UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

On the algebra of relevance logics

by

Johann Joubert Wannenburg

Submitted in partial fulfilment of the requirements for the degree

Master of Science

in the Faculty of Natural & Agricultural Sciences
University of Pretoria
Pretoria

November 2016

Declaration

I, Johann Joubert Wannenburg, declare that the dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

SIGNATURE: ....

DATE: ....

Acknowledgments

I would like to thank James Raftery for being a better supervisor than anyone could hope for; the entire time. *Grazie* Tommaso Moraschini for your collaboration, and your interest in this project. Lastly, I thank my family for their support, and especially my father for promising to try and read this.

Summary

After recalling some prerequisites from universal algebra in Chapter 1, we recount in Chapter 2 the general theory of *deductive* (logical) *systems*. As working examples, we consider the exponential-free fragment **CLL** of linear logic and some of its extensions, notably the *relevance logic* $\mathbf{R^t}$ and its fragment $\mathbf{R}$ (which lacks a sentential 'truth' constant t of $\mathbf{R^t}$). In Chapter 2, we focus on what it means for two deductive systems to be *equivalent* (in the sense of abstract algebraic logic). To be *algebraizable* is to be equivalent to the equational consequence relation $\models_{\mathsf{K}}$ of some class K of pure algebras. This phenomenon, first investigated in [11], is explored in detail in Chapter 3, and nearly all of the well-known algebraization results for familiar logics can be viewed as instances of it. For example, **CLL** is algebraized by the variety of involutive residuated lattices. The algebraization of stronger logics is then a matter of restriction. In particular, $\mathbf{R^t}$ corresponds in this way to the variety DMM of *De Morgan monoids*, which is studied in Chapter 4. Moreover, the subvarieties of DMM algebraize the axiomatic extensions of $\mathbf{R^t}$.

The lattice of axiomatic extensions of $\mathbf{R^t}$ is naturally of logical interest, but our perspective allows us to view its structure through an entirely algebraic lens: it is interchangeable with the subvariety lattice of DMM. The latter is susceptible to the methods of universal algebra. Exploiting this fact in Chapter 5, we determine (and axiomatize) the minimal subvarieties of DMM, of which, as it happens, there are just four. It follows immediately that $\mathbf{R^t}$ has just four maximal consistent axiomatic extensions; they are described transparently. These results do not appear to be in the published literature of relevance logic (perhaps for philosophical reasons relating to the status of the constant t).

The new findings of Chapter 5 allow us to give, in Chapter 6, a simpler proof of a theorem of K. Świrydowicz [59], describing the upper part of the lattice of axiomatic extensions of $\mathbf{R}$. Among the many potential applications of this result, we explain one that was obtained recently in [52]: the logic $\mathbf{R}$ has no structurally complete axiomatic consistent extension, except for classical propositional logic.

Contents

Introduction

The subject matter of this dissertation belongs to *algebraic logic*, i.e., we are motivated by problems concerning particular *logics*, which we seek to solve using the methods of universal algebra. In the last few decades, efforts of this kind have undergone a great deal of unification. The algebraization of the wide family of *substructural logics* by varieties of *residuated structures* bears witness to this; see [28].

Abstract algebraic logic goes further, in that it treats classes of logics with arbitrary signatures, i.e., special logical connectives such as $\neg, \wedge$ and $\to$ are not assumed present (see [12, 14, 22, 23, 24]). The familiar algebraization theorems for classical, intuitionistic, modal and substructural logics, involving Boolean, Heyting, interior and full Lambek algebras (respectively) are all encompassed by Blok and Pigozzi's influential general theory of algebraization, proposed in [11].

From a mathematical point of view, this is best explained by associating with each algebraizing class K a 'two-dimensional' logic, viz. the equational consequence relation $\models_{\mathsf{K}}$ of K. That object can be compared more directly with the (typically sentential) logic that it algebraizes. The term 'deductive system' accommodates multi-dimensional systems of this sort, as well as traditional sentential (one-dimensional) logics, considered as consequence relations. The algebraization phenomenon is then a special case of a natural notion of *equivalence* for arbitrary deductive systems. All of this is explained in Chapters 2 and 3 below.

An important consequence of the general theory is that, when a variety K algebraizes a sentential logic $\vdash$, there is a transparent lattice anti-isomorphism between the extensions of $\vdash$ and the subquasivarieties of K, taking *axiomatic* extensions onto sub*varieties*. The tools of universal algebra are well suited to investigating the structure of a subvariety lattice, especially in the widely applicable congruence distributive case. This often simplifies the study of the lattice of axiomatic extensions of $\vdash$, which is a natural logical problem.

These considerations apply especially to the family of *relevance logics*.

As we recount in Chapter 4, Dunn showed that the principal system $\mathbf{R^t}$ of Anderson and Belnap [2] is algebraized by the variety DMM of *De Morgan monoids*; see [19, 2]. Two striking results about De Morgan monoids deserve mention here: Slaney [54, 55] established that the free 0–generated De Morgan monoid is finite, and described all the subdirectly irreducible 0-generated De Morgan monoids, while Urquhart [64] proved that DMM has an undecidable equational theory.

Nevertheless, the algebraic analysis of relevance logics via De Morgan monoids is a rather neglected topic. Possible philosophical reasons for this are touched on in Section 4.2, where we discuss a fragment **R** of $\mathbf{R^t}$. In particular, the published literature seems to contain no algebraic analysis of the subvariety lattice of DMM.

In Chapters 5 and 6 of the present dissertation, we initiate an attempt to fill this gap. We identify the minimal varieties of De Morgan monoids, and hence the maximal consistent axiomatic extensions of $\mathbf{R^t}$, of which there are just four. The result yields a simpler proof of a description (due to Świrydowicz [59]) of the upper part of the lattice of axiomatic extensions of **R**. That description has led, in turn, to the identification (in [52]) of the structurally complete axiomatic extensions of **R**. In our account of this material, some findings of Slaney are presented in a more self-contained manner (and in more standard terminology), such as his implicit determination of the simple 0-generated De Morgan monoids.

Chapter 1

Preliminaries

1.1 Algebras

In the field of universal algebra, an *algebra* $\boldsymbol{A}$ is a non-empty set A, called the *universe* of $\boldsymbol{A}$, together with a set of *basic operations* (a.k.a. fundamental or distinguished operations) on A. Each basic operation is a function from A^n into A for some $n \in \omega$ and n is called the *arity* of the operation. (Here, ω denotes the set of non-negative integers. From now on, algebras denoted by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ are understood to have universes $A, B, C, \ldots$, respectively.)

Example 1.1. *Semigroups* are algebras with one *associative* binary operation. In other words, an algebra $\boldsymbol{A} = \langle A; \cdot \rangle$ is a semigroup if for all $a, b, c \in A$,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c. \tag{1.1}$$

An *(algebraic) signature* or *type* is a set $\mathcal{F}$ of symbols (called *operation symbols*), together with a function $\varphi : \mathcal{F} \to \omega$, called the *arity function* of $\mathcal{F}$. We say that $f \in \mathcal{F}$ is n*-ary* if $\varphi(f) = n$. (The arity function of a signature will always be denoted by φ in what follows.)

We say that an algebra $\boldsymbol{A}$ has signature (or type) $\mathcal{F}$ (and we write $\boldsymbol{A} = \langle A; \mathcal{F} \rangle$), provided that, to each symbol $f \in \mathcal{F}$ there corresponds exactly one basic operation of $\boldsymbol{A}$, denoted by $f^{\boldsymbol{A}}$, such that the arity of $f^{\boldsymbol{A}}$ is $\varphi(f)$. If, in addition, the set $\mathcal{F}$ is finite, we say that $\boldsymbol{A}$ has *finite type*. An algebra is said to be *finite* (resp. *trivial*) if its universe is a finite set (resp. a singleton). We often leave out the superscripts when denoting operations if the underlying algebra is clear. For example, every semigroup has a signature of $\mathcal{F} = \{\cdot\}$, with arity $\varphi(\cdot) = 2$. When two algebras have the same signature, they are said to be *similar*.

Example 1.2. An algebra $\boldsymbol{A} = \langle A; \cdot^{\boldsymbol{A}}, e^{\boldsymbol{A}} \rangle$ is a *monoid* if it is a semigroup with an element $e^{\boldsymbol{A}}$, called an *identity* (for $\cdot^{\boldsymbol{A}}$), and for every $a \in A$

$$a \cdot^{\boldsymbol{A}} e^{\boldsymbol{A}} = a \text{ and } e^{\boldsymbol{A}} \cdot^{\boldsymbol{A}} a = a. \tag{1.2}$$

So, the signature of a monoid is $\{\cdot, e\}$ with arity function

$$\begin{aligned} \varphi : \cdot &\mapsto 2 \\ e &\mapsto 0. \end{aligned}$$

As above, it is often convenient to write $a \cdot^{\boldsymbol{A}} b$ instead of $\cdot^{\boldsymbol{A}}(a, b)$ for binary operations. Operations with arity 0 are called *distinguished elements* or *constants*; they take no arguments and specify a certain element in an algebra. We often specify the signature using a list of symbols followed by a list of numbers, e.g., for monoids we could say that they have a signature $\cdot, e$ with type $(2, 0)$. Operation symbols with arity 0, such as e is this case, are called *constant symbols*.

Definition 1.3. A *lattice* is an algebra $\langle L; \wedge, \vee \rangle$ of type $(2, 2)$ such that for every $a, b, c \in L$,

$$\begin{aligned} a \wedge (b \wedge c) &= (a \wedge b) \wedge c & \qquad a \vee (b \vee c) &= (a \vee b) \vee c \\ a \wedge a &= a & a \vee a &= a \\ a \wedge b &= b \wedge a & a \vee b &= b \vee a \\ a \wedge (a \vee b) &= a & a \vee (a \wedge b) &= a \end{aligned}$$

The first condition is called *associativity*, as with semigroups. The second condition is called *idempotence*, the third *commutativity* and the last *absorption*. The operation $\wedge$ is called 'meet' and $\vee$ is called 'join'.

Lattices will be very important structures throughout this thesis, and we will focus on them in the next section, from a different point of view.

Definition 1.4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras in the same signature $\mathcal{F}$.

(i) $\boldsymbol{B}$ is a *subalgebra* of $\boldsymbol{A}$ if $B \subseteq A$ and $f^{\boldsymbol{B}} = f^{\boldsymbol{A}}|_{B^n}$ for every $f \in \mathcal{F}$, where $n = \varphi(f)$. Thus, the operations of $\boldsymbol{B}$ are just the operations of $\boldsymbol{A}$ restricted to the appropriate cartesian powers of B.

(ii) A function $h : A \to B$ is a *homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$* if, for every $f \in \mathcal{F}$ with $n = \varphi(f)$, and all $a_1, \ldots, a_n \in A$,

$$h(f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n)).$$

Let h be a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. If h is injective, we call it an *embedding*. If h is surjective (i.e., onto), then $\boldsymbol{B}$ is a *homomorphic image* of $\boldsymbol{A}$. If h is bijective then it is called an *isomorphism*, and we write $h\colon \boldsymbol{A} \cong \boldsymbol{B}$. When such an h exists, we say that $\boldsymbol{A}$ and $\boldsymbol{B}$ are *isomorphic*, and we write $\boldsymbol{A} \cong \boldsymbol{B}$. As in the case of familiar algebras, compositions of homomorphisms are homomorphisms, and inverse functions of isomorphisms are isomorphisms.

While we are talking about images of functions, let's introduce some notation. We denote the identity function on a set A by id_A. Let $f : A \to B$ be a function. For any $X \subseteq A$, we define $f[X] \coloneqq \{f(x) : x \in X\}$. If $Y \subseteq B$, then $\overleftarrow{f}[Y] = \{x \in A : f(x) \in Y\}$.

If $h : \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism of algebras and $\boldsymbol{X}$ (resp. $\boldsymbol{Y}$) is a subalgebra of $\boldsymbol{A}$ (resp. $\boldsymbol{B}$), then $h[X]$ (resp. $\overleftarrow{h}[Y]$) is the universe of a subalgebra of $\boldsymbol{B}$ (resp. $\boldsymbol{A}$), which we denote by $h[\boldsymbol{X}]$ (resp. $\overleftarrow{h}[\boldsymbol{Y}]$). In particular, if h is an embedding, then $\boldsymbol{A}$ is isomorphic to a subalgebra of $\boldsymbol{B}$.

Definition 1.5. Let I be some (index) set, and let $\boldsymbol{A}_i$ be an algebra for every $i \in I$, where all these algebras have signature $\mathcal{F}$. Recall that the *Cartesian product* of these sets, denoted $\prod_{i\in I} A_i$, is the set of all functions g from I into $\bigcup_{i\in I} A_i$ such that $g(i) \in A_i$ for all $i \in I$. We write the *direct product* $\boldsymbol{B}$ of these algebras as $\prod_{i\in I} \boldsymbol{A}_i$. The universe of $\boldsymbol{B}$ is defined to be $\prod_{i\in I} A_i$ and, for every $f \in \mathcal{F}$ such that $n = \varphi(f)$, and all $g_1, \ldots, g_n \in \boldsymbol{B}$, we define

$$(f^{\boldsymbol{B}}(g_1, \ldots, g_n))(i) = f^{\boldsymbol{A}_i}(g_1(i), \ldots, g_n(i)), \quad \text{for } i \in I.$$

For each $j \in I$, the j^{th} *projection* $p_j\colon \prod_{i\in I} A_i \to A_j$, sending each g to $g(j)$, is a (surjective) homomorphism from $\prod_{i\in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_j$.

As usual, when I is finite, with cardinality m, we can represent the elements of $\prod_{i\in I} \boldsymbol{A}_i$ as tuples of length m, and the operations simply act coordinate-wise.

If $\boldsymbol{B}$ is an algebra and h_i is a homomorphism from $\boldsymbol{B}$ to an algebra $\boldsymbol{A}_i$, for every i in some set I, then we can define a homomorphism

$$\prod_{i\in I} h_i : \boldsymbol{B} \to \prod_{i\in I} \boldsymbol{A}_i$$
$$b \mapsto f,$$

where $f : I \to \bigcup_{i\in I} A_i$, and $f(i) = h_i(b)$, for every $i \in I$.

Given a class of (similar) algebras K, we let $\mathbb{S}(\mathsf{K})$ stand for the class of all subalgebras of members of K. Similarly, $\mathbb{H}(\mathsf{K})$ and $\mathbb{P}(\mathsf{K})$ stand for the classes of homomorphic images and products, respectively. The direct product of the empty subfamily of K is understood to be a trivial algebra in the same signature. $\mathbb{I}(\mathsf{K})$ denotes the class to algebras isomorphic to members of K.

A binary relation θ on a set A is called an *equivalence relation* if it is

reflexive: for any $a \in A$, $\langle a, a\rangle \in \theta$;

symmetric: if $\langle a, b\rangle \in \theta$ then $\langle b, a\rangle \in \theta$;

transitive: if $\langle a, b\rangle, \langle b, c\rangle \in \theta$ then $\langle a, c\rangle \in \theta$.

Given an algebra $\boldsymbol{A}$, we say that θ is a *congruence* on $\boldsymbol{A}$ if it is an equivalence relation on A and it is compatible with the operations, i.e., for any $f \in \mathcal{F}$, such that $n = \varphi(f)$,

$$\text{if } \langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \theta \text{ then } \langle f^{\boldsymbol{A}}(a_1, \ldots, a_n), f^{\boldsymbol{A}}(b_1, \ldots, b_n)\rangle \in \theta.$$

We often write $a \equiv_\theta b$ instead of $\langle a, b\rangle \in \theta$. Let θ be an equivalence relation on set A. For any $a \in A$, the *equivalence class of a* is

$$a/\theta \coloneqq \{b \in A : a \equiv_\theta b\}.$$

We denote $A/\theta \coloneqq \{a/\theta : a \in A\}$. The following function is called the *canonical surjection*:

$$\begin{aligned} q_\theta : A &\to A/\theta \\ a &\mapsto a/\theta. \end{aligned}$$

Given a congruence θ of an algebra $\boldsymbol{A}$, we define an algebra $\boldsymbol{A}/\theta$, to have universe A/θ, and for any $f \in \mathcal{F}$ such that $n = \varphi(f)$,

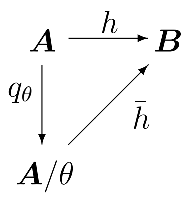
$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) \coloneqq f^{\boldsymbol{A}}(a_1, \ldots, a_n)/\theta.$$

It follows from the definition of congruence that the operations of $\boldsymbol{A}/\theta$ are well-defined, and that $\boldsymbol{A}/\theta$ is a homomorphic image of $\boldsymbol{A}$, under the homomorphism q_θ.

In fact, we can say more, but first we need to define the *kernel* of a homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$,

$$\ker h \coloneqq \{\langle a, b\rangle : h(a) = h(b)\}.$$

Theorem 1.6 (Homomorphism Theorem [5, Thm. 1.22])**.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar algebras, and let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism with kernel θ. Then θ is a congruence on $\boldsymbol{A}$ and there is a unique injective homomorphism $\bar{h} : \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $\bar{h} \circ q_\theta = h$. If h is surjective, then $\bar{h}$ is an isomorphism.*



1.2 Lattices

In Definition 1.3, we defined lattices as algebras that satisfy certain equations. Despite the fact that it will be important for us later that they can be defined in this way, it is more natural to think of them as special partially ordered sets.

Let A be a set. A binary relation $\leq$ on A is a *partial order* if $\leq$ is reflexive, transitive and *anti-symmetric*, i.e.,

$$\text{if } a \leq b \text{ and } b \leq a \text{ then } a = b.$$

In this case we say that $\langle A; \leq \rangle$ is a *poset* (short for partially ordered set), provided that $A \neq \emptyset$.

Let $\langle A; \leq^A \rangle$ and $\langle B; \leq^B \rangle$ be posets. A map $h : A \to B$ is said to be

- *order-preserving* or *isotone* if, whenever $a \leq^A b$, then $h(a) \leq^B h(b)$;
- *order-reversing* or *antitone* if, whenever $a \leq^A b$, then $h(b) \leq^B h(a)$;
- *order-reflecting* if, whenever $h(a) \leq^B h(b)$, then $a \leq^A b$.

Let $\leq$ be a partial order on A, and $X \subseteq A$. Then the *infimum* of X, if it exists, is an element $a \in A$ such that $a \leq b$ for any $b \in X$, and whenever $c \leq b$ for all $b \in X$, then $c \leq a$. In words, the infimum of X is the largest element of A that is a lower bound of every element of X. We denote the infimum of X by $\inf(X)$. Dually we can define the *supremum* of X, $\sup(X)$, as the smallest element of A that is an upper bound of every element of X, if it exists. Note that if $\inf(\emptyset)$ (resp. $\sup(\emptyset)$) exists, then it must be the greatest (resp. least) element of A, with respect to $\leq$.

Characterization 1.7 ([5, Def. 2.1])**.** Let $\boldsymbol{A}$ be a lattice. Define a binary relation $\leq$ on A as follows: for any $a, b \in A$,

$$a \leq b \text{ iff } a \wedge b = a.$$

Then $\leq$ is a partial order on A. Also, $a \wedge b = \inf(\{a, b\})$ and $a \vee b = \sup(\{a, b\})$, for all $a, b \in A$.

Conversely, let $\langle A, \leq \rangle$ be a poset, such that $\inf(\{a, b\})$ and $\sup(\{a, b\})$ exist for all $a, b \in A$. Then $\langle A; \wedge, \vee \rangle$ is a lattice, where we define $a \wedge b := \inf(\{a, b\})$ and $a \vee b := \sup(\{a, b\})$, for any $a, b \in A$.

Whenever we are working with a lattice, we will use the symbol $\leq$ to denote the partial order given above. We also define the notation $\bigwedge X := \inf(X)$ and $\bigvee X := \sup(X)$, and call these the meet and join of X, respectively. Similarly, we say that a poset $\langle A, \leq \rangle$ is a lattice if $\inf(\{a, b\})$ and $\sup(\{a, b\})$ exist for any $a, b \in A$.

Theorem 1.8 ([5, Prop. 2.3])**.** *Let h be a map from lattice $\boldsymbol{A}$ to lattice $\boldsymbol{B}$. Then h is a lattice isomorphism if and only if h is order-preserving, order-reflecting and surjective.*

Definition 1.9. A lattice $\boldsymbol{L}$ is called *complete* if, for every subset X of L, both $\inf(X)$ and $\sup(X)$ exist.

Let A be a set. We let $\mathcal{P}(A)$ denote the power-set of A, i.e., the set of all subsets of A. Then $\langle \mathcal{P}(A), \cap, \cup \rangle$ is a complete lattice, with partial order $\subseteq$.

Lemma 1.10. *Let $\langle A, \leq \rangle$ be a poset, such that* $\inf(X)$ *exists for any $X \subseteq A$. Then $\langle A; \leq \rangle$ is a complete lattice, where for any $X \subseteq A$,*

$$\sup(X) = \inf(\{a \in A : b \leq a \textit{ for every } b \in X\}).$$

Definition 1.11. Let A be a set. A subset $\mathcal{S}$ of $\mathcal{P}(A)$ is called a *closure system* over A if for every $\mathcal{X} \subseteq \mathcal{S}$, we have $\bigcap \mathcal{X} \in \mathcal{S}$. In this context, we interpret $\bigcap \emptyset$ as A (and $\bigcup \emptyset$ as $\emptyset$).

It is clear from Lemma 1.10 that if $\mathcal{S}$ is a closure system of A, then $\langle S, \subseteq \rangle$ is a complete lattice. It is interesting to note that the converse is also true, cf. [5, Thm. 2.26]:

Fact 1.12. *Every complete lattice is isomorphic to a closure system over some set.*

Definition 1.13. Let A be a set. A *closure operator* over A is a function $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that for all $X, Y \subseteq A$

(i) $X \subseteq C(X)$,

(ii) $C(C(X)) = C(X)$,

(iii) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$.

Theorem 1.14 ([29, pp. 24,31])**.** *Let A be a set.*

(i) If $\mathcal{S}$ is a closure system over A then

$$C_{\mathcal{S}} : X \mapsto \bigcap\{Y \in \mathcal{S} : X \subseteq Y\}$$

is a closure operator over A and $C_{\mathcal{S}}[\mathcal{P}(A)] = \mathcal{S}$.

(ii) If C is a closure operator over A, then $C[\mathcal{P}(A)]$ is a closure system over A and $C_{C[\mathcal{P}(A)]} = C$. Furthermore $\bigvee_{C[\mathcal{P}(A)]} \mathcal{X} = C\left(\bigcup \mathcal{X}\right)$, for any $\mathcal{X} \subseteq C[\mathcal{P}(A)]$.

Definition 1.15. A closure operator C over a set A is called *algebraic* if, for every $X \subseteq A$,

$$C(X) = \bigcup\{C(Z) : Z \text{ is a finite subset of } X\}.$$

Definition 1.16. Let $\boldsymbol{L}$ be a complete lattice.

(i) An element $a \in L$ is called *compact* if for every $X \subseteq L$, whenever $a \leq \bigvee X$, then there exists a finite $Z \subseteq X$ such that $a \leq \bigvee Z$.

(ii) $\boldsymbol{L}$ is called an *algebraic lattice* if every element is the join of a set of compact elements.

Theorem 1.17 ([5, Thm. 2.30])**.** *Let C be an algebraic closure operator over a set A. Then $C[\mathcal{P}(A)]$ is an algebraic lattice. The compact elements of $C[\mathcal{P}(A)]$ are those sets of the form $C(Y)$ for Y a finite subset of A.*

The following result is useful when exploiting Theorem 1.17.

Theorem 1.18 ([29, Lem. 3, p. 24])**.** *Let $\mathcal{S}$ be a closure system over a set A, and suppose that $\bigcup \mathcal{X} \in \mathcal{S}$ for every non-empty subset $\mathcal{X}$ of $\mathcal{S}$ that is* directed *(i.e., whenever $Y, Z \in \mathcal{X}$, then $Y \cup Z \subseteq W$ for some $W \in \mathcal{X}$). Then the closure operator $C_{\mathcal{S}}$ over A is algebraic.*

When the assumptions of Theorem 1.17 hold, we call $\mathcal{S}$ an *algebraic* closure system over A.

We will often make use of these theorems to show that certain structures are complete—possibly algebraic—lattices. In the interest of keeping all the lattice definitions in one place, we will introduce a bit of lattice terminology, before we get to examples of closure systems.

A lattice $\boldsymbol{L}$ is *bounded* is there exist $\bot, \top \in L$ such that $\bot \leq a \leq \top$ for all $a \in L$. The elements $\bot$ and $\top$ are called the *bounds* of $\boldsymbol{L}$.

An element a of a lattice $\boldsymbol{L}$ is called *meet-irreducible* provided that, whenever $a = b \wedge c$, with $b, c \in L$, then $a = b$ or $a = c$.

Let $\boldsymbol{L}$ be a complete lattice. An element $a \in L$ is said to be *completely meet-irreducible* (in $\boldsymbol{L}$) if, whenever $a = \bigwedge X$, where $X \subseteq L$, then $a \in X$.

Theorem 1.19 ([5, Lemma 3.22])**.** *Suppose that a is an element of a complete lattice $\boldsymbol{L}$. The following are equivalent.*

(i) a is completely meet-irreducible

(ii) There is an element $c \in L$ such that $a < c$ and for every $b \in L$, if $a < b$ then $c \leq b$.

Let b and c be elements of a lattice. We say that c is a *cover* of b, if $b < c$ and there are no elements strictly between b and c. In this case we also say that b is a *subcover* of c. If a lattice $\boldsymbol{L}$ has a least element $\bot$, then the covers of $\bot$ in $\boldsymbol{L}$ (if any) are called the *atoms* of $\boldsymbol{L}$.

We can define *join-irreducible* and *completely join-irreducible* elements dually.

Definition 1.20. A lattice $\boldsymbol{L}$ is *distributive* if, for every $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{1.3}$$

If a lattice $\boldsymbol{L}$ is distributive, then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1.4}$$

holds for all $a, b, c \in L$, and conversely. We refer to (1.3) and (1.4) as *distributive laws.*

Remark 1.21. An element a of a lattice $\boldsymbol{L}$ is said to be *join-prime* provided that, whenever $a \leq b \vee c$, with $b, c \in L$, then $a \leq b$ or $a \leq c$. In this case, clearly, a is join-irreducible. In a distributive lattice, the converse holds, i.e., an element is join-irreducible if and only if it is join-prime. (The reason is that the relation $a \leq b \vee c$ amounts to $a = a \wedge (b \vee c)$, which distributes.)

Definition 1.22. A *Boolean algebra* is a bounded distributive lattice $\boldsymbol{L}$, with bounds $\bot, \top$, in which every element $a \in L$ has a *complement*, i.e., there exists $a' \in L$ such that $a \vee a' = \top$ and $a \wedge a' = \bot$.

When we treat these structures as algebras (rather than posets), we normally give them the signature $\wedge, \vee, \,'$ (but note that $\top$ and $\bot$ are definable as $x \vee x'$ and $x \wedge x'$, respectively).

Definition 1.23. Let $\boldsymbol{L}$ be a lattice. A subset F of L is a *lattice filter* of $\boldsymbol{L}$ if F is *upward-closed* (i.e., whenever $a \in F$ and $a \leq b \in L$ then $b \in F$) and closed under meets (i.e., whenever $a, b \in F$ then $a \wedge b \in F$).

1.3 More on Algebras

Let us now return to congruences of algebras. Let $\boldsymbol{A}$ be an algebra. One can show easily that the intersection an arbitrary collection of congruences of $\boldsymbol{A}$ is still a congruence of A. Therefore, the set of congruences of $\boldsymbol{A}$ forms a closure system over $A \times A$. We let $\mathrm{Con}(\boldsymbol{A})$ denote the complete lattice of congruences of $\boldsymbol{A}$. We denote the corresponding closure operator by

$$\Theta^{\boldsymbol{A}} : X \mapsto \bigcap\{\theta \in \mathrm{Con}(\boldsymbol{A}) : X \subseteq \theta\}, \text{ for any } X \subseteq A \times A.$$

$\Theta^{\boldsymbol{A}}(X)$ is called the congruence of $\boldsymbol{A}$ *generated by* X. The least element of $\mathrm{Con}(\boldsymbol{A})$ is the *identity congruence* $\mathrm{id}_A \coloneqq \{\langle a, a\rangle : a \in A\}$ of $\boldsymbol{A}$. Notice that id_A is the same object as the identity function on A and that every congruence of $\boldsymbol{A}$ must contain id_A, because of reflexivity. The greatest element of $\mathrm{Con}(\boldsymbol{A})$, called the *total congruence*, is $A^2 \coloneqq A \times A$.

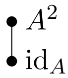
Congruence lattices are extremely important in universal algebra. We can characterize joins in $\mathrm{Con}(\boldsymbol{A})$ as follows:

Theorem 1.24 ([5, Prop. 2.16])**.** *Let Σ be a set of congruences of algebra $\boldsymbol{A}$. Then, in $\mathrm{Con}(\boldsymbol{A})$, we have $\langle a, b\rangle \in \bigvee \Sigma$ if and only if there exist $c_1, \ldots, c_n \in A$ and $\psi_1, \ldots, \psi_{n-1} \in \Sigma$ such that*

$$a = c_1 \equiv_{\psi_1} c_2 \equiv_{\psi_2} \cdots \equiv_{\psi_{n-1}} c_n = b.$$

For any algebra $\boldsymbol{A}$, the union of a directed non-empty subset of $\mathrm{Con}(\boldsymbol{A})$ is easily shown to be a congruence of $\boldsymbol{A}$, so $\mathrm{Con}(\boldsymbol{A})$ is both an algebraic closure system over $A \times A$ and an algebraic lattice (w.r.t. set inclusion), and $\Theta^{\boldsymbol{A}}$ is an algebraic closure operator over $A \times A$ (see Theorem 1.17 and Theorem 1.18). In particular, the compact elements of $\mathrm{Con}\, A$ are the *finitely generated* congruences of $\boldsymbol{A}$, i.e., the relations of the form $\Theta^{\boldsymbol{A}}(X)$, where X is a finite subset of $A \times A$.

An algebra $\boldsymbol{A}$ is said to be *simple* if it has just two congruences (as below), in which case $\boldsymbol{A}$ must be non-trivial.

$\mathrm{Con}(\boldsymbol{A})$: 

By the Homomorphism Theorem (Theorem 1.6), every homomorphic image of a simple algebra $\boldsymbol{A}$ is either trivial or isomorphic to $\boldsymbol{A}$. The Homomorphism Theorem is sometimes called *the First Isomorphism Theorem.*

Let θ and ψ be congruences of an algebra $\boldsymbol{A}$, such that $\theta \subseteq \psi$. We define a binary relation ψ/θ on A/θ by

$$\psi/\theta := \{\langle a/\theta, b/\theta\rangle : \langle a, b\rangle \in \psi\}.$$

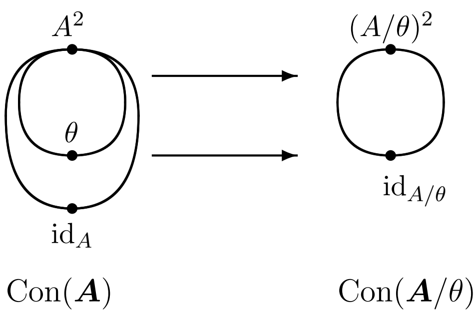
Theorem 1.25 (The Second Isomorphism Theorem [5, Thm. 3.5])**.** *Let $\theta \subseteq \psi$ be congruences on an algebra $\boldsymbol{A}$. Then ψ/θ is a congruence on $\boldsymbol{A}/\theta$. The algebras $(\boldsymbol{A}/\theta)/(\psi/\theta)$ and $\boldsymbol{A}/\psi$ are isomorphic.*

If $\boldsymbol{L}$ is a lattice with elements $a \leq b$, then the *interval* $I[a, b]$ is defined to be $\{c \in L : a \leq c \leq b\}$. This is obviously a *sublattice* of $\boldsymbol{L}$, i.e., a subalgebra of $\langle L; \wedge, \vee\rangle$.

Theorem 1.26 (The Correspondence Theorem [5, Thm. 3.6])**.** *Let $\boldsymbol{A}$ be an algebra and let θ be a congruence on $\boldsymbol{A}$. Then the map*

$$\begin{gathered} I[\theta, A^2] \to \mathrm{Con}(\boldsymbol{A}/\theta) \\ \psi \mapsto \psi/\theta \end{gathered}$$

is a lattice isomorphism.



When it comes to subalgebras, we wish to identify each subalgebra of an algebra $\boldsymbol{A}$ with its universe, so that we can take intersections of subalgebras.

But notice that it is possible (if $\boldsymbol{A}$ has no constants in its signature) that the intersection of the universes of two subalgebras of $\boldsymbol{A}$ may be empty. Our definition of an algebra does not allow the universe of an algebra to be empty. So, we need to define a *subuniverse* of an algebra $\boldsymbol{A}$ to be a subset of A that is closed with respect to the basic operations of $\boldsymbol{A}$. Notice that if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, then B is a subuniverse of $\boldsymbol{A}$. Conversely, if B is a *non-empty* subuniverse of $\boldsymbol{A}$, then we can can restrict the operations of $\boldsymbol{A}$ to the set B to form a subalgebra of $\boldsymbol{A}$ with universe B.

Now, intersections of subuniverses of $\boldsymbol{A}$ are again subuniverses of $\boldsymbol{A}$, as are the unions of directed non-empty families of subuniverses. For any $X \subseteq A$, therefore, we can define $\mathrm{Sg}^{\boldsymbol{A}}(X)$ to be the smallest subuniverse of $\boldsymbol{A}$ that contains X; we call it the *subuniverse of* $\boldsymbol{A}$ *generated by* X. The function $\mathrm{Sg}^{\boldsymbol{A}}$ is an algebraic closure operator over A. Notice that $\mathrm{Sg}^{\boldsymbol{A}}(X)$ can only be empty if X is empty and $\boldsymbol{A}$ has no constant in its signature; that combination of cases shall not arise here. We say that an algebra $\boldsymbol{A}$ is $\mathfrak{m}$-*generated*, where $\mathfrak{m}$ is a cardinal, if $\mathrm{Sg}^{\boldsymbol{A}}(X) = A$, for some $X \subseteq A$, where the cardinality of X is at most $\mathfrak{m}$. If, moreover, $\mathfrak{m} \in \omega$, we say that $\boldsymbol{A}$ is *finitely generated*. Note, therefore, that $\boldsymbol{A}$ cannot be 0-generated unless it has some constants in its signature. In this case, $\boldsymbol{A}$ is 0-generated if and only if it contains no proper subalgebra.

Theorem 1.27 ([5, Cor. 3.3])**.** *Suppose that* $h : \boldsymbol{A} \to \boldsymbol{B}$ *is a surjective homomorphism of algebras. If* $\boldsymbol{A}$ *is generated by* $X \subseteq A$*, then* $\boldsymbol{B}$ *is generated by* $h[X]$*.*

We will now introduce an extremely important class of algebras, namely *subdirectly irreducible* algebras. Their usefulness far outweighs their slightly complicated definition.

Definition 1.28. An algebra $\boldsymbol{B}$ is a *subdirect product* of algebras $\boldsymbol{A}_i$, for i in some index set I, if $\boldsymbol{B}$ is a subalgebra of $\prod_{i\in I} \boldsymbol{A}_i$ and for every $i \in I$, $p_i|_B : \boldsymbol{B} \to \boldsymbol{A}_i$ is surjective, where p_i is the i^{th} projection from $\prod_{j\in I} \boldsymbol{A}_j$ onto $\boldsymbol{A}_i$.

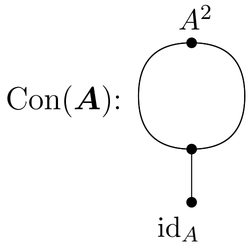
An embedding $h : \boldsymbol{B} \to \prod_{i\in I} \boldsymbol{A}_i$ is called *subdirect* if $h[\boldsymbol{B}]$ is a subdirect product the $\boldsymbol{A}_i$'s. In this case, each $\boldsymbol{A}_i$ is a homomorphic image of $\boldsymbol{B}$.

Definition 1.29. An algebra $\boldsymbol{A}$ is called *subdirectly irreducible (SI)* if for every subdirect embedding $h : \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i$, there is a $j \in I$ such that $p_j \circ h : \boldsymbol{A} \to \boldsymbol{A}_j$ is an isomorphism.

Thankfully there is a characterization which is much easier to visualise.

Theorem 1.30 ([5, Thm. 3.23])**.** *An algebra* $\boldsymbol{A}$ *is subdirectly irreducible if and only if* id_A *is completely meet-irreducible in* $\mathrm{Con}(\boldsymbol{A})$*.*

Notice that because of Theorem 1.19, an algebra $\boldsymbol{A}$ is SI iff it has a smallest non-identity congruence.



The main reason why subdirectly irreducible algebras are important is because of the following theorem; see [7] or [5, Thm. 3.24].

Theorem 1.31 (Birkhoff's Subdirect Decomposition Theorem)**.** *Every algebra* $\boldsymbol{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras (that are homomorphic images of* $\boldsymbol{A}$*).*

Definition 1.32. An algebra $\boldsymbol{A}$ is said to be *finitely subdirectly irreducible (FSI)* if id_A is meet-irreducible in $\mathrm{Con}(\boldsymbol{A})$.

Notice that simple algebras are subdirectly irreducible, since the identity congruence of a simple algebra has only one cover, namely the total congruence. Furthermore, it is clear from Theorem 1.30 that subdirectly irreducible algebras are finitely subdirectly irreducible.

Let K be a class of algebras. We let $\mathbb{P}_{\mathrm{s}}(\mathsf{K})$ denote the class of subdirect products of members of K. K_{SI} and K_{FSI} denote the subclasses of K consisting of subdirectly irreducible and finitely subdirectly irreducible algebras, respectively.

Let I be a set. A *filter over* I is a family $\mathcal{U}$ of subsets of I such that

- $I \in \mathcal{U}$;
- if $X \in \mathcal{U}$ and $X \subseteq Y$ then $Y \in \mathcal{U}$;
- if $X, Y \in \mathcal{U}$ then $X \cap Y \in \mathcal{U}$.

In other words $\mathcal{U}$ is a lattice filter of the lattice $\langle \mathcal{P}(I), \cap, \cup \rangle$. We call $\mathcal{U}$ a *proper* filter if $\emptyset \notin \mathcal{U}$.

A proper filter $\mathcal{U}$ over I is called an *ultrafilter* if

- whenever $X \cup Y \in \mathcal{U}$, then $X \in \mathcal{U}$ or $Y \in \mathcal{U}$.

Equivalently, an ultrafilter over I is a maximal proper filter over I.

Let $\boldsymbol{A}_i$ be an algebra for every i in set I, and $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i$. For $a, b \in A$ we define

$$[\![a = b]\!] := \{i \in I : a_i = b_i\},$$

where the subscript means the i^{th} coordinate. Let $\mathcal{U}$ be an ultrafilter over I. We define

$$\eta_{\mathcal{U}} = \{\langle a, b \rangle \in A^2 : [\![a = b]\!] \in \mathcal{U}\}.$$

$\eta_{\mathcal{U}}$ is a congruence of $\boldsymbol{A}$ (in fact we only need $\mathcal{U}$ to be a filter for this to be true). The algebra $\boldsymbol{A}/\eta_{\mathcal{U}}$ is called an *ultraproduct* of the $\boldsymbol{A}_i$'s, and we sometimes write $\boldsymbol{A}/\mathcal{U}$ instead of $\boldsymbol{A}/\eta_{\mathcal{U}}$. If all the $\boldsymbol{A}_i$'s are the same algebra $\boldsymbol{B}$, then $\boldsymbol{A}/\mathcal{U}$ is called an *ultrapower* of $\boldsymbol{B}$. If K is a class of algebras then we let $\mathbb{P}_{\mathrm{u}}(\mathsf{K})$ denote the class of ultraproducts of members of K.

Theorem 1.33 ([5, Ex. 5.1.3])**.** *Every algebra can be embedded into an ultraproduct of finitely generated subalgebras of itself.*

1.4 First-Order Languages and Free algebras

In the next chapter we will introduce some logical structures. They will be equipped with signatures in which the operation symbols resemble logical connectives, such as 'not', 'and', etc. At times, it will also be necessary to refer explicitly to syntactic aspects of the first-order theories of some of these logical structures. First-order languages are themselves equipped with logical connectives, again like 'not', 'and', etc., which must not be confused with the lower order operations of the logical structures, despite the formal similarities. In anticipation of this need, and to prevent potential confusion, some basic concepts of first-order logic are recalled briefly below.

In any *first-order language* we have following (first-order) logical connectives: $\forall$, $\exists$, $\Rightarrow$, &, $\sqcup$ and $\sim$. We choose to let $\sqcup$ and $\sim$ stand for (first-order) 'disjunction' and 'negation', respectively; the rest have their usual meanings.

In addition to these special symbols, a first-order language $\mathcal{L}$ comprises the following disjoint sets:

- an infinite set of *variables*;
- a set of *predicate* or *relation symbols*, each of which has a positive integer associated with it. If the integer n is associated with predicate symbol p, then p is said to be *n-ary*;

- a set of *operation symbols* (a.k.a. function symbols), each of which has a non-negative integer (its *arity*) assigned to it.

Expressions are finite strings of these symbols.

We recursively build the *terms* of $\mathcal{L}$ in the following way:

1. variables and 0-ary operation symbols (a.k.a. *constant symbols*) are terms;
2. for each positive integer n, and each n-ary operation symbol f, if $\alpha_1, \ldots, \alpha_n$ are terms, then the expression $f(\alpha_1, \ldots, \alpha_n)$ is a term;
3. no other expressions are terms.

Now, for each positive integer n, and each n-ary predicate symbol p, if $\alpha_1, \ldots, \alpha_n$ are terms then the expression $p(\alpha_1, \ldots, \alpha_n)$ is called an *atomic formula* of $\mathcal{L}$. We can then build up the *first-order formulas* of $\mathcal{L}$ recursively, using the atomic formulas and the first-order connectives in the following way:

- atomic formulas are first-order formulas;
- if Φ and Ψ are first-order formulas and x is a variable, then $\sim\Phi$, $\Phi \Rightarrow \Psi$, $\Phi \,\&\, \Psi$, $\Phi \sqcup \Psi$, $(\forall x)\Phi$ and $(\exists x)\Phi$ are first-order formulas.

A first-order formula Φ is called a *sentence* if every occurrence of a variable x in Φ lies within the scope of an appropriate quantifier, i.e., within a subformula of Φ of the form $(\forall x)\Psi$ or of the form $(\exists x)\Psi$. Every first-order formula Φ has a *universal closure*, which is the sentence obtained by placing appropriate universally quantified variables on the left of Φ.

Example 1.34. The first-order language $\mathcal{L}$ of algebras with signature $\mathcal{F}$, has $\mathcal{F}$ as its set of operation symbols and it has just one (binary) predicate symbol $\approx$, formalizing equality. Like every first-order theory, it has an infinite set *Var* of variables, disjoint from $\mathcal{F}$. The recursive definition of terms of $\mathcal{L}$ can be paraphrased as follows. We define

$$Fm_0 = \{x : x \in \mathit{Var} \text{ or } x \text{ is some } f \in \mathcal{F}, \text{ where } \varphi(f) = 0\}$$

and, for each $n \in \omega$,

$$Fm_{n+1} = Fm_n \cup \{f(\alpha_1, \ldots, \alpha_k) : f \in \mathcal{F},\, k = \varphi(f) \text{ and } \alpha_1, \ldots, \alpha_k \in Fm_n\}.$$

Then $Fm_{\mathcal{F}}(Var) = \bigcup_{n\in\omega} Fm_n$ is the set of terms of $\mathcal{L}$. We often shorten $Fm_{\mathcal{F}}(Var)$ to $Fm(Var)$ if the signature is clear from the context, and to Fm if Var is also clear.

Notice that for every term $\alpha \in Fm(Var)$, there is a smallest n such that $\alpha \in Fm_n$. We say that the *complexity of* α, denoted $\#\alpha$, is n. In other words, $\#\alpha$ is the number of occurrences of non-constant operation symbols in α. To prove that some statement holds for all terms, we will often perform induction on the complexity of a term.

The *atomic formulas* of $\mathcal{L}$ are then just *equations*, i.e., expressions of the form $\alpha \approx \beta$, where α and β are terms. Examples of first-order formulas of $\mathcal{L}$ are *quasi-equations*, i.e., expressions of the form

$$\big((\alpha_1 \approx \beta_1)\ \&\ \ldots\ \&\ (\alpha_n \approx \beta_n)\big) \Rightarrow (\alpha \approx \beta),$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \alpha, \beta$ are terms.

In the jargon of model theory, algebras with signature $\mathcal{F}$ are *structures for* $\mathcal{L}$. The demand that such an algebra *satisfy* a sentence of $\mathcal{L}$ is defined recursively, in a natural way. For example, to demand that a lattice $\boldsymbol{L}$ should satisfy the first-order sentence

$$(\exists y)((\forall x)(y \wedge x \approx y))$$

is just to demand that it should have a bottom element. For any $a, b \in L$, recall that $a \leq b$ if and only if $a \wedge b = a$. Similarly, we use the formal abbreviation $x \preceq y$ for $x \wedge y \approx x$.

Theorem 1.35 (Łoś' Theorem)**.** *Let* $\boldsymbol{A} = \prod_{i\in I} \boldsymbol{A}_i/\mathcal{U}$ *be an ultraproduct and* Ψ *a sentence of the first-order language of* $\boldsymbol{A}$*. Then* $\boldsymbol{A}$ *satisfies* Ψ *if and only if*

$$\{i \in I : \boldsymbol{A}_i \textit{ satisfies } \Psi\} \in \mathcal{U}.$$

In particular, if $\boldsymbol{A}$ *is an ultrapower of* $\boldsymbol{B}$*, then* $\boldsymbol{A}$ *satisfies exactly the same first-order sentences as* $\boldsymbol{B}$*.*

Corollary 1.36. *If* $\boldsymbol{B}$ *is a finite algebra and* $\boldsymbol{A}$ *is an ultrapower of* $\boldsymbol{B}$*, then* $\boldsymbol{A} \cong \boldsymbol{B}$*.*

Definition 1.37. Let K be a class of algebras and $\boldsymbol{U}$ an algebra in the same signature. Let $X \subseteq U$. We say that $\boldsymbol{U}$ is *free for* K *over* X if for every $\boldsymbol{A} \in \mathsf{K}$ and every function $h : X \to A$, there is a unique homomorphism $\bar{h} : \boldsymbol{U} \to \boldsymbol{A}$ which extends h. (This implies that $\boldsymbol{U} = \mathrm{Sg}^{\boldsymbol{U}}(X)$.)

Definition 1.38. Given a signature $\mathcal{F}$ and a disjoint set of variables Var, we define the *($\mathcal{F}$–) term algebra over* Var, denoted $\boldsymbol{Fm_{\mathcal{F}}(Var)}$ or $\boldsymbol{Fm}$ for short, as follows: The universe of $\boldsymbol{Fm}$ is Fm. For every $f \in \mathcal{F}$ such that $n = \varphi(f)$, we let $f^{\boldsymbol{Fm}}$ be the operation that maps an n-tuple $(\alpha_1, \ldots, \alpha_n)$ to the *term* $f(\alpha_1, \ldots, \alpha_n)$.

Let $\boldsymbol{A}$ be an algebra in the signature $\mathcal{F}$. Let $h : Var \to \boldsymbol{A}$ be a function. We will define a function $\bar{h} : Fm \to \boldsymbol{A}$ recursively on the complexity of a term $\alpha \in Fm$. Suppose that $\#\alpha = 0$. If $\alpha \in Var$ then $\bar{h}(\alpha) \coloneqq h(\alpha)$. If $\alpha = f \in \mathcal{F}$, with $\varphi(f) = 0$, then $\bar{h}(\alpha) \coloneqq f^{\boldsymbol{A}}$.

Now suppose that $\bar{h}(\beta)$ is defined for all terms β such that $\#\beta \leq n$. Let α be a term such that $\#\alpha = n+1$. Then $\alpha = f(\beta_1, \ldots, \beta_m)$ for some $f \in \mathcal{F}$ with $\varphi(f) = m$ and terms β_i with $\#\beta_i \leq n$ for every $i \leq m$. We define

$$\bar{h}(\alpha) \coloneqq f^{\boldsymbol{A}}(\bar{h}(\beta_1), \ldots, \bar{h}(\beta_m)).$$

It is not difficult to see that $\bar{h}$ is a homomorphism from $\boldsymbol{Fm}$ to $\boldsymbol{A}$. If we let $\mathsf{Alg}_{\mathcal{F}}$ denote the class of algebras with signature $\mathcal{F}$, then $\boldsymbol{Fm}_{\mathcal{F}}(Var)$ is free for $\mathsf{Alg}_{\mathcal{F}}$ over Var; see [5, Thm. 4.21]. For this reason we sometimes call $\boldsymbol{Fm}$ the *absolutely free algebra* in our given signature.

Since Var is infinite, we may assume that it contains the denumerable sequence of distinct variables $x_1, x_2, x_3, \ldots$. For $n \in \omega$, we define

$$Fm(n) = \{\alpha \in Fm : \text{ the variables occurring in } \alpha \text{ are among } x_1, \ldots, x_n\}.$$

If $\alpha \in Fm(n)$, we say that α is an n-*ary* term, and sometimes write α as $\alpha(x_1, \ldots, x_n)$ to indicate this. If $\boldsymbol{A}$ is an algebra, with $a_1, \ldots a_n \in A$, then $\alpha^{\boldsymbol{A}}(a_1, \ldots, a_n)$ denotes $h(\alpha)$, where $h : \boldsymbol{Fm} \to \boldsymbol{A}$ is a homomorphism such that $h(x_i) = a_i$ for $i = 1, \ldots, n$. The function $\alpha^{\boldsymbol{A}} \colon Fm^n \to A$ is called a *term operation*. We define an $\boldsymbol{A}$-*evaluation* to be any homomorphism from $\boldsymbol{Fm}$ to $\boldsymbol{A}$. By the freedom of $\boldsymbol{Fm}$, every $\boldsymbol{A}$-evaluation is determined by its action on Var.

When we say that an algebra $\boldsymbol{A}$ *satisfies* a (quasi-)equation Ψ, we mean that $\boldsymbol{A}$ satisfies the universal closure of Ψ. Thus, $\boldsymbol{A}$ satisfies $\alpha \approx \beta$ if $h(\alpha) = h(\beta)$ for every $\boldsymbol{A}$-evaluation h. More generally, $\boldsymbol{A}$ satisfies

$$\big((\alpha_1 \approx \beta_1) \ \& \ \ldots \ \& \ (\alpha_n \approx \beta_n)\big) \Rightarrow (\alpha \approx \beta)$$

provided that, for every $\boldsymbol{A}$-evaluation h,

$$\text{if } h(\alpha_i) = h(\beta_i) \text{ for every } i \leq n, \text{ then } h(\alpha) = h(\beta).$$

If Ψ is a sentence, equation or quasi-equation, then we use the notation $\boldsymbol{A} \models \Psi$ to signify that $\boldsymbol{A}$ satisfies Ψ.

A class K of algebras is said to *satisfy* Ψ if every algebra in K satisfies Ψ, in which case we write $\mathsf{K} \models \Psi$. We define a K-*evaluation* to be an $\boldsymbol{A}$-evaluation for some $\boldsymbol{A} \in \mathsf{K}$.

The compatibility of congruences and homomorphisms with basic operations extends to term operations (by induction on complexity). Moreover, the following holds.

Theorem 1.39 ([5, Thm. 4.32])**.** *Let* $\boldsymbol{A}$ *be an algebra and* $Y \subseteq A$. *Then*

$$\mathrm{Sg}^{\boldsymbol{A}}(Y) = \{\alpha^{\boldsymbol{A}}(a_1, \ldots, a_n) : \alpha \in Fm(n) \textit{ for } n \in \omega \textit{ and } a_1, \ldots, a_n \in Y\}.$$

1.5 Varieties and Other Classes of Algebras

In the exposition so far, we have introduced several basic *class operators*; they are $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{s}}$ and $\mathbb{P}_{\mathrm{u}}$. Although there are some set-theoretic difficulties with this, all of these are closure operators on the collection of all classes of similar algebras. (For any such class K, note that $\mathbb{I}(\mathsf{K})$ and $\mathbb{P}(\mathsf{K})$ are proper classes, not sets.) There are ways to overcome the foundational difficulties, as we will show in the case of varieties.

We can compose basic class operators to form yet more class operators and these form a certain hierarchy. For example, $\mathbb{I}\,\mathbb{S}(\mathsf{K})$ consist of all algebras that can be embedded into members of K. For class operators $\mathbb{O}_1$ and $\mathbb{O}_2$, we write $\mathbb{O}_1 \leq \mathbb{O}_2$ if, for every class K of similar algebras, we have $\mathbb{O}_1(\mathsf{K}) \subseteq \mathbb{O}_2(\mathsf{K})$.

The following relations in this hierarchy are well-known; see for instance [5, Lem. 3.41, Thm. 5.4].

Theorem 1.40.

(i) $\mathbb{S}\,\mathbb{H} \leq \mathbb{H}\,\mathbb{S}$

(ii) $\mathbb{P}\,\mathbb{S} \leq \mathbb{S}\,\mathbb{P}$

(iii) $\mathbb{P}\,\mathbb{H} \leq \mathbb{H}\,\mathbb{P}$

(iv) $\mathbb{P}_{\mathrm{s}} \leq \mathbb{S}\,\mathbb{P}$

(v) $\mathbb{I} \leq \mathbb{H}$

(vi) $\mathbb{O}\,\mathbb{I} \leq \mathbb{I}\,\mathbb{O}$, *where* $\mathbb{O}$ *is any of the basic class operators*

(vii) $\mathbb{P}_{\mathrm{u}} \leq \mathbb{H}\,\mathbb{P}$

(viii) $\mathbb{P}_u \mathbb{S} \leq \mathbb{S} \mathbb{P}_u$

(ix) $\mathbb{P}_u \mathbb{P} \leq \mathbb{S} \mathbb{P} \mathbb{P}_u$

(x) $\mathbb{S} \mathbb{P} \leq \mathbb{P}_s \mathbb{S}$, *hence* $\mathbb{I} \mathbb{S} \mathbb{P} \mathbb{P}_u \leq \mathbb{I} \mathbb{P}_s \mathbb{S} \mathbb{P}_u$

In particular, a class K of similar algebras is closed under all of $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}$ if and only if $\mathsf{K} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{K})$. Similarly, K is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ if and only if $\mathsf{K} = \mathbb{H}\mathbb{S}\mathbb{P}(\mathsf{K})$ [61]. Finally, K is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_u$, if and only if $\mathsf{K} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_u(\mathsf{K})$ [39, 30].

Definition 1.41. A class K of algebras is called a *variety* if it is *axiomatized* by equations, i.e., there exists a set of equations X such that K is the class of all algebras that satisfy all the equations in X.

Theorem 1.42 (Birkhoff's Theorem [6])**.** *A class K of algebras is variety if and only if $\mathsf{K} = \mathbb{H}\mathbb{S}\mathbb{P}(\mathsf{K})$. Furthermore, if an algebra $\boldsymbol{A}$ satisfies all equations satisfied by K, then $\boldsymbol{A} \in \mathbb{H}\mathbb{S}\mathbb{P}(\mathsf{K})$.*

Because of this theorem, we define the class operator $\mathbb{V} = \mathbb{H}\mathbb{S}\mathbb{P}$. We call $\mathbb{V}(\mathsf{K})$ the *variety generated* by K, as it is the smallest variety that contains K. The theorem establishes a bijection between varieties and the *sets* of equations that they satisfy. It is this correspondence that allows us to circumvent the set-theoretic difficulties we discussed earlier. In particular, the intersection of a set of varieties in a common signature is still a variety. If V is a variety, then we can talk about the *set* of subvarieties of V. It is clear that this set is the universe of a complete lattice, ordered by inclusion, called the *subvariety lattice* of V.

In Birkhoff's Subdirect Decomposition Theorem (Theorem 1.31), the subdirectly irreducible subdirect factors of an algebra $\boldsymbol{A}$ belong to every variety containing $\boldsymbol{A}$, as they belong to $\mathbb{H}(\boldsymbol{A})$. This yields the following more informative statement.

Theorem 1.43. *Every variety V is determined by its subdirectly irreducible members. In particular, $\mathsf{V} = \mathbb{I}\mathbb{P}_s(\mathsf{V}_{SI})$. More generally, for any two varieties V and W in the same signature, $\mathsf{V} \subseteq \mathsf{W}$ if and only if $\mathsf{V}_{SI} \subseteq \mathsf{W}_{SI}$.*

For every cardinal $\mathfrak{m}$ and class K of similar algebras, there is an algebra $\boldsymbol{F}_{\mathsf{K}}(\mathfrak{m})$ that is free for K over a set of cardinality $\mathfrak{m}$ (unless $\mathfrak{m} = 0$ and K has no constant symbol); this algebra is unique up to isomorphism and belongs to $\mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{K})$. See for instance [5, Sec. 4.3].

Theorem 1.44 ([6])**.** *For every variety K, we have $\mathsf{K} = \mathbb{V}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$.*

Definition 1.45. A class K of similar algebras is a *quasivariety* if it is axiomatized by some set of quasi-equations.

An analogue of Birkhoff's Theorem exists for quasivarieties:

Theorem 1.46 ([39, 30])**.** *A class K of similar algebras is a quasivariety if and only if it is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_{\mathrm{u}}$. Thus, the smallest such class containing K is $\mathbb{I}\,\mathbb{S}\,\mathbb{P}\,\mathbb{P}_{\mathrm{u}}(\mathsf{K})$.*

We refer to $\mathbb{Q}(\mathsf{K}) \coloneqq \mathbb{I}\,\mathbb{S}\,\mathbb{P}\,\mathbb{P}_{\mathrm{u}}(\mathsf{K})$ as the *quasivariety generated* by K.

Definition 1.47. A variety V is said to be *congruence distributive* if $\mathrm{Con}(\boldsymbol{A})$ is a distributive lattice for every $\boldsymbol{A} \in \mathsf{V}$.

Example 1.48 ([25])**.** If $\boldsymbol{A}$ is a lattice, possibly with additional operations, then $\mathrm{Con}(\boldsymbol{A})$ is a distributive lattice.

The following well-known theorem was proved in [32] (or see [34] or [5, Thm. 5.10]).

Theorem 1.49 (Jónsson's Theorem)**.** *Let V be a congruence distributive variety, and suppose that $\mathsf{V} = \mathbb{V}(\mathsf{K})$. Then $\mathsf{V}_{FSI} \subseteq \mathbb{H}\,\mathbb{S}\,\mathbb{P}_{\mathrm{u}}(\mathsf{K})$.*

Theorem 1.50 ([10])**.** *In a congruence distributive variety V, the following are equivalent:*

(i) for each $\boldsymbol{A} \in \mathsf{V}$, the intersection of any two compact (i.e., finitely generated) congruences of $\boldsymbol{A}$ is compact;

(ii) V_{FSI} is closed under $\mathbb{S}$ and $\mathbb{P}_{\mathrm{u}}$.

The following lemma is an immediate consequence of Theorem 1.46 and Theorem 1.40(x). (For a stronger result, see [15, Lem. 1.5].)

Lemma 1.51. *For any algebra $\boldsymbol{C}$, each subdirectly irreducible member of $\mathbb{Q}(\boldsymbol{C})$ can be embedded into an ultrapower of $\boldsymbol{C}$.*

1.6 Involutive Residuated Lattices

Residuated structures will be important examples in later chapters, because of their relationship with substructural logics (in particular, relevance logics). A standard text on this subject is [28].

Definition 1.52. An *involutive (commutative) residuated lattice*, or briefly, an *IRL*, is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \cdot, \neg, e \rangle$ comprising a commutative monoid $\langle A; \cdot, e\rangle$, a lattice $\langle A; \wedge, \vee\rangle$ and a unary operation $\neg$, called an *involution*, such that $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and

$$x \cdot y \leq z \text{ iff } \neg z \cdot y \leq \neg x. \tag{1.5}$$

Here, $\leq$ denotes the lattice order and $\neg$ binds more strongly than any other operation symbol.

Setting $y = e$ in (1.5), we see that $\neg$ is antitone (order-reversing).

Theorem 1.53. *Any IRL satisfies De Morgan's laws:*

$$\neg(x \wedge y) = \neg x \vee \neg y \tag{1.6}$$

$$\neg(x \vee y) = \neg x \wedge \neg y \tag{1.7}$$

Proof. We prove only (1.6), as (1.7) follows similar reasoning. Here we use Characterization 1.7 and show that $\neg(x \wedge y) = \sup\{\neg x, \neg y\}$. We first show that $\neg(x \wedge y)$ is an upper bound of $\neg x$ and $\neg y$. Recall that $x \wedge y \leq x$, so since $\neg$ is antitone, $\neg x \leq \neg(x \wedge y)$. Similarly $\neg y \leq \neg(x \wedge y)$.

We still need to show that $\neg(x \wedge y)$ is the *least* upper bound of $\neg x$ and $\neg y$. Suppose z is an upper bound of $\neg x$ and $\neg y$. But then $\neg z \leq x$ and $\neg z \leq y$, since $\neg$ is antitone and $\neg\neg x = x$. Therefore $\neg z \leq x \wedge y$, so $\neg(x \wedge y) \leq z$, as required. □

Theorem 1.54. *In an IRL,* $\cdot$ *is* compatible with the order *in the sense that, if* $x \leq y$ *then* $x \cdot z \leq y \cdot z$.

Proof. By the reflexivity of $\leq$ we know that $y \cdot z \leq y \cdot z$. Applying (1.5) we see that $\neg(y \cdot z) \cdot z \leq \neg y$. But $\neg y \leq \neg x$, since $\neg$ is antitone. So, by the transitivity of $\leq$, we have $\neg(y \cdot z) \cdot z \leq \neg x$. Applying (1.5) again, we see that $x \cdot z \leq y \cdot z$, as required. □

We define $x \to y := \neg(x \cdot \neg y)$ and $f := \neg e$, where $\to$ is called *residuation*.

Theorem 1.55. *Any IRL satisfies the* law of residuation*:*

$$x \cdot y \leq z \textit{ iff } y \leq x \to z \tag{1.8}$$

Proof. From (1.5) we know that $x \cdot y \leq z$ if and only if $x \cdot \neg z \leq \neg y$. The latter condition holds if and only if $y \leq \neg(x \cdot \neg z)$, since $\neg$ is antitone. But $\neg(x \cdot \neg z) = x \to z$ by definition. □

The law of residuation is a powerful tool that we will use often. In a sense, $\to$ compensates for the fact that we do not have multiplicative inverses (which would make $\langle A; \cdot, e\rangle$ a group). It is not difficult to show that if x has a multiplicative inverse, i.e., an element x^{-1} such that $x \cdot x^{-1} = e$, then $x \to y = x^{-1} \cdot y$.

The following properties will be used so often that they deserve to be proved here.

Theorem 1.56. *The following are properties of IRLs:*

(i) $x \cdot (x \to y) \leq y$

(ii) $x \leq y$ *iff* $e \leq x \to y$

Proof. (i) follows from the law of residuation, once we note that $x \to y \leq x \to y$ by reflexivity of $\leq$.

For (ii), notice that

$$x \leq y \text{ iff } x \cdot e \leq y \text{ iff } e \leq x \to y,$$

since $x \cdot e = x$ and by the law of residuation. □

The following properties of IRLs are well-known and not difficult to prove.

Theorem 1.57. *IRLs satisfy the following with regard to involution:*

(i) $\neg x = x \to f$

(ii) $x \to y = \neg y \to \neg x$

(iii) $x \cdot y = \neg(x \to \neg y)$

Theorem 1.58. *The following are properties of IRLs:*

(i) $e \leq x \to x$

(ii) If $x \leq y$ *then* $y \to z \leq x \to z$*, i.e.,* $\to$ *is order reversing in the first coordinate.*

(iii) If $x \leq y$ *then* $z \to x \leq z \to y$*, i.e.,* $\to$ *is order preserving in the second coordinate.*

(iv) $(x \cdot y) \to z = y \to (x \to z)$

(v) $(x \to y) \cdot (y \to z) \leq x \to z$

(vi) $x \cdot (y \wedge z) \leq (x \cdot y) \wedge (x \cdot z)$

(vii) $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$

We let IRL denote the class of all IRLs.

Theorem 1.59. *The class* IRL *is a variety.*

Proof. It can be shown that IRL is axiomatized by the following equations, together with those for lattices and commutative monoids; see [31].

$$
\begin{aligned}
& x \leq y \to (y \cdot x) \\
& x \cdot (x \to y) \leq y \\
& x \cdot (y \wedge z) \leq (x \cdot y) \wedge (x \cdot z) \\
& (x \to y) \wedge (x \to z) = x \to (y \wedge z) \\
& x = \neg\neg x \\
& x \to \neg y = y \to \neg x
\end{aligned}
$$

□

Chapter 2

Deductive Systems: unifying logic and algebra

The aim of this chapter is to explain what it means for a logic to be algebraized by a class of algebras. Following Blok and Pigozzi [11] (also see [12, 14, 22, 23, 24]), we will establish a common framework that unifies algebras and logics. In this framework, the basic notion is a mathematical structure called a *deductive system*. We proceed by first explaining what we mean by a *sentential logic*. We will then define the *equational consequence relation* of a class of algebras. It will then be shown that the notion of a deductive system encompasses both of these concepts. For a recent exposition of abstract algebraic logic see [49], which was the main source of the precise definitions and notation for this text.

2.1 Finitary Sentential Logics

To set the stage, all the structures that we are concerned with are in an algebraic language. An *algebraic language* consists of an algebraic signature and an infinite set of variables. We will assume throughout the chapter that a fixed but arbitrary algebraic signature $\mathcal{F}$ and an infinite set of variables Var are given. Elements of Var are often denoted by the letters $p, q, r, \ldots$, possibly with indices.

Let $\boldsymbol{Fm}_{\mathcal{F}}(Var)$ be the absolutely free algebra over Var in the given signature $\mathcal{F}$ (see Definition 1.38). When no confusion will arise, we will simply write $\boldsymbol{Fm}$. Recall that the elements of $\boldsymbol{Fm}$ are what an algebraist would call *terms*. We define a *substitution* to be a homomorphism from $\boldsymbol{Fm}$ to itself, i.e., an endomorphism of $\boldsymbol{Fm}$. Note the similarity between substitutions and evaluations (see Section 1.4). In fact, a substitution is just an $\boldsymbol{Fm}$-evaluation.

The logics $\mathbf{R^t}$ and $\mathbf{R}$ that will be studied in this thesis will be introduced in Chapter 4 as sentential formal systems. We start our exposition with a discussion about formal systems in general.

Each *formal system* $\mathbf{L}$ is defined in a certain language. In the present context the language is $\mathcal{F}$ and *Var*. In the logical context we call $\mathcal{F}$ the set of *(logical) connectives*. The elements of $\boldsymbol{Fm}$ are called the *formulas* of $\mathbf{L}$.[1] A *rule* $\Gamma \,/\, \alpha$ is a pair that consists of a set Γ of formulas, called the *premises* of the rule, together with a formula α, called the *conclusion* of the rule. We say that a rule $\Gamma \,/\, \alpha$ is *finite* if Γ is a finite set.

Definition 2.1. A *(sentential) formal system* $\mathbf{L}$ over $\mathcal{F}$ and *Var* consists of a set of formulas, called the *axioms of* $\mathbf{L}$, together with a set of rules, called the *inference rules of* $\mathbf{L}$. If all the inference rules of $\mathbf{L}$ are finite, then $\mathbf{L}$ is said to be *finitary*.

From now on, unless we say otherwise, all formal systems are assumed to be finitary.

Example 2.2. Let **CLL** be the following formal system. The connectives of **CLL** are $\wedge, \vee, \cdot, \to, \neg, t$ with type $(2, 2, 2, 2, 1, 0)$.

Axioms of CLL:

A1	$p \to p$	(identity)
A2	$(p \to q) \to ((r \to p) \to (r \to q))$	(prefixing)
A3	$(p \to (q \to r)) \to (q \to (p \to r))$	(exchange)
A4	$(p \wedge q) \to p$	
A5	$(p \wedge q) \to q$	
A6	$((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$	
A7	$p \to (p \vee q)$	
A8	$q \to (p \vee q)$	
A9	$((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$	(disjunction)
A10	t	
A11	$t \to (p \to p)$	
A12	$p \to (q \to (q \cdot p))$	
A13	$(p \to (q \to r)) \to ((q \cdot p) \to r)$	
A14	$(p \to \neg q) \to (q \to \neg p)$	(contraposition)
A15	$\neg\neg p \to p$	(double negation)

[1] Formulas can be defined in more generality (see Section 2.3) but in the sentential case formulas and terms are the same thing.

Inference rules of CLL:

MP	$p,\, p \to q \,/\, q$	(modus ponens)
AD	$p,\, q \,/\, p \wedge q$	(adjunction)

CLL is the exponential-free fragment of *classical linear logic* and it will be an important example throughout this and the next chapter.

A reader who is familiar with classical logic might be unfamiliar with the $\cdot$ symbol. To alleviate this curiosity we will mention here that $\cdot$ can be interpreted as 'co-tenability', i.e., one intuitive reading of $p \cdot q$ is 'p and q can be simultaneously true'. Another is that $p \cdot q$ is the strongest proposition r such that p implies $q \to r$. Classical logic (see Example 2.8 below) may be got from **CLL** by adding certain axioms. In classical logic, $\cdot$ and $\wedge$ (ordinary conjunction) coincide. Further philosophical discussion will be postponed until Chapter 4.

Let **L** be an arbitrary formal system. For any substitution $h : \boldsymbol{Fm} \to \boldsymbol{Fm}$ and any formula $\alpha \in Fm$ we say that $h(\alpha)$ is a *substitution instance* of α.

Given a set $\Gamma \cup \{\alpha\} \subseteq Fm$ of formulas, a *proof of* α *from* Γ *in* **L** is a finite sequence of formulas, terminating with α, such that each item in the sequence is one of the following:

1. an element of Γ,
2. a substitution instance of an axiom of **L**,
3. a substitution instance of the conclusion of an inference rule of **L**, where the same substitution turns the premises of the rule into previous items in the sequence.

If such a proof exists, we write $\Gamma \vdash_{\mathbf{L}} \alpha$ and we call $\Gamma \,/\, \alpha$ a *derivable rule* of **L**. We say that α is a *theorem* of **L** when $\emptyset \vdash_{\mathbf{L}} \alpha$, in which case we write $\vdash_{\mathbf{L}} \alpha$. The *deducibility relation* of **L**, denoted $\vdash_{\mathbf{L}}$, is just the set of derivable rules of **L**, so it is a binary relation between subsets of Fm and elements of Fm.

Following the abstract algebraic logic tradition we adopt the following definition.

Definition 2.3. A *sentential logic* is the deducibility relation $\vdash_{\mathbf{L}}$ of a (sentential) formal system **L** (over the same language). We say that $\vdash_{\mathbf{L}}$ is *axiomatized* by **L**.

This definition differs from an older tradition (see Definition 2.18) of identifying logics with their sets of theorems alone, which is less amenable to mathematical analysis. Note that our definition allows us to have different axiomatizations of a logic, so long as they produce the same derivable rules. To avoid notational clutter, we regularly attribute to a formal system **L** the significant properties of its deducibility relation $\vdash_{\mathbf{L}}$.

Example 2.4. The following constitutes a formal proof in **CLL** of the theorem

$$p \to \neg\neg p. \tag{2.1}$$

.

1	$\neg p \to \neg p$	A1
2	$(\neg p \to \neg p) \to (p \to \neg\neg p)$	A14
3	$p \to \neg\neg p$	1, 2, MP

The purpose of this example is to explain, once and for all, how the given definition of substitution corresponds exactly to our intuition. To show that 1 is a legitimate step in the proof, we observe that it is a substitution instance of axiom A1. Intuitively, we replace every occurrence of p in A1 with the term $\neg p$. Recall that there is an infinite set of variables Var, that generates $\boldsymbol{Fm}$, and that $p \in Var$. Now define a function $h : Var \to Fm$ that maps p to $\neg p$ and all other elements of Var to themselves. Since $\boldsymbol{Fm}$ is absolutely free, we know that h extends uniquely to a homomorphism, say $h' : \boldsymbol{Fm} \to \boldsymbol{Fm}$. Recall that A1 is $p \to p$. So, a substitution instance of it is $h'(p \to p) = h(p) \to h(p) = \neg p \to \neg p$.

In a similar way, we choose a function $g : Var \to Fm$ such that $g(p) = \neg p$ and $g(q) = p$ to show that 2 is a substitution instance of A14. The last step is of course, modus ponens with $\neg p \to \neg p$ substituted for p and $p \to \neg\neg p$ substituted for q.

Example 2.5 (suffixing)**.** The following is a theorem of **CLL**:

$$(p \to q) \to ((q \to r) \to (p \to r)) \tag{2.2}$$

Proof. The formal proof is as follows.

1	$(q \to r) \to ((p \to q) \to (p \to r))$	A2
2	$((q \to r) \to ((p \to q) \to (p \to r)))$ $\to ((p \to q) \to ((q \to r) \to (p \to r)))$	A3
3	$(p \to q) \to ((q \to r) \to (p \to r))$	1,2 MP □

Let's consider an example where we establish a rule.

Example 2.6. The following is a demonstration that

$$q \to p \vdash_{\mathbf{CLL}} \neg p \to \neg q. \tag{2.3}$$

To the proof of Example 2.4, we add:

4	$(p \to \neg\neg p) \to ((q \to p) \to (q \to \neg\neg p))$	A2
5	$(q \to p) \to (q \to \neg\neg p)$	3,4, MP
6	$q \to p$	
7	$q \to \neg\neg p$	5,6, MP
8	$(q \to \neg\neg p) \to (\neg p \to \neg q)$	A14
9	$\neg p \to \neg q$	7,8, MP

Note that 6 is a valid step in the proof, because it is one of the premises of the rule we are trying to establish.

When we justify the derivability of a rule, we will typically not present the actual sequence of steps that constitute a proof. Instead our steps will be of the form $\Gamma \vdash_{\mathbf{L}} \alpha$. Recall that this should be read as "there is a proof in $\mathbf{L}$ of α from Γ." When we reference such a step as motivation for another step, say $\Gamma' \vdash_{\mathbf{L}} \beta$ where $\Gamma \subseteq \Gamma'$, it is understood that a proof in $\mathbf{L}$ of β from Γ' could be exhibited, and would include (as a subsequence) a proof of α from Γ. Moreover, when we apply the same substitution to every step of a proof, the resulting sequence is still a proof. These two principles are exploited in step 2 of Example 2.7. Finally, we sometimes abbreviate proofs by combining two steps (at least) into one, as exemplified in steps 3 and 5 of Example 2.7.

Example 2.7 (converse of A13)**.** The following is a theorem of **CLL**:

$$((q \cdot p) \to r) \to (p \to (q \to r)) \tag{2.4}$$

Proof.

1	$\vdash_{\mathbf{CLL}} p \to (q \to (q \cdot p))$	A12
2	$\vdash_{\mathbf{CLL}} (q \to (q \cdot p)) \to (((q \cdot p) \to r) \to (q \to r))$	(2.2)
3	$\vdash_{\mathbf{CLL}} [p \to (q \to (q \cdot p))] \to [p \to (((q \cdot p) \to r) \to (q \to r))]$	2, A2, MP
4	$\vdash_{\mathbf{CLL}} p \to (((q \cdot p) \to r) \to (q \to r))$	1, 3, MP
5	$\vdash_{\mathbf{CLL}} ((q \cdot p) \to r) \to (p \to (q \to r))$	4, A3, MP ☐

Let $\mathbf{L}$ be a sentential logic. Whenever we add to $\mathbf{L}$ a set of axioms to get a formal system $\mathbf{L}'$, we say that $\vdash_{\mathbf{L}'}$ is an *axiomatic extension* of $\vdash_{\mathbf{L}}$. Notice that $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathbf{L}'}$.

Example 2.8. *Classical propositional logic* (**CPL**) is the axiomatic extension of **CLL** obtained by adding the axioms:

C1	$(p \to (p \to q)) \to (p \to q)$	(contraction)
C2	$p \to (q \to p)$	(weakening)

We abbreviate $(p \to q) \wedge (q \to p)$ as $p \leftrightarrow q$. The theorems of **CPL** include $(p \cdot q) \leftrightarrow (p \wedge q)$ and $(p \to p) \leftrightarrow (q \to q)$ and $(p \to p) \leftrightarrow t$. To this extent, $\cdot$ and t are redundant in **CPL** (but not in **CLL**). The formula $p \to p$ is often abbreviated in **CPL** by a constant symbol, such as $\top$ or 1, rather than t.

2.2 Equational Consequence Relations

In the previous section we defined the sentential logic axiomatized by a formal system. Recall that the aim of this chapter is to explain what it means for a logic to be algebraized by a certain class of algebras. In this section we return to our study of classes of algebras and establish notions and syntax that will allow us to view classes of algebras in a way that is comparable to our notion of a logic.

Let K be a class of algebras in our fixed signature $\mathcal{F}$. Recall that a K-evaluation is a homomorphism from $\boldsymbol{Fm}$ into some algebra $\boldsymbol{A} \in \mathsf{K}$ and it is determined entirely by its action on the elements of Var.

Definition 2.9. $\models_{\mathsf{K}}$ is the binary relation between sets of equations and equations (in our language), such that for any set of equations $\Sigma \cup \{\alpha \approx \beta\}$,

> $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ if and only if, for every K-evaluation h, if $h(\mu) = h(\nu)$ for all $\mu \approx \nu \in \Sigma$, then $h(\alpha) = h(\beta)$.

We call $\models_{\mathsf{K}}$ the *equational consequence relation of* K.

Notice the similarity between this definition and what it means for K to satisfy an equation or quasi-equation. Specifically, $\emptyset \models_{\mathsf{K}} \alpha \approx \beta$ if and only if K satisfies the equation $\alpha \approx \beta$. We often abbreviate $\emptyset \models_{\mathsf{K}} \alpha \approx \beta$ as $\models_{\mathsf{K}} \alpha \approx \beta$. Similarly, if $\Gamma \cup \{\alpha \approx \beta\}$ is a *finite* set of equations, then

$\Gamma \models_{\mathsf{K}} \alpha \approx \beta$ if and only if K satisfies the quasi-equation

$$\left(\underset{(\mu \approx \nu) \in \Gamma}{\&} \mu \approx \nu \right) \Rightarrow (\alpha \approx \beta).$$

The difference in emphasis here is that $\models_{\mathsf{K}}$ is defined as a binary relation, just like a sentential logic in the previous section.

Generalized quasi-equations are defined like quasi-equations, except that we allow them to have an infinite set of premises. Notice that, unlike quasi-equations, generalized quasi-equations are not generally formulas of the first-order language of K, owing to their possibly infinite length.

We say that $\models_{\mathsf{K}}$ is *finitary* if, whenever $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$, then there exists a finite $\Sigma' \subseteq \Sigma$, such that $\Sigma' \models_{\mathsf{K}} \alpha \approx \beta$.

Let's establish some important properties of $\models_{\mathsf{K}}$.

Theorem 2.10. *For any set of equations* $\Sigma \cup \Sigma' \cup \{\alpha \approx \beta\}$*,*

(i) if $(\alpha \approx \beta) \in \Sigma$ *then* $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$*.*

(ii) if $\Sigma' \models_{\mathsf{K}} \mu \approx \nu$ *for all* $(\mu \approx \nu) \in \Sigma$*, and* $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$*, then* $\Sigma' \models_{\mathsf{K}} \alpha \approx \beta$*.*

(iii) if $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ *, then* $h[\Sigma] \models_{\mathsf{K}} h(\alpha) \approx h(\beta)$ *for all* substitutions h*,*

where we define $h[\Sigma] := \{h(\mu) \approx h(\nu) : (\mu \approx \nu) \in \Sigma\}$*.*

Proof. (i) and (ii) follow simply from the definitions. To prove (iii), let g be any evaluation into some $\boldsymbol{A} \in \mathsf{K}$ such that

$$g(h(\mu)) \approx g(h(\nu)) \text{ for all } (\mu \approx \nu) \in \Sigma.$$

Notice that $g \circ h$ is also an $\boldsymbol{A}$-evaluation, so $gh(\alpha) = gh(\beta)$, as required. □

When dealing with $\models_{\mathsf{K}}$, we can assume, without loss of generality, that K is closed under isomorphisms, subalgebras and direct products, because of the following lemma.

Lemma 2.11. $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ *iff* $\Sigma \models_{\mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{K})} \alpha \approx \beta$*.*

Proof. The implication from right to left is obvious, as $\mathsf{K} \subseteq \mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{K})$.

For the other implication, suppose that $\Sigma \cup \{\alpha \approx \beta\}$ is a set of equations such that $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$. Let $g : \boldsymbol{A} \to \boldsymbol{B}$ be an embedding from some algebra $\boldsymbol{A}$ into some algebra $\boldsymbol{B} \in \mathsf{K}$. Let h be an $\boldsymbol{A}$-evaluation such that $h(\mu) = h(\nu)$ for any $\mu \approx \nu \in \Sigma$. Note that $g \circ h$ is a $\boldsymbol{B}$-evaluation and that $gh(\mu) = gh(\nu)$ for any $\mu \approx \nu \in \Sigma$. But then $gh(\alpha) = gh(\beta)$, because $\boldsymbol{B} \in \mathsf{K}$. Therefore $h(\alpha) = h(\beta)$, because g is an embedding.

Now suppose that $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{B_i}$, where I is some index set and $\boldsymbol{B_i} \in \mathsf{K}$ for all $i \in I$. Again, let h be an $\boldsymbol{A}$-evaluation such that $h(\mu) = h(\nu)$ for any $\mu \approx \nu \in \Sigma$. Let p_i be the i^{th} projection from $\boldsymbol{A}$ onto $\boldsymbol{B_i}$. Here $p_i \circ h$ is a $\boldsymbol{B_i}$-evaluation. Also, for any $\mu \approx \nu \in \Sigma$ and any $i \in I$ we have that $p_i h(\mu) = p_i h(\nu)$, so that $p_i h(\alpha) = p_i h(\beta)$. This means that $h(\alpha)$ and $h(\beta)$ agree in every coordinate, i.e. $h(\alpha) = h(\beta)$. □

A class K of algebras is called a *prevariety* if it is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. Recall that the smallest such class containing K is $\mathbb{I}\,\mathbb{S}\,\mathbb{P}(\mathsf{K})$. The result above can be generalised to a statement similar to Birkhoff's theorem for varieties. In particular:

Theorem 2.12. *If a class* K *of algebras is axiomatized by generalized quasi-equations, then it is a prevariety.*

The converse is problematic: If K is a prevariety, then it is axiomatized by the generalized quasi-equations that it satisfies, but these might not all be expressible in the fixed algebraic language that we are using, even if we allow for infinite expressions. The reason is that the cardinality of *Var* may be insufficient. Under certain assumptions, the question of whether every prevariety can be axiomatized using a *set* (as opposed to a proper class) of variables is independent of axiomatic set theory (with the axiom of choice); see [1].

The aforementioned problem with the cardinality of *Var* vanishes if $\models_{\mathsf{K}}$ is known to be finitary. This is essentially the content of the next result, which is a variant of Theorem 1.46.

Theorem 2.13. *Let* K *be a prevariety. Then the following conditions are equivalent:*

(i) K *is a quasivariety (axiomatized by quasi-equations).*

(ii) $\models_{\mathsf{K}}$ *is finitary.*

(iii) K *is closed under ultraproducts.*

We can uniquely associate every formal equation $\alpha \approx \beta$ with a pair of terms in our language, namely $\langle \alpha, \beta \rangle \in Fm \times Fm$. Although this is an obvious observation, it leads to an important connection between congruences and equational consequence.

A congruence θ of an algebra $\boldsymbol{A}$ is called a K-*congruence* if $\boldsymbol{A}/\theta \in \mathsf{K}$. The following facts are easily verified. When K is a prevariety, the set $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ of K-congruences of $\boldsymbol{A}$ is closed under arbitrary intersections, whence it becomes a complete lattice (with intersections as meets) when ordered by set inclusion. When K is a quasivariety, then $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ is also closed under the unions of non-empty directed subsets, so by Theorems 1.17 and 1.18, it is an algebraic lattice.

Theorem 2.14. *Suppose that* K *is a quasivariety and let* $\theta \subseteq Fm \times Fm$. *Then* θ *is a* K-*congruence of* $\boldsymbol{Fm}$ *if and only if, whenever* $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ *and* $\Sigma \subseteq \theta$, *then* $\langle \alpha, \beta \rangle \in \theta$.

Proof. Suppose that $\theta \in \mathrm{Con}(\boldsymbol{Fm})$ such that $\boldsymbol{Fm}/\theta \in \mathsf{K}$. Now suppose that $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ and $\Sigma \subseteq \theta$. Let q be the canonical surjection from $\boldsymbol{Fm}$ to $\boldsymbol{Fm}/\theta$. Note that q is an $\boldsymbol{Fm}/\theta$-evaluation. Also, $q(\mu_i) = q(\nu_i)$ for any $\mu_i \approx \nu_i \in \Sigma$ since $\Sigma \subseteq \theta$. But then $q(\alpha) = q(\beta)$ (because $\boldsymbol{Fm}/\theta \in \mathsf{K}$), i.e., $\langle \alpha, \beta \rangle \in \theta$.

Conversely, suppose that θ is a subset of $Fm \times Fm$ that satisfies the second condition. Let $\alpha, \beta, \gamma \in Fm$. It clear from the definition that $\models_{\mathsf{K}} \alpha \approx \alpha$ so that $\langle \alpha, \alpha \rangle \in \theta$. Therefore θ is reflexive.

Now suppose that $\langle \alpha, \beta \rangle \in \theta$. It is also clear from the definitions that $\alpha \approx \beta \models_{\mathsf{K}} \beta \approx \alpha$, so that $\langle \beta, \alpha \rangle \in \theta$. Therefore θ is symmetric. Similarly, transitivity of θ follows easily from the fact that $\alpha \approx \beta, \beta \approx \gamma \models_{\mathsf{K}} \alpha \approx \gamma$.

Now, let f be any operation symbol in $\mathcal{F}$, such that $\varphi(f) = n$. Suppose that $\langle \alpha_1, \beta_1 \rangle, \ldots, \langle \alpha_n, \beta_n \rangle \in \theta$. We want to show that

$$\langle f(\alpha_1, \ldots, \alpha_n), f(\beta_1, \ldots, \beta_n) \rangle \in \theta.$$

This will follow if we can show that

$$\alpha_1 \approx \beta_1, \ldots, \alpha_n \approx \beta_n \models_{\mathsf{K}} f(\alpha_1, \ldots, \alpha_n) \approx f(\beta_1, \ldots, \beta_n).$$

Let h be any evaluation into an algebra $\boldsymbol{A} \in \mathsf{K}$, such that $h(\alpha_i) = h(\beta_i)$ for all $i \leq n$. Then

$$\begin{aligned} h(f(\alpha_1, \ldots, \alpha_n)) &= f^{\boldsymbol{A}}(h(\alpha_1), \ldots, h(\alpha_n)) \\ &= f^{\boldsymbol{A}}(h(\beta_1), \ldots, h(\beta_n)) \\ &= h(f(\beta_1, \ldots, \beta_n)). \end{aligned}$$

So far we have shown that θ is a congruence of $\boldsymbol{Fm}$. To show that it is a K-congruence, suppose that $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$. It suffices to show that $\Sigma \models_{\{\boldsymbol{Fm}/\theta\}} \alpha \approx \beta$, because K is axiomatized by quasi-equations. Let h be any $\boldsymbol{Fm}/\theta$-evaluation such that $h(\mu) = h(\nu)$ for any $(\mu \approx \nu) \in \Sigma$. For any $x \in Var$ we know that $h(x)$ is some (non-empty) θ-equivalence class. So, for any $x \in Var$ we choose some $\alpha_x \in h(x)$. We uniquely extend the mapping $x \mapsto \alpha_x$ to a substitution g. If we let q denote the canonical surjection from $\boldsymbol{Fm}$ to $\boldsymbol{Fm}/\theta$, then $q \circ g = h$, since these functions agree on Var. From Theorem 2.10(iii), we know that $g[\Sigma] \models_{\mathsf{K}} g(\alpha) \approx g(\beta)$. For each $(\mu \approx \nu) \in \Sigma$,

$$qg(\mu) = h(\mu) = h(\nu) = qg(\nu),$$

so $\langle g(\mu), g(\nu)\rangle \in \ker q = \theta$. Therefore $\langle g(\alpha), g(\beta)\rangle \in \theta$, by assumption. But then $h(\alpha) = qg(\alpha) = qg(\beta) = h(\beta)$ as required. □

In the next section we will introduce a mathematical structure, called a *deductive system*, which encompasses equational consequence relations and sentential logics. We will then be in a position to finally say what it means for a logic to be *algebraized* by a class of algebras. When we explore the details of algebraization in Chapter 3, the intuition that formal equations are related to congruences, will act as our guiding principle.

2.3 Deductive Systems

Let us now turn to the promised unifying structure.

A *consequence relation* on a set A is a binary relation $\vdash$ from subsets of A to elements of A satisfying the following postulates, for all $B \cup C \cup \{a\} \subseteq A$:

- if $a \in B$ then $B \vdash a$ (reflexivity);
- if $C \vdash b$ for all $b \in B$ and $B \vdash a$, then $C \vdash a$ (transitivity).

From these postulates it follows easily that

- if $B \vdash a$ and $B \subseteq C$, then $C \vdash a$ (monotonicity).

We say that $\vdash$ is *finitary* if, in addition to the two postulates above, it satisfies

- if $B \vdash a$ then $B' \vdash a$ for some finite $B' \subseteq B$ (finitarity).

Recall that in Section 2.1 we fixed an algebraic language, $\mathcal{F}$ and Var. Also recall that we took *formulas* for a sentential formal system or logic to be the same thing as *terms* in our language, i.e., elements of Fm. We now wish to broaden our definition of formulas.

Let R be the set of predicate symbols of some first-order language (see Section 1.4) with operation symbols $\mathcal{F}$ and variables Var. An *R–formula* is an atomic formula of the same first-order language, i.e., a formula of the form $r(\alpha_1, \ldots, \alpha_k)$ where $r \in R$ is a k-ary predicate symbol and $\alpha_1, \ldots, \alpha_k$ are terms in Fm.

Definition 2.15. An *R-deductive system* is a consequence relation $\vdash$ over the set of R–formulas, such that for any set of R–formulas $\Gamma \cup \{\alpha\}$,

- if $\Gamma \vdash \alpha$ then $h[\Gamma] \vdash h(\alpha)$ for every substitution h (substitution-invariance),

where the *substitution instance* $h(r(\alpha_1, \ldots, \alpha_k))$ of an R–formula $r(\alpha_1, \ldots, \alpha_k)$ is defined to be $r(h(\alpha_1), \ldots, h(\alpha_n))$.

From now on, a *deductive system* will mean an R-deductive system for some suitable R as above, and its *formulas* will be the set of R–formulas.

For example, consider the first-order language of algebras in $\mathcal{F}$ and Var, where equality is the only (binary) predicate symbol. The atomic formulas of this language are just formal equations. An R-deductive system is called a *2-dimensional* deductive system (briefly, a *2-deductive system*) if R contains a single binary predicate symbol. Theorem 2.10 shows that equational consequence relations $\models_{\mathsf{K}}$ are, in fact, 2-deductive systems, where we interpret $h(\alpha \approx \beta)$ as $h(\alpha) \approx h(\beta)$ for every substitution h. Here, 'formulas' are equations, not terms.

A *sentential deductive system* can be defined abstractly as an R-deductive system where R contains only a single unary predicate symbol, say r. In this case, 'hiding' the r and identifying each atomic formula $r(\alpha)$ with the term α, we can take the formulas of the system to be just terms. In the finitary case, this clearly encompasses the notion of a sentential logic $\vdash_{\mathbf{L}}$, defined earlier. Conversely, we understand the *first-order language of a sentential logic* $\vdash_{\mathbf{L}}$ to be the first-order language equipped with $\mathcal{F}$, Var and just one unary predicate symbol r, called the *truth-predicate*. Intuitively $r(\alpha)$ means that term α is *asserted*.

In the abstract algebraic logic tradition, there is actually no distinction between 'logic' and 'deductive system'. We follow this tradition in our informal remarks, but in Definition 2.18, we will define a 'logic *over* a deductive system'. In that context, the words have distinct meanings.

From here on, we will use the same symbols as in Section 2.1, α and Γ, to denote formulas and sets of formulas of deductive systems. But in the present context these need not refer to terms, except in the sentential case.

Let $\vdash$ be a deductive system. As with sentential systems, the *theorems* of $\vdash$ are the formulas α such that $\emptyset \vdash \alpha$, which we abbreviate as $\vdash \alpha$. Γ/Π will denote $\{\Gamma/\alpha : \alpha \in \Pi\}$. If $\Gamma \vdash \alpha$ for all $\alpha \in \Pi$, we often write $\Gamma \vdash \Pi$. Also $\Gamma, \alpha \vdash \Pi$ means $\Gamma \cup \{\alpha\} \vdash \Pi$, and $\Gamma \dashv\vdash \Pi$ means '$\Gamma \vdash \Pi$ and $\Pi \vdash \Gamma$'.

We can now broaden our definition of *formal system* to include our wider notion of formulas, by repeating essentially verbatim Definition 2.1, and only changing what we mean by 'formulas' to include R–formulas.

Fact 2.16 ([35]). *Every* finitary *deductive system $\vdash$ is the deducibility relation of a formal system (in this wider sense).*

Indeed, the theorems of $\vdash$ can serve as axioms in one such formal system, with the other finite derivable rules serving as inference rules.

We could further broaden our definition of a formal system so that the fact above holds even for non-finitary deductive systems, but this would for instance require a discussion about infinite proofs, which falls outside the scope of this text.

As an aside, our definition also allows formulas to be *sequents*, such as $\alpha_1, \ldots, \alpha_n \rhd \beta_1, \ldots, \beta_m$, i.e., pairs of finite sequences of terms. Thus, $\Gamma \vdash \alpha$ might abbreviate an instance of a so-called 'cut-rule', such as,

$$\frac{\gamma \rhd \beta \qquad \sigma, \beta, \pi \rhd \delta}{\sigma, \gamma, \pi \rhd \delta}$$

In this way, *sequential systems* (a.k.a. *Gentzen systems*) may be regarded as deductive systems, amply justifying the extra generality in the definition. In this dissertation, however, we shall be concerned, almost exclusively, with sentential deductive systems and equational consequence relations.

Now that we have set up our common framework, we can define what it means for a deductive system $\vdash$ to be algebraized by a class of algebras. Given a class of algebras K in the same algebraic language, we will say that $\vdash$ is *algebraized* by K if the deductive systems $\vdash$ and $\models_{\mathsf{K}}$ are *equivalent*, in a certain natural sense. We will now discuss what it means for two deductive systems to be equivalent.

Suppose that $\vdash$ is a deductive system. A set T of formulas of $\vdash$ is called a $\vdash$–*theory* provided that

$$\text{whenever } \Gamma \vdash \alpha \text{ and } \Gamma \subseteq T \text{ then } \alpha \in T.$$

Notice that Theorem 2.14 states that the theories of an equational consequence relation $\models_{\mathsf{K}}$ are exactly the K-congruences θ of $\boldsymbol{Fm}$.

Example 2.17. The set of theorems of a deductive system $\vdash$ is a $\vdash$–theory.

Proof. Let T be the set of theorems of $\vdash$, i.e., $T = \{\alpha : \vdash \alpha\}$. Suppose that $\Gamma \vdash \beta$ and $\Gamma \subseteq T$. Notice that $\emptyset \vdash \gamma$ for every $\gamma \in \Gamma$, so by transitivity, $\emptyset \vdash \beta$, i.e., $\beta \in T$. □

As already mentioned, there is a tendency to refer to any deductive system informally as a 'logic'. However, the following definition is more orthodox:

Definition 2.18. A *logic over* $\vdash$ is a $\vdash$–theory that is closed under substitution.

It is easy to check that intersections of theories are again theories, so that the set of theories of $\vdash$ becomes a complete lattice when ordered by set inclusion; see Lemma 1.10. The corresponding closure operator is denoted by $\mathrm{Cn}_\vdash$, standing for 'consequences of'. So, for any set Γ of formulas, the *theory generated* by Γ is

$$\mathrm{Cn}_\vdash \Gamma = \{\alpha : \Gamma \vdash \alpha\}.$$

Therefore, T is a theory if and only if $\mathrm{Cn}_\vdash T = T$. Also,

$$\Gamma \vdash \alpha \text{ iff } \mathrm{Cn}_\vdash\{\alpha\} \subseteq \mathrm{Cn}_\vdash \Gamma. \tag{2.5}$$

This means that $\vdash$ can be recovered from its *lattice of theories* $\langle \mathrm{Th}_\vdash; \cap, \vee\rangle$. However, the lattice operations do not express the substitution-invariance of $\vdash$, which amounts to this:

> whenever T is a theory of $\vdash$, then so is $h^{-1}[T]$, for every substitution h.

This leads us to the following definition of Jónsson [8, 9].

Definition 2.19. Two deductive systems with the same language are *equivalent* if there is a lattice isomorphism Λ between their lattices of theories such that

$$\Lambda(h^{-1}[T]) = h^{-1}[\Lambda(T)]$$

for all theories T and substitutions h.

We can reword this definition by defining the *theory algebra* of $\vdash$ to be $\langle \mathrm{Th}_\vdash; \cap, \vee, h^{-1}\,(h \in \mathit{Sub})\rangle$, which is a lattice with extra unary operations h^{-1} indexed by the set *Sub* of all substitutions h. Then two deductive systems are equivalent if and only if their theory algebras are *isomorphic*.

Thus, the notion of equivalence is symmetric. The isomorphism Λ in Definition 2.19 is called *an equivalence.*

This definition of equivalence is algebraically natural and elegant, but is rather abstract from the point of view of general logic. Thankfully, a syntactic characterisation is available that does not venture into the realm of theory algebras.

Let $\vdash_1$ and $\vdash_2$ be deductive systems in the same language. A *translation from* $\vdash_1$ *to* $\vdash_2$ is understood here to be a function $\boldsymbol{\tau}$ from formulas of $\vdash_1$ to sets of formulas of $\vdash_2$; it is said to be *definable* if

$$\boldsymbol{\tau}(h(\alpha)) = h[\boldsymbol{\tau}(\alpha)] \text{ for any substitution } h.$$

In the latter case, $\boldsymbol{\tau}$ is determined by its action on 'atoms', such as variables in the sentential case, and pairs of variables in the equational case. Indeed, the formulas to which the atoms are sent *explicitly define* the action of $\boldsymbol{\tau}$ on *all* formulas α.

Specifically, a definable translation $\boldsymbol{\tau}$ from a sentential system $\vdash_1$ to a 2-deductive system $\vdash_2$ can be specified as a set of pairs $\langle \tau^l(p), \tau^r(p)\rangle$, where each coordinate is a unary term. (Recall that p denotes a variable.) When $\vdash_2$ is an equational consequence relation, it is natural to treat $\langle \tau^l(p), \tau^r(p)\rangle$ as an equation $\tau^l(p) \approx \tau^r(p)$. Then

$$\boldsymbol{\tau}(\alpha) = \{\tau^l(\alpha) \approx \tau^r(\alpha) : \langle \tau^l(p), \tau^r(p)\rangle \in \boldsymbol{\tau}\}.$$

Similarly, if $\vdash_1$ is a 2-deductive system and $\vdash_2$ is a sentential system, and $\boldsymbol{\rho}$ is a definable translation from $\vdash_1$ to $\vdash_2$, then $\boldsymbol{\rho}$ is determined by a set of binary terms, $\boldsymbol{\rho}(p, q)$.

Example 2.20. Recall $\vdash_{\mathbf{CPL}}$ is the deducibility relation of classical proposition logic (see Example 2.8), and let BA denote the variety of Boolean algebras (see Definition 1.22). A definable translation $\boldsymbol{\tau}$ from $\vdash_{\mathbf{CPL}}$ to $\models_{\mathsf{BA}}$ is given by

$$\boldsymbol{\tau} : \alpha \mapsto \{\alpha \approx t\}.$$

Here t is interpreted, in any Boolean algebra, as the top element, and $\cdot$ as $\wedge$. We interpret $\neg x$ as the complement of x in the context of BA. Lastly we define $x \to y \coloneqq \neg x \vee y$ in BA. Under these conventions, the signatures of **CPL** and BA become the same.

When a translation $\boldsymbol{\tau}$ of a sentential system is known to be definable, it can be specified by its action on the variable p. Indeed, for any formula α, letting h be a substitution with $\alpha = h(p)$, we have $\boldsymbol{\tau}(\alpha) = \boldsymbol{\tau}(h(p)) = h[\boldsymbol{\tau}(p)]$. When introducing such a $\boldsymbol{\tau}$, therefore, we often write, for example: $\boldsymbol{\tau} = \{p \approx t\}$ or $\boldsymbol{\tau} = \{\langle p, t\rangle\}$.

In general, in a 2-deductive system with binary relation r, we can identify every formula $r(\alpha, \beta)$ with the pair $\langle \alpha, \beta \rangle$. A definable translation $\boldsymbol{\rho}$ from $\models_{\mathsf{BA}}$ to $\vdash_{\mathbf{CPL}}$ is given by

$$\boldsymbol{\rho} : \langle \alpha, \beta \rangle \mapsto \{\alpha \to \beta, \beta \to \alpha\}.$$

Again, if we write $\boldsymbol{\rho} = \{p \to q, q \to p\}$, then, for every substitution h, we have $\boldsymbol{\rho}(h(p), h(q)) = h[\boldsymbol{\rho}(p, q)] = \{h(p) \to h(q), h(q) \to h(p)\}$.

For any definable translation $\boldsymbol{\tau}$, we shall abbreviate

$$\bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma) \text{ as } \boldsymbol{\tau}[\Gamma], \text{ and } \{\alpha : \boldsymbol{\tau}(\alpha) \subseteq \Gamma\} \text{ as } \boldsymbol{\tau}^{-1}[\Gamma].$$

The next result is the main theorem on equivalence. It follows from [27, Sec. 4], which generalizes both [11, Thm. 3.7(ii)] and [9, Thm. 5.5].

Theorem 2.21. *Deductive systems $\vdash_1$ and $\vdash_2$ are equivalent iff there are definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ such that the following hold for any set $\Gamma \cup \{\alpha\}$ of formulas of $\vdash_1$ and any formula ψ of $\vdash_2$:*

(1) $\Gamma \vdash_1 \alpha$ *iff* $\boldsymbol{\tau}[\Gamma] \vdash_2 \boldsymbol{\tau}(\alpha)$*;*

(2) $\psi \dashv\vdash_2 \boldsymbol{\tau}[\boldsymbol{\rho}(\psi)]$.

In this case, an equivalence Λ from $\vdash_1$ to $\vdash_2$ is given by

$$\Lambda : T \mapsto \mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T] = \boldsymbol{\rho}^{-1}[T], \text{ for any theory } T \text{ of } \vdash_1 .$$

Remark. By the symmetry of equivalence, the conjunction of (1) and (2) could be replaced, in Theorem 2.21, by that of

(3) $\Sigma \vdash_2 \psi$ iff $\boldsymbol{\rho}[\Sigma] \vdash_1 \boldsymbol{\rho}(\psi)$;

(4) $\gamma \dashv\vdash_1 \boldsymbol{\rho}[\boldsymbol{\tau}(\gamma)]$,

where $\Sigma \cup \{\psi\}$ is any set of formulas of $\vdash_2$ and γ is any formula of $\vdash_1$.

Proof of Theorem 2.21. Here we prove only that if definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ satisfying (1)–(4) exist, then $\vdash_1$ and $\vdash_2$ are equivalent.

We begin by showing that $\mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T] = \boldsymbol{\rho}^{-1}[T]$ for any theory T of $\vdash_1$. Let $\psi \in \boldsymbol{\rho}^{-1}[T]$, so that $\boldsymbol{\rho}(\psi) \subseteq T$. Therefore,

$$\boldsymbol{\tau}[\boldsymbol{\rho}[\psi]] \subseteq \boldsymbol{\tau}[T] \subseteq \mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T].$$

Furthermore, from (2) we have that $\boldsymbol{\tau}[\boldsymbol{\rho}[\psi]] \vdash_2 \psi$. But then $\psi \in \mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T]$, since the latter is a $\vdash_2$–theory.

Conversely, suppose that $\psi \in \mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T]$. This means that $\boldsymbol{\tau}[T] \vdash_2 \psi$. From (2) we know that $\psi \vdash_2 \boldsymbol{\tau}[\boldsymbol{\rho}(\psi)]$, which implies that $\boldsymbol{\tau}[T] \vdash_2 \boldsymbol{\tau}[\boldsymbol{\rho}(\psi)]$. By (1) it follows that $T \vdash_1 \boldsymbol{\rho}(\psi)$. Therefore $\boldsymbol{\rho}(\psi) \subseteq T$, as required.

This shows, in particular, that $\boldsymbol{\rho}^{-1}[T]$ is a $\vdash_2$–theory, for every $\vdash_1$–theory T. By symmetry, $\boldsymbol{\tau}^{-1}[T]$ is a $\vdash_1$–theory, for every $\vdash_2$–theory T.

Let Λ be as in the theorem's statement. Now, to show that Λ is order-preserving, suppose that T and T' are $\vdash_1$-theories such that $T \subseteq T'$. It is clear that $\boldsymbol{\tau}[T] \subseteq \boldsymbol{\tau}[T']$, so that $\mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T] \subseteq \mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T']$.

To show that Λ is order-reflecting, suppose that $\mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T] \subseteq \mathrm{Cn}_{\vdash_2} \boldsymbol{\tau}[T']$. It follows that $\boldsymbol{\tau}[T'] \vdash_2 \boldsymbol{\tau}[T]$. By (1) it follows that $T' \vdash_1 T$, hence $T \subseteq T'$ (as T' is a theory).

To prove surjectivity, let T_2 be any theory of $\vdash_2$. Let $T_1 = \boldsymbol{\tau}^{-1}[T_2]$. As noted above, T_1 is a $\vdash_1$–theory. For any $\psi \in T_2$, we know from (2) that $\psi \vdash_2 \boldsymbol{\tau}[\boldsymbol{\rho}(\psi)]$, so $\boldsymbol{\tau}[\boldsymbol{\rho}(\psi)] \subseteq T_2$. It follows that $\boldsymbol{\rho}(\psi) \subseteq \boldsymbol{\tau}^{-1}[T_2] = T_1$, i.e., $\psi \in \boldsymbol{\rho}^{-1}[T_1]$. This shows that $T_2 \subseteq \Lambda(T_1)$.

Now suppose that $\psi \in \Lambda(T_1) = \boldsymbol{\rho}^{-1}[T_1]$. It follows that $\boldsymbol{\rho}(\psi) \subseteq T_1 = \boldsymbol{\tau}^{-1}[T_2]$. But then $\boldsymbol{\tau}[\boldsymbol{\rho}(\psi)] \subseteq T_2$. Again from (2) it follows that $\psi \in T_2$. Therefore $T_2 = \Lambda(T_1)$, showing that Λ is onto.

Let h be any substitution and let T be any $\vdash_1$ theory. Let $\psi \in \Lambda(h^{-1}[T]) = \boldsymbol{\rho}^{-1}[h^{-1}[T]]$. Then $h[\boldsymbol{\rho}(\psi)] \subseteq T$. Since $\boldsymbol{\rho}$ is a definable translation we have that $h[\boldsymbol{\rho}(\psi)] = \boldsymbol{\rho}[h(\psi)]$. But this implies that $h(\psi) \in \boldsymbol{\rho}^{-1}[T]$, showing that $\psi \in h^{-1}[\Lambda(T)]$. The reverse inclusion is similar. □

The definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ in Theorem 2.21 are unique up to inter-derivability in $\vdash_2$ and $\vdash_1$, respectively, i.e., if $\boldsymbol{\tau}'$ and $\boldsymbol{\rho}'$ also witness the equivalence between $\vdash_1$ and $\vdash_2$ then,

$$\boldsymbol{\tau}(\alpha) \dashv\vdash_2 \boldsymbol{\tau}'(\alpha) \text{ and } \boldsymbol{\rho}(\beta) \dashv\vdash_1 \boldsymbol{\rho}'(\beta),$$

for all $\vdash_1$–formulas α and $\vdash_2$–formulas β, cf. [11, Thm. 2.15]. Moreover, in Theorem 2.21, if $\vdash_1$ is finitary, then $\boldsymbol{\tau}$ can be chosen *finite*, in the sense that $\boldsymbol{\tau}(\alpha)$ is a finite set of formulas, for every α. By symmetry, if $\vdash_2$ is finitary, then $\boldsymbol{\rho}$ can be chosen finite; see the proof of [9, Thm. 6.3].

It is worthwhile for us to put these observations together into an explicit characterization of algebraization for the structures that we care about.

Characterization 2.22. A sentential deductive system $\vdash$ is algebraized by a class of algebras K iff there exists a set $\boldsymbol{\tau}$ consisting of pairs of the form $\langle \tau^l(p), \tau^r(p) \rangle$, where τ^l and τ^r are unary terms, and a set $\boldsymbol{\rho}$ of binary terms, such that for any set of terms $\Gamma \cup \{\alpha\}$ and any equation $\beta \approx \gamma$ we have

(1) $\Gamma \vdash \alpha$ iff $\boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}} \boldsymbol{\tau}(\alpha)$;

(2) $\beta \approx \gamma \;=\!\!|\!\models_{\mathsf{K}} \boldsymbol{\tau}[\boldsymbol{\rho}(\beta, \gamma)]$.

The conjunction of (1) and (2) can be replaced by

(3) $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ iff $\boldsymbol{\rho}[\Sigma] \vdash \boldsymbol{\rho}(\alpha, \beta)$;

(4) $\gamma \dashv\vdash \boldsymbol{\rho}[\boldsymbol{\tau}(\gamma)]$.

We say that $\vdash$ is *elementary algebraizable* if it is equivalent to a finitary equational consequence relation. In this case we know that $\vdash$ is algebraized by a quasivariety K, by Lemma 2.11 and Theorem 2.13. Moreover, K can be shown unique [11], so it is called the *equivalent quasivariety* of $\vdash$.

Example 2.23. The simplest example of algebraization is that of **CPL** by BA. We know that a formula is a theorem of **CPL** if an only if it evaluates to the top element t of any Boolean algebra. In fact, we only need to consider evaluations into the 2-element Boolean algebra. (The reason for this is proved in Theorem 5.23.) The same is true more generally for the rules of **CPL**. For every set of formulas $\Gamma \cup \{\alpha, \beta\}$,

$$\Gamma \vdash_{\mathbf{CPL}} \alpha \text{ iff } \{\gamma \approx t : \gamma \in \Gamma\} \models_{\mathsf{BA}} \alpha \approx t.$$

Furthermore, it is easy to see that

$$\alpha \approx \beta \models_{\mathsf{BA}} \{t \approx \alpha \to \beta,\ t \approx \beta \to \alpha\}$$

and

$$\{t \approx \alpha \to \beta,\ t \approx \beta \to \alpha\} \models_{\mathsf{BA}} \alpha \approx \beta.$$

Thus, if we let $\boldsymbol{\tau} = \{\langle t, p\rangle\}$ and $\boldsymbol{\rho} = \{p \to q, q \to p\}$, as in Example 2.20, then $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ witness the algebraization of **CPL** by BA, in the sense of Characterization 2.22.

2.4 Extensions

We say that a deductive system $\vdash'$ is an *extension* of $\vdash$ (a deductive system in the same language), when $\vdash' \supseteq \vdash$. It is an *axiomatic extension* of $\vdash$ if there is a set Δ of formulas, closed under substitution, such that for any set $\Gamma \cup \{\alpha\}$ of formulas, we have

$$\Gamma \vdash' \alpha \text{ iff } \Gamma \cup \Delta \vdash \alpha.$$

If a deductive system $\vdash$ is itself finitary and axiomatized by **L**, then all the finitary extensions of $\vdash$ can be obtained by adding axioms and/or inference rules to **L**. (The proof uses Fact 2.16.) Similarly, the axiomatic extensions of $\vdash$ all arise by adding axioms to the formal system **L**, but not adding new inference rules. Thus, in the sentential case, the above definition coincides with the one before Example 2.8. Obviously, the axiomatic extensions of a finitary system are finitary.

Given a deductive system $\vdash$, it is clear that intersections of extensions of $\vdash$ are again extensions of $\vdash$, which means that the extensions of $\vdash$ form a complete lattice, ordered by $\subseteq$, in which meets are intersections.

For the rest of this section, we will consider a formal system to simply be a set of finite inference rules, where its axioms are simply special inference rules with an empty set of premises.

Theorem 2.24. *The finitary extensions of a deductive system $\vdash$ form a lattice ordered by inclusion.*

Proof. Let $\vdash'$ and $\vdash''$ be finitary extensions of $\vdash$. We claim that $\vdash' \cap \vdash''$ is a finitary extension of $\vdash$. Clearly $\vdash' \cap \vdash''$ is an extension of $\vdash$, so it remains to show that it is finitary. Let $\Gamma/\alpha \in \vdash' \cap \vdash''$. Then $\Gamma \vdash' \alpha$ and $\Gamma \vdash'' \alpha$. By finitarity, there exist finite $\Pi, \Pi' \subseteq \Gamma$ such that $\Pi \vdash' \alpha$ and $\Pi' \vdash'' \alpha$. If we let $\Gamma' = \Pi \cup \Pi'$ then Γ' is a finite subset of Γ and $\Gamma'/\alpha \in \vdash' \cap \vdash''$.

By Fact 2.16, we know that $\vdash'$ and $\vdash''$ can be axiomatized by formal systems **L** and **K** respectively. We claim that $\vdash_{\mathbf{L}\cup\mathbf{K}}$ is smallest finitary extension containing both $\vdash'$ and $\vdash''$. It is clear that $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathbf{L}\cup\mathbf{K}}$ and $\vdash_{\mathbf{K}} \subseteq \vdash_{\mathbf{L}\cup\mathbf{K}}$. Now suppose that $\vdash'''$ is a finitary extension of $\vdash$ which contains both $\vdash'$ and $\vdash''$. In particular, $\vdash'''$ will contain every inference rule of both **K** and **L** (including the axioms, thought of as rules of the form $\emptyset/\beta$). It is easy to see, once we recall that $\vdash'''$ is itself axiomatized by a formal system, that a proof of α from Γ in $\vdash_{\mathbf{L}\cup\mathbf{K}}$ can be transformed into a proof in $\vdash'''$.

So, the finitary extensions of $\vdash$ forms a lattice in which the meet is just intersection and the join is as described. □

We can show that the set of axiomatic extensions of $\vdash$ forms a lattice by associating each axiomatic extension with its set of theorems. Specifically, referring to Definition 2.18, we have the following.

Theorem 2.25. *A set of formulas is a logic over a deductive system $\vdash$ if and only if it is the set of all theorems of some axiomatic extension of $\vdash$.*

Proof. Suppose that T is a logic over $\vdash$, i.e., T is a $\vdash$–theory closed under substitutions. We define a binary relation $\vdash'$ from sets of $\vdash$–formulas to $\vdash$–formulas by

$$\Gamma \vdash' \alpha \text{ iff } \Gamma \cup T \vdash \alpha,$$

and we claim that $\vdash'$ is an axiomatic extension of $\vdash$.

It is clear that if $\Gamma \vdash \alpha$ then $\Gamma \vdash' \alpha$, by the monotonicity of $\vdash$. If we can show that $\vdash'$ is a deductive system, then it will follow from its definition that it is an axiomatic extension. The fact that $\vdash'$ is transitive and reflexive follows easily from the monotonicity of $\vdash$. To prove substitution-invariance, suppose that $\Gamma \vdash' \alpha$, and let h be any substitution. Then

$$\Gamma \cup T \vdash \alpha, \ \therefore \ h[\Gamma] \cup h[T] \vdash h(\alpha), \ \therefore \ h[\Gamma] \cup T \vdash h(\alpha), \ \therefore \ h[\Gamma] \vdash' h(\alpha).$$

Conversely, suppose that $\vdash'$ is an axiomatic extension of $\vdash$. Let T be the set of theorems of $\vdash'$. Then, as in Example 2.17, T is a $\vdash'$–theory. It is easy to see from the definition of a theory that T is then also a $\vdash$–theory. Furthermore, it follows from the substitution-invariance of $\vdash'$ that T is closed under substitutions. □

The fact that the axiomatic extensions of $\vdash$ form a lattice then follows easily from the fact that arbitrary intersections of logics over $\vdash$ are again logics over $\vdash$. Theorem 2.25 also explains why logicians are specifically interested in the axiomatic extensions of deductive systems.

Theorem 2.26. *Let $\vdash_{\mathbf{L}}$ and $\vdash_{\mathbf{S}}$ be finitary deductive systems, axiomatized by formal systems $\mathbf{L}$ and $\mathbf{S}$ respectively. If $\vdash_{\mathbf{L}}$ and $\vdash_{\mathbf{S}}$ are equivalent, then the definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$, from Theorem 2.21, induce mutually inverse lattice isomorphisms between the finitary extensions of $\vdash_{\mathbf{L}}$ and those of $\vdash_{\mathbf{S}}$. The isomorphism in one direction is given by*

$$h : \vdash_{\mathbf{L}'} \mapsto \vdash_{\mathbf{S}'},$$

where $\vdash_{\mathbf{L}'}$ is any finitary extension of $\vdash_{\mathbf{L}}$ and

$$\mathbf{S}' = \mathbf{S} \cup \bigcup \big\{ \boldsymbol{\tau}[\Gamma]/\boldsymbol{\tau}(\alpha) : \Gamma \vdash_{\mathbf{L}'} \alpha \textit{ and } \Gamma \textit{ is finite} \big\}.$$

The isomorphism g in the other direction is defined in a similar way, but uses $\boldsymbol{\rho}$ instead of $\boldsymbol{\tau}$.

Furthermore, each $\vdash_{\mathbf{L}'}$ is equivalent to $h(\vdash_{\mathbf{L}'})$, witnessed by the same translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$.

Proof. Without loss of generality $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ can be assumed finite (see the remarks before Characterization 2.22).

It is clear that $h(\vdash_{\mathbf{L}'})$ is an extension of $\vdash_{\mathbf{S}}$ for every extension $\vdash_{\mathbf{L}'}$ of $\vdash_{\mathbf{L}}$, since $\mathbf{S} \subseteq \mathbf{S}'$.

Let's show that h is order-preserving. Suppose that $\vdash_{\mathbf{L}'} \subseteq \vdash_{\mathbf{L}''}$. It follows from the definition that $\mathbf{S}' \subseteq \mathbf{S}''$, so $h(\vdash_{\mathbf{L}'}) = \vdash_{\mathbf{S}'} \subseteq \vdash_{\mathbf{S}''} = h(\vdash_{\mathbf{L}''})$.

We can show in a similar way that g is order-preserving. It remains to show, by Theorem 1.8, that h and g are mutually inverse functions.

Let $\vdash_{\mathbf{L}'}$ be a finitary extension of $\vdash_{\mathbf{L}}$. Suppose that $\Gamma \vdash_{\mathbf{L}'} \alpha$, where Γ might be infinite. By finitarity, there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathbf{L}'} \alpha$, so $\boldsymbol{\tau}[\Gamma'] \vdash_{\mathbf{S}'} \boldsymbol{\tau}(\alpha)$, by definition. By monotonicity and the fact that $\boldsymbol{\tau}[\Gamma'] \subseteq \boldsymbol{\tau}[\Gamma]$, we have $\boldsymbol{\tau}[\Gamma] \vdash_{\mathbf{S}'} \boldsymbol{\tau}(\alpha)$. Let $\mathbf{L}''$ be the axiomatization of $g(\vdash_{\mathbf{S}'})$ indicated in the theorem statement, so $g(\vdash_{\mathbf{S}'}) = \vdash_{\mathbf{L}''}$. Then $\boldsymbol{\rho}[\boldsymbol{\tau}[\Gamma]] \vdash_{\mathbf{L}''} \boldsymbol{\rho}[\boldsymbol{\tau}(\alpha)]$. But since $\mathbf{L}''$ is an extension of $\mathbf{L}$, and by Theorem 2.21(4), we can conclude that $\Gamma \vdash_{\mathbf{L}''} \alpha$. Therefore, $\vdash_{\mathbf{L}'} \subseteq gh(\vdash_{\mathbf{L}'})$.

Before we prove the reverse inclusion, take note of the following claim, which we will prove after the main argument is completed.

Claim 1: Whenever $\Sigma \vdash_{\mathbf{S}'} \psi$, then $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi)$.

Suppose that $\Gamma \vdash_{\mathbf{L}''} \alpha$. Then there is a proof $\vec{\alpha} = \alpha_1, \ldots, \alpha_n$ of α from Γ in $\mathbf{L}''$. We will now replace certain α_i's in $\vec{\alpha}$ with proofs of α_i from Γ in $\mathbf{L}'$, so that the new sequence will be a proof of α from Γ in $\mathbf{L}'$. Let α_i be a step in $\vec{\alpha}$. If α_i is a member of Γ or is a substitution instance of the conclusion of an (axiom or) inference rule of $\mathbf{L}$, we leave it unchanged. Otherwise, α_i is an f–substitution instance of the conclusion of an inference rule of $\mathbf{L}''$, where the rule belongs to $\boldsymbol{\rho}[\Sigma]/\boldsymbol{\rho}(\psi)$ and $\Sigma \vdash_{\mathbf{S}'} \psi$. By Claim 1, $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi)$, so, since $\vdash_{\mathbf{L}'}$ is substitution-invariant, $f[\boldsymbol{\rho}[\Sigma]] \vdash_{\mathbf{L}'} f[\boldsymbol{\rho}(\psi)]$. Now, since $\alpha_i \in f[\boldsymbol{\rho}(\psi)]$, there is a proof $\vec{\beta_i}$ of α_i from $f[\boldsymbol{\rho}[\Sigma]]$ in $\mathbf{L}'$. Notice that every element of $f[\boldsymbol{\rho}[\Sigma]]$ is α_j for some $j < i$. If we replace every $\alpha_i \in \vec{\alpha}$ by $\vec{\beta_i}$, then the resulting sequence is a proof of α from Γ in $\mathbf{L}'$.

Therefore, $gh(\vdash_{\mathbf{L}'}) \subseteq \vdash_{\mathbf{L}'}$. We have shown that the composition of g and h is the identity function on the lattice of finitary extensions of $\vdash_{\mathbf{L}}$. By symmetry the composition of h and g is the identity function on the lattice of finitary extensions of $\vdash_{\mathbf{S}}$. This shows that h is a lattice isomorphism, with g as its inverse, as required.

We still need to show that $\vdash_{\mathbf{L}'}$ is equivalent to $h(\vdash_{\mathbf{L}'}) = \vdash_{\mathbf{S}'}$. If $\Gamma \vdash_{\mathbf{L}'} \alpha$, then, as we showed at the start of the proof, $\boldsymbol{\tau}[\Gamma] \vdash_{\mathbf{S}'} \boldsymbol{\tau}(\alpha)$.

Conversely, suppose that $\boldsymbol{\tau}[\Gamma] \vdash_{\mathbf{S}'} \boldsymbol{\tau}(\alpha)$. By finitarity there is a finite subset X of $\boldsymbol{\tau}[\Gamma]$ from which we can derive $\boldsymbol{\tau}(\alpha)$ in $\mathbf{S}'$. In particular, since $\boldsymbol{\tau}$ is finite and by monotonicity, we can choose X to be $\boldsymbol{\tau}[\Gamma']$ for some finite $\Gamma' \subseteq \Gamma$. Notice that $\boldsymbol{\rho}[\boldsymbol{\tau}[\Gamma']]/\boldsymbol{\rho}[\boldsymbol{\tau}(\alpha)] \subseteq gh(\vdash_{\mathbf{L}'}) = \vdash_{\mathbf{L}'}$, so that $\boldsymbol{\rho}[\boldsymbol{\tau}[\Gamma']] \vdash_{\mathbf{L}'} \boldsymbol{\rho}[\boldsymbol{\tau}(\alpha)]$. But then $\Gamma' \vdash_{\mathbf{L}'} \alpha$, by Theorem 2.21(4), since $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathbf{L}'}$. Therefore, $\Gamma \vdash_{\mathbf{L}'} \alpha$, by monotonicity. Lastly, it is clear from the relation $\vdash_{\mathbf{S}} \subseteq \vdash_{\mathbf{S}'}$ that, for any $\vdash_{\mathbf{S}}$–formula ψ,

$$\psi \dashv\vdash_{\mathbf{S}'} \boldsymbol{\tau}[\boldsymbol{\rho}(\psi)].$$

Proof of Claim 1. Suppose that $\Sigma \vdash_{\mathbf{S}'} \psi$. Let $\vec{\psi} = \psi_1, \ldots, \psi_n$ be a proof of ψ from Σ in $\mathbf{S}'$. We will show that $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi)$ by induction on n. Suppose that $n = 1$. If $\psi (= \psi_1)$ is an element of Σ, then $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi)$, by reflexivity. Suppose that ψ is an f–substitution instance of an $\mathbf{S}'$ axiom ψ'. If ψ' is an axiom of $\mathbf{S}$, then $\vdash_{\mathbf{L}} \boldsymbol{\rho}(\psi')$, by Theorem 2.21(3). Since $\vdash_{\mathbf{L}}$ is substitution-invariant and $\boldsymbol{\rho}$ is definable, $\vdash_{\mathbf{L}} \boldsymbol{\rho}(f(\psi'))$. Therefore $\vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi)$, since $\vdash_{\mathbf{L}'}$ is an extension of $\vdash_{\mathbf{L}}$. To complete the basis step we need to consider the case where $\psi' \in \boldsymbol{\tau}(\gamma)$, where γ is some theorem of $\mathbf{L}'$. Since $\vdash_{\mathbf{L}'}$ is an extension of $\vdash_{\mathbf{L}}$, we know that $\gamma \vdash_{\mathbf{L}'} \boldsymbol{\rho}[\boldsymbol{\tau}(\gamma)]$, by Theorem 2.21(3). Therefore, $\vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi')$. Again, the fact that $\vdash_{\mathbf{L}'} \boldsymbol{\rho}(\psi)$ follows from substitution-invariance and the definability of $\boldsymbol{\rho}$.

Now suppose that $\boldsymbol{\rho}[\Sigma] \vdash \boldsymbol{\rho}(\psi_i)$ for all $i < n$. In the cases where ψ_n is either a member of Σ or an axiom of $\mathbf{S}'$ we can use an argument similar to the base case. Since $\vdash_{\mathbf{L}'}$ is substitution-invariant and $\boldsymbol{\rho}$ is definable, we can assume without loss of generality that $\psi (= \psi_n)$ is the conclusion of an inference rule Σ'/ψ of $\mathbf{S}'$, where $\Sigma' \subseteq \{\psi_1, \ldots, \psi_{n-1}\}$. If Σ'/ψ is an inference rule of $\mathbf{S}$, then $\boldsymbol{\rho}[\Sigma'] \vdash_{\mathbf{L}} \boldsymbol{\rho}(\psi)$. Therefore $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}} \boldsymbol{\rho}(\psi)$, by transitivity and the induction hypothesis. It remains to consider the case where Σ'/ψ is in $\boldsymbol{\tau}[\Gamma]/\boldsymbol{\tau}(\gamma)$ and $\Gamma \vdash_{\mathbf{L}'} \gamma$. By the induction hypothesis and Theorem 2.21(4),

$$\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}[\Sigma'] \text{ and } \boldsymbol{\rho}[\boldsymbol{\tau}[\Gamma]] \vdash_{\mathbf{L}'} \Gamma \text{ and } \Gamma \vdash_{\mathbf{L}'} \gamma \text{ and } \gamma \vdash_{\mathbf{L}'} \boldsymbol{\rho}[\boldsymbol{\tau}(\gamma)].$$

Therefore $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}[\boldsymbol{\tau}(\gamma)]$, by the fact that $\Sigma' = \boldsymbol{\tau}[\Gamma]$ and transitivity. In particular, since $\psi \in \boldsymbol{\tau}(\gamma)$, we have $\boldsymbol{\rho}[\Sigma] \vdash_{\mathbf{L}'} \boldsymbol{\rho}[\psi]$. □

Corollary 2.27 (Blok & Pigozzi [11])**.** *Let $\vdash$ be a finitary deductive system (elementarily) algebraized by quasivariety* K*. Then the lattice of finitary extensions of $\vdash$ is anti-isomorphic to the lattice of subquasivarieties of* K.[2]

Proof. By Theorem 2.26, the *finitary* extensions of $\vdash$ map onto those of $\models_{\mathsf{K}}$. As K is a quasivariety, the quasi-equations that hold in K are exactly the finite rules belonging to $\models_{\mathsf{K}}$. So, there is an obvious lattice anti-isomorphism

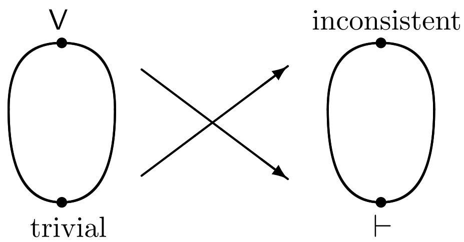
[2] An *anti-isomorphism* between lattices is an isomorphism from one to the order-dual of the other.

between the finitary extensions of $\models_{\mathsf{K}}$ and the subquasivarieties of K. Composing these two maps we see that the lattice of finitary extensions of $\vdash$ is anti-isomorphic to the lattice of subquasivarieties of K. □

Corollary 2.28. *Let $\vdash$ be a finitary deductive system (elementarily) algebraized by variety V. Then the lattice of axiomatic extensions of $\vdash$ is anti-isomorphic to the lattice of subvarieties of V.*

Proof. Recall that the axiomatic extensions of $\vdash$ are obtained by adding axioms to any formal system axiomatizing $\vdash$. From Theorem 2.26, notice that the lattice isomorphism h will map axiomatic extensions of $\vdash$ to axiomatic extensions of $\models_{\mathsf{V}}$. But rules with an empty set of premises, in the language of $\models_{\mathsf{V}}$, are just equations. So, the lattice of axiomatic extensions of $\models_{\mathsf{V}}$ is anti-isomorphic to the lattice of subvarieties of V. Therefore, composing the restriction of h to the axiomatic extensions of $\vdash$ with this lattice anti-isomorphism, we get the desired result. □

The following picture explains the situation. Notice that the trivial variety satisfies all equations, and so is contained in any subvariety of V. We say that a deductive system is *inconsistent* if each of its formulas is a theorem of it. Clearly, any sentential system has an inconsistent system as its largest extension.



Subvarieties of V **Axiomatic extensions of $\vdash$**

Recall, from the introduction, that we will be exploring 'maximal' extensions of the 'logics' $\mathbf{R^t}$ and $\mathbf{R}$ (which we will define in Chapter 4). They are finitary sentential logics that are algebraized by varieties. The central point to take away from this discussion is that we can study the finitary [resp. axiomatic] extensions of these logics by studying the subquasivarieties [resp. subvarieties] of their corresponding algebras.

2.5 Fragments

Let $\vdash$ be a sentential deductive system. Recall that $\vdash$ has a signature $\mathcal{F}$ which was fixed at the start of this chapter. For every $\mathcal{F}' \subseteq \mathcal{F}$ the $\mathcal{F}'$-*fragment* of $\vdash$ is the set of rules Γ/α in $\vdash$, such that the formulas in $\Gamma \cup \{\alpha\}$ contain only the connectives from $\mathcal{F}'$.

For every algebra $\boldsymbol{A}$ with signature $\mathcal{F}$ and every $\mathcal{F}' \subseteq \mathcal{F}$, the $\mathcal{F}'$-*reduct* of $\boldsymbol{A}$ is the algebra $\langle A; \mathcal{F}' \rangle$, i.e., it is the algebra obtained when the operations indexed by $\mathcal{F} - \mathcal{F}'$ are deleted from $\boldsymbol{A}$. (We also call $\boldsymbol{A}$ an *expansion* of $\langle A; \mathcal{F}' \rangle$.) Given a class of algebras K, the class of $\mathcal{F}'$-*subreducts of* K is the class of all algebras in the signature $\mathcal{F}'$ that embed into the $\mathcal{F}'$-reduct of an algebra in K, i.e., if K' is the class of $\mathcal{F}'$-reducts of members of K, then the $\mathcal{F}'$-subreduct class of K is $\mathbb{I}\,\mathbb{S}(\mathsf{K}')$.

Lemma 2.29 (Mal'cev [38]). *If K' is the class of $\mathcal{F}'$-reducts of quasivariety K, then $\mathbb{Q}(\mathsf{K}') = \mathbb{I}\,\mathbb{S}(\mathsf{K}')$.*

Proof. Recall that $\mathbb{Q} = \mathbb{I}\,\mathbb{S}\,\mathbb{P}\,\mathbb{P}_{\mathrm{u}} \geq \mathbb{I}\,\mathbb{S}$; see Theorem 1.40. To show that $\mathbb{I}\,\mathbb{S}\,\mathbb{P}\,\mathbb{P}_{\mathrm{u}}(\mathsf{K}') \subseteq \mathbb{I}\,\mathbb{S}(\mathsf{K}')$, it suffices to demonstrate that K' is already closed under $\mathbb{P}_{\mathrm{u}}$ and $\mathbb{P}$.

If $\boldsymbol{A'} = \prod_{i \in I} \boldsymbol{A'}_i$, where $\boldsymbol{A'}_i \in \mathsf{K}'$ for every $i \in I$, then there exist $\boldsymbol{A}_i \in \mathsf{K}$ ($i \in I$), such that each $\boldsymbol{A'}_i$ is a reduct of $\boldsymbol{A}_i$. But then $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i \in \mathsf{K}$, since K is a quasivariety. And it is easy to see that $\boldsymbol{A'}$ is the $\mathcal{F}'$-reduct of $\boldsymbol{A}$.

Similarly, it is easy to see that any ultraproduct of members in K' is the reduct of an ultraproduct in K, where we use the same ultrafilter in the latter case. $\square$

Theorem 2.30 (Blok & Pigozzi [11, Cor. 2.12]). *Let $\vdash$ be a sentential deductive system, with signature $\mathcal{F}$. If $\vdash$ is algebraizable, then so is any $\mathcal{F}'$-fragment of $\vdash$, where $\mathcal{F}'$ contains all the connectives that occur in the definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. If, moreover, $\vdash$ is algebraized by quasivariety K, then the $\mathcal{F}'$-fragment of $\vdash$ is algebraized by the $\mathcal{F}'$-subreduct class of K.*

Proof. If we let K' be the class of $\mathcal{F}'$-reducts of K, then it is clear from Characterization 2.22, that (1) and (2) continue to hold when $\vdash$ is replaced by its $\mathcal{F}'$-fragment and K by K'. Notice that $\models_{\mathsf{K}'}$ is still finitary. So, by Lemma 2.11 and Theorem 2.13, K' can be replaced by $\mathbb{Q}(\mathsf{K}')$ in 2.22, and the latter is equal to $\mathbb{S}(\mathsf{K}')$, by Lemma 2.29. $\square$

Chapter 3

Algebraization

In the previous chapter we established what it means for a deductive system $\vdash$ to be algebraized by a class of algebras K. This notion entails the existence of two definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. In this chapter we will discuss the details of how we might go about finding these translations (and how we might show that they do not exist for non-algebraizable deductive systems). The strategy is to explore some of the properties of $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$, and apply these insights to the specific example of the logic **CLL** (Example 2.2). The goal is to show that **CLL** is algebraized by the class IRL of involutive residuated lattices; see Section 1.6. Once this algebraization is established we will be able to use the same $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$ to algebraize the logic $\mathbf{R^t}$ in the next chapter.

3.1 Equivalence Formulas

Recall that in Section 2.2 we noticed that formal equations, like elements of congruences of $\boldsymbol{Fm}$, are essentially pairs of terms. This identification allowed us to prove Theorem 2.14. In the language of the framework established in Section 2.3, Theorem 2.14 states that the theories of an equational consequence relation $\models_{\mathsf{K}}$ are exactly the K-congruences of $\boldsymbol{Fm}$. Moreover, the theory lattice of $\models_{\mathsf{K}}$ is same thing as the K-congruence lattice of $\boldsymbol{Fm}$, as meets are intersections in both cases.

Let $\vdash$ be a sentential logic that is algebraized by a class of algebras K, via definable translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. Recall, from Characterization 2.22, that we can suppose that $\boldsymbol{\rho}$ is a set of binary terms and that $\boldsymbol{\tau}$ consists of pairs of unary terms. Also, recall that the equivalence from (the theory algebra of) $\vdash$ to (that of) $\models_{\mathsf{K}}$ is given by

$$\Lambda : T \mapsto \mathrm{Cn}_{\vdash}\, \boldsymbol{\tau}[T] = \boldsymbol{\rho}^{-1}[T].$$

We can use the connection between theories and congruences to see that $\boldsymbol{\rho}$ will satisfy certain properties.

Definition 3.1. A set $\boldsymbol{\rho}$ of binary formulas of a sentential logic $\vdash$ is called a set of *equivalence formulas* for $\vdash$ if

(i) $\vdash \boldsymbol{\rho}(p, p)$

(ii) $\{p\} \cup \boldsymbol{\rho}(p, q) \vdash q$

(iii) $\boldsymbol{\rho}(p_1, q_1) \cup \cdots \cup \boldsymbol{\rho}(p_n, q_n) \vdash \boldsymbol{\rho}(\alpha(\vec{p}), \alpha(\vec{q}))$ for all n-ary formulas α,

where $\vec{p}$ abbreviates $p_1, \ldots, p_n$ and similarly for $\vec{q}$. We say that $\vdash$ is *[finitely] equivalential* if it has a [finite] set of equivalence formulas.

An intuitive reading of $\boldsymbol{\rho}(p, q)$ is 'p is equivalent to q'. The first condition captures the idea that statements should be equivalent to themselves. The second is a form of modus ponens, i.e., if p is asserted and p is equivalent to q, then we can assert q. The last condition is called the *replacement property*: if p_i is equivalent to q_i for $i \leq n$, then a statement α is equivalent to the statement that is got by replacing some occurrences in α of p_i with q_i.

Theorem 3.2. *The definable translation $\boldsymbol{\rho}$ from $\models_{\mathsf{K}}$ to $\vdash$ is a set of equivalence formulas.*

Proof. Let T be the set of theorems of $\vdash$. By Example 2.17, T is a theory of $\vdash$. Therefore $\Lambda(T) = \boldsymbol{\rho}^{-1}[T]$ is a $\models_{\mathsf{K}}$–theory, i.e., a K-congruence of $\boldsymbol{Fm}$ (see Theorem 2.14). Since congruences are reflexive we have that $\langle p, p\rangle \in \boldsymbol{\rho}^{-1}[T]$. But this means that $\boldsymbol{\rho}(p, p) \subseteq T$, proving the first condition of Definition 3.1.

To prove the second condition, recall that $\{p\} \cup \boldsymbol{\rho}(p, q) \vdash q$ will hold if and only if

$$\mathrm{Cn}_{\vdash}\{q\} \subseteq \mathrm{Cn}_{\vdash}\{p\} \cup \boldsymbol{\rho}(p, q).$$

Let $T = \mathrm{Cn}_{\vdash}\{p\} \cup \boldsymbol{\rho}(p, q)$. The result will follow if we can show that $q \in T$. First notice that $\langle p, q\rangle \in \boldsymbol{\rho}^{-1}[T] = \Lambda(T)$. Secondly, because $T \vdash p$, we have that $\boldsymbol{\tau}(p) \subseteq \mathrm{Cn}_{\models_{\mathsf{K}}} \boldsymbol{\tau}[T] = \Lambda(T)$. Consequently, since $\Lambda(T)$ is a congruence,

$$\tau^l(q) \equiv_{\Lambda(T)} \tau^l(p) \equiv_{\Lambda(T)} \tau^r(p) \equiv_{\Lambda(T)} \tau^r(q),$$

for any $\langle \tau^l, \tau^r\rangle \in \boldsymbol{\tau}$. Therefore $\boldsymbol{\tau}(q) \subseteq \Lambda(T) = \boldsymbol{\rho}^{-1}[T]$. In other words $\boldsymbol{\rho}[\boldsymbol{\tau}(q)] \subseteq T$. From the definition of theory we have that $T \vdash \boldsymbol{\rho}[\boldsymbol{\tau}(q)]$, from which it follows, by Theorem 2.21(4), that $T \vdash q$, as required.

To prove the last condition of Definition 3.1 we use a similar method. Let

$$T = \mathrm{Cn}_{\vdash}\, \boldsymbol{\rho}(p_1, q_1) \cup \cdots \cup \boldsymbol{\rho}(p_n, q_1).$$

Then $\langle p_1, q_1\rangle, \ldots, \langle p_n, q_n\rangle \in \boldsymbol{\rho}^{-1}[T]$. As $\boldsymbol{\rho}^{-1}[T]$ is a congruence, and hence compatible with terms operations, we have $\langle \alpha(\overrightarrow{p}), \alpha(\overrightarrow{q})\rangle \in \boldsymbol{\rho}^{-1}[T]$. $\square$

Items (i) and (ii) of Definition 3.1 suggest that implication is the prototype for $\boldsymbol{\rho}$. Taking (iii) into consideration, however, we see that $\boldsymbol{\rho}$ better simulates a bi-conditional $\leftrightarrow$. This intuition holds up in the case of **CLL**.

Example 3.3. $\{p \to q, q \to p\}$ is a set of equivalence formulas for **CLL**.

Proof. The first condition of Definition 3.1 follows simply from axiom A1 and the second from modus ponens. Let's prove the third condition. Let α be an n-ary formula. Let

$$\Gamma = \{p_1 \to q_1, q_1 \to p_1, \ldots, p_n \to q_n, q_n \to p_n\}.$$

We want to show that $\Gamma \vdash \alpha(\overrightarrow{p}) \to \alpha(\overrightarrow{q})$ and $\Gamma \vdash \alpha(\overrightarrow{q}) \to \alpha(\overrightarrow{p})$. We proceed by induction on the complexity $\#\alpha$ of α.

First, if $\alpha(p_1, \ldots, p_n) = p_i$ for some $i = 1, \ldots, n$, then clearly $\Gamma \vdash p_i \to q_i$ and $\Gamma \vdash q_i \to p_i$, because $\{p_i \to q_i, q_i \to p_i\} \subseteq \Gamma$. Secondly, if α is t, then by a substitution instance of A1 we have $\vdash t \to t$.

For the inductive step suppose that β and γ are terms such that

$$\Gamma \vdash \{\beta(\overrightarrow{p}) \to \beta(\overrightarrow{q}), \beta(\overrightarrow{q}) \to \beta(\overrightarrow{p}), \gamma(\overrightarrow{p}) \to \gamma(\overrightarrow{q}), \gamma(\overrightarrow{q}) \to \gamma(\overrightarrow{p})\}.$$

To simplify the notation, let's say $r_1 = \beta(\overrightarrow{p})$, $r_2 = \gamma(\overrightarrow{p})$, $s_1 = \beta(\overrightarrow{q})$ and $s_2 = \gamma(\overrightarrow{q})$. This allows us to rewrite the above as

IH $\Gamma \vdash \{r_1 \to s_1, s_1 \to r_1, r_2 \to s_2, s_2 \to r_2\}$. (induction hypothesis)

Suppose that $\alpha = \neg\beta$. Recall that in Example 2.6 we proved that

$$q \to p \vdash_{\mathbf{CLL}} \neg p \to \neg q.$$

Using the induction hypothesis and an appropriate substitution we can say that $\Gamma \vdash_{\mathbf{CLL}} \neg r_1 \to \neg s_1$. In our original notation this means that

$$\Gamma \vdash_{\mathbf{CLL}} \neg\beta(\overrightarrow{p}) \to \neg\beta(\overrightarrow{q}) \quad \text{i.e.} \quad \Gamma \vdash_{\mathbf{CLL}} \alpha(\overrightarrow{p}) \to \alpha(\overrightarrow{q})$$

as required. The arrow in the other direction follows simply from the symmetry of the problem. The proofs for the other connectives follow a similar pattern to this one.

Suppose that $\alpha = \beta \wedge \gamma$.

1	$\vdash_{\mathbf{CLL}} (r_1 \wedge r_2) \to r_1$	A4
2	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \wedge r_2) \to s_1$	1, IH, A2, MP×2
3	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \wedge r_2) \to s_2$	A5, IH, A2, MP×2
4	$\Gamma \vdash_{\mathbf{CLL}} ((r_1 \wedge r_2) \to s_1) \wedge ((r_1 \wedge r_2) \to s_2)$	1, 2, AD
5	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \wedge r_2) \to (s_1 \wedge s_2)$	3, A6, MP

Now suppose that $\alpha = \beta \vee \gamma$.

1	$\vdash_{\mathbf{CLL}} s_1 \to (s_1 \vee s_2)$	A7
2	$\Gamma \vdash_{\mathbf{CLL}} r_1 \to (s_1 \vee s_2)$	IH, 1, A2, MP×2
3	$\Gamma \vdash_{\mathbf{CLL}} r_2 \to (s_1 \vee s_2)$	IH, A8, A2, MP×2
4	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \to (s_1 \vee s_2)) \wedge (r_2 \to (s_1 \vee s_2))$	2, 3, AD
5	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \vee r_2) \to (s_1 \vee s_2)$	4, A9, MP

Suppose that $\alpha = \beta \cdot \gamma$.

1	$\vdash_{\mathbf{CLL}} s_2 \to (s_1 \to (s_1 \cdot s_2))$	A12
2	$\Gamma \vdash_{\mathbf{CLL}} r_2 \to (s_1 \to (s_1 \cdot s_2))$	IH, 1, A2, MP×2
3	$\Gamma \vdash_{\mathbf{CLL}} s_1 \to (r_2 \to (s_1 \cdot s_2))$	2, A3, MP
4	$\Gamma \vdash_{\mathbf{CLL}} r_1 \to (r_2 \to (s_1 \cdot s_2))$	IH, 3, A2, MP
5	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \cdot r_2) \to (s_1 \cdot s_2)$	4, A13, MP

Suppose that $\alpha = \beta \to \gamma$. Here we use suffixing, (2.2), from Example 2.5.

1	$\vdash_{\mathbf{CLL}} (r_2 \to s_2) \to ((s_1 \to r_2) \to (s_1 \to s_2))$	A2
2	$\Gamma \vdash_{\mathbf{CLL}} (s_1 \to r_2) \to (s_1 \to s_2)$	IH, 1, MP
3	$\vdash_{\mathbf{CLL}} (s_1 \to r_1) \to ((r_1 \to r_2) \to (s_1 \to r_2))$	(2.2)
4	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \to r_2) \to (s_1 \to r_2)$	IH, 3, MP
5	$\Gamma \vdash_{\mathbf{CLL}} (r_1 \to r_2) \to (s_1 \to s_2)$	4, 2, A2, MP×2 □

It turns out (see notes after Theorem 3.15), that a logic has at most one set of equivalence formulas, up to inter-derivability. This means that we can be confident that, in our algebraization of **CLL**, $\boldsymbol{\rho}$ will be $\{p \to q, q \to p\}$. However, this is only half of what is needed to establish the algebraization of **CLL**; we still need to find $\boldsymbol{\tau}$. This is what we will do in the next section.

3.2 Semantics

Let $\vdash$ be a sentential logic algebraized by K. In this section we will explore some properties of the definable translation $\boldsymbol{\tau}$, which translates formulas of $\vdash$ into equations of $\models_{\mathsf{K}}$. In particular, $\boldsymbol{\tau}$ should translate theorems of $\vdash$ into equations that hold in K. If we consider the theorems of $\vdash$ to be 'true' statements, then $\boldsymbol{\tau}$ will tell us which elements of a particular algebra $\boldsymbol{A} \in \mathsf{K}$ can be considered 'true'; they are the elements of $\boldsymbol{A}$ which satisfy the equations of $\boldsymbol{\tau}$. This intuition leads us to the study of *semantics*, which will give us a useful perspective from which to approach $\boldsymbol{\tau}$. A semantics for a logic $\vdash$ consists of a class of mathematical objects, called *models*, together with a precise definition of what it means for a rule to be *valid* in a model. Furthermore we require that the *soundness* and *completeness* relations,

$$\Gamma \vdash \alpha \text{ iff the rule } \Gamma/\alpha \text{ is valid in every model,}$$

hold for all sets $\Gamma \cup \{\alpha\}$ of formulas of $\vdash$.

This is exactly what condition (1) of Characterization 2.22 specifies:

$$\Gamma \vdash \alpha \text{ iff } \boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}} \boldsymbol{\tau}(\alpha).$$

In this case the models are the algebras of K. Recall that $\boldsymbol{\tau}$ is a set of equations. We specify that a rule Γ/α is *valid* in an algebra $\boldsymbol{A} \in \mathsf{K}$, if for every $\boldsymbol{A}$-evaluation h, whenever $h(\tau^l(\gamma)) = h(\tau^r(\gamma))$ for every $\gamma \in \Gamma$, then $h(\tau^l(\alpha)) = h(\tau^r(\alpha))$, for every $\langle \tau^l, \tau^r \rangle \in \boldsymbol{\tau}$. This is the definition of an *algebraic semantics* for $\vdash$, which was made precise by Blok and Pigozzi [11].

For deductive systems, there are three types of structure that are commonly used for semantics. In one, the structures include topologies. For example, Kripke semantics for modal or intuitionistic logics can (sometimes) be extended to a *duality* which incorporates topology. In another, they are algebras (as above). In the last case they are structures called *matrices*.

Recall that, with every deductive system, we have associated a first-order language. In the case of a sentential deductive system there is a hidden unary predicate symbol, called the truth predicate. For every first-order theory there is a canonical semantics. In this semantics, each n-ary predicate symbol is associated with an n-ary relation in the models. For the first-order theory of a sentential system $\vdash$ the models are pairs $\langle \boldsymbol{A}; F\rangle$ where $\boldsymbol{A}$ is an algebra in the same signature as $\vdash$ and F is a unary relation, associated with the truth predicate. In other words, F is just a subset of A. Such a pair is called a *matrix*. The elements of F are called the *designated* elements of this matrix.

We will soon see that every sentential system has a non-trivial semantics of this kind. The aim of this section is, roughly speaking, to show how we can change the matrix model semantics of an algebraizable sentential logic to an algebraic semantics, by eventually defining F using equations. These equations will be those contained in $\boldsymbol{\tau}$. Along the way we will apply our observations to the logic **CLL** in the form of examples, and end this section with a proof that **CLL** is algebraized by IRL. It will be convenient for us to change the definition of IRL to distinguish the operation $\to$ and stipulate the axiom $x \to y = \neg(x \cdot \neg y)$. As every IRL has the same terms, congruences and subalgebras as its $\to$ expansion, we may systematically confuse IRL with the resulting variety.

Let $\vdash$ be a fixed but arbitrary sentential logic throughout.

We say that matrix $\langle \boldsymbol{A}, F\rangle$ (in the signature of $\vdash$) *validates* a rule Γ/α, written $\langle \boldsymbol{A}, F\rangle \models \Gamma/\alpha$, if, for every $\boldsymbol{A}$-evaluation h,

$$\text{whenever } h[\Gamma] \subseteq F, \text{ then } h(\alpha) \in F.$$

If K is a class of matrices, then '$\mathsf{K} \models \Gamma/\alpha$' means '$\langle \boldsymbol{A}, F\rangle \models \Gamma/\alpha$ for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{K}$'.

A *matrix model of* $\vdash$ is a matrix $\langle \boldsymbol{A}, F\rangle$ that validates every derivable rule of $\vdash$, that is to say, if $\Gamma \vdash \alpha$ then $\langle \boldsymbol{A}, F\rangle \models \Gamma/\alpha$. In this case, we also say that F is a $\vdash$*–filter* of $\boldsymbol{A}$. We let $\mathrm{Mod}(\vdash)$ denote the class of all matrix models of $\vdash$.

If $\vdash$ is axiomatized by a formal system $\mathbf{L}$, then, to show that $\langle \boldsymbol{A}, F\rangle$ is a matrix model of $\vdash$, it is only necessary to check that $\langle \boldsymbol{A}, F\rangle$ validates all the axioms and inference rules in $\mathbf{L}$. This follows by induction on the length of a proof in $\mathbf{L}$.

Example 3.4. A subset T of Fm is a $\vdash$–theory if and only it is a $\vdash$–filter of $\boldsymbol{Fm}$, i.e., $\langle \boldsymbol{Fm}, T\rangle \in \mathrm{Mod}(\vdash)$.

Proof. Let Γ/α be any rule such that $\Gamma \vdash \alpha$. Suppose that T is a $\vdash$–theory. We need to show that $\langle \boldsymbol{Fm}, T\rangle \models \Gamma/\alpha$. Let h be any $\boldsymbol{Fm}$-evaluation, i.e., any substitution. Then $h[\Gamma] \vdash h[\alpha]$, since $\vdash$ is substitution invariant. Now, if $h[\Gamma] \subseteq T$, then $h(\alpha) \in T$, since T is a theory.

Conversely, let T be a $\vdash$–filter of $\boldsymbol{Fm}$. Suppose that $\Gamma \vdash \alpha$ and $\Gamma \subseteq T$. But then $\alpha \in T$, as witnessed by the identity function on $\boldsymbol{Fm}$. □

It is easy to show that $\mathrm{Mod}(\vdash)$ forms a semantics for $\vdash$ in the following sense.

Theorem 3.5. $\Gamma \vdash \alpha$ *if and only if* $\mathrm{Mod}(\vdash) \models \Gamma/\alpha$.

Proof. The forward direction is clear from the definition. For the reverse direction, let Γ/α be a rule such that $\mathrm{Mod}(\vdash) \models \Gamma/\alpha$. Recall that $\Gamma \vdash \alpha$ if and only if $\mathrm{Cn}_\vdash(\{\alpha\}) \subseteq \mathrm{Cn}_\vdash(\Gamma)$. Let $T = \mathrm{Cn}_\vdash(\Gamma)$. From Example 3.4 above we know that $\langle \boldsymbol{Fm}, T\rangle \in \mathrm{Mod}(\vdash)$. In particular this means that $\langle \boldsymbol{Fm}, T\rangle \vdash \Gamma/\alpha$. From the definition of T we know that $\Gamma \subseteq T$. If we let h be the identity map from $\boldsymbol{Fm}$ to itself, then we see that $h(\alpha) = \alpha \in T$. Therefore $\mathrm{Cn}_\vdash(\{\alpha\}) \subseteq T$ as required. □

Given formal system $\mathbf{L}$, we often shorten $\mathrm{Mod}(\vdash_{\mathbf{L}})$ to $\mathrm{Mod}(\mathbf{L})$.

Example 3.6. Let $\boldsymbol{A}$ be an IRL, with lattice order $\leq$, and define $F_\leq :=\{a \in A : e \leq a\}$. Then $\langle \boldsymbol{A}, F_\leq\rangle \in \mathrm{Mod}(\mathbf{CLL})$.

Proof. Notice from the definition that we require matrix models of a sentential system to have the same signature as the sentential system. It is for this reason that we included $\to$ into the signature of IRLs for this section. There is still a slight inconsistency in this regard, since **CLL** has a constant symbol t and IRLs have a constant symbol e. We prefer to keep this ambiguity, as e is a natural symbol for a multiplicative identity, whereas t has a natural logical meaning. We will henceforth use whichever symbol is more appropriate without comment.

We need to show that $\langle \boldsymbol{A}, F_\leq\rangle \models \Gamma/\alpha$, whenever $\Gamma \vdash_{\mathbf{CLL}} \alpha$. Recall that we have an axiomatization of **CLL** given in Example 2.2. Therefore, we only have to check that $\langle \boldsymbol{A}, F_\leq\rangle$ validates all the axioms and inference rules of **CLL**. For brevity, we define $F := F_\leq$.

Let h be any $\boldsymbol{A}$-evaluation. In particular, let $a := h(p)$, $b := h(q)$ and $c := h(r)$.

By Theorem 1.58(i), $e \leq a \to a = h(p \to p)$. This means that $h(p \to p) \in F$. Since h was chosen arbitrarily, we conclude that $\langle \boldsymbol{A}, F\rangle$ validates A1. From Theorem 1.56(ii), it follows that $e \leq e \to (a \to a)$, validating A11. While we're at it, A10 is validated by the simple fact that $e \leq e$. We will use Theorem 1.56(ii) so often in this proof that we will do so without comment.

Certainly $a \wedge b \leq a$ and $a \wedge b \leq b$, so that $e \leq (a \wedge b) \to b$ and $e \leq (a \wedge b) \to b$. This respectively validates A4 and A5. The validity of A7 and A8 follows similarly from $a \leq a \vee b$ and $b \leq a \vee b$.

While we are catching low hanging fruit, notice that A15 follows from the fact that $\neg\neg a \leq a$. Similarly, A14 follows from Theorem 1.57(ii), i.e., $a \to \neg b \leq b \to \neg a$.

For A12 we need to increase the complexity slightly. We know that $a \cdot b \leq b \cdot a$, since $\cdot$ is commutative. If we now apply the law of residuation we get $a \leq b \to (b \cdot a)$, which shows that $e \leq a \to (b \to (b \cdot a))$.

To prove the validity of A13 we notice that

$$e \leq (a \to (b \to c)) \to ((b \cdot a) \to c)$$

will follow it we can show that $a \to (b \to c) \leq (b \cdot a) \to c$. This will in turn follow from the law of residuation if we can show $(b \cdot a) \cdot (a \to (b \to c)) \leq c$.

But this last statement is true, because

$$\begin{aligned}(b \cdot a) \cdot (a \to (b \to c)) &= b \cdot (a \cdot (a \to (b \to c))) && \because \text{associativity}\\ &\leq b \cdot (b \to c) && \because \text{Theorem 1.56(i)}\\ &\leq c && \because \text{Theorem 1.56(i)}\end{aligned}$$

The second-to-last inequality also uses the compatability of $\cdot$ with the lattice order (Theorem 1.54). Using similar reasoning, we can show that A3 is valid, by successive applications of the law of residuation (in reverse), which show that

$$e \leq (a \to (b \to c)) \to (b \to (a \to c)) \text{ iff } a \cdot (b \cdot (a \to (b \to c))) \leq c.$$

The right-hand inequality is true, because

$$\begin{aligned}a \cdot b \cdot (a \to (b \to c)) &= b \cdot a \cdot (a \to (b \to c)) && \because \text{commutativity}\\ &\leq b \cdot (b \to c) && \because \text{Theorem 1.56(i)}\\ &\leq c && \because \text{Theorem 1.56(i).}\end{aligned}$$

By a similar method we can show for A2 that

$$e \leq (a \to b) \to ((c \to a) \to (c \to b)) \text{ iff } c \cdot (c \to a) \cdot (a \to b) \leq b.$$

The right-hand-side clearly follows from two successive applications of Theorem 1.56(i). For A6:

$$e \leq ((a \to b) \wedge (a \to b)) \to (a \to (b \wedge c)) \text{ iff } a \cdot ((a \to b) \wedge (a \to b)) \leq b \wedge c.$$

To prove this we use Theorem 1.58(vi), i.e., the fact that IRL's satisfy $x \cdot (y \wedge z) \leq (x \cdot y) \wedge (x \cdot z)$.

$$\begin{aligned} a \cdot ((a \to b) \wedge (a \to b)) &\leq (a \cdot (a \to b)) \wedge (a \cdot (a \to c)) \\ &\leq b \wedge c \end{aligned}$$

Before we get to the inference rules, we need to prove A9 which will follow from

$$e \leq ((a \to c) \wedge (b \to c)) \to ((a \vee b) \to c) \iff (a \vee b) \cdot ((a \to c) \wedge (b \to c)) \leq c$$

Here we use Theorem 1.58(vii), i.e., $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$.

Notice that

$$\begin{aligned} a \cdot ((a \to c) \wedge (b \to c)) &\leq (a \cdot (a \to c)) \wedge (a \cdot (b \to c)) \\ &\leq (a \cdot (a \to c)) \\ &\leq c. \end{aligned}$$

Similarly, $b \cdot ((a \to c) \wedge (b \to c)) \leq c$. Therefore the join of these two terms falls below c, as required.

Let's now show that MP is valid in $\boldsymbol{A}$. Suppose $e \leq a$ and $e \leq a \to b$. From the last inequality it follows that $a \leq b$. So by transitivity of $\leq$ we get $e \leq b$ as required.

Finally, for AD, suppose $e \leq a$ and $e \leq b$. Then $e \leq a \wedge b$, since $a \wedge b$ is the greatest lower bound of a and b. □

To justify all the work that went into proving this example, recall that we will eventually show that **CLL** is algebraized by IRL. We are trying to find the definable translations that witness this. We already mentioned in Section 3.1 that $\boldsymbol{\rho}$ will be $\{p \to q, q \to p\}$. We will eventually show that $\boldsymbol{\tau}(\alpha) = \{e \wedge \alpha \approx e\}$, or using the notation introduced in Sections 1.4 and 2.3, $\boldsymbol{\tau} = \{e \preceq p\}$. Example 3.6 shows that if $\Gamma \vdash_{\mathbf{CLL}} \alpha$, then $\boldsymbol{\tau}[\Gamma] \models_{\mathsf{IRL}} \boldsymbol{\tau}(\alpha)$. The goal for the rest of this section is to establish the converse. In particular, what hinders us is that the class of matrix models of **CLL**, Mod(**CLL**), is too 'large'. For instance, Example 3.4 tells us that Mod(**CLL**) contains all the matrices of the form $\langle \boldsymbol{Fm}, T \rangle$ for all **CLL**-theories T. Notice that $\boldsymbol{Fm}$ is not contained in IRL, and that for any particular algebra (such as $\boldsymbol{Fm}$) there are many filters that qualify a matrix for membership of Mod(**CLL**). In an effort to reduce the size of Mod(**CLL**), in such a way that (since we hope that **CLL** is algebraizable) each algebra has only one **CLL**–filter, we need to consider the *reduced matrix models* of **CLL**.

We say that $\langle \boldsymbol{A}, F\rangle$ is *reduced* if it is model-theoretically simple. This means that it is impossible to map $\boldsymbol{A}$ onto an algebra $\boldsymbol{B}$ with a homomorphism that preserves *and reflects* the subset F, unless the map is an isomorphism.

Equivalently, in algebraic terms:

Characterization 3.7. A matrix $\langle \boldsymbol{A}, F\rangle$ is reduced iff no congruence relation on $\boldsymbol{A}$ makes F a union of congruence classes, except for the identity congruence.

Let $\mathrm{Mod}^*(\vdash)$ denote the class of reduced matrix models of $\vdash$. As with Mod, we often shorten $\mathrm{Mod}^*(\vdash_{\mathbf{L}})$ to $\mathrm{Mod}^*(\mathbf{L})$, for a given formal system $\mathbf{L}$. Let $\langle \boldsymbol{A}, F\rangle$ be some fixed matrix, in the same language as $\vdash$, for the rest of this section. Let $\theta \in \mathrm{Con}(\boldsymbol{A})$.

We say that θ *is compatible with* F if F is a union of θ-classes, i.e.,

$$\text{whenever } a \in F \text{ and } a \equiv_\theta b, \text{ then } b \in F.$$

Lemma 3.8. *Let* $\Sigma = \{\psi \in \mathrm{Con}\,\boldsymbol{A} : \psi \text{ is compatible with } F\}$. *Then the join of* Σ *in* $\mathbf{Con}\,\boldsymbol{A}$ *is compatible with* F.

Proof. Notice that the identity congruence of $\boldsymbol{A}$, denoted id_A, is compatible with every subset of A. In particular, $\mathrm{id}_A \in \Sigma$ so that Σ is non-empty.

Let $a \in F$ and let $\theta = \bigvee \Sigma$. Let $b \in A$ such that $a \equiv_\theta b$. By Theorem 1.24, there exist $c_1, \ldots, c_n \in A$ such that

$$a = c_1 \equiv_{\psi_1} c_2 \equiv_{\psi_2} \cdots \equiv_{\psi_{n-1}} c_n = b,$$

where $\psi_i \in \Sigma$ for $i = 1, \ldots, n-1$.

From this we can infer that $c_2 \in F$ since ψ_1 is compatible with F and $c_1 = a \in F$. By a similar argument we can conclude that $c_3, \ldots, c_n$ are in F, so that $b = c_n \in F$. $\square$

This lemma allows us to define $\Omega^{\boldsymbol{A}}F$ as the largest congruence of $\boldsymbol{A}$ that is compatible with F.

Blok and Pigozzi [11] called $\Omega^{\boldsymbol{A}}F$ the *Leibniz congruence* of F, because Leibniz proposed that two entities are equal if they have the same properties. The Leibniz congruence identifies the elements of A that have the same properties definable in the first-order language of $\boldsymbol{A}$. This follows from the observation that, $a \equiv_{\Omega^{\boldsymbol{A}}F} b$ if and only if,

$$\alpha(a, c_1, \ldots, c_{k-1}) \in F \iff \alpha(b, c_1, \ldots, c_{k-1}) \in F,$$

for every k-ary term $\alpha \in Fm$ and all $c_1, \ldots, c_{k-1} \in A$ (see for instance [11, Thm. 1.5]). The *Leibniz operator* of $\vdash$ is constituted by the maps

$$F \mapsto \Omega^{\boldsymbol{A}} F \quad (F \text{ a } \vdash\text{–filter of } \boldsymbol{A})$$

taken over all algebras in the signature of $\vdash$. Recall that the development of equivalence formulas in Section 3.1 was motivated by following our intuition about the notion of 'equivalence'. The characterization above suggests that the Leibniz operator will play some part in this narrative. In fact, we will see after Theorem 3.15 that the Leibniz operator is the end goal of that particular story.

One of the reasons for introducing the Leibniz operator is that it allows us to characterise what it means for a matrix to be reduced. It is clear from Characterization 3.7 that $\langle \boldsymbol{A}, F\rangle$ is reduced if and only if $\Omega^{\boldsymbol{A}} F = \mathrm{id}_A$.

Define $F/\Omega^{\boldsymbol{A}} F = \{a/\Omega^{\boldsymbol{A}} F : a \in F\}$, where $a/\Omega^{\boldsymbol{A}} F$ denotes the $\Omega^{\boldsymbol{A}} F$-equivalence class of a, that is $a/\Omega^{\boldsymbol{A}} F = \{b \in A : a \equiv_{\Omega^{\boldsymbol{A}} F} b\}$. The following is an important property of the Leibniz operator that we use regularly in this section.

Lemma 3.9. *If $b/\Omega^{\boldsymbol{A}} F \in F/\Omega^{\boldsymbol{A}} F$, then $b \in F$.*

Proof. Suppose $b/\Omega^{\boldsymbol{A}} F \in F/\Omega^{\boldsymbol{A}} F$ for some $b \in A$. Then there exists $a \in F$ such that $b/\Omega^{\boldsymbol{A}} F = a/\Omega^{\boldsymbol{A}} F$. This means that $a \equiv_{\Omega^{\boldsymbol{A}} F} b$, but since $\Omega^{\boldsymbol{A}} F$ is compatible with F, we have that $b \in F$. □

To simplify our notation, we abbreviate $\langle \boldsymbol{A}/\Omega^{\boldsymbol{A}} F, F/\Omega^{\boldsymbol{A}} F\rangle$ as $\langle \boldsymbol{A}^*, F^*\rangle$.

Theorem 3.10. *$\langle \boldsymbol{A}^*, F^*\rangle$ is reduced.*

Proof. Suppose that $\Omega^{\boldsymbol{A}^*} F^* \supsetneq \mathrm{id}_{A^*}$. Let $\theta := \Omega^{\boldsymbol{A}^*} F^*$.

Now, by the Correspondence Theorem (Theorem 1.26), there exists $\phi \in \mathrm{Con}(\boldsymbol{A})$ such that $\phi/\Omega^{\boldsymbol{A}} F = \theta$. In particular $\Omega^{\boldsymbol{A}} F \subsetneq \phi$.

Now we show that ϕ is compatible with F. Let $a \in F$ and $b \in A$ such that $a \equiv_\phi b$. We have that $a/\Omega^{\boldsymbol{A}} F \equiv_\theta b/\Omega^{\boldsymbol{A}} F$. Now $a/\Omega^{\boldsymbol{A}} F \in F^*$ by definition. But then $b/\Omega^{\boldsymbol{A}} F \in F^*$ since θ is compatible with F^*. So, by Lemma 3.9 we have that $b \in F$. Therefore ϕ is compatible with F.

This contradicts the maximality of $\Omega^{\boldsymbol{A}} F$. □

Another consequence of Lemma 3.9 is:

Theorem 3.11. *$\langle \boldsymbol{A}, F\rangle \models \Gamma/\alpha$ if and only if $\langle \boldsymbol{A}^*, F^*\rangle \models \Gamma/\alpha$.*

Proof. First suppose that $\langle \boldsymbol{A}, F\rangle \models \Gamma/\alpha$. Let h be an $\boldsymbol{A}^*$-evaluation such that $h[\Gamma] \subseteq F^*$.

For any variable $x \in \mathit{Var}$ choose an element a_x from $h(x) \in A^*$, i.e. $h(x) = a_x/\Omega^{\boldsymbol{A}}F$. Define a mapping

$$\begin{aligned} g : \mathit{Var} &\to A \\ x &\mapsto a_x. \end{aligned}$$

We can extend g uniquely to a homomorphism $g' : \boldsymbol{Fm} \to \boldsymbol{A}$, since $\boldsymbol{Fm}$ is absolutely free over Var. Let $q : \boldsymbol{A} \to \boldsymbol{A}^*$ be the canonical surjection. Note that $q \circ g' = h$, since these functions agree on Var. In particular, for any $\gamma \in \Gamma$, we have $g'(\gamma)/\Omega^{\boldsymbol{A}}F = h(\gamma) \in F^*$. Therefore, by Lemma 3.9, it follows that $g'(\gamma) \in F$ for any $\gamma \in \Gamma$. By assumption, since g' is an $\boldsymbol{A}$-evaluation, $g'(\alpha) \in F$. Therefore $h(\alpha) = g'(\alpha)/\Omega^{\boldsymbol{A}}F \in F^*$.

Conversely, suppose that $\langle \boldsymbol{A}^*, F^*\rangle \models \Gamma/\alpha$. Let h be an $\boldsymbol{A}$-evaluation such that $h[\Gamma] \subseteq F$. Then $q \circ h$ is an $\boldsymbol{A}^*$-evaluation, and $qh[\Gamma] \subseteq F^*$, so that by assumption $qh(\alpha) \in F^*$, i.e. $h(\alpha)/\Omega^{\boldsymbol{A}}F \in F^*$. By Lemma 3.9 we have that $h(\alpha) \in F$. □

The following is an immediate corollary of Theorems 3.10 and 3.11:

Corollary 3.12. *If* $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash)$ *then* $\langle \boldsymbol{A}^*, F^*\rangle \in \mathrm{Mod}^*(\vdash)$.

Now we get to the main theorem of this discussion so far. In our model-theoretic semantics, established in Theorem 3.5, we only have to focus our attention on the reduced matrix models.

Theorem 3.13. $\Gamma \vdash \alpha$ *iff* $\mathrm{Mod}^*(\vdash) \models \Gamma/\alpha$.

Proof. Necessity follows from Theorem 3.5, since $\mathrm{Mod}^*(\vdash) \subseteq \mathrm{Mod}(\vdash)$. For sufficiency, use Theorems 3.10 and 3.11. □

Now that we have shown why we can restrict our attention to the reduced matrix models of $\vdash$, it is worthwhile to make precise some of the intuitions we established at the start of this section.

Definition 3.14. A logic $\vdash$ is *truth-equational* if there exists a set of unary equations,

$$\boldsymbol{\tau} = \{\delta_i(x) \approx \epsilon_i(x) : i \in I\},$$

such that for all reduced matrix models $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and all $a \in A$, we have

$$a \in F \text{ iff } (\delta_i^{\boldsymbol{A}}(a) = \epsilon_i^{\boldsymbol{A}}(a) \text{ for all } i \in I).$$

The reason for this name is that $\boldsymbol{\tau}$ allows us to define the 'true' elements of $\boldsymbol{A}$ (these are the elements of F), using equations. It turns out that a sentential logic is algebraizable if and only if it is truth-equational and equivalential (see Definition 3.1).

If $\boldsymbol{B}$ is an IRL and $a \in B$, then we write $[a) := \{b \in B : a \leq b\}$ and $(a] := \{b \in B : b \leq a\}$. For our algebraization of **CLL**, we need to prove that if $\boldsymbol{\tau}[\Gamma] \models_{\mathsf{IRL}} \boldsymbol{\tau}(\alpha)$ then $\Gamma \vdash_{\mathbf{CLL}} \alpha$. We shall show that the *reduced* matrix models of **CLL** are exactly the matrices of the form $\langle \boldsymbol{B}, [e^{\boldsymbol{B}})\rangle$, where $\boldsymbol{B} \in \mathsf{IRL}$. This will allow us to conclude that **CLL** is truth-equational with $\boldsymbol{\tau} = \{e \wedge x \approx e\}$. To demonstrate this, we first need to show that these matrices are reduced, which will follow from the remarkable theorem below.

Theorem 3.15 ([14, Thm. 3.1.2])**.** *Suppose that $\boldsymbol{\rho}$ be a set of equivalence formulas for $\vdash$ and $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash)$. Then $\Omega^{\boldsymbol{A}}F = \{\langle a, b\rangle \in A^2 : \boldsymbol{\rho}^{\boldsymbol{A}}(a, b) \subseteq F\}$.*

Proof. Let $\theta = \{\langle a, b\rangle \in A^2 : \boldsymbol{\rho}^{\boldsymbol{A}}(a, b) \subseteq F\}$. Let's first show that $\theta \in \mathrm{Con}\,\boldsymbol{A}$. We have that $\vdash \boldsymbol{\rho}(p, p)$, therefore $\boldsymbol{\rho}(a, a) \subseteq F$ for any $a \in A$. Therefore θ is reflexive.

Let $\rho_j \in \boldsymbol{\rho}$. Note that by an instance of Definition 3.1(iii) we have that

$$\boldsymbol{\rho}(p, q), \boldsymbol{\rho}(p, p) \vdash \boldsymbol{\rho}(\rho_j(p, p), \rho_j(q, p)).$$

Furthermore, from Definition 3.1(ii) we can deduce that

$$\rho_j(p, p), \boldsymbol{\rho}(\rho_j(p, p), \rho_j(q, p)) \vdash \rho_j(q, p).$$

Using properties of consequence relations and Definition 3.1(i) we infer

$$\boldsymbol{\rho}(p, q) \vdash \rho_j(q, p).$$

Suppose that $\boldsymbol{\rho}^{\boldsymbol{A}}(a, b) \subseteq F$. We then have that $\rho_j(b, a) \in F$. Since ρ_j was chosen arbitrarily we can conclude that θ is symmetric.

Using a similar argument we have that

$$\boldsymbol{\rho}(p, p), \boldsymbol{\rho}(q, r) \vdash \boldsymbol{\rho}(\rho_j(p, q), \rho_j(p, r))$$

and

$$\rho_j(p, q), \boldsymbol{\rho}(\rho_j(p, q), \rho_j(p, r)) \vdash \rho_j(p, r),$$

so that

$$\rho_j(p, q), \boldsymbol{\rho}(q, r) \vdash \rho_j(p, r).$$

Since ρ_j was chosen arbitrarily we can say that

$$\boldsymbol{\rho}(p,q), \boldsymbol{\rho}(q,r) \vdash \boldsymbol{\rho}(p,r).$$

Suppose that $\boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F$ and $\boldsymbol{\rho}^{\boldsymbol{A}}(b,c) \subseteq F$. From the argument above can conclude that $\boldsymbol{\rho}^{\boldsymbol{A}}(a,c) \subseteq F$. So, θ is transitive.

Let f be a connective of $\vdash$ with arity n. Suppose that

$$a_1, \ldots, a_n, b_1, \ldots, b_n \in A,$$

such that $\boldsymbol{\rho}^{\boldsymbol{A}}(a_i, b_i) \subseteq F$ for every $i \leq n$. Then it follows simply from Definition 3.1(iii) that $\boldsymbol{\rho}^{\boldsymbol{A}}(f^{\boldsymbol{A}}(a_1, \ldots, a_n), f^{\boldsymbol{A}}(b_1, \ldots, b_n)) \subseteq F$, since $f(p_1, \ldots, p_n)$ is an n-ary formula.

Therefore $\theta \in \mathrm{Con}(\boldsymbol{A})$, and it remains to show that θ is the largest congruence compatible with F. Let $a \in F$ and let $b \in A$ such that $\boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F$. We can see clearly from Definition 3.1(ii) that $b \in F$. Therefore θ is compatible with F.

Let ϕ be any congruence of $\boldsymbol{A}$ that is compatible with F. Let $\langle a, b\rangle \in \phi$ and let $\rho_j \in \boldsymbol{\rho}$. We know that $\rho_j^{\boldsymbol{A}}(a,a) \in F$. Furthermore, we have that $\rho_j^{\boldsymbol{A}}(a,a) \equiv_\phi \rho_j^{\boldsymbol{A}}(a,b)$. But then $\rho_j^{\boldsymbol{A}}(a,b) \in F$, since ϕ is compatible with F. Since ρ_j was chosen arbitrarily, $\boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F$, which means that $\phi \subseteq \theta$, and we are done. □

Recall that in Section 3.1 we found an explicit set of equivalence formulas for **CLL**. Theorem 3.15 allows us to use these to calculate explicitly the Leibniz congruences of the matrix models of **CLL**. In particular, we can show that matrices of the form $\langle \boldsymbol{B}, [e^{\boldsymbol{B}})\rangle$, where $\boldsymbol{B} \in \mathsf{IRL}$, are reduced.

Example 3.16. Let $\boldsymbol{A} \in \mathsf{IRL}$, and let $F_{\leq} = \{a \in A : e \leq a\}$ (as in Example 3.6). Then $\langle \boldsymbol{A}, F_{\leq}\rangle \in \mathrm{Mod}^*(\mathbf{CLL})$.

Proof. From Example 3.6 we already know that $\langle \boldsymbol{A}, F_{\leq}\rangle \in \mathrm{Mod}(\mathbf{CLL})$. It remains to check that $\langle \boldsymbol{A}, F_{\leq}\rangle$ is reduced. This will follow if we can show that $\Omega^{\boldsymbol{A}} F_{\leq} = \mathrm{id}_A$. By Theorem 3.15 and Example 3.3,

$$\Omega^{\boldsymbol{A}} F_{\leq} = \{\langle a, b\rangle \in A^2 : \{a \to b, b \to a\} \subseteq F_{\leq}\}.$$

Suppose that $e \leq a \to b$ and $e \leq b \to a$. Then $a \leq b$ and $b \leq a$, so that $a = b$, as required. □

Theorem 3.15 shows us that the Leibniz operator and equivalence formulas are intrinsically linked. But notice that the Leibniz congruence is defined for any matrix, not just the matrix models of equivalential systems, i.e., it does not rely on the existence of certain terms. In particular, the Leibniz operator is defined for every sentential logic. From the way we defined the Leibniz operator of $\vdash$, we can see that its action is totally determined by the signature of $\vdash$ alone; it is only its domain (the $\vdash$–filters of an algebra) which depends on $\vdash$. In this way, the Leibniz operator justifies the 'abstract' in abstract algebraic logic.

Another consequence of Theorem 3.15 is that it guarantees the uniqueness (up to inter-derivability) of equivalence formulas for a given sentential deductive system, and partially proves the converse of Characterization 2.22.

A natural question we might ask is: how does the Leibniz operator of $\vdash$ act on $\boldsymbol{Fm}$? In Example 3.4 we showed that the $\vdash$–filters of $\boldsymbol{Fm}$ are exactly the theories of $\vdash$. Also notice that the target of $\Omega^{\boldsymbol{Fm}}$ is a set of congruences of $\boldsymbol{Fm}$, just like the theories of an equational consequence relation in the same language as $\boldsymbol{Fm}$. It turns out that, if $\vdash$ is algebraized by class K, then $\Omega^{\boldsymbol{Fm}}$ will witness the *equivalence* (isomorphism) between (the theory algebras of) $\vdash$ and $\models_{\mathsf{K}}$. In fact, we can characterize the algebraizability of $\vdash$ purely by means of its Leibniz operator.

Fact 3.17. *The sentential logic $\vdash$ is algebraizable if and only if the Leibniz operator of $\vdash$ is order-preserving, injective and commutes with homomorphic inverse images. (See [14] and its references.)*

Notice that there is no mention of the equivalent class of algebras in this characterization. If we wanted to show that $\vdash$ is *not* algebraizable, then we would simply have to find an algebra, with one or more $\vdash$–filters, that contradict any of the three conditions above. There are many interesting metalogical properties of deductive systems that can be characterised purely by means of their Leibniz operators. As with algebraizability, we can use these characterizations to falsify such properties. Examples include showing that a certain sentential logic is not equivalential or truth-equational [11, 48].

Before we return to the algebraization of **CLL**, we present a final bit of abstract algebraic logic theory, which we will need later.

Notice that intersections of $\vdash$–filters are again $\vdash$–filters. So, for any algebra $\boldsymbol{B}$ with the same signature as $\vdash$, the set of $\vdash$–filters of $\boldsymbol{B}$ forms a complete lattice.

The following theorem proves some of the claims made earlier about $\Omega^{\boldsymbol{Fm}}$.

Theorem 3.18 (Blok & Pigozzi [11, Thm. 5.1])**.** *Suppose the sentential deductive system $\vdash$ is algebraized by quasivariety K, with $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ as in*

Characterization 2.22. For every algebra $\boldsymbol{B}$, in the signature of $\vdash$, the Leibniz operator $\Omega^{\boldsymbol{B}}$ is an isomorphism between the lattices of $\vdash$–filters and K*-congruences of $\boldsymbol{B}$. In particular,*

$$\Omega^{\boldsymbol{B}} F = \{\langle a, b\rangle \in B^2 : \boldsymbol{\rho}^{\boldsymbol{B}}(a, b) \subseteq F\}$$

and the inverse isomorphism is given by

$$H^{\boldsymbol{B}} : \theta \mapsto \{a \in B : \boldsymbol{\tau}^{\boldsymbol{B}}(a) \subseteq \theta\}.$$

Proof. Recall from Theorem 3.2 that $\boldsymbol{\rho}$ is a set of equivalence formulas, so that by Theorem 3.15 it is indeed true that

$$\Omega^{\boldsymbol{B}} F = \{\langle a, b\rangle \in B^2 : \boldsymbol{\rho}^{\boldsymbol{B}}(a, b) \subseteq F\}. \tag{3.1}$$

Let $\boldsymbol{B}$ be an algebra with the same signature as $\vdash$ and F a $\vdash$–filter on $\boldsymbol{B}$. We show that $\Omega^{\boldsymbol{B}} F$ is a K-congruence. Recall from the definition that $\Omega^{\boldsymbol{B}} F$ is a congruence of $\boldsymbol{B}$. Let $\Sigma \cup \{\alpha \approx \beta\}$ be a set of equations, such that $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$. Let h be any $\boldsymbol{B}$-evaluation. Suppose that $\langle h(\mu), h(\nu)\rangle \in \Omega^{\boldsymbol{B}} F$ for every $\langle \mu, \nu\rangle \in \Sigma$. So by the observation above,

$$\boldsymbol{\rho}^{\boldsymbol{B}}(h(\mu), h(\nu)) \subseteq F \text{ for every } \langle \mu, \nu\rangle \in \Sigma.$$

Now by Characterization 2.22(3), $\boldsymbol{\rho}[\Sigma] \vdash \boldsymbol{\rho}(\alpha, \beta)$. Thus, $\boldsymbol{\rho}^{\boldsymbol{B}}(h(\alpha), h(\beta)) \subseteq F$, since F is a $\vdash$–filter, i.e., $\langle h(\alpha), h(\beta)\rangle \in \Omega^{\boldsymbol{B}} F$.

Therefore $\Sigma \models_{\{\boldsymbol{B}/\Omega^{\boldsymbol{B}} F\}} \alpha \approx \beta$, whenever $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$. So, because K is axiomatized by quasi-equations, $\boldsymbol{B}/\Omega^{\boldsymbol{B}} F \in \mathsf{K}$, as required.

Now let θ be an arbitrary K-congruence of $\boldsymbol{B}$, and let $H^{\boldsymbol{B}}\theta$ be defined as in the statement of the theorem. We can use the dual of the argument above, with Characterization 2.22(1) in place of Characterization 2.22(3), to show that $H^{\boldsymbol{B}}\theta$ is a $\vdash$–filter of $\boldsymbol{B}$.

Now we prove that $\Omega^{\boldsymbol{B}} H^{\boldsymbol{B}}\theta = \theta$. For all $a, b \in B$ we have $\langle a, b\rangle \in \Omega^{\boldsymbol{B}} H^{\boldsymbol{B}}\theta$ iff $\boldsymbol{\tau}^{\boldsymbol{B}}(\boldsymbol{\rho}^{\boldsymbol{B}}(a, b)) \subseteq \theta$. But by Characterization 2.22(2),

$$p \approx q \mathrel{=\!\mid\models}_{\mathsf{K}} \boldsymbol{\tau}[\boldsymbol{\rho}(p, q)],$$

so we have $\boldsymbol{\tau}^{\boldsymbol{B}}(\boldsymbol{\rho}^{\boldsymbol{B}}(a, b)) \subseteq \theta$ iff $\langle a, b\rangle \in \theta$. Thus $\Omega^{\boldsymbol{B}} H^{\boldsymbol{B}}\theta = \theta$.

Dually, we can show that $H^{\boldsymbol{B}}\Omega^{\boldsymbol{B}} F = F$ in a similar way, using Characterization 2.22(4) instead of Characterization 2.22(2). Therefore $\Omega^{\boldsymbol{B}}$ and $H^{\boldsymbol{B}}$ are mutually inverse bijections.

It remains to show that $\Omega^{\boldsymbol{B}}$ is order-preserving and order-reflecting. From Characterization 2.22(4), we get that $a \in F$ iff $\boldsymbol{\rho}^{\boldsymbol{B}}[\boldsymbol{\tau}^{\boldsymbol{B}}(a)] \in F$ iff $\boldsymbol{\tau}(a) \subseteq \Omega^{\boldsymbol{B}} F$. So for all $\vdash$–filters F and G, if $\Omega^{\boldsymbol{B}} F \subseteq \Omega^{\boldsymbol{B}} G$ then $F \subseteq G$. The converse follows from (3.1). □

Let us return now to the algebraization of **CLL**. Recall that we have already shown that the matrices $\langle \boldsymbol{B}, [e^{\boldsymbol{B}})\rangle$, where $\boldsymbol{B} \in \mathsf{IRL}$, are reduced models of **CLL**. We still need to show that they are exactly the reduced matrix models of **CLL**. The following fact will bring us part of the way.

Example 3.19. Let $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^*(\mathbf{CLL})$. Define a binary relation $\leq_F$ on A by $a \leq_F b$ iff $a \to b \in F$. Then $\boldsymbol{A}$ is an IRL whose lattice order is $\leq_F$.

Proof. We first show that $\leq_F$ is a lattice order.

Let us start by proving that $\leq_F$ is anti-symmetric. Suppose $a, b \in A$ such that $a \leq_F b$ and $b \leq_F a$. In other words $a \to b \in F$ and $b \to a \in F$. By Example 3.3 and Theorem 3.15 we have that $\langle a, b\rangle \in \Omega^{\boldsymbol{A}} F$. Now, since $\langle \boldsymbol{A}, F\rangle$ is reduced, we know that $\Omega^{\boldsymbol{A}} F = \mathrm{id}_A$. Therefore $a = b$.

Secondly, we know that $\vdash_{\mathbf{CLL}} p \to p$ by A1. Therefore, $a \to a \in F$, for any $a \in A$, since F is a $\vdash_{\mathbf{CLL}}$–filter. This means that $a \leq_F a$, i.e., $\leq_F$ is reflexive.

Lastly, suppose $a, b, c \in A$ such that $a \leq_F b$ and $b \leq_F c$, i.e., $a \to b \in F$ and $b \to c \in F$. By A2,

$$(b \to c) \to ((a \to b) \to (a \to c)) \in F,$$

because F is a $\vdash_{\mathbf{CLL}}$ filter. Then $a \to c \in F$ by two applications of modus ponens. Therefore $\leq_F$ is transitive.

So far, we have shown that $\leq_F$ is a partial order. We claim that $a \wedge b$ is the infimum of $\{a, b\}$ with respect to $\leq_F$. By A4 and A5 we respectively have that $(a \wedge b) \to a \in F$ and $(a \wedge b) \to b \in F$. So $a \wedge b \leq_F a$ and $a \wedge b \leq_F b$, i.e., $a \wedge b$ is a lower bound of a and b. Now consider any $c \in A$ such that $c \leq_F a$ and $c \leq_F b$, i.e., $c \to a \in F$ and $c \to b \in F$. By adjunction we have that $(c \to a) \wedge (c \to b) \in F$. Then by A6 and modus ponens we get $c \to (a \wedge b) \in F$, i.e., $c \leq_F a \wedge b$. Therefore $a \wedge b$ is the greatest lower bound of a and b.

Now we show that $a \vee b$ is the supremum of a and b. By A7 and A8 we have that $a \leq_F a \vee b$ and $b \leq_F a \vee b$. Let $c \in A$ such that $a \to c \in F$ and $b \to c \in F$. By adjunction we see that $(a \to c) \wedge (b \to c) \in F$. By applying modus ponens to A9 we obtain $(a \vee b) \to c \in F$. Therefore, $a \vee b$ is the least upper bound of a and b.

We can therefore conclude that $\leq_F$ is a lattice order of $\boldsymbol{A}$ with lattice operations $\vee$ and $\wedge$. We make use of this fact when we prove the rest of the requirements that show $\boldsymbol{A}$ is an IRL.

Let's move on to the monoid operation. We start by showing commutativity.

1	$\vdash_{\mathbf{CLL}} p \to (q \to (q \cdot p))$	A12
2	$\vdash_{\mathbf{CLL}} (p \cdot q) \to (q \cdot p)$	1, A13, MP

By symmetry we see that $(a \cdot b) \to (b \cdot a) \in F$ and $(b \cdot a) \to (a \cdot b) \in F$. Therefore, $a \cdot b = b \cdot a$, by the antisymmetry of $\leq_F$.

Now we show that t is the identity element for $\cdot$.

1	$\vdash_{\mathbf{CLL}} t \to (p \to p)$	A11
2	$\vdash_{\mathbf{CLL}} (p \cdot t) \to p$	1, A13, MP

and

1	$\vdash_{\mathbf{CLL}} t \to (p \to (p \cdot t))$	A12
2	$\vdash_{\mathbf{CLL}} p \to (p \cdot t)$	1, A10, MP

Since we have already shown that $\cdot$ is commutative we see that t is indeed the identity element for $\cdot$ (and we call it e in the context of $\boldsymbol{A}$).

To show the associativity of $\cdot$ we use the converse of A13, i.e., (2.4) from Example 2.7.

1	$\vdash_{\mathbf{CLL}} (q \cdot r) \to (p \to (p \cdot (q \cdot r)))$	A12
2	$\vdash_{\mathbf{CLL}} r \to (q \to (p \to (p \cdot (q \cdot r))))$	1, (2.4), MP
3	$\vdash_{\mathbf{CLL}} (q \to (p \to (p \cdot (q \cdot r)))) \to ((p \cdot q) \to (p \cdot (q \cdot r)))$	A13
4	$\vdash_{\mathbf{CLL}} r \to ((p \cdot q) \to (p \cdot (q \cdot r)))$	3, 2, A2, MP×2
5	$\vdash_{\mathbf{CLL}} ((p \cdot q) \cdot r) \to (p \cdot (q \cdot r))$	4, A13, MP

The converse uses a similar argument.

1	$\vdash_{\mathbf{CLL}} r \to ((p \cdot q) \to ((p \cdot q) \cdot r))$	A12
2	$\vdash_{\mathbf{CLL}} ((p \cdot q) \to ((p \cdot q) \cdot r)) \to (q \to (p \to ((p \cdot q) \cdot r)))$	(2.4)
3	$\vdash_{\mathbf{CLL}} r \to (q \to (p \to ((p \cdot q) \cdot r)))$	2, 1, A2, MP×2
4	$\vdash_{\mathbf{CLL}} (p \cdot (q \cdot r)) \to ((p \cdot q) \cdot r)$	3, A13×2, MP×2

That $\boldsymbol{A}$ satisfies the law of double-negation follows from $\vdash_{\mathbf{CLL}} \neg\neg p \to p$ and $\vdash_{\mathbf{CLL}} p \to \neg\neg p$ (see A15 and Example 2.4).

We will now show that $\to$ satisfies the law of residuation:

$$a \cdot b \leq_F c \text{ iff } b \leq_F a \to c.$$

If $(a \cdot b) \to c \in F$, then $b \to (a \to c) \in F$ by (2.4) and modus ponens. Conversely, if $b \to (a \to c) \in F$, then $(a \cdot b) \to c \in F$ by axiom A13 and modus ponens.

We need to check (1.5):

$$a \cdot b \leq_F c \text{ iff } \neg c \cdot a \leq_F \neg b.$$

Suppose that $a \cdot b \leq_F c$. By the law of residuation $b \leq_F a \to c$, i.e., $b \to (a \to c) \in F$. It follows that $a \to (b \to c) \in F$ using axiom A3 and modus ponens. We can adapt the argument in Example 2.6 to prove that $(a \to c) \to (\neg c \to \neg a) \in F$. By axiom A2 and modus ponens $b \to (\neg c \to \neg a) \in F$. We can then use A3, A13, A2 and multiple executions of modus ponens to conclude that $(\neg c \cdot b) \to \neg a \in F$. The converse uses similar tools.

It remains to show that $a \to b = \neg(a \cdot \neg b)$, because we took this as an axiom in order to include $\to$ in the signature of IRLs. Since $\leq_F$ is reflexive $a \cdot \neg b \leq_F a \cdot \neg b$. If we then apply (1.5), we get $a \cdot \neg(a \cdot \neg b) \leq_F b$. But then $\neg(a \cdot \neg b) \leq_F a \to b$ by the law of residuation. It is easy to see that $a \cdot (a \to b) \leq_F b$, since the argument proving Theorem 1.56(i) only uses the law of residuation. But then $a \cdot \neg b \leq_F \neg(a \to b)$, by (1.5). We know that $\neg$ is antitone with respect to $\leq_F$ by setting $b = e$ in (1.5), so $a \to b \leq_F \neg(a \cdot \neg b)$. □

From this point on, the discussion will change perspective, from the general to the specific. What have been considered examples up to this point will now become the main focus of our discussion. We can now finally characterize the reduced models of **CLL**.

Theorem 3.20. $\mathrm{Mod}^*(\mathbf{CLL}) = \{\langle \boldsymbol{A}, [e^{\boldsymbol{A}}) \rangle : \boldsymbol{A} \in \mathsf{IRL}\}$.

Proof. Let K be the class on the right. Example 3.16 shows that $\mathsf{K} \subseteq \mathrm{Mod}^*(\mathbf{CLL})$.

For the reverse inclusion, let $\langle \boldsymbol{A}, F \rangle \in \mathrm{Mod}^*(\mathbf{CLL})$. From Example 3.19 we know that $\boldsymbol{A} \in \mathsf{IRL}$, with lattice order $\leq_F$. It remains to show that $F = [e^{\boldsymbol{A}})$, or using the notation established in Example 3.6, that $F = F_{\leq_F}$. For all $a \in A$,

$$a \in F \text{ iff } e \to a \in F \text{ iff } e \leq_F a \text{ iff } a \in F_{\leq_F}. \quad \square$$

Finally, as promised, we can establish the algebraizability of **CLL**.

Theorem 3.21. $\vdash_{\mathbf{CLL}}$ *is (elementarily) algebraizable, with* IRL *as its equivalent quasivariety.*

Proof. We need to show that the deductive systems $\vdash_{\mathbf{CLL}}$ and $\models_{\mathsf{IRL}}$ are equivalent. We let $\boldsymbol{\rho}(\alpha, \beta) = \{\alpha \to \beta, \beta \to \alpha\}$ and $\boldsymbol{\tau}(\alpha) = \{\alpha \wedge e \approx e\}$.

From Characterization 2.22, we need to show that, for any set of formulas $\Gamma \cup \{\alpha, \beta\}$,

$$\Gamma \vdash_{\mathbf{CLL}} \alpha \text{ iff } \{e \preceq \gamma : \gamma \in \Gamma\} \models_{\mathsf{IRL}} e \preceq \alpha \tag{3.2}$$

$$\alpha \approx \beta \dashv\models_{\mathsf{IRL}} \{e \preceq \alpha \to \beta,\ e \preceq \beta \to \alpha\}. \tag{3.3}$$

It follows from the definitions that $\{e \preceq \gamma : \gamma \in \Gamma\} \models_{\mathsf{IRL}} e \preceq \alpha$ means the same as $\{\langle \boldsymbol{A}, [e^{\boldsymbol{A}})\rangle : \boldsymbol{A} \in \mathsf{IRL}\} \models \Gamma/\alpha$. But from Theorem 3.20, $\{\langle \boldsymbol{A}, [e^{\boldsymbol{A}})\rangle : \boldsymbol{A} \in \mathsf{IRL}\} = \mathrm{Mod}^*(\mathbf{CLL})$. By Theorem 3.13,

$$\mathrm{Mod}^*(\mathbf{CLL}) \models \Gamma/\alpha \text{ iff } \Gamma \vdash_{\mathbf{CLL}} \alpha.$$

Therefore, statement (3.2) holds.

To show statement (3.3), let $\boldsymbol{A} \in \mathsf{IRL}$. For any $a, b \in A$,

$$a = b \text{ iff } a \leq b \text{ and } b \leq a \text{ iff } e \leq a \to b \text{ and } e \leq b \to a. \quad \square$$

Chapter 4

Relevance Logics and Their Algebras

In this chapter we will finally introduce the relevance logics $\mathbf{R^t}$ and $\mathbf{R}$. Using the tools we developed in Chapters 2 and 3 it will be easy to show that they are algebraized by the varieties of De Morgan monoids and relevant algebras respectively. Once this is established we will explain some of the history and motivation behind relevance logics.

4.1 $\mathbf{R^t}$ and $\mathbf{R}$

Definition 4.1. Let $\mathbf{R^t}$ be the sentential logic obtained by adding the following axioms to the axiomatization of **CLL**, given in Example 2.2: First, we add contraction (axiom C1 from Example 2.8), i.e.,

$$(p \to (p \to q)) \to (p \to q).$$

Secondly, we add the *distribution* axiom:

$$\mathbf{R1} \quad (p \wedge (q \vee r)) \to ((p \wedge q) \vee (p \wedge r)) \quad \text{(distribution)}$$

Notice that $\mathbf{R^t}$ is an axiomatic extension of **CLL**. From Corollary 2.28, we know that $\mathbf{R^t}$ will be algebraized by a subvariety of IRL.

Definition 4.2. *De Morgan monoids* are IRLs that are distributive (as lattices) and that satisfy the *square-increasing law*,

$$x \leq x \cdot x. \tag{4.1}$$

Let DMM denote the variety of all De Morgan monoids. The following was essentially established by Dunn [19] (see [2]).

Theorem 4.3. $\mathbf{R^t}$ *is algebraized by* DMM.

Proof. The proof follows exactly the form of Theorem 3.21, where we prove the algebraization of **CLL**. However, we need to expand Theorem 3.20, to include our extra axioms. Specifically, we need to prove that $\text{Mod}^*(\mathbf{R^t})$ is exactly the class of matrices of the form $\langle \boldsymbol{A}, [e^{\boldsymbol{A}})\rangle$, where $\boldsymbol{A}$ is a De Morgan monoid.

First we let $\langle \boldsymbol{A}, F\rangle \in \text{Mod}^*(\mathbf{R^t})$ and we will show that $\boldsymbol{A}$ is a De Morgan monoid. Because of Example 3.19, we know that $\boldsymbol{A} \in \mathsf{IRL}$, where the lattice order is defined as $a \leq_F b$ iff $a \to b \in F$. It remains to check that $\boldsymbol{A}$ is distributive and satisfies the square-increasing law.

For any $a \in F$, by contraction,

$$(a \to (a \to (a \cdot a))) \to (a \to (a \cdot a)) \in F,$$

and by A12, $a \to (a \to (a \cdot a)) \in F$. So by modus ponens, $a \to (a \cdot a) \in F$, hence, $a \leq_F a \cdot a$.

Let $a, b, c \in A$. Since we already know that $\leq_F$ is a lattice order, we have that $(a \wedge b) \vee (a \wedge c) \leq_F a \wedge (b \vee c)$; see Section 1.2. The reverse inequality follows directly from the distribution axiom.

Conversely, let $\boldsymbol{A}$ be any square-increasing distributive IRL, with lattice order $\leq$. Define $F_{\leq} \coloneqq \{a \in A : e \leq a\}$. We need to show that $\langle \boldsymbol{A}, F_{\leq}\rangle \in \text{Mod}^*(\mathbf{R^t})$. Because of Example 3.6, we just need to prove that $\langle \boldsymbol{A}, F_{\leq}\rangle$ validates C1 and R1. Let $a, b, c \in A$.

Note that by Theorem 1.58(iv), $a \to (a \to b) = (a \cdot a) \to b$. By the square-increasing law $a \leq a \cdot a$. Now, since $\to$ is order-reversing in the first coordinate (Theorem 1.58(ii)), $(a \cdot a) \to b \leq a \to b$. Therefore,

$$a \to (a \to b) \leq a \to b. \tag{4.2}$$

By Theorem 1.56(ii), $e \leq (a \to (a \to b)) \to (a \to b)$.

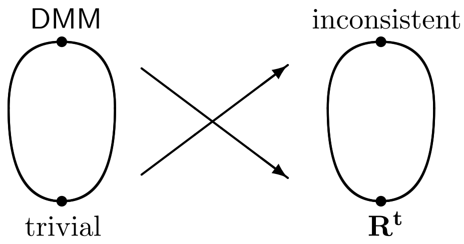
Since $\boldsymbol{A}$ is distributive we know that $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$. So by Theorem 1.56(ii), $e \leq (a \wedge (b \vee c)) \to ((a \wedge b) \vee (a \wedge c))$. Therefore, $\langle \boldsymbol{A}, F_{\leq}\rangle \in \text{Mod}^*(\mathbf{R^t})$.

It is true that if $\langle \boldsymbol{A}, F\rangle \in \text{Mod}^*(\mathbf{R^t})$, then $F = F_{\leq_F}$, using the same argument as in Theorem 3.20. Therefore,

$$\text{Mod}^*(\mathbf{R^t}) = \{\langle \boldsymbol{A}, [e^{\boldsymbol{A}})\rangle : \boldsymbol{A} \in \mathsf{DMM}\}. \quad \square$$

Recall from the Introduction that the main theorem we are building towards, Theorem 5.33, characterizes the *maximal consistent axiomatic extensions* of $\mathbf{R^t}$. Now that we have established the algebraization of $\mathbf{R^t}$ by

DMM, we can say by Corollary 2.28, that the lattice of axiomatic extensions of $\mathbf{R}^{\mathbf{t}}$ is anti-isomorphic to the lattice of subvarieties of DMM. The following picture depicts this anti-isomorphism.



Subvarieties of DMM **Axiomatic extensions of $\mathbf{R}^{\mathbf{t}}$**

It is clear that the *maximal consistent axiomatic* extensions of $\mathbf{R}^{\mathbf{t}}$ are the subcovers (see Section 1.2) of the inconsistent logic in the lattice of axiomatic extensions of $\mathbf{R}^{\mathbf{t}}$. These extensions will correspond with the covers of the trivial variety in the lattice of subvarieties of DMM, via the anti-isomorphism of Corollary 2.28. Recall that covers of the bottom element of a lattice are called atoms. Therefore, if we can find the atoms of the subvariety lattice of DMM, then we have found the consistent axiomatic extensions of $\mathbf{R}^{\mathbf{t}}$. This is the strategy that we will follow in the following chapter.

Once we prove our main result, we will use it in Chapter 6 to describe the upper part of the lattice of axiomatic extensions of the relevance logic $\mathbf{R}$, defined below.

Definition 4.4. Let $\mathbf{R}$ be the sentential logic obtained from $\mathbf{R}^{\mathbf{t}}$ by deleting the connective t from the signature and also removing axioms A10 and A11 (the axioms containing t) from the axiomatization of $\mathbf{R}^{\mathbf{t}}$.

The top of the lattice of axiomatic extensions of $\mathbf{R}$ was described by Świrydowicz in [59]. Our contribution here is to give an alternative proof of his characterization, which exploits a deeper connection between $\mathbf{R}$ and $\mathbf{R}^{\mathbf{t}}$, proved in the next theorem.

The following is an example of a *separation theorem*; see [51]. It shows that the axioms for $\mathbf{R}^{\mathbf{t}}$ were chosen in such a way that we can axiomatize the t-free fragment of $\mathbf{R}^{\mathbf{t}}$ by removing the axioms that do not contain t.

Theorem 4.5. $\mathbf{R}$ *is the t-free fragment of* $\mathbf{R}^{\mathbf{t}}$.

Proof. Suppose $\Gamma \vdash_{\mathbf{R}^{\mathbf{t}}} \alpha$, where t does not occur in any member of $\Gamma \cup \{\alpha\}$. We must show that $\Gamma \vdash_{\mathbf{R}} \alpha$.

Since $\mathbf{R^t}$ is finitary, we can assume, without loss of generality, that Γ is finite. Therefore, we may assume that the variables occurring in members of $\Gamma \cup \{\alpha\}$ are among $p_1, \ldots, p_n$. Define

$$r_n \coloneqq |p_1| \wedge \cdots \wedge |p_n|,$$

where $|\beta| \coloneqq \beta \to \beta$, for all formulas β.

Consider a proof in $\mathbf{R^t}$ of α from Γ. Replacing every occurrence of t in this proof by r_n, we obtain a finite sequence of formulas of $\mathbf{R}$, terminating in α. It's therefore enough to show that every formula in this sequence is provable from Γ in $\mathbf{R}$.

Given that t and $t \to |p|$ are the only axioms of $\mathbf{R^t}$ involving t (A10 and A11), and that no member of Γ or inference rule of $\mathbf{R^t}$ involves t, it's enough to show that, for any formula β in (at most) the variables $p_1, \ldots, p_n$, the formulas r_n and $r_n \to |\beta|$ are theorems of $\mathbf{R}$.

First, by A1, $|p_i|$ is a theorem of $\mathbf{R}$ for every $i \leq n$, so by AD, r_n is a theorem of $\mathbf{R}$.

We show that $\vdash_{\mathbf{R}} r_n \to |\beta|$ by induction on the complexity of β. It is true in the base case (where β is a variable necessarily among $p_1, \ldots, p_n$), in view of the axioms A4 and A5.

Suppose γ and δ are formulas, such that $\#\gamma, \#\delta < \#\beta$, and

IH $\quad \vdash_{\mathbf{R}} r_n \to |\gamma|$ and $\vdash_{\mathbf{R}} r_n \to |\delta|$. (induction hypothesis)

We need to prove that $\vdash_{\mathbf{R}} r_n \to |\neg\gamma|$ and $\vdash_{\mathbf{R}} r_n \to |\gamma \Box \delta|$ for every connective $\Box$ among $\cdot, \to, \wedge, \vee$.

We can obtain the results for $\cdot$ and $\vee$ from the others, by noting that $(p \cdot q) \leftrightarrow \neg(p \to \neg q)$ and $(p \vee q) \leftrightarrow \neg(\neg p \wedge \neg q)$ are theorems of $\mathbf{R}$ and that $\{p \to q, q \to p\}$ is a set of equivalence formulas for $\mathbf{R}$. (See Definition 3.1(iii) and note that the proof of Example 3.3 makes no use of t and its postulates.) So, we need only consider the connectives $\neg, \wedge$ and $\to$.

Consider the case where $\beta = \neg\gamma$. Notice that our proof of $p \to \neg\neg p$ (2.1) in Example 2.4 does not use A10 or A11, so it is still a theorem of $\mathbf{R}$. Therefore,

1	$\vdash_{\mathbf{R}} \gamma \to \neg\neg\gamma$	(2.1)
2	$\vdash_{\mathbf{R}} (\gamma \to \gamma) \to (\gamma \to \neg\neg\gamma)$	1, A2, MP
3	$\vdash_{\mathbf{R}} (\gamma \to \neg\neg\gamma) \to (\neg\gamma \to \neg\gamma)$	A14
4	$\vdash_{\mathbf{R}} (\gamma \to \gamma) \to (\neg\gamma \to \neg\gamma)$	3, 2, A2, MP×2
5	$\vdash_{\mathbf{R}} r_n \to (\neg\gamma \to \neg\gamma)$	IH, 4, A2, MP ×2

Now suppose that $\beta = \gamma \wedge \delta$.

1	$\vdash_{\mathbf{R}} \gamma \to (r_n \to \gamma)$	IH, axiom A3, MP
2	$\vdash_{\mathbf{R}} (\gamma \wedge \delta) \to \gamma$	A4
3	$\vdash_{\mathbf{R}} (\gamma \wedge \delta) \to (r_n \to \gamma)$	1, 2, A2, MP×2
4	$\vdash_{\mathbf{R}} (\gamma \wedge \delta) \to (r_n \to \delta)$	similar
5	$\vdash_{\mathbf{R}} (\gamma \wedge \delta) \to ((r_n \to \gamma) \wedge (r_n \to \delta))$	3, 4, AD, A6, MP
6	$\vdash_{\mathbf{R}} ((r_n \to \gamma) \wedge (r_n \to \delta)) \to (r_n \to (\gamma \wedge \delta))$	A6
7	$\vdash_{\mathbf{R}} (\gamma \wedge \delta) \to (r_n \to (\gamma \wedge \delta))$	6, 5, A2, MP×2
8	$\vdash_{\mathbf{R}} r_n \to ((\gamma \wedge \delta) \to (\gamma \wedge \delta))$	7, A3, MP

Lastly, suppose $\beta = \gamma \to \delta$. Then,

1	$\vdash_{\mathbf{R}} (\delta \to \delta) \to ((\gamma \to \delta) \to (\gamma \to \delta))$	A2
2	$\vdash_{\mathbf{R}} r_n \to ((\gamma \to \delta) \to (\gamma \to \delta))$	1, IH, A2, MP×2 □

At this point we might hope to use Theorem 2.30 to algebraize **R**. But notice that the algebraization of $\mathbf{R^t}$ is witnessed by the same definable translations as **CLL**. In particular, one of the translations is $\boldsymbol{\tau}(\alpha) = \{e \preceq \alpha\}$, which contains the symbol t, in the guise of e. Therefore, our situation does not satisfy the conditions of Theorem 2.30. Thankfully, we can overcome this situation using the following lemma:

Lemma 4.6. *For any element a of an IRL, $e \leq a$ if and only if $a \to a \leq a$.*

Proof. First suppose that $e \leq a$. Then $a \to a \leq e \to a = a$, because $\to$ is order-reversing in the first coordinate.

Now suppose that $a \to a \leq a$. We know that $e \leq a \to a$, by Theorem 1.58(i), so the result follows by transitivity of $\leq$. □

Thus in the algebraization of $\mathbf{R^t}$ by DMM, the translation $\boldsymbol{\tau} = \{e \preceq x\}$ can be replaced by $\boldsymbol{\tau} = \{|x| \preceq x\}$.

Definition 4.7. Let RA denote the class of all e-free subreducts of De Morgan monoids. RA is called the class of *relevant algebras*.

Corollary 4.8. **R** *is algebraized by* RA*, as witnessed by* $\boldsymbol{\rho} = \{p \to q, q \to p\}$ *and* $\boldsymbol{\tau} = \{|x| \preceq x\}$*.*

Proof. By Lemma 4.6, $\boldsymbol{\rho} = \{p \to q, q \to p\}$ and $\boldsymbol{\tau} = \{|x| \preceq x\}$ witness the algebraization of $\mathbf{R^t}$ by DMM. Notice that t is no longer part of the definition of $\boldsymbol{\tau}$. Therefore, we may use Theorem 2.30 to conclude that the t-free fragment of $\mathbf{R^t}$ is algebraized by the t-free subreduct class of DMM. By Theorem 4.5, the former is $\mathbf{R}$ and the latter is RA, by definition. □

4.2 Motivation and History of R and $\mathbf{R^t}$

Now that we have introduced the logics $\mathbf{R^t}$ and $\mathbf{R}$, as well as their algebraizations, it is a good time to interrupt the mathematics and discuss their origins and motivations. After this digression we will analyse De Morgan monoids, which will lead to the main new results of this thesis.

Both $\mathbf{R}$ and $\mathbf{R^t}$ are part of a family of logics called *relevance logics*. The first such logic was called *Entailment*, normally denoted $\mathbf{E}$, and was introduced by Anderson and Belnap [2].[1] Interest in $\mathbf{E}$ soon shifted to the more well-behaved logic $\mathbf{R}$, which is sometimes called the principal relevance logic. Originally these logics were intended to be alternatives to classical logic, with the express purpose of avoiding some of the so-called paradoxes of material implication (which identifies $p \to q$ with $\neg p \vee q$). These include the *weakening axiom* $p \to (q \to p)$, which, when interpreted intuitively, states that if p is true then q implies p. The paradox lies in the fact that q can be any statement, even something unrelated to p. The relevance logicians wanted a logic where α would imply β only if α is *relevant* to β. This demand found its expression in the form of a variable-sharing principle, called the *relevance principle*, which states that for a formula $\alpha \to \beta$ to be a theorem, α and β must at least share a variable. $\mathbf{R}$ satisfies this demand; see [3]. Unfortunately $\mathbf{R^t}$ does not share this principle, as witnessed by A11. However, $\mathbf{R^t}$ does of course satisfy the principle for theorems that do not contain t, as these are the theorems of $\mathbf{R}$, by Theorem 4.5.

As we have noted, the relationship between $\mathbf{R^t}$ and DMM was essentially established by Dunn [19]. In the 1960's, this was not an obvious connection, because implicational postulates—as opposed to properties of fusion—had been emphasized by the Anderson-Belnap school of relevance logicians. As we mentioned before, $\cdot$ can be interpreted as co-tenability, i.e., one intuitive reading of $p \cdot q$ is 'p and q can be simultaneously true'. Another way to understand fusion is by analogy with proof-theoretic Gentzen systems, for the reader who is familiar with them. A typical rule of many Gentzen systems is:

[1]Although, see [18, 46] for an interesting independent development.

$$\frac{\alpha, \Gamma \rhd \beta}{\Gamma \rhd \alpha \to \beta}$$

The analogy with the law of residuation is clear. We can intuitively identify $\cdot$ with the 'comma' in sequents used in Gentzen systems.

Relevance logicians attach more philosophical significance to a relational (Kripke-like) semantics for $\mathbf{R^t}$, supplied by Routley and Meyer [53], than to the algebraic semantics. Urquhart [65] extended the former to a categorical duality between DMM and a category of enriched topological spaces, whose intelligibility owes much to the distributive law for $\wedge$ and $\vee$. Slaney and Meyer [58] have re-interpreted the relational semantics as a philosophically neutral tool for the analysis of multiple-agent reasoning, thus freeing it from prior commitments to 'relevance' and from objections to material implication. For further semantic insights, see [62, 63].

The name 'De Morgan monoid' is due to Dunn [19], and the analogy with De Morgan lattices/algebras is clear, where De Morgan lattices are distributive lattices with an involution, and De Morgan algebras are bounded De Morgan lattices. Felicitously, according to Pratt [47], the law of residuation is also implicit in Augustus De Morgan's paper [17] of 1860, perhaps its earliest appearance.

The relevance logic literature is equivocal, however, as to the precise definition of a De Morgan monoid. Our definition conforms with Dunn and Restall [20], Meyer and Routley [44, 53], Slaney [54] and Urquhart [64], yet other papers by some of the same authors entertain a discrepancy. In all sources, the identity element of a De Morgan monoid $\boldsymbol{A}$ is assumed to exist but, in [55, 56, 57] for instance, it is not distinguished, i.e., the symbol for e (and likewise f) is absent from the signature of $\boldsymbol{A}$. That locally innocuous convention has significant global effects: it would prevent DMM from being a variety, as it would cease to be closed under subalgebras, and the tight correspondence between axiomatic extensions of $\mathbf{R^t}$ and subvarieties of DMM would disappear. The discrepancy is no oversight: some 'subalgebras' named in [55] really do omit the identity element of a parent De Morgan monoid, but only when a different identity element is available in the subalgebra (as is guaranteed if the subalgebra is finitely generated—see below—but not generally). The practice of not distinguishing identity elements harks back to the relevance principle of $\mathbf{R}$.

This may explain why we have found in the literature no statement of our Theorem 5.33, identifying the only four maximal consistent axiomatic extensions of $\mathbf{R^t}$, although the algebras defining these extensions were well known.

Because RA *is* closed under subalgebras, its study accommodates the relevance principle without sacrificing the benefits of accurate algebraization. For the algebraist, however, De Morgan monoids have much in common with ordered abelian groups (residuals being a partial surrogate for multiplicative inverses), whereas relevant algebras are less intuitive, being semigroup-based, rather than monoid-based. In fact RA has some algebraically forbidding features. For instance, RA lacks the congruence extension property (unlike DMM); see [16, p.289].

Further work on relevant algebras can be found in [21, 36, 37, 52, 60].

It is sometimes easiest to obtain a result about relevant algebras indirectly, as a corollary of a more swiftly established property of De Morgan monoids. This is exactly what we will do in Section 6.1. Thus, De Morgan monoids are a useful tool, even for those logicians who which to eschew sentential constants, such as t.

In 1996, Urquhart [65, p.263] observed that

> [t]he algebraic theory of relevant logics is relatively unexplored, particularly by comparison with the field of algebraic modal logic.

In the same vein, in a paper in 2001, Dunn and Restall [20, Sec. 3.5] wrote:

> Not as much is known about the algebraic properties of De Morgan monoids as one would like.

These remarks pre-date many papers on residuated structures—see the bibliography of [28], for instance. But the latter have concentrated mainly on varieties incomparable with DMM (e.g., Heyting and MV-algebras), larger than DMM (e.g., IRLs and full Lambek algebras) or smaller (e.g., Sugihara monoids), so the remarks remain pertinent.

In particular, an algebraic analysis of the axiomatic extensions of $\mathbf{R^t}$ (via the interchangeable subvariety of DMM) is still missing, perhaps because of the equivocal formal status of e (and despite interest in the problem descernable in [40, 41]). Here we initiate an attempt to fill this gap, using algebraic methods.

Chapter 5

The Variety of De Morgan Monoids

In this chapter we will shift our focus away from the logics $\mathbf{R^t}$ and $\mathbf{R}$, and instead focus on their algebraic counter-parts. Specifically, in this chapter we do an algebraic analysis of variety of De Morgan monoids, which algebraizes $\mathbf{R^t}$, as we established in Theorem 4.3. The main result, Theorem 5.30, at the end of this chapter, is a characterization of the atoms of the subvariety lattice of De Morgan monoids. As discussed in Section 4.1, we can then use the tools we developed in Chapters 2 and 3 to easily obtain a characterization of the maximal consistent axiomatic extensions of $\mathbf{R^t}$. We will also use the results in this chapter to obtain easier proofs of results about relevant algebras in Chapter 6.

5.1 De Morgan Monoids

We start with an algebraic analysis of De Morgan monoids. Recall that, in Section 1.6, we defined a term $f \coloneqq \neg e$. Because of Theorem 1.57(i) ($\neg x = x \to f$), we know that the involution of an IRL is determined by f. We did not find much use for this symbol so far, but it will play a larger role in this chapter.

Theorem 5.1. *The following are basic properties of De Morgan monoids:*

(i) $x \wedge y \leq x \cdot y$;

(ii) *If* $x, y \leq e$, *then* $x \wedge y = x \cdot y$;

(iii) $x \cdot \neg x \leq f$.

(iv) $e \leq x \vee \neg x$*;*

Proof. By the square-increasing law and Theorem 1.54, we have

$$x \wedge y \leq (x \wedge y) \cdot (x \wedge y) \leq x \cdot y,$$

which proves (i).

For (ii), suppose that $x, y \leq e$. We already know that $x \cdot y \leq x \wedge y$ by (i). For the other inequality, notice that $x \cdot y \leq x \cdot e = x$ and similarly $x \cdot y \leq y$, so $x \cdot y \leq x \wedge y$.

To prove (iii), notice that, $x \cdot \neg x = x \cdot (x \to f) \leq f$.

By (i) and (iii), $x \wedge \neg x \leq f$, from which (iv) follows by De Morgan's laws and the fact that $\neg$ is antitone. □

The following generalizes a result of Slaney [54, T39, p. 491] (where only the case $a = f$ was discussed).

Lemma 5.2. *Let* $\boldsymbol{A}$ *be a De Morgan monoid with* $f \leq a \in A$*. Then* $a^3 = a^2$*.*

Proof. As $f \leq a$, we have $\neg a \leq \neg f = e \leq a \to a$, by Theorem 1.58(i). So, from the fact that $\to$ is order preserving in the second coordinate and Theorem 1.58(iv), we obtain

$$a \to \neg a \leq a \to (a \to a) = a^2 \to a.$$

Using Theorem 1.58(v) and Theorem 5.1(i), we infer that

$$a \to \neg a = (a^2 \to a) \wedge (a \to \neg a) \leq (a^2 \to a) \cdot (a \to \neg a) \leq a^2 \to \neg a.$$

Thus, $\neg(a^2 \to \neg a) \leq \neg(a \to \neg a)$, i.e., $a^2 \cdot a \leq a \cdot a$, i.e., $a^3 \leq a^2$. The reverse inequality follows from the square-increasing law and the fact that $\cdot$ is compatible with the lattice order. □

Unless we say otherwise, if $\bot$ and $\top$ denote elements of a De Morgan monoid $\boldsymbol{A}$, then they denote the least and greatest element, respectively. If these exist, then $\boldsymbol{A}$ is said to be *bounded*, and $\bot, \top$ are called its *bounds*. Note, however, that $\bot$ and $\top$ are not generally *distinguished* elements of $\boldsymbol{A}$, so they are not always retained in subalgebras. Notice that $\neg\top = \bot$ and $\neg\bot = \top$.

Theorem 5.3. *For any bounded De Morgan monoid* $\boldsymbol{A}$*,*

(i) $\bot \cdot a = \bot$ *for any* $a \in A$*;*

(ii) $\top \cdot \top = \top$.

Proof. For (i) it suffices to show that $\bot \cdot a \leq \bot$, which will follow from the law of residuation if $\bot \leq a \to \bot$. This is clearly true, since $\bot$ is the bottom element.

(ii) follows from the fact that $\top = e \cdot \top \leq \top \cdot \top$. □

A De Morgan monoid is said to be *integral* if e is its greatest element.

Theorem 5.4. *A De Morgan monoid is integral iff it is a Boolean algebra in which the operation $\wedge$ is duplicated by $\cdot$.*

Proof. It suffices to prove necessity. Suppose $\boldsymbol{A}$ is an integral De Morgan monoid. Since e is the top element of $\boldsymbol{A}$, it is easy to see that f is its least element. Therefore, $\boldsymbol{A}$ is bounded. All the elements of $\boldsymbol{A}$ are below e, so, by Theorem 5.1(ii), $\cdot$ coincides with $\wedge$. Since we know that $\boldsymbol{A}$ is distributive (by definition), it remains to show that $\boldsymbol{A}$ is complemented. For every element $a \in A$, we will show that its complement is $\neg a$. By Theorem 5.1(iv), $e \leq a \vee \neg a$. But e is the top element so $e = a \vee \neg a$. This is enough, since it follows from De Morgan's laws that $f = \neg a \wedge a$. □

Although the statement of the following theorem is not new, the proof given here has not yet been published, and was found by our collaborator Tommaso Moraschini. Also see the notes before Corollary 6.6, where we use this theorem to show that finitely generated relevant algebras are also bounded.

Theorem 5.5. *Every finitely generated De Morgan monoid is bounded.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a finite set of generators for a De Morgan monoid $\boldsymbol{A}$. Let

$$c \coloneqq e \vee f \vee \bigvee_{i \leq n} (a_i \vee \neg a_i), \text{ and } b = c^2.$$

We will show that $\neg b \leq a \leq b$ for all $a \in A$.

Notice that in any De Morgan monoid we can define $\vee$ in terms of $\wedge$ and $\neg$, by De Morgan's laws. This implies, by Theorem 1.39, that every element of $\boldsymbol{A}$ has the form $\alpha^{\boldsymbol{A}}(a_1, \ldots, a_n)$ for some n-ary term $\alpha(x_1, \ldots, x_n)$ in the signature $\cdot, \wedge, \neg, e$. The proof is by induction on the complexity $\#\alpha$ of α. We shall write $\vec{x}$ and $\vec{a}$ for the respective sequences $x_1, \ldots x_n$ and $a_1, \ldots, a_n$.

For the case $\#\alpha = 0$, note that $e, a_1, \ldots, a_n \leq c \leq b$, by the square-increasing law. Likewise, $f, \neg a_1, \ldots, \neg a_n \leq c \leq b$, so that by involution properties, $\neg b \leq e, a_1, \ldots, a_n$.

Now, suppose that $\#\alpha \geq 1$ and that $\neg b \leq \beta^{\boldsymbol{A}}(\vec{a}) \leq b$ for all terms β with $\#\beta < \#\alpha$.

It follows from the induction hypothesis and basic properties of IRLs that, if α has the form $\neg\beta(\vec{x})$ or $\beta_1(\vec{x}) \wedge \beta_2(\vec{x})$, then $\neg b \leq \alpha^{\boldsymbol{A}}(\vec{a}) \leq b$. It remains to prove that the last assertion holds when α is $\beta_1(\vec{x}) \cdot \beta_2(\vec{x})$ for some less complex terms β_1 and β_2.

By the induction hypothesis and the order-compatibility of $\cdot$, it is clear that $(\neg b)^2 \leq \beta_1(\vec{a}) \cdot \beta_2(\vec{a}) \leq b^2$. Recall that $f \leq c \leq b$. Then, since $\neg$ is antitone, $\neg b \leq e$. But then $(\neg b)^2 = \neg b \wedge \neg b = \neg b$, by Theorem 5.1(ii). On the other hand, $c^3 = c^2$, by Lemma 5.2, so $b^2 = c^4 = c^2 = b$. Therefore, $\neg b \leq \alpha^{\boldsymbol{A}}(\vec{a}) \leq b$, as required. $\square$

Given a De Morgan monoid $\boldsymbol{A}$, we say that a set $F \subseteq A$ is a *deductive filter* of $\boldsymbol{A}$ if F is a lattice filter containing e. Recall from Definition 1.23 that F is a lattice filter if it is upward closed and, whenever $a, b \in F$, then $a \wedge b \in F$.

Owing to Theorem 5.1(i), if $a, b \in F$, then $a \cdot b \in F$. In fact, we can define the deductive filters of an IRL to be its lattice filters that contain e and are also closed under $\cdot$. Many of the results to follow also hold for the deductive filters of IRLs. Note that the upper bounds of e (including e itself) constitute the smallest deductive filter of any IRL.

It is easy to see that arbitrary intersections of deductive filters are again deductive filters. We will let $\mathrm{DFil}(\boldsymbol{A})$ denote the lattice of deductive filters of $\boldsymbol{A}$ and let $\mathrm{DFg}^{\boldsymbol{A}}$ denote the corresponding closure operator. The *deductive filter generated by* $X \subseteq A$ is $\mathrm{DFg}^{\boldsymbol{A}}(X)$, and it is the smallest deductive filter of $\boldsymbol{A}$ containing X. It is also easy to see that the unions of directed non-empty subfamilies of $\mathrm{DFil}(\boldsymbol{A})$ are deductive filters. Therefore, $\mathrm{DFil}(\boldsymbol{A})$ and the operator $\mathrm{DFg}^{\boldsymbol{A}}$ are algebraic; see Theorem 1.18 and Theorem 1.17.

Theorem 5.6. *The deductive filters of a De Morgan monoid $\boldsymbol{A}$ are precisely the $\mathbf{R^t}$–filters of $\boldsymbol{A}$.*

Proof. Let $F \in \mathrm{DFil}(\boldsymbol{A})$. Let h be any $\boldsymbol{A}$-evaluation. For any axiom α of $\mathbf{R^t}$, we know from the algebraization of $\mathbf{R^t}$, in Theorem 4.3, that $e \leq h(\alpha)$. In particular, $h(\alpha) \in F$ since $e \in F$ and F is upward-closed. Let's now consider the inference rules of $\mathbf{R^t}$. Suppose that $h(p), h(q) \in F$. Then, $h(p) \wedge h(q) \in F$, which takes care of AD. Lastly, suppose that

$$h(p), h(p) \to h(q) \in F.$$

As mentioned above, $h(p) \cdot (h(p) \to h(q)) \in F$. But, by Theorem 1.56(i), $h(p) \cdot (h(p) \to h(q)) \leq h(q)$, so $h(q) \in F$. Therefore, F is an $\mathbf{R^t}$–filter of $\boldsymbol{A}$.

Conversely, let F be an $\mathbf{R^t}$–filter of $\boldsymbol{A}$. By AD it is clear that if $a, b \in F$, then $a \wedge b \in F$. Suppose that $a \in F$ and $a \leq b$. Recall that by A6 $a \to (a \vee b) \in F$, so by modus ponens, $a \vee b = b \in F$. Lastly, since t is a theorem of $\mathbf{R^t}$ (A10), we have that $e \in F$. □

Recall that $\mathbf{R^t}$ is algebraized by the variety DMM, as witnessed by $\boldsymbol{\rho} = \{p \to q, q \to p\}$ and $\boldsymbol{\tau} = \{\langle e \wedge x, e\rangle\}$. Let $\boldsymbol{A}$ be a De Morgan monoid. By Theorem 3.18, we know that the lattice of $\mathbf{R^t}$–filters of $\boldsymbol{A}$ is isomorphic to the lattice of DMM-congruences of $\boldsymbol{A}$. But since DMM is a variety, all congruences of $\boldsymbol{A}$ are DMM-congruences. Therefore, by Theorem 5.6 above, the lattices $\mathrm{DFil}(\boldsymbol{A})$ and $\mathrm{Con}(\boldsymbol{A})$ are isomorphic. This observation lead to the following theorem. (Recall that $p \leftrightarrow q$ abbreviates $(p \to q) \wedge (q \to p)$.)

Theorem 5.7. *Let $\boldsymbol{A}$ be a De Morgan monoid.*

(i) If F is a deductive filter of $\boldsymbol{A}$ then $\Omega^{\boldsymbol{A}} F$, the Leibniz congruence of F, is given by $\{\langle a, b\rangle \in A : a \leftrightarrow b \in F\}$.

(ii) If $\theta \in \mathrm{Con}(\boldsymbol{A})$ then the set $\{b \in A : b \geq a,$ for some $a \in e/\theta\}$ is a deductive filter of $\boldsymbol{A}$ coinciding with $H^{\boldsymbol{A}}(\theta)$, from Theorem 3.18.

(iii) The maps $\Omega^{\boldsymbol{A}}$ and $H^{\boldsymbol{A}}$ are mutually inverse lattice isomorphisms between $\mathrm{Con}(\boldsymbol{A})$ and $\mathrm{DFil}(\boldsymbol{A})$.

Proof. By the discussion preceding the theorem we know that $\mathrm{DFil}(\boldsymbol{A})$ and $\mathrm{Con}(\boldsymbol{A})$ are isomorphic. Theorem 3.18 says that this isomorphism is

$$\Omega^{\boldsymbol{A}} : F \mapsto \{\langle a, b\rangle \in A : \boldsymbol{\rho}^{\boldsymbol{A}}(a, b) \subseteq F\},$$

where $\boldsymbol{\rho} = \{p \to q, q \to p\}$ and its inverse is

$$H^{\boldsymbol{A}} : \theta \mapsto \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \theta\}$$

where $\boldsymbol{\tau} = \{\langle x \wedge e, e\rangle\}$. Clearly, $\boldsymbol{\rho}$ is inter-derivable, in $\mathbf{R^t}$, with $\{(p \to q) \wedge (q \to p)\}$.

Let $\theta \in \mathrm{Con}(\boldsymbol{A})$ and $F = \{b \in A : b \geq a,$ for some $a \in e/\theta\}$. We will show that $F = H^{\boldsymbol{A}}(\theta)$. For every $b \in F$, there exists $a \in e/\theta$ such that $a \leq b$. Notice that $a \in H^{\boldsymbol{A}}(\theta)$, because $\langle a, e\rangle \in \theta$, which implies that $\langle a \wedge e, e\rangle \in \theta$. But since $H^{\boldsymbol{A}}(\theta)$ is a deductive filter, and hence upward-closed, $b \in H^{\boldsymbol{A}}(\theta)$.

Conversely, suppose that $a \in H^{\boldsymbol{A}}(\theta)$, i.e., $\langle a \wedge e, e\rangle \in \theta$. Then $a \wedge e \in e/\theta$ and $a \geq a \wedge e$, so $a \in F$. □

It is interesting to note the similarities between the role that deductive filters play in the theorem above, and how normal subgroups behave in group theory. In particular, the normal subgroups of a group form a lattice, which is isomorphic to its congruence lattice. In most introductory abstract algebra courses, congruences are left out entirely. For example, the Homomorphism Theorem (Theorem 1.6) can be rephrased for groups, by defining the kernel of a group homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$, to be the normal subgroup of $\boldsymbol{A}$ consisting of all the elements of A that are mapped to the identity of $\boldsymbol{B}$. In our case, the kernel of a De Morgan monoid homomorphism corresponds to the upward-closure of all the elements that are mapped to e. It is a striking feature of the Leibniz operator that, whenever a sentential logic $\vdash$ is algebraized by a variety K, then for any $\boldsymbol{A} \in \mathsf{K}$, $\Omega^{\boldsymbol{A}}$ is an isomorphism between the $\vdash$–filters of $\boldsymbol{A}$ and $\mathrm{Con}(\boldsymbol{A})$. This means that we can think of the $\vdash$–filters as analogous to normal subgroups for groups.

Recall that, by Birkhoff's Subdirect Decomposition Theorem (Theorem 1.31), any variety is determined by its subdirectly irreducible members. Because of this, much of our algebraic analysis aims to gain insight into the structure of subdirectly irreducible De Morgan monoids. Furthermore, subdirectly irreducible algebras are characterized by the structure of their congruence lattices; see Theorem 1.30. The correspondence between deductive filters and congruences is an indispensable tool in this regard.

Lemma 5.8. *Let $\boldsymbol{A} \in \mathsf{DMM}$. For all $a_1, \ldots, a_m \in A$, where $m \in \omega$,*

$$\mathrm{DFg}^{\boldsymbol{A}}(a_1, \ldots, a_m) = [e \wedge a_1 \wedge \cdots \wedge a_m).$$

Thus, every finitely generated deductive filter of $\boldsymbol{A}$ is principal *(i.e., generated by a single element).*

Proof. Let $F = [e \wedge a_1 \wedge \cdots \wedge a_m)$. Clearly, F is a lattice filter containing $\{a_1, \ldots, a_m\}$. It remains to show that F is the smallest such filter. So, let G be any deductive filter containing $\{a_1, \ldots, a_m\}$. By definition G contains e. Also, G is closed with respect to meets, so $e \wedge a_1 \wedge \cdots \wedge a_m \in G$. But then $F \subseteq G$, since G is upward-closed. $\square$

In the relevance logic literature a De Morgan monoid is said to be 'prime' if it is finitely subdirectly irreducible. The reason is item (i) of the next theorem. We choose to not follow this convention, because of the universal algebraic significance of finitely subdirectly irreducible algebras.

Theorem 5.9. *Let $\boldsymbol{A} \in \mathsf{DMM}$.*

(i) $\boldsymbol{A}$ is finitely subdirectly irreducible (FSI) iff e is join-irreducible iff e is join-prime.

(ii) $\boldsymbol{A}$ is subdirectly irreducible (SI) iff there is a greatest element strictly below e.

(iii) $\boldsymbol{A}$ is simple iff e has just one strict lower bound.

Proof. (i): It suffices to prove the first equivalence, in view of Remark 1.21.

$\Rightarrow$: Suppose that $\boldsymbol{A}$ is FSI. Now suppose that $e = a \vee b$ for some $a, b \in A$. In particular $a, b \leq e$. Therefore, by Lemma 5.8, $\mathrm{DFg}^{\boldsymbol{A}}(a) = [a)$ and $\mathrm{DFg}^{\boldsymbol{A}}(b) = [b)$. For every $c \in \mathrm{DFg}^{\boldsymbol{A}}(a) \cap \mathrm{DFg}^{\boldsymbol{A}}(b)$, we see that $a, b \leq c$. But then $c \geq a \vee b = e$, i.e., $c \in [e)$. Recall that $[e)$ is the smallest deductive filter of $\boldsymbol{A}$. Therefore $\mathrm{DFg}^{\boldsymbol{A}}(a) \cap \mathrm{DFg}^{\boldsymbol{A}}(b) = [e)$. Since $\boldsymbol{A}$ is FSI, by Definition 1.32, the identity congruence of $\boldsymbol{A}$ is meet-irreducible in $\mathrm{Con}(\boldsymbol{A})$. By Theorem 5.7, since $[e)$ corresponds to the identity congruence, it is meet-irreducible in $\mathrm{DFil}(\boldsymbol{A})$. Therefore $\mathrm{DFg}^{\boldsymbol{A}}(a) = [e)$ or $\mathrm{DFg}^{\boldsymbol{A}}(b) = [e)$, i.e., $a = e$ or $b = e$.

(i) $\Leftarrow$: Assume that e is join-irreducible in $\boldsymbol{A}$, and let F and G be deductive filters of $\boldsymbol{A}$, such that $F \cap G = [e)$. We must show that $F = [e)$ or $G = [e)$. Suppose, on the contrary, that $a \in F$ and $b \in G$, where $e \not\leq a$ and $e \not\leq b$, i.e., $a \wedge e < e$ and $b \wedge e < e$. For every $d \in A$ such that $a \wedge e, b \wedge e \leq d$, we have $d \in F \cap G = [e)$, i.e., $e \leq d$. This shows that e is the join of $a \wedge e$ and $b \wedge e$, contradicting the join-irreducibility of e.

(ii): To show the forward direction, the proof of (i) can be adapted; the details are left to the reader. For the other direction, suppose that c is the largest element strictly below e and let F be any deductive filter of $\boldsymbol{A}$. We show that $[c) \subseteq F$, whenever $[e) \subsetneq F$. If $b \in F$, but not in $[e)$, then b cannot be an upper bound of e. In particular, $b \wedge e < e$, so there exists $a \in F$, such that $a < e$. But then $a \leq c$, hence,

$$[c) \subseteq [a) \subseteq F.$$

(iii) follows easily from (ii), by noticing that, if c is the only strict lower bound of e, then $[c) = A$. So, $[c)$ corresponds to the total congruence A^2. Conversely, if e has more than one lower bound, then there is a chain $e > a > b$, so $[e), [a), [b)$ are distinct deductive filters of $\boldsymbol{A}$, contradicting simplicity. □

We proceed by giving some results about the structure of FSI De Morgan monoids.

Theorem 5.10. *Let* $\boldsymbol{A}$ *be a FSI De Morgan monoid, and* $a \in A$. *Then* $e \leq a$ *or* $a \leq f$. *Thus,* $A = [e) \cup (f]$.

Proof. By Theorem 5.1(iv) $e \leq a \vee \neg a$. So, because $\boldsymbol{A}$ is FSI, $e \leq a$ or $e \leq \neg a$, by Theorem 5.9(i). In the latter case, $a \leq f$, because $\neg$ is order-reversing. □

The next lemma is a straightforward consequence of the law of residuation (i.e., (1.8) from Theorem 1.55).

Lemma 5.11. *The following conditions on a bounded De Morgan monoid* $\boldsymbol{A}$ *are equivalent (where* $\bot \leq a \leq \top$ *for all* $a \in A$*).*

(i) $\top \cdot a = \top$ *whenever* $\bot \neq a \in A$.

(ii) $a \to \bot = \bot$ *whenever* $\bot \neq a \in A$.

(iii) $\top \to b = \bot$ *whenever* $\top \neq b \in A$.

Definition 5.12. Following Meyer [41], we say that a De Morgan monoid is *rigorously compact* if it is bounded and satisfies the equivalent conditions of Lemma 5.11.

Theorem 5.13. *Let* $\boldsymbol{A}$ *be a bounded FSI De Morgan monoid. Then* $\boldsymbol{A}$ *is rigorously compact.*

Proof. Let $\bot \neq a \in A$. It suffices to show that $\top \cdot a = \top$. By the law of residuation,

$$\top \cdot a \leq f \text{ iff } a \leq \top \to f = \neg \top = \bot \text{ iff } a = \bot,$$

which is false. Therefore, $\top \cdot a \not\leq f$, so by Theorem 5.10,

$$e \leq \top \cdot a.$$

Using this and the fact that $\top = \top^2$ (by Theorem 5.3(ii)) we infer that

$$\top = \top \cdot e \leq \top^2 \cdot a = \top \cdot a \leq \top,$$

whence $\top \cdot a = \top$. □

Corollary 5.14. *Finitely generated FSI De Morgan monoids are rigorously compact.*

Proof. Let $\boldsymbol{A}$ be a finitely generated FSI De Morgan monoid. Since $\boldsymbol{A}$ is finitely generated, it is bounded, by Theorem 5.5. But then, since $\boldsymbol{A}$ is also FSI, $\boldsymbol{A}$ is rigorously compact, by Theorem 5.13. □

An element a of an IRL $\boldsymbol{A}$ is said to be *idempotent* if $a \cdot a = a$. We say that $\boldsymbol{A}$ is *idempotent* if all of its elements are. In the next result, the implication (i) $\Rightarrow$ (ii) is well known, but we have not found (ii) $\Rightarrow$ (iii) in the literature.

Theorem 5.15. *Let $\boldsymbol{A}$ be a De Morgan monoid. The following are equivalent.*

(i) $f^2 = f$.

(ii) $f \leq e$.

(iii) $\boldsymbol{A}$ *is idempotent.*

Proof. (i) $\Rightarrow$ (ii): It suffices to show that if $\boldsymbol{A}$ is an IRL in which $f^2 \leq f$, then $f \leq e$ in $\boldsymbol{A}$. Suppose that $f^2 \leq f$. By Theorem 1.57(iii),

$$f \cdot f = \neg(f \rightarrow \neg f) = \neg(f \rightarrow e).$$

Then $\neg(f \rightarrow e) \leq f$, hence, since $\neg$ is antitone, we obtain $e = \neg f \leq f \rightarrow e$. Therefore, $f \leq e$, by Theorem 1.56(ii).

(ii) $\Rightarrow$ (iii): Let $a \in A$. Using properties of involution, we see that

$$a = \neg\neg a = \neg a \rightarrow f.$$

If $f \leq e$, then $f \leq \neg a \rightarrow \neg a$, by Theorem 1.58(i), and since $\rightarrow$ preserves order in its second coordinate, $a \leq \neg a \rightarrow (\neg a \rightarrow \neg a)$. By (4.2) in Theorem 4.3,

$$\neg a \rightarrow (\neg a \rightarrow \neg a) \leq \neg a \rightarrow \neg a,$$

the last of which is equal to $a \rightarrow a$, because of Theorem 1.57(ii). Therefore, $a \leq a \rightarrow a$, so that, $a \cdot a \leq a$, by the law of residuation. But then $a^2 = a$, as De Morgan monoids are square-increasing.

(iii) $\Rightarrow$ (i) is trivial. □

5.2 Sugihara Monoids

Idempotent De Morgan monoids are called *Sugihara monoids*. The class SM of Sugihara monoids is obviously a subvariety of DMM. The extension $\mathbf{RM^t}$ of $\mathbf{R^t}$ by the *mingle axiom* $p \rightarrow (p \rightarrow p)$ is algebraized by SM. This is an easy consequence of Corollary 2.28 and its proof.

Sugihara monoids are very well understood, cf. Dunn's contributions to [2]. Because of this, we will recount only what pertains directly to Theorem 5.30 below, despite the fact that a full understanding of the subvarieties of SM contributes to our understanding of the subvarieties of DMM.

Definition 5.16. An IRL is said to be *semilinear* if it is a subdirect product of totally ordered IRLs. (Recall that an IRL $\boldsymbol{A}$ is *totally ordered* if for any $a, b \in A$, we have $a \leq b$ or $b \leq a$.)

Theorem 5.17 ([31])**.** *A De Morgan monoid is semilinear if and only if it satisfies*

$$e \leq (x \to y) \vee (y \to x). \tag{5.1}$$

Proof. ($\Rightarrow$): Notice that (5.1) abbreviates an equation. So, it suffices to show that every totally ordered IRL satisfies (5.1), as equations persist in subdirect products. So, let $\boldsymbol{A}$ be a totally ordered IRL, and let $a, b \in A$. By symmetry, we may assume that $a \leq b$. Then $e \leq a \to b$, so

$$e \leq (a \to b) \vee (b \to a).$$

($\Leftarrow$): Let $\boldsymbol{A}$ be a De Morgan monoid that satisfies (5.1). By Birkhoff's subdirect decomposition theorem, $\boldsymbol{A}$ is a subdirect product of SI De Morgan monoids $\boldsymbol{A}_i$, for $i \in I$. It suffices to show that each $\boldsymbol{A}_i$ is totally ordered. Fix $i \in I$. Now, $\boldsymbol{A}_i$ satisfies (5.1), because (5.1) is an abbreviated equation and $\boldsymbol{A}_i$ is a homomorphic image of $\boldsymbol{A}$. So, for each $a, b \in A$, because (5.1) gives $e \leq (a \to b) \vee (b \to a)$, we have $e \leq a \to b$ or $e \leq b \to a$, by Theorem 5.9(i), i.e., $a \leq b$ or $b \leq a$. □

Corollary 5.18. *Every Sugihara monoid is semilinear.*

Proof. Let $\boldsymbol{A}$ be a Sugihara monoid, and let $a, b \in A$. It suffices, by Theorem 5.17, to show that $\boldsymbol{A}$ satisfies $e \leq (a \to b) \vee (b \to a)$. By Theorem 5.1(iii), $a \cdot \neg a \leq f$ and $b \cdot \neg b \leq f$, so, by commutativity and idempotence, $(a \cdot \neg b) \cdot (b \cdot \neg a) \leq f \cdot f = f$. But then, $(a \cdot \neg b) \wedge (b \cdot \neg a) \leq f$, by Theorem 5.1(i). Finally, by De Morgan's law, we get

$$e \leq \neg((a \cdot \neg b) \wedge (b \cdot \neg a)) = \neg(a \cdot \neg b) \vee \neg(b \cdot \neg a) = (a \to b) \vee (b \to a),$$

as required. □

We say that a Sugihara monoid is *odd* if $e = f$. The reason for this name will become clear in view of the next theorem. Notice that if a De Morgan monoid satisfies $e = f$ then it is an odd Sugihara monoid, by Theorem 5.15.

Example 5.19. The set of all integers is the universe of an odd Sugihara monoid $\boldsymbol{S}$ with identity $e = 0$ in which the lattice order $\leq$ is the usual total order. In $\boldsymbol{S}$, $x \cdot y$ is the integer in $\{x, y\}$ with the larger absolute value if $|x| \neq |y|$; otherwise $x \cdot y = x \wedge y$. The involution is just additive inversion.

For each odd positive integer $2n+1$, we use $\boldsymbol{S_{2n+1}}$ to denote the subalgebra of $\boldsymbol{S}$ with universe

$$S_{2n+1} = \{-n, -n+1, \dots, -2, -1, 0, 1, 2, \dots, n-1, n\}.$$

Theorem 5.20. *For each $n \in \omega$, every n-generated subdirectly irreducible odd Sugihara monoid is isomorphic to $\boldsymbol{S_{2m+1}}$ for some $m \leq n$.*

Proof. Let $\boldsymbol{A}$ be a subdirectly irreducible odd Sugihara monoid and X a generating set for $\boldsymbol{A}$, with $|X| \leq n$. Let

$$Y \coloneqq X \cup \{e\} \cup \{\neg a : a \in X\}.$$

Notice that Y has $2m+1$ elements, for some $m \leq n$. Now, since $\boldsymbol{A}$ is SI and semilinear (by Corollary 5.18), $\boldsymbol{A}$ is totally ordered. It follows that Y is closed under $\wedge$ and $\vee$.

Let $a, b \in A$. We claim that, $a \cdot b = a$ or $a \cdot b = b$. By symmetry we may assume that $a \leq b$. Then $a = a \cdot a \leq a \cdot b \leq b \cdot b = b$. If $e \leq a \cdot b$, then

$$a \cdot b = a \cdot (b \cdot b) = (a \cdot b) \cdot b \geq e \cdot b = b,$$

so $a \cdot b = b$. Similarly, if $a \cdot b \leq e$ then $a \cdot b = a$. Since $\boldsymbol{A}$ is totally ordered, this completes the proof of the claim.

The claim shows that Y is closed under $\cdot$. Since $\boldsymbol{A}$ is odd, i.e., $e = \neg e$, Y is also closed under $\neg$. It follows that Y is a subuniverse of $\boldsymbol{A}$, hence $A = Y$ (because $Y \supseteq X$).

It is clear that there is a bijection between $\boldsymbol{A}$ and $\boldsymbol{S_{2m+1}}$ which preserves $\wedge$, $\vee$ and $\neg$.

Recall that for any $a \in A$ we defined $|a| = a \to a$. First we establish that $|a|$ is the largest element of $\{a, \neg a\}$, or equivalently, whichever one of a or $\neg a$ is an upper bound e.

Suppose that $e \leq a$. Since a is idempotent, $a \cdot a = a$, so by the law of residuation $a \leq a \to a$. Furthermore, $\neg a = e \cdot \neg a \leq a \cdot \neg a$, so by properties of involution,

$$a \to a = \neg(a \cdot \neg a) \leq \neg\neg a = a.$$

Therefore, $a = a \to a$. The result for $e \leq \neg a$ follows by symmetry, once we note that $a \to a = \neg a \to \neg a$.

It remains to show, for all $a, b \in A$, that $a \cdot b$ is the element of $\{a, b\}$ with the larger 'absolute value', whenever $|a| \neq |b|$, and that $a \cdot b = a \wedge b$, whenever $|a| = |b|$.

Suppose that $|a| = |b|$. If $a = b$ then clearly $a \cdot b = a \cdot a = a$. If $b = \neg a$, then,

$$a \cdot b = a \cdot \neg a = \neg\neg(a \cdot \neg a) = \neg(a \to a) = \neg|a| = a \wedge \neg a = a \wedge b.$$

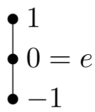
Lastly, suppose that $|a| < |b|$. We are required to prove that $a \cdot b = b$. Suppose not. Then $a \cdot b = a$ by the claim above. Notice that $a \neq b$, so either $a < b$ or $b < a$, since $\boldsymbol{A}$ is totally ordered. If $a < b$, then $a \cdot b \leq b \cdot b = b < a$, a contradiction. Therefore $b < a$. If $a \leq e$, then $a \cdot b \leq e \cdot b = b < a$, again contradicting the fact that $a \cdot b = a$. Therefore $e < a$. Notice that $b < \neg a \leq e$, since $|a| \leq |b|$. But then $b \cdot \neg a = b$, otherwise $\neg a = b \cdot \neg a \leq b \cdot e = b < \neg a$, a contradiction. Also, notice that $a \cdot \neg a = \neg a$, since $|a| = |\neg a|$ and $\neg a \leq a$. Therefore,

$$a = a \cdot b = a \cdot \neg a \cdot b = \neg a \cdot b = b,$$

contradicting the assumption that $|a| \neq |b|$. □

It is worthwhile to mention the following, even though it will not be needed directly. We say that a Sugihara monoid is *even*, when $f < e$. It is easy to expand the proof above to show that every finitely generated even SI Sugihara monoid is isomorphic to some $\boldsymbol{S_{2m+2}}$, which is defined in a similar way to $\boldsymbol{S_{2m+1}}$ in Example 5.19, except that we remove 0, so that 1 becomes the identity element e. In this way we have a complete description of the finitely generated SI Sugihara monoids. Notice that $\boldsymbol{S_2}$ is the 2-element Boolean algebra.

$\boldsymbol{S_3}$:



Corollary 5.21. *$\boldsymbol{S_3}$ is, up to isomorphism, the only simple finitely generated odd Sugihara monoid.*

Proof. By Theorem 5.9(iii), $\boldsymbol{S_3}$ is simple, owing to the fact that -1 is the only strict lower bound of 0 (the identity). Also, $\boldsymbol{S_3}$ is clearly generated by 1.

Conversely, suppose that $\boldsymbol{A}$ is a simple finitely generated odd Sugihara monoid. By Theorem 5.20, $\boldsymbol{A}$ is isomorphic to $\boldsymbol{S_{2n+1}}$ for some $n \in \omega$. Since $\boldsymbol{A}$ is non-trivial, $n \neq 0$. Furthermore, if $n > 1$, then there is more than one strict lower bound of 0 in $\boldsymbol{S_{2n+1}}$. The only remaining option is $\boldsymbol{S_3}$. Therefore, $\boldsymbol{A} \cong \boldsymbol{S_3}$. □

Theorem 5.22. *Every non-Boolean Sugihara monoid has a non-trivial odd Sugihara monoid as a homomorphic image.*

Proof. Let $\boldsymbol{A}$ be a non-Boolean Sugihara monoid. Since SM is a variety and by Birkhoff's Subdirect Decomposition Theorem, $\boldsymbol{A}$ is a subdirect product of subdirectly irreducible members of SM. At least one of these members $\boldsymbol{B}$ is non-Boolean and there exists a homomorphism h from $\boldsymbol{A}$ onto $\boldsymbol{B}$. Note that if $\boldsymbol{B}$ is odd, then we are done. So suppose that $e \neq f$ in $\boldsymbol{B}$, i.e., $f < e$ by Theorem 5.15.

By Theorem 5.10, $\boldsymbol{B}$ is the union of $(f]$ and $[e)$, which are disjoint. Furthermore, since $\boldsymbol{B}$ is non-Boolean, there exists an element strictly above e, by Theorem 5.4.

We define $\theta \coloneqq \mathrm{id}_B \cup \{(e, f), (f, e)\}$ and claim that θ is a congruence of $\boldsymbol{B}$. It is easy to see that θ is an equivalence relation. It is also easy to see that θ is compatible with $\neg$ and e. The fact that θ is compatible with $\wedge$ and $\vee$ follows from the fact that there are no elements between e and f.

To show that θ is compatible with $\cdot$, the only non-trivial part we need to show is that $a \cdot e \equiv_\theta a \cdot f$, for every $a \in A$. We show that $a = a \cdot f$ for any $a \neq e$. First notice that

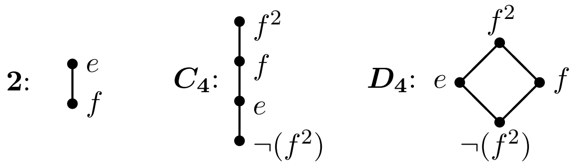
$$a \cdot f \leq a \cdot e = a.$$

Recall that either $a \leq f$ or $e < a$. In the first case $a = a \cdot a \leq a \cdot f$, so $a = a \cdot f$. In the second case, notice that $e \not\leq a \to e$, otherwise $a \leq e$, by Theorem 1.56(ii). So, since $\boldsymbol{B}$ is a totally ordered, $a \to e \leq f$, hence $e \leq a \cdot f$ by laws of involution. Therefore,

$$a = a \cdot e \leq a \cdot a \cdot f = a \cdot f. \quad \square$$

5.3 Varieties of De Morgan Monoids

In this section we will finally characterize the atoms of the subvariety lattice of DMM. We start by characterizing all the simple 0-generated De Morgan monoids. In [54, 55], Slaney describes all the 0-generated finitely subdirectly irreducible members of DMM (he called them prime constant De Morgan monoids). Only three of these algebras (depicted below) are simple. For our purposes, only the simple ones will prove important, so it is easier to identify them via a direct argument.



Here $\mathbf{2}$ ($= \boldsymbol{S}_2$) is just the 2-element Boolean algebra. Notice that, by Lemma 5.2 and Theorem 5.3, the information given in the pictures is enough to determine the entire algebra in each case; the details will become clear in the proof of Theorem 5.24.

It is now easy for us to give a short proof of the following very well known fact.

Theorem 5.23. *Every finitely subdirectly irreducible Boolean algebra is isomorphic to* $\mathbf{2}$*. Consequently, every Boolean algebra is isomorphic to a subdirect product of copies of* $\mathbf{2}$*, whence* $\mathsf{BA} = \mathbb{Q}(\mathbf{2})$*.*

Proof. It is clear from Theorem 5.9(iii) that $\mathbf{2}$ is simple, and hence finitely subdirectly irreducible.

Suppose that $\boldsymbol{A}$ is a finitely subdirectly irreducible Boolean algebra. So, e is the top element of $\boldsymbol{A}$, and $\neg e = f$ is the bottom element. Suppose that $c \in A$, such that $f < c < e$. Notice that $f < \neg c < e$, by properties of involution. Also $c \vee \neg c = e$, since $\neg c$ is the complement of c. But this contradicts the fact that e is join-irreducible in $\boldsymbol{A}$, by Theorem 5.9(i). Therefore, e and f are the only elements of $\boldsymbol{A}$. □

In particular, $\mathbf{2}$ is, up to isomorphism, the only simple Boolean algebra.

Theorem 5.24. *Every simple 0-generated De Morgan monoid is isomorphic to* $\mathbf{2}$*,* $\boldsymbol{C_4}$ *or* $\boldsymbol{D_4}$*.*

Proof. Let $\boldsymbol{A}$ be a simple 0-generated De Morgan monoid. By definition, $\boldsymbol{A}$ is non-trivial. Suppose that $\boldsymbol{A} \ncong \mathbf{2}$. Then $\boldsymbol{A}$ cannot be integral, otherwise it would be a Boolean algebra by Theorem 5.4, and hence isomorphic to $\mathbf{2}$, by Theorem 5.23.

Suppose that $\boldsymbol{A}$ is idempotent. Then $f \leq e$, by Theorem 5.15. We cannot have $e = f$, as then $\{e\}$ would be a proper subuniverse of (the non-trivial) $\boldsymbol{A}$, contradicting the fact that $\boldsymbol{A}$ is 0-generated. Therefore, $f < e$. Now, since $\boldsymbol{A}$ is simple, e has just one strict lower bound, by Theorem 5.9, so f is the least element of $\boldsymbol{A}$. But then e is the greatest element of $\boldsymbol{A}$, i.e., $\boldsymbol{A}$ is integral, a contradiction. Therefore $\boldsymbol{A}$ is not idempotent.

By Theorem 5.15, $f \nleq e$ and $f \neq f^2$. But since $\boldsymbol{A}$ is square-increasing, $f < f^2$, hence $\neg(f^2) < e$. Since $\boldsymbol{A}$ is simple, we have that $\neg(f^2)$ is the bottom element of $\boldsymbol{A}$, so f^2 is the top element. Consequently, $a \cdot \neg(f^2) = \neg(f^2)$ for all $a \in A$, by Theorem 5.3(i). Also, since $\boldsymbol{A}$ is bounded and FSI, it is rigorously compact, by Theorem 5.13, so $a \cdot f^2 = f^2$, whenever $\neg(f^2) \neq a \in A$. It follows that $\{e, f, f^2, \neg(f^2)\} \subseteq A$ is closed with respect to $\cdot$ and $\neg$.

There are two possibilities for the order: $e < f$ or $e \nleq f$. If $e \nleq f$, then $e \wedge f < e$, whence $e \wedge f$ is the bottom element $\neg(f^2)$, by simplicity of $\boldsymbol{A}$. If we then apply De Morgan's law, we see that $e \vee f = f^2$. In the case where $e < f$, we see that $\neg(f^2) < e < f < f^2$. Either way, $\{e, f, f^2, \neg(f^2)\}$ is the universe of a four-element subalgebra of $\boldsymbol{A}$. Therefore, since $\boldsymbol{A}$ is 0-generated, $A = \{e, f, f^2, \neg(f^2)\}$. Thus, $\boldsymbol{A} \cong \boldsymbol{C_4}$ if $e < f$ and $\boldsymbol{A} \cong \boldsymbol{D_4}$ if $e \nleq f$. □

Now that we have found all the simple 0-generated De Morgan monoids, we can use universal algebraic tools, which we will develop now, to prove our characterization of the minimal subvarieties of DMM.

Theorem 5.25 (Jónsson [33])**.** *If a non-trivial algebra* $\boldsymbol{A}$ *of finite type is finitely generated, then* $\boldsymbol{A}$ *has a simple homomorphic image.*

Proof. Let $\boldsymbol{A} = \mathrm{Sg}^{\boldsymbol{A}}(\{a_1, \dots, a_n\})$ have finite type $\mathcal{F}$, where, without loss of generality, $1 \leq n \in \omega$. By Theorem 1.39, we know that $A = \{\alpha^{\boldsymbol{A}}(a_1, \dots, a_n) : \alpha \text{ is an } n\text{-ary term}\}$. Let

$$X = \{a_1, \dots, a_n\} \cup \{g^{\boldsymbol{A}}(a_1, \dots, a_1) : g \in \mathcal{F}\}.$$

Let $Y = X \times X$. Notice that Y is finite. We claim that $A \times A = \Theta^{\boldsymbol{A}}(Y)$.

Let $b \in A$. Then $b = \alpha^{\boldsymbol{A}}(a_1, \dots, a_n)$ for some n-ary term α. We first show, using induction on the complexity of α, that $\langle b, a_1 \rangle \in \Theta^{\boldsymbol{A}}(Y)$. Once this is shown then the claim follows easily from the transitivity and symmetry of $\Theta^{\boldsymbol{A}}(Y)$.

Suppose that $\#\alpha = 0$, i.e., α is a variable or a constant. If it is a variable, then clearly

$$\langle b, a_1 \rangle = \langle a_i, a_1 \rangle \in Y \subseteq \Theta^{\boldsymbol{A}}(Y), \text{ for some } i \leq n.$$

If α is a constant symbol, then $\langle \alpha^{\boldsymbol{A}}, a_1 \rangle \in Y \subseteq \Theta^{\boldsymbol{A}}(Y)$. Now suppose that α is $g(\beta_1, \dots, \beta_m)$, where $\langle \beta_i^{\boldsymbol{A}}(\overrightarrow{a}), a_1 \rangle \in \Theta^{\boldsymbol{A}}(Y)$, for some $g \in \mathcal{F}$ and each $i \leq m$. Then

$$g^{\boldsymbol{A}}(\beta_1^{\boldsymbol{A}}(\overrightarrow{a}), \dots, \beta_m^{\boldsymbol{A}}(\overrightarrow{a})) \equiv_{\Theta^{\boldsymbol{A}}(Y)} g^{\boldsymbol{A}}(a_1, \dots, a_1) \equiv_{\Theta^{\boldsymbol{A}}(Y)} a_1.$$

We have shown that the congruence $A \times A$ is finitely generated. We still need to show that $\boldsymbol{A}$ has a simple homomorphic image. We will prove, using Zorn's Lemma, that $\boldsymbol{A}$ has a maximal proper congruence, say θ. By the Correspondence Theorem, $\boldsymbol{A}/\theta$ will then have only two congruences, namely the trivial and total congruences. In other words, $\boldsymbol{A}/\theta$ is simple.

Let $\Sigma = \{\theta \in \mathrm{Con}(\boldsymbol{A}) : \theta \subsetneq A \times A\}$. Notice that Σ is non-empty, since the trivial congruence belongs to Σ, because $\boldsymbol{A}$ is non-trivial. Let Σ_0 be a non-empty chain in Σ. Then clearly $\bigcup \Sigma_0$ is an upper bound of Σ_0. It remains to prove that $\bigcup \Sigma_0 \in \Sigma$.

As $\mathrm{Con}(\boldsymbol{A})$ is algebraic, and as chains are directed, $\bigcup \Sigma_0$ is a congruence, but we still need to show that $\bigcup \Sigma_0$ is *proper* subset of $A \times A$. Suppose, towards a contradiction, that $\bigcup \Sigma_0 = A \times A$. This means that $Y \subseteq \bigcup \Sigma_0$. Since Y is finite and Σ_0 is a chain, $Y \subseteq \theta$ for some $\theta \in \Sigma_0$. But then $\theta = A \times A$, a contradiction. □

Corollary 5.26. *Every non-trivial variety of finite type has a simple finitely generated member.*

Proof. Let K be such a non-trivial variety. Then there exists a non-trivial algebra $\boldsymbol{A} \in \mathsf{K}$. In particular, there exist $a, b \in A$ such that $a \neq b$. Now let $\boldsymbol{B} = \mathrm{Sg}^{\boldsymbol{A}}(\{a, b\})$, so $\boldsymbol{B}$ is a finitely generated subalgebra of $\boldsymbol{A}$. Notice that $\boldsymbol{B} \in \mathsf{K}$, since K is closed under subalgebras. By Theorem 5.25, $\boldsymbol{B}$ has a simple homomorphic image $\boldsymbol{C}$, which is contained in K since K is closed under homomorphic images. Furthermore, by Theorem 1.27, $\boldsymbol{C}$ is generated by the images of the generators of $\boldsymbol{B}$. □

For varieties of finite type, Corollary 5.26 strengthens *Magari's Theorem* (cf. [13, Thm. II.10.13]), which states that *every* non-trivial variety has a simple member.

Definition 5.27. A variety K has the *congruence extension property (CEP)* provided that, for every $\boldsymbol{A} \in \mathsf{K}$ and every subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$,

if $\theta \in \mathrm{Con}(\boldsymbol{B})$, then there exists $\alpha \in \mathrm{Con}(\boldsymbol{A})$ such that $\alpha \cap B^2 = \theta$.

Theorem 5.28. *The variety* DMM *has the congruence extension property.*

Proof. Let $\boldsymbol{A} \in \mathsf{DMM}$ and $\boldsymbol{B}$ a subalgebra of $\boldsymbol{A}$. We shall show that for every deductive filter F of $\boldsymbol{B}$, there exists a deductive filter G of $\boldsymbol{A}$ such that $B \cap G = F$.

Let F be a deductive filter of $\boldsymbol{B}$. So $\mathrm{DFg}^{\boldsymbol{A}}(F)$ is a deductive filter of $\boldsymbol{A}$. Clearly, $F \subseteq B \cap \mathrm{DFg}^{\boldsymbol{A}}(F)$. For the reverse inclusion, let $b \in B \cap \mathrm{DFg}^{\boldsymbol{A}}(F)$. As $\mathrm{DFil}(\boldsymbol{A})$ is an algebraic closure system, $\mathrm{DFg}^{\boldsymbol{A}}(b)$ is compact in $\mathrm{DFil}(\boldsymbol{A})$, hence there is a finite subset F' of F such that $\mathrm{DFg}^{\boldsymbol{A}}(b) \subseteq \mathrm{DFg}^{\boldsymbol{A}}(F')$, i.e., $b \in \mathrm{DFg}^{\boldsymbol{A}}(F')$. By Lemma 5.8, $\mathrm{DFg}^{\boldsymbol{A}}(F') = [a)^{\boldsymbol{A}}$, where a is either e or the meet (in $\boldsymbol{A}$) of the elements of F'. Thus, $a \in F$, because F' is a finite subset of F and because $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$. Therefore $b \in F$, since $a \leq b$, which shows that $B \cap \mathrm{DFg}^{\boldsymbol{A}}(F) \subseteq F$.

Finally, let θ be a congruence of $\boldsymbol{B}$. Let $F = H^{\boldsymbol{B}}(\theta)$, from Theorem 5.7. Then $F \in \mathrm{DFil}(\boldsymbol{B})$ and $\theta = \{\langle a, b\rangle \in B^2 : a \leftrightarrow b \in F\}$, by Theorem 5.7. By the above argument, if we let $G = \mathrm{DFg}^{\boldsymbol{A}}(F)$, then $F = B \cap G$. By Theorem 5.7, the relation

$$\theta_G = \{\langle a, b\rangle \in A^2 : a \leftrightarrow b \in G\}$$

is a congruence of $\boldsymbol{A}$, and we claim that $B^2 \cap \theta_G = \theta$. It suffices to show that for all $a, b \in B$, we have $\langle a, b\rangle \in \theta_G$ iff $\langle a, b\rangle \in \theta$. Let $a, b \in B$. Then $\langle a, b\rangle \in \theta_G$ iff $a \leftrightarrow b \in G$ iff $a \leftrightarrow b \in F$ (because $a \leftrightarrow b \in B$) iff $\langle a, b\rangle \in \theta$. □

One reason for introducing the congruence extension property is to prove the following theorem. Its proof can easily be extended to every variety with the CEP.

Corollary 5.29. *Non-trivial subalgebras of simple De Morgan monoids are simple.*

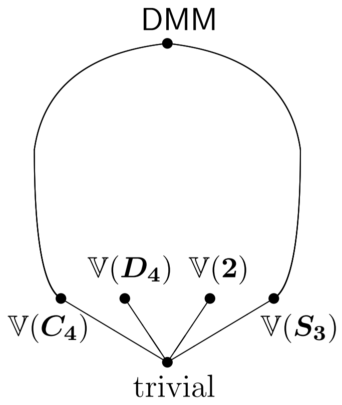
Proof. Let $\boldsymbol{A}$ be a simple De Morgan monoid and $\boldsymbol{B}$ a non-trivial subalgebra of $\boldsymbol{A}$. Suppose that $\boldsymbol{B}$ is not simple, i.e., there exists $\theta \in \mathrm{Con}(\boldsymbol{B})$ such that $\mathrm{id}_B \subsetneq \theta \subsetneq B^2$. By Theorem 5.28, there exists $\theta' \in \mathrm{Con}(\boldsymbol{A})$ such that $\theta' \cap B^2 = \theta$. Notice that $\theta' \neq A^2$, otherwise $\theta = B^2$. Also, $\theta' \neq \mathrm{id}_A$, otherwise $\theta = \mathrm{id}_B$. Therefore, $\mathrm{id}_A \subsetneq \theta' \subsetneq A^2$, contradicting the fact that $\boldsymbol{A}$ is simple. □

Now we can finally prove the main new result of this thesis.

Theorem 5.30. *The distinct classes $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S_3})$, $\mathbb{V}(\boldsymbol{C_4})$ and $\mathbb{V}(\boldsymbol{D_4})$ are precisely the minimal (non-trivial) varieties of De Morgan monoids.*

Proof. Let $\boldsymbol{X}$ be any of the four algebras $\mathbf{2}$, $\boldsymbol{S_3}$, $\boldsymbol{C_4}$ or $\boldsymbol{D_4}$. Notice that $\mathbb{V}(\boldsymbol{X})$, being a class of lattices, is congruence distributive. Therefore, by Jónsson's Theorem (Theorem 1.49), the SI members of $\mathbb{V}(\boldsymbol{X})$ are contained in $\mathbb{H}\,\mathbb{S}\,\mathbb{P}_{\mathrm{u}}(\boldsymbol{X})$. Since $\boldsymbol{X}$ is finite, by Corollary 1.36, any ultrapower of $\boldsymbol{X}$ is isomorphic to $\boldsymbol{X}$. Furthermore, in the case where $\boldsymbol{X} = \boldsymbol{S_3}$, the only proper subalgebra of $\boldsymbol{X}$ is the trivial subalgebra. In the other cases, $\boldsymbol{X}$ is 0-generated, i.e., has no proper subalgebra. Lastly, since $\boldsymbol{X}$ is simple, the homomorphic images of $\boldsymbol{X}$ are either trivial or isomorphic to $\boldsymbol{X}$. Since SI algebras are non-trivial, all the SI members of $\mathbb{V}(\boldsymbol{X})$ are isomorphic to $\boldsymbol{X}$. Now, since varieties are determined by their SI members, $\mathbb{V}(\boldsymbol{X})$ has no proper non-trivial subvariety, and $\mathbb{V}(\boldsymbol{X}) \neq \mathbb{V}(\boldsymbol{Y})$, whenever $\boldsymbol{Y}$ is any one of the four algebras except $\boldsymbol{X}$.

Conversely, let K be a minimal (non-trivial) subvariety of DMM. Then it is easy to see that K is generated by any one of its non-trivial members. By Corollary 5.26, K has a finitely generated simple (hence non-trivial) member $\boldsymbol{A}$. So $\mathbb{V}(\boldsymbol{A}) = \mathsf{K}$. Let $\boldsymbol{B}$ be the smallest subalgebra of $\boldsymbol{A}$, so $\boldsymbol{B}$ is 0-generated. Now, if $\boldsymbol{B}$ is non-trivial, then $\mathsf{K} = \mathbb{V}(\boldsymbol{B})$. Furthermore, $\boldsymbol{B}$ is simple, by Theorem 5.28. In this case, by Theorem 5.24, $\boldsymbol{B}$ is isomorphic to $\mathbf{2}$, $\boldsymbol{C_4}$ or $\boldsymbol{D_4}$. On the other hand, if $\boldsymbol{B}$ is trivial, then the identity $e = f$ holds in $\boldsymbol{A}$. So, $\boldsymbol{A}$ is a finitely generated simple odd Sugihara monoid, by Theorem 5.15. Therefore, by Corollary 5.21, $\boldsymbol{A} \cong \boldsymbol{S_3}$, so $\mathsf{K} = \mathbb{V}(\boldsymbol{S_3})$. □



Subvarieties of DMM

Before we axiomatize these varieties, we prove the following lemma.

Lemma 5.31. *If a non-trivial De Morgan monoid $\boldsymbol{A}$ satisfies $e \leq f$ and $x \leq f^2$, then $\boldsymbol{C_4}$ can be embedded into $\boldsymbol{A}$.*

Proof. Suppose $\boldsymbol{A}$ satisfies $e \leq f$ and $x \leq f^2$. Then $\boldsymbol{A}$ satisfies $\neg(f^2) \leq x$, by involution properties. In $\boldsymbol{A}$, if $e = f$, then $f^2 = e^2 = e$, whence $\neg(f^2) = \neg e = f$, yielding $\neg(f^2) = f^2$, i.e., $\boldsymbol{A}$ is trivial. This is a contradiction, so $e < f$. By Theorem 5.15, therefore, $f < f^2$, i.e., $\neg(f^2) < e$. Thus, $\boldsymbol{C_4} \in \mathbb{I}\,\mathbb{S}(\boldsymbol{A})$. □

Theorem 5.32. *Consider the following equations.*

$$e \leq (x \to (y \vee \neg y)) \vee (y \wedge \neg y) \tag{5.2}$$
$$e \leq (f^2 \to x) \vee (x \to e) \vee \neg x \tag{5.3}$$
$$x \wedge (x \to f) \leq (f \to x) \vee (x \to e) \tag{5.4}$$
$$x \to e \leq x \vee (f^2 \to \neg x) \tag{5.5}$$

(i) $\mathbb{V}(\mathbf{2})$ *is axiomatized adding $x \leq e$ to the axioms of* DMM*;*

(ii) $\mathbb{V}(\boldsymbol{S_3})$ *by adding* $e = f$, (5.1) *and* (5.2)*;*

(iii) $\mathbb{V}(\boldsymbol{D_4})$ *by adding* $x \le f^2$, $x \wedge \neg x \le y$ *and* (5.3)*;*

(iv) $\mathbb{V}(\boldsymbol{C_4})$ *by adding* $x \le f^2$, $e \le f$, (5.1), (5.4) *and* (5.5).

Proof. Let $\boldsymbol{X} \in \{\boldsymbol{2}, \boldsymbol{S_3}, \boldsymbol{C_4}, \boldsymbol{D_4}\}$. It can be verified mechanically that $\boldsymbol{X}$ satisfies the proposed axioms for $\mathbb{V}(\boldsymbol{X})$. Let $\boldsymbol{A}$ be an SI algebra satisfying the putative axioms for $\mathbb{V}(\boldsymbol{X})$. It suffices to show that $\boldsymbol{A} \cong \boldsymbol{X}$.

Suppose $\boldsymbol{X}$ is $\boldsymbol{2}$. Since e is the greatest element of $\boldsymbol{A}$, we have $\boldsymbol{A} \cong \boldsymbol{2}$, by Theorems 5.4 and 5.23.

Let a be the largest element of A strictly below e, which exists by Theorem 5.9(ii), because $\boldsymbol{A}$ is SI. Then, by properties of involution, $\neg a$ is the smallest element of $\boldsymbol{A}$ strictly above f. Furthermore, if $\boldsymbol{X}$ is $\boldsymbol{S_3}$ or $\boldsymbol{C_4}$, then $\boldsymbol{A}$ is totally ordered, by Theorem 5.17, because (5.1) signifies that $\boldsymbol{A}$ is semilinear.

Suppose that $\boldsymbol{X} = \boldsymbol{S_3}$. In $\boldsymbol{A}$, since $e = f$, we have $a < e < \neg a$, and there is no other element in this interval. We claim that there is no element of $\boldsymbol{A}$ strictly above $\neg a$. Suppose, with a view to contradiction, that $b > \neg a$. By (5.2) and the fact that e is join-prime (Theorem 5.9(i)), we know that $e \le b \to (a \vee \neg a)$ or $e \le a \wedge \neg a$. Since $a \wedge \neg a = a < e$, we have $b \le a \vee \neg a = \neg a$, a contradiction. This vindicates the above claim. Now suppose $b \in A - \{a, e, \neg a\}$. Since $\boldsymbol{A}$ is totally ordered, the claim shows that $b < a$, but then $\neg b > \neg a$, contradicting the claim. Therefore, $A = \{a, e, \neg a\}$, from which it follows easily that $\boldsymbol{A} \cong \boldsymbol{S_3}$.

If $\boldsymbol{X}$ is $\boldsymbol{C_4}$ or $\boldsymbol{D_4}$, then $\boldsymbol{A}$ satisfies $x \le f^2$. In particular, $\boldsymbol{A}$ is bounded (with lower bound $\neg(f^2)$), and therefore rigorously compact, by Theorem 5.13.

Suppose that $\boldsymbol{X} = \boldsymbol{D_4}$. Notice that $b \wedge \neg b = \neg(f^2)$ for any $b \in A$. First we show that e is incomparable with f in $\boldsymbol{A}$. If $e \le f$, then $e = e \wedge f = \neg(f^2)$, i.e., e is the bottom element of $\boldsymbol{A}$, forcing $\boldsymbol{A}$ to be trivial by Theorem 1.56(ii). On the other hand, if $f \le e$ then $\boldsymbol{A}$ is idempotent by Theorem 5.15, so again $\neg(f^2) = \neg f = e$. Either way, this contradicts the fact that $\boldsymbol{A}$ is SI.

Recall that a is the greatest element strictly less than e. Thus, $a < f$, by Theorem 5.10. But then $a \le e \wedge f = \neg(f^2)$, showing that $a = \neg(f^2)$. This means that there is no element between $\neg(f^2)$ and e in $\boldsymbol{A}$. It follows by properties of involution that there is also no element between f and f^2. Now suppose that $b \in A$, such that $\neg(f^2) < b < f$. Then, by (5.3),

$$e \le (f^2 \to \neg b) \vee (\neg b \to e) \vee b.$$

Since $\boldsymbol{A}$ is rigorously compact and $\neg b \neq f^2$, we have $f^2 \to \neg b = \neg(f^2)$. So, because e is join-prime, $e \leq \neg b \to e$ or $e \leq b$. The last condition is false, for otherwise $e \leq b < f$. Therefore, $\neg b \leq e$, i.e., $f \leq b$, contrary to assumption. Therefore, there is no element between $\neg(f^2)$ and f. Then there is also no element between e and f^2 in $\boldsymbol{A}$, by properties of involution. But then $A = \{\neg(f^2), e, f, f^2\}$, because, by Theorem 5.10, every element of A is either above e or below f. In this case $\boldsymbol{A} \cong \boldsymbol{D_4}$.

Lastly, suppose $\boldsymbol{X} = \boldsymbol{C_4}$. Notice that $\boldsymbol{C_4}$ embeds into $\boldsymbol{A}$, by Lemma 5.31. Recall that $a < e$, so, by (5.5),

$$e \leq a \to e \leq a \vee (f^2 \to \neg a).$$

So, since e is join-prime and $a < e$, we have $f^2 \leq \neg a$. But then $a = \neg(f^2)$. It follows that there is no element between $\neg(f^2)$ and e in $\boldsymbol{A}$, nor is there an element between f and f^2.

Suppose, with a view to contradiction, that $b \in A - \{\neg(f^2), e, f, f^2\}$. By Theorem 5.10, $e \leq b$ or $e \leq \neg b$, so we may suppose, without loss of generality, that $e < b$. In particular, by the previous paragraph and since $\boldsymbol{A}$ is totally ordered, $e < b < f$. But then $e \leq b \to f$, so by (5.4),

$$e \leq b \wedge (b \to f) \leq (f \to b) \vee (b \to e).$$

So, since e is join-prime, $b \leq e$ or $f \leq b$, a contradiction. Thus, $\boldsymbol{A} \cong \boldsymbol{C_4}$. □

Now that we have found the atoms of the subvariety lattice of DMM and produced axiomatizations for them, we can use the strategy set out in Section 4.1 to give a transparent description of the maximal consistent axiomatic extensions of $\mathbf{R^t}$.

Theorem 5.33. *Let Λ be the lattice anti-isomorphism from Corollary 2.28, from the subvariety lattice of* DMM *onto the lattice of axiomatic extensions of $\mathbf{R^t}$. The maximal consistent axiomatic extensions of $\mathbf{R^t}$ are exactly*

$$\Lambda(\mathbb{V}(\mathbf{2})) = \vdash_{\mathbf{CPL}}, \quad \Lambda(\mathbb{V}(\boldsymbol{S_3})), \quad \Lambda(\mathbb{V}(\boldsymbol{C_4})) \quad \textit{and} \quad \Lambda(\mathbb{V}(\boldsymbol{D_4})).$$

This means that, for each axiomatic consistent extension $\mathbf{L}$ of $\mathbf{R^t}$, there exists $\boldsymbol{B} \in \{\mathbf{2}, \boldsymbol{S_3}, \boldsymbol{C_4}, \boldsymbol{D_4}\}$ such that the theorems of $\mathbf{L}$ all take values $\geq e$ on any $\boldsymbol{B}$-evaluation. Axioms for these extensions of $\mathbf{R^t}$ follow systematically from Theorem 5.32, in view of Theorem 2.26. For example, after the obvious simplifications, (5.1) becomes the axiom $(p \to q) \vee (q \to p)$, while (5.5) becomes $(p \to t) \to (p \vee (f^2 \to \neg p))$.

Chapter 6

Some Applications

In this chapter we will use some of the results from the previous chapter to prove results about relevant algebras. The results in this chapter are not new, but we believe that the proofs given here are easier than the published versions. In particular, we will use our characterization of the minimal subvarieties of DMM to describe the lower part of the subvariety lattice of RA. The outcome is a result of Świrydowicz [59]. Our approach, unlike his, does not employ the somewhat complex ternary relation structures of [53] that provide a relational semantics for **R**. Following Raftery and Świrydowicz [52], we then show how the result can be exploited to identify the structurally complete axiomatic extensions of **R**.

6.1 Relevant Algebras

Recall that we use the notation $|x|$ to abbreviate $x \to x$, and that the class RA was introduced in Definition 4.7.

Theorem 6.1. *Let $\boldsymbol{A} \in \mathsf{RA}$, with $a, b \in A$.*

(i) $||a|| = |a|$;

(ii) $||a| \wedge |b|| \leq |a| \wedge |b|$.

Proof. We know that $\boldsymbol{A}$ is a subalgebra of some algebra $\boldsymbol{B}$ which is a reduct of a De Morgan monoid. In $\boldsymbol{B}$ we know that $e \leq |a|$, by Theorem 1.58(i), so by Lemma 4.6, $||a|| \leq |a|$. Furthermore, by two applications of Theorem 1.56(i),

$$a \cdot (a \to a) \cdot (a \to a) \leq a.$$

So, by two applications of the law of residuation, $|a|^2 \leq |a|$, and then $|a| \leq |a| \to |a|$. Therefore, $||a|| = |a|$ in $\boldsymbol{B}$, but since $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$, the equality holds in $\boldsymbol{A}$ as well.

For (ii), notice that $e \leq |a|$ and $e \leq |b|$ in $\boldsymbol{B}$. Therefore, $e \leq |a| \wedge |b|$. But then $||a| \wedge |b|| \leq |a| \wedge |b|$, by Lemma 4.6. □

The following theorem provides one of the main tools for transitioning between relevant algebras and De Morgan monoids.

Theorem 6.2. *Every finitely generated relevant algebra is the e-free* reduct *of a De Morgan monoid.*

Proof. Let $\boldsymbol{A} \in \mathsf{RA}$, such that $\boldsymbol{A}$ is generated by $a_1, \ldots, a_n \in A$. We claim that $\bar{e} \coloneqq |a_1| \wedge \cdots \wedge |a_n|$ is an identity element of $\boldsymbol{A}$. Let $b \in A$. We will show that $b \cdot \bar{e} = b$.

We know that $\boldsymbol{A}$ is a subalgebra of some algebra $\boldsymbol{B}$, which is a reduct of a De Morgan monoid. So, $\boldsymbol{B}$ has an identity e, and $e \leq |a_i|$ for every $i \leq n$, because of Theorem 1.58(i). Therefore, $e \leq \bar{e}$, and multiplying throughout by b, we have $b \leq b \cdot \bar{e}$ in $\boldsymbol{B}$, hence, also in $\boldsymbol{A}$.

Notice that, since De Morgan's laws are equations that do not contain e, they also hold in relevant algebras. So, by Theorem 1.39, there is some n-ary term α, in the signature $\wedge, \cdot, \neg$, such that $b = \alpha^{\boldsymbol{A}}(a_1, \ldots, a_n)$. Now we show that $b \cdot \bar{e} \leq b$ by induction on the complexity of α.

If b is one of the generators, say a_j, then

$$b \cdot \bar{e} \leq b \cdot |a_j| = b \cdot (b \to b) \leq b.$$

Assume that $c, d \in A$, such that $c \cdot \bar{e} \leq c$ and $d \cdot \bar{e} \leq d$. If $b = \neg c$, then $(\neg c) \cdot \bar{e} \leq \neg c$, by (1.5). Suppose that $b = c \wedge d$. Then

$$(c \wedge d) \cdot \bar{e} \leq c \cdot \bar{e} \leq c.$$

Similarly, $(c \wedge d) \cdot \bar{e} \leq d$. Therefore $(c \wedge d) \cdot \bar{e} \leq c \wedge d$.

Now, suppose that $b = c \cdot d$. Then, by the square-increasing law,

$$c \cdot d \cdot \bar{e} \leq c \cdot \bar{e} \cdot d \cdot \bar{e} \leq c \cdot d. \quad \square$$

We will use the following notation: If $\boldsymbol{A}$ is a De Morgan monoid, then $\boldsymbol{A}^-$ will denote the e-free reduct of $\boldsymbol{A}$. Furthermore, whenever we use the term 'reduct' in this section, we really mean 'e-free reduct'.

Also, if K is a variety of De Morgan monoids, then K^- will denote the class of reducts of the members of K. Note that

$$\text{if } \boldsymbol{A} \in \mathsf{DMM} \text{ then } \mathbb{V}(\boldsymbol{A})^- \subseteq \mathbb{V}(\boldsymbol{A}^-). \tag{6.1}$$

Indeed, every equation satisfied by $\boldsymbol{A}^-$ is an e-free identity of $\boldsymbol{A}$, and therefore of $\mathbb{V}(\boldsymbol{A})$, and therefore of $\mathbb{V}(\boldsymbol{A})^-$.

The operation of taking e-free reducts preserves many structural properties, as exemplified by the following lemma. Crucially however, the subalgebras of a reduct of a De Morgan monoid need not contain e, and they need not be reducts of De Morgan monoids themselves, unless they are finitely generated.

Lemma 6.3. *If* $\boldsymbol{A}$ *is a De Morgan monoid, then* $\mathrm{Con}(\boldsymbol{A}^-) = \mathrm{Con}(\boldsymbol{A})$. *In particular, if* $\boldsymbol{A}$ *is simple, FSI or SI, then so is* $\boldsymbol{A}^-$.

Proof. It is clear from the definition of a congruence that θ is a congruence of $\boldsymbol{A}^-$ if and only if θ is a congruence of $\boldsymbol{A}$, because the demand that θ should be compatible with e is already captured by the reflexivity of θ. □

Theorem 6.4. RA *is a variety.*

Proof. We already know from Lemma 2.29 that RA is a quasivariety, i.e., $\mathsf{RA} = \mathbb{I}\,\mathbb{S}\,\mathbb{P}\,\mathbb{P}_{\mathrm{u}}(\mathsf{RA})$. It remains to show that RA is closed with respect to homomorphic images.

Let $\boldsymbol{A} \in \mathsf{RA}$ and let h be a homomorphism from $\boldsymbol{A}$ onto an algebra $\boldsymbol{B}$. We need to show that $\boldsymbol{B}$ is a subreduct of a De Morgan monoid.

By Theorem 1.33, $\boldsymbol{B}$ can be embedded into an ultraproduct $\prod_{i\in I} \boldsymbol{B}_i/\mathcal{U}$, where $\boldsymbol{B_i}$ is a finitely generated subalgebra of $\boldsymbol{B}$ for every $i \in I$.

Fix $i \in I$. Let $\{b_1, \dots, b_n\}$ be a set of generators for $\boldsymbol{B}_i$. Choose $a_j \in \overleftarrow{h}[\{b_j\}]$ for every $j \leq n$. Let $\boldsymbol{A}_i = \mathrm{Sg}^{\boldsymbol{A}}(a_1, \dots, a_n)$, so that $\boldsymbol{A}_i$ is a finitely generated subalgebra of $\boldsymbol{A}$. By Theorem 6.2, $\boldsymbol{A}_i$ is a reduct of some De Morgan monoid, which by Lemma 6.3 has exactly the same congruences as $\boldsymbol{A}$. If we let $h_i = h|_{A_i}$, then h_i is a homomorphism onto $\boldsymbol{B}_i$. So, by the Homomorphism Theorem, if we let $\theta = \ker h_i$, then $\boldsymbol{A}_i/\theta \cong \boldsymbol{B}_i$. Therefore $\boldsymbol{B}_i$ is a reduct of a De Morgan monoid, hence $\boldsymbol{B_i} \in \mathsf{RA}$.

Therefore $\boldsymbol{B} \in \mathbb{I}\,\mathbb{S}\,\mathbb{P}_{\mathrm{u}}(\mathsf{RA}) = \mathsf{RA}$, as required. □

Although we will not need to rely on it, the following theorem provides an axiomatization of RA (cf. [50]), and it can be rephrased in terms of pure equations, like those in Theorem 1.59.

Theorem 6.5. *An algebra* $\langle A; \cdot, \wedge, \vee, \neg\rangle$ *is a relevant algebra if and only if the following hold:* $\langle A; \wedge, \vee\rangle$ *is a distributive lattice,* $\langle A; \cdot\rangle$ *is a commutative semigroup and for all* $a, b, c \in A$,

1. $\neg\neg a = a \leq a \cdot a$,

2. $a \leq b$ *iff* $\neg b \leq \neg a$,

3. $a \cdot b \leq c$ *iff* $a \cdot \neg c \leq \neg b$,

4. $a \leq a \cdot (\neg(b \cdot \neg b) \wedge \neg(c \cdot \neg c))$.

The only published proof of the next corollary, viz. [60, Prop. 5], is quite complicated. The boundedness of finitely generated relevant algebras has been known for a long time. Świrydowicz attributes it in [60] to Meyer and to Dziobiak (independently). This is the first of several situations in which results for De Morgan monoids are used to simplify investigations into relevant algebras.

Corollary 6.6. *Every finitely generated relevant algebra is bounded.*

Proof. Let $\boldsymbol{A}$ be a finitely generated relevant algebra. By Theorem 6.2, $\boldsymbol{A}$ is a reduct of a De Morgan monoid, which is also finitely generated (by the same set). Therefore, by Theorem 5.5, $\boldsymbol{A}$ is bounded. □

Theorem 6.7. *Every non-trivial relevant algebra* $\boldsymbol{A}$ *has a copy of* $\mathbf{2}^-$ *as a subalgebra.*

Proof. As $\boldsymbol{A}$ is non-trivial, there exist $a, b \in A$ such that $a \neq b$. Let $\boldsymbol{B} = \mathrm{Sg}^{\boldsymbol{A}}(a, b)$, so that $\boldsymbol{B}$ is a non-trivial finitely generated member of RA. By Corollary 6.6, $\boldsymbol{B}$ is bounded. Let $\top$ and $\bot$ denote the respective upper and lower bounds of $\boldsymbol{B}$. It is an easy consequence of Theorem 5.3 that $\{\top, \bot\}$ is the universe of a subalgebra of $\boldsymbol{B}$ that is isomorphic to $\mathbf{2}^-$. □

Clearly, when a Boolean algebra $\boldsymbol{A}$ is thought of as an integral De Morgan monoid, it has the same term operations as its e-free reduct $\boldsymbol{A}^-$, because e is definable as $x \to x$. Thus, by Theorem 5.23, the variety BA of Boolean algebras can be identified with $\mathbb{V}(\mathbf{2}^-) = \mathbb{Q}(\mathbf{2}^-)$.

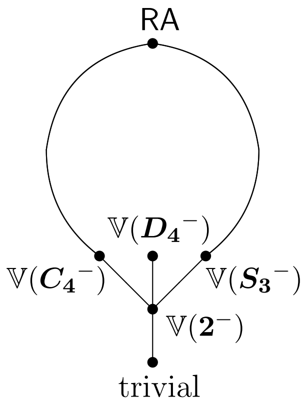
Corollary 6.8. *Let* K *be any non-trivial subquasivariety of* RA*. Then* $\mathbb{V}(\mathbf{2}^-) \subseteq \mathsf{K}$*. In other words, Boolean algebras constitute the smallest non-trivial (quasi)variety of relevant algebras.*

Proof. Let $\boldsymbol{A}$ be a non-trivial member of K. It follows from Theorem 6.7 that $\mathbf{2}^- \in \mathbb{I}\,\mathbb{S}(\boldsymbol{A}) \subseteq \mathbb{I}\,\mathbb{S}(\mathsf{K}) = \mathsf{K}$, since K is a quasivariety. □

Corollary 6.9. **CPL** *is the only maximal consistent finitary extension of* **R**.

Proof. Since **R** is algebraized by RA, its finitary extensions are algebraized, via the same definable translations, by the subquasivarieties of RA, as in Corollary 2.27. This process, applied to $\mathbb{V}(\mathbf{2}^-) = \mathsf{BA}$, yields **CPL**, by Example 2.23. The result therefore follows from Corollary 6.8, because the correspondence in Corollary 2.27 is a lattice anti-isomorphism. □

We shall show that $\mathbb{V}(\boldsymbol{S_3}^-)$, $\mathbb{V}(\boldsymbol{C_4}^-)$ and $\mathbb{V}(\boldsymbol{D_4}^-)$ are exactly the covers of $\mathbb{V}(\mathbf{2}^-)$ in the subvariety lattice of RA.



Subvarieties of RA

Lemma 6.10. $\mathbb{V}(\boldsymbol{S_3}^-)$, $\mathbb{V}(\boldsymbol{C_4}^-)$ *and* $\mathbb{V}(\boldsymbol{D_4}^-)$ *are distinct covers of* $\mathbb{V}(\mathbf{2}^-)$ *in the subvariety lattice of* RA.

Proof. Let $\boldsymbol{X}$ be one of $\boldsymbol{S_3}^-$, $\boldsymbol{C_4}^-$ or $\boldsymbol{D_4}^-$. By Lemma 6.3, the algebras $\boldsymbol{X}$ and $\mathbf{2}^-$ are simple. As in the proof of Theorem 5.30, Jónsson's Theorem allows us to conclude that every subdirectly irreducible member of $\mathbb{V}(\boldsymbol{X})$ is isomorphic to $\mathbf{2}^-$ or to $\boldsymbol{X}$. The only difference in the present argument is the fact that $\boldsymbol{X}$ has just one non-trivial proper subalgebra, and it is isomorphic to $\mathbf{2}^-$.

Since varieties are determined by their subdirectly irreducible members, there are no subvarieties of RA between $\mathbb{V}(\mathbf{2}^-)$ and $\mathbb{V}(\boldsymbol{X})$, and $\mathbb{V}(\boldsymbol{X}) \neq \mathbb{V}(\boldsymbol{Y})$, for any $\boldsymbol{Y} \in \{\boldsymbol{S_3}^-, \boldsymbol{C_4}^-, \boldsymbol{D_4}^-\}$ such that $\boldsymbol{Y} \neq \boldsymbol{X}$. □

To show that these varieties are exactly the covers of $\mathbb{V}(\mathbf{2}^-)$ within RA requires a bit more work, essentially because RA lacks the congruence extension property. We start by defining deductive filters for relevant algebras and exploring some of their properties.

For any $\boldsymbol{A} \in \mathsf{RA}$, we say that $F \subseteq A$ is a *deductive filter* of $\boldsymbol{A}$ if F is a lattice filter of $\boldsymbol{A}$ and $|a| \in F$ for all $a \in A$.

Theorem 6.11. *F is a deductive filter of $\boldsymbol{A} \in \mathsf{RA}$ if and only if F is an $\mathbf{R}$–filter of $\boldsymbol{A}$.*

Proof. Suppose that F is a deductive filter of $\boldsymbol{A}$. Let α be an axiom of $\mathbf{R}$. In particular, $\vdash_{\mathbf{R}} \alpha$. From the algebraization of $\mathbf{R}$, Corollary 4.8, $\boldsymbol{A} \models |\alpha| \preceq \alpha$, because $\boldsymbol{A} \in \mathsf{RA}$. So, for any $\boldsymbol{A}$-evaluation h, we have $|h(\alpha)| = h(|\alpha|) \leq h(\alpha)$. But $|h(\alpha)| \in F$, so $h(\alpha) \in F$.

Let h be any $\boldsymbol{A}$-evaluation. If $h(p), h(q) \in F$ then, since F is a lattice filter, $h(p \wedge q) = h(p) \wedge h(q) \in F$. Therefore, F validates adjunction. Now suppose that $h(p), h(p \to q) \in F$. $\boldsymbol{A}$ is a subalgebra of some $\boldsymbol{B} \in \mathsf{RA}$ such that $\boldsymbol{B}$ is a reduct of De Morgan monoid. In $\boldsymbol{B}$ we know that the following hold, because of Theorem 5.1(ii) and Theorem 1.56(i):

$$h(p) \wedge h(p \to q) \leq h(p) \cdot (h(p) \to h(q)) \leq h(q).$$

But then $h(q) \in F$, since F is a lattice filter and all the above are elements of $\boldsymbol{A}$.

Conversely, suppose that F is an $\mathbf{R}$–filter. To establish the fact that F is a lattice filter, we can use the same reasoning as in Theorem 5.6. Lastly, for any $a \in A$, we know that $a \to a \in F$, because of A1. □

Let $\boldsymbol{A}$ be a relevant algebra. As with De Morgan monoids, we let $\mathrm{DFil}(\boldsymbol{A})$ denote the lattice of deductive filters of $\boldsymbol{A}$ and $\mathrm{DFg}(X)$ the smallest deductive filter of $\boldsymbol{A}$ that contains X, for any $X \subseteq A$.

Theorem 6.11 implies, via Theorem 3.18, that $\mathrm{DFil}(\boldsymbol{A}) \cong \mathrm{Con}(\boldsymbol{A})$, since RA is a variety (Theorem 6.4).

Theorem 6.12. *Let $\boldsymbol{A}$ be a relevant algebra, with $a, b \in A$.*

(i) $\mathrm{DFg}^{\boldsymbol{A}}(a) = \{d \in A : a \wedge |c| \leq d$ *for some* $c \in A\}$*;*

(ii) $\mathrm{DFg}(a) \cap \mathrm{DFg}(b) = \mathrm{DFg}(a \vee b)$.

Proof. For (i), let $F = \{d \in A : a \wedge |c| \leq d$ for some $c \in A\}$. Then $a \in F$, since $a \wedge |a| \leq a$. First we show that F is a deductive filter of $\boldsymbol{A}$. Suppose that $b \in F$ and $b \leq c$. Then there exists $d \in A$ such that $a \wedge |d| \leq b$. But then $a \wedge |d| \leq c$, from which we can conclude that $c \in F$.

Suppose $b, c \in F$. Then there exist $d, d' \in A$, such that $a \wedge |d| \leq b$ and $a \wedge |d'| \leq c$. Therefore,

$$b \wedge c \geq a \wedge |d| \wedge |d'| \geq a \wedge ||d| \wedge |d'||,$$

by Theorem 6.1(ii). Consequently, $b \wedge c \in F$. Lastly, for any $b \in A$, we know $a \wedge |b| \leq |b|$, so $|b| \in F$. Therefore F is a deductive filter of $\boldsymbol{A}$.

It remains to show that F is the smallest deductive filter containing a. So, let G be a deductive filter containing a. Let $b \in F$. Then, there exists $d \in A$, such that $a \wedge |d| \leq b$. Since $a \in G$ and $|d| \in G$, we have that $a \wedge |d| \in G$. But then $b \in G$, as required.

Now we show (ii). Let $a, b \in A$. Since $a, b \leq a \vee b$, we know that $a \vee b \in \mathrm{DFg}(a)$ and $a \vee b \in \mathrm{DFg}(b)$, hence, $a \vee b \in \mathrm{DFg}(a) \cap \mathrm{DFg}(b)$.

Let $c \in \mathrm{DFg}(a) \cap \mathrm{DFg}(b)$. So, there exist $d, d' \in A$, such that $a \wedge |d| \leq c$ and $b \wedge |d'| \leq c$. In particular, $c \geq a \wedge |d| \wedge |d'|$ and $c \geq b \wedge |d| \wedge |d'|$. Therefore,

$$\begin{aligned} c &\geq (a \wedge |d| \wedge |d'|) \vee (b \wedge |d| \wedge |d'|) \\ &= (a \vee b) \wedge (|d| \wedge |d'|) \\ &\geq (a \vee b) \wedge ||d| \wedge |d'||. \end{aligned}$$

The second step follows from distributivity, and the last inequality follows from Theorem 6.1(ii). Therefore $c \in \mathrm{DFg}(a \vee b)$. □

Corollary 6.13. *The class of FSI members of* RA *is closed under subalgebras and ultraproducts.*

Proof. Let $\boldsymbol{A} \in \mathsf{RA}$. Note that the isomorphism between $\mathrm{DFil}(\boldsymbol{A})$ and $\mathrm{Con}(\boldsymbol{A})$ preserves compactness (like any isomorphism between complete lattices). Therefore $\mathrm{DFil}(\boldsymbol{A})$ is algebraic, so by Theorem 1.17, a deductive filter F is compact in $\mathrm{DFil}(\boldsymbol{A})$ if and only if it is finitely generated.

Clearly, $\mathrm{DFg}^{\boldsymbol{A}}(a_1, \ldots, a_n) = \mathrm{DFg}^{\boldsymbol{A}}(a_1 \wedge \cdots \wedge a_n)$, so every finitely generated deductive filter of $\boldsymbol{A}$ is principal (as with De Morgan monoids). Therefore, by Theorem 6.12(ii), the intersection of any two compact deductive filters is compact.

We can therefore use Theorem 1.50 to conclude that RA_{FSI} is closed under subalgebras and ultraproducts. □

Before we show that our three algebras are exactly the covers of $\mathbb{V}(\mathbf{2}^-)$, we need the following theorem about De Morgan monoids. It is a weaker version of [55, Thm. 1]. (The full version does not require the image to be simple.)

Theorem 6.14 (Slaney [55, Thm. 1])**.** *Every homomorphism from an FSI De Morgan monoid into a simple 0-generated De Morgan monoid is an isomorphism or has* $\boldsymbol{C_4}$ *as its image.*

Proof. Suppose that $\boldsymbol{A}$ is an FSI De Morgan monoid and h is a non-bijective homomorphism into a simple 0-generated algebra $\boldsymbol{B}$. Then $h[A] = B$, as

$\boldsymbol{B}$ is 0-generated. Thus, h is not injective, i.e., there exist $a, b \in A$, such that $h(a) = h(b)$ and $a \neq b$. By Theorem 1.56(ii), $e \not\leq a \leftrightarrow b$, hence $\neg(a \leftrightarrow b) \not\leq f$. But since $A = (f] \cup [e)$ (by Theorem 5.10), $e \leq \neg(a \leftrightarrow b)$.

But then $e \leq h(a) \leftrightarrow h(b)$ and $e \leq \neg(h(a) \leftrightarrow h(b))$. It follows that,

$$e \leq h(a) \leftrightarrow h(b) \leq f.$$

If $e = f$ in $\boldsymbol{B}$, then $\{e\}$ is the universe of a trivial subalgebra of $\boldsymbol{B}$. So, since $\boldsymbol{B}$ is 0-generated, $\boldsymbol{B}$ is trivial, contradicting the simplicity assumption. Therefore, $e < f$ in $\boldsymbol{B}$. From Theorem 5.24, we know that the only simple 0-generated De Morgan monoid which satisfies this property is (isomorphic to) $\boldsymbol{C_4}$. □

Lemma 6.15. *Let* K *be a subvariety of* RA*, not consisting entirely of Boolean algebras. Then there exists* $\boldsymbol{X} \in \{\boldsymbol{S_3}^-, \boldsymbol{C_4}^-, \boldsymbol{D_4}^-\}$ *such that* $\mathbb{V}(\boldsymbol{X}) \subseteq \mathsf{K}$*.*

Proof. By Birkhoff's Subdirect Decomposition Theorem, there exists $\boldsymbol{A} \in \mathsf{K}$ such that $\boldsymbol{A}$ is non-Boolean and subdirectly irreducible.

By Theorem 1.33, $\boldsymbol{A}$ can be embedded into an ultraproduct of finitely generated non-trivial subalgebras of itself. If each of these subalgebras were Boolean, then $\boldsymbol{A}$ would be Boolean, a contradiction. We may therefore choose a non-Boolean finitely generated subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$. Note that $\boldsymbol{B}$ is FSI, by Corollary 6.13, and non-trivial. Then there exists $\boldsymbol{B}^+ \in \mathsf{DMM}$ such that $\boldsymbol{B}$ is a reduct of $\boldsymbol{B}^+$, by Theorem 6.2. Now $\boldsymbol{B}^+$ is finitely generated, finitely subdirectly irreducible (by Lemma 6.3) and non-trivial.

Notice that, by (6.1),

$$\mathbb{V}(\boldsymbol{B}^+)^- \subseteq \mathbb{V}(\boldsymbol{B}) \subseteq \mathbb{V}(\boldsymbol{A}) \subseteq \mathsf{K}.$$

Since the varieties generated by $\mathbf{2}, \boldsymbol{S_3}, \boldsymbol{C_4}$ and $\boldsymbol{D_4}$ are minimal subvarieties of DMM, by Theorem 5.30, at least one of these algebras is a member of $\mathbb{V}(\boldsymbol{B}^+)$. If any of $\boldsymbol{S_3}, \boldsymbol{C_4}$ or $\boldsymbol{D_4}$ is a member of $\mathbb{V}(\boldsymbol{B}^+)$, then we are done.

Now consider the case when $\mathbf{2} \in \mathbb{V}(\boldsymbol{B}^+)$. By Jónsson's Theorem, $\mathbf{2} \in \mathbb{H}\,\mathbb{S}\,\mathbb{P}_{\mathrm{u}}(\boldsymbol{B}^+)$, since $\mathbf{2}$ is subdirectly irreducible. This means that there exists an algebra $\boldsymbol{C}$ which has $\mathbf{2}$ as a homomorphic image, and can be embedded into an ultrapower $\prod \boldsymbol{B}^+/\mathcal{U}$. By Corollary 6.13, $\boldsymbol{C}$ is FSI. But then, by Theorem 6.14, $\boldsymbol{C} \cong \mathbf{2}$, because $\mathbf{2}$ simple and 0-generated. So $\mathbf{2}$ embeds into $\prod \boldsymbol{B}^+/\mathcal{U}$. In particular, $\prod \boldsymbol{B}^+/\mathcal{U}$ is idempotent, by Theorem 5.15, since $f \leq e$ in $\mathbf{2}$. Then by Łoś' Theorem (Theorem 1.35), $\boldsymbol{B}^+$ is idempotent. Since $\boldsymbol{B}^+$ is also non-Boolean, Theorem 5.22 tells us that $\boldsymbol{B}^+$ has a homomorphic image $\boldsymbol{D}$ which is a non-trivial odd Sugihara monoid. In particular, $\mathbb{V}(\boldsymbol{D})$ has a simple finitely generated (odd) member, by Corollary 5.26, which must be isomorphic to $\boldsymbol{S_3}$, by Corollary 5.21. So,

$$\boldsymbol{S_3}^- \in \mathbb{V}(\boldsymbol{D})^- \subseteq \mathbb{V}(\boldsymbol{B}^+)^- \subseteq \mathsf{K}. \quad \square$$

We can now infer the following result of Świrydowicz [59].

Theorem 6.16. *$\mathbb{V}(\boldsymbol{S_3}^-), \mathbb{V}(\boldsymbol{C_4}^-)$ and $\mathbb{V}(\boldsymbol{D_4}^-)$ are exactly the covers of $\mathbb{V}(\mathbf{2}^-)$ in the subvariety lattice of* RA.

Proof. This follows from Lemma 6.10 and Lemma 6.15. □

Corollary 6.17. *The sentential deductive systems corresponding to $\mathbb{V}(\boldsymbol{S_3}^-)$, $\mathbb{V}(\boldsymbol{C_4}^-)$ and $\mathbb{V}(\boldsymbol{D_4}^-)$ are exactly the three maximal non-classical axiomatic extensions of* **R**.

6.2 Structural Completeness in R

In this last section we present a recent result by Raftery and Świrydowicz, in [52], where they characterize the *structurally complete* axiomatic extensions of **R**. As their proof relies heavily on Theorem 6.16, which was the main result of the previous section, it is of interest to recount it here. This result also suggests an avenue for future research on $\mathbf{R^t}$ and De Morgan monoids.

A finite rule Γ/α in the language of a deductive system $\vdash$ is said to be *admissible* in $\vdash$ provided that, for every substitution h,

$$\text{if } \vdash h(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then } \vdash h(\alpha).$$

Clearly all finite derivable rules of a deductive system are admissible, but the converse need not hold. For example, Meyer and Dunn [42] proved that, whenever α and $\neg\alpha \vee \beta$ are theorems of **R**, then so is β. The rule $p, \neg p \vee q/q$, which is called the *disjunctive syllogism*, is therefore admissible in **R**. It is not derivable in **R**, however, as witnessed by $\boldsymbol{C_4}^-$, evaluating p to e and q to $\neg(f^2)$.

Definition 6.18. A finitary deductive system $\vdash$ is said to be *structurally complete* if it contains all of its admissible rules.

Equivalently, $\vdash$ is structurally complete if and only if each of its proper finitary extensions has some new theorem—as opposed to having nothing but new rules of derivation. This property has an algebraic characterization if a deductive system is algebraizable. Its statement is anticipated in [4]; a proof can be found in [45, Sec. 7].

Theorem 6.19. *A finitary deductive system algebraized by a variety* K *is structurally complete if and only if every proper subquasivariety of* K *generates a proper subvariety of* K.

Notice that **R** is not structurally complete, as the disjunctive syllogism is not derivable in **R**. It is then natural to ask which axiomatic extensions of **R** are structurally complete. The following theorem provides a satisfying answer, especially since structural completeness is not generally inherited by subvarieties [4, Ex. 2.14.4].

Theorem 6.20 (Raftery & Świrydowicz [52, Thm. 4])**.** *No consistent axiomatic extension of* **R** *is structurally complete, except for* **CPL**.

Proof. Let K be a non-Boolean subvariety of RA. Then $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$, where $\boldsymbol{A}$ is a free $\aleph_0$-generated algebra in K, by Theorem 1.44. By Theorem 6.16, K includes an algebra $\boldsymbol{B}$ that is one of $\boldsymbol{S_3}^-$, $\boldsymbol{C_4}^-$ or $\boldsymbol{D_4}^-$. Furthermore, by Corollary 6.8, $\mathbf{2}^- \in \mathsf{K}$. So if we let $\boldsymbol{C} \coloneqq \boldsymbol{A} \times \boldsymbol{B} \times \mathbf{2}^-$, then $\boldsymbol{C} \in \mathsf{K}$. Because $\mathbb{V}(\boldsymbol{C})$ is closed under homomorphic images, $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{C}) \subseteq \mathbb{V}(\boldsymbol{C})$, whence $\mathsf{K} = \mathbb{V}(\boldsymbol{C})$, because $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$.

We claim that $\mathsf{K} \neq \mathbb{Q}(\boldsymbol{C})$. This will follow if $\boldsymbol{B} \notin \mathbb{Q}(\boldsymbol{C})$. As $\boldsymbol{B}$ is simple, and hence subdirectly irreducible, it suffices, by Lemma 1.51, to show that $\boldsymbol{B}$ can't be embedded into an ultrapower of $\boldsymbol{C}$. Now, $\boldsymbol{B}$ is a finite algebra of finite type, so the attribute of lacking a subalgebra isomorphic to $\boldsymbol{B}$ is first-order definable. Thus, by Łoś' Theorem, $\boldsymbol{B}$ won't embed into an ultrapower of $\boldsymbol{C}$ unless it embeds into $\boldsymbol{C}$ itself.

Suppose, with a view to contradiction, that h is an embedding from $\boldsymbol{B}$ into $\boldsymbol{C}$. Let p be the projection from $\boldsymbol{C}$ onto $\mathbf{2}^-$. The homomorphism $p \circ h$ maps e either to e or to f.

Notice that for all three possible choices of $\boldsymbol{B}$, $f \to e \leq f$. So, since homomorphisms preserve order, $\neg ph(e) \to ph(e) \leq \neg ph(e)$. In the first case where $ph(e) = e$, we have that in $\mathbf{2}^-$, $f \to e \leq f$, but

$$f \to e = \neg(f \cdot f) = \neg(f \wedge f) = \neg f = e.$$

Then $e \leq f$ in $\mathbf{2}^-$, a contradiction.

In the second case, where $ph(e) = f$,

$$ph(e) \to ph(e) = f \to f = \neg f = e.$$

But in $\boldsymbol{B}$, we have $e \to e = e$, so $e = f$ in $\mathbf{2}^-$ a contradiction.

Therefore, $\boldsymbol{B}$ cannot be embedded into $\boldsymbol{C}$, which vindicates the claim that $\mathbb{Q}(\boldsymbol{C}) \neq \mathsf{K} = \mathbb{V}(\boldsymbol{C})$. Thus, $\mathbb{Q}(\boldsymbol{C})$ is a proper subquasivariety of K

which fails to generate a *proper* subvariety of K. That completes the proof, in view of Theorem 6.19. □

The argument of Meyer and Dunn can also be used to show that the disjunctive syllogism is admissible in $\mathbf{R^t}$ (see for instance [43]). In particular, $\mathbf{R^t}$ is not structurally complete. It would be an interesting research question to determine the structurally complete axiomatic extensions of $\mathbf{R^t}$. This open problem is the topic of an ongoing investigation by T. Moraschini, J. Raftery and the author. The situation for $\mathbf{R^t}$ is known to be more complicated than for $\mathbf{R}$. For example, it is known that there are infinitely many structurally complete axiomatic extensions of the logic $\mathbf{RM^t}$. They include all the varieties of odd Sugihara monoids; see for instance [26, Thm. 7.3]. It is possible that our new understanding of the bottom part of the subvariety lattice of DMM, in Theorem 5.30, could give us some insight into this question.

Bibliography

[1] J. Adámek, *How many variables does a quasivariety need?*, Algebra Universalis **27** (1990), 44–48.

[2] A. R. Anderson, N. D. Belnap, 'Entailment: The Logic of Relevance and Necessity, Vol. I', Princeton University Press, 1975.

[3] N. D. Belnap, *Entailment and Relevance*, Journal of Symbolic Logic **25** (1960), 144–146.

[4] C. Bergman, 'Structural completeness in algebra and logic', Algebraic Logic, ed. by H Anderéka, J. D. Monk, I Nemeti, Colloquia, vol. 54, North-Holland, Amsterdam, 1991, pp. 59–73.

[5] C. Bergman, 'Universal Algebra', CRC Press, New York, 2012.

[6] G. Birkhoff, *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society **29** (1935), 433–454.

[7] G. Birkhoff, *Subdirect unions in universal algebra*, Bulletin of the American Mathematical Society **50** (1944), 764–768.

[8] W. J. Blok, B. Jonsson, *Algebraic structures for logic*, New Mexico State University, 1999.

[9] W. J. Blok, B. Jonsson, *Equivalence of consequence operations*, Studia Logica **83** (2006), 91–110.

[10] W. J. Blok, D. Pigozzi, *A finite basis theorem for quasivarieties*, Algebra Universalis **22** (1986), 1–13.

[11] W. J. Blok, D. Pigozzi, 'Algebraizable Logics', *Memoirs of the American Mathematical Society*, vol. 396, American Mathematical Society, Providence, 1989.

[12] W. J. Blok, D. Pigozzi, *Abstract algebraic logic and the deduction theorem*, manuscript, 1997. [See http://orion.math.iastate.edu/dpigozzi/ for updated version, 2001.]

[13] S. Burris, H. P. Sankappanavar, 'A Course in Universal Algebra', Springer-Verlag, New York, 1981.

[14] J. Czelakowski, 'Protoalgebraic Logics', Springer Netherlands, Dordrecht, 2001.

[15] J. Czelakowski, W. Dziobiak, *Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class*, Algebra Universalis **27** (1990), 128–149.

[16] J. Czelakowski, W. Dziobiak, *Deduction theorems within RM and its extensions*, Journal of Symbolic Logic **64** (1999), 279–290.

[17] A. De Morgan, *On the syllogism, no. IV, and on the logic of relations*, Transactions of the Cambridge Philosophical Society **10** (1864), 331–358.

[18] K. Došen, *The first axiomatization of relevant logic*, Journal of Philosophical Logic **21** (1992), 339–356.

[19] J. M. Dunn, "The Algebra of Intensional Logics", PhD thesis, University of Pittsburgh, 1966.

[20] J. M. Dunn, G. Restall, 'Relevance logic and entailment', Handbook of Philosophical Logic, Vol 8, ed. by D Gabbay, 2nd ed., Kluwer, Dordrecht, 2001, pp. 1–128.

[21] W. Dziobiak, *There are* $2^{\aleph_0}$ *logics with the relevance principle between R and RM*, Studia Logica **43** (1983), 49–61.

[22] J. M. Font, 'Abstract Algebraic Logic : An Introductory Textbook', College Publications, London, 2016.

[23] J. M. Font, R. Jansana, D. Pigozzi, *A survey of abstract algebraic logic*, Studia Logica **74** (2003), 13–97.

[24] J. M. Font, R. Jansana, D. Pigozzi, *Update to "A survey of abstract algebraic logic"*, Studia Logica **91** (2009), 125–130.

[25] N. Funayama, T. Nakayama, *On the distributivity of a lattice of lattice-congruences*, Proceedings of the Japan Academy **18** (1942), 553–554.

[26] N. Galatos, J. G. Raftery, *A category equivalence for odd Sugihara monoids and its applications*, Journal of Pure and Applied Algebra **216** (2012), 2177–2192.

[27] N. Galatos, C. Tsinakis, *Equivalence of consequence relations: An order-theoretic and categorical perspective*, Journal of Symbolic Logic **74** (2009), 780–810.

[28] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, 'Residuated Lattices: An Algebraic Glimpse at Substructural Logics', Studies in Logic and the Foundations of Mathematics 151, Elsevier, 2007.

[29] G. Grätzer, 'Universal Algebra', 2nd ed., Springer-Verlag, New York, 1979.

[30] G. Grätzer, H. Lakser, *A note on the implicational class generated by a class of structures*, Canadian Mathematical Bulletin **16** (1973), 603–605.

[31] J. Hart, L. Rafter, C. Tsinakis, *The structure of commutative residuated lattices*, International Journal of Algebra and Computation **12** (2002), 509–524.

[32] B. Jónsson, *Algebras whose congruence lattices are distributive*, Mathematica Scandinavica **21** (1967), 110–121.

[33] B. Jónsson, 'Topics in Universal Algebra', Lecture Notes in Mathematics, Vol. 250, Springer-Verlag, Berlin/New York, 1972.

[34] B. Jónsson, *Congruence distributive varieties*, Mathamatica Japonica **42** (1995), 353–401.

[35] J. Łoś, R. Suszko, *Remarks on sentential logics*, Indagationes Mathematicae (Proceedings) **61** (1958), 177–183.

[36] L. L. Maksimova, *An interpretation and separation theorems for the logical systems E and R*, Algebra and Logic **10** (1971), 232–241.

[37] L. L. Maksimova, *Implication lattices*, Algebra and Logic **12** (1973), 249–261.

[38] A. I. Mal'cev, *On the general theory of algebraic systems*, Matematicheskii Sbornik **35** (1954), 3–20.

[39] A. I. Mal'cev, *Several remarks on quasivarieties of algebraic systems*, Algebra i Logika **5** (1966), 3–9.

[40] R. K. Meyer, *A note on* $R_{\rightarrow}$ *matrices*, Notre Dame Journal of Formal Logic **24** (1983), 450–472.

[41] R. K. Meyer, *Sentential constants in* R *and* $R^{\neg}$, Studia Logica **45** (1986), 301–327.

[42] R. K. Meyer, J. M. Dunn, *E , R , and* γ, Journal of Symbolic Logic **34** (1969), 460–474.

[43] R. K. Meyer, J. M. Dunn, H. Leblanc, *Completeness of relevant quantification theories*, Notre Dame Journal of Formal Logic **15** (1974), 97–121.

[44] R. K. Meyer, R. Routley, *Algebraic analysis of entailment, I*, Logique et Analyse N.S. **15** (1972), 407–428.

[45] J. S. Olson, J. G. Raftery, C. J. Van Alten, *Structural completeness in substructural logics*, Logic Journal of IGPL **16** (2008), 453–495.

[46] I. E. Orlov, *The calculus of compatibility of propositions*, Matematicheskii Sbornik **35** (1928), 263–286.

[47] V. R. Pratt, 'Origins of the calculus of binary relations', *Proceedings of the Seventh Annual IEEE Symposium on Logic in Computer Science*, Santa Cruz, CA, 1992, pp. 248–254.

[48] J. G. Raftery, *The equational definability of truth predicates*, Reports on Mathematical Logic **41** (2006), 95–149.

[49] J. G. Raftery, *A perspective on the algebra of logic*, Quaestiones Mathematicae **34** (2011), 275–325.

[50] J. G. Raftery, *Contextual deduction theorems*, Studia Logica **99** (2011), 279–319.

[51] J. G. Raftery, C. J. van Alten, *Rule separation and embedding theorems for logics without weakening*, Studia Logica **76** (2004), 241–274.

[52] J. G. Raftery, K. Świrydowicz, *Structural completeness in relevance logics*, Studia Logica **104** (2016), 381–387.

[53] R. Routley, R. K. Meyer, 'The Semantics of Entailment', Truth, Syntax and Modality, ed. by H. Leblanc, North-Holland, Amsterdan, 1973, pp. 199–243.

[54] J. K. Slaney, *3088 varieties a solution to the Ackermann constant problem*, Journal of Symbolic Logic **50** (1985), 487–501.

[55] J. K. Slaney, *On the structure of De Morgan monoids with corollaries on relevant logic and theories*, Notre Dame Journal of Formal Logic **30** (1989), 117–129.

[56] J. K. Slaney, *The Ackermann constant theorem: A computer-assisted investigation*, Journal of Automated Reasoning **7** (1991), 453–474.

[57] J. K. Slaney, *Sentential constants in systems near R*, Studia Logica **52** (1993), 443–455.

[58] J. K. Slaney, R. K. Meyer, *Logic for two: The semantics of distributive substructural logics*, Qualitative and Quantitative Practical Reasoning (1997), 554–567.

[59] K. Świrydowicz, *A remark on the maximal extensions of the relevant logic R*, Reports on Mathematical Logic **29** (1995), 19–33.

[60] K. Świrydowicz, *There exists exactly two maximal strictly relevant extensions of the relevant Logic R*, Journal of Symbolic Logic **64** (1999), 1125–1154.

[61] A. Tarski, *A remark on functionally free algebras*, Annals of mathematics **47** (1946), 163–165.

[62] A. Urquhart, *Semantics for relevant logics*, Journal of Symbolic Logic **37** (1972), 159–169.

[63] A. Urquhart, *Relevant implication and projective geometry*, Logique et Analyse **26** (1983), 345–357.

[64] A. Urquhart, *The undecidability of entailment and relevant implication*, Journal of Symbolic Logic **49** (1984), 1059–1073.

[65] A. Urquhart, *Duality for algebras of relevant logics*, Studia Logica **56** (1996), 263–276.

Symbols and Abbreviations

Index