Magnetohydrodynamic Turbulent Flows for Viscous Incompressible Fluids Through the Lenses of Harmonic Analysis

by

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Declaration

I, Tesfalem Abate Tegegn declare that the thesis, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature:

Name: Tesfalem Abate Tegegn

Date: 15 November, 2016

IN LOVING MEMORY OF MY LATE GRAND FATHER, ABYEA AND MY LATE COUSIN, HABTAMU.

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Abstract

This thesis is divided into three main parts devoted to the study of magnetohydrodynamics (MHD) turbulence flows.

Part I consists of introduction and background (or preliminary) materials which were crucially important in the process. The main body of the thesis is included in parts II and III.

In Part II, new regularity results for stochastic heat equations in probabilistic evolution spaces of Besov type are established, which in turn were used to establish global and local in time existence and uniqueness results for stochastic MHD equations. The existence result holds with positive probability which can be made arbitrarily close to one. The work is carried out by blending harmonic analysis tools, such as Littlewood-Paley decomposition, Jean-Micheal Bony paradifferential calculus and stochastic calculus. Our global existence result is new in three-dimensional spaces and is published in [148](Sango and Tegegn, *Harmonic analysis tools for stochastic magnetohydrodynamics equations in Besov spaces*, International Journal of Modern Physics B, World Scientific, 2016, **30**). Our results in this part are novel; they introduced Littlewood-Paley theory and paradifferential calculus for stochastic partial differential equation.

In Part III, we studied Kolmogorov's spectral theory for MHD equations with reasonably smooth external forces applied to both velocity and magnetic fields. It was

shown that, the spectral energy function of our MHD system, given by

$$E(k,t) = \int_{|\xi|=k} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t|^2) \mathrm{d}S(\xi), \quad k \in [0,\infty),$$

satisfies the Kolmogorov's 5/3 law over a range of wave numbers, say $[\bar{k}_1, \bar{k}_2]$. We have also established bounds for the spectral energy function, explicitly calculated value for the inertial range, minimum possible rate of dissipation and maximum possible time so that the MHD flow exhibits Kolmogorov's phenomenon. These results are new in the framework of magnetohydrodynamic turbulent flows.

Contents

Part I

Introduction and Background Materials

Chapter 1

Introduction and Main Results

1.1 Introduction

According to Dieter Biskamp, in [19], magnetohydrodynamics (MHD) is a macroscopic behavior of flow of electrically conducting fluids such as plasma, liquid metal and electrolytes. The basic principle behind MHD is that moving electric charges produce magnetic field and a changing magnetic field induces electric current; hence a magnetic field can induce current in a moving conductive fluid, and the induced electric current in turn acts on the magnetic field forcing it to vary, and so on.

The human civilization has had the basic knowhow on the dynamics of fluids since antiquities, [129, Ch. 1], and the rules of electromagnetism have been well known from the early 19th century (see [85]) so that one can foresee MHD phenomenon. For instance, in 1832, M. Faraday did an experiment on Thames river, London, to measure the induced electromotive force (emf) due to the flow of its salty water and André-Marie Ampère also did an experiment on liquid mercury for same purpose. Both of the experiments were not successful in detecting the phenomenon; the first is mostly due to the incapability of the measuring device to respond to very small data and the latter was due to the very low conductivity of mercury, see [114].

The term magnetohydrodynamics was first used by Hannes Alfvén in his celebrated work [4] where he formulated the basic principles of magnetohydrodynamics. Even if he was not the first person to realize the phenomenon, due to his breakthrough results and contributions, Alfvén was known as the father of magnetohydrodynamics. Some of his finest works are [4, 5, 6, 7, 8, 9]. Therefore, one may consider MHD as the youngest child of fluid dynamics and electromagnetism which came to life in the midst twentieth century. We refer to [114, 125] for a good reading on the evolution and development of MHD theory.

As mentioned earlier, MHD as a phenomenon involving electromagnetism and fluid dynamics; it is, therefore, clear that MHD models can be derived by combining the rules of fluid dynamics such as Navier-Stokes equations (1.1), and the laws of electromagnetism, or Maxwell equations (1.2). Navier-Stokes equations for incompressible fluid flows are given by

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \\ \mathrm{div} u = 0, \\ u(\cdot,0) = u_0, \end{aligned} \tag{1.1}$$

where u is the velocity field, p is the pressure and μ is viscosity of the fluid, and Maxwell equations of electromagnetism are given by

$$\begin{aligned} \oint_S \vec{E}\cdot\hat{n}\mathrm{d}S = \tfrac{q_{enc}}{\varepsilon_0}, \\ \oint_S \vec{b}\cdot\hat{n}\mathrm{d}S = 0, \\ \oint_C \vec{E}\cdot\mathrm{d}\vec{l} = \tfrac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{b}\cdot\hat{n}\mathrm{d}S, \\ \oint_C \vec{b}\cdot\mathrm{d}\vec{l} = \mu_0\left(I_{enc} + \varepsilon_0\tfrac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{E}\cdot\hat{n}\mathrm{d}S\right), \end{aligned} \tag{1.2}$$

where $\vec{E}$ stands for electric field in a system, S is a surface enclosing a portion of interest, $\hat{n}$ is an outward unit normal vector to surface S, q_{enc} is the amount of charge enclosed by S, ε_0 is the electric permittivity of free space, $\vec{b}$ is the magnetic field, and I_{enc} is the enclosed current. We refer to [10, 36, 105, 108, 109, 159] for a thorough reading on Navier-Stokes equations and [63, 66, 84, 139] on Maxwell equations.

In the most ideal case of MHD flow, the fluid is assumed to have so little resistivity that it is treated as a perfect conductor, see [70, 76]. Depending on area of specialization of different authors, one may find several formulations for MHD models; see for instance [19, 25, 59]. The standard MHD model for density independent incompressible fluids which is universally acceptable among the mathematics community is given by

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla\Pi - (b\cdot\nabla)b - \nu\Delta u &= f_1, && (0,\infty)\times D,\\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u - \eta\Delta b &= f_2, && (0,\infty)\times D,\\ \operatorname{div} u = \operatorname{div} b &= 0, && D,\\ u|_{t=0} = u_0, \quad b|_{t=0} &= b_0, && D, \end{aligned} \tag{1.3}$$

where $u = u(t,x)$ is the flow velocity, $b = b(t,x)$ is the magnetic field, Π is the total pressure, $\nu > 0$ is the kinetic viscosity of the fluid, $\eta > 0$ is the resistivity of the fluid and D is the spatial domain which will be elaborated in chapter 4.

In the present work, we are concerned with the standard model (1.3) and its stochastic version, given by (1.4) which constitutes chapter 3.

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla\Pi - (b\cdot\nabla)b - \Delta u &= g_1\dot{W}, && \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n,\\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u - \Delta b &= g_2\dot{W}, && \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n,\\ \operatorname{div} u = \operatorname{div} b &= 0, && \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n,\\ u|_{t=0} = u_0, \quad b|_{t=0} &= b_0, && \text{in } \Omega\times\mathbb{R}^n, \end{aligned} \tag{1.4}$$

where $u = u(\omega,t,x)$ is the flow velocity, $b = b(\omega,t,x)$ is the magnetic field, Π is the total pressure, and $g_1\dot{W}$ and $g_2\dot{W}$ are random external forces; W is an infinite dimensional Wiener process where its components are independently, identically distributed one dimensional Wiener processes and the stochastic differential is understood in the Itô sense. One can find a more general formulation of the model (1.4) in [147].

This thesis has three main parts. The first part consists of introduction and preliminary background material where we include all the necessary results on Fourier transform, Littlewood-Paley theory, stochastic analysis, real analysis and so on.

In the second part of the thesis we initiate the study of a class of stochastic MHD equations, particularly of the type (1.4), in Besov and Sobolev spaces by developing the necessary framework; which can also be used for the study of models such as stochastic Navier-Stokes equations, Schrödinger equations and even for a more general family of stochastic partial differential equations.

Indeed, stochastic partial differential equations are natural extensions of deterministic partial differential equations and have proven their effectiveness in studying processes which involve a random phenomenon and noise. For instance, Navier-Stokes equations have long been used to investigate turbulent flows, which up to now, is not understood completely. In the process of addressing the issue, stochastic Navier-Stokes equations are introduced by Bensoussan and Temam in [14]. Ever since their introduction, stochastic Navier-Stokes equations attracted lots of attentions as an alternative means to address the issue of turbulence in fluid flows, see [13, 14, 15, 16, 22, 55, 65, 72, 98, 116, 117, 120, 121, 140, 146, 147, 166, 167, 168], just to cite few.

However, due to the complexity of the overall process, turbulent flows are not completely understood until now and remained to be one of the most difficult problems of our millennium, [29]. Therefore, one may expect MHD turbulence to be even more complex phenomenon than the hydrodynamic turbulence due to the involvement of the electromagnetic component. It is, therefore, expected to update models of MHD so that it goes along with the developments in Navier-Stokes equations. For instance, Zeldovich, Ruzmaikin, and Sokiloff used the approach to numerically model MHD turbulent flows, see [172, p. 188].

Stochastic MHD equations have attracted a considerable attention. The following works in this direction are of great importance to our purpose; Sundar in [154] established existence and uniqueness result of a mild solution for two dimensional stochastic MHD model in the presence of multiplicative noise or additive fractional Brownian noise, Sango in [147] presented a very detailed investigation, using the Galerkin approximation, on the problem of existence of weak solutions for three dimensional stochastic MHD model with multiplicative noises, Deugoué et al. in

[56] proved existence of weak solution for three dimensional stochastic MHD alpha model, and Sritharan and Sundar in [153] proved existence and uniqueness of space time statistical solutions by means of weak convergence method. Recently, Motyl in [128], and Tan et al. also considered the three dimensional stochastic MHD with multiplicative noise. In [155] they used the contraction mapping principle to establish existence and uniqueness of strong local solution and strong global solution with small data.

In this work, we use a completely different approach which blends harmonic analysis tools such as Littlewood-Paley theory, Jean-Michel Bony Paradifferential calculus and stochastic calculus. Our main concern is to deal with the question of existence and uniqueness of strong solutions (in a probabilistic sense) in suitable spaces. We established local existence and uniqueness of a strong (in probabilistic sense) and large time unique solution with small data.

We first reduce (1.4) to a more symmetric form by introducing a relevant transformation. Then we drop the pressure term by applying the Leray projector expressed in terms of Riesz transforms. Finally, we study the reduced problem by seeking a solution that can be written as a sum of solutions of systems of heat equations of type (1.5) and (1.6) given by

$$\begin{aligned} \mathrm{d}v - \Delta v \mathrm{d}t &= f \mathrm{d}W_t, \\ v|_{t=0} &= v_0, \end{aligned} \tag{1.5}$$

$$\begin{aligned} \tilde{v}\mathrm{d}t - \Delta \tilde{v} \mathrm{d}t &= \tilde{f} \mathrm{d}t, \\ \tilde{v}|_{t=0} &= 0. \end{aligned} \tag{1.6}$$

such that v, f, $\tilde{v}$, $\tilde{f}$ defined in appropriate domain with certain conditions imposed on initial datum, v_0, f, $\tilde{f}$. The stochastic heat equation is studied by making use of Littlewood-Paley theory and Itô's calculus. To the best of our knowledge, our approach to the stochastic heat equation is also new both in methodology and result. In the deterministic case, this approach is proven to be very handy way to address issues of existence and uniqueness both for Navier-Stokes and MHD equations, see [2, 23, 25, 30, 34, 37, 38, 47, 48, 49, 50, 173], to cite few. Indeed [10, 25, 37, 147, 173] have greatly influenced our work to take this direction.

On the other hand, turbulence theory is built up on statistical methods, such as averaging of a flow process, are of greater importance in the practical use like in engineering applications and weather forecast. The famous statement of Leonardo da Vinci as quoted in [58]; " Observe the motion of a surface of water, whcih resembles that of hair, which as two motions, of which one is caused by the weight of the hair and the other by the direction of of the furls; thus the water ha eddying motions, one part of which is due to the principal current, the other to the random and reverse motion" can be taken as an evidence to argue that the Italian genius was the first to insight the averaging method which later was advanced by Osborne Reynolds in [141, 142]. In his work, Reynolds divided fluid motion into two components; the mean flow and fluctuating flow, leading to the system of equations called Reynolds-Averaged-Navier-Stokes equations which are of greater in importance in engineering applications. He is also known for a setting a criterion which measures velocity fluctuations between two locations in a stream of flow, which later named after him, called Reynolds numbers, usually denoted as Re. Generically Reynold's numbers are used as a standard tool for labeling whether a flow is turbulent or laminar; accordingly, a flow is laminar if $Re < 1900$ and turbulent if $Re > 2000$, see [142]. Following the progress by Reynold, Richardson crafted a cascade theory, which he summarized it in passion[1].

A very important development in this direction has come into light due to Kolmogorov and his team in the early 1940's, see [91, 92, 93, 94, 131]. In these works they developed a theory which explains how energy dissipates and turbulent flow decays into laminar flow. The theory is based on a series of hypothesis, called Kolmogorov's hypothesis, which enabled them to study the phenomenon over a range of scales, such as energy-containing range, inertial range and dissipation range, see [136, Ch. 6]. In the larger scale (or energy scale) the geometry of the large eddies

[1]Big whirls have little whirls, that feed on their velocity. Little whirls have lesser whirls and so on to viscosity- in the molecular sense. ([51, p. 199],[136, P. 183])

determine the flow. The geometry of large eddies is determined by the mean flow of the velocity field and the boundary conditions. In the small scales, which encompasses the inertial range and the dissipation range; flow (motion) in the inertial range is determined by inertial effects while the effect of viscosity is negligible and finally in the smallest scale, i.e. dissipation range, viscosity takes responsibly to shape the flow.

The very important issue at this point is the rate of energy decay in the inertial range. Kolmogorov used dimensional analysis to come up with an explicit formulation, also known as the-5/3 spectral law, which states that in the inertial range energy decays proportional to $k^{-5/3}$ over an inertial range of, say, $k_1 \leq k \leq k_2$. Several researchers have focused on this topic in order to investigate the validity of the Kolmogorov theory, see [71, 126, 127] and the references therein.

It is worth noting that Kolmogorov's theory is not devised for MHD turbulent flows as it has not taken a magnetic field into consideration.

Our concern in the third part of this thesis is to investigate Kolmogorov's theory for magnetohydrodynamics flows governed by equations of the type (1.3). In fact, since the mid of 20th century several works on the energy spectral function for MHD equations has been done. From the earliest works one can mention [57, 86, 99, 100]. Indeed, these works lead to a phenomenological theory, which Verma in [165] mentioned it as KID phenomenon. In the KID phenomenon, the spectral energy decays proportional to $k^{-3/2}$ unlike Kolmogorov's phenomenon. However, later investigations suggest that over the inertial range of MHD turbulent flow the spectral energy decay agrees more closely to Kolmogorov than KID; for a good reading on this issue, we refer to [18, 164, 165] and the references therein. Further discussion in this area is given in chapter 4. For a very detail and elaborative reading on MHD turbulent flow we refer to [19, 21, 114].

Therefore, our duty in this part of the thesis is to investigate the spectral behavior of a general MHD flow rigorously through mathematical techniques such as harmonic analysis. The work is motivated by the 2012 paper of Biryuk and Craig, [17] where they used harmonic analysis techniques to investigate the spectral behavior of flows governed by Navier-Stokes equations. In our work, we have given a detailed analysis on energy dissipation rate and the bounds of the spectral range.

As mentioned earlier, in this thesis we address two issues; the first one is existence and uniqueness of solution for the stochastic MHD system, (1.4), and the second one is investigating Kolmogorov's spectral behavior of the deterministic MHD system, (1.3).

1.2 Main Results

1.2.1 Existence and uniqueness results for Stochastic MHD

The problem we have considered in chapter 3 is the system of stochastic MHD equations, given by (1.4). In this work we have proved existence and uniqueness of strong global solution (in a probabilistic sense) (1.3) with small data and local in time strong solutions in certain spaces. The work is done by blending harmonic analysis such as Littlewood-Paley theory, Jean-Michel Bony paradifferential calculus and stochastic calculus, such as Itô integral. Our result is based on estimates on stochastic heat equations. These estimates are new both in result and approach. We broadly used Littlewood-Paley theory, Burkholder-Davis-Gundy inequality, Itô calculus, stopping time and other stochastic estimates to get our a priori estimates.

The proof of our result is based on the fixed point argument and Bony paradifferential calculus. Bony's paradifferential calculus is proved to be an essential tool to study nonlinear partial differential equations, such as Navier-Stokes equations and MHD equations, see [2, 10, 24, 25, 30, 34, 35, 39, 40, 41, 47, 48, 74, 75, 108, 109] to cite just few of them. The very advantage of this method is that, the application of Littlewood-Paley decomposition gives the freedom to treat distributions and smooth functions to the same standard.

However, we have not come across any work where Littlewood-Paley theory and

Bony paradifferential calculus are used to tackle stochastic models of Navier-Stokes equations or MHD equations [2] To take advantage of the niceness of this method, we need a pathwise estimate for the solutions of stochastic heat equation of the type

$$\mathrm{d}v - \Delta v \mathrm{d}t = f \mathrm{d}W,$$
$$v|_{t=0} = v_0,$$

defined in appropriate domain with certain essential conditions imposed on the initial data, v_0 and f. This is done by making use of Tchebychev's inequality. The pathwise estimate holds with a positive probability close to one; indeed the probability can be made as close enough to 1 as one desires.

We have two main results; the first result is on existence and uniqueness of local in time strong solution (in probabilistic sense) and the second result is on existence and uniqueness of global in time mild solution for (1.4). The global existence and uniqueness result is published in [148].

Definitions of terminologies appearing in these results and their details are given in chapter 3.

Theorem 1. *Given a probability basis* $(\Omega, \mathscr{F}, P, \{\mathbf{F}_t\}_{0\le t\le T}, W)$*, let* u_0*,* b_0 *be* $\mathcal{F}_0$*-measurable with* $\operatorname{div} u_0 = 0$*,* $\operatorname{div} b_0 = 0$ *and* $G_1, G_2 \in \mathcal{M}_T$*. If there exist constants* C_1 *and* C_2 *such that*

$$P\left\{\omega : \left\|\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix}(\omega,\cdot)\right\|_{\dot H^{\frac{n}{2}-1}} \le C_1, \qquad \left\|\begin{pmatrix}G_1\\ G_2\end{pmatrix}\right\|_{L^4_T(\dot H^{\frac{n}{2}-1})} \le C_2\right\} = 1,$$

then there exists a random set $\tilde\Omega$ *with* $P(\tilde\Omega) > 0$*, a random time* $\tau(\omega) > 0$*, and a*

[2] At the time of our work we were not aware of the works [3, 45], which were communicated to us by one of examiners of the thesis. They use Bony paradifferential calculus to study Navier-Stokes equations.

process

$$\begin{pmatrix} u \\ b \end{pmatrix}(\omega, \cdot) \in L^4_\tau(\dot{H}^{\frac{n-1}{2}}) \cap \mathcal{M}_\tau$$

for all ω *in* $\tilde{\Omega}$, *and* $\binom{u}{b}$ *is a local solution of problem in the sense of definition 76.*

Theorem 2. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\le t\le T}, \mathbf{P}, W)$ *be a probability basis. Let* u_0, b_0 *be* $\mathscr{F}_0-$*measurable with* $div\, u_0 = div\, b_0 = 0$, *and* $G_1, G_2 \in \mathcal{M}_T$. *Assume that for any positive* T *we have,*

$$(1+T)\left\|\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right\|_{L^4_\Omega L^4_T\left(\dot{H}^{\frac{n}{2}-1}\right)} + \left\|\begin{pmatrix} \beta_0 \\ \theta_0 \end{pmatrix}\right\|_{L^4_\Omega\left(\dot{H}^{\frac{n}{2}-1}\right)} < \infty.$$

Then there is a random set $\tilde{\Omega}$ *with positive probability and a unique global mild-solution of* (3.1) *in a ball centered at the origin in the space* $L^4_T(\dot{H}^{\frac{n-1}{2}})$, *for all* ω *in* $\tilde{\Omega}$. *Furthermore, for* $s = \frac{n}{2} - 1$, *if* $f_i \in L^2_\Omega L^2_T \dot{H}^s(\mathbb{R}^n)$ *for* $i = 1, 2$, *and* u_0, $b_0 \in L^2_\Omega \dot{H}^s(\mathbb{R}^n)$, *the solution* u, b *of* (3.1) *belongs to the space* $L^2_\Omega L^\infty_T \dot{H}^s(\mathbb{R}^n) \cap L^2_\Omega L^2_T \dot{H}^{s+1}(\mathbb{R}^n)$.

1.2.2 Result on Komogorov's spectral theory and Inertial range bounds for magnetohydrodynamic flows

The problem we have considered in the second part of this work is to analyze Kolmogorov's spectral behavior for the MHD flows governed by the system of equations (1.3) from a fully mathematical perspective. The 1941 and 1962 works of A.N. Kolmogorov on turbulence theory for incompressible hydrodynamic flows, [91, 92, 93, 94, 95], are the very important developments in the attempt to understand the not yet fully understood phenomenon of turbulence.

The core of this theory is Kolmogorov's 5/3 law, which gives an explanation on how energy decays in the inertial range. The law roughly states that there is a range of wave numbers, say $[k_1, k_2]$, such that the spectral energy $E_K(k)$ of the fluid decays according to the rule

$$C_0\epsilon^{2/3}k^{-5/3},$$

for $k \in [k_1, k_2]$. In developing his theory, Kolmogorov used similarity hypothesis and dimensional analysis.

As we have noted earlier, our concern is to analyze this theory through a direct mathematical approach. For our purpose instead of using dimensional analysis and similarity hypothesis, we combined harmonic analysis tools like the Fourier transform, complex analysis and functional analysis by imposing certain necessary conditions on the data, u_0, b_0, f_1 and f_2.

Leray in [110, 112] considered weak solutions of the Navier-Stokes equations as a turbulent solution for a flow governed by the system of Navier-Stokes equations. Therefore, our motivation is that if Leray's weak solutions are considered as turbulent solutions then they must satisfy Kolmogorov's spectral law. The work of Biryuk and Craig in [17] for Navier-Stokes equation is our eye-opener to try the method for MHD turbulence.

We define the spectral energy of (1.3), denoted by $E(k,t)$, by

$$E(k,t) := \int_{|\xi|=k} \left(|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2\right) \, \mathrm{d}S(\xi), \quad k \in [0,\infty).$$

The function $E(k,t)$ is throughly analyzed for the Leray weak solution (defined in the sequel) u, b of (1.3). We have shown that in the absence of external forces to the system, i.e., when $f_i \equiv 0$ for $i = 1, 2$, there is a universal constant such that $E(k,t)$ does not exceed that value, and for the case when $f_i \not\equiv 0$ a similar analysis shows existence of a constant which at most depends on the time that bounds $E(k,t)$ from above. Furthermore, for a given positive time T, the time average of the spectral energy is given by,

$$\frac{1}{T}\int_0^T E(k,t)\mathrm{d}t,$$

decays at the rate of k^{-2}, which indeed is faster than the decay rate in Kolmogorov's phenomenon and twice as fast as the rate in KID phenomenon. The detail of the analysis is given in section 4.3.

We may consider the following two theorems are the main results for this part of the thesis.

Theorem 3. *Suppose that* u_0, b_0 *belong to* $A_{R_1} \bigcap B_R(0)$, *and the external forces* $f_i \equiv 0$ *for each* $i = 1, 2$ *or* $f_i \in L^{\infty}_{loc}([0, \infty]; H^{-1}(D) \cap L^2(D))$ *for* $i = 1, 2$; *where* R, R_1, A_{R_1} *and* $B_R(0)$ *will be defined in the sequel. Then, the following are true about the Kolmogorov's inertial range for* (1.3)*:*

(i) Kolmogorov's parameters must satisfy

$$\left(\min(\nu, \eta)\right)^{5/6} C_0 \epsilon^{2/3} \leq 4\pi \left(\frac{R_2(T)}{\sqrt{T}}\right)^{5/3} R_1^{\frac{1}{3}}(T). \tag{1.7}$$

(ii) An absolute lower bound for the inertial range is given by

$$\bar{k}_1 = \frac{C_0^{3/5} \epsilon^{2/5}}{(4\pi R_1^2)^{3/5}}. \tag{1.8}$$

(iii) An absolute upper bound for the inertial range is given by

$$\bar{k}_2 = \left(\frac{4\pi}{C_0 \min(\nu, \eta)}\right)^3 \frac{1}{\epsilon^2} \frac{R_2^6(T)}{T^3}. \tag{1.9}$$

Theorem 4. *Let* (u, b) *be a solution of* (1.3) *with initial data* $u_0(x), b_0(x)$ *in* $A_{R_1} \cap B_R(0)$. *If* (u, b) *exhibits a spectral behavior of* E_K *uniformly over* $[k_1, k_2] \times [0, T]$, *then either*

(i). $\bar{k}_1 \leq k_1 \leq k_2 \leq \bar{k}_2$,

or

(ii). if $k_1 < \bar{k}_1$ *or* $\bar{k}_2 < k_2$, *then there is a small neighborhood of* $\bar{k}_j$, *for each* $j = 1, 2$, *to which* k_j *belongs;*

where a solution (u, b) *of* (1.3) *is said to have the spectral behavior of* $E_k(k)$ *uniformly over* $[k_1, k_2] \times [0, T]$ *if its energy spectral function* $E(k, t)$ *satisfies*

$$\sup_{\substack{t \in [0,T] \\ k \in [k_1, k_2]}} (1 + k^{5/3})|E(k,t) - E_K(k)| < C_1 \epsilon^{2/3},$$

and $C_1 \ll C_0$.

Proof of these theorems is quite involving, almost spans through the whole of part III.

Plan of the Thesis

The rest of the thesis is organized as follows. In the next chapter we will give necessary background materials which are used through out the thesis and indeed are corner stones of the work.

Chapter 3 is devoted to the study of the stochastic MHD model (1.4). A brief discussion on evolution spaces of Besov type or Chemin-Lerner spaces is also given. We derived new estimates for stochastic heat equation in probabilistic Chemin-Lerner spaces. These estimates indeed are of great importance in proving the first two main results.

In chapter 4, we have paid special attention to MHD turbulence. We attempt to verify Kolmogorov's spectral theory for the MHD system (1.3). The detail proofs of our main results on the topic are given in this section.

We conclude the thesis by giving a highlight on our future work and list of reference materials.

Chapter 2

Background Materials

2.1 Littlewood-Paley Theory

In this chapter we will provide the basic construction of Littlewood-Paley theory and its basic properties. We will not give any detailed proof of the results we mention, rather we will refer to the materials from where we got them.

The Littlewood-Paley theory provides us the opportunity to treat functions or distributions as a countable sum of smooth functions whose Fourier transforms are compactly supported in a ball or an annulus. We have organized the section as follows; firstly we will briefly discuss distributions and the Fourier transform, next we will give Bernstein lemma then we discuss the dyadic blocks of unity and finally we will briefly look at functions spaces such as Sobolev and Besov spaces in this framework.

2.1.1 Fourier Analysis

Definition 5 (Schwartz Space)**.** *Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the set of smooth functions in $\mathbb{R}^n$ whose derivatives are rapidly decreasing. That is*

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty : \|f\|_{k,\mathcal{S}} < \infty, \quad \textit{for all } k \in \mathbb{N} \right\},$$

where $C^\infty(\mathbb{R}^n)$ *is the set of smooth functions from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *and*

$$\|f\|_{k,\mathcal{S}} = \sup_{\substack{|\alpha|\leq k \\ x\in\mathbb{R}^n}} \left|(1+|x|)^k D^\alpha f(x)\right|.$$

Generally speaking, a function $f(x)$ is rapidly decreasing if all its derivatives $Df(x)$, $D^2f(x), \dots$ exist every where on $\mathbb{R}^n$ and go to zero as $|x| \longrightarrow \pm\infty$ faster than any inverse power of x.

Example

1. $x^j e^{-a|x|^2} \in \mathcal{S}(\mathbb{R}^n)$ for all $j \in \mathbb{Z}, x \in \mathbb{R}^n$ and a positive real number.
2. Any compactly supported smooth function.

Remark 6. *The set* $\mathcal{S}(\mathbb{R}^n)$ *equipped with the family of seminorms* $\big(\|\cdot\|_{k,\mathcal{S}}\big)_{k\in\mathbb{N}}$ *is a Fréchet space and the space* $\mathcal{D}(\mathbb{R}^n)$ *of smooth compactly supported functions on* $\mathbb{R}^n$ *is dense in* $\mathcal{S}(\mathbb{R}^n)$*. For the detail one can see [10, 171].*

Definition 7. *A tempered distribution* f *is a mapping* $f : \mathcal{S} \mapsto \mathbb{C}$ *having the following properties of linearity and continuity;*

(i) $f(c_1\varphi_1 + c_2\varphi_2) = c_1 f(\varphi_1) + c_2 f(\varphi_2)$ *for all* $\varphi_k \in \mathcal{S}, c_k \in \mathbb{C},\ k = 1, 2,$

(ii) if $\{\varphi_j\}$ *is a sequence in* $\mathcal{S}(\mathbb{R}^n)$ *which converges to* φ *in* $\mathcal{S}$*, then the sequence* $\{f(\varphi_j)\}$ *in* $\mathbb{C}$ *converges to* $f(\varphi)$ *in* $\mathbb{C}$*.*

The space of tempered distributions, denoted by $\mathcal{S}'$ *is the continuous dual of Schwartz spaces; or more simply* $\mathcal{S}'$ *is the set of all tempered distributions.*

We now define the Fourier transform.

Definition 8. *Let* u *be in* $\mathcal{S}$*. The Fourier transform of* u *denoted by* $\hat{u}$ *or* $\mathcal{F}u$ *is defined by*

$$\hat{u}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\ \mathrm{d}x \quad \textit{for } \xi \in \mathbb{R}^n,$$

where $x \cdot \xi$ denotes an inner product in $\mathbb{R}^n$. The inverse Fourier transform of u denoted by $\mathcal{F}^{-1}u$ is defined by

$$\mathcal{F}^{-1}u(x) := (2\pi)^{-1} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\ \mathrm{d}\xi = (2\pi)^{-1}\bar{\mathcal{F}} \quad \text{for } x \in \mathbb{R}^n,$$

where $\bar{\mathcal{F}}$ denotes the complex conjugate of $\mathcal{F}$.

We next extend the definition of Fourier transform from $\mathcal{S}$ to $\mathcal{S}'$ by duality as follows;

Definition 9. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform $\hat{u}$ of u is defined by duality as*

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle_{\mathcal{S}',\mathcal{S}}$$

whenever $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$.

The inverse Fourier can also be defined in the same way.

The Fourier transform has the following important properties which makes it one of favorite tools for tackling non-linear partial differential equations.

Properties of Fourier Transform

1. Derivatives: for all multi-index $\alpha \in \mathbb{N}^n$, we have

$$\mathcal{F}(\partial_x^\alpha u) = (i\xi)^\alpha \mathcal{F}u \quad \text{and} \quad \mathcal{F}(x^\alpha u) = (-i)^{|\alpha|}\partial_\xi^\alpha \mathcal{F}u$$

2. Algebraic properties: for $(u, v) \in \mathcal{S} \times \mathcal{S}$, we have $u * v \in \mathcal{S}$ and

$$\mathcal{F}(u * v) = \mathcal{F}u\mathcal{F}v,$$

   where the operation $*$ stands for convolution.

3. Fourier transform is a continuous linear map from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. This is clear from the fact that that $\left|\hat{u}(\xi)\right| \leq \|u\|_{L^1}$.

4. For any function $\phi \in L^1(\mathbb{R}^n)$ and an automorphism L on $\mathbb{R}^n$, we have

$$\mathcal{F}(\phi \circ L) = \frac{1}{|\det L|}\hat{\phi} \circ L^{-1},$$

where $\det L$ is the determinant of L.

5. The Fourier transform continuously maps $\mathcal{S}$ to $\mathcal{S}$: i.e., for any integer k, there exists a constant C and an integer d such that

$$\forall \phi \in \mathcal{S}, \|\hat{\phi}\|_{k,\mathcal{S}} \leq C\|\phi\|_{d,\mathcal{S}}$$

6. [Fourier-Plancherel formula] The Fourier transform is an automorphism of $\mathcal{S}'$ with inverse $(2\pi)^{-n}\bar{\mathcal{F}}$. Moreover, $\mathcal{F}$ is also an automorphism of $L^2(\mathbb{R}^n)$ which satisfies, for any function f in L^2,

$$\|\hat{f}\|_{L^2} = (2\pi)\|f\|_{L^2}$$

2.1.2 Bernstein Lemma

This section is devoted to one of the most important inequalities of the whole theory of Littlewood-Paley, and is one of the most frequently used inequalities in this work. The proof can be found in [10, 36].

Lemma 10 (Bernstein Lemma)**.** *Let $k \in \mathbb{N}$. Let (R_1, R_2) satisfy $0 < R_1 < R_2$. There exsits a constant C depending only on R_1, R_2, n such that for all $1 \leq a \leq b \leq \infty$ and $u \in L^a$, we have*

$$\text{If } \operatorname{supp} \hat{u} \subset \mathcal{B}(0, R_1\lambda) \text{ then } \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1}\lambda^{k+n(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a} \text{ and} \tag{2.1}$$

$$\text{if } \operatorname{supp} \hat{u} \subset \mathcal{C}(0, \lambda R_1, \lambda R_2) \text{ then } C^{-k-1}\lambda^k\|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1}\lambda^k\|u\|_{L^a} \tag{2.2}$$

where $\mathcal{B}(0, \lambda R_1)$ *is an open ball with center* 0 *and radius* λR_1 *and* $\mathcal{C}(0, \lambda R_1, \lambda R_2)$ *is a shell centered at* 0 *with inner radius* λR_1 *and outer radius* λR_2*.*

2.1.3 Littlewood-Paley Decomposition

The purpose of this section is to find a way of decomposing distributions into smooth functions in a Fourier space such that each component is supported either in a ball or a shell of size proportional to 2^q, for some q in $\mathbb{Z}$.

The following result with its detailed proof can be found in [36, p. 17].

Proposition 11. *Denote by* $\mathcal{C}$ *the annulus of center* 0*, shorter radius* $\frac{4}{3}$ *and long radius* $\frac{8}{3}$*.* [1] *Then there exist two positive radial functions* χ *and* φ *belonging respectively to* $C_0^\infty(\mathcal{B}(0, 4/3))$ *and* $C_0^\infty(\mathcal{C})$ *such that*[2]

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \tag{2.3}$$

$$|p - q| \geq 2 \textit{ implies } \operatorname{supp} \varphi(2^{-q}\cdot) \cap \operatorname{supp} \varphi(2^{-p}\cdot) = \emptyset \tag{2.4}$$

$$q \geq 1 \textit{ implies } \operatorname{supp} \chi\cdot \cap \operatorname{supp} \varphi(2^{-q}\cdot) = \emptyset. \tag{2.5}$$

Furthermore, if we denote $\tilde{\mathcal{C}} = \mathcal{B}(0, \frac{2}{3}) + \mathcal{C}$*, then* $\tilde{\mathcal{C}}$ *is an annulus and we have that*

$$\textit{if } |p - q| \geq 5 \textit{ then } 2^p\tilde{\mathcal{C}} \cap 2^p\mathcal{C} = \emptyset \textit{and} \tag{2.6}$$

$$1/3 \leq \chi^2(\xi) + \sum_{q \geq 0} \varphi^2(2^{-q}\xi) \leq 1 \tag{2.7}$$

[1] for any $\alpha > 1$, the shell $\mathcal{C}(o, \frac{1}{\alpha}, 2\alpha)$ and the ball $\mathcal{B}(0, \alpha)$ would work

[2] C_0^∞ is the space of compactly supported infinitely differentiable functions

Next we define the following Fourier multipliers,

$$\begin{aligned}
\Delta_j u &= \begin{cases} 0 & : \ j \leq -2 \\ \varphi(2^{-j}D)u & : \ j \geq 0 \\ \chi(D)u & : \ j = -1 \end{cases}, \\
S_j u &= \textstyle\sum_{k \leq j-1} \Delta_k u = \chi(2^{-j}D)u, \quad j \in \mathbb{Z}, \\
\dot{\Delta}_j u &= \varphi(2^{-j}D)u, \qquad j \in \mathbb{Z}, \\
\dot{S}_j u &= \chi(2^{-j}D)u, \quad j \in \mathbb{Z}.
\end{aligned} \tag{2.8}$$

The pair $\{S_j, \Delta_j\}$ is called non-homogeneous Littlewood-Paley decomposition, and the pair $(\dot{\Delta}_j u, \dot{S}_j u)$ is called a homogeneous Littlewood-Paley decomposition of u. It is clear from the definition that, the operators in (2.8) are supported in dyadic blocks or shells of size proportional to 2^q. Then the following proposition follows from Proposition 11.

Proposition 12. *Let $u, v \in \mathcal{S}'$. Then*

1. $u = \sum_{j \in \mathbb{Z}} \Delta_j u,$
2. $\Delta_p \Delta_q u \equiv 0$ *if* $p - q \geq 2,$
3. $\Delta_q(S_{p-1}\Delta_p v) \equiv 0$ *if* $|p - q| \geq 5.$

Proof. The proof of the first statement is given in detail in [10, 36]. Properties 2 and 3 follow from proposition 11. Further reading on this topic can also be found in [134, 145, 162]. □

Concerning homogeneous Littlewood-Paley decomposition, Proposition 12 holds with the only exception of property 1. Indeed, for instance, when $u = 1$, $\dot{\Delta}_j u = 0$ for any $j \in \mathbb{Z}$, thus it fails. However property 1 holds modulo some polynomial. Peetre in [134, p. 52–54] proved that for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, there is a natural number $N = N(u)$ and a polynomial $P_k(u)$ of degree at most N such that

$$\lim_{k \to -\infty} \dot{S}_k u - P_k(u) = 0 \tag{2.9}$$

uniformly in the topology of $S'(\mathbb{R}^n)$. Note that a sequence $(u_n)_{n\in\mathbb{N}}$ of tempered distributions is said to converge to u in $\mathcal{S}'(\mathbb{R}^n)$ if

$$\forall \phi \in \mathcal{S}(\mathbb{R}^n), \lim_{n\to\infty} \langle u_n, \phi\rangle = \langle u, \phi\rangle.$$

For a detail reading on the topic we refer to [10, 134, 145, 162, 163].

2.1.4 Littlewood-Paley Decomposition and Besov Spaces

Hans Triebel in [162] said " .... If smoothness is expressed via the scales of Besov and Triebel-Lizorkin spaces respectively (to be defined in the sequel) then one has an armada of different devices at hand." [3]

In this section we present Besov spaces and its relation to other spaces, such as Sobolev in the frame work of Littlewood-Paley theory.

Definition 13. *Let $1 \le p, r \le \infty$ and $s \in \mathbb{R}$. For $u \in \mathcal{S}'(\mathbb{R}^n)$, we set*

$$\|u\|_{B^s_{p,r}} := \left(\sum_{q\in\mathbb{Z}} (2^{qs}\|\Delta_q u\|_{L^p})^r\right)^{\frac{1}{r}}.$$

The non-homogeneous Besov space, denoted by $B^s_{p,r}$, is the set of tempered distributions u such that $\|u\|_{B^s_{p,r}}$ is finite, i.e.,

$$B^s_{p,r} := \left\{ u \in \mathcal{S}' : \left(\sum_{q\in\mathbb{Z}} (2^{qs}\|\Delta_q u\|_{L^p})^r\right)^{\frac{1}{r}} < \infty \right\}$$

[3] Let $1 \le p, r \le \infty$ and $s \in \mathbb{R}$. Triebel-Lizorkin spaces, denoted by $F^s_{p,r}$ (non-homogeneous) or $\dot{F}^s_{p,r}$ (homogeneous), are the set of temperate distributions u such that

$$\|u\|_{F^s_{p,r}} := \left(\int_{\mathbb{R}^n} \left(\sum_{q\in\mathbb{Z}} (2^{qs}\Delta_q u)^r\right)^{\frac{p}{r}}\right)^{\frac{1}{p}} \tag{2.10}$$

is finite. The homogeneous space $\dot{F}^s_{p,r}$ is defined in a similar way using the homogeneous blocks $\dot{\Delta}_j u$.

Before we give the definition of the homogeneous Besov spaces we would like to recall that the series $\sum_q \dot{\Delta}_q u$ converges to u only modulo a polynomial. However it is not suitable to deal with distribution modulo polynomials specially when we are playing with nonlinear PDEs. Therefore, we introduce the following space due to Chemin [38].

Definition 14. *We denote by $\mathcal{S}'_h$ the space of tempered distribution u such that,*

$$\lim_{j\to-\infty} S_j u = 0 \quad in\ \mathcal{S}'.$$

Remark 15. *We note that the space $\mathcal{S}'_h$ is exactly the space of tempered distributions for which we may write*

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u.$$

For the detail of this and more we again refer to [38, 49].

Definition 16. *Let $1 \le p, r \le \infty$ and $s \in \mathbb{R}$. For $u \in \mathcal{S}'(\mathbb{R}^n)$, we set*

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_{q\in\mathbb{Z}} (2^{qs}\|\dot{\Delta}_q u\|_{L^p})^r \right)^{\frac{1}{r}}.$$

The homogeneous Besov space, denoted by $\dot{B}^s_{p,r}$, is the set of temprate distributions $u \in \mathcal{S}'_h$ such that $\|u\|_{\dot{B}^s_{p,r}}$ is finite, i.e.,

$$\dot{B}^s_{p,r} := \left\{ u \in \mathcal{S}' : \left(\sum_{q\in\mathbb{Z}} (2^{qs}\|\dot{\Delta}_q u\|_{L^p})^r \right)^{\frac{1}{r}} < \infty \right\}$$

Besov spaces exhibit interesting relations with classical spaces such as Sobolev, Hölder spaces, Lebesgue spaces and so on; see [161, p. 34],[145, p. 14]. Moreover, based on the choice of parameters p, r and s, Besov spaces possess features which give them priorities over other spaces for our purpose. Below is some of the basic properties both for homogeneous and non-homogeneous cases.

We next give basic some important properties of $B^s_{p,r}$ and $\dot{B}^s_{p,r}$; the proofs can be found in [10].

Proposition 17 (Properties of $B^s_{p,r}$)**.** *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. Then the following statements are true*

1. *The space $B^s_{p,r}$ does not depend on the choice of the functions χ and φ.*

2. *The space $B^s_{p,r}$ is a Banach space and satisfies the Fatou property, namely, if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$, then an element u of $B^s_{p,r}$ and a subsequence $u_{\phi(n)}$ exists such that*

$$\lim_{n\to\infty} u_{\phi(n)} = u \text{ in } \mathcal{S}' \text{ and } \|u\|_{B^s_{p,r}} \leq \liminf_{n\to\infty} \|u_{\phi(n)}\|_{B^s_{p,r}}. \tag{2.11}$$

3. *For all $s \in \mathbb{R}$ and $1 < p, r < \infty$, the space $B^{-s}_{p'r'}$ is the dual space of $B^s_{p,r}$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = 1$. If $1 \leq p < \infty$, the completion $\mathcal{B}^s_{p,\infty}$ of C^∞_0 for the norm $\|\cdot\|_{B^s_{p,\infty}}$ is the predual of $B^{-s}_{p',1}$.*

4. *We also have the following embedding property;*

   (a) *$B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,\tilde{r}}$ whenever $\tilde{s} < s$ or $\tilde{s} = s$ and $\tilde{r} \geq r$,*

   (b) *$B^s_{p,r} \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$ whenever $\tilde{p} \geq p$,*

   (c) *we have $B^0_{\infty,1} \hookrightarrow \mathcal{C} \cap L^\infty$. If $p < \infty$ then the space $B^{\frac{N}{p}}_{p,1}$ is continuously embedded in the space $\mathcal{C}_0$ of continuous bounded functions which decay at infinity.*

5. *There exists a constant $C > 0$ satisfying the following properties; if s_1 and s_2 are real numbers such that $s_1 < s_2$, $\theta \in (0,1)$ and $1 \leq p, r \leq \infty$, then we have*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^\theta_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}} \tag{2.12}$$

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) \|u\|^\theta_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}} \tag{2.13}$$

Proposition 18 (Properties of homogeneous Besov spaces)**.** *Let $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$*

1. *Let u be a tempered distribution and N be any positive integer. Define a tempered distribution u_N by $u_N := u(2^N\cdot)$. Then the following statement is true: if $\|u\|_{\dot{B}^s_{p,r}}$ is finite, so it is for u_N and we have*

$$\|u_N\|_{\dot{B}^s_{p,r}} = 2^{N(s-\frac{d}{p})}\|u\|_{\dot{B}^s_{p,r}} \tag{2.14}$$

2. *The space $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a normed space. Moreover, if $s < \frac{d}{p}$, then $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a Banach space. For any p, the space $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^{\frac{d}{p}}_{p,1}})$ is also a Banach space.*

3. *The two spaces $\dot{H}^s$ and $\dot{B}^s_{2,2}$ are equal and the corresponding norms satisfy*

$$\frac{1}{C^{|s|+1}}\|u\|_{B^s_{2,2}} \leq \|u\|_{\dot{H}^s} \leq C^{|s|+1}\|u\|_{B^s_{2,2}}. \tag{2.15}$$

4. *There exists a constant $C > 0$ such that if $\theta \in (0,1)$, $r \in [1,\infty]$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, then for any $u \in \mathcal{S}'_h$,*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}, \quad \text{and} \tag{2.16}$$

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}. \tag{2.17}$$

5. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$ satisfy $s < \frac{n}{p}$, or $s = \frac{n}{p}$ and $r = 1$.*

   (a) *Let $(u_q)_{q\in\mathbb{Z}}$ be a sequence of functions such that $\left(\sum_q \left(2^{qs}\|u_q\|_{L^p}\right)^r\right)^{\frac{1}{r}} < \infty$. If $\operatorname{supp}\hat{u}_q \subset \mathcal{C}(0, 2^qR_1, 2^qR_2)$ for some $0 \leq R_1 \leq R_2$ then $u := \sum_{q\in\mathbb{Z}} u_q$ belongs to $\dot{B}^s_{p,r}$ and there exists a constant C such that*

$$\|u\|_{\dot{B}^s_{p,r}} \leq C^{1+|s|}\left(\sum_q \left(2^{qs}\|u_q\|_{L^p}\right)^r\right)^{\frac{1}{r}}.$$

   (b) *Let $(u_q)_{q\in\mathbb{Z}}$ be a sequence of functions such that $\left(\sum_q \left(2^{qs}\|u_q\|_{L^p}\right)^r\right)^{\frac{1}{r}} < \infty$. If $\operatorname{supp}\hat{u}_q \subset \mathcal{B}(0, 2^qR)$ for some positive R and if in addition s is* *positive then* $u := \sum_{q\in\mathbb{Z}} u_q$ *belongs to* $\dot{B}^s_{p,r}$ *and there exists a constant* C *such that*

$$\|u\|_{\dot{B}^s_{p,r}} \leq \frac{C^{1+s}}{s}\left(\sum_q \left(2^{qs}\|u_q\|_{L^p}\right)^r\right)^{\frac{1}{r}}$$

For the proof of these statements we refer to [38, 49].

Paradifferential Calculus and Bony decomposition

In dealing with product of tempered distributions such as, uv, unlike product of two functions we do not have pointwise evaluation. We formally define the product using the Littlewood-Paley decomposition as follows,

$$uv = \sum_{p,q\in\mathbb{Z}} \Delta_q u \Delta_p v. \tag{2.18}$$

Paradifferential calculus is a very useful tool to deal with products of the type (2.18), by which the product will be split into three parts depending on the size of p and q; the first part consists of blocks Δ_p, Δ_q such that $p \leq q - N$ for some appropriate positive integer N, the second part consists of blocks Δ_p, Δ_q such that $q \leq p - N$ and the third part consists of blocks Δ_p, Δ_q such that $|p - q| < N$.

Definition 19. *The non-homogeneous (respectively homogeneous) paraproduct of* v *by* u*, denoted by* $T_u v$*, (respectively* $\dot{T}_u v$*) is defined respectively as;*

$$T_u v := \sum_{p\leq q-2} \Delta_p u \Delta_q v,$$

and

$$\dot{T}_u v := \sum_{p\leq q-2} \dot{\Delta}_p u \dot{\Delta}_q v.$$

Definition 20. *The non-homogeneous (respectively homogeneous) remainder of* u

and v is defined respectively by

$$R(u,v) = \sum_{|p-q|\leq 1} \Delta_p u \Delta_q v. \tag{2.19}$$

$$\dot{R}(u,v) = \sum_{|p-q|\leq 1} \dot{\Delta}_p u \dot{\Delta}_q v. \tag{2.20}$$

Remark 21. *The operators $T_u v$, $\dot{T}_u v$, $R(u,v)$ and $\dot{R}(u,v)$ exhibit several interesting properties, such as; bilinearity and continuity; for the detail we refer to [10, 49].*

Once equipped with these two definitions, we now define the product uv by the Bony-decomposition; for $u,\ v \in B^s_{p,r}$ (respectively $u,\ v \in \dot{B}^s_{p,r}$)

$$uv = T_u v + T_v u + R(u,v), \tag{2.21}$$

and

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v) \tag{2.22}$$

respectively.

Next, we state few theorems on continuity of the product the remainder; their proofs can be found in [10, Ch. 2],[49].

Let $\tilde{L}(V,W)$ denote the set of all continuous linear operators from the normed space V to another normed space W, then we have the following results:

Theorem 22. *There exists a constant $C > 0$ such that for any $(p, r_1, r_2) \in [1,\infty]^3$ and $(s,t) \in \mathbb{R} \times (-\infty, 0)$;*

$$\|T\|_{\tilde{L}(L^\infty \times B^s_{p,r}; B^s_{p,r})} \leq C^{|s|+1}, \tag{2.23}$$

$$\|T\|_{\tilde{L}(B^t_{\infty,r_1} \times B^{s+t}_{p,r_2})} \leq \frac{C^{|s+t|+1}}{-t} \quad \textit{with} \quad \frac{1}{r} = \min(1, \frac{1}{r_1} + \frac{1}{r_2}). \tag{2.24}$$

Furthermore, for $k \in \mathbb{N}$, we have

$$\|T_u v\|_{B^s_{p,r}} \le C\|u\|_{L^\infty}\|D^k v\|_{B^{s-k}_{p,r}} \text{ and } \|T_u v\|_{B^{s+t}_{p,r}} \le C\|u\|_{B^t_{\infty,r_1}}\|D^k v\|_{B^{s-k}_{p,r_2}} \tag{2.25}$$

Theorem 23. *There exists a constant $C > 0$ satisfying the following inequalities. Let $(s_1, s_2) \in \mathbb{R}^2$ and $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1 \text{ and } \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \le 1. \tag{2.26}$$

If $s_1 + s_2 > 0$, then we have, for any (u, v) in $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$,

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}} \le \frac{C^{|s_1+s_2|+1}}{s_1+s_2}\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}. \tag{2.27}$$

If $r = 1$ and $s_1 + s_2 = 0$, then we have, for any (u, v) in $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$,

$$\|R(u,v)\|_{B^0_{p,\infty}} \le C^{|s_1+s_2|+1}\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}. \tag{2.28}$$

Theorem 24. *For any positive real number s and any $(p, r) \in [1, \infty]^2$, the space $L^\infty \bigcap B^s_{p,r}$ is an algebra, and a constant C exists such that*

$$\|uv\|_{B^s_{p,r}} \le \frac{C^{s+1}}{s}\Big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}}\|v\|_{L^\infty}\Big) \tag{2.29}$$

Theorem 25. *Let s, p, r_1 such that $\dot{B}^s_{p,r_1}$ is a Banach space. Then the paraproduct maps continuously $L^\infty \times \dot{B}^s_{p,r_1}$ into $\dot{B}^s_{p,r}$. Moreover, if t is negative and r_2 such that*

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \le 1, \tag{2.30}$$

and if $\dot{B}^{s+t}_{p,r}$ is a Banach space, then $\dot{T}$ maps continuously $\dot{B}^t_{\infty,r_1} \times \dot{B}^s_{p,r_2}$ into $\dot{B}^{s+t}_{p,r}$.

Theorem 26. *Let p_k, r_k (for $k \in \{1, 2\}$) such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \le 1, \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \le 1. \tag{2.31}$$

Let $(s_1, s_2) \in \mathbb{R}^2$ such that $s_1 + s_2 \in (0, d/p)$, the operator $\dot{R}$ maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$

into $\dot{B}^{s_1+s_2}_{p,r}$. *Moreover, if* $s_1 + s_2 = 0$ *and* $r = 1$, *the operator* $\dot{R}$ *maps* $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ *into* $\dot{B}^{0}_{p,1}$.

2.2 Stochastic Analysis

In this section we give definitions and classical results from probability theory and stochastic analysis which are of interest to us. Similarly to previous sections, we refer to sources for detail readings and proofs. Definitions and notations are mostly taken from [90, 152].

2.2.1 Probability and Random Variables

Definition 27. *An ordered triplet* $(\Omega, \mathscr{F}, \mathbf{P})$ *where,*

(a) Ω *is a set of points* ω,

(b) $\mathscr{F}$ *is a* $\sigma-$*algebra of subsets of* Ω,

(c) $\mathbf{P}$ *is a probability measure on* $\mathscr{F}$,

is called a probabilistic model or a probability space. Here, Ω *is the sample space or space of elementary events, the sets* A *in* $\mathscr{F}$ *are events, and* $\mathbf{P}(A)$ *is the probability of the event* A. *A probability measure or a probability* $\mathbf{P}$ *of* A, *where* A *belongs to an algebra* $\mathscr{A}$ *of subsets of* Ω, *is a countably additive measure such that* $\mathbf{P}(\Omega) = 1$.

Definition 28. *Let* $(\Omega, \mathscr{F})$ *be a measurable space and let* $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ *be the real line with the system* $\mathscr{B}(\mathbb{R})$ *of Borel sets. A real function* $\zeta = \zeta(\omega)$ *defined on* $(\Omega, \mathscr{F})$ *is an* $\mathscr{F}-$*measurable function, or a random variable, if*

$$\{\omega : \zeta(\omega) \in B\} \in \mathscr{F}$$

for every $B \in \mathscr{B}(\mathbb{R})$; *or equivalently, if the inverse image*

$$\zeta^{-1}(B) \equiv \{\omega : \zeta(\omega) \in B\}$$

is a measurable set in Ω.[4]

If the random variable ζ *takes the form*

$$\zeta(\omega) = \sum_{i=1}^{\infty} x_i I_{A_i}(\omega), \tag{2.32}$$

where $\bigcup A_i = \Omega$, $A_i \in \mathscr{F}$, *then we call it discrete. Furthermore, if* (2.32) *is finite then* ζ *is simple.*

Definition 29. *We define the probability distribution of the random variable* ζ *on* $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, *denoted by* $\mathbf{P}_\zeta$, *as*

$$\mathbf{P}_\zeta(B) = \mathbf{P}\{\omega : \zeta(\omega) \in B\}; \tag{2.33}$$

and the distribution function of ζ *is given by*

$$F_\zeta(x) = \mathbf{P}(\omega : \zeta(\omega) \leq x), \quad x \in \mathbb{R}. \tag{2.34}$$

A random variable is determined by the its distribution function. For instance, a random variable ζ is called continuous if its distribution function F_ζ is continuous for all $x \in \mathbb{R}$; and for each random variable ζ there is a nonnegative function $f = f_\zeta(x)$, called its density, such that

$$F_\zeta(x) = \int_{-\infty}^{x} f_\zeta(y)\mathrm{d}y, \quad x \in \mathbb{R}, \tag{2.35}$$

the integral can be taken in the Riemann sense or Lebesgue sense.

[4] When $(\Omega, \mathscr{F}) = (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, then $\mathscr{B}(\mathbb{R}^n)-$ measurable functions are called Borel functions.

2.2.2 Expectation

Definition 30. *For* $\Omega, \mathscr{F}, \mathbf{P}$ *a finite probability space and* $\zeta = \zeta(\omega)$ *a simple random variable,*

$$\zeta(\omega) = \sum_{j=1}^{k} x_k I_{A_k}(\omega), \tag{2.36}$$

the expectation $\mathbf{E}\zeta$ *is defined as*

$$\mathbf{E}\zeta = \sum_{j=1}^{k} x_j \mathbf{P}(A_j). \tag{2.37}$$

Remark 31. *In general, expectation of a random variable can merely be treated as the Lebesgue integral of an* $\mathscr{F}$ *measurable function* $\zeta = \zeta(\omega)$ *with respect to the probability measure* $\mathbf{P}$; *in this case, we define*

$$\mathbf{E}\zeta = \int_\Omega \zeta(\omega)\mathbf{P}(\mathrm{d}\omega), \quad \textit{or} \;\; \mathbf{E}\zeta = \int_\Omega \zeta \mathrm{d}\mathbf{P}. \tag{2.38}$$

Remark 31 tells us that $\mathbf{E}$ enjoys properties of Lebesgue integrals; such as linearity, monotonicity and the likes. [152] is a rich source of information on expectation. The following highly celebrated results are of great importance to our work.

Theorem 32 (Chebyshev's Inequality)**.** *Let* ζ *be a nonnegative random variables. Then for every* $\epsilon > 0$ *we have*

$$\mathbf{P}(\zeta \geq \epsilon) \leq \frac{\mathbf{E}\zeta}{\epsilon}. \tag{2.39}$$

Theorem 33 (The Cauchy-Bunyakovskii Inequality)**.** *Let* ξ *and* η *be random variables and satisfy* $\mathbf{E}\xi^2 < \infty$, $\mathbf{E}\eta^2 < \infty$. *Then* $\mathbf{E}|\xi\eta| < \infty$ *and*

$$(\mathbf{E}|\xi\eta|)^2 \leq \mathbf{E}\xi^2 \cdot \mathbf{E}\eta^2. \tag{2.40}$$

Theorem 34 (Jensen's Inequality)**.** *Let the Borel function* $g = g(x)$ *be convex* *downward and* $\mathbf{E}|\xi| < \infty$. *Then*

$$g(\mathbf{E}\xi) \leq \mathbf{E}g(\xi). \tag{2.41}$$

Theorem 35 (Lyapunov's Inequality)**.** *If* $0 < s < t$

$$(\mathbf{E}|\zeta|^s)^{1/s} \leq (\mathbf{E}|\zeta|^t)^{1/t} \tag{2.42}$$

Theorem 36 (Hölder's Inequality)**.** *Let* $1 < p, q < \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\mathbf{E}|\zeta|^p < \infty$, $\mathbf{E}|\eta|^q < \infty$, *then* $\mathbf{E}|\zeta\eta| < \infty$ *and*

$$\mathbf{E}|\zeta\eta| \leq (\mathbf{E}|\zeta|^p)^{1/p}(\mathbf{E}|\eta|^q)^{1/q} \tag{2.43}$$

Theorem 37 (Minkowski's Inequality)**.** *If* $\mathbf{E}|\zeta|^p < \infty$, $\mathbf{E}|\eta|^p < \infty$, $1 \leq p < \infty$, *then we have* $\mathbf{E}|\zeta + \eta|^p < \infty$ *and*

$$(\mathbf{E}|\zeta + \eta|^p)^{1/p} \leq (\mathbf{E}|\zeta|^p)^{1/p} + (\mathbf{E}|\eta|^p)^{1/p} \tag{2.44}$$

2.2.3 Conditional Probabilities and Conditional Expectations

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space, $\mathscr{G}$ be a $\sigma-$algebra, $\mathscr{G} \subset \mathscr{F}$ be a sub $\sigma-$algebra and $\zeta = \zeta(\omega)$ be a random variable. The conditional expectation of a non-negative random variable with respect to the $\sigma-$algebra $\mathscr{G}$ is a non negative extended random variable, denoted by $\mathbf{E}(\zeta|\mathscr{G})$ or $\mathbf{E}(\zeta|\mathscr{G})(\omega)$, such that;

(a) $\mathbf{E}(\zeta|\mathscr{G})$ is $\mathscr{G}-$measurable;

(b) for every $A \in \mathscr{G}$

$$\int_A \zeta \mathrm{d}\mathbf{P} = \int_A \mathbf{E}(\zeta|\mathscr{G}) \mathrm{d}\mathbf{P}. \tag{2.45}$$

And the conditional expectation $\mathbf{E}(\zeta|\mathscr{G})$ of any random variable ζ with respect to the $\sigma-$algebra $\mathscr{G}$ is defined in terms of non-negative random variable ζ^+ and ζ^-

such that, if

$$\min(\mathbf{E}(\zeta^+|\mathscr{G}), \mathbf{E}(\zeta^-|\mathscr{G})) < \infty \tag{2.46}$$

$\mathbf{P}-$a.s then $\mathbf{E}(\zeta|\mathscr{G})$ is defined by the formula

$$\mathbf{E}(\zeta|\mathscr{G}) = \mathbf{E}(\zeta^+|\mathscr{G}) - \mathbf{E}(\zeta^-|\mathscr{G}). \tag{2.47}$$

On the set (of probability zero) of sample points for which $\mathbf{E}(\zeta^+|\mathscr{G}) = \mathbf{E}(\zeta^-|\mathscr{G}) = \infty$, the difference $\mathbf{E}(\zeta^+|\mathscr{G}) - \mathbf{E}(\zeta^-|\mathscr{G})$ is given an arbitrary value, for example zero.

Here we would like to note that all the properties of expectation hold for conditional expectations as well. A very nice further reading can be found in [152, P. 210–232].

2.2.4 Stochastic Processes and Filtrations

Definition 38. *A stochastic process is a mathematical model for the occurrence at each moment after the initial time, of a random phenomenon; or more generally stochastic process is a collection of random variables $X = \{X_t : 0 \leq t < \infty\}$ on the sample space $(\Omega, \mathscr{F})$ which takes values in a second measurable space $(S, \mathscr{S})$, called state space. For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega); t \geq 0$ is the sample path (realization, trajectory) of the process X associated with ω.*

Definition 39. *Let X and Y be two stochastic processes defined on the same probability spaces $(\Omega, \mathscr{F}, \mathbf{P})$. Then we say*

1. *Y is a modification of X, if for every $t \geq 0$, we have $\mathbf{P}[X_t = Y_t] = 1$.*
2. *X and Y have the same finite dimensional distributions if, for any integer $n \geq 1$, real numbers $0 \leq t_1 \leq t_2 \leq \ldots t_n < \infty$, and $A \in \mathscr{B}(\mathbb{R}^n)$ we have*

$$\mathbf{P}[(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \in A] = \mathbf{P}[(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \in A] \tag{2.48}$$

3. *X and Y are called indistinguishable if almost all their sample paths agree:*

$$\mathbf{P}[X_t = Y_t; \forall 0 \leq t < \infty] = 1. \tag{2.49}$$

4. *The stochastic process X is said to be measurable if for every $A \in \mathscr{B}(\mathbb{R}^n)$, the set $\{(t,\omega) : X_t(\omega) \in A\}$ belongs to the product $\sigma-$field $\mathscr{B}([0,\infty) \otimes \mathscr{F}$; i.e., when the mapping*

$$(t,\omega) \mapsto X_t(\omega) : ([0,\infty) \times \Omega, \mathscr{B}([0,\infty)) \otimes \mathscr{F}) \to (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)) \tag{2.50}$$

is measurable.

Definition 40. *A filtration of a sample space $(\Omega, \mathscr{F})$ is a non-decreasing family $\{\mathscr{F}_t : t \geq 0\}$ of sub$-\sigma-$fields of $\mathscr{F}$: $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$ for $0 < s < t < \infty$. We set $\mathscr{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathscr{F}_t\right)$.*

Remark 41. *Given a stochastic process, X, the simplest choice of filtration is that generated by the process itself, i.e.,*

$$\mathscr{F}_t^X := \sigma\left(X_s : 0 \leq s \leq t\right) \tag{2.51}$$

the simplest $\sigma-$field with respect to which X_s is measurable for every $s \in [0,t]$.

Definition 42. *Let $\{\mathscr{F}_t : t \geq 0\}$ be a filtration and X be a stochastic process. Define*

$\mathscr{F}_{t^-} := \sigma\left(\bigcup_{s<t} \mathscr{F}_s\right)$ to be the $\sigma-$field of events strictly prior to $t > 0$;

$\mathscr{F}_{t^+} := \bigcap_{\epsilon>0} \mathscr{F}_{t+\epsilon}$ to be the $\sigma-$field of events immediately after $t \geq 0$.

$\mathscr{F}_{0^-} := \mathscr{F}_0$

Then, we say

1. *the filtration $\{\mathscr{F}_t\}$ is right (left) continuous if $\mathscr{F}_t = \mathscr{F}_{t^+}$ (respectively $\mathscr{F}_t = \mathscr{F}_{t^-}$) holds for $t \geq 0$; and filtration $\{\mathscr{F}_t\}$ is continuous when it is both right continuous and left continuous.*

2. *the filtration* $\{\mathscr{F}_t\}$ *satisfies the usual conditions if it is right continuous and* $\mathscr{F}_0$ *contains all* $\mathbf{P}$*-negligible events, or events with zero probability.*

3. *the stochastic process* X *is adapted to* $\{\mathscr{F}_t\}$ *if for each* $t \geq 0$, X_t *is an* $\mathscr{F}_t-$*measurable random variable.*

4. *the stochastic process* X *is progressively measurable with respect to the filtration* $\{\mathscr{F}_t\}$ *if, for each* $t \geq 0$ *and* $A \in \mathscr{B}(\mathbb{R}^n)$, *the set* $\{(s,\omega); 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ *belongs to the product* $\sigma-$*field* $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$; *in other words if the mapping* $(s,\omega) \mapsto X_s(\omega) : \big([0,t] \times \Omega, \mathscr{B}([0,t]) \otimes \mathscr{F}_t\big) \to (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ *is measurable, for each* $t \geq 0$.

Proposition 43. *If the stochastic process* X *is measurable and adapted to the filtration* $\{\mathscr{F}_t\}$, *then it has a progressively measurable modification.*

For the detail on these definitions and the proof of the above result we refer to [44, 90]. Hereafter, thanks to proposition 43, we will not make any difference between a process X and its modification.

A random time T is an $\mathscr{F}-$measurable random variable, with values in $[0,\infty]$. For a stochastic process X and random time T, we define a function X_T on the event $T < \infty$ by

$$X_T(\omega) := X_{T(\omega)}(\omega). \tag{2.52}$$

If $X_T(\omega)$ is defined for all $\omega \in \Omega$, then X_∞ can also be defined on Ω, by setting $X_T(\omega) := X_\infty(\omega)$ on $\{T = \infty\}$.

Definition 44. *Let* $(\Omega, \mathscr{F})$ *be a measurable space equipped with the filtration* $\{\mathscr{F}_t\}$. *A random time* T *is called a stopping time with respect to the filtration, if the event* $\{T \leq t\} = \{\omega : T(\omega) \leq t\}$ *belongs to the* $\sigma-$*field* $\mathscr{F}_t$, *for all* $t \geq 0$. *A random time* T *is an optional time of the filtration* $\{\mathscr{F}_t\}$, *if* $\{\omega : T(\omega) < t\} \in \mathscr{F}_t$ *for all* $t \geq 0$.

Remark 45. *Every random time equal to a non-negative constant is a stopping time, [90, p. 6].*

Again, we consider the process $\{X_t : 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ adapted to a a given filtration $\{\mathscr{F}_t\}$ and such that $\mathbf{E}|X_t| < \infty$ holds for every $t \geq 0$. The process $\{X_t : 0 \leq t < \infty\}$ is said to be a submartingale (respectively, a supermartingale) if, for every $0 \leq s < t < \infty$, we have, a.s. $\mathbf{P}$: $\mathbf{E}(X_t|\mathscr{F}_s) \geq X_s$ (respectively, $\mathbf{E}(X_t|\mathscr{F}_s) \leq X_s$). $\{X_t : 0 \leq t < \infty\}$ is a martingale if it is both submaritingale and a supermartingale.

Now let $X = \{X_t, \mathscr{F}_t; 0 \leq t < \infty\}$ be a (continuous) process with $X_0 = 0$ a.s. If there exists a nondecreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\{\mathscr{F}_t\}$, such that $\{X_t^{(n)} := X_{t\wedge T_n}, \mathscr{F}_t; 0 \leq t < \infty\}$ is a martingale for each $n \geq 1$ and $\mathbf{P}[\lim_{n\to\infty} T_n = \infty] = 1$, then we say that X is a (*continuous*) local martingale and write $X \in \mathscr{M}^{loc}$ (respectively, $X \in \mathscr{M}^{c,loc}$ if X is continuous).

Remark 46. *Every martingale is a local martingale, but not the converse,[44, 90].*

Before we pass to other issues, we give one interesting result on continuous martingales;

Theorem 47. *Let $\{X_t, \mathscr{F}_t : 0 \leq t < \infty\}$ be right continuous submartingale such that $\sup\limits_{t\geq 0} \mathbf{E}(X_t^+) < \infty$. Then*

$$X_\infty(\omega) := \lim_{t\to\infty} X_t(\omega) \tag{2.53}$$

exists for a.e. $\omega \in \Omega$ and $\mathbf{E}|X_\infty| < \infty$.

Definition 48. *Consider a probability space $(\Omega, \mathscr{F}, \mathbf{P})$.*

(1) An adapted process A is called increasing if for $\mathbf{P} - a.e. \, \omega \in \Omega$ we have

(i) $A_0(\omega) = 0$

(ii) $t \mapsto A_t(\omega)$ is non-decreasing, right continuous function and $\mathbf{E}(A_t) < \infty$ for all $t \in [0, \infty)$.

(2) An increasing process A, is integrable if $\mathbf{E}(A_\infty) < \infty$, where $A_\infty := \lim\limits_{t\to\infty} A_t$.

(3) An increasing process A is natural, if for every bounded right continuous martingale $\{M_t, \mathscr{F}_t; 0 \leq t < \infty\}$ we have

$$\mathbf{E}\int_{(0,t]} M_s \mathrm{d}A_s = \mathbf{E}\int_{(0,t]} M_{s^-} \mathrm{d}A_s = \mathbf{E}\int_{(0,t]} M_{s^+} \mathrm{d}A_s = \mathbf{E}(M_t A_t) \tag{2.54}$$

Definition 49. *Let $\mathscr{S}(\mathscr{S}_a)$ be a class of stoping times T of a filtration $\{\mathscr{F}_t\}$ such that $\mathbf{P}(T < \infty) = 1$ (respectively $\mathbf{P}(T \leq a) = 1$ for a given finite number $a > 0$). The right continuous process $\{X_t, \mathscr{F}_t; 0 \leq t < \infty\}$ is said to be of class D, if the family $\{X_T\}_{T\in\mathscr{S}}$ is uniformly integrable (respectively of class DL, if the family $\{X_T\}_{T\in\mathscr{S}_a}$ is uniformly integrable, for every $0 < a < \infty$).*

The following result is due to Joseph L. Doob and Paul-André Meyer, which is a crucial transition towards stochastic integration.

Theorem 50 (Doob-Meyer Theorem)**.** *Let $\{\mathscr{F}_t\}$ satisfy the usual conditions. If the right continuous submartingale $X = \{X_t, \mathscr{F}_t : 0 \leq t < \infty\}$ is of class DL, then it admits the decomposition*

$$X_t = M_t + A_t, \qquad 0 \leq t < \infty, \tag{2.55}$$

where $M := \{M_t, \mathscr{F}_t : 0 \leq t < \infty\}$ is a right continuous martingale, $A = \{A_t, \mathscr{F}_t : 0 \leq t < \infty\}$ is an increasing process. If X is of class D, then M is a uniformly integrable martingale and A is integrable.

From now on, we refer to (2.55) as Doob's decomposition. Indeed, similar decomposition property, such as (2.59), of stochastic processes can be used as an alternative definition for continuous semimartingale, see [90, pp. 149]

Definition 51. *Let $X = \{X_t, \mathscr{F}_t\}$ be a right continuous martingale. We say that X is square integrable if $E(X_t^2) < \infty$ for all $t \geq 0$. If $X_0 = 0$ $\mathbf{P}$-a.s, we write $X \in \mathscr{M}_2$ (or $X \in \mathscr{M}_2^c$ if X is continuous.)*

Remark 52. *If $X \in \mathscr{M}_2$, then $X^2 = \{X_t^2, 0 \leq t < \infty\}$ is a submartingale of class*

DL, and therefore has the following Doob's decomposition;

$$X_t^2 = M_t + A_t, \qquad 0 \le t < \infty \tag{2.56}$$

with M_t a right continuous martingale and A_t a natural increasing process.

For a detailed discussion and proofs of the above results, we refer to [20, 44, 90, 138].

2.2.5 Brownian Motion

Definition 53. *A Brownian motion (standard one dimensional) is a continuous adapted process $B = \{B_t, \mathscr{F}_t : 0 \le t < \infty\}$ defined on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$, with properties that $B_0 = 0$ a.s. and for $0 \le s < t$, the increment $B_t - B_s$ is independent of $\mathscr{F}_s$ and is normally distributed with mean zero and variance $t - s$; where the variance of a random variable, say ζ, denoted by $V\zeta$ is given by*

$$V\zeta := \mathbf{E}(\zeta - \mathbf{E}\zeta)^2. \tag{2.57}$$

Definition 54. *Let n be a positive integer and μ a probability measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. Let $B = \{B_t, \mathscr{F}_t; t \ge 0\}$ be a continuous, adapted process with values in $\mathbb{R}^n$, defined on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$. This process is called a $n-$dimensional Brownian motion with initial distribution μ, if*

(i) $\mathbf{P}[B_0 \in \Gamma] = \mu(\Gamma), \forall \Gamma \in \mathscr{B}(\mathbb{R}^n)$;

(ii) for $0 \le s < t$, the increment $B_t - B_s$ is independent of $\mathscr{F}_s$ and is normally distributed with mean zero and covariance matrix equal to $(t - s)I_n$, where I_n is the $n \times n$ identity matrix.

Remark 55. *B is a square integrable martingale with $\langle B \rangle_t = t$ for all $t \ge 0$; where $\langle B \rangle_t = A_t$ is the increasing process in the Doob Meyer's decomposition of B^2 and is called the quadratic variation of B.*

A very detailed work on existence and construction of Brownian motions can be found in [44, 90].

2.2.6 Stochastic Integral

We introduce the Itô's integral formula for a process, say X. For the construction and other details we refer to [87, 88, 90, 102, 118]. From now on we denote a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ endowed with a filtration $\{\mathscr{F}_t\}$ by $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P})$. Now let $T > 0$, $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leq t\leq T}, \mathbf{P})$ be a probability basis and $\mathcal{H}$ be a Banach space. For an $\{\mathscr{F}_t\}$-adapted process X, we define

$$[X]_T^2 := \mathbf{E}\int_0^T X_t^2 \mathrm{d}\langle M\rangle_t,$$

when the RHS is finite, where $M \in \mathscr{M}_2^c$ is a continuous square integrable martingale.

Let $\mathscr{L}$ denote the set of equivalence classes of all measurable $\{\mathscr{F}_t\}$ adapted processes X, for which $[X]_T < \infty$ for all $T > 0$, and $\mathscr{L}^*$ denote the set of equivalence classes of progressively measurable processes satisfying $[X]_T < \infty$ for all $T > 0$.

Definition 56. *For $X \in \mathscr{L}^*$, we define the stochastic integral of X with respect to the martingale $M \in \mathscr{M}_2^c$ by the process $I(X) = \{I_t(X), \mathscr{F}_t; 0 \leq t < \infty\}$ which satisfies $\lim_{n\to\infty} \|I(X^{(n)}) - I(X)\| = 0$ for every sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathscr{L}_0$ with $\lim_{n\to 0}[X^{(n)} - X] = 0$; and we write*

$$I_t(X) = \int_0^t X_s \mathrm{d}M_s, \qquad 0 \leq t < \infty; \tag{2.58}$$

where $\mathscr{L}_0$ is the set of simple processes.

Let X, $Y \in \mathscr{L}^*$ $I(X)$, $M \in \mathscr{M}_2^c$. Then $I(X)$ defined above is a square integrable martingale and has a quadratic variation given by $\langle I(X)\rangle_t = \int_0^t X_u^2 \mathrm{d}\langle M\rangle_u$. Furthermore, for any two stopping times S, T such that $S \leq T$ of filtration $\{\mathscr{F}_t\}$ and any

$t > 0$ we have

$$\mathbf{E}[I_{t\wedge T}|\mathscr{F}_S] = I_{t\wedge S}, \quad \mathbf{P} - a.s,$$

$$\mathbf{E}\left[(I_{t\wedge T}(X) - I_{t\wedge S}(X))(I_{t\wedge T}(Y)) - (I_{t\wedge S}(Y))|\mathscr{F}_S\right] = \mathbf{E}\left[\int_{t\wedge S}^{t\wedge T} X_u Y_u \mathrm{d}\langle M\rangle_u \middle| \mathscr{F}_S\right],$$

and in particular for any $s \in [0, t]$,

$$\mathbf{E}[(I_t(X) - I_s(X))(I_t(Y) - I_s(Y))|\mathscr{F}_s] = \mathbf{E}\left[\int_s^t X_u Y_u \mathrm{d}\langle M\rangle_u \middle| \mathscr{F}_s\right].$$

Finally, $I_{t\wedge T}(X) = I_t(\tilde{X})$, where $\tilde{X}_t(\omega) := X_t(\omega)1_{\{t\leq T(\omega)\}}$.

One can find detailed proofs on properties of stochastic integrals and more in [90].

Lemma 57 (Kunita-Watanabe, 1967)**.** *For $M, N \in \mathscr{M}_2^c$, $X \in \mathscr{L}^*(M)$, and $Y \in \mathscr{L}^*(N)$, the following holds a.s.*

$$\int_0^t |X_s Y_s| \mathrm{d}\check{\xi}_s \leq \left(\int_0^t X_s^2 \mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\int_0^t Y_s^2 \mathrm{d}\langle N\rangle_s\right)^{1/2}$$

where $\check{\xi}$ is the cross variation of the processes $\xi := \langle M, N\rangle$ on $[0, s]$.

The following property of a continuous semimartingale is a crucial concept in the study of stochastic integrals: for $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P})$ a basic probability space, a continuous semimartingale $X = \{X_t, \mathscr{F}_t; 0 \leq t < \infty\}$ is an adapted process which has a unique decomposition of the type

$$X_t = X_0 + M_t + B_t; \qquad 0 \leq t < \infty \mathbf{P} - a.s, \tag{2.59}$$

where $M = \{M_t, \mathscr{F}_t; 0 \leq t < \infty\} \in \mathscr{M}^{c,loc}$ and $B = \{B_t, \mathscr{F}_t; 0 \leq t < \infty\}$ is the difference of continuous, nondecreasing , adapted processes $\{A_t^{\pm}, \mathscr{F}_t; 0 \leq t < \infty\}$

such that

$$B_t = A_t^+ - A_t^-; \quad 0 \le t < \infty,$$

with $A_0^{\pm} = 0$, $\mathbf{P} - a.s$, [90]

The following result is called Itô's integral formula, and, some times the chain rule of stochastic calculus, due to Kiyosi Itô in [87]; for the proof we refer to [87, 90, 102].

Theorem 58. *For $f : \mathbb{R} \to \mathbb{R}$ in C^2 and $X = \{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ a continuous semimartingale with decomposition* (2.59)*, we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}Ms + \int_0^t f'(X_s)\mathrm{d}Bs + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle M\rangle_s, \quad 0 \le t < \infty. \tag{2.60}$$

The multidimensional version of Theorem 58 is stated as follows, see [90]:

Theorem 59. *Let $\{M_t := (M_t^{(1)}, M_t^{(2)}, \dots, M_t^{(n)}), \mathscr{F}_t; 0 \le t < \infty\}$ be a vector of local martingales in $\mathscr{M}^{c,loc}$, $\{B := (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(n)}), \mathscr{F}_t; 0 \le t < \infty\}$ a vector of adapted processes of bounded variation with $B_0 = 0$, and set $X_t = X_0 + M_t + B_t$; $0 \le t < \infty$, where X_0 is an $\mathscr{F}_0-$measurable random vector in $\mathbb{R}^n$. Let $f(t,x) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be of class $C^{1,2}([0,\infty) \times \mathbb{R}^n$. Then $\mathbf{P}$ a.s.,*

$$\begin{aligned} f(t,X_t) =& f(0,X_0) + \int_0^t \frac{\partial}{\partial t} f(s,X_s)\mathrm{d}s + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(s,X_s)\mathrm{d}B_s^{(i)} \\ &+ \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(s,X_s)\mathrm{d}M_s^{(i)} \\ &+ \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s,X_s)\mathrm{d}\langle M^{(i)}, M^{(j)}\rangle_s, \quad 0 \le t < \infty \end{aligned} \tag{2.61}$$

One can also write the Itô integral in differential form as follows,

$$\begin{aligned} \mathrm{d}f(X_t) &= f'(X_t)\mathrm{d}M_t + f'(X_t)\mathrm{d}B_t + \frac{1}{2}f''(X_t)\mathrm{d}\langle M\rangle_t \\ &= f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}\langle M\rangle_t, \quad 0 \le t < \infty, \end{aligned} \tag{2.62}$$

for one dimensional process, and

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)dt + Df(t, X_t) \cdot dB_t + Df(t, X_t) \cdot dM_t + \frac{1}{2} D^2 f(t, X_t) \cdot d\langle M \rangle_t \tag{2.63}$$

where $D^2 f(t, X_t) \cdot d\langle M \rangle_t = \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t) d\langle M^{(i)}, M^{(j)} \rangle$

We conclude the section by the following three results, which are repeatedly used in the process of writing the thesis. The results are taken respectively from [133], [90] and [60].

Theorem 60 (Itô Isometry)**.** *Let $M = \{M_t, \mathscr{F}_t; 0 \leq t < \infty\}$ be in $\mathscr{M}_2^c$ and suppose $X \in \mathscr{L}_\infty^*(M)$. Then we have*

$$\mathbf{E}\left(\int_0^\infty X_t dM_t\right)^2 = \mathbf{E}\int_0^\infty X_t^2 d\langle M \rangle_t. \tag{2.64}$$

Theorem 61 (The Burkholder-Davis-Gundy Inequalities)**.** *Let $M \in \mathcal{M}^{c,loc}$. For every $m > 0$ there exist universal positive constants k_m, K_m (depending only on m), such that*

$$k_m \mathbf{E}(\langle M \rangle_T^m) \leq \mathbf{E}[(M_T^*)^{2m}] \leq K_m \mathbf{E}(\langle M \rangle_T^m)$$

holds for every stopping time T; where $M_t^* := \max_{0 \leq s \leq t} |M_s|$.

Theorem 62 (Young's Inequality with ϵ)**.** *Let a, b positive real numbers and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\epsilon > 0$*

$$ab \leq \epsilon a^p + C_\epsilon b^q,$$

where $C_\epsilon = \dfrac{1}{(\epsilon p)^{q/p} q}(\epsilon p)^{-q/p}$.

Part II

Stochastic Magnetohydrodynamics Equations

Chapter 3

Stochastic Magnetohydrodynamics Equations

3.1 Introduction

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$ of right-continuous $\sigma-$algebra. Let $W = (\omega_i(t))_{i\in\mathbb{N}}$ be an infinite dimensional Wiener process on this probability space; the components ω_i are independently, identically distributed standard one dimensional Wiener Processes. We assume that the filtration $\{\mathscr{F}\}_{t\geq 0}$ is generated by W.

Our focus is towards investigating the stochastic magnetohydrodynamics (SMHD) equation;

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\Pi - (b\cdot\nabla)b - \Delta u = g_1\dot{W} & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n \\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u - \Delta b = g_2\dot{W} & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n \\ \text{div } u = \text{div } b = 0 & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0 & \text{in } \Omega\times\mathbb{R}^n \end{cases} \tag{3.1}$$

where $n \geq 2$ is a natural number, $u = u(\omega, t, x)$ is the flow velocity, $b = b(\omega, t, x)$ is the magnetic field, Π is the total pressure, and $g_1\dot{W}$ and $g_2\dot{W}$ are random external forces; W is a standard infinite dimensional Wiener process discussed above, and the stochastic differential is understood in the Itô sense. The Laplace, the gradient and divergence operators are defined respectively as

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$
$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\rangle,$$
$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i},$$

where n is the dimension of the space variable.

The system of equations (3.1) at least in principle governs the flow of incompressible electrically conducting fluids, such as plasma, where the random (or Brownian) movement of fluid particles is taken into consideration. Magnetohydrodynamics equations are obtained by coupling the stochastic Navier-Stokes and Maxwell equations in a certain way and plays a crucial role in the fields of astrophysics, cosmology, geophysics, plasma physics and medicine. Despite their tremendous importance, fully understanding fluid and plasma flows is one of the most challenging tasks of our time (see [62]), it is widely accepted that the difficulty arises from their turbulent nature. Nevertheless, since from the end of the 15th century or beginning of the 16th century when a very systematic observation of Da Vinci and Richter, a lot of research has been done to unlock the mystery of turbulent flows; one can mention the following pioneering works of Reynolds [141, 142], Poincaré and Magini [135], Taylor [157, 158], Leray [110, 111, 113], Heisenberg [78], De Karman and Howarth [52], Millionshchikov [124], Kolmogorov [91, 92, 93, 94, 95], Obukhov [130, 131, 132], Kraichnan [96, 97, 98, 99, 100, 101], Ladyzhenskaya [103], Hopf [79, 81], Sritharan and Sundar [153], Bensoussan and Temam [14], Fujita and Kato [73] and so on.

As it was mentioned in chapter 1, generally, we have two approaches to tackle the problem of turbulent flows. The first is a direct approach, in which one analyzes properties like existence, uniqueness, regularity of solutions for system equations like Navier-Stokes equations for non conductive fluid and magnetohydrodynamics equations for conductive fluids and plasma, to cite few from the works in this direction [73, 79, 80, 104, 105, 106, 110, 111, 113, 147, 150, 159]. The second approach has evolved from the very observation of Da Vinci and Richter, and later materialized by Reynolds where he split the velocity field into chaotic (random) and regular components (see [141, 142]). In fact this approach of Reynolds gave rise to a whole theory of turbulence. To see some of the works in this direction we refer to [19, 52, 91, 92, 93, 94, 95, 96, 97, 99, 115, 124, 130, 131, 132, 164] and the references in there. The book by Davidson, Kaneda, Moffatt, and Sreenivasan [51] takes us through the process of evolution of turbulent theory and short biography of the main contributors.

In this part of the thesis we will be focusing on the first method, where a direct approach is applied to analyze long time and short time behavior of the solution field, and the second method is a subject of Part III.

The fascinating link between Navier-Stokes Equations (NSE) and turbulence could be traced back to the pioneering work of Leray on incompressible viscous fluids [110, 112, 113] where weak solutions are referred to as turbulent solutions. In these papers Leray established the mathematical study of NSE on firm grounds as far as weak (variational) solutions are concerned. The monographs of Ladyzhenskaya and Silverman [105] and Temam [159] cover most of the achievements in that direction. To see some of the results in this direction for MHD equations, we refer to [2, 25, 26, 77, 106, 150, 173] and the references in there.

Following the development of stochastic differential equations, stochastic Navier-Stokes equations have become a central tool in the still unfinished journey to unlock the mystery of turbulence. The mathematical study of stochastic Navier-Stokes equations began in the pioneering work of Bensoussan and Temam in [14] where the external force is driven by a white noise. This work was immediately followed by the works of Bensoussan and Temam [15] and Frisch et al. [72]. Indeed, the approach has attracted lots of attention and tremendous amount of research is carried out, for instance to cite few [13, 16, 22, 27, 28, 54, 55, 64, 119, 120, 121, 122, 123, 146]. The work of Mikulevicius and Rozovskii in [120] is the first where the model was derived rigorously under reasonable physical assumptions. It is therefore not an exaggeration to claim that SNSE are no longer an hypothetical model for turbulent flows of fluids but a very credible tool for the investigation of turbulence.

Modeling turbulence in MHD flows by stochastic MHD equations is also a well accepted approach. For instance; Zeldovich, Ruzmaikin, and Sokiloff in [172] have used random external forces which depend nonlinearly on velocity and magnetic fields in their treatment of the numerical simulation of MHD turbulence, Sritharan and Sundar in [153] proved existence and uniqueness of space time statistical solutions by means of weak convergence method, Barbu and Da Prato in [11] proved existence of solutions to stochastic MHD equations of dimension two driven by random exterior forcing terms both in the velocity and in the magnetic field, Sundar in [154] established existence and uniqueness result for two dimensional stochastic MHD model in the presence of multiplicative noise or additive fractional Brownian noise, Sango in [147] presented a very detailed investigation, using the Galerkin approximation, on the problem of existence of weak solutions for three dimensional stochastic MHD model with multiplicative noises, Deugoué et al. in [56] proved existence of weak solution for three dimensional stochastic MHD alpha model. Recently, Motyl in [128], and Tan et al. also considered the three dimensional stochastic MHD with multiplicative noise. Tan et al. in [155] used the contraction mapping principle to establish existence and uniqueness of strong local solution and strong global solution with small data.

One of our goals in this thesis is to investigate existence and uniqueness of global and local solutions (strong in probabilistic sense) to the system of stochastic partial differential equations given by (3.1). In recent years the use of Harmonic analysis tools such as the Littlewood-Paley theory blended with semigroup theory and fixed point theory has lead to promising results towards unlocking the secrets of Navier-Stokes equations and MHD equations. For instance the works of Chemin [36, 37, 38], Bahouri et al. [10], Danchin [48, 48, 50], Chae and Lee [30], Gallagher and Planchon [74, 75] on Navier-Stokes equations and Cannone et al. [25], Abidi and Hmidi [2] and Zhang [173] on MHD equations can be mentioned as pioneering examples. The present work is in this direction, i.e., application of Littlewood-Paley theory to stochastic MHD equations.

We established local and global existence and uniqueness of strong solution (in probabilistic sense) for (3.1); the global result holds with sufficiently small initial data. The work is done through the following main steps. Firstly, we reduce (3.1) in to a more symmetric form by introducing transformations, $\theta = u+b$ and $\beta = u-b$. Then we drop the pressure term by applying the Leray projector expressed in terms of Riesz transforms. Finally we study the reduced problem by seeking a solution that can be written as a sum of solutions of system of two heat equations, where the first has a random external force driven by Brownian noise attached to it and the later takes a form of deterministic heat equation, see (1.5) and (1.6). The stochastic heat equation is studied by making use of Littlewood-Paley theory and Itô's calculus and for the deterministic component we use results from [10], which will be given in the process. To the best of our knowledge this is the first work which blends Littlewood-Paley theory, Bony's paradifferential calculus and stochastic calculus to treat stochastic MHD equations.

By being able to make use of the Littlewood-Paley theory together with Tchebychev's inequality and Itô calculus, we managed to get novel results which are given in subsection 1.2.1. The global existence and uniqueness result is published in [148]. In fact, Tan et al. in [155] used the the contraction mapping principle to establish global existence result for stochastic MHD equations when the initial data is sufficiently small.

The rest of Part III is organized as follows; in the next section, section 3.2, we reduce (3.1) into a simpler form by applying linear transformations θ and β and the Leray projector $\mathcal{P}$ which is defined in the sequel. In section 3.3 we make a necessary mathematical preparation and give two estimates on stochastic heat equations; in fact these estimates played a central role. In section 3.4 we state our main results and finally in section 3.5 we give the detail proof of the results.

3.2 Reduction of the problem

We start tackling the problem by reducing (3.1) to a relatively simpler but equivalent form. This will be achieved by the following procedures.

Firstly we apply the transformation $\beta = u - b$, $\theta = u + b$, $\tilde{G}_1 = g_1 - g_2$ and $\tilde{G}_2 = g_1 + g_2$ to (3.1) and get

$$\begin{cases} \partial_t\beta - \Delta\beta + (\theta \cdot \nabla)\,\beta + \nabla\Pi = \tilde{G}_1\dot{W} & \text{in } \Omega \times (0,\infty) \times \mathbb{R}^n \\ \partial_t\theta - \Delta\theta + (\beta \cdot \nabla)\,\theta + \nabla\Pi = \tilde{G}_2\dot{W} & \text{in } \Omega \times (0,\infty) \times \mathbb{R}^n \\ \operatorname{div}\beta = 0,\ \operatorname{div}\theta = 0 & \text{in } \Omega \times (0,\infty) \times \mathbb{R}^n \\ \theta|_{t=0} = u_0 + b_0,\ \beta|_{t=0} = u_0 - b_0 & \text{in } \Omega \times \mathbb{R}^n \end{cases}. \tag{3.2}$$

For further simplification we introduce the Leray projector $\mathcal{P}$, named after the French mathematician Jean Leray (1906-1998), defined by

$$\mathcal{P}\cdot := Id - \nabla\Delta^{-1}\text{div}\cdot\,.$$

Clearly $\mathcal{P}$ is linear, homogeneous differential operator of order zero. Moreover, the Fourier transform of $\mathcal{P}$ takes the form

$$\mathcal{F}(\mathcal{P}f)^j(\xi) = \sum_{k=1}^{n}\left(\delta_{kj} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\hat{f}^j(\xi), \tag{3.3}$$

where

$$\delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases},$$

which makes $\mathcal{P}$ a pseudo-differential operator, see [43].

Now apply $\mathcal{P}$ to eliminate the pressure term from (3.2), the resulting equation is

$$\begin{cases} \partial_t\beta - \Delta\beta + \mathcal{P}(\theta\cdot\nabla\beta) = G_1\dot{W} & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n \\ \partial_t\theta - \Delta\theta + \mathcal{P}(\beta\cdot\nabla\theta) = G_2\dot{W} & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^n \\ \beta|_{t=0} = \beta_0 & \text{in } \Omega\times\mathbb{R}^n \\ \theta|_{t=0} = \theta_0 & \text{in } \Omega\times\mathbb{R}^n \end{cases} \tag{3.4}$$

where $G_i = \mathcal{P}\tilde{G}_i$ for $i = 1, 2$.

Thus we have a reduced system of equations (3.4) which is an equivalent formulation of (3.1).

This can actually be rewritten in a much simplified form using matrix notation and by introducing an operator $\mathbf{Q}$, defined by

$$\mathbf{Q}\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) := -\frac{1}{2}\left(\begin{pmatrix}\mathcal{P}(\theta_1\cdot\nabla\beta_2)\\ \mathcal{P}(\beta_1\cdot\nabla\theta_2)\end{pmatrix} + \begin{pmatrix}\mathcal{P}(\theta_2\cdot\nabla\beta_1)\\ \mathcal{P}(\beta_2\cdot\nabla\theta_1)\end{pmatrix}\right). \tag{3.5}$$

It is not difficult to see that the $\mathbf{Q}$ is a symmetric bilinear operator. For instance, if $\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}$, $\begin{pmatrix}\beta_1'\\ \theta_1'\end{pmatrix}$, $\begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}$ and $\begin{pmatrix}\beta_2'\\ \theta_2'\end{pmatrix}$ in an appropriate space, we have

$$\begin{aligned}
\mathbf{Q}&\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix} + \begin{pmatrix}\beta_1'\\ \theta_1'\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) = \mathbf{Q}\left(\begin{pmatrix}\beta_1+\beta_1'\\ \theta_1+\theta_1'\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) \\
&= -\frac{1}{2}\left(\begin{pmatrix}\mathcal{P}((\theta_1+\theta_1')\cdot\nabla\beta_2)\\ \mathcal{P}((\beta_1+\beta_1')\cdot\nabla\theta_2)\end{pmatrix} + \begin{pmatrix}\mathcal{P}(\theta_2\cdot\nabla(\beta_1+\beta_1'))\\ \mathcal{P}(\beta_2\cdot\nabla(\theta_1+\theta_1'))\end{pmatrix}\right) \\
&= -\frac{1}{2}\left(\begin{pmatrix}\mathcal{P}(\theta_1\cdot\nabla\beta_2)+\mathcal{P}(\theta_1'\cdot\nabla\beta_2)\\ \mathcal{P}(\beta_1\cdot\nabla\theta_2)+\mathcal{P}(\beta_1'\cdot\nabla\theta_2)\end{pmatrix} + \begin{pmatrix}\mathcal{P}(\theta_2\cdot\nabla\beta_1)+\mathcal{P}(\theta_2\cdot\nabla\beta_1')\\ \mathcal{P}(\beta_2\cdot\nabla\theta_1)+\mathcal{P}(\beta_2\cdot\nabla\theta_1')\end{pmatrix}\right) \\
&= -\frac{1}{2}\left(\begin{pmatrix}\mathcal{P}(\theta_1\cdot\nabla\beta_2)\\ \mathcal{P}(\beta_1\cdot\nabla\theta_2)\end{pmatrix} + \begin{pmatrix}\mathcal{P}(\theta_2\cdot\nabla\beta_1)\\ \mathcal{P}(\beta_2\cdot\nabla\theta_1)\end{pmatrix}\right) - \frac{1}{2}\left(\begin{pmatrix}\mathcal{P}(\theta_1'\cdot\nabla\beta_2)\\ \mathcal{P}(\beta_1'\cdot\nabla\theta_2)\end{pmatrix} + \begin{pmatrix}\mathcal{P}(\theta_2\cdot\nabla\beta_1')\\ \mathcal{P}(\beta_2\cdot\nabla\theta_1')\end{pmatrix}\right)
\end{aligned}$$

$$= \mathbf{Q}\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) + \mathbf{Q}\left(\begin{pmatrix}\beta_1'\\ \theta_1'\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right).$$

Note that the purpose of $\mathbf{Q}$ is not only to simplify our model, but also it enables us use the fixed point argument, [10, Ch. 5]. Now combining (3.5) with (3.4) we get

$$\begin{cases} \mathrm{d}\begin{pmatrix}\beta\\ \theta\end{pmatrix} - \Delta\begin{pmatrix}\beta\\ \theta\end{pmatrix}\mathrm{d}t = \mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}, \begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\mathrm{d}t + \begin{pmatrix}G_1\\ G_2\end{pmatrix}\mathrm{d}W_t \\ \begin{pmatrix}\beta\\ \theta\end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix} \end{cases}, \tag{3.6}$$

which is an equivalent but more handy formulation for (3.1) than (3.4) is.

3.3 A Priori Results

It is well known that for a Banach space B of distributions on $\mathbb{R}^n$, the space-time Banach space $L^q(0,T;B)$ for appropriate B plays an important role in the study of partial differential equations. However, in particular for $B = \dot{B}^s_{p,r}$, the usual space-time Banach space $L^q(0,T;\dot{B}^s_{p,r})$ for $p,q,r \in [1,+\infty], s \in \mathbb{R}$ does not have a structure which is natural to the structure of Besov spaces due to the fact that time integration will be performed before the l^r-norm summation. For this very technical reason we give the following definition due to Chemin and Lerner, [33].

Definition 63 (Chemin-Lerner)**.** *For $T > 0, s \in \mathbb{R}, 1 \le p,r,q \le +\infty$, we set*

$$\|u\|_{\mathcal{L}^q_T(\dot{B}^s_{p,r})} = \left\| 2^{js}\|\dot{\Delta}_j u\|_{L^q_T(L^p)} \right\|_{l^r}. \tag{3.7}$$

The space $\mathcal{L}^q_T(\dot{B}^s_{p,r})$ is defined as the set of tempered distributions u over $[0,T]\times\mathbb{R}^n$ such that $u(t) \in \mathcal{S}'_h$ for each $t \in [0,T]$ and $\|u\|_{\mathcal{L}^q_T(\dot{B}^s_{p,r})} < \infty$.

Here we would like to note the following important relation between the ordinary evolution space and the Chemin-Lerner space regarding Besov spaces. Now observe that from (3.7) we have

$$\left\| 2^{js}\|\dot{\Delta}_j u\|_{L_T^q(L^p)} \right\|_{l^r} = \left(\sum_{j\in\mathbb{Z}} \left(\int_0^T 2^{jsq} \|\dot{\Delta}_j u\|_{L^p}^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}$$
$$= \left(\left(\sum_{j\in\mathbb{Z}} \left(\int_0^T 2^{jsq} \|\dot{\Delta}_j u\|_{L^p}^q \right)^{\frac{r}{q}} \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}.$$

Thus we have the following remark

Remark 64. *According to Minkowski inequality, we have*

$$\|u\|_{\mathcal{L}_T^q(\dot{B}_{p,r}^s)} \leq \|u\|_{L_T^q(\dot{B}_{p,r}^s)} \quad \textit{if } r \geq q, \|u\|_{\mathcal{L}_T^q(\dot{B}_{p,r}^s)} \geq \|u\|_{L_T^q(\dot{B}_{p,r}^s)} \quad \textit{if } r \leq q.$$

Particularly for $p = r = q = 2$ and $s < \frac{n}{2}$ we have $\|u\|_{\mathcal{L}_T^2(\dot{B}_{2,2}^s)} \equiv \|u\|_{L_T^2(\dot{H}^s)}$; for the detail on relations between Besov and Sobolev spaces we refer to [10, p. 63-102].

Theorem 65. *For $1 \leq p, r < \infty$ and $s < \frac{n}{p}$ the space $\mathcal{L}_T^q(\dot{B}_{p,r}^s)$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{L}_T^q(\dot{B}_{p,r}^s)$. Then for every $\varepsilon > 0$, there is a positive integer N such that

$$\|u_m - u_n\|_{\mathcal{L}_T^q(\dot{B}_{p,r}^s)} = \left(\sum_{j\in\mathbb{Z}} \left(\int_0^T 2^{jsq} \|\dot{\Delta}_j u_m - \dot{\Delta}_j u_n\|_{L^p}^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} < \varepsilon, \quad \text{for } n, m \geq N.$$

Then the sequence

$$\left\{ \left(\int_0^T 2^{jsq} \|\dot{\Delta}_j u_m - \dot{\Delta}_j u_n\|_{L^p}^q \right)^{\frac{1}{q}} \right\}_{j\in\mathbb{Z}}$$

converges to zero for all $m, n \geq N$. Hence, $\left\{ 2^{js}\dot{\Delta}_j u_m \right\}_m$ is a Cauchy sequence in $L^q(0, T; L^p)$. The space $L^q(0, T; L^p)$ being a complete space for the given values of p and q, then for each j we have a u^j such that $\dot{\Delta}_j u_m$ converges to u^j as m goes to

∞. Now appealing to Proposition 18, we have a $u \in \dot{B}^s_{p,r}$ such that

$$u = \sum_j u^j$$

Next we show that u belongs to the space $\mathcal{L}^q_T(\dot{B}^s_{p,r})$. By definition, u^j is supported in a cell of size 2^j. Therefore, it is logical to assume that $u^j = \dot{\Delta}_j u$. Thus we have $\dot{\Delta}_j u_m \to \dot{\Delta}_j u$ in $L^q(0,T;L^p)$ as $m \to \infty$. Therefore, there is a positive integer $N' \in \mathbb{N}$ such that whenever $m \geq N'$ we have

$$\left(\int_0^T 2^{jsq}\|\dot{\Delta}_j u\|_p^q \mathrm{d}t\right)^{\frac{1}{q}} \leq \left(\int_0^T 2^{jsq}\|\dot{\Delta}_j u_m\|_p^q \mathrm{d}t\right)^{\frac{1}{q}} + \varepsilon_j$$

where ε_j is a positive real number such that $\varepsilon_j \lll 2^{-|j|}$ for each j. Now taking the l^r norm of the sequence

$$\left\{\left(\int_0^T 2^{jsq}\|\dot{\Delta}_j u_m\|_p^q \mathrm{d}t\right)^{\frac{1}{q}} + \varepsilon_j\right\}_j$$

we get

$$\begin{aligned}&\left(\sum_{j\in\mathbb{Z}}\left(\left(\int_0^T 2^{jsq}\|\dot{\Delta}_j u_m\|_p^q \mathrm{d}t\right)^{\frac{1}{q}} + \varepsilon_j\right)^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{j\in\mathbb{Z}}\left(\int_0^T 2^{jsq}\|\dot{\Delta}_j u_m\|_p^q \mathrm{d}t\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} + \left(\sum_{j\in\mathbb{Z}}\varepsilon_j^r\right)^{\frac{1}{r}}.\end{aligned}$$

The right hand side is finite as u_m belongs to $\mathcal{L}^q_T(\dot{B}^s_{p,r})$ and the series $\sum_{j\in\mathbb{Z}}\varepsilon_j^r$ is convergent. Thus,

$$\left(\sum_{j\in\mathbb{Z}}\left(\int_0^T 2^{jsq}\|\dot{\Delta}_j u\|_{L^p}^q \mathrm{d}t\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} < \infty.$$

Hence $u \in \mathcal{L}^q_T(\dot{B}^s_{p,r})$. This completes the proof. □

We next give an adaptation of Chemin-Lerner spaces to probabilistic Besov type evolution space. Before that, we give the usual definition of probabilistic evolution spaces.

Definition 66. *Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P})$ be a stochastic basis and B a separable Banach space. For any $p, r \in [1, \infty]$ we denote by $L^p(\Omega, \mathbf{P}; L^r(0, T; B))$ the space of processes $u = u(\omega, t)$ with values in B defined on $\Omega \times [0, T]$ such that;*

(i) $u(\cdot, t)$ is progressively measurable,

(ii) $u(\omega, t) \in B$ for almost all (ω, t) and

(iii)

$$\|u\|_{L^p(\Omega,\mathbf{P};L^r(0,T;B))} = \left(\mathbf{E}\left(\int_0^T \|u\|_B^r \mathrm{d}t\right)^{\frac{p}{r}}\right)^{\frac{1}{p}}, \tag{3.8}$$

where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$.

Definition 67. *Given a filtered probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P}\right)$ with the expectation $\mathbf{E}$ and $T > 0$, we denote by $\mathcal{M}_T$ set of all functions $f : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ $f(\cdot, x) : \Omega \times [0, T] \to \mathbb{R}^n$ is progressively measurable.*

Definition 68. *Let $p, r \in [1, \infty]$, σ, $\rho \in [1, \infty)$, $s \in \mathbb{R}$ and $T > 0$. We denote by $\mathcal{L}^\sigma_\Omega \mathcal{L}^\rho_T(\dot{B}^s_{p,r})$, the space of distribution-processes $f \in \mathcal{M}_T$ such that $f(t, \omega) \in \mathcal{S}'_h$, $\mathbf{P} - a.s$, and the quasinorm,*

$$\|f\|_{\mathcal{L}^\sigma_\Omega \mathcal{L}^\rho_T(\dot{B}^s_{p,r})} = \begin{cases} \left(\sum\limits_{j\in\mathbb{Z}} 2^{jsr}[\mathbf{E}(\int_0^T \|\dot{\Delta}_j f(t)\|^\rho_{L^p} dt)^{\sigma/\rho}]^{r/\sigma}\right)^{1/r} & \text{if } 1 \le r < \infty, \\ \sup\limits_{j\in\mathbb{Z}} 2^{js}[\mathbf{E}(\int_0^T \|\dot{\Delta}_j f(t)\|^\rho_{L^p} dt)^{\sigma/\rho}]^{1/\sigma} & \text{if } r = \infty. \end{cases}$$

is finite.

Remark 69. *Again from Minkowski inequality and Remark 2 we have that $\mathcal{L}^\sigma_\Omega \mathcal{L}^\rho_T(\dot{B}^s_{p,r}) = L^\sigma_\Omega L^\rho_T(\dot{B}^s_{p,r})$ under the condition that $\sigma = \rho = r = p$. The space $L^\sigma_\Omega L^\sigma_T\left(\dot{B}^s_{p,r}\right)$ is* *defined as a Bochner space (modulo usual measurability conditions) with the norm*

$$\|f\|_{L^\sigma_\Omega L^\sigma_T(\dot{B}^s_{p,r})} = \left(\mathbf{E}\int_0^T \|f(t)\|^\sigma_{\dot{B}^s_{p,r}}\,dt\right)^{1/\sigma}.$$

By Minkowski inequality, the inequalities in Remark 2 are preserved for $\mathcal{L}^q_\Omega \mathcal{L}^q_T\left(\dot{B}^s_{p,r}\right)$ *and* $L^q_\Omega L^q_T\left(\dot{B}^s_{p,r}\right)$ *with the obvious relations between* q *and* r*.*

Once equipped with the necessary definitions, we return to discuss our simplified model (3.6).

Our aim is to find a solution $\binom{\beta}{\theta}$ of (3.6) such that

$$\begin{pmatrix}\beta\\ \theta\end{pmatrix} = \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}, \begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) \tag{3.9}$$

where $\binom{\alpha}{\gamma}$ satisfies the system

$$\begin{cases} \partial_t \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} - \Delta \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} = \begin{pmatrix}G_1\\ G_2\end{pmatrix}\dot{W} \\ \left.\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\right|_{t=0} = \begin{pmatrix}u_0 - b_0\\ u_0 + b_0\end{pmatrix} \end{cases} \tag{3.10}$$

and B is a bilinear form satisfying the heat equation

$$\begin{cases} \partial_t B\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) - \Delta B\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) = \mathbf{Q}\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right) \\ \left.B\left(\begin{pmatrix}\beta_1\\ \theta_1\end{pmatrix}, \begin{pmatrix}\beta_2\\ \theta_2\end{pmatrix}\right)\right|_{t=0} = \mathbf{0} \end{cases} \tag{3.11}$$

for $\binom{\beta_1}{\theta_1}, \binom{\beta_2}{\theta_2}$ in appropriate spaces.

Therefore our main task becomes investigating stochastic heat equation of the type (3.12) and deterministic heat equation of the type (3.13) given by;

$$\begin{cases} dv - \Delta v dt &= f dW \\ v|_{t=0} &= u_0 \end{cases}, \tag{3.12}$$

$$\begin{cases} \partial_t u - \Delta u &= f \\ u|_{t=0} &= 0. \end{cases}. \tag{3.13}$$

Since we have plenty of literatures done on the model of heat equation of type (3.13) in the required frame work of Besov spaces (see [10, 37, 49] and references in there), the only duty left to us is to investigate the stochastic heat equation model of type (3.12).

Estimates for stochastic heat equation in Besov and Sobolev spaces

This section is the central part of the work. We establish key a priori estimates for the solution of the initial value problem (3.12). Indeed, taking the Fourier transform of (3.12) with respect to the spatial variable yields,

$$\begin{cases} \mathrm{d}\hat{v}(t,\xi) &= -|\xi|^2 \hat{v}(t,\xi)\ \mathrm{d}t + \hat{f}(t,\xi)\ \mathrm{d}W \\ \hat{v}|_{t=0}(\xi) &= \hat{u}_0(\xi) \end{cases}. \tag{3.14}$$

This is a linear stochastic differential equation and it has a unique solution

$$\hat{v}(t,\xi) = H\left(t, \hat{f}(t,\xi), \hat{u}_0(\xi), W(t)\right),$$

for some integral function H in the space distribution-valued processes on $\mathbb{R}^{n+1}$. Hence $v(t,x) = \mathcal{F}^{-1}\left(H\left(t, \hat{f}(t,\xi), \hat{u}_0(\xi), W(t)\right)\right)$ is a formal solution of (3.12). Therefore, our task is to establish the regularity in the framework of Besov spaces. We shall give two regularity results which are crucially important for proving Theorem 78 and Theorem 79.

Theorem 70. *Let u_0 be $\mathscr{F}_0$ measurable and f progressively measurable on $\Omega \times [0,T] \times \mathbb{R}^n$ and for $q \in [2,\infty], \sigma \in [2,\infty), s \in \mathbb{R}$*

$$u_0 \in \mathcal{L}^\sigma_\Omega(\dot{B}^{\frac{s}{\sigma}}_{2,q}), f \in \mathcal{L}^\sigma_\Omega \mathcal{L}^\sigma_T(\dot{B}^{\frac{s}{\sigma}}_{2,q}),$$

then the solution v of (3.12) is in the space

$$\mathcal{L}^\sigma_\Omega \mathcal{L}^\infty_T(\dot{B}^{\frac{s}{\sigma}}_{2,q}) \cap \mathcal{L}^\sigma_\Omega \mathcal{L}^\sigma_T(\dot{B}^{\frac{s+2}{\sigma}}_{2,q}),$$

and

$$\|v\|_{\mathcal{L}^\sigma_\Omega \mathcal{L}^\infty_T(\dot{B}^{\frac{s}{\sigma}}_{2,q})} + \|v\|_{\mathcal{L}^\sigma_\Omega \mathcal{L}^\sigma_T(\dot{B}^{\frac{s+2}{\sigma}}_{2,q})} \leq C\left[(1+T^{\frac{\sigma-2}{2}})\|f\|_{\mathcal{L}^\sigma_\Omega \mathcal{L}^\sigma_T(\dot{B}^{\frac{s}{\sigma}}_{2,q})} + \|u_0\|_{\mathcal{L}^\sigma_\Omega(\dot{B}^{\frac{s}{\sigma}}_{2,q})}\right] \quad (3.15)$$

the constant C is independent of T.

Proof. We apply the dyadic block $\dot{\Delta}_j$ to the system (3.12) and we get the following result.

$$\mathrm{d}\dot{\Delta}_j v - \Delta\dot{\Delta}_j v \ \mathrm{d}t = \dot{\Delta}_j f \ \mathrm{d}W, \quad \dot{\Delta}_j v(0) = \dot{\Delta}_j u_0. \quad (3.16)$$

We note that since f, v are in $\mathcal{S}'(\mathbb{R}^d)$, their Fourier transforms $\mathcal{F}f, \mathcal{F}v \in \mathcal{S}'(\mathbb{R}^d)$. The function φ_j have compact supports thus both $\varphi_j\mathcal{F}f, \varphi_j\mathcal{F}v \in \mathcal{S}'(\mathbb{R}^d)$ and have compact support. Since $(\mathcal{F}\dot{\Delta}_j v)(\xi) = \varphi_j(\xi)\mathcal{F}v, (\mathcal{F}\dot{\Delta}_j f)(\xi) = \varphi_j(\xi)\mathcal{F}f$, it follows from Paley-Wiener-Schwartz's Theorem (see [83, p. 181]) that $\dot{\Delta}_j f$ and $\dot{\Delta}_j v$ are smooth functions with compact supports. Hence these functions make sense in (3.16).

Consider the sequence of stopping times

$$\tau_N = \begin{cases} \inf\{t \geq 0, \|\dot{\Delta}_j v\| > N\}, & \text{if the set } \{\omega : \|\dot{\Delta}_j v\| > N\} \neq \varnothing \\ T, & \text{if } \{\omega : \|\dot{\Delta}_j v\| > N\} = \varnothing \end{cases}$$

$N = 1, 2, \ldots$. By means of Itô's formula applied to $\|\dot{\Delta}_j v\| =: \|\dot{\Delta}_j v\|_{L^2}$, we have

$$\mathrm{d}\|\dot{\Delta}_j v\|^2 = 2(\dot{\Delta}_j v, \Delta\dot{\Delta}_j v)\ \mathrm{d}t + 2(\dot{\Delta}_j f, \dot{\Delta}_j v)\ \mathrm{d}W + \|\dot{\Delta}_j f\|^2\ \mathrm{d}t, \quad P - a.s. \tag{3.17}$$

on the interval $[0, \min\{T, \tau_N\}]$.

Applying the Itô's formula to $(\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2}$ for $\sigma \geq 2, \epsilon > 0$ it follows from (3.17) that

$$\begin{aligned} &\mathrm{d}(\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2} \\ =&\frac{\sigma}{2}(\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2-1}\left[2(\dot{\Delta}_j v, \Delta\dot{\Delta}_j v)\ \mathrm{d}t + 2(\dot{\Delta}_j f, \dot{\Delta}_j v)\ \mathrm{d}W + \|\dot{\Delta}_j f\|^2\ \mathrm{d}t\right] \\ &+ \frac{\sigma}{2}(\frac{\sigma-2}{2})4(\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2-2}(\dot{\Delta}_j f, \dot{\Delta}_j v)^2\ \mathrm{d}t. \end{aligned} \tag{3.18}$$

We introduce this regularization with ϵ in order to avoid dealing with a potential zero with a negative power. We will get rid of ϵ through a passage to the limit. We integrate (3.18) over the interval $[0, t]$ for $t < \min(T, \tau_N)$, take the expectation in the resulting relation and estimate the terms.

Cauchy-Schwartz's inequality followed by Young's inequality (with $\varepsilon > 0$) gives

$$\begin{aligned} &\mathbf{E}\int_0^t (\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2-2}(\dot{\Delta}_j f, \dot{\Delta}_j v)^2\ \mathrm{d}r \\ &\leq \mathbf{E}\sup_{r\in[0,t]}(\|\dot{\Delta}_j v(r)\|^2 + \epsilon)^{\sigma/2-2}\|\dot{\Delta}_j v\|^2 \int_0^t \|\dot{\Delta}_j f\|^2\ \mathrm{d}r \\ &\leq \varepsilon\mathbf{E}\sup_{r\in[0,t]}\{(\|\dot{\Delta}_j v(r)\|^2 + \epsilon)^{\sigma/2-2}\|\dot{\Delta}_j v\|^2\}^{\sigma/(\sigma-2)} + C_\varepsilon\mathbf{E}(\int_0^t \|\dot{\Delta}_j f\|^2\ \mathrm{d}r)^{\sigma/2} \\ &\leq \varepsilon\mathbf{E}\sup_{r\in[0,t]}\{(\|\dot{\Delta}_j v(r)\|^2 + \epsilon)^{\sigma/2-2}\|\dot{\Delta}_j v\|^2\}^{\sigma/(\sigma-2)} + C_\varepsilon t^{(\sigma-2)/\sigma}\mathbf{E}\int_0^t \|\dot{\Delta}_j f\|^\sigma\ \mathrm{d}r \end{aligned} \tag{3.19}$$

where we have used the fact that

$$\int_0^t \|\dot{\Delta}_j f\|^2 \ \mathrm{d}r \le t^{(\sigma-2)/\sigma}(\int_0^t \|\dot{\Delta}_f\|^\sigma \ \mathrm{d}r)^{2/\sigma} \tag{3.20}$$

By integration by parts, we have

$$\mathbf{E}\int_0^t (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}(\dot{\Delta}_j v, \Delta\dot{\Delta}_j v) \ \mathrm{d}r = -\mathbf{E}\int_0^t (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}\|\nabla\Delta_j v\|^2 \ \mathrm{d}r. \tag{3.21}$$

Next Young's inequality with $\varepsilon > 0$ gives

$$\begin{aligned}&\mathbf{E}\int_0^t (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}\|\dot{\Delta}_j f\|^2 \ \mathrm{d}r\\ &\le \varepsilon\mathbf{E}\sup_{r\in[0,t]}\{(\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}\}^{\sigma/(\sigma-2)} + C_\varepsilon t^{(\sigma-2)/2}\mathbf{E}\int_0^t \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r.\end{aligned} \tag{3.22}$$

We now proceed to estimate the stochastic integral

$$\mathbf{E}\int_0^t (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}(\dot{\Delta}_j f, \dot{\Delta}_j v) \ \mathrm{d}W$$

We have by Burkholder-Davis-Gundy inequality

$$\begin{aligned}&\mathbf{E}\sup_{r'\in[0,t]}\left|\int_0^{r'} (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}(\dot{\Delta}_j f, \dot{\Delta}_j v) \ \mathrm{d}W\right|\\ &\le C\mathbf{E}\left(\int_0^t \left|(\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}(\dot{\Delta}_j f, \dot{\Delta}_j v)\right|^2 \ \mathrm{d}r\right)^{1/2}\\ &\le C\mathbf{E}\left(\int_0^t (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma-2}\|\dot{\Delta}_j v\|^2\|\dot{\Delta}_j f\|^2 \ \mathrm{d}r\right)^{1/2}\\ &\le C\mathbf{E}\sup_{r\in[0,t]}(\|\dot{\Delta}_j v\|^2+\epsilon)^{(\sigma-2)/2}\|\dot{\Delta}_j v\|\left(\int_0^t \|\dot{\Delta}_j f\|^2 \ \mathrm{d}r\right)^{1/2}.\end{aligned} \tag{3.23}$$

Using that inequality in (3.23) and applying Young's inequality we get

$$\mathbf{E}\sup_{r'\in[0,t]}\left|\int_0^{r'} (\|\dot{\Delta}_j v\|^2+\epsilon)^{\sigma/2-1}(\dot{\Delta}_j f, \dot{\Delta}_j v) \ \mathrm{d}W\right|$$

$$\leq C\varepsilon \mathbf{E} \sup_{r\in[0,t]} \left[(\|\dot{\Delta}_j v\|^2 + \epsilon)^{(\sigma-2)/2} \|\dot{\Delta}_j v\| \right]^{\sigma/(\sigma-1)} + CC_\varepsilon \mathbf{E} t^{(\sigma-2)/\sigma} \int_0^t \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r. \tag{3.24}$$

Combining the inequalities (3.19), (3.21), (3.22),(3.24) we get

$$\begin{aligned}
&\mathbf{E} \sup_{t\in[0,T\wedge\tau_N]} (\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2} + 2\mathbf{E} \int_0^{T\wedge\tau_N} (\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2-1} \|\nabla \dot{\Delta}_j v\|^2 \ \mathrm{d}r \\
\leq& \mathbf{E}(\|\dot{\Delta}_j u_0\|^2 + \epsilon)^{\sigma/2} + C\varepsilon \mathbf{E} \sup_{t\in[0,T\wedge\tau_N]} \left[(\|\dot{\Delta}_j v\|^2 + \epsilon)^{(\sigma-2)/2} \|\dot{\Delta}_j v\| \right]^{\sigma/(\sigma-1)} \\
&+ \varepsilon \mathbf{E} \sup_{r\in[0,t]} \{ (\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2-1} \}^{\sigma/(\sigma-2)} + \varepsilon \mathbf{E} \sup_{r\in[0,t]} \{ (\|\dot{\Delta}_j v\|^2 + \epsilon)^{\sigma/2-2} \|\dot{\Delta}_j v\|^2 \}^{\sigma/(\sigma-2)} \\
&+ C_\varepsilon \left(1 + (T\wedge\tau_N)^{(\sigma-2)/\sigma} \right) \mathbf{E} \int_0^{T\wedge\tau_N} \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r.
\end{aligned}$$

Passing to the limit as $\epsilon \longrightarrow 0$ and choosing $\varepsilon > 0$ sufficiently small, we get

$$\begin{aligned}
&\mathbf{E} \sup_{t\in[0,T\wedge\tau_N]} \|\dot{\Delta}_j v(t)\|^\sigma + \mathbf{E} \int_0^{T\wedge\tau_N} \|\dot{\Delta}_j v\|^{\sigma-2} \|\nabla \dot{\Delta}_j v\|^2 \ \mathrm{d}r \\
\leq& C \left(\mathbf{E}\|\dot{\Delta}_j u_0\|^\sigma + \left(1 + (T\wedge\tau_N)^{(\sigma-2)/\sigma} \right) \mathbf{E} \int_0^T \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r \right)
\end{aligned} \tag{3.25}$$

In view of the conditions on u_0 and f, we see that $\mathbf{E} \sup_{t\in[0,T\wedge\tau_N]} \|\dot{\Delta}_j v(t)\|^\sigma$ is bounded by a constant independent of N, thus passing to the limit in (3.25) as $N \longrightarrow \infty$ and using the fact that $\tau_N \longrightarrow T$, $P-$a.s.

$$\begin{aligned}
&\mathbf{E} \sup_{t\in[0,T]} \|\dot{\Delta}_j v(t)\|^\sigma + \mathbf{E} \int_0^T \|\dot{\Delta}_j v\|^{\sigma-2} \|\nabla \dot{\Delta}_j v\|^2 \ \mathrm{d}r \\
\leq& C \left(\mathbf{E}\|\dot{\Delta}_j u_0\|^\sigma + \left(1 + T^{(\sigma-2)/\sigma} \right) \mathbf{E} \int_0^T \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r \right).
\end{aligned} \tag{3.26}$$

We now recall Bernstein's result which stipulated that if the support of $(\mathcal{F}\dot{\Delta}_j v)(\xi)$ lies in the annulus $\{\xi : A_1 2^{j-1} \leq |\xi| \leq A_2 2^{j+1}\}, 0 < A_1 < A_2$, then there exists a positive constant $\tilde{C}$ such that

$$\tilde{C}^{-2} 2^j \|\dot{\Delta}_j v\| \leq \|\nabla \dot{\Delta}_j v\| \leq \tilde{C}^2 2^j \|\dot{\Delta}_j v\|$$

using this fact we get

$$\begin{aligned} &\mathbf{E}\sup_{t\in[0,T]}\|\dot{\Delta}_j v(t)\|^\sigma + 2^{2j}\mathbf{E}\int_0^T \|\dot{\Delta}_j v\|^\sigma \ \mathrm{d}r \\ \leq & C\left(\mathbf{E}\|\dot{\Delta}_j u_0\|^\sigma + \left(1+T^{(\sigma-2)/\sigma}\right)\mathbf{E}\int_0^T \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r\right) \end{aligned} \tag{3.27}$$

Multiplying both sides of (3.27) by 2^{js} we have the following inequality

$$\begin{aligned} &2^{js}\mathbf{E}\sup_{t\in[0,T]}\|\dot{\Delta}_j v(t)\|^\sigma + 2^{j(s+2)}\mathbf{E}\int_0^T \|\dot{\Delta}_j v\|^\sigma \ \mathrm{d}r \\ \leq & C\left(2^{js}\mathbf{E}\|\dot{\Delta}_j u_0\|^\sigma + \left(1+T^{(\sigma-2)/\sigma}\right)2^{js}\mathbf{E}\int_0^T \|\dot{\Delta}_j f\|^\sigma \ \mathrm{d}r\right) \end{aligned} \tag{3.28}$$

Next raising to the power $\frac{q}{\sigma}$, summing up over $j \in \mathbb{Z}$, raising ot he power $\frac{1}{q}$ and applying Serrin's inequality, (see [151, Lemma 1, p. 252]), completes the proof. □

The result proved is new both in terms of the approach, based on Littlewood-Paley theory, and conclusion. We shall however need a stronger version of Theorem 70 convenient for the pathwise arguments which we shall use for the proof of our main results. This will happen at some cost; namely with positive probability less than one.

Theorem 71. *Assuming the conditions of Theorem 70 to hold , the solution v of* (3.12) *satisfies the following statement for $\sigma = 4$, $q = 2$, $s = 2n - 4$: there exists a set $\tilde{\Omega}$ with positive probability, such that*

$$v(\omega,\cdot) \in L_T^4(\dot{H}^{\frac{n-1}{2}})$$

and there exists a constant $\tilde{C}$ such that

$$\left\|v\left(\omega,\cdot\right)\right\|_{\mathcal{L}_T^4\left(\dot{H}^{\frac{n-1}{2}}\right)} \leq \bar{C}\left[\left(1+T^{\frac{1}{2}}\right)\|f\|_{\mathcal{L}_\Omega^4\mathcal{L}_T^4\left(\dot{H}^{\frac{n}{2}-1}\right)} + \|u_0\|_{\mathcal{L}_\Omega^4\left(\dot{H}^{\frac{n}{2}-1}\right)}\right], \tag{3.29}$$

for all $\omega \in \tilde{\Omega}$.

Proof. Consider the set Ω^* defined by,

$$\Omega^* := \left\{ \omega \in \Omega : \|v(\omega, \cdot)\|_{L_T^4\left(\dot{H}^{\frac{n-1}{2}}\right)} > \bar{C}\left[\left(1 + T^{\frac{1}{2}}\right)\|f\|_{\mathcal{L}_\Omega^4 \mathcal{L}_T^4\left(\dot{H}^{\frac{n}{2}-1}\right)} + \|u_0\|_{\mathcal{L}_\Omega^4\left(\dot{H}^{\frac{n}{2}-1}\right)}\right]\right\},$$

for some positive $\bar{C}$. Using Remark 64 and Minkowski's inequality , we have

$$\|v\|_{L_\Omega^4 L_T^4\left(\dot{B}_{2,2}^{\frac{n-1}{2}}\right)} \leq \|v\|_{\mathcal{L}_\Omega^4 \mathcal{L}_T^4\left(\dot{B}_{2,2}^{\frac{n-1}{2}}\right)}.$$

By Tchebychev's inequality we have that

$$\mathbf{P}(\Omega^*) \leq \frac{\mathbf{E}\|v(\omega, \cdot)\|^4_{L_T^4\left(\dot{H}^{\frac{n-1}{2}}\right)}}{\bar{C}^4\left[\left(1 + T^{\frac{1}{2}}\right)\|f\|_{\mathcal{L}_\Omega^4 \mathcal{L}_T^4\left(\dot{H}^{\frac{n}{2}-1}\right)} + \|u_0\|_{\mathcal{L}_\Omega^4\left(\dot{H}^{\frac{n}{2}-1}\right)}\right]^4}.$$

This implies

$$\mathbf{P}(\Omega^*) \leq \left(\frac{C}{\bar{C}}\right)^4,$$

where C is the constant in Theorem 70.

Now set $\tilde{\Omega}$ to be the complement of Ω^* and take $\bar{C} > C$ so that,

$$\mathbf{P}(\tilde{\Omega}) = 1 - \mathbf{P}(\Omega^*) \geq 1 - \left(\frac{C}{\bar{C}}\right)^4 > 0. \tag{3.30}$$

This concludes our proof. □

Remark 72. *From* (3.30) *we see that by making a wise choice on* $\bar{C}$ *one can improve the result with a probability close to one.*

In the case when right hand side of (3.12) is a function not involving the noise, better results are available. For instance the problem

$$\begin{cases} \mathrm{d}v - \Delta v \ \mathrm{d}t = f \ \mathrm{d}t \\ v(0) = v_0 \end{cases} \tag{3.31}$$

is studied in [10, p. 210] and we found the following result suitable to us.

Lemma 73. *Let v be the solution in $\mathcal{C}([0,T];\mathcal{S}'(\mathbb{R}^n))$ of the Cauchy problem* (3.31) *with f in $L^2([0,T];\dot{H}^{s-1})$ and v_0 in $\dot{H}^s(\mathbb{R}^n)$. Then,*

$$v \in \left(\bigcap_{p=2}^{\infty} L^p([0,T];\dot{H}^{s+\frac{2}{p}}) \right) \cap \mathcal{C}([0,T];\dot{H}^s).$$

Moreover, we have the following estimates:

$$\|v(t)\|^2_{\dot{H}^s} + 2\int_0^t \|\nabla v(t')\|^2_{\dot{H}^s} \ \mathrm{d}t' = \|v_0\|^2_{\dot{H}^s} + 2\int_0^t \langle f(t'), v(t')\rangle_s \ \mathrm{d}t',$$

$$\left(\int_{\mathbb{R}^n} |\xi|^{2s} \left(\sup_{0\le t'\le t} |\hat{v}(t',\xi)| \right)^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \le \|v_0\|_{\dot{H}^s} + \frac{1}{\sqrt{2}}\|f\|_{L^2_T(\dot{H}^{s-1})},$$

$$\|v(t)\|_{L^p_T(\dot{H}^{s+\frac{2}{p}})} \le \|v_0\|_{\dot{H}^s} + \|f\|_{L^2_T(\dot{H}^{s-1})}$$

with $\langle a,b\rangle_s := \int |\xi|^{2s}\hat{a}(\xi)\overline{\hat{b}(\xi)} \ \mathrm{d}\xi$.

For a bilinear operator Q defined by;

$$Q(a,b) = -\frac{1}{2}\mathcal{P}\left((a\cdot\nabla)b + (b\cdot\nabla)a\right), \tag{3.32}$$

we have the following result from [10, p. 210];

Lemma 74. *A constant C exists such that*

$$\|Q(a,b)\|_{\dot{H}^{\frac{n}{2}-2}} \le C\|a\|_{\dot{H}^{\frac{n-1}{2}}}\|b\|_{\dot{H}^{\frac{n-1}{2}}},$$

where n is the dimension of the physical space $\mathbb{R}^n$.

We note that the structure of (3.32) and (3.5) being same the result in Lemma 74 perfectly works for (3.5).

We close the section by proving the following theorem;

Theorem 75. *Consider the stochastic heat equation of the type*

$$\begin{aligned} \mathrm{d}v - \Delta v \mathrm{d}t &= g\mathrm{d}t + f\mathrm{d}W_t \quad \text{in } \Omega \times \mathbb{R}^n \times [0,T], \\ v|_{t=0} &= v_0 \quad \text{in } \Omega \times \mathbb{R}^n. \end{aligned} \tag{3.33}$$

If $f \in L^2_\Omega L^2_T \dot{H}^s(\mathbb{R}^n)$, $g \in L^2_\Omega L^2_T \dot{H}^{s-1}(\mathbb{R}^n)$ *and* $v_0 \in L^2_\Omega \dot{H}^s(\mathbb{R}^n)$ *then,* (3.33) *has a solution in* $L^2_\Omega L^2_T \dot{H}^{s+1}(\mathbb{R}^n) \cap L^2_\Omega L^\infty_T \dot{H}^s(\mathbb{R}^n)$.

Proof. We begin by applying the Fourier transform to (3.33) which gives

$$\begin{aligned} \mathrm{d}\hat{v} + |\xi|^2 \hat{v} \mathrm{d}t = \hat{g}\mathrm{d}t + \hat{f}\mathrm{d}W_t \\ \hat{v}|_{t=0} = \hat{v}_0 \end{aligned}. \tag{3.34}$$

Now multiplying the first statement of (3.34) by $|\xi|^s$ we get

$$\mathrm{d}(|\xi|^s \hat{v}) + (|\xi|^{s+2}\hat{v})\mathrm{d}t = (|\xi|^s \hat{g})\mathrm{d}t + (|\xi|^s \hat{f})\mathrm{d}W_t.$$

Next applying Itô's integral formula, (2.60), we get

$$\mathrm{d}(|\xi|^{2s}\hat{v}^2) = |\xi|^{2s}\hat{f}^2\mathrm{d}t - 2|\xi|^{2(s+1)}\hat{v}^2\mathrm{d}t + 2|\xi|^{2s}\hat{v}\hat{g}\mathrm{d}t + 2|\xi|^{2s}\hat{v}\hat{f}\mathrm{d}W_t,$$

Which is equivalent to

$$\mathrm{d}(|\xi|^{2s}\hat{v}^2) + 2|\xi|^{2(s+1)}\hat{v}^2\mathrm{d}t = |\xi|^{2s}\hat{f}^2\mathrm{d}t + 2|\xi|^{2s}\hat{v}\hat{g}\mathrm{d}t + 2|\xi|^{2s}\hat{v}\hat{f}\mathrm{d}W_t$$

Next we integrate with respect to time variable followed by integration with respect to the space variable and finally applying Fubini, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |\xi|^{2s}\hat{v}^2(t)\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^n} 2|\xi|^{2(s+1)}\hat{v}^2 \, \mathrm{d}\xi\mathrm{d}t' \\ &\quad = \int_{\mathbb{R}^n} |\xi|^{2s}\hat{v}_0^2\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^n} |\xi|^{2s}\hat{f}^2 \, \mathrm{d}\xi\mathrm{d}t' \\ &\quad + 2\int_0^t \int_{\mathbb{R}^n} |\xi|^{2s}\hat{v}\hat{g} \, \mathrm{d}\xi\mathrm{d}t' + 2\int_0^t \int_{\mathbb{R}^n} |\xi|^{2s}\hat{v}\hat{f} \, \mathrm{d}\xi\mathrm{d}W_{t'}. \end{aligned} \tag{3.35}$$

But,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{g}\hat{v}| \mathrm{d}\xi \mathrm{d}t' \\
&\leq \int_0^t \left(\int_{\mathbb{R}^n} |\xi|^{2(s-1)} |\hat{g}|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\xi|^{2(s+1)} |\hat{v}|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \mathrm{d}t' \\
&\leq \varepsilon_1 \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(s+1)} |\hat{v}|^2 \mathrm{d}\xi \mathrm{d}t' + C_{\varepsilon_1} \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(s-1)} |\hat{g}|^2 \mathrm{d}\xi \mathrm{d}t'. \qquad (3.36)
\end{aligned}$$

Here we used Young's and Hölder's inequalities.

Now we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}^2(t) \mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^n} 2|\xi|^{2(s+1)} \hat{v}^2 \,\mathrm{d}\xi \mathrm{d}t' \leq \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}_0^2 \mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^n} |\xi|^{2s} \hat{f}^2 \,\mathrm{d}\xi \mathrm{d}t' \\
&\quad + \varepsilon_1 \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(s+1)} |\hat{v}|^2 \mathrm{d}\xi \mathrm{d}t' + C_{\varepsilon_1} \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(s-1)} |\hat{g}|^2 \mathrm{d}\xi \mathrm{d}t' \\
&\quad + \int_0^t \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}\hat{f} \,\mathrm{d}\xi \mathrm{d}W_{t'} \qquad (3.37)
\end{aligned}$$

Now taking the supremum in time of (3.37) over the range $[0, t]$, we get

$$\begin{aligned}
&\sup_{t'' \in [0,t]} \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}^2(t'') \mathrm{d}\xi + \sup_{t'' \in [0,t]} \int_0^{t''} \int_{\mathbb{R}^n} 2|\xi|^{2(s+1)} \hat{v}^2 \,\mathrm{d}\xi \mathrm{d}t' \\
&\qquad \leq \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}_0^2 \mathrm{d}\xi + \sup_{t'' \in [0,t]} \int_0^{t''} \int_{\mathbb{R}^n} |\xi|^{2s} \hat{f}^2 \,\mathrm{d}\xi \mathrm{d}t' \\
&\qquad + \varepsilon_1 \sup_{t'' \in [0,t]} \int_0^{t''} \int_{\mathbb{R}^n} |\xi|^{2(s+1)} |\hat{v}|^2 \mathrm{d}\xi \mathrm{d}t' + C_{\varepsilon_1} \sup_{t'' \in [0,t]} \int_0^{t''} \int_{\mathbb{R}^n} |\xi|^{2(s-1)} |\hat{g}|^2 \mathrm{d}\xi \mathrm{d}t' \\
&\qquad + \sup_{t'' \in [0,t]} \int_0^{t''} \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}\hat{f} \,\mathrm{d}\xi \mathrm{d}W_{t'},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\sup_{t'' \in [0,t]} \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}^2(t'') \mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^n} 2|\xi|^{2(s+1)} \hat{v}^2 \,\mathrm{d}\xi \mathrm{d}t' \\
&\qquad \leq \int_{\mathbb{R}^n} |\xi|^{2s} \hat{v}_0^2 \mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^n} |\xi|^{2s} \hat{f}^2 \,\mathrm{d}\xi \mathrm{d}t' \\
&\qquad + \varepsilon_1 \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(s+1)} |\hat{v}|^2 \mathrm{d}\xi \mathrm{d}t' + C_{\varepsilon_1} \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(s-1)} |\hat{g}|^2 \mathrm{d}\xi \mathrm{d}t'
\end{aligned}$$

$$+\sup_{t''\in[0,t]}\int_0^{t''}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}\hat{f}\,\mathrm{d}\xi\mathrm{d}W_{t'}. \tag{3.38}$$

We now take the expectation of (3.38) to get

$$\begin{aligned}
&\mathbf{E}\sup_{t''\in[0,t]}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}^2(t'')\mathrm{d}\xi+\mathbf{E}\int_0^t\int_{\mathbb{R}^n}2|\xi|^{2(s+1)}\hat{v}^2\,\mathrm{d}\xi\mathrm{d}t'\\
&\quad\le\mathbf{E}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}_0^2\mathrm{d}\xi+\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2s}\hat{f}^2\,\mathrm{d}\xi\mathrm{d}t'\\
&\quad+\varepsilon_1\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2(s+1)}|\hat{v}|^2\mathrm{d}\xi\mathrm{d}t'+C_{\varepsilon_1}\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2(s-1)}|\hat{g}|^2\mathrm{d}\xi\mathrm{d}t'\\
&\quad+\mathbf{E}\sup_{t''\in[0,t]}\int_0^{t''}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}\hat{f}\,\mathrm{d}\xi\mathrm{d}W_{t'}.
\end{aligned} \tag{3.39}$$

Now we estimate the stochastic integral;

$$\begin{aligned}
\mathbf{E}\sup_{t''\in[0,t]}\int_0^{t''}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}\hat{f}\mathrm{d}\xi\mathrm{d}W_{t'}&\le C\mathbf{E}\left(\int_0^t\left(\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{v}\hat{f}|\mathrm{d}\xi\right)^2\mathrm{d}t'\right)^{\frac{1}{2}}\\
&\le C\mathbf{E}\left(\int_0^t\left(\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{v}|^2\mathrm{d}\xi\right)\left(\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{f}|^2\mathrm{d}\xi\right)\mathrm{d}t'\right)^{\frac{1}{2}}\\
&\le C\ \mathbf{E}\sup_{[0,t]}\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{v}|^2\mathrm{d}\xi\left(\int_0^t\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{f}|^2\mathrm{d}\xi\mathrm{d}t'\right)^{\frac{1}{2}}\\
&\le\varepsilon C\mathbf{E}\sup_{[0,t]}\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{v}|^2\mathrm{d}\xi+C_\varepsilon C\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{f}|^2\mathrm{d}\xi\mathrm{d}t',
\end{aligned} \tag{3.40}$$

Here we used Burkholder-Davis-Gundy and Young's inequalities. Next putting (3.40) back in (3.39) we get

$$\begin{aligned}
&\mathbf{E}\sup_{t''\in[0,t]}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}^2(t'')\mathrm{d}\xi+\mathbf{E}\int_0^t\int_{\mathbb{R}^n}2|\xi|^{2(s+1)}\hat{v}^2\,\mathrm{d}\xi\mathrm{d}t'\\
&\quad\le\mathbf{E}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}_0^2\mathrm{d}\xi+\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2s}\hat{f}^2\,\mathrm{d}\xi\mathrm{d}t'\\
&\quad+\varepsilon_1\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2(s+1)}|\hat{v}|^2\mathrm{d}\xi\mathrm{d}t'+C_{\varepsilon_1}\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2(s-1)}|\hat{g}|^2\mathrm{d}\xi\mathrm{d}t'
\end{aligned}$$

$$+\varepsilon C\mathbf{E}\sup_{[0,t]}\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{v}|^2\mathrm{d}\xi + C_\varepsilon C\mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{f}|^2\mathrm{d}\xi\mathrm{d}t'. \qquad (3.41)$$

Finally, taking ε and ε_1 small enough, we get

$$\mathbf{E}\sup_{t''\in[0,t]}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}^2(t'')\mathrm{d}\xi + \mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2(s+1)}\hat{v}^2\,\mathrm{d}\xi\mathrm{d}t'$$
$$\leq C\left(\mathbf{E}\int_{\mathbb{R}^n}|\xi|^{2s}\hat{v}_0^2\mathrm{d}\xi + \mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2s}\hat{f}^2\,\mathrm{d}\xi\mathrm{d}t' + \mathbf{E}\int_0^t\int_{\mathbb{R}^n}|\xi|^{2(s-1)}|\hat{g}|^2\mathrm{d}\xi\mathrm{d}t'\right),$$

for some positive constant C. Thus, if $f\in L^2_\Omega L^2_T\dot{H}^s(\mathbb{R}^n)$, $v_0\in L^2_\Omega\dot{H}^s(\mathbb{R}^n)$ and $g\in L^2_\Omega L^2_T\dot{H}^{s-1}(\mathbb{R}^n)$, the solution v of (3.33) is in $L^2_\Omega L^\infty_T\dot{H}^s(\mathbb{R}^n)\cap L^2_\Omega L^2_T\dot{H}^{s+1}(\mathbb{R}^n)$.

□

3.4 Main Result

It is time to give our main results. The first result is on local existence and uniqueness of a strong solution and our second result is on existence and uniqueness of a global solution for (3.1). The result on global solution has appeared on [148] and the local strong solution result is new.

For this purpose, we first formulate our notion of solution with the following two definitions.

Definition 76. *For a fixed probability basis $\left(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in[0,T]},\mathbf{P},W\right)$, a divergence free process $\binom{\beta}{\theta}$ is a local strong solution of (3.6), if there exists a positive random time τ, such that $\binom{\beta}{\theta}(\omega)\in L^2_\tau\left(\dot{H}^{\frac{n-1}{2}}\right)\bigcap\mathcal{M}_\tau$ and satisfies the relation*

$$\begin{pmatrix}\beta\\ \theta\end{pmatrix}(t)=\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix}+\int_0^t\left(\Delta\begin{pmatrix}\beta\\ \theta\end{pmatrix}+\mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right)(s)\;\mathrm{d}s+\int_0^t\begin{pmatrix}G_1\\ G_2\end{pmatrix}\;\mathrm{d}W_s,$$

P-a.s., for any $t\in[0,\tau]$

Definition 77. *For a fixed probability basis* $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P}, W\big)$, *a divergence free process* $\binom{\beta}{\theta}$ *is a global mild-solution of problem* (3.6), *if* $\begin{pmatrix}\beta(\omega,\cdot)\\ \theta(\omega,\cdot)\end{pmatrix} \in L^4_t(\dot{H}^{\frac{n-1}{2}}) \bigcap \mathcal{M}_t$ *for all* $t \geq 0$ *and* P*-a.s.*

$$\begin{pmatrix}\beta\\ \theta\end{pmatrix}(t) = e^{t\Delta}\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix} + \int_0^t e^{(t-s)\Delta}\begin{pmatrix}G_1\\ G_2\end{pmatrix}\mathrm{d}W_s + \int_0^t e^{(t-s)\Delta}\mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}, \begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\mathrm{d}s$$

where $e^{t\Delta}$ *is the heat semi-group and* $\operatorname{div}\beta_0 = 0$, $\operatorname{div}\theta_0 = 0$.

The following two theorems are our main results;

Theorem 78. *Given a probability basis* $(\Omega, \mathscr{F}, P, \{\mathbf{F}_t\}_{0\leq t\leq T}, W)$, *let* u_0, b_0 *be* $\mathcal{F}_0$*-measurable with* $\operatorname{div} u_0 = 0$, $\operatorname{div} b_0 = 0$ *and* $G_1, G_2 \in \mathcal{M}_T$. *We assume that there exists a positive constant* K *such that*

$$(1+T^{\frac{1}{2}})\left\|\begin{pmatrix}G_1\\ G_2\end{pmatrix}\right\|_{\mathcal{L}^4_\Omega\mathcal{L}^4_T(\dot{H}^{\frac{n}{2}-1})} + \left\|\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix}\right\|_{\mathcal{L}^4_\Omega\dot{H}^{\frac{n}{2}-1}} \leq K$$

Then there exists a random set $\tilde{\Omega}$ *with* $P(\tilde{\Omega}) > 0$, *a random time* $\tau(\omega) > 0$, *and a process*

$$\begin{pmatrix}\beta\\ \theta\end{pmatrix}(\omega,\cdot) \in L^4_\tau(\dot{H}^{\frac{n-1}{2}}) \cap \mathcal{M}_\tau \tag{3.42}$$

for all ω *in* $\tilde{\Omega}$, *and* $\binom{\beta}{\theta}$ *is a local solution of problem in the sense of definition 76.*

Theorem 79. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leq t\leq T}, \mathbf{P}, W)$ *be a probability basis. Let* u_0, b_0 *be* $\mathscr{F}_0$*−measurable with* $\operatorname{div}u_0 = \operatorname{div}b_0 = 0$, *and* G_1, $G_2 \in \mathcal{M}_T$. *Assume that for any positive* T *we have,*

$$\left(1+T^{\frac{1}{2}}\right)\left\|\begin{pmatrix}G_1\\ G_2\end{pmatrix}\right\|_{\mathcal{L}^4_\Omega\mathcal{L}^4_T\left(\dot{H}^{\frac{n}{2}-1}\right)} + \left\|\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix}\right\|_{\mathcal{L}^4_\Omega\left(\dot{H}^{\frac{n}{2}-1}\right)} < \infty.$$

Then there is a random set $\tilde{\Omega}$ *with positive probability and a unique global mild-*

solution of (3.1) in a ball centered at the origin in the space $L^4_T(\dot{H}^{\frac{n-1}{2}})$*, for all* ω *in* $\tilde{\Omega}$*. Furthermore, for* $s = \frac{n}{2} - 1$*, if* $G_i \in L^2_\Omega L^2_T \dot{H}^s(\mathbb{R}^n)$ *for* $i = 1, 2$*, and* $u_0, b_0 \in L^2_\Omega \dot{H}^s(\mathbb{R}^n)$*, then the solution* $u(\omega, \cdot)$*,* $b(\omega, \cdot)$ *of (3.1) belongs to the space* $L^\infty_T \dot{H}^s(\mathbb{R}^n) \cap L^2_T \dot{H}^{s+1}(\mathbb{R}^n)$*.*

3.5 Proof of Main Results

The proof relies on the following version of fixed point theorem. For the proof we refer to [10, p. 207].

Lemma 80. *Let* $(Y, \|\cdot\|_Y)$ *be a Banach space and* $\Phi : Y \times Y \longrightarrow Y$ *a bilinear continuous map with the norm*

$$\|\Phi\| = \sup_{\|\phi\|_Y, \|\psi\|_Y \leq 1} \|\Phi(\phi, \psi)\|_Y.$$

Then for all $\phi \in Y$*, such that*

$$\|\phi\|_Y < \frac{1}{4\|\Phi\|},$$

the equation

$$\psi = \phi + \Phi(\psi, \psi)$$

has a unique solution ψ *in the ball* $\{\varphi \in Y : \|\varphi\|_Y < 1/(2\|\Phi\|)\}$*.*

Proof of Theorem 79

Proof. Recalling the discussion in page 54, we look for $\binom{\beta}{\theta}$ such that

$$\begin{pmatrix} \beta \\ \theta \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix} \tag{3.43}$$

where $\binom{\alpha}{\gamma}$ satisfies (3.10) and $\binom{\alpha'}{\gamma'} = B\left(\binom{\beta}{\theta}, \binom{\beta}{\theta}\right)$ satisfies (3.11). This enables us to work pathwise in the implementation of fixed point argument, Lemma 80, on the map

$$\Psi\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) = \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}, \begin{pmatrix}\beta\\ \theta\end{pmatrix}\right),$$

where

$$\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}(t) = e^{t\Delta}\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix} + \int_0^t e^{(t-s)\Delta}\begin{pmatrix}G_1\\ G_2\end{pmatrix} dW_s,$$

$$B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}(t), \begin{pmatrix}\beta\\ \theta\end{pmatrix}(t\right) = \int_0^t e^{(t-s)\Delta}\mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}, \begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) ds.$$

and $e^{t\Delta}$ is a heat semigroup.

It clear that

$$\begin{pmatrix}\beta\\ \theta\end{pmatrix}(t) = e^{t\Delta}\begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix} + \int_0^t e^{(t-s)\Delta}\begin{pmatrix}G_1\\ G_2\end{pmatrix} dW_s + \int_0^t e^{(t-s)\Delta}\mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix}, \begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) ds$$

is a fixed point of Ψ.

We next show that B satisfies the condition in Lemma 80 for an appropriate space

Y to be determined in the process . Since $B\left(\binom{\beta}{\theta},\binom{\beta}{\theta}\right)$ satisfies the equation

$$\begin{cases} \partial_t B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) - \Delta B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) = \mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right) \\ B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\Bigg|_{t=0} = \mathbf{0} \end{cases}.$$

Lemma 73 implies that

$$\left\| B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right\|_{L^p_T(\dot{H}^{s+\frac{2}{p}})} \leq \left\| \mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right\|_{L^2_T(\dot{H}^{s-1})}. \tag{3.44}$$

Now set $s = \frac{n}{2} - 1$. Then Lemma 74 with Hölder's inequality implies that

$$\left\| \mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right\|_{L^2_T(\dot{H}^{\frac{n}{2}-2})} \leq \left\|\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \left\|\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})}. \tag{3.45}$$

Next we choose $p = 4$ and apply this to (3.44) to get

$$\left\| B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \leq \left\| \mathbf{Q}\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right\|_{L^2_T(\dot{H}^{\frac{n}{2}-2})}. \tag{3.46}$$

Then combining (3.45) and (3.46) gives

$$\left\| B\left(\begin{pmatrix}\beta\\ \theta\end{pmatrix},\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right)\right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \leq \left\|\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \left\|\begin{pmatrix}\beta\\ \theta\end{pmatrix}\right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})}. \tag{3.47}$$

Thus if we set $Y = L^4_T(\dot{H}^{\frac{n-1}{2}})$, then B satisfies the condition that $\|B\|_Y \leq C_B$, for some constant C_B independent of T. Therefore by Lemma 80 the Theorem will be proved if

$$\left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\omega, \cdot) \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \leq \frac{1}{4C_B}, \quad \text{with positive probability.} \tag{3.48}$$

For this to hold we appeal to Theorem 71 to deduce the corresponding restrictions on the data. Indeed by (3.29) we have that

$$\left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\omega, \cdot) \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \leq \bar{C} \left((1 + T^{\frac{1}{2}}) \left\| \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \right\|_{L^4_\Omega L^4_T(\dot{H}^{\frac{n}{2}-1})} + \left\| \begin{pmatrix} \beta_0 \\ \theta_0 \end{pmatrix} \right\|_{L^4_\Omega(\dot{H}^{\frac{n}{2}-1})} \right),$$

for all ω in $\tilde{\Omega}$ of positive probability. Therefore for the condition (3.48) to hold, our data should satisfy

$$(1 + T^{\frac{1}{2}}) \left\| \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \right\|_{L^4_\Omega L^4_T(\dot{H}^{\frac{n}{2}-1})} + \left\| \begin{pmatrix} \beta_0 \\ \theta_0 \end{pmatrix} \right\|_{L^4_\Omega(\dot{H}^{\frac{n}{2}-1})} \leq \frac{1}{4\bar{C}C_B}.$$

This proves the first part of our theorem. To prove continuity, we use Theorem 75; observe from (3.46) that $\mathbf{Q}\left(\left(\begin{smallmatrix} \beta \\ \theta \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta \\ \theta \end{smallmatrix}\right)\right) \in L^2_T(\dot{H}^{\frac{n}{2}-1})$, $\left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right)(\omega, \cdot) \in L^2_T(\dot{H}^s)$, $\left(\begin{smallmatrix} \beta_0 \\ \theta_0 \end{smallmatrix}\right)(\omega, \cdot) \in \dot{H}^s$ P-a.s., we have $\left(\begin{smallmatrix} \beta(\omega,\cdot) \\ \theta(\omega,\cdot) \end{smallmatrix}\right)$ belongs to the space $L^\infty(0, T; \dot{H}^{\frac{n}{2}-1}) \cap L^2(0, T; \dot{H}^{\frac{n}{2}})$. Hence, for each $\omega \in \tilde{\Omega}$ the solution $u(\omega, \cdot), b(\omega, \cdot)$ of (3.4) belongs to the space $L^\infty(0, T; \dot{H}^{\frac{n}{2}-1}) \cap L^2(0, T; \dot{H}^{\frac{n}{2}})$.

This concludes the proof of of Theorem 79. □

Proof of Theorem 78

Proof. Let

$$\begin{pmatrix} \beta \\ \theta \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix} \tag{3.49}$$

be the decomposition of the solution $\binom{\beta}{\theta}$ of (3.6) (as discussed earlier) such that $\binom{\alpha}{\gamma}$ satisfies

$$\begin{cases} \partial_t \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} - \Delta \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} = \begin{pmatrix}G_1\\ G_2\end{pmatrix} \dot{W} \\ \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix}\beta_0\\ \theta_0\end{pmatrix} \end{cases} \tag{3.50}$$

and $\binom{\alpha'}{\gamma'}$ satisfies

$$\begin{cases} \partial_t \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix} - \Delta \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix} = \mathbf{Q}\left(\begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}, \begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}\right) \\ \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{t=0} = 0 \end{cases} . \tag{3.51}$$

We fix $\tilde{\Omega}$ in Theorem 71. We shall solve (3.51) by using an iterative scheme. Let $\left(\binom{\alpha'}{\gamma'}_k\right)_{k=0,1,\dots}$ be a sequence defined recursively by; $\binom{\alpha'}{\gamma'}_0 = 0$, $\binom{\alpha'}{\gamma'}_{k+1}$ is defined in terms of $\binom{\alpha'}{\gamma'}_k$ as a solution of the initial value problem

$$\begin{cases} \partial_t \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k+1} - \Delta \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k+1} = \mathbf{Q}\left(\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k + \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}, \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k + \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\right) \\ \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k+1}\Bigg|_{t=0} = 0 \end{cases} . \tag{3.52}$$

We now apply Lemma 73 to (3.52) to get

$$\left\| \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k+1} \right\|_{L^p_T(\dot{H}^{s+\frac{2}{p}})}$$

$$\leq \left[\left\| \mathbf{Q}\left(\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k, \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right) \right\|_{L^2_T(\dot{H}^{s-1})} + 2 \left\| \mathbf{Q}\left(\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right) \right\|_{L^2_T(\dot{H}^{s-1})} \right.$$
$$\left. + \left\| \mathbf{Q}\left(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right) \right\|_{L^2_T(\dot{H}^{s-1})} \right]. \tag{3.53}$$

Here we used advantage of bilinearity of $\mathbf{Q}$. Now set $s = \frac{n}{2} - 1$ and apply Theorem 74 together with Hölder's inequality to get

$$\left\| \mathbf{Q}\left(\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right) \right\|_{L^2_T(\dot{H}^{\frac{n}{2}-2})} \leq C \left\| \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})}$$

Similarly for the remaining terms in (3.53) we have

$$\left\| \mathbf{Q}\left(\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k, \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right) \right\|_{L^2_T(\dot{H}^{\frac{n}{2}-2})} \leq C \left\| \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \left\| \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})}$$

$$\left\| \mathbf{Q}\left(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right) \right\|_{L^2_T(\dot{H}^{\frac{n}{2}-2})} \leq C \left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})}$$

Thus for $p = 4$ in Lemma 73 we get

$$\left\| \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_{k+1} \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \leq C \left[\left\| \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} + \left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right\|_{L^4_T(\dot{H}^{\frac{n-1}{2}})} \right]^2. \tag{3.54}$$

For a positive number δ, let $\tau(\omega, \delta)$ be the stopping time defined by

$$\tau(\omega,\delta) = \begin{cases} \inf A(\omega), & \text{if } A(\omega) \neq \emptyset \\ T, & \text{if } A(\omega) = \emptyset \end{cases}, \tag{3.55}$$

where

$$A(\omega) = \left\{ t \in [0,T] : \left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\omega, \cdot) \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} \geq \delta \, P - \text{a.s.} \right\}. \tag{3.56}$$

Since $\binom{\alpha}{\gamma}$ solves problem (3.50), Theorem 71 tells us that if

$$C \left[\left\| \begin{pmatrix} \beta_0 \\ \theta_0 \end{pmatrix} (\omega, \cdot) \right\|_{\dot{H}^{\frac{n}{2}-1}} + \left\| \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (\omega, \cdot) \right\|_{L^4_t(\dot{H}^{\frac{n}{2}-1})} \right] \leq \delta, \; P - \text{a.s.}, \tag{3.57}$$

then

$$\left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\omega, \cdot) \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} \leq \delta, \; \forall \omega \in \tilde{\Omega}. \tag{3.58}$$

We note that $\|\cdot\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})}$ is continuous and non decreasing as a function of t. Thus $\tau(\omega, \delta)$ exists and is positive for all ω in $\tilde{\Omega}$.

Given sufficiently small $\delta > 0$, such that $4C\delta < 1$ (C is a constant from Theorem 71, let τ the corresponding $\tau(\omega, \delta)$ for which

$$\left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\omega, \cdot) \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} \leq \delta, \; \forall \omega \in \tilde{\Omega}. \tag{3.59}$$

We show by induction that

$$\left\| \begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}_k \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} \leq \delta, \text{ for any } n \in \mathbb{N}, \; \forall \omega \in \tilde{\Omega}. \tag{3.60}$$

Since $\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_0 = 0$, we get from (3.54) and (3.59), that

$$\left\| \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_1 (\omega,\cdot) \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} < C\delta^2 < \delta, \forall\omega \in \tilde{\Omega}. \tag{3.61}$$

Assume that

$$\left\| \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k (\omega,\cdot) \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} < \delta, \forall\omega \in \tilde{\Omega}. \tag{3.62}$$

Then it follows from (3.54) and (3.59) that

$$\left\| \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k+1} (\omega,\cdot) \right\|_{L^4_t(\dot{H}^{\frac{n-1}{2}})} < 4C\delta^2 < \delta, \forall\omega \in \tilde{\Omega},\ \delta < \frac{1}{4C},\ k \in \mathbb{N}. \tag{3.63}$$

Thus (3.60) is proved.

We next show that $\left(\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k (\omega,\cdot)\right)_{n=1,2,\ldots}$ is a Cauchy sequence in $L^4_\tau(\dot{H}^{\frac{n-1}{2}})$, for all $\omega \in \tilde{\Omega}$.

Letting $Z_k := \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k+1} - \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k$. Then we have the following equation for Z_k.

$$\begin{aligned}
&\partial_t Z_k - \Delta Z_k = \mathbf{Q}\left(Z_{k-1}, \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k + \begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k-1} \right) + 2\mathbf{Q}\left(Z_{k-1}, \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\right)\\
&Z_k(0) = 0
\end{aligned} \tag{3.64}$$

As earlier, we apply Lemma 73 to this problem and get

$$\left\| Z_k(\omega,\cdot)\right\|_{L^p_\tau(\dot{H}^{s+\frac{2}{p}})} \leq$$

$$\left[\left\|\mathbf{Q}\left(Z_{k-1},\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k+\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k-1}\right)\right\|_{L^2_\tau(\dot{H}^{s-1})}+2\left\|\mathbf{Q}\left(Z_{k-1},\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\right)\right\|_{L^2_\tau(\dot{H}^{s-1})}\right]. \tag{3.65}$$

Next Lemma 74 with $s=\frac{n}{2}-1$ implies

$$\begin{aligned}
&\left\|\mathbf{Q}\left(Z_{k-1},\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k+\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k-1}\right)\right\|_{L^2_\tau(\dot{H}^{\frac{n}{2}-2})}\\
\leq&\left\|\mathbf{Q}\left(Z_{k-1},\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k\right)\right\|_{L^2_\tau(\dot{H}^{\frac{n}{2}-2})}+\left\|\mathbf{Q}\left(Z_{k-1},\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k-1}\right)\right\|_{L^2_\tau(\dot{H}^{\frac{n}{2}-2})}\\
\leq&C\left\|Z_{k-1}\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}\left[\left\|\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}+\left\|\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_{k-1}\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}\right]
\end{aligned} \tag{3.66}$$

and

$$\left\|\mathbf{Q}\left(Z_{k-1},\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\right)\right\|_{L^2_\tau(\dot{H}^{\frac{n}{2}-2})}\leq C\left\|Z_{k-1}\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}\left\|\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}. \tag{3.67}$$

These estimates together with (3.59), (3.60) and (3.65) imply that

$$\left\|Z_k(\omega,\cdot)\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}\leq 4C\delta\left\|Z_{k-1}(\omega,\cdot)\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}, \tag{3.68}$$

for all ω in $\tilde{\Omega}$ and all n in $\mathbb{N}$.

Since δ is an arbitrary small positive number, choosing it such that $C\delta^2<1$, we see that $\left\|Z_k(\omega,\cdot)\right\|_{L^4_\tau(\dot{H}^{\frac{n-1}{2}})}$ is contractive and hence $\left\{\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}(\omega,\cdot)\right)_k\right\}_{k=1,2,\ldots}$ is a Cauchy sequence in $L^4_\tau(\dot{H}^{\frac{n-1}{2}})$, for all $\omega\in\tilde{\Omega}$. In view of the completeness of $L^4_\tau(\dot{H}^{\frac{n-1}{2}})$, we can then extract a subsequence $\left(\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}_k\right)_{k=1,2,\ldots}$ denoted by the same symbol, which converges to a limit $\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}\in L^4_\tau(\dot{H}^{\frac{n-1}{2}})$ in the norm of $L^4_\tau(\dot{H}^{\frac{n-1}{2}})$. And $\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}$

solves (3.51) uniquely, for all $\omega \in \tilde{\Omega}$.

We have thus constructed with positive probability, the unique pathwise solution $\binom{\beta}{\theta}$ of problem (3.6) in the space $L^4_\tau(\dot{H}^{\frac{n-1}{2}})$ as the sum of solutions $\binom{\alpha}{\gamma}$ and $\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix}$ of the problems (3.50) and (3.51) respectively. This completes the proof Theorem 78.

$\square$

Part III

Magnetohydrodynamics Turbulence and Kolmogorov Spectral Law

Chapter 4

Inertial range bounds on Kolmogorov Spectra for Magnetohydrodynamics Equations

4.1 Introduction

At high Reynolds number fluid and plasma flows exhibit a complex random behavior called turbulence. This phenomenon is observed in a great majority of fluids both in nature such as atmosphere, river currents, oceans, solar wind and interstitial bodies, and technical devices such as laboratory installations, nuclear power plants, etc. Its importance in industry and physical sciences, such as making predictions about heat transfer in nuclear power plants, drag in oil pipelines and the weather is tremendous. Besides these real life relevant issues, the study of turbulence can assist mathematical researchers in understanding some aspect of Navier-Stokes equation and MHD equations, such as regularity problems [42].

According to literature, many generation of scientists passed through the struggle to unlock the mysteries of turbulent flows ever since the very systematic observation by Leonardo da Vinci, at the beginning of 16th century. Very long after L. da Vinci, the discovery of Euler equations in the mid of the 18th century (to describe motion of non viscous fluids) and Navier-Stokes equations in the first half of the 19th century (for viscous fluids) are the major breakthrough developments in terms of having governing rule for certain type of fluid flows.

Towards the end of 19th century Osborne Reynolds in his attempt to solve Euler and Navier-Stokes equations laid a foundation for the theory of turbulence, see [89, 141, 142], [160, p. 488]. Reynolds work was based on decomposing the velocity field $u(x,t)$ into average velocity $\bar{u}(x,t)$ over a time interval and fluctuation velocity $u'(x,t) = u(x,t) - \bar{u}(x,t)$. He studied the dynamical system in terms of the average velocity $\bar{u}(x,t)$, and the resulting equation is called Reynolds equations or Reynolds averaged Navier-Stokes (RANS) equations. Furthermore, his analysis on the kinetic energy of a turbulent flow was another big contribution [107, 143]. Other great contributions include the works of Ludwig Prandtl (1875-1953) on boundary layer problems [137], Theodore von Kármán (1881-1963) on isotropic turbulence [169, 170], Geoffrey Ingram Taylor (1886-1975) on isotropic turbulence and turbulent diffusion [157, 158], etc. For further reading in this regard we refer to the book "A Voyage through turbulence" [51] and an excellent review on the works of Onsager by Eyink and Sreenivasan [61].

Despite these important developments in the study of turbulent flows, it was in the early 1940's that Kolmogorov and his students, Millionshchikov and Obukhov, brought the whole theory to qualitatively new level which has essentially stood the test of time. Particularly the 1941 works of Kolmogorov [91, 92, 93, 94] and Obukhov[131], usually referred as K41 theory or Kolmogorov theory, which later improved following the critics of E Landau [95] play a central role in the area. The main achievement of the theory is Kolmogorov's 5/3 law which postulates decay of the spectral energy according to the function $E_K(k)$ called Kolmogorov spectral function, defined by

$$C_0 \epsilon^{2/3} k^{-5/3} \tag{4.1}$$

over a range of wave numbers $k \in [k_1, k_2]$; where ϵ is energy dissipation rate and C_0 is a universal constant called Kolmogorov constant. The exponents in (4.1) are determined by dimensional analysis. The state of the art exposition of Komogorov's school of turbulence can be found in the seminal monographs of Monin and Yaglom [1, 127], sometimes referred to as the bible of turbulence. Besides the breakthrough results they come up with, their approach was found to be more convenient than that of Reynolds in the sense that Reynolds averaging approach complicates the dynamical equation and to simplify the complexity arising from averaging over a given time interval (or a spatial region) it was required to set certain general conditions that hold only approximately and therefore inconvenient. On this aspect, A. M. Yaglom in his commentary about the contributions of Kolmogorov theory said "Before these papers appeared, nobody had guessed that random turbulent fluctuations obey some simple quantitative relationships of a quite universal character, that is, they remain valid for all flows sufficiently distinct from laminar flows " [160, p. 489]. For a detail reading on Kolmogorov's theory in particular and turbulence in general, see [12, 71, 126, 127, 136, 149].

We have seen in Part II that direct approach to turbulence via Navier-Stokes could be traced back to Leray's ground breaking work [110, 112], where weak solutions are referred to as turbulent solutions. Fundamental contributions are also due to Hopf [79, 81, 82], see also [67, 68, 69, 159], just to cite a few. The emergence of the modern theory stochastic processes also led to modeling of turbulence through stochastic Navier-Stokes equations; the relevant mathematical studies were pioneered by Bensoussan and Temam in [14]. See also [22, 119, 120, 121] for recent important contributions.

Our aim in this part of the thesis is to investigate Kolmogorov's theory for electrically conductive or MHD flows. Indeed, several works on the energy spectral function of MHD flows have been done since the mid of 20th century. From the earliest, the works of Kraichnan [99, 100], Iroshnikov [86] can be mentioned. Unlike Kolmogorov, in their studies Kraichnan and Iroshnikov concluded that the spectral energy of MHD flows is proportional to $k^{-3/2}$ over a certain range of wave numbers, k; this was later on supported by Dobrowolny et al. in [57]. Verma in [165] mentioned these works to be the first to establish phenomenological theory on MHD turbulence, which he referred to as KID phenomenon.

Despite the clear difference between KID and Kolmogorov theory a lot is done to verify their validity. For instance, Verma in [164] has noted that his observational results and calculations agreed with Kolmogorov's theory more than with the KID phenomenon; and Biskamp in [18] concluded that in the general magnetohydrodynamics case the behavior of fully developed MHD turbulence is close to Kolmogorov's $k^{-5/3}$ theory rather than KID.

Therefore, in this work we investigate the spectral behavior of general MHD flow rigorously through mathematical techniques such as Harmonic Analysis. The work was motivated by the 2012 paper of Biryuk and Craig [17], where they used weak solution of Navier stokes equation to give rigorous upper and lower bounds on the inertial range. Even if Kolmogorov spectral theory roughly holds for a fully developed MHD turbulence, practically, MHD flows are different from hydrodynamic flows at least for a reason that MHD flows are controlled by a combined effect of velocity and magnetic fields. S. Chandrasekhar in [32] described the situation as follows, "...the amplification of the magnetic field by the turbulent motions and the suppression of the motions by the magnetic field will balance each other and one may expect that an equipartition between the two forms of energy will result."

The MHD model under consideration is

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla\Pi - (b\cdot\nabla)b - \nu\Delta u &= f_1, && (0,\infty)\times D,\\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u - \eta\Delta b &= f_2, && (0,\infty)\times D,\\ \operatorname{div}\, u = \operatorname{div}\, b &= 0, && (0,\infty)\times D,\\ u|_{t=0} = u_0, \quad b|_{t=0} &= b_0, && D, \end{aligned} \tag{4.2}$$

where $u(x,t)$ is the flow velocity, $b(x,t)$ is the magnetic field, Π is the total pressure, $\nu > 0$ is the kinetic viscosity of the fluid, $\eta > 0$ is the resistivity of the fluid. The spatial domain D is either the whole of the Euclidean space $\mathbb{R}^3$, or the compact boundary-less torus, $\mathbb{T}^3 := \left(\mathbb{R}/L\mathbb{Z}\right)^3$ of length L (with Lebesgue measure $\mathrm{d}x$; note that the total measure of this torus is L^3). The time domain is $0 < t < \infty$, and the inhomogeneous external forces f_1, f_2 are assumed to be divergence-free and satisfy

$f_1, f_2 \in L^\infty_{loc}([0,\infty); H^{-1}(D) \cap L^2(D))$.

The energy spectral function, $E(k,t)$ for our MHD flow model is given by

$$E(k,t) := \int_{|\xi|=k} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2)\ \mathrm{d}S(\xi), \quad k \in [0,\infty), \{|\xi| = k\} \subset D. \tag{4.3}$$

We replace the integral in (4.3) with an appropriate sum when D is periodic. Through the process we will give global estimates in Fourier spaces for a weak solution of (4.2) when subject to a reasonably smooth data. Similarly to Biryuk and Craig in [17], these estimates are of high importance in getting the estimates on the spectral function. The spectral function $E(k,t)$ in (4.3) is found to be bounded uniformly in k, and pointwise in time in the presence of external forces f_i for $i = 1, 2$ and uniform otherwise. Furthermore, $E(k,t)$ obeys Kolmogorov's spectral theory over a small neighborhood of explicitly calculated inertial range over a finite time interval which depends on the energy dissipation rate of the flow.

As noted earlier, weak solutions can be taken as turbulent solutions. Thus, our aim is to investigate Kolmogorov's spectral theory for the weak solution of the system (4.2). Before giving the definition of weak solution, we introduce some function spaces and their notations as they appear in [25]. We denote by $C^\infty_{0,\sigma}$ the set of all divergence free smooth functions with compact support in $\mathbb{R}^n$. L^p_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^p norm in the usual sense. For $1 \le p \le \infty$ the space L^p stands for the usual (vector-valued) Lebesgue space over $\mathbb{R}^n$. For $s \in \mathbb{R}$, we denote by H^s_σ the closure of $C^\infty_{0,\sigma}$ with respect to the H^s norm. A weak solution for (4.2) in the sense of Leray and Hopf is defined as:

Definition 81. *Let $(u_0(x), b_0(x)) \in L^2_\sigma(\mathbb{R}^n)$. A vector (u,b) is said to be a weak solution to (4.2) on $D \times [0,\infty)$ if it satisfies the following conditions:*

(i) for any $T > 0$ the vector function (u,b) lies in the following function space:

$$u,\ b \in L^\infty([0,T); L^2_\sigma(D)) \cap L^2([0,T); H^1_\sigma(D)),$$

(ii) the pair (u,b) is a distributional solution of (4.2); i.e., for every $(\Phi,\Psi) \in$

$H^1((0,T);H^1_\sigma \cap L^2)$ *with* $\Phi(T) = \Psi(T) = 0$,

$$\int_0^T \{-(u,\partial_t\Phi) + \nu(\nabla u,\nabla\Phi) + (u\cdot\nabla u,\Phi) - (b\cdot\nabla b,\Phi)\}\ \mathrm{d}t$$
$$= -(u_0,\Phi(0)) + \int_0^T (f_1,\Phi)\ \mathrm{d}t$$

and

$$\int_0^T \{-(b,\partial_t\Psi) + \eta(\nabla b,\nabla\Psi) + (u\cdot\nabla b,\Psi) - (b\cdot\nabla u,\Psi)\}\ \mathrm{d}t$$
$$= -(b_0,\Psi(0)) + \int_0^T (f_2,\Psi)\ \mathrm{d}t$$

where n *is the dimension of spatial domain. Furthermore,* $\lim_{t\to 0^+} u(\cdot,t) = u_0(\cdot)$ *exists in the strong* L^2 *sense.*

(iii) the following energy inequality is satisfied,

$$\frac{1}{2}\int_D |u(x,t)|^2 + |b(x,t)|^2\ \mathrm{d}x + \min(\nu,\eta)\int_0^t\int_D |\nabla u(x,s)|^2 + |\nabla b(x,s)|^2\ \mathrm{d}x\ \mathrm{d}s$$
$$-\int_0^t\int_D u(x,s)\cdot f_1(x,s) + b(x,s)\cdot f_2(x,s)\ \mathrm{d}x\ \mathrm{d}s \le \frac{1}{2}\int_D |u_0(x)|^2 + |b_0(x)|^2\ \mathrm{d}x \tag{4.4}$$

for all $0 < t < \infty$.

The rest of this chapter is organized as follows: in section 4.2 we give estimates for the weak solution of (4.2) in Fourier spaces. Section 4.3 is devoted to analyzing spectral behavior of the MHD flow governed by (4.2). It is, therefore, in this section that we state and prove our main results.

4.2 Fourier estimates for the solution field (u, b)

4.2.1 The Fourier transform and reduction of the problem

The Fourier transform of u denoted either by $\hat{u}$ or $\mathcal{F}u$ is defined as,

$$\hat{u}(k) = \int_D e^{-ik\cdot x} u(x) \ \mathrm{d}x,$$

where D is either $\mathbb{R}^3$ or $\mathbb{T}^3$. The Fourier transform has several interesting properties, of which Parseval-Plancherel identity, given by (4.5), is of huge importance in our work. This is due to the fact that energy of our system in Fourier space is the same as energy of the system in Cartesian space.

$$\|u\|^2_{L^2(D)} = \|\hat{u}\|^2_{L^2(D)}. \tag{4.5}$$

For the detail of this and other properties of the Fourier transform we refer to [156, 171] and [83].

We next give an equivalent formulation for (4.2) in Fourier space. The first step is to eliminate the pressure term, and this will be done by making use of the Leray projector, $\mathcal{P}$, defined by

$$\mathcal{P}\cdot := Id - \nabla\Delta^{-1}\operatorname{div}\cdot. \tag{4.6}$$

The application of $\mathcal{P}$ reduces the system (4.2) to

$$\begin{cases} \partial_t u - \nu\Delta u = \mathcal{P}((b\cdot\nabla)b) - \mathcal{P}((u\cdot\nabla)u) + f_1 \\ \partial_t b - \eta\Delta b = \mathcal{P}((b\cdot\nabla)u) - \mathcal{P}((u\cdot\nabla)b) + f_2 \\ u|_{t=0} = u_0 \quad b|_{t=0} = b_0 \end{cases}. \tag{4.7}$$

Now taking the Fourier transform of (4.7) we get

$$\begin{cases} \hat{u}_t + \nu|k|^2\hat{u} = \mathcal{F}(\mathcal{P}((b\cdot\nabla)b)) - \mathcal{F}(\mathcal{P}((u\cdot\nabla)u)) + \hat{f}_1 \\ \hat{b}_t + \eta|k|^2\hat{b} = \mathcal{F}(\mathcal{P}((b\cdot\nabla)u)) - \mathcal{F}(\mathcal{P}((u\cdot\nabla)b)) + \hat{f}_2 \\ \hat{u}|_{t=o} = \hat{u}_0, \quad \hat{b}|_{t=o} = \hat{b}_0 \end{cases}. \tag{4.8}$$

Thus (4.8) is a reduced but equivalent formulation of (4.2) in Fourier space.

4.2.2 Estimates in $\mathbb{T}^3$

Note that in $\mathbb{R}^3$ we have $(u\cdot\nabla)b = \sum_{j=1}^{3} u^j\partial_j b$. Therefore from linearity of Fourier transform we get

$$\mathcal{F}((u\cdot\nabla)b) = \sum_{j=1}^{3} \hat{u}^j * (ik^j\hat{b}).$$

Now for $k \in \hat{\mathbb{T}}^3$, a Pontryagin dual of $\mathbb{T}^3$,

$$(\hat{u} * \hat{b})(k) = \sum_{k_1\in\hat{\mathbb{T}}^3} \hat{u}(k-k_1)\hat{b}(k_1).$$

[1]Then it follows that

$$\mathcal{F}((u\cdot\nabla)b) = i\sum_{j=1}^{3}\sum_{k_1} \hat{u}^j(k-k_1)k_1^j\hat{b}(k_1) = i\sum_{k_1} k_1\cdot\hat{u}(k-k_1)\hat{b}(k_1).$$

This implies

$$\mathcal{P}(\widehat{(u\cdot\nabla)}b) = i\sum_{k_1} k_1\cdot\hat{u}(k-k_1)\hat{b}(k_1) - i\frac{k}{|k|^2}k\cdot\sum_{k_1} k_1\cdot\hat{u}(k-k_1)\hat{b}(k_1). \tag{4.9}$$

Now defining a linear homogeneous operator by

$$\Pi_k(z) := z - (z\cdot k)\frac{k}{|k|^2}$$

[1]This after, by writing $\sum_{k_1}\cdot$ we mean $\sum_{k_1\in\hat{\mathbb{T}}^3}\cdot$

for z in an appropriate domain we get

$$\mathcal{P}(\widehat{(u \cdot \nabla)b}) = i\Pi_k \sum_{k_1} k_1 \cdot \hat{u}(k - k_1)\hat{b}(k_1). \tag{4.10}$$

Now substituting (4.10) in (4.8) yields

$$\begin{cases} \hat{u}_t + \nu|k|^2\hat{u} = i\Pi_k \sum_{k_1} k_1 \cdot \hat{u}(k - k_1)\hat{u}(k_1) - i\Pi_k \sum_{k_1} k_1 \cdot \hat{b}(k - k_1)\hat{b}(k_1) + \hat{f}_1 \\ \hat{b}_t + \eta|k|^2\hat{b} = i\Pi_k \sum_{k_1} k_1 \cdot \hat{b}(k - k_1)\hat{u}(k_1) - i\Pi_k \sum_{k_1} k_1 \cdot \hat{u}(k - k_1)\hat{b}(k_1) + \hat{f}_2 \\ \hat{u}|_{t=o} = \hat{u}_0, \quad \hat{b}|_{t=o} = \hat{b}_0 \end{cases} . \tag{4.11}$$

Thus we have (4.11) as an equivalent formulation of (4.2) in Fourier space, $\hat{\mathbb{T}}^3$.

We now define the following terminologies, which are used repeatedly throughout the work; notations and definitions are adopted from [17].

Definition 82. *Set A is said to be a (future) invariant set with respect to a function φ or family of functions $\{\varphi(t) : t \in [0, \infty)\}$, if*

$$\varphi(0) \in A \textit{ implies } \varphi(t) \in A \quad \textit{for all } t \geq 0.$$

Remark 83. *We denote by*

(i) $B_R(0)$ *a ball in* $L^2(D)$ *of radius* R.

(ii) $A_{R_1} := \{(u, b) : \forall k \in \hat{\mathbb{T}}^3, |k|(|\hat{u}(k)| + |\hat{b}(k)|) \leq R_1\}$ [2]

Remark 84. *Note that when $f_i \equiv 0$ for $i = 1, 2$ and $\|u_0\|_{L^2(D)} + \|b_0\|_{L^2(D)} \leq R$ from energy equation (4.4) $B_R(0)$ can be taken as a future invariant set for weak solutions of (4.2).*

Our first result in this section is;

[2] When $D = \mathbb{R}^3$, then $k \in \mathbb{R}^3$ in Fourier space.

Theorem 85. *Let $f_i \equiv 0$, $i = 1, 2$, and R, R_1 be non-negative real numbers such that*

$$R^2 \leq \frac{1}{4}\min(\nu, \eta) R_1. \tag{4.12}$$

Then for (u, b), a weak solution of (4.2), we have

(i) $A_{R_1} \cap B_R(0)$ is a future invariant set of (u, b),

(ii) If u_0, $b_0 \in A_{R_1} \cap B_R(0)$ then

$$\sup_{0<t<\infty}\left(\left|\hat{u}(k,t)\right| + \left|\hat{b}(k,t)\right|\right) \leq \frac{R_1}{|k|}, \quad \forall k \in \hat{\mathbb{T}}^3 \tag{4.13}$$

Proof of Theorem 85. Suppose (u, b) is a weak solution of (4.2). Let $\mathbb{C}_k^2$ be a set defined by,

$$\mathbb{C}_k^2 := \{w \in \mathbb{C}^3 | w \cdot k = 0\}.$$

Since vectors u and b are divergence free, and $\mathcal{F}(\operatorname{div} u) = k \cdot \hat{u} = 0$, $\mathcal{F}(\operatorname{div} b) = k \cdot \hat{b} = 0$ we have $\hat{u}, \hat{b} \in \mathbb{C}_k^2$. Then for $\hat{u}$, $\hat{b} \in \mathbb{C}_k^2$, (4.11) is equivalent to

$$\begin{cases} \hat{u}_t = -\nu|k|^2\hat{u} + i\Pi_k k \cdot \sum_{k_1} \hat{u}(k-k_1)\hat{u}(k_1) - i\Pi_k k \cdot \sum_{k_1} \hat{b}(k-k_1)\hat{b}(k_1) + \hat{f}_1 \\ \hat{b}_t = -\eta|k|^2\hat{b} + i\Pi_k k_1 \cdot \sum_{k_1} \hat{b}(k-k_1)\hat{u}(k_1) - i\Pi_k k_1 \cdot \sum_{k_1} \hat{u}(k-k_1)\hat{b}(k_1) + \hat{f}_2 \\ \hat{u}|_{t=o} = \hat{u}_0, \quad \hat{b}|_{t=o} = \hat{b}_0 \end{cases}. \tag{4.14}$$

But Holder's and Young's inequalities imply that

$$\left|\Pi_k k \cdot \sum_{k_1} \hat{u}(k-k_1)\hat{b}(k_1)\right| \leq |k| \left\|\sum_{k_1} \hat{u}(k-k_1)\hat{b}(k_1)\right\|_\infty \leq |k| \|\hat{u}\|_{L^2} \|\hat{b}\|_{L^2} \tag{4.15}$$

Now denote the RHS of first two equations of (4.14) by $X_1(\hat{u}, \hat{b})$ and $X_2(\hat{u}, \hat{b})$ respectively. Then following the arguments of Biryuk and Craig in [17, p. 429], the radial component of $X_1(\hat{u}, \hat{b})$ is given by $\Re\left(\frac{\bar{\hat{u}}}{|\hat{u}|} \cdot X_1(\hat{u}, \hat{b})\right)$ and radial component of

$X_2(\hat{u},\hat{b})$ is given by $\Re\left(\frac{\bar{\hat{b}}}{|\hat{b}|}\cdot X_2(\hat{u},\hat{b})\right)$. [3]

Now suppose (u, b) is on the boundary of A_{R_1} for some (k, t), i.e.,
$|k|\left(|\hat{u}(k,t)|+|\hat{b}(k,t)|\right)=R_1$. Now computing the dot product $\frac{\bar{\hat{u}}}{|\hat{u}|}\cdot X_1(\hat{u},\hat{b})$,

$$\left(-\nu|k|^2\hat{u}+i\Pi_k k\cdot\sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1)-i\Pi_k k\cdot\sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1)\right)\cdot\frac{\bar{\hat{u}}}{|\hat{u}|}$$
$$=-\nu|k|^2\frac{|\hat{u}|^2}{|\hat{u}|}+i\frac{\bar{\hat{u}}}{\hat{u}}\cdot\left(\Pi_k k\cdot\sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1)-\Pi_k k\cdot\sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1)\right).$$

This implies,

$$\begin{aligned}
&\Re(X_1\cdot\bar{\hat{u}})\\
&=-\nu|k|^2|\hat{u}|^2+\Im\left(\bar{\hat{u}}\cdot(\Pi_k\sum_{k_1}k_1\cdot\hat{u}(k-k_1)\hat{u}(k_1))-\bar{\hat{u}}\cdot(\Pi_k\sum_{k_1}k_1\cdot\hat{b}(k-k_1)\hat{b}(k_1))\right)\\
&\le-\nu|k|^2|\hat{u}|^2+|\hat{u}|\left|\Pi_k\sum_{k_1}k_1\cdot\hat{u}(k-k_1)\hat{u}(k_1)\right|+|\hat{u}|\left|\Pi_k\sum_{k_1}k_1\cdot\hat{b}(k-k_1)\hat{b}(k_1)\right|\\
&\le-\nu|k|^2|\hat{u}|^2+|\hat{u}||k|\|\hat{u}\|_{L^2}^2+|\hat{u}||k|\|\hat{b}\|_{L^2}^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\Re(X_2\cdot\bar{\hat{b}})\\
&=-\eta|k|^2|\hat{b}|^2+\Im\left(\bar{\hat{b}}\cdot(\Pi_k\sum_{k_1}k_1\cdot\hat{b}(k-k_1)\hat{u}(k_1))-\bar{\hat{b}}\cdot(\Pi_k\sum_{k_1}k_1\cdot\hat{u}(k-k_1)\hat{b}(k_1))\right)\\
&\le-\eta|k|^2|\hat{b}|^2+|\hat{b}||k|\|\hat{b}\|_{L^2}\|\hat{u}\|_{L^2}+|\hat{b}||k|\|\hat{u}\|_{L^2}\hat{b}\|_{L^2}\\
&=-\eta|k|^2|\hat{b}|^2+2|\hat{b}||k|\|\hat{b}\|_{L^2}\|\hat{u}\|_{L^2}.
\end{aligned}$$

Then it follows that,

$$\begin{aligned}
&\Re(X_1\cdot\bar{\hat{u}})/|\hat{u}|+\Re(X_2\cdot\bar{\hat{b}})/\hat{b}\\
&\le-\nu|k|^2|\hat{u}|+|k|\|\hat{u}\|_{L^2}^2+|k|\|\hat{b}\|_{L^2}^2-\eta|k|^2|\hat{b}|+2|k|\|\hat{b}\|_{L^2}\|\hat{u}\|_{L^2}
\end{aligned}$$

[3] $\Re(\cdot)$ denotes the real component and $\Im(\cdot)$ denotes the imaginary component.

$$\leq -\min(\nu,\eta)|k|^2(|\hat{u}|+|\hat{b}|)+|k|\left(\|\hat{u}\|_{L^2}^2+\|\hat{b}\|_{L^2}^2+2\|\hat{b}\|_{L^2}\|\hat{u}\|_{L^2}\right)$$

$$\leq -\frac{1}{2}\min(\nu,\eta)|k|R_1+2|k|\left(\|\hat{u}\|_{L^2}^2+\|\hat{b}\|_{L^2}^2\right). \tag{4.16}$$

When $\big(u(\cdot,t),b(\cdot,t)\big)\in B_R(0)$ and $R^2<\frac{1}{4}\min(\nu,\eta)R_1$, the RHS of (4.16) is negative. This implies that the solution (u,b) never leaves the region $A_{R_1}\cap B_R(0)$. Hence, $A_{R_1}\cap B_R(0)$ is a future invariant set of the weak solution of (4.2). Therefore, we have that

$$\sup_{0<t<\infty}|\hat{u}(k,t)|+|\hat{b}(k,t)|\leq\frac{R_1}{|k|}$$

□

Let $(u_0,b_0)\in B_R(0)\subset L^2(D)$ and $f_1,f_2\in L^\infty_{loc}([0,\infty);H^{-1}(D)\cap L^2(D))$. Suppose that an appropriate frame is chosen and the total pressure Π is suitably normalized so that $\int_D u(x,t)\cdot f_1(x,t)+b(x,t)\cdot f_2(x,t)\ \mathrm{d}x$ is bounded. This implies that for any $T>0$ we can have a non negative function $R(T)$ such that

$$\|u(\cdot,T)\|_{L^2}^2+\|b(\cdot,T)\|_{L^2}^2+\min(\nu,\eta)\int_0^T(\|\nabla u(\cdot,s)\|_{L^2}^2+\|\nabla b(\cdot,s)\|_{L^2}^2)\ \mathrm{d}s\leq R^2(T). \tag{4.17}$$

Define $F^2(T):=\int_0^T\|f_1(\cdot,t)\|_{\dot{H}^{-1}}^2+\|f_2(\cdot,t)\|_{\dot{H}^{-1}}^2\ \mathrm{d}t$. Then the function $R^2(T)$ becomes an upper bound for the LHS of energy inequality (4.4).

Theorem 86. *Suppose $f_i\not\equiv 0$ for $i=1,2$ and $R(t)$ is a priori upper bound for $\|u(\cdot,t)\|_{L^2}+\|b(\cdot,t)\|_{L^2}$. If $R_1(t)$ is a non decreasing function such that for all k,t,*

$$2R^2(t)+\frac{|\hat{f}_1|+|\hat{f}_2|}{|k|}\leq\min(\nu,\eta)R_1(t) \tag{4.18}$$

is satisfied, then whenever the initial data u_0, b_0 belongs to $A_{R_1(0)}\cap B_{R(0)}(0)$ we have that for any $0<t<\infty$ the Fourier coefficient of any weak solution generated by

u_0, b_0 *subjected to external forces* f_1, f_2 *satisfies*

$$|\hat{u}(k,t)| + |\hat{b}(k,t)| \leq \frac{R_1(t)}{|k|}. \tag{4.19}$$

Proof of Theorem 86. For $f_i \not\equiv 0$, for $i = 1,2$ we have an equivalent equation to (4.2) in the Fourier space

$$\begin{cases} \hat{u}_t = -\nu|k|^2\hat{u} + i\Pi_k \sum_{k_1} k_1 \cdot \hat{u}(k-k_1)\hat{u}(k_1) - i\Pi_k \sum_{k_1} k_1 \cdot \hat{b}(k-k_1)\hat{b}(k_1) + \hat{f}_1 \\ \hat{b}_t = -\eta|k|^2\hat{b} + i\Pi_k \sum_{k_1} k_1 \cdot \hat{b}(k-k_1)\hat{u}(k_1) - i\Pi_k \sum_{k_1} k_1 \cdot \hat{u}(k-k_1)\hat{b}(k_1) + \hat{f}_2 \\ \hat{u}|_{t=o} = \hat{u}_0, \quad \hat{b}|_{t=o} = \hat{b}_0 \end{cases} \tag{4.20}$$

By abuse of notation we denote the RHS of (4.20) by $X_1(u,b)(k)$ for the first and $X_2(u,b)(k)$ for the second one as in (4.14). Therefore the radial component of $X_1(u,b)(k)$ is the real component of $X_1(u,b)(k) \cdot \dfrac{\bar{\hat{u}}}{|\hat{u}|}$ and the radial component of $X_2(u,b)(k)$ is the real component of $X_2(u,b)(k) \cdot \dfrac{\bar{\hat{b}}}{|\hat{b}|}$. Since

$$\begin{aligned} \Re(X_1 \cdot \bar{\hat{u}}) &= -\nu|k|^2|\hat{u}|^2 + \Re(\hat{f}_1 \cdot \hat{u}) \\ &+ \Im\left(\bar{\hat{u}} \cdot (\Pi_k \sum_{k_1} k_1 \cdot \hat{u}(k-k_1)\hat{u}(k_1)) - \bar{\hat{u}} \cdot (\Pi_k \sum_{k_1} k_1 \cdot \hat{b}(k-k_1)\hat{b}(k_1))\right) \\ &\leq -\nu|k|^2|\hat{u}|^2 + |\hat{u}||k|\|\hat{u}\|^2_{L^2} + |\hat{u}||k|\|\hat{b}\|^2_{L^2} + |\hat{f}_1||\hat{u}|, \end{aligned}$$

and similarly

$$\Re(X_2 \cdot \bar{\hat{b}}) \leq -\eta|k|^2|\hat{b}|^2 + 2|\hat{b}||k|\|\hat{b}\|_{L^2}\|\hat{u}\|_{L^2} + |\hat{f}_2||\hat{b}|.$$

Thus radial component of the sum $X_1 + X_2$ is given by

$$\begin{aligned} \Re(X_1 \cdot \frac{\bar{\hat{u}}}{|\hat{u}|}) + \Re(X_2 \cdot \frac{\bar{\hat{b}}}{|\hat{b}|}) &\leq -\nu|k|^2|\hat{u}| - \eta|k|^2|\hat{b}| + |k|\|\hat{u}\|^2_{L^2} + |k|\|\hat{b}\|^2_{L^2} \\ &+ 2|k|\|\hat{b}\|_{L^2}\|\hat{u}\|_{L^2} + |\hat{f}_1| + |\hat{f}_2| \\ &\leq -\min(\nu,\eta)|k|^2\left(|\hat{u}| + |\hat{b}|\right) + |k|(\|\hat{u}\|^2 + \|\hat{b}\|^2) + 2|k|\|\hat{u}\|\|\hat{b}\| + |\hat{f}_1| + |\hat{f}_2| \end{aligned}$$

$$\begin{aligned}
&\leq -\min(\nu,\eta)|k|^2\left(|\hat{u}|+|\hat{b}|\right)+2|k|\left(\|\hat{u}\|^2+\|\hat{b}\|^2\right)+|\hat{f}_1|+|\hat{f}_2| \\
&\leq -\min(\nu,\eta)|k|^2\left(|\hat{u}|+|\hat{b}|\right)+2|k|\left(\|\hat{u}\|^2+\|\hat{b}\|^2\right)+|\hat{f}_1|+|\hat{f}_2| \\
&= -\min(\nu,\eta)|k|^2\left(|\hat{u}|+|\hat{b}|\right)+|k|\left(2(\|\hat{u}\|^2+\|\hat{b}\|^2)+\frac{|\hat{f}_1|+|\hat{f}_2|}{|k|}\right).
\end{aligned} \tag{4.21}$$

For $(u, b)(k, t)$ on the boundary of $A_{R_1(t)}$ we have $|k|\left(|\hat{u}|+|\hat{b}|\right) = R_1(t)$ and if $2(\|\hat{u}\|^2+\|\hat{b}\|^2)+\frac{|\hat{f}_1|+|\hat{f}_2|}{|k|} < \min(\nu,\eta)R_1(t)$, then the RHS of (4.21) is negative, i.e, the radial component of the resultant vector of u and b is negative. This implies that the solution vector (u, b) will never skip set $A_{R_1(t)} \cap B_R(t)$. Hence (4.19) follows. □

Theorem 87. *Let* (u, b) *be a weak solution of* (4.2)*, with initial condition* (u_0, b_0) *in* A_{R_1}*, which satisfies* (4.17)*. Then we have*

$$\int_0^T |\hat{u}(k,t)|^2+|\hat{b}(k,t)|^2 dt \leq \frac{R_2^2(T)}{\min(\nu,\eta)|k|^4}, \tag{4.22}$$

where

$$\begin{aligned}
R_2(T) &= \frac{1}{2}\left(R_4(T)+\sqrt{2\left(R_1^2(0)+R_1^2(T)\right)+R_4^2(T)}\right), \\
R_4(T) &= \frac{2}{\sqrt{\min(\nu,\eta)}}R^2(T)+\sqrt{\frac{2}{\min(\nu,\eta)}}F_1(k,T), \\
F_1(k,T) &= \left(\int_0^T |\hat{f}_1(k,t)|^2+|\hat{f}_2(k,t)|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

If $\sup_{k\in\mathbb{Z}^3} F_1(k,T) := F_\infty(T) < \infty$*, the constant* $R_2(T)$ *will be independent of* k*.*

Proof of Theorem 87. From elementary theory of complex numbers and elementary calculus we have

$$|\hat{u}|^2 = \hat{u}\bar{\hat{u}} \qquad \text{and} \qquad \frac{\partial}{\partial t}|\hat{u}|^2 = \frac{\partial \hat{u}}{\partial t}\bar{\hat{u}} + \hat{u}\frac{\partial \bar{\hat{u}}}{\partial t}.$$

Now combining this with (4.20) we have,

$$
\begin{aligned}
\frac{\partial}{\partial t}|\hat{u}|^2 = &\left(-\nu|k|^2\hat{u} + i\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1) - i\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1) + \hat{f}_1\right)\bar{\hat{u}} \\
&+ \hat{u}\overline{-\nu|k|^2\hat{u} + i\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1) - i\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1) + \hat{f}_1} \\
= &- 2\nu|k|^2|\hat{u}|^2 + 2\Re\left(i\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1)\right) \\
&- 2\Re\left(i\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1)\right) + 2\Re\left(\bar{\hat{u}}\hat{f}_1\right).
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}|\hat{u}|^2 + \nu|k|^2|\hat{u}|^2 = &-\Im\left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1)\right) \\
&+\Im\left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1)\right) + \Re\left(\bar{\hat{u}}\hat{f}_1\right). \qquad (4.23)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}|\hat{b}|^2 + \eta|k|^2|\hat{b}|^2 = &-\Im\left(\bar{\hat{b}}\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{u}(k_1)\right) \\
&+\Im\left(\bar{\hat{b}}\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{b}(k_1)\right) + \Re\left(\bar{\hat{b}}\hat{f}_2\right). \qquad (4.24)
\end{aligned}
$$

Finally summing up (4.23) and (4.24) we get

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}(|\hat{u}|^2 + |\hat{b}|^2) + \left(\nu|k|^2|\hat{u}|^2 + \eta|k|^2|\hat{b}|^2\right) = &-\Im\left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{u}(k_1)\right) \\
&+ \Im\left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{b}(k_1)\right) - \Im\left(\bar{\hat{b}}\Pi_k k \cdot \sum_{k_1}\hat{b}(k-k_1)\hat{u}(k_1)\right) \\
&+ \Im\left(\bar{\hat{b}}\Pi_k k \cdot \sum_{k_1}\hat{u}(k-k_1)\hat{b}(k_1)\right) + \Re\left(\bar{\hat{u}}\hat{f}_1\right) + \Re\left(\bar{\hat{b}}\hat{f}_2\right). \qquad (4.25)
\end{aligned}
$$

Taking the time integral of (4.25) over the interval $[0, T]$ and rearranging some of the terms gives

$$|k|^4 \int_0^T (\nu|\hat{u}|^2 + \eta|\hat{b}|^2)\ \mathrm{d}t = \frac{1}{2}|k|^2[|\hat{u}_0|^2 + |\hat{b}_0|^2] - \frac{1}{2}|k|^2[|\hat{u}(T)|^2 + |\hat{b}(T)|^2]$$
$$- \Im|k|^2 \int_0^T \left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1} \hat{u}(k-k_1)\hat{u}(k_1)\right) dt + \Im|k|^2 \int_0^T \left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1} \hat{b}(k-k_1)\hat{b}(k_1)\right)\ \mathrm{d}t$$
$$- \Im|k|^2 \int_0^T \left(\bar{\hat{b}}\Pi_k k \cdot \sum_{k_1} \hat{b}(k-k_1)\hat{u}(k_1)\right)\ \mathrm{d}t + \Im \int_0^T \left(\bar{\hat{b}}\Pi_k k \cdot \sum_{k_1} \hat{u}(k-k_1)\hat{b}(k_1)\right)$$
$$+ \Re|k|^2 \int_0^T \left(\bar{\hat{u}}\hat{f}_1 + \bar{\hat{b}}\hat{f}_2\right)\ \mathrm{d}t. \tag{4.26}$$

Next we deal with terms in the RHS terms (4.26).

$$\frac{1}{2}\left(|k|^2|\hat{u}_0|^2 + |k|^2|\hat{b}_0|^2\right) \le \frac{1}{2}\left(|k||\hat{u}_0| + |k||\hat{b}_0|\right)^2 \le \frac{1}{2}R_1^2(0) \tag{4.27}$$

$$\frac{1}{2}\left(|k|^2|\hat{u}(T)|^2 + |k|^2|\hat{b}(T)|^2\right) \le \frac{1}{2}\left(|k||\hat{u}(T)| + |k||\hat{b}(T)|\right)^2 \le \frac{1}{2}R_1^2(T). \tag{4.28}$$

$$\left|\Im|k|^2 \int_0^T \left(\bar{\hat{u}}\Pi_k k \cdot \sum_{k_1} \hat{u}(k-k_1)\hat{u}(k_1)\right) dt\right|$$
$$\le |k|^2 \left|\int_0^T \left(\bar{\hat{u}}\Pi_k \cdot \sum_{k_1} \hat{u}(k-k_1)\cdot k_1\hat{u}(k_1)\right) dt\right|$$
$$\le |k|^2 (\int_0^T |\bar{\hat{u}}|^2)^{1/2} (\int_0^T |\Pi_k \cdot \sum_{k_1} \hat{u}(k-k_1)\cdot k_1\hat{u}(k_1)|^2 dt)^{1/2}$$
$$\le |k|^2 (\int_0^T |\bar{\hat{u}}|^2)^{1/2} (\int_0^T \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 dt)^{1/2}$$

$$\leq |k|^2 (\int_0^T |\bar{\hat{u}}|^2)^{1/2} \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2} (\int_0^T \|\nabla u(\cdot, t)\|_{L^2}^2 dt)^{1/2} \tag{4.29}$$

Here we used Holder's and Young's inequalities and divergence-free condition of vector u.

Similarly we have

$$\left| \Im |k|^2 \int_0^T \left(\bar{\hat{u}} \Pi_k k \cdot \sum_{k_1} \hat{b}(k - k_1) \hat{b}(k_1) \right) dt \right|$$

$$\leq |k|^2 (\int_0^T |\bar{\hat{u}}|^2)^{1/2} \sup_{0 \leq t \leq T} \|b(\cdot, t)\|_{L^2} (\int_0^T \|\nabla b(\cdot, t)\|_{L^2}^2 dt)^{1/2}. \tag{4.30}$$

$$\left| \Im |k|^2 \int_0^T \left(\bar{\hat{b}} \Pi_k k \cdot \sum_{k_1} \hat{b}(k - k_1) \hat{u}(k_1) \right) dt \right|$$

$$\leq |k|^2 (\int_0^T |\bar{\hat{b}}|^2)^{1/2} \sup_{0 \leq t \leq T} \|b(\cdot, t)\|_{L^2} (\int_0^T \|\nabla u(\cdot, t)\|_{L^2}^2 dt)^{1/2}. \tag{4.31}$$

$$\left| \Im |k|^2 \int_0^T \left(\bar{\hat{b}} \Pi_k k \cdot \sum_{k_1} \hat{u}(k - k_1) \hat{b}(k_1) \right) dt \right|$$

$$\leq |k|^2 (\int_0^T |\bar{\hat{b}}|^2)^{1/2} \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2} (\int_0^T \|\nabla b(\cdot, t)\|_{L^2}^2 dt)^{1/2}. \tag{4.32}$$

Finally, for the terms involving non-homogeneous forces;

$$\left| |k|^2 \Re \int_0^T \hat{f}_1 \bar{\hat{u}} + \hat{f}_2 \bar{\hat{b}} \, dt \right| \leq |k|^2 \left| \int_0^T \hat{f}_1 \bar{\hat{u}} \, dt \right| + |k|^2 \left| \int_0^T \hat{f}_2 \bar{\hat{b}} \, dt \right|$$

$$\leq |k|^2 (\int_0^T |\hat{f}_1|^2 dt)^{1/2} (\int_0^T |\hat{u}|^2 dt)^{1/2} + |k|^2 (\int_0^T |\hat{f}_2|^2 dt)^{1/2} (\int_0^T |\hat{b}|^2 dt)^{1/2}$$

$$\leq |k|^2 \left((\int_0^T |\hat{f}_1|^2 dt)^{1/2} + (\int_0^T |\hat{f}_2|^2 dt)^{1/2} \right) \left((\int_0^T |\hat{u}|^2 dt)^{1/2} + (\int_0^T |\hat{b}|^2 dt)^{1/2} \right)$$

$$\leq \sqrt{2}|k|^2 \left(\int_0^T |\hat{f}_1|^2 dt + \int_0^T |\hat{f}_2|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |\hat{u}|^2 dt + \int_0^T |\hat{b}|^2 dt \right)^{\frac{1}{2}}. \quad (4.33)$$

Now define

$$I^2(k,T) := \min(\nu,\eta)|k|^4 \int_0^T |\hat{u}(k,t)|^2 + |\hat{b}(k,t)|^2 dt.$$

Combining (4.26) -(4.33) gives

$$\begin{aligned}
&I^2(k,T) \\
&\leq \frac{1}{2}R_1^2(0) + \frac{1}{2}R_1^2(T) + |k|^2 (\int_0^T |\bar{\hat{u}}|^2 \ dt)^{1/2} \sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2} (\int_0^T \|\nabla u(\cdot,t)\|_{L^2}^2 dt)^{1/2} \\
&+ |k|^2 (\int_0^T |\bar{\hat{u}}|^2)^{1/2} \sup_{0\leq t\leq T} \|b(\cdot,t)\|_{L^2} (\int_0^T \|\nabla b(\cdot,t)\|_{L^2}^2 dt)^{1/2} \\
&+ |k|^2 (\int_0^T |\bar{\hat{b}}|^2)^{1/2} \sup_{0\leq t\leq T} \|b(\cdot,t)\|_{L^2} (\int_0^T \|\nabla u(\cdot,t)\|_{L^2}^2 dt)^{1/2} \\
&+ |k|^2 (\int_0^T |\bar{\hat{b}}|^2)^{1/2} \sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2} (\int_0^T \|\nabla b(\cdot,t)\|_{L^2}^2 dt)^{1/2} \\
&+ \sqrt{2}|k|^2 \left(\int_0^T |\hat{f}_1|^2 dt + \int_0^T |\hat{f}_2|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |\hat{u}|^2 dt + \int_0^T |\hat{b}|^2 dt \right)^{\frac{1}{2}} \\
&\leq 2|k|^2 \left(\int_0^T |\hat{u}|^2 + |\hat{b}|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
&\times \left(\sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2} + \sup_{0\leq t\leq T} \|b(\cdot,t)\|_{L^2} \right) \\
&+ \frac{1}{2}\left[R_1^2(0) + R_1^2(T)\right] + \sqrt{2}|k|^2 \left(\int_0^T |\hat{f}_1|^2 + |\hat{f}_2|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |\hat{u}|^2 + |\hat{b}|^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq 2\frac{I(k,T)R^2(T)}{\sqrt{\min(\nu,\eta)}} + \frac{1}{2}\left[R_1^2(0) + R_1^2(T)\right] + \sqrt{2}F_1(k,T)\frac{I(k,T)}{\sqrt{\min(\nu,\eta)}}. \tag{4.34}$$

This implies that

$$I^2(k,T) - \left(\frac{2}{\sqrt{\min(\nu,\eta)}}R^2(T) + \frac{\sqrt{2}}{\sqrt{\min(\nu,\eta)}}F_1(k,T)\right)I(k,T) - \frac{1}{2}\left(R_1^2(0) + R_1^2(T)\right) \leq 0. \tag{4.35}$$

Since $I(k,T)$ is positive, then it can not exceed the largest positive root of the quadratic equation associated to (4.35). Hence,

$$I(k,T) \leq \frac{1}{2}\left(\frac{2}{\sqrt{\min(\nu,\eta)}}R^2(T) + \sqrt{\frac{2}{\min(\nu,\eta)}}F_1(k,T)\right) + \frac{1}{2}\sqrt{\left(\frac{2}{\sqrt{\min(\nu,\eta)}}R^2(T) + \sqrt{\frac{2}{\min(\nu,\eta)}}F_1(k,T)\right)^2 + 2\left(R_1(0)^2 + R_1(T)^2\right)}$$

This concludes the theorem. □

4.2.3 Estimates in $\mathbb{R}^3$

This section is devoted to finding estimates in Fourier space for solutions of (4.2) for the case when $D = \mathbb{R}^3$. To begin with, we take the Fourier of (4.7) in $\mathbb{R}^3$ to get

$$\begin{cases} \hat{u}_t + \nu|\xi|^2\hat{u} = \mathcal{F}(\mathcal{P}((b\cdot\nabla)b)) - \mathcal{F}(\mathcal{P}((u\cdot\nabla)u)) + \hat{f}_1 \\ \hat{b}_t + \eta|\xi|^2\hat{b} = \mathcal{F}(\mathcal{P}((b\cdot\nabla)u)) - \mathcal{F}(\mathcal{P}((u\cdot\nabla)b)) + \hat{f}_2 \\ \hat{u}|_{t=o} = \hat{u}_0, \quad \hat{b}|_{t=o} = \hat{b}_0 \end{cases}. \tag{4.36}$$

An approach similar to the one used in the derivation of (4.10), yields,

$$\mathcal{F}(\mathcal{P}((b\cdot\nabla)b)) = i\Pi_\xi\left(\int_D \zeta\cdot\hat{u}^j(\xi-\zeta)\hat{b}(\zeta)\ \mathrm{d}\zeta\right). \tag{4.37}$$

Thus

$$\begin{cases} \hat{u}_t = -\nu|\xi|^2\hat{u} + i\Pi_\xi(\int_D \zeta \cdot \hat{b}(\xi-\zeta)\hat{b}(\zeta)d\zeta) - i\Pi_\xi(\int_D \zeta \cdot \hat{u}(\xi-\zeta)\hat{u}(\zeta)d\zeta) + \hat{f}_1 \\ \hat{b}_t = -\eta|\xi|^2\hat{b} + i\Pi_\xi(\int_D \zeta \cdot \hat{b}(\xi-\zeta)\hat{u}(\zeta)d\zeta) - i\Pi_\xi(\int_D \zeta \cdot \hat{u}(\xi-\zeta)\hat{b}(\zeta)d\zeta) + \hat{f}_2 \\ \hat{u}|_{t=o} = \hat{u}_0, \quad \hat{b}|_{t=o} = \hat{b}_0 \end{cases} \tag{4.38}$$

Without loss of generality we may assume that $\|u(\cdot,t)\|^2_{L^2} + \|b(\cdot,t)\|^2_{L^2} \leq R^2(t)$ (and in the presence of no external forces, i.e., $f_i \equiv 0$ for $i = 1,\ 2$, $R(t) = R(0)$ suffices). In the case when $D = \mathbb{R}^3$, it is well known that $\hat{u}(\xi,t)$ and $\hat{b}(\xi,t)$ are members of a Hilbert space however their values at a particular point (ξ,t) are not well defined. The issue can be addressed by taking filtered values of $u(x,t)$ and $b(x,t)$.

Let $k(\neq 0) \in \mathbb{R}^3$, $0 < \delta < \frac{|k|}{2\sqrt{3}}$. Define $\hat{\chi}_k(\xi)$ to be a smooth cut off function of the cube Q_k about k of side length 2δ such that $\hat{\chi}_k(\xi) = 1$ on a cube of the same center with side δ and

$$\operatorname{supp}\hat{\chi} = \{\xi \in \mathbb{R}^3 : \frac{|k|}{2} \leq |\xi| \leq \frac{3}{2}|k|\}.$$

We now define the following three functions;

$$\left(\hat{\chi}_k(D)u\right)(x,t) := \mathcal{F}^{-1}\left(\hat{\chi}_k(\xi)\hat{u}(\xi,t)\right) = (\chi_k * u)(x,t), \tag{4.39}$$

$$e_p(k,t) := \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p d\xi\right)^{\frac{1}{p}}, \tag{4.40}$$

$$h_p(k,t) := \sup_{0\leq s\leq t}\left(\int [|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p]/|\xi|^p d\xi\right)^{\frac{1}{p}}, \tag{4.41}$$

To read further on these functions and their properties we refer to [17, p. 430].

Next we state the two main results of this section, which indeed are equivalences of Theorems 85 and 86 in $D = \mathbb{R}^3$.

Theorem 88. *Suppose that a weak solution* (u, b) *of* (4.2) *satisfies* $\|u(\cdot, t)\|_{L^2} + \|b(\cdot, t)\|_{L^2} \leq R(t)$. *And also suppose that there exists a non-decreasing function* $R_1(t)$ *such that for* $2 \leq p < \infty$ *and* $t \in \mathbb{R}+$

$$(2\delta)^{3/p} R^2(t) + h_p(k, t) \leq \frac{\nu}{6} R_1(t). \tag{4.42}$$

where $\delta < \frac{|k|}{2\sqrt{3}}$. *If* u, b *initially satisfies*

$$\sup_{2 \leq p < \infty} e_p(k, 0) < \frac{R_1(0)}{|k|},$$

then for all positive t

$$\sup_{2 \leq p < \infty} e_p(k, t) < \frac{R_1(t)}{|k|}$$

holds.

Theorem 89. *Suppose the weak solution of* (4.2) *satisfies* (4.17) *and* $\sup_{2 \leq p < \infty} e_p(k, 0) < \frac{R_1(0)}{|k|}$. *Then for all* $T \in \mathbb{R}^+$

$$\int_0^T \sup_{2 \leq p < \infty} e_p(k, t) \ \mathrm{d}t \leq \frac{R_2^2(T)}{\nu |k|^4}, \tag{4.43}$$

and

$$R_2(T) := \frac{1}{2}(R_5(T) + \sqrt{4R_1^2(0) + R_5^2(T)}) \tag{4.44}$$

where

$$\begin{aligned} R_5(T) &= \frac{2R^2(T)}{\nu} + \frac{2F_\infty(T)}{\sqrt{\nu}} \\ F_\infty(T) &= \sup_{k \in \mathbb{R} \ \{0\}} \left(\int_0^T |\hat{\chi}_k(\xi) \hat{f}_1|^2_{L_\infty} + |\hat{\chi}_k(\xi) \hat{f}_2|^2_{L_\infty} \mathrm{d}t \right) \end{aligned} \tag{4.45}$$

To prove these results we first need to establish estimates on e_p. This will be done in two steps; first by bounding $e_2(k,t)$ which is done in Lemma 90 followed by estimate on $e_p(k,t)$ which is done in Lemma 91.

Lemma 90. *Suppose that* $\|u(\cdot,t)\|_{L^2} + \|b(\cdot,t)\|_{L^2} \le R(t)$ *and there exists a non-decreasing function* $R_1(t)$ *such that*

$$(2\delta)^{3/2}\sqrt{2}R^2(t) + 2h_2(k,t) < \frac{\min(\nu,\eta)}{6}R_1(t) \tag{4.46}$$

for all $t \in \mathbb{R}^+$ *and* $\delta < \frac{|k|}{2\sqrt{3}}$.
If $e_2(k,0) < \frac{R_1(0)}{|k|}$, *then for any* $t \in (0,\infty)$ *we have*

$$e_2(k,t) \le \frac{R_1(t)}{|k|}. \tag{4.47}$$

Proof of Lemma 90. By definition

$$e_2^2(k,t) = \int_D \hat{\chi}_k(\xi)\hat{u}(\xi,t)\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)} + \hat{\chi}_k(\xi)\hat{b}(\xi,t)\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\ \mathrm{d}\xi.$$

Differentiating this relation with respect to time and using equation (4.38), we get

$$\frac{d}{dt}e_2^2(k,t)$$

$$= \int_D \left[\hat{\chi}_k(\xi)\frac{d}{dt}\hat{u}(\xi,t)(\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}) + (\hat{\chi}_k(\xi)\hat{u}(\xi,t))\overline{\hat{\chi}_k(\xi)}\frac{d}{dt}\overline{\hat{u}(\xi,t)}\right.$$

$$\left. + \hat{\chi}_k(\xi)\frac{d}{dt}\hat{b}(\xi,t)\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)} + \hat{\chi}_k(\xi)\hat{b}(\xi,t)\overline{\hat{\chi}_k(\xi)}\frac{d}{dt}\overline{\hat{b}(\xi,t)}\right]\mathrm{d}\xi$$

$$= \int_D \left[\hat{\chi}_k(\xi)\left(-\nu|\xi|^2\hat{u} + i\Pi_\xi(\int_D \zeta\cdot\hat{b}(\xi-\zeta)\hat{b}(\zeta)\ \mathrm{d}\zeta) - i\Pi_\xi(\int_D \zeta\cdot\hat{u}(\xi-\zeta)\hat{u}(\zeta)\ \mathrm{d}\zeta) + \hat{f}_1\right)\right.$$

$$(\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}) + (\hat{\chi}_k(\xi)\hat{u}(\xi,t))$$

$$\overline{\hat{\chi}_k(\xi)\left(-\nu|\xi|^2\hat{u} + i\Pi_\xi(\int_D \zeta\cdot\hat{b}(\xi-\zeta)\hat{b}(\zeta)\ \mathrm{d}\zeta) - i\Pi_\xi(\int_D \zeta\cdot\hat{u}(\xi-\zeta)\hat{u}(\zeta)\ \mathrm{d}\zeta) + \hat{f}_1\right)}$$

$$+ \hat{\chi}_k(\xi)\left(-\eta|\xi|^2\hat{b} + i\Pi_\xi(\int_D \zeta\cdot\hat{b}(\xi-\zeta)\hat{u}(\zeta)\ \mathrm{d}\zeta) - i\Pi_\xi(\int_D \zeta\cdot\hat{u}(\xi-\zeta)\hat{b}(\zeta)\ \mathrm{d}\zeta) + \hat{f}_2\right)$$

$$\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)} + \hat{\chi}_k(\xi)\hat{b}(\xi,t)$$

$$\left.\overline{\hat{\chi}_k(\xi) - \eta|\xi|^2\hat{b} + i\Pi_\xi(\int_D \zeta\cdot\hat{b}(\xi-\zeta)\hat{u}(\zeta)\ \mathrm{d}\zeta) - i\Pi_\xi(\int_D \zeta\cdot\hat{u}(\xi-\zeta)\hat{b}(\zeta)\ \mathrm{d}\zeta) + \hat{f}_2}\right]\ \mathrm{d}\xi.$$

Applying elementary properties of complex numbers we deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}e_2^2(k,t) =& -\nu\int_D |\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^2\ \mathrm{d}\xi - \eta\int_D |\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^2\ \mathrm{d}\xi\\
&+ \int_D \Re\left(i\Pi_\xi\left(\int_D \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\ \mathrm{d}\zeta\right)\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\right)\ \mathrm{d}\xi\\
&+ \int_D \Re\left(i\Pi_\xi\left(\int_D \hat{b}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\ \mathrm{d}\zeta\right)\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\right)\ \mathrm{d}\xi\\
&- \int_D \Re\left(i\Pi_\xi\left(\int_D \hat{b}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta)\ \mathrm{d}\zeta\right)\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\right)\ \mathrm{d}\xi\\
&- \int_D \Re\left(i\Pi_\xi\left(\int_D \hat{u}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta)\ \mathrm{d}\zeta\right)\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\right)\ \mathrm{d}\xi\\
&+ \Re\int_D \hat{\chi}_k(\xi)\hat{f}_1(\xi,t)\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\ \mathrm{d}\xi + \Re\int_D \hat{\chi}_k(\xi)\hat{f}_2(\xi,t)\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\ \mathrm{d}\xi\\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{4.48}$$

Let us deal with terms on the RHS of (4.48) in more detail, we have

$$\begin{aligned}
I_1 + I_2 &= -\nu\int_D |\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^2\ \mathrm{d}\xi - \eta\int_D |\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^2\ \mathrm{d}\xi\\
&\le -\min(\nu,\eta)\frac{|k|^2}{4}\int_D \left(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^2 + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^2\right)\ \mathrm{d}\xi
\end{aligned} \tag{4.49}$$

In (4.49) we used the fact $\xi \in \operatorname{supp}\hat{\chi}_k$; that is $\frac{|k|}{2} \le |\xi| \le \frac{3}{2}|k|$.

$$I_3 = -\Im\int_D \left(\Pi_\xi\left(\int_D \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\ \mathrm{d}\zeta\right)\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\right)\ \mathrm{d}\xi,$$

which implies

$$\begin{aligned}|I_3| \leq & \left| \int_D \left(\Pi_\xi \left(\int_D \hat{u}(\xi - \zeta) \cdot \zeta \hat{u}(\zeta) \ \mathrm{d}\zeta \right) \overline{\hat{\chi}_k(\xi) \hat{u}(\xi, t)} \right) \ \mathrm{d}\xi \right| \\ \leq & \|\hat{\chi}_k \hat{u}(\cdot, t)\|_{L^2} \|\hat{\chi}_k \Pi_\xi \left(\xi \cdot \int_D \hat{u}(\xi - \zeta) \hat{u}(\zeta) \ \mathrm{d}\zeta \right) \|_{L^2} \\ \leq & \|\hat{\chi}_k \hat{u}(\cdot, t)\|_{L^2} \|\hat{\chi}_k \xi\|_{L^2} \|\hat{u}(\cdot, t)\|_{L^2}^2.\end{aligned}$$

This estimate is due to divergence freeness of the vectors u and b, and relevant properties of complex numbers. Holder's and Young's inequalities are also used.

We know from construction of χ, that

$$\begin{aligned}\|\hat{\chi}_k \xi\|_{L^2} \leq \|\xi\|_{L^4} \|\chi_k\|_{L^4} = & \left(\int_{Q_k} |\xi|^4 \ \mathrm{d}\xi \right)^{\frac{1}{4}} \left(\int_{Q_k} |\chi|^4 \ \mathrm{d}\xi \right)^{\frac{1}{4}} \\ & \leq \frac{3}{2} |k| (2\delta)^{\frac{3}{4}} (2\delta)^{\frac{3}{4}} = \frac{3}{2} |k| (2\delta)^{\frac{3}{2}}.\end{aligned}$$

Thus we have

$$|I_3| \leq \frac{3}{2} |k| (2\delta)^{\frac{3}{2}} \|\hat{\chi}_k \hat{u}(\cdot, t)\|_{L^2} \|\hat{u}(\cdot, t)\|_{L^2}^2. \tag{4.50}$$

Proceeding similarly with I_4, I_5 and I_6 we get

$$|I_4| \leq \frac{3}{2} |k| (2\delta)^{\frac{3}{2}} \|\hat{\chi}_k \hat{b}(\cdot, t)\|_{L^2} \|\hat{u}(\cdot, t)\|_{L^2} \|\hat{b}(\cdot, t)\|_{L^2} \tag{4.51}$$

$$|I_5| \leq \frac{3}{2} |k| (2\delta)^{\frac{3}{2}} \|\hat{\chi}_k \hat{u}(\cdot, t)\|_{L^2} \|\hat{b}(\cdot, t)\|_{L^2}^2 \tag{4.52}$$

$$|I_6| \leq \frac{3}{2} |k| (2\delta)^{\frac{3}{2}} \|\hat{\chi}_k \hat{b}(\cdot, t)\|_{L^2} \|\hat{u}(\cdot, t)\|_{L^2} \|\hat{b}(\cdot, t)\|_{L^2}. \tag{4.53}$$

Next we estimate I_7 as follows.

$$\begin{aligned}|I_7| = & \left| \Re \int_D \hat{\chi}_k(\xi) \hat{f}_1(\xi, t) \overline{\hat{\chi}_k(\xi) \hat{u}(\xi, t)} d\xi \right| \\ \leq & \left| \int_D \overline{\hat{\chi}_k(\xi) \hat{u}(\xi, t)} \hat{\chi}_k(\xi) |\xi| \hat{f}_1(\xi, t) / |\xi| d\xi \right| \\ \leq & \| |\xi| \hat{\chi}_k \hat{u}(\cdot, t)\|_{L^2} \|\hat{\chi}_k \hat{f}_1(\cdot, t) |\xi|^{-1}\|_{L^2}\end{aligned}$$

$$\leq \frac{3}{2}|k| \|\hat{\chi}_k \hat{u}(\cdot,t)\|_{L^2} \|\hat{\chi}_k \hat{f}_1(\cdot,t)|\xi|^{-1}\|_{L^2}; \tag{4.54}$$

thanks to Holder's inequality.

Similarly we get

$$|I_8| \leq \frac{3}{2}|k| \|\hat{\chi}_k \hat{b}(\cdot,t)\|_{L^2} \|\hat{\chi}_k \hat{f}_2(\cdot,t)|\xi|^{-1}\|_{L^2} \tag{4.55}$$

Now combining the estimates (4.49)-(4.55) we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}e_2^2(k,t) \leq &- \min(\nu,\eta)\frac{|k|^2}{4}\int_D \left(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^2 + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^2\right) d\xi \\
&+ \frac{3}{2}|k|(2\delta)^{\frac{3}{2}} \|\hat{\chi}_k \hat{u}(\cdot,t)\|_{L^2} \left(\|\hat{u}(\cdot,t)\|_{L^2}^2 + \|\hat{b}(\cdot,t)\|_{L^2}^2\right) \\
&+ 3|k|(2\delta)^{\frac{3}{2}} \|\hat{\chi}_k \hat{b}(\cdot,t)\|_{L^2} \|\hat{u}(\cdot,t)\|_{L^2} \|\hat{b}(\cdot,t)\|_{L^2} \\
&+ \frac{3}{2}|k| \left[\|\hat{\chi}_k \hat{u}(\cdot,t)\|_{L^2} \|\hat{\chi}_k \hat{f}_1(\cdot,t)|\xi|^{-1}\|_{L^2} + \|\hat{\chi}_k \hat{b}(\cdot,t)\|_{L^2} \|\hat{\chi}_k \hat{f}_2(\cdot,t)|\xi|^{-1}\|_{L^2}\right] \\
\leq &- \min(\nu,\eta)\frac{|k|^2}{4}e_2^2(k,t) + \frac{3}{2}|k|(2\delta)^{3/2}\left(\|\hat{\chi}_k \hat{u}\| + \|\hat{\chi}_k \hat{b}\|\right)\left(\|\hat{u}\|^2 + \|\hat{b}\|^2\right) \\
&+ \frac{3}{2}|k| \left(\|\hat{\chi}_k \hat{u}\| + \|\hat{\chi}_k \hat{b}\|\right)\left(\|\hat{\chi}_k \hat{f}_1/|\xi|\| + \|\hat{\chi}_k \hat{f}_2/|\xi|\|\right) \\
\leq &- \min(\nu,\eta)\frac{|k|^2}{4}e_2^2(k,t) + \frac{3}{2}|k|(2\delta)^{3/2}\sqrt{2}e_2(k,t)\left(\|\hat{u}\|^2 + \|\hat{b}\|^2\right) \\
&+ \frac{3}{2}|k|\sqrt{2}e_2(k,t)\left(\|\hat{\chi}_k \hat{f}_1/|\xi|\| + \|\hat{\chi}_k \hat{f}_2/|\xi|\|\right) \\
\leq &- \min(\nu,\eta)\frac{|k|^2}{4}e_2^2(k,t) + \frac{3}{2}|k|e_2(k,t)\left((2\delta)^{3/2}\sqrt{2}R^2(t) + 2h_2(k,t)\right)
\end{aligned} \tag{4.56}$$

Here we have used Serine's inequality to get an upper bound for $\|\hat{\chi}_k \hat{u}\| + \|\hat{\chi}_k \hat{b}\|$ and $\|\hat{\chi}_k \hat{f}_1/|\xi|\| + \|\hat{\chi}_k \hat{f}_2/|\xi|\|$; i.e.,

$$\begin{aligned}
\|\hat{\chi}_k \hat{u}\| + \|\hat{\chi}_k \hat{b}\| &\leq \sqrt{2}e_2(k,t) \\
\|\hat{\chi}_k \hat{f}_1/|\xi|\| + \|\hat{\chi}_k \hat{f}_2/|\xi|\| &\leq \sqrt{2}h_2.
\end{aligned}$$

Now define the set B_{R_1} by

$$B_{R_1} = \left\{e : e \leq R_1/|k|\right\} = \left\{e(k,t) : e(k,t) \leq R_1(t)/|k|\right\}.$$

When $e(k,t) = e_2(k,t)$ is on the boundary of B_{R_1}, i.e., when $e_2(k,t) = \frac{R_1(t)}{|k|}$, then we have

$$\frac{1}{2}\frac{d}{dt}e_2^2(k,t) < \frac{-\min(\nu,\eta)}{4}R_1^2(t) + \frac{3}{2}R_1(t)\frac{\min(\nu,\eta)}{6}R_1(t) \leq 0.$$

This implies that

$$e_2(k,t)\frac{d}{dt}e_2(k,t) < 0$$

But since $e_2(k,t) \geq 0$, it follows that

$$\frac{d}{dt}e_2(k,t) < 0$$

This means that B_{R_1} is an attracting[4] set for $e_2(k,t)$.

Therefore, if $e_2(k,0) < \frac{R_1(0)}{|k|}$, then $e_2(k,t) < \frac{R_1(t)}{|k|}$ for all $t \in (0,\infty)$.

This completes the proof of Lemma 90.

□

Lemma 91. *Suppose that for a given* $k \in \mathbb{R}^3$ *and* $2 \leq p < \infty$ *there is a nondecreasing function* $R_1(t)$ *that satisfies the condition*

$$2^{\frac{1}{p}}(2\delta)^{3/p}R^2(t) + 2h_p(k,t) < \frac{\nu}{6}R_1(t)$$

for $0 < \delta < |k|/2\sqrt{3}$.

If a solution to (4.2) *initially satisfies*

$$e_p(k,0) < R_1(0)/|k|,$$

then for all $0 < t < \infty$

$$e_p(k,t) < \frac{R_1(t)}{|k|}. \tag{4.57}$$

[4]We refer to [53] on attracting sets.

Proof of Lemma 91. The proof of this lemma is done in the same way as that of lemma 90. Thus taking the time derivative of $e_p^p(k,t)$

$$
\begin{aligned}
\frac{d}{dt}e_p^p(k,t) =&\partial_t \int \left|\hat{\chi}_k(\xi)\hat{u}(\xi,t)\right|^p + \left|\hat{\chi}_k(\xi)\hat{b}(\xi,t)\right|^p \ \mathrm{d}\xi \\
=&\Re\Bigg\{\int \Bigg(p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\left((\hat{\chi}_k(\xi)\partial_t\hat{u}(\xi,t))(\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)})\right) \\
&+p|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\left((\hat{\chi}_k(\xi)\partial_t\hat{b}(\xi,t))(\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)})\right)\Bigg) \ \mathrm{d}\xi\Bigg\} \\
=&-\nu\int p|\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi - \eta\int p|\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi \\
&+\Re\int ip|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\int \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+\Re\int ip|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\int \hat{b}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+\Re\int ip|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\int \hat{b}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+\Re\int ip|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\int \hat{u}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+\Re\int p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_1(\xi,t) \ \mathrm{d}\xi \\
&+\Re\int p|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_2(\xi,t) \ \mathrm{d}\xi \\
=&: I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8
\end{aligned}
\tag{4.58}
$$

In the derivation of (4.58) we have used the following fact;

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\hat{\chi}_k(\xi)\hat{b}(\xi,t)| =&\frac{d}{dt}\sqrt{\hat{\chi}_k(\xi)\hat{b}(\xi,t)\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}} \\
=&\frac{1}{2}|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{-1}\Big[(\hat{\chi}_k(\xi)\partial_t\hat{b}(\xi,t))(\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}) \\
&+\overline{(\hat{\chi}_k(\xi)\partial_t\hat{b}(\xi,t))(\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)})}\Big].
\end{aligned}
$$

We now estimate the RHS of (4.58), i.e., the I_j's for $j=1,\dots,8$ as follows;

$$
I_1+I_2 = -\nu\int p|\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi - \eta\int p|\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi
$$

$$\leq \frac{-\nu p|k|^2}{4}\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \;\mathrm{d}\xi + \frac{-\eta p|k|^2}{4}\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \;\mathrm{d}\xi$$
$$\leq -\frac{\min(\nu,\eta)}{4}p|k|^2 e_p^p(k,t). \tag{4.59}$$

Here we have used the fact that for $\xi \in \operatorname{supp}\chi$, $\frac{|k|}{2} \leq |\xi| \leq \frac{3}{2}|k|$. Thanks to Hölder's and Young's inequalities, we have

$$\begin{aligned}|I_3| =& \left|\Im \int p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\xi \cdot \int \hat{u}(\xi-\zeta)\hat{u}(\zeta)\;\mathrm{d}\zeta\;\mathrm{d}\xi\right| \\ \leq& \left|\int p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\xi \cdot \int \hat{u}(\xi-\zeta)\hat{u}(\zeta)\;\mathrm{d}\zeta\;\mathrm{d}\xi\right| \\ \leq& p(\int (|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-1})^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}(\int |\hat{\chi}_k(\xi)\Pi_\xi\xi \cdot \int \hat{u}(\xi-\zeta)\hat{u}(\zeta)d\zeta|^p\;\mathrm{d}\xi)^{\frac{1}{p}} \\ \leq& p(\int (|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-1})^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}(\int |\xi\hat{\chi}_k(\xi)|^p\;\mathrm{d}\xi)^{1/p}\|\int \hat{u}(\xi-\zeta)\hat{u}(\zeta)\;\mathrm{d}\zeta\|_{L^\infty} \\ \leq& p(\int (|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-1})^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\;\mathrm{d}\xi)^{1/p}(\int |\hat{u}(\xi,t)|^p\;\mathrm{d}\xi)^{1/p} \\ &(\int |\hat{u}(\xi,t)|^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}\end{aligned} \tag{4.60}$$

Following a similar approach, we get

$$\begin{aligned}|I_4| \leq& p(\int (|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-1})^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\;\mathrm{d}\xi)^{1/p}(\int |\hat{b}(\xi,t)|^p\;\mathrm{d}\xi)^{1/p} \\ &(\int |\hat{b}(\xi,t)|^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}\end{aligned} \tag{4.61}$$

$$\begin{aligned}|I_5| \leq& p(\int (|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-1})^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\;\mathrm{d}\xi)^{1/p}(\int |\hat{b}(\xi,t)|^p\;\mathrm{d}\xi)^{1/p} \\ &(\int |\hat{u}(\xi,t)|^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}\end{aligned} \tag{4.62}$$

$$\begin{aligned}|I_6| \leq& p(\int (|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-1})^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\;\mathrm{d}\xi)^{1/p}(\int |\hat{u}(\xi,t)|^p\;\mathrm{d}\xi)^{1/p} \\ &(\int |\hat{b}(\xi,t)|^{\frac{p}{p-1}}\;\mathrm{d}\xi)^{\frac{p-1}{p}}\end{aligned} \tag{4.63}$$

To estimate I_7 and I_8 we follow a different approach as follows;

$$\begin{aligned}
|I_7| &= \left|\Re\int p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)\ \mathrm{d}\xi\right| \\
&\leq \left|\int p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)\ \mathrm{d}\xi\right| \\
&\leq p\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\xi|^p|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p}{|\xi|^p}\ \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\leq \frac{3p}{2}|k|\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p}{|\xi|^p}\ \mathrm{d}\xi\right)^{1/p}. \qquad (4.64)
\end{aligned}$$

Similarly,

$$|I_8| \leq \frac{3p}{2}|k|\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p}{|\xi|^p}\ \mathrm{d}\xi\right)^{1/p} \qquad (4.65)$$

Now plugging the estimates (4.59)-(4.65) in (4.58) and rearranging the terms we get,

$$\begin{aligned}
\frac{d}{dt}e_p^p(k,t) &\leq -\frac{\min(\nu,\eta)}{4}p|k|^2e_p^p(k,t) + \left(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{1/p} \\
&\Bigg[\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|_{L^2}^2 + \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{b}\|_{L^2}^2 \\
&+ \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|_{L^2}\|\hat{b}\|_{L^2} + \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|_{L^2}\|\hat{b}\|_{L^2}\Bigg] \\
&+ \frac{3p}{2}|k|\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p}{|\xi|^p}\right)^{1/p} \\
&+ \frac{3p}{2}|k|\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p}{|\xi|^p}\ \mathrm{d}\xi\right)^{1/p}
\end{aligned}$$

We know from the definition of $\hat{\chi}_k$ that $\left(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{1/p}$ is bounded from above as

$$\left(\int |\xi|^p|\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{1/p} \leq \frac{3|k|}{2}(2\delta)^{3/p}. \qquad (4.66)$$

Furthermore, we have

$$\begin{aligned}
&\frac{3|k|p}{2}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p}{|\xi|^p}\right)^{1/p}+\\
&\frac{3|k|p}{2}\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p}{|\xi|^p} \ \mathrm{d}\xi\right)^{1/p}\\
\leq&\frac{3|k|p}{2}\left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}+\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\right)\\
&\left(\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p}{|\xi|^p}\right)^{1/p}+\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p}{|\xi|^p}\right)^{1/p} \mathrm{d}\xi\right)\\
\leq&\frac{3|k|p}{2}2^{\frac{1}{p}}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi+\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\\
&2^{\frac{p-1}{p}}\left(\int \frac{|\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p}{|\xi|^p}+\int \frac{|\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p}{|\xi|^p} \ \mathrm{d}\xi\right)^{1/p}\\
\leq&2\frac{3|k|p}{2}e_p^{p-1}(k,t)h_p(k,t), \qquad (4.67)
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg[\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|_{L^2}^2+\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{b}\|_{L^2}^2\\
&+\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|_{L^2}\|\hat{b}\|_{L^2}+\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|_{L^2}\|\hat{b}\|_{L^2}\Bigg]\\
=&\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\left(\|\hat{u}\|^2+\|\hat{b}\|^2\right)+2\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\|\hat{u}\|\|\hat{b}\|\\
\leq&\left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}+\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\right)\left(\|\hat{u}\|^2+\|\hat{b}\|^2\right)\\
\leq&\sqrt[p]{2}e_p^{p-1}(k,t)R^2(t). \qquad (4.68)
\end{aligned}$$

Now combining (4.66), (4.67) and (4.68) we get

$$\begin{aligned}
\frac{d}{dt}e_p^p(k,t)\leq&-\frac{\min(\nu,\eta)}{4}p|k|^2e_p^p(k,t)+\frac{3|k|}{2}(2\delta)^{3/p}2^{\frac{1}{p}}e_p^{p-1}(k,t)R^2(t)\\
&+2\frac{3|k|p}{2}e_p^{p-1}(k,t)h_p(k,t)
\end{aligned}$$

$$= -\frac{\min(\nu,\eta)}{4} p|k|^2 e_p^p(k,t) + \frac{3|k|}{2} e_p^{p-1}(k,t)\left(2^{\frac{1}{p}}(2\delta)^{3/p}R^2(t) + 2ph_p(k,t)\right)$$
$$\leq -\frac{\min(\nu,\eta)}{4} p|k|^2 e_p^p(k,t) + \frac{3|k|}{2} p e_p^{p-1}(k,t)\left(2^{\frac{1}{p}}(2\delta)^{3/p}R^2(t) + 2h_p(k,t)\right).$$

Now again consider the set

$$B_{R_1} = \left\{ e(k,t) : 0 \leq e(k,t) \leq \frac{R_1(t)}{|k|} \right\}.$$

Setting $e(k,t) = e_p(k,t)$ on the boundary, that is when $e_p(k,t) = \frac{R_1(t)}{|k|}$, therefore $|k|e_p(k,t) = R_1(t)$, we have

$$\frac{d}{dt} e_p^p(k,t) \leq -\frac{\nu}{4} p|k|^2 \frac{R_1^p(t)}{|k|^p} + p\frac{3|k|}{2}\frac{R_1^{p-1}(t)}{|k|^{p-1}}\left(2^{\frac{1}{p}}(2\delta)^{3/p}R^2(t) + 2h_p(k,t)\right)$$
$$< -\frac{\min(\nu,\eta)}{4} p|k|^2 \frac{R_1^p(t)}{|k|^p} + p\frac{3|k|}{2}\frac{R_1^{p-1}(t)}{|k|^{p-1}}\frac{\min(\nu,\eta)}{6} R_1(t) = 0$$

Here we have used the condition that $2^{\frac{1}{p}}(2\delta)^{3/p}R^2(t) + 2h_p(k,t) < \frac{\nu}{6}R_1(t)$. This implies B_{R_1} is an attracting set for $e_p(k,t)$. Therefore, if $e_p(k,0) < \frac{R_1(0)}{|k|}$, then $e_p(k,t) < \frac{R_1(t)}{|k|}$ for all $t \in \mathbb{R}^+$. □

Lemmas 90 and 91 give us all the necessary tools to prove Theorem 88 and Theorem 89.

Proof of Theorem 88. Lemma 91 implies that $e_p(k,t)$ is bonded uniformly in p. Then taking the supremum over all $2 \leq p < \infty$ concludes the proof of 88. Indeed the proof of Theorem 88 is very direct. □

Proof of Theorem 89. Recalling the definition of $e_p(k,t)$ from (4.40), we have

$$e_p^2(k,t) = \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}}.$$

This implies

$$\frac{\partial}{\partial_t} e_p^2(k,t) = \frac{2}{p}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \frac{\partial}{\partial_t} e_p^p(k,t). \qquad (4.69)$$

Now plugging (4.58) in (4.69) gives,

$$
\begin{aligned}
\frac{\partial}{\partial_t} e_p^2(k,t) = &\left[\frac{2}{p}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\right] \\
&\times \left[-\nu \int p|\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi - \eta \int p|\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right. \\
&+ \Re \int ip|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+ \Re \int ip|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{b}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+ \Re \int ip|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{b}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+ \Re \int ip|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta) \ \mathrm{d}\zeta \ \mathrm{d}\xi \\
&+ \Re \int p|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_1(\xi,t) \ \mathrm{d}\xi \\
&\left.+\Re \int p|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_2(\xi,t) \ \mathrm{d}\xi\right] \qquad (4.70)
\end{aligned}
$$

For the sake of calculation simplicity, we split the RHS of (4.70) into the following integrals[5].

$$
\begin{aligned}
I_1 := -2\nu &\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \\
&\left(\int |\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)
\end{aligned}
$$

$$
\begin{aligned}
I_2 :=2 &\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \\
&\Re \int \left(i|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta) \ \mathrm{d}\zeta\right) \ \mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
I_3 :=2 &\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \\
&\Re \int \left(i|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{b}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta) \ \mathrm{d}\zeta\right) \ \mathrm{d}\xi
\end{aligned}
$$

[5] We use notations I_1, $I_2, \cdots, I_7$ similar to the one in the proof of Lemma 91 .

$$I_4 := 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$
$$\Re\int\left(i|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\int \hat{b}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\; \mathrm{d}\zeta\right)\; \mathrm{d}\xi$$

$$I_5 := 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$
$$\Re\int\left(i|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi\int \hat{u}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta)\; \mathrm{d}\zeta\right)\; \mathrm{d}\xi$$

$$I_6 := 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$
$$\Re\int\left(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)\right)\; \mathrm{d}\xi$$

$$I_7 := 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$
$$\Re\int\left(|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{b}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)\right)\; \mathrm{d}\xi.$$

We now proceed to estimating each of these integrals;

$$\begin{aligned}I_1 =& -2\nu\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}-1}\\ &\left(\int |\xi|^2|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\xi|^2|\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}}\\ =& -2\nu\left(\frac{\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi}{\int |\xi|^2(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p) \; \mathrm{d}\xi}\right)^{\frac{2}{p}-1}\\ &\left(\int |\xi|^2(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p) \; \mathrm{d}\xi\right)\end{aligned} \tag{4.71}$$

$$|I_2| = \Bigg|2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \; \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$

$$\int \left(i|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\ \mathrm{d}\zeta \right)\ \mathrm{d}\xi \Big|$$
$$\le 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-2}{p}}$$
$$\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int |\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\|\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\ \mathrm{d}\zeta\|_{L^\infty}. \tag{4.72}$$

Here we repeatedly used Holder's inequality. Similar calculations give

$$|I_3| \le 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-2}{p}}$$
$$\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int |\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\|\Pi_\xi \int \hat{b}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta)\ \mathrm{d}\zeta\|_{L^\infty} \tag{4.73}$$

$$|I_4| \le 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-2}{p}}$$
$$\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int |\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\|\Pi_\xi \int \hat{b}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta)\ \mathrm{d}\zeta\|_{L^\infty} \tag{4.74}$$

$$|I_5| \le 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-2}{p}}$$
$$\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int |\hat{\chi}_k(\xi)|^p\ \mathrm{d}\xi\right)^{\frac{1}{p}}\|\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta)\ \mathrm{d}\zeta\|_{L^\infty} \tag{4.75}$$

For integrals involving the inhomogeneous forces,

$$|I_6| \le 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$
$$\left|\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^{p-2}\overline{\hat{\chi}_k(\xi)\hat{u}(\xi,t)}\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)\ \mathrm{d}\xi\right|$$
$$\le 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p\ \mathrm{d}\xi\right)^{\frac{p-2}{p}}$$

$$\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \left(\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \tag{4.76}$$

Similarly,

$$\begin{aligned}|I_7| \leq & 2\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-2}{p}} \\ & \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \left(\int |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}}. \end{aligned} \tag{4.77}$$

Now taking the time integral of (4.70) over the interval $[0, T]$ we get

$$e_p^2(k,T) - e_p^2(k,0) = \int_0^T \sum_{j=1}^{7} I_j \ \mathrm{d}t.$$

Then it follows from (4.71) that,

$$\begin{aligned} & 2\min(\nu,\eta)\int_0^T \left(\frac{\int |\xi|^2(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p) \ \mathrm{d}\xi}{\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi}\right)^{1-\frac{2}{p}} \\ & \qquad \left(\int |\xi|^2(|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p) \ \mathrm{d}\xi\right)^{2/p} \ \mathrm{d}t \\ & \leq e_p^2(k,0) - e_p^2(k,T) + \sum_{j=2}^{7}\int_0^T |I_j| \ \mathrm{d}t. \end{aligned} \tag{4.78}$$

Once again making use of the Young's inequality gives,

$$\|\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{b}(\zeta)d\zeta\|_{L^\infty} \leq \|\hat{u}(\cdot,t)\|_{L^2}\|\xi\hat{b}(\cdot,t)\|_{L^2}.$$

Therefore,

$$\int_0^T |I_2| \ \mathrm{d}t$$

$$\leq 2\int_0^T \left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-2}{p}}\right.$$

$$\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \quad \left(\int |\hat{\chi}_k(\xi)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \|\Pi_\xi \int \hat{u}(\xi-\zeta)\cdot\zeta\hat{u}(\zeta) \ \mathrm{d}\zeta\|_{L^\infty}\Bigg) \ \mathrm{d}t$$

$$\leq 2\int_0^T \left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1} \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}}\right.$$
$$\left.\left(\int |\hat{\chi}_k(\xi)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \|\hat{u}(\cdot,t)\|_{L^2}\|\xi\hat{u}(\cdot,t)\|_{L^2}\right) \ \mathrm{d}t \tag{4.79}$$

$$\int_0^T |I_3| \ \mathrm{d}t \leq 2\int_0^T \left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\right.$$
$$\left.\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}} \left(\int |\hat{\chi}_k(\xi)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \|\hat{b}(\cdot,t)\|_{L^2}\|\xi\hat{b}(\cdot,t)\|_{L^2}\right) \ \mathrm{d}t \tag{4.80}$$

$$\int_0^T |I_4| \leq 2\int_0^T \left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\right.$$
$$\left.\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}} \left(\int |\hat{\chi}_k(\xi)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \|\hat{u}(\cdot,t)\|_{L^2}\|\xi\hat{b}(\cdot,t)\|_{L^2}\right) \ \mathrm{d}t \tag{4.81}$$

$$\int_0^T |I_5| \ \mathrm{d}t \leq 2\int_0^T \left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1}\right.$$
$$\left.\left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}} \left(\int |\hat{\chi}_k(\xi)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \|\hat{b}(\cdot,t)\|_{L^2}\|\xi\hat{u}(\cdot,t)\|_{L^2}\right) \ \mathrm{d}t \tag{4.82}$$

$$\int_0^T |I_6| \ \mathrm{d}t \leq 2\int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{2}{p}-1}$$
$$\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{p-1}{p}} \left(\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p \ \mathrm{d}\xi\right)^{\frac{1}{p}} \ \mathrm{d}t \tag{4.83}$$

$$\int_0^T |I_7| \, \mathrm{d}t \leq 2 \int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{2}{p}-1} \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{p-1}{p}} \left(\int |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} \mathrm{d}t \tag{4.84}$$

Now putting estimates (4.79)-(4.82) together we have,

$$\begin{aligned}
&\int_0^T |I_2| \, \mathrm{d}t + \int_0^T |I_3| \, \mathrm{d}t + \int_0^T |I_4| \, \mathrm{d}t + \int_0^T |I_5| \, \mathrm{d}t \\
\leq & 2 \left(\int |\hat{\chi}_k(\xi)|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} \int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{2}{p}-1} \\
& \left[\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{p-1}{p}} \left(\|\hat{u}(\cdot,t)\|_{L^2} \|\xi\hat{u}(\cdot,t)\|_{L^2} + \|\hat{b}(\cdot,t)\|_{L^2} \|\xi\hat{b}(\cdot,t)\|_{L^2} \right) \right. \\
& \left. + \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \right)^{\frac{p-1}{p}} \left(\|\hat{u}(\cdot,t)\|_{L^2} \|\xi\hat{b}(\cdot,t)\|_{L^2} + \|\hat{b}(\cdot,t)\|_{L^2} \|\xi\hat{u}(\cdot,t)\|_{L^2} \right) \right] \mathrm{d}t \\
\leq & 2 \left(\int |\hat{\chi}_k(\xi)|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} \int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{2}{p}-1} \\
& \left[\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{p-1}{p}} \right. \\
& \left. \left(\|\xi\hat{u}(\cdot,t)\|_{L^2} + \|\xi\hat{b}(\cdot,t)\|_{L^2} \right) \left(\|\hat{u}(\cdot,t)\|_{L^2} + \|\hat{b}(\cdot,t)\|_{L^2} \right) \right] \mathrm{d}t \\
\leq & 2(2\delta)^{\frac{3}{p}} \int_0^T \left(\int |\hat{\chi}_k\hat{u}|^p + |\hat{\chi}_k\hat{b}|^p \, \mathrm{d}\xi \right)^{\frac{1}{p}} \left(\|\hat{u}\| + \|\hat{b}\| \right) \left(\|\xi\hat{u}\| + \|\xi\hat{b}\| \right) \mathrm{d}t \\
\leq & 2(2\delta)^{\frac{3}{p}} \left(\int_0^T \left(\int |\hat{\chi}_k\hat{u}|^p + |\hat{\chi}_k\hat{b}|^p \, \mathrm{d}\xi \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \\
& \sup_{0 \leq t \leq T} \left(\|\hat{u}\| + \|\hat{b}\| \right) \left(\int_0^T \left(\|\xi\hat{u}\| + \|\xi\hat{b}\| \right)^2 \mathrm{d}t \right)^{\frac{1}{2}}
\end{aligned}$$

$$\le 2(2\delta)^{\frac{3}{p}} \sup_{0\le t\le T} \left(\|\hat{u}\| + \|\hat{b}\|\right) \left(\int_0^T \left(\int |\hat{\chi}_k \hat{u}|^p + |\hat{\chi}_k \hat{b}|^p \ \mathrm{d}\xi \right)^{\frac{2}{p}} \right)^{\frac{1}{2}}$$

$$\left(\int_0^T \left(\|\nabla u\| + \|\nabla b\|\right)^2 \ \mathrm{d}t \right)^{\frac{1}{2}}$$

$$\le 2(2\delta)^{\frac{3}{p}} R^2(T) \left(\int_0^T \left(\int |\hat{\chi}_k \hat{u}|^p + |\hat{\chi}_k \hat{b}|^p \ \mathrm{d}\xi \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \tag{4.85}$$

and, from (4.83) and (4.84) we have,

$$\int_0^T |I_6| \ \mathrm{d}t + \int_0^T |I_7| \ \mathrm{d}t$$

$$\le 2 \int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{2}{p}-1} \left[\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{p-1}{p}} \right.$$

$$\left. (\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p \ \mathrm{d}\xi)^{\frac{1}{p}} + (\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi)^{\frac{p-1}{p}} (\int |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \ \mathrm{d}\xi)^{\frac{1}{p}} \right] \ \mathrm{d}t$$

$$\le 2 \int_0^T \left\{ \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{2}{p}-1} \right.$$

$$\left(\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{p-1}{p}} + \left(\int |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{p-1}{p}} \right)$$

$$\left. \left(\left(\int |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{1}{p}} + \left(\int |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{1}{p}} \right) \right\} \ \mathrm{d}t$$

$$\le 2 \int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{2}{p}-1}$$

$$\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{p-1}{p}}$$

$$\left(\int |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \ \mathrm{d}\xi \right)^{\frac{1}{p}} \ \mathrm{d}t$$

$$\leq 2\left(\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\left(\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t\right)^{\frac{1}{2}}. \tag{4.86}$$

Therefore putting (4.78), (4.85) and (4.86) together and using the fact that $|\xi| \geq \dfrac{|k|}{2}$ in the support of $\hat{\chi}_k$ gives,

$$\frac{1}{2}\nu|k|^2\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t$$

$$\leq 2\left(\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t\right)^{\frac{1}{2}} \Bigg[2(2\delta)^{\frac{3}{p}}R^2(t) +$$

$$\left.\left(\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t\right)^{\frac{1}{2}}\right] + e_p^2(k,0) - e_p^2(k,T) \tag{4.87}$$

Now multiplying (4.87) by $|k|^2$

$$\nu|k|^4\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t$$

$$\leq 4\left(\int_0^T |k|^4\left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\left((2\delta)^{\frac{3}{p}}R^2(t) + \left(\int_0^T\left(\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \;\mathrm{d}\xi\right)^{\frac{2}{p}} \;\mathrm{d}t\right)^{\frac{1}{2}}\right)$$

$$+ |k|^2\left[e_p^2(k,0) - e_p^2(k,T)\right].$$

Now define

$$I_p^2(k,T) = \int_0^T |k|^4 \left(\int |\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{2}{p}} \, \mathrm{d}t$$

$$\mathbb{F}_p(T) = \left(\int_0^T \left(\int |\hat{\chi}_k(\xi)\hat{f}_1(\xi,t)|^p + |\hat{\chi}_k(\xi)\hat{f}_2(\xi,t)|^p \, \mathrm{d}\xi \right)^{\frac{2}{p}} \, \mathrm{d}t \right)^{\frac{1}{2}}.$$

This together with the assumption $e_p(k,0) \leq \frac{R_1(0)}{|k|}$ imply that

$$\begin{aligned} & I_p^2(k,T) \leq R_1^2(0) - |k|^2 e_p^2(k,T) + 4I_p(k,T)\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right] \\ \Longrightarrow\ & I_p^2(k,T) \leq R_1^2(0) + 4I_p(k,T)\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right] \\ \Longrightarrow\ & I_p^2(k,T) - 4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right] I_p(k,T) - R_1^2(0) \leq 0 \end{aligned} \tag{4.88}$$

Solving the associated quadratic equation gives

$$\begin{aligned} & \frac{4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right] \pm \sqrt{\left(4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right]\right)^2 + 4R_1^2(0)}}{2} \\ & = \frac{1}{2}\left(4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right] \pm \sqrt{\left(4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right]\right)^2 + 4R_1^2(0)}\right). \end{aligned}$$

Elementary mathematics tells us that $I_p(k,t)$ cannot exceed the largest positive root of the associated quadratic equation, i.e

$$\frac{1}{2}\left(4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right] + \sqrt{\left(4\left[(2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T)\right]\right)^2 + 4R_1^2(0)}\right)$$

Now set,

$$R_{5,p}(T) := (2\delta)^{\frac{3}{p}} R^2(T) + \mathbb{F}_p(T).$$

Now letting $p \longrightarrow \infty$ completes the proof of Theorem 89. $\square$

4.3 Estimates on the spectral Energy function and Inertial Ranges

In his largely celebrated works of 1883 and 1894, O. Reynolds described (quantitatively) fluid flows as laminar and turbulent depending on their Reynolds number, Re, see [141, 142]. Accordingly, a flow is laminar if $Re < 1900$ and turbulent if $Re > 2000$. Turbulent flows can further be classified as large scale turbulent flows and small scale turbulent flows. The motion of large scale turbulence is determined by the geometry of the flow (that is the boundary condition) while small scale turbulence flow is largely influenced by the rate of energy they receive from large scales and viscosity of the fluid. Indeed, large scale turbulent flows get transformed to small scale turbulent by losing their energy. Richardson summarized this cascade of energy with his famous rhyme, " Big whorls have little whorls, which feed on their velocity; And little whorls have lesser whorls, and so on to viscosity ", see [144]. Nevertheless, the very important question in turbulence theory is the rate of energy transfer from bigger scale to lesser scales. In this regard the works of Kolmogorov and his students, Obukhov and Millionshchikov, [91, 92, 93, 94, 95, 124, 131] are of tremendous importance. Particularly the 1941 works of Kolmogorov permitted prediction of a number of laws for turbulent flows of sufficiently large Reynolds numbers, see [130] and references in there.

As discussed on page 7, the most interesting result of Kolmogorov's and Obukhov's work is their estimate on the energy decay rate of the turbulent flow. They argued that in the inertial range the spectral energy decays according to

$$E(k, \cdot) \sim C_0 \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{4.89}$$

where C_0 is a universal constant called Kolmogorov constant, ϵ is the energy dissipation rate and k belongs to regime of wave numbers $[k_1, k_2]$ of the inertial range. This argument is based on dimensional analysis and very sound similarity hypothesis of Kolmogorov, see [94]. This result is universally accepted as long as hydrodynamic turbulence is concerned. However, for the case of MHD turbulence, we see few arguments which do not favor Kolmogorov, for instance the KID phenomenon (see the discussion on page 8).

Despite the odds to KID phenomenon, several works actually support Kolmogorov's theory for MHD turbulence, see [18, 164, 165]. And therefore it is completely plausible to establish this fact through a rigorous mathematical proof.

In this section, we aim to establish a range of wave numbers such that Kolmogorov's phenomenon holds for Magnetohydrodynamics turbulence. We use the approach of Biryuk and Craig in [17] where they used Fourier analysis methods to establish Kolmogorov's phenomenon for Navier Stokes equations.

To begin with we define the spectral energy function for (4.2) by the spherical integral

$$E(k,t) = \int_{|\xi|=k} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2) \ \mathrm{d}S(\xi) \tag{4.90}$$

when $D = \mathbb{R}^3$; where $0 \le k < \infty$ is the radial coordinate in Fourier transform variable. When $D = \mathbb{T}^3$, we define the spectral energy function over the dual $\hat{\mathbb{T}}^3$ as a sum over Fourier space annuli of given thickness, say a by

$$E(k,t) = \frac{1}{a} \sum_{k \le |\xi| < k+a} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2). \tag{4.91}$$

We now give the first two results on bounds of $E(k,t)$; in the first result we show $E(k,t)$ is bounded uniformly if the external forces, $f_i \equiv 0$, for every $i = 1, 2$ and point-wise otherwise; and in the second result we give the time average bound over a finite interval of time $[0, T]$.

Theorem 92. *If for all i, $f_i \equiv 0$ and the initial data u_0, $b_0 \in A_{R_1} \cap B_R(0)$, where R and R_1 satisfy* (4.12)*. Then, the estimate*

$$E(k,t) \le 4\pi R_1^2, \tag{4.92}$$

holds for all k and all t. Furthermore, when $f_i \not\equiv 0$ for some $i = 1, 2$, there is a *finite but possibly growing upper bound given by*

$$E(k,t) \leq 4\pi R_1^2(t). \tag{4.93}$$

Proof of Theorem 92. We prove this in two cases,

Case 1: When $D = \mathbb{T}^3$. From (4.91) we have that

$$\begin{aligned} E(k,t) &= \frac{1}{a} \sum_{k \leq |\xi| < k+a} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2) \\ &\leq \frac{1}{a} \sum_{k \leq |\xi| < k+a} [|\hat{u}(\xi,t)| + |\hat{b}(\xi,t)|]^2 \\ &\leq \frac{1}{a} \sum_{k \leq |\xi| < k+a} \frac{R_1^2}{k^2} \end{aligned} \tag{4.94}$$

Now we appeal to the result of Chamizo [31, p. 9] to estimate (4.94)[6]. Accordingly it follows that

$$E(k,t) \leq \frac{1}{a}\frac{R_1^2}{k^2} 4\pi k^2 a = 4\pi R_1^2$$

Case 2: When $D = \mathbb{R}^3$. The proof for this case is quite similar to the first case. From definition (4.90) and the fact that $\hat{u}$, $\hat{b} \in A_{R_1(t)}$ we have

$$\begin{aligned} E(k,t) &= \int\limits_{|\xi|=k} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2) dS(\xi) \\ &\leq \int\limits_{|\xi|=k} \frac{R_1^2}{k^2} dS(\xi) \\ &= 4\pi R_1^2. \end{aligned}$$

Here we used the fact that the surface area of a sphere with radius k is equal to $4\pi k^2$.

[6]According to this result we see that the number of lattice points in a d-sphere of radius r is given by $\frac{4}{3}\pi r^3 + O(r^{3/2})$. Hence the number of lattice points in the annulus $k \leq |\xi| < k + a$ can roughly be approximated by $4\pi k^2 a$.

Note that, conditionally on existence and non existence of external forces on the system, the given upper bound may or may not depend on time. In the later case the bound is a uniform bound as R_1 is independent of time. With this we completes the proof. □

Theorem 93. *Suppose the initial data* u_0, b_0 *is in* $A_{R_1} \cap B_R(0)$, *where* R_1, R *satisfy* (4.12) *and the forces* $f_i \in L^{\infty}_{loc}([0,\infty]; H^{-1}(D) \cap L^2(D))$ *for* $i = 1, 2$ *is bounded as it appears in* (4.42), (4.45). *Then for every* T, *the energy spectral function satisfies*

$$\frac{1}{T}\int_0^T E(k,t)dt \leq \frac{4\pi R_2^2(T)}{\min(\nu,\eta)Tk^2}. \tag{4.95}$$

Proof of Theorem 93. As we have seen in theorem 92, the proofs when $D = \mathbb{T}^3$ and $D = \mathbb{R}^3$ are analogous. We therefore limit ourselves to the case when $D = \mathbb{R}^3$.

$$\begin{aligned}
\frac{1}{T}\int_0^T E(k,t)dt &= \frac{1}{T}\int_0^T \int_{|\xi|=k} (|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2)dS(\xi)dt \\
&= \frac{1}{T}\int_0^T \int_{|\xi|=k} (|\hat{\chi}_k(\xi)\hat{u}(\xi,t)|^2 + |\hat{\chi}_k(\xi)\hat{b}(\xi,t)|^2)dS(\xi)dt \\
&\leq \int_{|\xi|=k} \frac{1}{T}\frac{R_2^2(T)}{\min(\nu,\eta)k^4}dS(\xi) \\
&= \int_{|\xi|=k} \frac{R_2^2(T)}{\min(\nu,\eta)Tk^4}dS(\xi) \leq 4\pi k^2 \frac{R_2^2(T)}{\min(\nu,\eta)Tk^4} \\
&= \frac{4\pi R_2^2(T)}{\min(\nu,\eta)Tk^2};
\end{aligned}$$

where we have used (4.44). □

Next we establish Kolmogorov's inertial range to (4.2). Based on theorems 92 and 93, we define a set S of all wave numbers k such that both conditions (4.92) and (4.95) are satisfied by

$$S := \left\{(k,E) : E \leq E(k,\cdot) \leq 4\pi R_1^2,\ E \leq E(k,\cdot) \leq \frac{4\pi R_2^2}{\min(\nu,\eta)Tk^2}\right\} \tag{4.96}$$

Considering Kolmogorov's spectral law as an ideal case and stating as

$$E_K(k) := C_0 \epsilon^{2/3} k^{-5/3}, \tag{4.97}$$

we give estimates on the parameters and the inertial range. The parameters are those variable which determine the turbulent flow in the lower scale or inertial range. A further analysis on the maximum time of observation for the phenomenon in the inertial range is also done.

Theorem 94. *Assuming the conditions in Theorem 92 and Theorem 93 we have the following are true about the Kolmogorov's inertial range for* (4.2)*:*

(i) Kolmogorov's parameters must satisfy

$$\big(\min(\nu,\eta)\big)^{5/6} C_0 \epsilon^{2/3} \leq 4\pi \left(\frac{R_2(T)}{\sqrt{T}}\right)^{5/3} R_1^{\frac{1}{3}}(T). \tag{4.98}$$

(ii) An absolute lower bound for the inertial range is given by

$$\bar{k}_1 = \frac{C_0^{3/5} \epsilon^{2/5}}{(4\pi R_1^2)^{3/5}}. \tag{4.99}$$

(iii) An absolute upper bound for the inertial range is given by

$$\bar{k}_2 = \left(\frac{4\pi}{C_0 \min(\nu,\eta)}\right)^3 \frac{1}{\epsilon^2} \frac{R_2^6(T)}{T^3}. \tag{4.100}$$

Proof of Theorem 94. Let $A := \{(k,E) : E_K(k) = E\} \bigcap S$; i.e., A is part of the graph of $E_K(k)$ that lies in region S. Due to Theorem 92 we know that the spectral energy of our system bounded from above by the $4\pi R_1^2$ in the absence of external force and $4\pi R_1^2(T)$ in the presence of external force. Further more from Theorem 93 the time average is bounded by $\frac{4\pi R_2^2(T)}{\min(\nu,\eta) T k^2}$. Therefore Kolmogorov's spectral function must satisfy both cases simultaneously. Hence $A \neq \emptyset$.

Note that for $A \neq \emptyset$ the point where graphs of $E_K(k)$ and $\frac{4\pi R_2^2}{\min(\nu,\eta) T k^2}$ must intersect

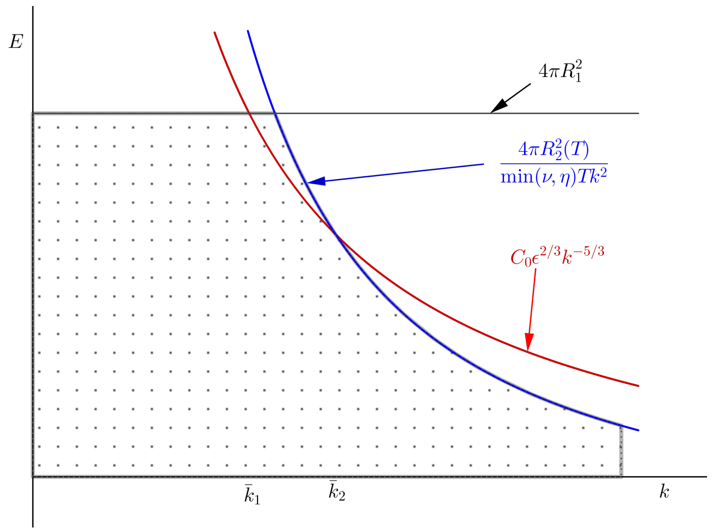

Figure 4.1: Set S and spectral function

below the line $E = 4\pi R_1^2$. But this point of intersection is when

$$C_0\epsilon^{2/3}k^{-5/3} = \frac{4\pi R_2^2}{\min(\nu,\eta)Tk^2}.$$

That is when

$$k = \left(\frac{4\pi R_2^2(T)}{\min(\nu,\eta)TC_0\epsilon^{2/3}}\right)^3$$

On the other hand, the graph of $E_K(k)$ intersects the line $E = 4\pi R_1^2$ when,

$$C_0\epsilon^{2/3}k^{-5/3} = 4\pi R_1^2.$$

Therefore, $E_K(k)$ enters region S at $k = \left(\frac{4\pi R_1^2}{C_0\epsilon^{2/3}}\right)^{-3/5}$ and leaves at $k = \left(\frac{4\pi R_2^2(T)}{\min(\nu,\eta)TC_0\epsilon^{2/3}}\right)^3$.

Now set,

$$\bar{k}_1 = \left(\frac{4\pi R_1^2}{C_0\epsilon^{2/3}}\right)^{-3/5}, \qquad \bar{k}_2 = \left(\frac{4\pi R_2^2(T)}{\min(\nu,\eta)TC_0\epsilon^{2/3}}\right)^3.$$

Thus the portion of the graph of $E_K(k)$ remains in region S as long as k is between $\bar{k}_1$ and $\bar{k}_2$. Moreover, the fact that $E_K(k)$ is decreasing tells us that $\bar{k}_1 \leq \bar{k}_2$.

Therefore, we have

$$\left(\frac{4\pi R_1^2}{C_0\epsilon^{2/3}}\right)^{-3/5} \leq \left(\frac{4\pi R_2^2(T)}{\min(\nu,\eta)TC_0\epsilon^{2/3}}\right)^3.$$

Hence,

$$C_0\min(\nu,\eta)^{5/6}\epsilon^{2/3} \leq 4\pi\left(\frac{R_2(T)}{\sqrt{T}}\right)^{5/3} R_1^{\frac{1}{3}}(T).$$

Hence our system exhibits Kolmogorov's phenomenon on the range of wave numbers $[\bar{k}_1, \bar{k}_2]$ the parameters such as viscosity and dissipation satisfy (4.98). This completes the proof. □

The theorem above tells us that the upper bound of the inertial range is decreasing in time. This is because for $f_1, f_2 \in L^\infty_{loc}([0,\infty); H^{-1}(D) \cap L^2(D))$ the growth of $R_2(T)$ is at most linear in time. But when $f_i \equiv 0$ for $i = 1, 2$ R_1 and R_2 are constants. In the later case at time $T = T_0$ such that,

$$T_0 := \frac{(4\pi)^{6/5}R_2^2(T)R_1^{2/5}(T)}{\epsilon^{4/5}C_0^{6/5}\min(\nu,\eta)} \tag{4.101}$$

we get $\bar{k}_1 = \bar{k}_2$ and This means that for any time $T \geq T_0$, the spectral range is empty, in other words the intersection of sets A and S is empty. Consequently, time T_0 appears to be the maximal time to have a Kolmogorov's phenomenon in the system.

If the dissipation rate is time dependent then (4.101) gives

$$\epsilon(T_0) = \frac{(4\pi)^{3/2}R_1^{1/2}R_2^{5/2}}{T_0^{5/4}\min(\nu,\eta)^{5/4}C_0^{3/2}}. \tag{4.102}$$

The time T_0 being the maximal time, (4.102) must be the minimum dissipation rate to maintain a spectral behavior.

We conclude our work by quantifying a spectral behavior of a flow based on the solution of (4.2) accordingly with definition 81. Borrowing from the works of Biryuk and Craig in [17] for Navier-Stokes equations, we define spectral behavior for our system as follows.

Definition 95. *A solution* (u, b, p) *to* (4.2) *is said to have the spectral behavior of* $E_k(k)$*, uniformly over the range* $[k_1, k_2]$ *and for the time interval* $[0, T]$*, if its energy spectral function* $E(k, t)$ *satisfies*

$$\sup_{\substack{t\in[0,T]\\ k\in[k_1,k_2]}} (1 + k^{5/3})|E(k, t) - E_K(k)| < C_1\epsilon^{2/3}, \tag{4.103}$$

where $C_1 \ll C_0$.[7]

Theorem 96. *Let* (u, b) *be a solution of* (4.2) *with initial data* $u_0(x), b_0(x)$ *in* $A_{R_1} \cap B_R(0)$*. If* (u, b) *exhibits a spectral behavior of* E_K *uniformly over* $[k_1, k_2] \times [0, T]$ *in the sense of definition 95, then either*

(i). $\bar{k}_1 \le k_1 \le k_2 \le \bar{k}_2$,

or

(ii). if $k_1 < \bar{k}_1$ *or* $\bar{k}_2 < k_2$*, then there is a small neighborhood of* $\bar{k}_j$*, for each* $j = 1, 2$*, to which* k_j *belongs.*

Proof of Theorem 96. From Theorems 92 and 93 we know that when the initial data $u_0(x), b_0(x)$ in $A_{R_1} \cap B_R(0)$ the spectral energy is bounded. And Theorem (94) tells us that gives us over the range $[\bar{k}_1, \bar{k}_2]$ the spectral energy has a uniform spectral behavior.

To show the case *ii*, we again consider two separate cases; first we analyze the situation at the left of $\bar{k}_1$ and then at the right of $\bar{k}_2$.

Case 1: $k_1 < \bar{k}_1$. From (94) we know that when $k_1 < \bar{k}_1$, $E_K(k) \ge 4\pi R_1^2$ but

[7] $C_1 \ll C_0$ is to mean that C_1 is a very small constant in comparison to C_0.

$E(k,t) \leq 4\pi R_1^2$ for all $k \in [k_1, \bar{k}_1]$. Therefore[8],

$$C_0\epsilon^{2/3}k_1^{-5/3} - 4\pi R_1^2 \leq E_k(k_1) - E(k_1,t) < o(1)C_0\epsilon^{2/3}$$

But $4\pi R_1^2$ is a constant and hence from (4.99) we get

$$C_0\epsilon^{2/3}k_1^{-5/3} - C_0\epsilon^{2/3}\bar{k}_1^{-5/3} = C_0\epsilon^{2/3}(k_1^{-5/3} - \bar{k}_1^{-5/3})$$

From elementary calculus

$$k_1^{-5/3} - \bar{k}_1^{-5/3} = \int_{k_1}^{\bar{k}_1} \frac{5}{3}k^{-8/3}dk \geq (\bar{k}_1 - k_1)\frac{5}{3}\bar{k}_1^{-8/3}.$$

This implies

$$\frac{5}{3}C_0\epsilon^{2/3}(\bar{k}_1 - k_1)\bar{k}_1^{-8/3} \leq C_0\epsilon^{2/3}(k_1^{-5/3} - \bar{k}_1^{-5/3}) \leq o(1)C_0\epsilon^{2/3}.$$

Therefore,

$$\bar{k}_1 - k_1 \leq o(1)\frac{3}{5}\bar{k}_1^{8/3}. \tag{4.104}$$

Thus, k_1 is at a finite and very close distance from $\bar{k}_1$. Moreover, multiplying (4.104) by the reciprocal of $\bar{k}_1$ gives $1 - \frac{k_1}{\bar{k}_1} \leq o(1)\frac{3}{5}\bar{k}_1^{5/3}$, that is $1 - o(1)\frac{3}{5}\bar{k}_1^{5/3} \leq \frac{k_1}{\bar{k}_1} \leq 1$.

This in turn implies that

$$4\pi R_1^2 = C_0\epsilon^{2/3}\bar{k}_1^{-5/3} = C_0\epsilon^{2/3}k_1^{-5/3}\left(\frac{k_1}{\bar{k}_1}\right)^{5/3}.$$

Thus our estimate holds for k_1 with a negligibly small change in on the constant C_0. Hence (4.103) holds with a very small change in the constant.

[8] $o(1)$ is the little-O notation which is to indicate the constant denoted is very small relative to 1.

Case 2 : $k_2 > \bar{k}_2$. It is clear from (4.103) that

$$\frac{1}{T}\int_0^T k^{5/3}|E_k(k) - E(k,t)|dt \leq o(1)C_0\epsilon^{2/3}$$

But from (4.97) we have that

$$C_0\epsilon^{2/3}k^{-5/3} - \frac{4\pi R_2^2(T)}{\min(\nu,\eta)Tk^2} \leq |E_k(k) - E(k,t)| \quad \text{for } k \geq \bar{k}_2.$$

Thus

$$k^{5/3}(C_0\epsilon^{2/3}k^{-5/3} - \frac{4\pi R_2^2(T)}{\min(\nu,\eta)Tk^2}) \leq o(1)C_0\epsilon^{2/3},$$

and

$$C_0\epsilon^{2/3} - \frac{4\pi R_2^2(T)}{\min(\nu,\eta)Tk^{1/3}}) \leq o(1)C_0\epsilon^{2/3}$$

for all k in $[k_1, k_2]$. Now Solving for k gives

$$k^{1/3} \leq \frac{4\pi R_2^2}{\min(\nu,\eta)C_0\epsilon^{2/3}T(1-o(1))}.$$

Now making use of (4.100) we get

$$k^{1/3} \leq (\bar{k}_2)^{1/3}\frac{1}{1-o(1)}.$$

Hence, $1 - o(1) \leq \left(\frac{\bar{k}_2}{k}\right)^{1/3} \leq 1$. If we set $k = \bar{k}_2$, then we have that

$$1 - o(1) \leq \left(\frac{\bar{k}_2}{k_2}\right)^{1/3} \leq 1.$$

Therefore k_2 satisfies (4.100) with a very small change on the constant.

Thus, for each $j = 1, 2$ there is a small neighborhood $\mathcal{B}(\bar{k}_j, \delta)$ of $\bar{k}_j$ such that $k_j \in \mathcal{B}(\bar{k}_j, \delta)$. This completes the proof of Theorem 96. □

Chapter 5

Conclusion and Future Perspectives

5.1 Conclusion

In this thesis we have studied two models of MHD flows, where the first model is the stochastic MHD model given by (3.1) and the second case is the deterministic MHD model given by (4.2) in two parts.

In the first case we provided a detailed investigation of the stochastic MHD equation (3.1) and the following results were established:

- New estimates for the solution of stochastic heat equation (3.12) in Besov like evolution and probabilistic spaces.
- New pathwise estimate in evolution spaces of Sobolev type were given for the system of equations (3.12), the pathwise estimate holds with positive probability less than one. The positive probability can be made as close to 1 as one desires but not 1.
- Existence and uniqueness of global and local strong solutions (in probabilistic sense) for the system (3.1) were established. The global result holds when certain smallness conditions are imposed on the initial data.

The method we used and results obtained in the part, where stochastic MHD equation was considered are pioneering, in the sense that, we are the first to use

Littlewood-Paley theory and Besov spaces for the purpose of investigating stochastic MHD equations.

In the second case the deterministic model (4.2) was investigated; a rigorous mathematical proof was used to establish Kolmogorov's spectral theory for the MHD flows when the initial data is reasonably smooth. We have also established conditions that allow the MHD flow to exhibit Kolmogorov's phenomenon. The following are some of the results given in the thesis.

- The spectral energy function $E(k,t)$ is always bounded in the inertial range,
- There is an interval of wave numbers such that $E(k,t)$ decays proportional to $k^{-5/3}$.
- There is a minimum rate of energy dissipation for an MHD turbulent flow to exhibit Kolmogorov's phenomenon,
- Given an estimate on the energy dissipation rate, there is a time limit to exhibit Kolmogorov's phenomenon.

These results are novel, in the sense that, this is the first work which totally used mathematical theories to establish Kolmogorov theory for MHD flows. Indeed, the work adopted the approach of [17] for Navier-Stokes equations: notations and terminologies are taken unchanged for their credit but their definitions were modified to fit into the MHD theory.

5.2 Future work perspectives

Since we are using a new methodology, we have considered a fairly simple model of stochastic MHD equations, where the external force is driven by an infinite dimensional Brownian motion. Our next task is to use our method for the case where external forces depend not only on the Brownian motion, but also the velocity and magnetic fields as well, for instance, the model considered in [147]. Additional problem is to consider the stochastic MHD system with variable density; where the deterministic version was done by Abidi and Hmidi in [2]. We will also be using this same technique to study Navier-Stokes equations with variety of conditions on the data and nature of the fluid. We will also consider studying regularity problem using Kolmogorov theory for MHD systems, for instance, for the case of Navier-Stokes one may look at [42].

Bibliography

[1] A.M. Yaglom A. S. Monin. *Statistical Fluid Mechanics: Mechanics of Turbulence*. Number 2006053473. Dover Publications, Inc., Mineola, New York 11501, 1975.

[2] Hammadi Abidi and Taoufik Hmidi. Résultats d'existence dans des espaces critiques pour le systéme de la MHD inhomogéne. In *Annales mathématiques Blaise Pascal*, volume 14, pages 103–148, 2007.

[3] S Albeverio, B Ferrario, et al. Uniqueness of solutions of the stochastic Navier–Stokes equation with invariant measure given by the enstrophy. *The Annals of Probability*, 32(2):1632–1649, 2004.

[4] Hannes Alfvén. On the cosmogony of the solar system. *Stockholms Observatoriums Annaler*, 14:2–1, 1942.

[5] Hannes Alfvén. Existence of electromagnetic-hydrodynamic waves. *Nature*, 150:405–406, 1942.

[6] Hannes Alfvén. Remarks on the rotation of a magnetized sphere with application to solar rotation. *Arkiv for Astronomi*, 28:1–9, 1942.

[7] Hannes Alfvén. On the cosmogony of the solar system iii. *Stockholms Observatoriums Annaler*, 14:9–1, 1946.

[8] Hannes Alfvén. On the theory of magnetic storms and aurorae. *Tellus A*, 10 (1), 1958.

[9] Hannes Alfvén. The origin of the solar system. In *Evolution in the Universe*, page 31, 1982.

[10] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer Science & Business Media, 2011.

[11] Viorel Barbu and Giuseppe Da Prato. Existence and ergodicity for the two-dimensional stochastic magneto-hydrodynamics equations. *Applied Mathematics and Optimization*, 56(2):145–168, 2007.

[12] George Keith Batchelor. *The theory of homogeneous turbulence.* Cambridge university press, 1953.

[13] A Bensoussan. Stochastic Navier-Stokes equations. *Acta Applicandae Mathematica*, 38(3):267–304, 1995.

[14] A Bensoussan and R Temam. Équations aux dérivées partielles stochastiques non linéaires. *Israel Journal of Mathematics*, 11(1):95–129, 1972.

[15] A Bensoussan and R Temam. Equations stochastiques du type Navier-Stokes. *Journal of Functional Analysis*, 13(2):195–222, 1973.

[16] Alain Bensoussan and Jens Frehse. Local solutions for stochastic Navier-Stokes equations. *ESAIM: Modélisation Mathématique et Analyse Numérique*, 34(2):241–273, 2000.

[17] Andrei Biryuk and Walter Craig. Bounds on Kolmogorov spectra for the Navier–Stokes equations. *Physica D: Nonlinear Phenomena*, 241(4):426–438, 2012.

[18] D Biskamp. Cascade models for magnetohydrodynamic turbulence. *Physical Review E*, 50(4):2702, 1994.

[19] Dieter Biskamp. *Magnetohydrodynamic turbulence.* Cambridge University Press, 2003.

[20] Andrei N Borodin and Paavo Salminen. *Handbook of Brownian motion-facts and formulae*. Birkhäuser, 2012.

[21] Thomas James Morrow Boyd and Jeffrey John Sanderson. *The physics of plasmas*. Cambridge University Press, 2003.

[22] Z Brzeźniak, M Capiński, and F Flandoli. Stochastic Navier-Stokes equations with multiplicative noise. *Stochastic Analysis and Applications*, 10(5):523–532, 1992.

[23] Marco Cannone. *Ondelettes, paraproduits et Navier-Stokes*. PhD thesis, Université de Paris 09, Paris, FRANCE, 1995.

[24] Marco Cannone. Harmonic analysis tools for solving the incompressible Navier-Stokes equations. *Handbook of mathematical fluid dynamics*, 3:161–244, 2004.

[25] MARCO Cannone, CX Miao, NICOLAS Prioux, and BQ Yuan. The Cauchy problem for the magneto-hydrodynamic system. *Banach Center Publications*, 74:59–93, 2006.

[26] Chongsheng Cao and Jiahong Wu. Two regularity criteria for the 3d mhd equations. *Journal of Differential Equations*, 248(9):2263–2274, 2010.

[27] Marek Capinski and Dariusz Gatarek. Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension. *Journal of Functional Analysis*, 126(1):26–35, 1994.

[28] Marek Capiński and Szymon Peszat. Local existence and uniqueness of strong solutions to 3-D stochastic Navier-Stokes equations. *Nonlinear Differential Equations and Applications NoDEA*, 4(2):185–200, 1997.

[29] James A Carlson, Arthur Jaffe, and Andrew Wiles. *The millennium prize problems*. American Mathematical Soc., 2006.

[30] Dongho Chae and Jihoon Lee. On the global well-posedness and stability of the Navier–Stokes and the related equations. In *Contributions to current challenges in mathematical fluid mechanics*, pages 31–51. Springer, 2004.

[31] F Chamizo. Lattice point counting and harmonic analysis. *Proceedings of the "Segundas Jornadas de Teoria de Números" (Madrid, 2007)*, 1:18, 2008.

[32] S Chandrasekhar. Hydromagnetic turbulence. i. a deductive theory. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 233, pages 322–330. The Royal Society, 1955.

[33] J-Y Chemin and Nicolas Lerner. Flot de champs de vecteurs non lipschitziens et équations de navier-stokes. *Journal of Differential Equations*, 121(2):314–328, 1995.

[34] Jean-Yves Chemin. Remarques sur l'existence globale pour le système de Navier-Stokes incompressible. *SIAM journal on mathematical analysis*, 23 (1):20–28, 1992.

[35] Jean-Yves Chemin. About Navier-Stokes system. *Prépublication du Laboratoire d'Analyse Numérique de l'Université Paris*, 6:R96023, 1996.

[36] Jean-Yves Chemin. *Perfect incompressible fluids.* Number 14. Oxford University Press, 1998.

[37] Jean-Yves Chemin. Théorémes d'unicité pour le systéme de Navier-Stokes tridimensionnel. *Journal d'Analyse Mathématique*, 77(1):27–50, 1999.

[38] Jean-Yves Chemin. Localization in Fourier space and Navier-stokes system. *Phase space analysis of partial differential equations*, 1:53–136, 2004.

[39] Jean-Yves Chemin and Isabelle Gallagher. Wellposedness and stability results for the Navier–Stokes equations in $\mathbb{R}^3$. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 26, pages 599–624. Elsevier, 2009.

[40] Jean-Yves Chemin, Benoît Desjardins, Isabelle Gallagher, and Emmanuel Grenier. Fluids with anisotropic viscosity. *ESAIM: Mathematical Modelling and Numerical Analysis*, 34(2):315–335, 2000.

[41] Jean-Yves Chemin, Isabelle Gallagher, and Marius Paicu. Global regularity for some classes of large solutions to the Navier-Stokes equations. *arXiv preprint arXiv:0807.1265*, 2008.

[42] Gui-Qiang Chen and James Glimm. Kolmogorov's theory of turbulence and inviscid limit of the Navier-Stokes equations in $\mathbb{R}^3$. *Communications in Mathematical Physics*, 310(1):267–283, 2012.

[43] Golse F Chen G.-Q, Fila M et al. *Handbook of Differential Equations; Evolutionary Equations*, volume 2. Elsevier, 2005.

[44] Kai L Chung and Zhongxin Zhao. *From Brownian motion to Schrödinger's equation*, volume 312. Springer Science & Business Media, 2001.

[45] Giuseppe Da Prato and Arnaud Debussche. Two-dimensional Navier–Stokes equations driven by a space–time white noise. *Journal of Functional Analysis*, 196(1):180–210, 2002.

[46] Leonardo Da Vinci and Jean Paul Richter. The notebooks of leonardo da vinci, compiled and edited from the original manuscripts by jean paul richter, 1970.

[47] Raphaël Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Inventiones Mathematicae*, 141(3):579–614, 2000.

[48] Raphaël Danchin. Local theory in critical spaces for compressible viscous and heat-conductive gases. *Communications in Partial Differential Equations*, 26 (7-8):1183–1233, 2001.

[49] Raphaël Danchin. Fourier analysis methods for PDEs. *Lecture notes*, 14, 2005.

[50] Raphaël Danchin. A survey on fourier analysis methods for solving the compressible Navier-Stokes equations. *Science China Mathematics*, 55(2):245–275, 2012.

[51] Peter A Davidson, Yukio Kaneda, Keith Moffatt, and Katepalli R Sreenivasan. *A voyage through turbulence*. Cambridge University Press, 2011.

[52] Theodore De Karman and Leslie Howarth. On the statistical theory of isotropic turbulence. In *Proceedings of the Royal Society of London A: Math-*

ematical, Physical and Engineering Sciences, volume 164, pages 192–215. The Royal Society, 1938.

[53] Michael Dellnitz and Oliver Junge. *Handbook of Dynamical Systems*, volume 2, chapter Set Oriente Numerical Methods for Dynamical Systems, pages 221–264. Elsevier Science B.V., 2002.

[54] Gabriel Deugoué and Mamadou Sango. On the strong solution for the 3D stochastic Leray-alpha model. *Boundary Value Problems*, 2010:28, 2010.

[55] Gabriel Deugoué and Mamadou Sango. Weak solutions to stochastic 3D Navier–Stokes-α model of turbulence: α-asymptotic behavior. *Journal of Mathematical Analysis and Applications*, 384(1):49–62, 2011.

[56] Gabriel Deugoué, Paul André Razafimandimby, and Mamadou Sango. On the 3-D stochastic magnetohydrodynamic-α model. *Stochastic processes and their applications*, 122(5):2211–2248, 2012.

[57] M Dobrowolny, A Mangeney, and P Veltri. Fully developed anisotropic hydromagnetic turbulence in interplanetary space. *Physical Review Letters*, 45 (2):144, 1980.

[58] Robert Ecke. The turbulence problem. *An Experimentalist's Perspective (Los Alamos Science, Number 29, 2005)*, 2005.

[59] MA El Tawil and MH Kamel. MHD flow under stochastic porous media. *Energy conversion and management*, 35(11):991–997, 1994.

[60] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, second edition, 2010.

[61] Gregory L Eyink and Katepalli R Sreenivasan. Onsager and the theory of hydrodynamic turbulence. *Reviews of modern physics*, 78(1):87–135, 2006.

[62] Charles L Fefferman. *The millennium prize problems*, chapter Existence and Smoothness of the Navier-Stokes equations, pages 57–67. American Mathematical Society, 2006.

[63] Richard Feynman. The Feynman lectures on physics. Online, October 2016. URL http://www.feynmanlectures.caltech.edu/.

[64] Franco Flandoli. Dissipativity and invariant measures for stochastic Navier-Stokes equations. *Nonlinear Differential Equations and Applications NoDEA*, 1(4):403–423, 1994.

[65] Franco Flandoli and Dariusz Gatarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probability Theory and Related Fields*, 102 (3):367–391, 1995.

[66] Daniel Fleisch. *A Student's guide to Maxwell's equations*. Cambridge University Press, 2008.

[67] C Foias, OP Manley, and R Temam. Bounds for the mean dissipation of 2-D enstrophy and 3-D energy in turbulent flows. *Physics Letters A*, 174(3): 210–215, 1993.

[68] Ciprian Foias. What do the Navier-Stokes equations tell us about turbulence? *Contemporary Mathematics*, 208:151–180, 1997.

[69] Ciprian Foias, Darryl D Holm, and Edriss S Titi. The Navier–Stokes-alpha model of fluid turbulence. *Physica D: Nonlinear Phenomena*, 152:505–519, 2001.

[70] Jeffrey P. Freidberg. *Ideal MHD*. Cambridge Univ Press, 2014.

[71] U Frisch. *Turbulence: the legacy of A. N. Kolmogorov*. Cambridge University Press, 1995.

[72] U Frisch, M Lesieur, and A Brissaud. A Markovian random coupling model for turbulence. *Journal of Fluid Mechanics*, 65(01):145–152, 1974.

[73] Hiroshi Fujita and Tosio Kato. On the Navier-Stokes initial value problem. I. *Archive for rational mechanics and analysis*, 16(4):269–315, 1964.

[74] Isabelle Gallagher and Fabrice Planchon. On global infinite energy solutions to the Navier-Stokes equations in two dimensions. *Archive for Rational Mechanics and Analysis*, 161(4):307–337, 2002.

[75] Isabelle Gallagher, Dragos Iftimie, and Fabrice Planchon. Asymptotics and stability for global solutions to the Navier-Stokes equations. In *Annales de l'Institut fourier*, volume 53, pages 1387–1424, 2003.

[76] Johan Peter Goedbloed. Lecture notes on ideal magnetohydrodynamics. Technical report, Stichting voor Fundamenteel Onderzoek der Materie, 1983.

[77] Cheng He and Zhouping Xin. On the regularity of weak solutions to the magnetohydrodynamic equations. *Journal of Differential Equations*, 213(2): 235–254, 2005.

[78] W Heisenberg. On the theory of statistical and isotropic turbulence. In *Original Scientific Papers Wissenschaftliche Originalarbeiten*, pages 115–119. Springer, 1985.

[79] Eberhard Hopf. A mathematical example displaying features of turbulence. *Communications on Pure and Applied Mathematics*, 1(4):303–322, 1948.

[80] Eberhard Hopf. The partial differential equation $u_t + uu_x = \mu_{xx}$. *Communications on Pure and Applied mathematics*, 3(3):201–230, 1950.

[81] Eberhard Hopf. Statistical hydromechanics and functional calculus. *Journal of rational Mechanics and Analysis*, 1(1):87–123, 1952.

[82] Eberhard Hopf. Repeated branching through loss of stability, an example. In *Proceedings of the conference on differential equations, Maryland*, pages 49–56, 1955.

[83] Lars Hörmander. *The Analysis of Linear Partial Differential Operators: Vol.: 1.: Distribution Theory and Fourier Analysis*. Springer-Verlag, 1983.

[84] Paul G Huray. *Maxwell's equations*. John Wiley & Sons, 2011.

[85] Igor' Evgen'evich Irodov, Natasha Deineko, and Ram Wadhwa. *Basic laws of electromagnetism.* Imported Pubn, 1986.

[86] PS Iroshnikov. Turbulence of a conducting fluid in a strong magnetic field. *Soviet Astronomy*, 7:566, 1964.

[87] Kiyosi Itô. Stochastic Integral. *Proceedings of the Imperial Academy*, 20(8): 519–524, 1944.

[88] Kiyosi Itô, editor. *Proceedings of the International Symposium on Stochastic Differential Equations*, Kyoto, 1976. A Wiley-Interscience Publication, JOHN WILEY & SONS.

[89] Derek Jackson and Brian Launder. Osborne Reynolds and the publication of his papers on turbulent flow. *Annu. Rev. Fluid Mech.*, 39:19–35, 2007.

[90] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics.* Springer Science & Business Media, 2012.

[91] Andrej Nikolaevich Kolmogorov. On the degeneration of isotropic turbulence in an incompressible viscous fluid. In *Dokl. Akad. Nauk SSSR*, volume 31, pages 319–323, 1941.

[92] Andrej Nikolaevich Kolmogorov. Equations of turbulent motion in an incompressible fluid. In *Dokl. Akad. Nauk SSSR*, volume 30, pages 299–303, 1941.

[93] Andrey Nikolaevich Kolmogorov. Dissipation of energy in locally isotropic turbulence. In *Akademiia Nauk SSSR Doklady*, volume 32, page 16, 1941.

[94] Andrey Nikolaevich Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers [in Russian]. In *Dokl. Akad. Nauk SSSR*, volume 30, pages 299–303, 1941.

[95] Andrey Nikolaevich Kolmogorov. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *Journal of Fluid Mechanics*, 13(01):82–85, 1962.

[96] Robert H Kraichnan. Irreversible statistical mechanics of incompressible hydromagnetic turbulence. *Physical Review*, 109(5):1407, 1958.

[97] Robert H Kraichnan. The structure of isotropic turbulence at very high Reynolds numbers. *Journal of Fluid Mechanics*, 5(04):497–543, 1959.

[98] Robert H Kraichnan. Dynamics of nonlinear stochastic systems. *Journal of Mathematical Physics*, 2(1):124–148, 1961.

[99] Robert H Kraichnan. Inertial-range spectrum of hydromagnetic turbulence. *Physics of Fluids (1958-1988)*, 8(7):1385–1387, 1965.

[100] Robert H Kraichnan. Inertial-range transfer in two-and three-dimensional turbulence. *J. Fluid Mech*, 47(3):525–535, 1971.

[101] Robert H Kraichnan. On Kolmogorov's inertial-range theories. *Journal of Fluid Mechanics*, 62(02):305–330, 1974.

[102] Hiroshi Kunita and Shinzo Watanabe. On square integrable martingales. *Nagoya Mathematical Journal*, 30:209–245, 1967.

[103] O Ao Ladyzhenskaya. Mathematical problems in the dynamics of a viscous incompressible fluid, 1970.

[104] OA Ladyzhenskaya and GA Seregin. Regularity of solutions of two-dimensional equations in fluid dynamics models with nonlinear viscosity. *Journal of Mathematical Sciences*, 109(5):1911–1927, 2002.

[105] Olga A Ladyzhenskaya and Richard A Silverman. *The mathematical theory of viscous incompressible flow*, volume 76. Gordon and Breach New York, 1969.

[106] Olga Aleksandrovna Ladyzhenskaya and Vsevolod Alekseevich Solonnikov. Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid. *Trudy Matematicheskogo Instituta im. VA Steklova*, 59:115–173, 1960.

[107] Brian E Launder. First steps in modelling turbulence and its origins: a commentary on Reynolds (1895)'On the dynamical theory of incompressible viscous fluids and the determination of the criterion'. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 373(2039):20140231, 2015.

[108] Pierre Gilles Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. CRC Press, 2002.

[109] Pierre Gilles Lemarié-Rieusset. *The Navier-Stokes Problem in the 21st Century*. CRC Press, 2016.

[110] J Leray. Sur le systeme d'équations aux dérivés partielles qui régit l'écoulement premanant des fluids visqueux. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 192:1180–1182, 1931.

[111] Jean Leray. *Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*. PhD thesis, École Normale Supérieure, 1933.

[112] Jean Leray. Etude de diverses équations intégrales non linéaires et de quelques problémes que pose l'hydrodynamique. *Théses françaises de l'entre-deux-guerres*, 142:1–82, 1933.

[113] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta mathematica*, 63(1):193–248, 1934.

[114] Alexander E Lifshits. *Magnetohydrodynamics and spectral theory*, volume 4. Springer Science & Business Media, 2012.

[115] Lev Gerasimovich Loitsianskii. Some basic laws of isotropic turbulent flow. Technical Report 1079, National Advisory Committee for Aeronautics, NACA, Washington, September 1945. This work seems to be first published in Russian in 1939, on Trudy TsAGI (This information is found in the commentary of Yaglom on the works of Kolmogorov).

[116] SV Lototskii and Boris L Rozovskii. Passive scalar equation in a turbulent incompressible Gaussian velocity field. *Russian Mathematical Surveys*, 59(2): 297–312, 2004.

[117] Sergey Lototsky and Boris Rozovskii. Stochastic differential equations: a Wiener chaos approach. In *From stochastic calculus to mathematical finance*, pages 433–506. Springer, 2006.

[118] PA Meyer. Decomposition of supermartingales: the uniqueness theorem. *Illinois Journal of Mathematics*, 7(1):1–17, 1963.

[119] R Mikulevicius and BL Rozovskii. Martingale problems for stochastic PDEs. *Stochastic partial differential equations: six perspectives*, 64:243–326, 1999.

[120] R Mikulevicius and BL Rozovskii. Stochastic Navier–Stokes equations for turbulent flows. *SIAM Journal on Mathematical Analysis*, 35(5):1250–1310, 2004.

[121] R Mikulevicius and BL Rozovskii. Global L_2-solutions of stochastic Navier–Stokes equations. *The Annals of Probability*, 33(1):137–176, 2005.

[122] R Mikulevicius and BL Rozovskii. On unbiased stochastic Navier–Stokes equations. *Probability Theory and Related Fields*, 154(3-4):787–834, 2012.

[123] Remigijus Mikulevicius and Boris L Rozovskii. Normalized stochastic integrals in topological vector spaces. In *Séminaire de probabilités XXXII*, pages 137–165. Springer, 1998.

[124] MD Millionshchikov. Decay of homogeneous isotropic turbulence in viscous incompressible fluids. In *Dokl. Akad. Nauk SSSR*, volume 22, page 236, 1939.

[125] Sergei S Molokov, René Moreau, and H Keith Moffatt. *Magnetohydrodynamics: Historical evolution and trends*, volume 80. Springer Science & Business Media, 2007.

[126] A. S. Monin and A. M. Yaglom. *Statistical fluid mechanics, volume II: mechanics of turbulence*, volume 2. Dover Publications, 2007.

[127] A. S. Monin and A. M. Yaglom. *Statistical fluid mechanics, volume II: mechanics of turbulence*, volume 2. Courier Corporation, 2013.

[128] Elżbieta Motyl. Stochastic hydrodynamic-type evolution equations driven by Lévy noise in 3D unbounded domains – abstract framework and applications. *Stochastic Processes and their Applications*, 124(6):2052–2097, 2014.

[129] Yasuki Nakayama and Robert Boucher. *Introduction to fluid mechanics*. Butterworth-Heinemann, 1998.

[130] AM Obukhoff and AM Yaglom. The microstructure of turbulent flow. *Prikladnaya Matematika i Mekhanika*, XV, 1951. Translated to English in 1953 by National Advisory Committee for Aeronautics, NACA, Technical Memorandum 1350.

[131] AM Obukhov. On the distribution of energy in the spectrum of turbulent flow. In *Dokl. Akad. Nauk SSSR*, volume 32, pages 22–24, 1941.

[132] AM Obukhov. Some specific features of atmospheric turbulence. *Journal of Geophysical Research*, 67(8):3011–3014, 1962.

[133] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[134] Jaak Peetre. *New thoughts on Besov spaces*, volume 1. Mathematics Department, Duke University, 1976.

[135] Henri Poincaré and R Magini. Les méthodes nouvelles de la mécanique céleste. *Il Nuovo Cimento (1895-1900)*, 10(1):128–130, 1899.

[136] Stephen B Pope. *Turbulent flows*. Cambridge University Press, 2001.

[137] L. Prandtl. *Erly developments of modern aerodynamics*, chapter On the motion of fluids with very little friction, pages 77–87. Amer Inst of Aeronautics &, 1904.

[138] Philip E. Protter. *Stochastic Integration and Differential Equations.* Number 21 in Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, 2005.

[139] Simon Ramo, John R Whinnery, and Theodore Van Duzer. *Fields and waves in communication electronics.* John Wiley & Sons, 2008.

[140] Paul André Razafimandimby and Mamadou Sango. Strong solution for a stochastic model of two-dimensional second grade fluids: existence, uniqueness and asymptotic behavior. *Nonlinear Analysis: Theory, Methods & Applications*, 75(11):4251–4270, 2012.

[141] Osborne Reynolds. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. *Proceedings of the royal society of London*, 35(224-226):84–99, 1883.

[142] Osborne Reynolds. On the dynamical theory of incompressible viscous fluids andthe determination of the criterion. *Proceedings of the Royal Society of London*, 56(336-339):40–45, 1894.

[143] Osborne Reynolds, Arthur William Brightmore, and William Henry Moorby. *Papers on Mechanical and Physical Subjects: 1881-1900*, volume 2. The University Press, 1901.

[144] Lewis Fry Richardson. *Weather prediction by numerical process.* Cambridge University Press, second edition, 2007.

[145] Thomas Runst and Winfried Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3. Walter de Gruyter, 1996.

[146] Mamadou Sango. Density dependent stochastic Navier–Stokes equations with non-lipschitz random forcing. *Reviews in Mathematical Physics*, 22(06):669–697, 2010.

[147] Mamadou Sango. Magnetohydrodynamic turbulent flows: existence results. *Physica D: Nonlinear Phenomena*, 239(12):912–923, 2010.

[148] Mamadou Sango and Tesfalem Abate Tegegn. Harmonic analysis tools for stochastic magnetohydrodynamics equations in Besov spaces. *International Journal of Modern Physics B*, 30(28 & 29), August 2016. DOI: http://dx.doi.org/10.1142/S0217979216400257.

[149] George R Sell, Ciprian Foias, and Roger Temam. *Turbulence in Fluid Flows: A Dynamical Systems Approach*, volume 55. Springer Science & Business Media, 2012.

[150] Michel Sermange and Roger Temam. Some mathematical questions related to the MHD equations. *Communications on Pure and Applied Mathematics*, 36(5):635–664, 1983.

[151] James Serrin. Local behavior of solutions of quasi-linear equations. *Acta Mathematica*, 111(1):247–302, 1964.

[152] A. N. Shiryayev. *Probability*. Number 95 in Graduate Texts in Mathematics. Springer Science & Business Media, 1984.

[153] SS Sritharan and P Sundar. The stochastic magneto-hydrodynamic system. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 2 (02):241–265, 1999.

[154] P Sundar. Stochastic magnetohydrodynamic system perturbed by general noise. *Commun. Stoch. Anal*, 4(2):253–269, 2010.

[155] Zhong Tan, Dehua Wang, and Huaqiao Wang. Global strong solution to the three-dimensional stochastic incompressible magnetohydrodynamic equations. *Mathematische Annalen*, pages 1–38, 2015.

[156] Terry Tao. The fourier transform. https://terrytao.wordpress.com/2009/04/06/the-fourier-transform/, April 2009.

[157] Geoffrey Ingram Taylor. Experiments with rotating fluids. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 100(703):114–121, 1921.

[158] Geoffrey Ingram Taylor. Statistical theory of turbulence. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 151, pages 421–444. The Royal Society, 1935.

[159] Roger Temam. *Navier-Stokes equations: theory and numerical analysis*, volume 343. American Mathematical Soc., 2001.

[160] Vladimir M Tikhomirov. *Selected Works of AN Kolmogorov: Volume I: Mathematics and Mechanics*, volume 25. Springer Science & Business Media, 1991.

[161] Hans Triebel. *Theory of Function Spaces*, volume I. Birkhäuser, 1983.

[162] Hans Triebel. *Theory of function spaces. II, volume 84 of Monographs in Mathematics*. Birkhauser Verlag, Basel, 1992.

[163] Hans Triebel. *Theory of function spaces. III, volume 100 of Monographs in Mathematics*. Birkhauser Verlag, Basel, 2006.

[164] Mahendra K Verma. Mean magnetic field renormalization and Kolmogorov's energy spectrum in magnetohydrodynamic turbulence. *Physics of Plasmas (1994-present)*, 6(5):1455–1460, 1999.

[165] Mahendra K Verma. Statistical theory of magnetohydrodynamic turbulence: recent results. *Physics Reports*, 401(5):229–380, 2004.

[166] M Viot. *Solutions faibles d'équations aux derives partielles stochastiques non linéaires*. PhD thesis, Thése Doct. Sci. Univ. Pierre Marie Curie, 1976.

[167] Mark Iosifovich Vishik and Andrej V Fursikov. *Mathematical problems of statistical hydromechanics*, volume 9. Springer Science & Business Media, 2012.

[168] Marko Iosifovich Vishik, Aleksandr Il'ich Komech, and Andrei Vladimirovich Fursikov. Some mathematical problems of statistical hyromechanics. *Russian Mathematical Surveys*, 34(5):149–234, 1979.

[169] Theodore Von Karman. Progress in the statistical theory of turbulence. *Proceedings of the National Academy of Sciences of the United States of America*, 34(11):530, 1948.

[170] Theodore Von Karman and CC Lin. On the concept of similiarity in the theory of isotropic turbulence. *Reviews of Modern Physics*, 21(3):516, 1949.

[171] Thomas H Wolff, Izabella Laba, and Carol Shubin. *Lectures on harmonic analysis*, volume 29. American Mathematical Society, 2003.

[172] Ya.B. Zeldovich, A.A. Ruzmaikin, and D.D. Sokiloff. *Magnetic Fields in Astrophysics*. New York, London, Paris. Gordon and Breach Publishers, 1983.

[173] Junyong Zhang. Wellposedness for the magnetohydrodynamics equation in critical space. *Applicable Analysis*, 87(7):773–785, 2008.