2t + 1$, we have $A_b(y_{2t+1}, y_{2t+1}, \ldots, y_{2t+1}, y_{2t+2}) \le hA_b(y_{2t}, y_{2t}, \ldots, y_{2t}, y_{2t+1})$. Thus,

$$A_b(y_t, y_t, \ldots, y_t, y_{t+1}) \le hA_b(y_{t-1}, y_{t-1}, \ldots, y_{t-1}, y_t) \quad \text{for each } t \in \mathbb{N}.$$

Note that

$$A_b(y_t, y_t, \ldots, y_t, y_{t+1}) \le h A_b(y_{t-1}, y_{t-1}, \ldots, y_{t-1}, y_t)$$

$$\le h^2 A_b(y_{t-2}, y_{t-2}, \ldots, y_{t-2}, y_{t-1})$$

$$\vdots$$

$$\le h^t A_b(y_0, y_0, \ldots, y_0, y_1) \quad \text{for each } t \in \mathbb{N}.$$

By Lemma 1.7 we have

$$A_b(y_0, y_0, \ldots, y_0, y_{t+1}) \le s\big[(n-1)A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$+ h(n-1)A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$+ h^2(n-1)A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$\vdots$$

$$+ (n-1)h^{t-1}A_b(y_0, y_0, \ldots, y_0, y_1) + h^t A_b(y_0, y_0, \ldots, y_0, y_1)\big]$$

$$\le s(n-1)\big(h^0 + h^1 + h^2 + \cdots + h^t\big)A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$= s(n-1)\frac{1-h^{t+1}}{1-h}A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$\le r \quad \text{for each } t \in \mathbb{N}.$$

That is, $y_{t+1} \in Y$. By induction, $\{y_k\} \subseteq Y$. Now we show that $\{y_k\}$ is a Cauchy sequence in Y.

For this, let $m, k \in \mathbb{N}$ with $m > k$. By Lemma 1.7 we have

$$A_b(y_k, y_k, \ldots, y_k, y_m) \le s\big[(n-1)A_b(y_k, y_k, \ldots, y_k, y_{k+1})$$

$$+ (n-1)A_b(y_{k+1}, y_{k+1}, \ldots, y_{k+1}, y_{k+2})$$

$$\vdots$$

$$+ (n-1)A_b(y_{m-2}, y_{m-2}, \ldots, y_{m-2}, y_{m-1})$$

$$+ A_b(y_{m-1}, y_{m-1}, \ldots, y_{m-1}, y_m)\big]$$

$$\le s\big[(n-1)h^k A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$+ (n-1)h^{k+1}A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$\vdots$$

$$+ (n-1)h^{m-2}A_b(y_0, y_0, \ldots, y_0, y_1) + h^{m-1}A_b(y_0, y_0, \ldots, y_0, y_1)\big]$$

$$\le (n-1)s\big(h^k + h^{k+1} + \cdots + h^{m-1}\big)A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$\le (n-1)s\big(h^k + h^{k+1} + \cdots\big)A_b(y_0, y_0, \ldots, y_0, y_1)$$

$$= (n-1)s\left(\frac{h^k}{1-h}\right)A_b(y_0, y_0, \ldots, y_0, y_1).$$

Hence, $\lim_{k,m \to \infty} A_b(y_k, y_k, y_k, \ldots, y_k, y_m) = 0$, that is, $\{y_k\}$ is a Cauchy sequence in Y.

Next, we assume that $T(Y)$ is a complete subspace of Y. We choose a point x^* in $T(Y)$ such that $\lim_{k\to\infty} y_k = x^*$. Then it follows that

$$\lim_{k\to\infty} Sx_{2k} = \lim_{k\to\infty} gx_{2k+1} = \lim_{k\to\infty} Tx_{2k+1} = \lim_{k\to\infty} fx_{2k+2} = x^*.$$

Since $T(Y) \subseteq f(Y)$, there exists a point y in Y such that $f(y) = x^*$. It follows from (2.1) that

$$
\begin{aligned}
A_b\big(Sy, Sy, \ldots, Sy, x^*\big) &\le (n-1)sA_b(Sy, Sy, \ldots, Sy, Tx_{2k+1}) \\
&\quad + sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big) \\
&\le \gamma\big(M(y, x_{2k+1})\big) + sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big),
\end{aligned}
$$

where

$$
\begin{aligned}
M(y, x_{2k+1}) &= \max\big\{ A_b(fy, fy, \ldots, fy, gx_{2k+1}), A_b(fy, fy, \ldots, fy, Sy), \\
&\qquad A_b(gx_{2k+1}, gx_{2k+1}, \ldots, gx_{2k+1}, Tx_{2k+1}), A_b(Sy, Sy, \ldots, Sy, gx_{2k+1}), \\
&\qquad A_b(fy, fy, \ldots, fy, Tx_{2k+1})\big\} \\
&= \max\big\{ A_b\big(x^*, x^*, \ldots, x^*, gx_{2k+1}\big), A_b\big(x^*, x^*, \ldots, x^*, Sy\big), \\
&\qquad A_b(gx_{2k+1}, gx_{2k+1}, \ldots, gx_{2k+1}, Tx_{2k+1}), A_b(Sy, Sy, \ldots, Sy, gx_{2k+1}), \\
&\qquad A_b\big(x^*, x^*, \ldots, x^*, Tx_{2k+1}\big)\big\}.
\end{aligned}
$$

We consider the following cases:

(i) If $M(y, x_{2k+1}) = A_b(x^*, x^*, \ldots, x^*, gx_{2k+1})$, then we have

$$
\begin{aligned}
&A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \\
&\quad \le \gamma A_b\big(x^*, x^*, \ldots, x^*, gx_{2k+1}\big) + sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big).
\end{aligned}
$$

Taking the limit as $k \to \infty$, we have

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \le \gamma A_b\big(x^*, x^*, \ldots, x^*, x^*\big) = 0.$$

Hence, $A_b(Sy, Sy, \ldots, Sy, x^*) = A_b(x^*, x^*, \ldots, x^*, Sy) = 0$ implies that $Sy = x^*$.

(ii) If $M(y, x_{2k+1}) = A_b(x^*, x^*, \ldots, x^*, Sy)$, then we obtain that

$$
\begin{aligned}
&A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \\
&\quad \le \gamma A_b\big(Sy, Sy, \ldots, Sy, x^*\big) + sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big).
\end{aligned}
$$

Taking the limit as $k \to \infty$, we have

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \le \gamma A_b\big(Sy, Sy, \ldots, Sy, x^*\big),$$

which further implies that $Sy = x^*$.

(iii) When $M(y, x_{2k+1}) = A_b(gx_{2k+1}, gx_{2k+1}, \ldots, gx_{2k+1}, Tx_{2k+1})$, we have

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \leq \gamma A_b(gx_{2k+1}, gx_{2k+1}, \ldots, gx_{2k+1}, Tx_{2k+1})$$
$$+ sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big),$$

which by taking the limit as $k \to \infty$ gives that

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \leq \gamma d\big(x^*, x^*, \ldots, x^*, x^*\big) = 0,$$

and hence $Sy = x^*$.

(iv) If $M(y, x_{2k+1}) = A_b(Sy, Sy, \ldots, Sy, gx_{2k+1})$, then we have

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big)$$
$$\leq \gamma A_b(Sy, Sy, \ldots, Sy, gx_{2k+1}) + sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big).$$

Taking the limit as $k \to \infty$, we obtain that

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \leq \gamma A_b\big(Sy, Sy, \ldots, Sy, x^*\big),$$

which implies that $Sy = x^*$.

(v) When $M(y, x_{2k+1}) = A_b(x^*, x^*, \ldots, x^*, Tx_{2k+1})$, we have

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big)$$
$$\leq \gamma A_b\big(x^*, x^*, \ldots, x^*, Tx_{2k+1}\big) + sA_b\big(Tx_{2k+1}, Tx_{2k+1}, \ldots, Tx_{2k+1}, x^*\big),$$

which by taking the limit as $k \to \infty$ implies that

$$A_b\big(Sy, Sy, \ldots, Sy, x^*\big) \leq \gamma A_b\big(x^*, x^*, \ldots, x^*, x^*\big),$$

and hence $Sy = x^*$.

Thus, in all cases, we have $Sy = x^*$, and hence $Sy = fy = x^*$ is a unique coincidence point of (f, S) in Y.

Since $S(Y) \subseteq g(Y)$, we choose a point z in Y such that $g(z) = x^*$. Note that

$$A_b\big(Tz, Tz, \ldots, Tz, x^*\big) \leq (n-1)sA_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, Tz)$$
$$+ sA_b\big(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*\big)$$
$$\leq \gamma\big(M(x_{2k}, z)\big) + sA_b\big(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*\big),$$

where

$$M(x_{2k}, z) = \max\big\{A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, gz), A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Sx_{2k}),$$
$$A_b(gz, gz, \ldots, gz, Tz), A_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, gz),$$
$$A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Tz)\big\}$$

$$= \max\{A_b(x^*, x^*, \ldots, x^*, fx_{2k}), A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Sx_{2k}),$$
$$A_b(x^*, x^*, \ldots, x^*, Tz), A_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*),$$
$$A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Tz)\}.$$

We again consider the following cases:

(i) If $M(x_{2k}, z) = A_b(x^*, x^*, \ldots, x^*, fx_{2k})$, then we have

$$A_b(x^*, x^*, \ldots, x^*, Tz) \leq \gamma A_b(x^*, x^*, \ldots, x^*, fx_{2k}) + sA_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*),$$

which by taking the limit as $k \to \infty$ implies $A_b(x^*, x^*, \ldots, x^*, Tz) \leq \gamma A_b(x^*, x^*, \ldots, x^*, x^*) + sA_b(x^*, x^*, \ldots, x^*, x^*)$, and hence $Tz = x^*$.

(ii) If $M(x_{2k}, z) = A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Sx_{2k})$, then we have

$$A_b(x^*, x^*, \ldots, x^*, Tz)$$
$$\leq \gamma A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Sx_{2k}) + sA_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*),$$

which by taking the limit as $k \to \infty$ implies that

$$A_b(x^*, x^*, \ldots, x^*, Tz)$$
$$\leq \gamma A_b(x^*, x^*, \ldots, x^*, x^*) + sA_b(x^*, x^*, \ldots, x^*, x^*),$$

which further implies that $Tz = x^*$.

(iii) If $M(x_{2k}, z) = A_b(x^*, x^*, \ldots, x^*, Tz)$, then we have

$$A_b(x^*, x^*, \ldots, x^*, Tz)$$
$$\leq \gamma A_b(x^*, x^*, \ldots, x^*, Tz) + sA_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*),$$

which by taking the limit as $k \to \infty$ gives

$$A_b(x^*, x^*, \ldots, x^*, Tz) \leq \gamma A_b(x^*, x^*, \ldots, x^*, Tz) + sA_b(x^*, x^*, \ldots, x^*, x^*),$$

and hence $Tz = x^*$.

(iv) When $M(x_{2k}, z) = A_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*)$, we have

$$A_b(x^*, x^*, \ldots, x^*, Tz)$$
$$\leq \gamma A_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*) + sA_b(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*).$$

Taking the limit as $k \to \infty$, we obtain that

$$A_b(x^*, x^*, \ldots, x^*, Tz)$$
$$\leq \gamma A_b(x^*, x^*, \ldots, x^*, x^*) + sA_b(x^*, x^*, \ldots, x^*, x^*),$$

and so $Tz = x^*$.

(v) If $M(x_{2k}, z) = A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Tz)$, then we have

$$A_b\big(x^*, x^*, \ldots, x^*, Tz\big)$$
$$\leq \gamma A_b(fx_{2k}, fx_{2k}, \ldots, fx_{2k}, Tz) + s A_b\big(Sx_{2k}, Sx_{2k}, \ldots, Sx_{2k}, x^*\big),$$

which by taking the limit as $k \to \infty$ gives

$$A_b\big(x^*, x^*, \ldots, x^*, Tz\big)$$
$$\leq \gamma A_b\big(x^*, x^*, \ldots, x^*, Tz\big) + s A_b\big(x^*, x^*, \ldots, x^*, x^*\big),$$

and hence $Tz = x^*$.

Thus, in all cases, we have $Tz = gz = x^*$, a unique coincidence point of (g, T) in Y.

Suppose that (f, S) is weakly compatible. Then $Sfy = fSy$ implies that $Sx^* = fx^*$. Note that

$$A_b\big(Sx^*, Sx^*, \ldots, Sx^*, x^*\big) = A_b\big(Sx^*, Sx^*, \ldots, Sx^*, Tz\big) \leq \alpha\big(M\big(x^*, z\big)\big),$$

where

$$\begin{aligned}
M\big(x^*, z\big) = \max\big\{ & A_b\big(fx^*, fx^*, \ldots, fx^*, gz\big), A_b\big(fx^*, fx^*, \ldots, fx^*, Sx^*\big), \\
& A_b(gz, gz, \ldots, gz, Tz), A_b\big(Sx^*, Sx^*, \ldots, Sx^*, gz\big), \\
& A_b\big(fx^*, fx^*, \ldots, fx^*, Tz\big)\big\} \\
= \max\big\{ & A_b\big(Sx^*, Sx^*, \ldots, Sx^*, x^*\big), A_b\big(fx^*, fx^*, \ldots, fx^*, Sx^*\big), \\
& A_b\big(x^*, x^*, \ldots, x^*, x^*\big), A_b\big(Sx^*, Sx^*, \ldots, Sx^*, x^*\big), \\
& A_b\big(Sx^*, Sx^*, \ldots, Sx^*, x^*\big)\big\}.
\end{aligned}$$

Thus, $A_b(Sx^*, Sx^*, \ldots, Sx^*, x^*) \leq \alpha A_b(Sx^*, Sx^*, \ldots, Sx^*, x^*)$ implies that $x^* = Sx^* = fx^*$.

Similarly, if (g, T) is weakly compatible, then $Tgz = gTz$ implies that $Tx^* = gx^*$. By (2.1) we have

$$A_b\big(x^*, x^*, \ldots, x^*, Tx^*\big) = A_b\big(Sx^*, Sx^*, \ldots, Sx^*, Tx^*\big) \leq \alpha\big(M\big(x^*, x^*\big)\big),$$

where

$$\begin{aligned}
M\big(x^*, x^*\big) = \max\big\{ & A_b\big(fx^*, fx^*, \ldots, fx^*, gx^*\big), A_b\big(fx^*, fx^*, \ldots, fx^*, Sx^*\big), \\
& A_b\big(gx^*, gx^*, \ldots, gx^*, Tx^*\big), A_b\big(Sx^*, Sx^*, \ldots, Sx^*, gx^*\big), \\
& A_b\big(fx^*, fx^*, \ldots, fx^*, Tx^*\big)\big\} \\
= \max\big\{ & A_b\big(x^*, x^*, \ldots, x^*, Tx^*\big), A_b\big(fx^*, fx^*, \ldots, fx^*, Sx^*\big), \\
& A_b\big(Tx^*, Tx^*, \ldots, Tx^*, Tx^*\big), A_b\big(x^*, x^*, \ldots, x^*, Tx^*\big), \\
& A_b\big(x^*, x^*, \ldots, x^*, Tx^*\big)\big\}.
\end{aligned}$$

Thus, we have $A_b(x^*, x^*, \ldots, x^*, Tx^*) \leq \alpha A_b(x^*, x^*, \ldots, x^*, Tx^*)$, which implies that $x^* = Tx^* = gx^*$, and hence x^* is a common fixed point of f, T, S, and g in Y.

To prove the uniqueness, let $u \in Y$ be such that $fu = gu = Su = Tu = u$. Note that

$$A_b\big(x^*, x^*, \ldots, x^*, u\big) = A_b\big(Sx^*, Sx^*, \ldots, Sx^*, Tu\big) \leq \alpha\big(M\big(x^*, u\big)\big),$$

where

$$\begin{aligned}
M\big(x^*, u\big) = \max\big\{ &A_b\big(fx^*, fx^*, \ldots, fx^*, gu\big), A_b\big(fx^*, fx^*, \ldots, fx^*, Sx^*\big), \\
&A_b(gu, gu, \ldots, gu, Tu), A_b\big(Sx^*, Sx^*, \ldots, Sx^*, gu\big), \\
&A_b\big(fx^*, fx^*, \ldots, fx^*, Tu\big)\big\} \\
= \max\big\{ &A_b\big(x^*, x^*, \ldots, x^*, u\big), A_b\big(x^*, x^*, \ldots, x^*, x^*\big), A_b(u, u, \ldots, u, u), \\
&A_b\big(x^*, x^*, \ldots, x^*, u\big), A_b\big(u, u, \ldots, u, x^*\big)\big\}.
\end{aligned}$$

Hence, $x^* = u$.　　　　　　　　　　　　　　　　　　　　　　　　　　　　　□

Corollary 2.2 *Let f, g, T, and S be self-mappings of a dislocated A_b-quasi-metric space X with coefficient $s \geq 1$. Assume that $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$, and one of $S(X)$, $g(X)$, $T(X)$, or $f(X)$ is a complete subspace of X. Suppose that there exists $\alpha \in [0, \frac{1}{2})$ such that for any $x, y \in X$, we have*

$$A_b(Sx, Sx, \ldots, Sx, Ty) \leq \alpha\big(M(x, y)\big), \tag{2.2}$$

where

$$\begin{aligned}
M(x, y) = \max\big\{ &A_b(fx, fx, \ldots, fx, gy), A_b(fx, fx, \ldots, fx, Sx), A_b(gy, gy, \ldots, gy, Ty), \\
&A_b(Sx, Sx, \ldots, Sx, gy), A_b(fx, fx, \ldots, fx, Ty)\big\}.
\end{aligned}$$

Then (f, S) and (g, T) have a unique point of coincidence in X. Furthermore, if (f, S) and (g, T) are weakly compatible, then there exists a unique common fixed point of f, T, S, and g in X.

Example 2.3 Let X and A_b be as in Example 1.3. Define the mappings f, g, S, and $T : X \longrightarrow X$ by

$$f(x) = \begin{cases} x, & x < 2, \\ 2, & x \geq 2; \end{cases} \qquad S(x) = \begin{cases} \frac{2x}{(1+x)} & \text{if } x \neq -1, \\ 0, & x = -1; \end{cases} \qquad g(x) = T(x) = \begin{cases} \frac{x}{6} & \text{if } x \neq 1, \\ 1, & x = 1. \end{cases}$$

Clearly, (f, S) and (g, T) are pairwise weakly compatible on X with $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$, and $S(X)$ a complete subspace of X. We now show that for all $x, y \in X$, condition (2.2) is satisfied. For this, we consider the following cases:

(i) If $x \neq 1$ and $y = 1$, then

$$\begin{aligned}
A_b(Sx, Sx, \ldots, Sx, Ty) &= \left|\frac{2x}{1+x}\right|^2 + (n-1)\left|2\left(\frac{2x}{1+x}\right) - 1\right|^2 \\
&\leq \frac{1}{36} A_b(fx, fx, \ldots, fx, Ty) \leq \alpha M(x, y).
\end{aligned}$$

(ii) If $x \neq 1$ and $y \neq 1$, then

$$A_b(Sx, Sx, \ldots, Sx, Ty) = \left| \frac{2x}{1+x} \right|^2 + (n-1) \left| 2 \left(\frac{2x}{1+x} \right) - \frac{y}{6} \right|^2$$

$$\leq \frac{1}{36} A_b(fx, fx, \ldots, fx, Ty) \leq \alpha M(x, y).$$

(iii) If $x = -1$ and $y = 1$, then

$$A_b(Sx, Sx, \ldots, Sx, Ty) = (n-1)|0 - 1|^2$$

$$\leq \frac{1}{36} A_b(fx, fx, \ldots, fx, Ty) \leq \alpha M(x, y).$$

(iv) If $x = -1$ and $y \neq 1$, then

$$A_b(Sx, Sx, \ldots, Sx, Ty) = (n-1) \left| 0 - \frac{y}{6} \right|^2$$

$$\leq \frac{1}{36} A_b(fx, fx, \ldots, fx, Ty) \leq \alpha M(x, y).$$

Thus, all the conditions of Corollary 2.2 are satisfied. Moreover, $x = 1$ is a common fixed point of f, T, S, and g.

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