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Nonlinear theories of generalised functions

by

Dennis Ferdinand Agbebaku

Submitted in partial fulfilment of the requirements for the degree

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Declaration

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not previously been submitted by me or any other person for any degree at this or any other University.

Name: Dennis Ferdinand Agbebaku
Date: August 2015

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Dedication

"...memory is one gift of God that death cannot destroy..." - A. K Rowswell

To the memory of my Dad, Supro AGBEBAKU, Marcus Eguagi, who passed on during the course of my study.

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Title	Nonlinear theories of generalised functions
Name	Dennis Ferdinand Agbebaku
Supervisor	Dr JH van der Walt
Department	Mathematics and Applied Mathematics
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Summary

In this thesis, the Order Completion Method for nonlinear partial differential equation, in the setting of convergence spaces, is interpreted in terms of the algebraic theory of generalised functions. In particular, certain spaces of generalised functions that are involved in the construction of generalised solutions for nonlinear partial differential equations through the Order Completion Method are identified with a differential chain of algebras of generalise functions. By so doing, the generalised solutions for smooth nonlinear partial differential equation obtained through Order Completion Method are interpreted as chain generalised solutions. Moreover, the mentioned differential chain is shown to be related to the Rosinger's chain of nowhere dense algebras of generalised functions. This leads to an interpretation of the existence result for the solution of smooth nonlinear partial differential equations obtained through the order completion method in the chain of nowhere dense algebras.

Using techniques introduced by Verneave, we construct a chain of almost everywhere algebras of generalised functions and show how the chain of algebras of generalised functions associated with the order completion method is related to this chain of almost everywhere algebras of generalised functions. We also discuss the embedding of the distributions into the chain of almost everywhere algebras of generalised functions. We further show that the generalised solutions of nonlinear partial differential equations obtained through the order completion method corresponds to a chain generalised solution in the chain of nowhere dense algebras of generalized functions.

Finally, using the theory of chains of algebras of generalized functions, we construct algebras of generalised functions that can handle certain types of singularities occurring on sets of first Baire category, so called, space-time foam algebras.

Preface

Ever since Newton and, independently, Leibnitz introduced the differential and integral calculus, ordinary differential equations (ODEs) and partial differential (PDEs) have been one of the central tools by which the laws o nature are given exact mathematical formulation. While, initially, the equations used were mostly linear and of second order, general nonlinear equations have become increasingly important with the emergence of sophisticated scientific theorems of technologies, particularly in the second half of the twentieth century.

For more than one hundred and fifty years from now there has been a general and type independent existence result for solution of nonlinear PDEs, namely, the Cauchy - Kovalevskaia Theorem [30, 50]. While the Cauchy - Kovalevskaia Theorem is completely type independent it suffers from two major deficiencies. Firstly, it is restricted to the class of analytic PDEs, with analytic initial data specified on a non characteristic analytic hyperplane. Secondly, the solutions obtained through the theorem are local in nature. That is, the solution may fail to exist on the entire domain of definition of the respective PDE. Recall [1, 33, 71] that the initial value problem

$$\begin{aligned} &u_t + uu_x = 0 \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R} \end{aligned}$$

does not admit a classical solution in $\mathbb{R} \times [0, +\infty)$ whenever, $u_0'(x) < 0$ at even a single $x \in \mathbb{R}$.

The local nature of classical solutions of PDEs, in general, has lead to the interest in global nonclassical solutions of partial differential equations. Such solutions, usually referred to as generalized solutions, are obtained as elements of suitable spaces of generalized functions, that is, objects which retain certain essential features of the usual real or complex valued functions. Initial attempts at the exact formulation of the concepts of generalized function and generalized solutions of PDEs, culminated in the work of Schwartz [57], where the space $\mathcal{D}'(\Omega)$ of distributions is introduced, we mention the Ehrenpreis-Malgrange Theorem [21, 38] which states that any nonzero, constant coefficient differential operator

$$P(D) = \sum_{|\alpha| \leq m} C_\alpha D^\alpha$$

admits a solution of

$$P(D) = \delta$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\delta \in \mathcal{D}'(\mathbb{R}^n)$ is the dirac distribution. As a consequence of this result, the equation

$$P(D) = f$$

is solvable in $\mathcal{D}'(\mathbb{R}^n)$ for any $f \in \mathcal{D}'(\mathbb{R}^n)$. It is therefore clear that the theory of distribution is highly clear in the context of linear constant coefficient PDEs.

In spite of the power of $\mathcal{D}'(\Omega)$ - distributions in the context of linear, constant coefficient PDEs, the theory of distributions suffers from two major deficiencies. Firstly, $\mathcal{D}'(\Omega)$ does not admit solutions of linear variable coefficient PDEs, even in the case where the coefficients are smooth, [26, 34]. Secondly, the pointwise multiplication of smooth functions cannot be extended to $\mathcal{D}'(\Omega)$ is a reasonable way so that $\mathcal{D}'(\Omega)$ becomes and algebra, [51]. Thus the concept of solution of a generalised, nonlinear PDE cannot be formulated in terms of the $\mathcal{D}'$ - distributions alone.

In view of the mentioned limitations of the linear distribution theory of distributions, in particular from the point of view of nonlinear PDEs, alternative, nonlinear theories of generalized functions have been introduced, see for instance [41, 48]. In this thesis, two such theories are investigated, namely, theory of algebras of generalized functions [48, 51, 53, 73], focussing on so-called nowhere dense algebras, and the Order Completion Method (OCM) [41, 68, 72]. We show that the OCM can be interpreted in terms of the algebraic theory of generalized functions. The relationship between spaces of generalized functions on which the OCM is based with certain nowhere dense algebras is established. We also investigate the extent to which the mentioned nowhere dense algebra are able to deal with singularities occurring in closed nowhere dense sets, and construct algebras able to handle more general types of singularities.

The thesis is divided into two parts. Part I contains a concise introduction to the general theories mentioned in the receding paragraph. The results presented here are from the literature. It is made up of two chapters.

- Chapter one, contains a discussion on the main ideas involve in the algebraic nonlinear theory of generalized functions. In particular, the general method of constructing an algebra of generalized functions containing the distribution as a linear subspace is discussed. In addition, we discussed the general method for constructing a chain of algebra of generalized functions that contained the distributions.

- Chapter two contains the main ideas involve in the Order Completion Methods as well the enrichment through convergence spaces. The structure and regularity results for generalized solutions of partial differential equations, obtained through the order completion method, is discussed.

Part II contains the original contribution of this work. That is, the algebraic interpretations of the spaces involved in the order Completion Method. It contains four chapters. We briefly highlight the contents of these chapters below.

- In Chapter three we discuss the algebraic and chain structure of the spaces generalized functions involved in the Order Completion Method. In particular we show how these spaces can be represented as algebras of generalized functions forming a differential chain of algebra of generalised functions. The existence result for generalised solutions of $\mathcal{C}^\infty$-smooth PDEs obtained through the OCM is interpreted in the context of chains of algebras of generalised functions.

- Chapter four deals with the nowhere dense algebras of generalised functions. We recall the construction of Rosinger's nowhere dense chain, Vernaeve's almost everywhere algebras, see [49, 51, 73, 74]. Based on Vernaeve's construction [73], we

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construct a chain of almost everywhere algebras and study some of properties of these chains. The relationship between the spaces involve in the OCM and the nowhere dense and almost everywhere chains is established, leading to an existence result for generalized solutions of $\mathcal{C}^{\infty}$-smooth PDEs in these chains. Motivated by the problem of constructing so-called space - time foam algebras [53, 65], we use the nowhere dense and almost everywhere chains to construct algebras admitting certain densely singular functions.

- Concluding remarks which sets out the main results of the research work as well as suggestions for future research in this areas are contained in Chapter five.

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Part I

Introduction

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Chapter 1

Algebraic nonlinear theories of generalised functions

1.1 Deficiencies of $\mathcal{D}'(\Omega)$

The linear theory of distributions, as well as certain generalization of it, has proven to be very useful in the analysis of constant coefficient linear partial differential equations, see for instance [26]. In this regard, we may recall [21, 38] that each linear, constant coefficient partial differential operator

$$P(D)u(x) = \sum_{|\alpha| \leq m} A_\alpha D^\alpha u(x), \ x \in \mathbb{R}^n$$

admits a fundamental solution. That is, there exists a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ so that

$$P(D)T = \delta,$$

where $\delta \in \mathcal{D}'(\mathbb{R}^n)$ is the Dirac delta distribution. From this it follows that for any $\varphi \in \mathcal{D}(\Omega)$, the equation

$$P(D)u(x) = \varphi(x), \ x \in \Omega$$

has a solution in $\mathcal{D}(\Omega)$.

Notwithstanding the usefulness of the $\mathcal{D}'$-distribution in the analysis of linear constant coefficient PDEs, the Schwartz distributions have some major deficiencies. One such deficiency, referred to as the Lewy insufficiency, see [34], is the inability of the Schwartz distribution to solve linear PDEs with nonconstant coefficient. Indeed, Hörmander [26] showed that there exist C^∞-smooth functions h for which the equation

$$u_x + iu_y - 2i(x + iy)u_t = h,$$

has no distributional solution in any neighborhood of any point in $\mathbb{R}^3$.

A second and well known deficiency of the Schwartz distributions concerns the definition of nonlinear operations, such as multiplication, on $\mathcal{D}'(\Omega)$. Each distribution $T \in \mathcal{D}'(\Omega)$ can be multiplied with any C^∞-smooth function u, the product uT being defined by

$$Tu : \mathcal{D}(\Omega) \ni \varphi \mapsto T(u\varphi) \in \mathbb{R}. \tag{1.1}$$

Clearly (1.1) extends the usual multiplication of smooth functions. Indeed, for $v \in \mathcal{C}^\infty(\Omega)$, the product of the distribution T_v associated with v through

$$T_v : \mathcal{D}(\Omega) \ni \varphi \mapsto \int_\Omega \varphi v dx \in \mathbb{R}$$

and a $\mathcal{C}^\infty$-smooth function u, as given by (1.1), is given by

$$uT_v(\varphi) = T_v(u\varphi) = \int_\Omega \varphi uv dx = T_{uv}(\varphi).$$

It is not possible to extend the multiplication (1.1) to all of $\mathcal{D}'(\Omega)$ in such a way that $\mathcal{D}'(\Omega)$, together with the usual vector space operations, is an algebra, see [50, 51]. In this regard, Schwartz [58], see also [51], proved a version of the following result.

Theorem 1.1. *Let $\mathcal{A}$ be an associative algebra so that $C(\mathbb{R}) \subset \mathcal{A}$, and uv is the usual product of functions for each $u, v \in C(\mathbb{R})$. If $D : \mathcal{A} \longrightarrow \mathcal{A}$ is a differential operator, that is, D is linear and satisfies Leibnitz rule for derivatives of products of functions, so that D restricted to $C^1(\mathbb{R}) \subset \mathcal{A}$ is the usual differentiation operation, then there is no $\delta \in \mathcal{A}, \delta \neq 0$, such that $x\delta = 0$.*

Theorem 1.1 is often interpreted in the following way. If $\delta \in \mathcal{D}'(\mathbb{R})$ is the Dirac delta distribution, then in view of (1.1), for any $u \in C^\infty(\mathbb{R})$ we have

$$(u\delta)(\varphi) = \delta(u\varphi) = u(0)\varphi, \ \varphi \in \mathcal{D}(\Omega).$$

Therefore, if $x \in C^\infty(\mathbb{R})$ is the identity function on $\mathbb{R}$, then

$$(x\delta)\varphi = \delta(x\varphi) = 0, \ \varphi \in \mathcal{D}(\Omega).$$

Hence, $x\delta$ is the additive identity in $\mathcal{D}'(\mathbb{R})$. Therefore, if $\mathcal{D}'(\Omega)$ were an algebra with multiplication extending the usual pointwise product of smooth functions, then it follows from Theorem 1.1 that $\delta = 0$, which is not the case. Therefore there cannot be reasonable concept of multiplication on $\mathcal{D}'(\Omega)$.

In view of the above mentioned Lewy insufficiency and Schwartz impossibility results, it is a common belief that there is no general and convenient nonlinear theory of generalized function. In particular, it is widely believed that there is no general and type independent theory for generalized solutions of nonlinear PDEs. As a result, various ad hoc methods and techniques have been developed which are applicable only to the particular types of nonlinear PDEs they were developed to handle, see for instance [10, 44, 59, 61]. However in the late 1960s an alternative approach to dealing with the Schwartz impossibility problem was introduced. This approach, summarized in the slogan 'algebra first', is to construct suitable algebras of generalized functions that contain the $\mathcal{D}'(\Omega)$-distributions as a linear subspace. The main ideas involved in this approach is the subject of the remainder of this chapter.

1.2 Vector spaces of generalised functions

Generalized solutions to *linear* PDEs are typically constructed as elements of the completion of a suitably chosen locally convex, often metrizable, topological vector space.

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Indeed, the Sobolev space $W^{m,p}(\Omega)$ as well as the space $\mathcal{D}'(\Omega)$ of distributions may be constructed as the completion of $\mathcal{C}^\infty(\Omega)$ with respect to a suitable locally convex topology.

In this section, as a motivation for the construction of algebras of generalised functions presented in Section 1.3, we recall briefly the abstract construction of the completion of a metrizable topological vector space.

Let $\mathcal{X}$ be a metrizable, locally convex topological vector space. Consider the set

$$\mathcal{X}^{\mathbb{N}} = \left\{ u = (u_n) \;\middle|\; \begin{array}{ll} \forall & n \in \mathbb{N} \;: \\ & u_n \in \mathcal{X} \end{array} \right\} \tag{1.2}$$

of sequences in X. With the usual termwise operations on sequences of function, the set $\mathcal{X}^{\mathbb{N}}$ is in a natural way vector space. Let $\mathcal{S}$ be the set of all Cauchy sequences in $\mathcal{X}$ and let $\mathcal{V}$ be the set of all sequences converging to zero in $\mathcal{X}$. Clearly $\mathcal{S}$ and $\mathcal{V}$ are linear subspaces of $\mathcal{X}^{\mathbb{N}}$, and $\mathcal{V} \subset \mathcal{S}$. The completion $\mathcal{X}^\sharp$ of $\mathcal{X}$ may the constructed as the quotient vector space

$$\mathcal{X}^\sharp = \mathcal{S}/\mathcal{V}, \tag{1.3}$$

equipped with a suitable metrizable, locally convex topology. Furthermore, one has a vector space embedding

$$\mathcal{X} \subset \mathcal{X}^\sharp = \mathcal{S}/\mathcal{I} \tag{1.4}$$

which is defined by the linear injective mapping

$$\iota_{\mathcal{X}} : \mathcal{X} \ni u \mapsto i_{\mathcal{X}}(u) = (u) + \mathcal{V} \in \mathcal{X}^\sharp, \tag{1.5}$$

where $(u) \in \mathcal{S}$ is the constant sequence $(u) = (u_n) = (u, \cdots, u \cdots)$.

The existence of the linear injection (1.4) - (1.5) depends on the *neutrix condition*

$$\mathcal{U}_{\mathcal{X},\mathbb{N}} \subseteq \mathcal{S},\; \mathcal{V} \cap \mathcal{U}_{\mathcal{X},\mathbb{N}} = \mathcal{O}, \tag{1.6}$$

see [51]. Here

$$\mathcal{U}_{\mathcal{X},\mathbb{N}}(\Omega) = \left\{ (u_n) \in \mathcal{X}^{\mathbb{N}} \;\middle|\; \begin{array}{ll} \exists & v \in \mathcal{X} \;: \\ & u_n = v,\; n \in \mathbb{N} \end{array} \right\},$$

and

$$\mathcal{O} = \{(0)\}$$

is the null vector subspace in $\mathcal{X}^{\mathbb{N}}$. The condition (1.6) is satisfied since the topology on $\mathcal{X}$ is Hausdorff.

The inclusion $\mathcal{U}_{\mathcal{X},\mathbb{N}} \subseteq \mathcal{S}$ gives rise to a linear surjection

$$\Delta : \mathcal{X} \ni u \mapsto (u) \in \mathcal{U}_{\mathcal{X},\mathbb{N}},$$

while the condition $\mathcal{V} \cap \mathcal{U}_{\mathcal{X},\mathbb{N}} = \mathcal{O}$ implies that the canonical quotient map

$$q_{\mathcal{V}} : \mathcal{S} \ni u \mapsto u + \mathcal{V} \in \mathcal{X}^\sharp$$

is injective, when restricted to $\mathcal{U}_{\mathcal{X},\mathbb{N}}$. Therefore

$$\iota_{\mathcal{X}} : \mathcal{X} \ni u \mapsto q_{\mathcal{V}} \circ \Delta(u) \in \mathcal{X}^\sharp$$

defines a linear injection.

1.3 Differential algebras of generalised functions

We now proceed to present the main points involved in the construction of algebras of generalised functions, see for instance [49, 50, 51] for details. The construction is essentially a variation of the construction of the completion of a metrizable topological vector space outlined in Section 1.2.

Consider the Cartesian product

$$\mathcal{C}^\infty(\Omega)^\Lambda = \left\{ u = (u_\lambda)_{\lambda\in\Lambda} \left| \begin{array}{ll} \forall & \lambda \in \Lambda \ : \\ & u_\lambda \in \mathcal{C}^\infty(\Omega) \end{array} \right. \right\}, \tag{1.7}$$

where Λ is some infinite index set. Since the space $\mathcal{C}^\infty(\Omega)$ of infinitely differentiable functions on Ω is a unital and commutative algebra with respect to the usual point-wise operations on functions, so is space $\mathcal{C}^\infty(\Omega)^\Lambda$, when considered with componentwise operations.

Let $\mathcal{S}$ be a subalgebra of $\mathcal{C}^\infty(\Omega)^\Lambda$ and $\mathcal{I}$ an ideal in $\mathcal{S}$. The quotient algebra

$$\mathcal{A}(\Omega) = \mathcal{S}/\mathcal{I} \tag{1.8}$$

can, to some extent, be interpreted as an algebra of *generalized functions*. The algebras constructed in (1.8) are further particularised by introducing the following natural requirements. Firstly, we require that the space $\mathcal{C}^\infty(\Omega)$ is embedded as a subalgebra into $\mathcal{A}(\Omega)$. Furthermore, the partial differential operators $D^p : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$, should be extendable to $\mathcal{A}(\Omega)$ in a canonical way.

Concerning the first of these requirements, we note that

$$\Delta_\Lambda^\infty : \mathcal{C}^\infty(\Omega) \ni u \mapsto \Delta_\Lambda^\infty(u) \in \mathcal{C}^\infty(\Omega)^\Lambda, \tag{1.9}$$

where $\Delta_\Lambda^\infty(u)_\lambda = u$ for each $\lambda \in \Lambda$, defines a natural injective algebra homomorphism. Let

$$\mathcal{U}_\Lambda^\infty = \left\{ u = (u_\lambda)_{\lambda\in\Lambda} \left| \begin{array}{ll} \exists & v \in \mathcal{C}^\infty(\Omega) : \\ \forall & \lambda \in \Lambda \ : \\ & u_\lambda = v \end{array} \right. \right\}.$$

Then the mapping Δ_Λ^∞ maps $\mathcal{C}^\infty(\Omega)$ bijectively onto $\mathcal{U}_\Lambda^\infty$. It is clear that

$$\mathcal{E}_\infty : \mathcal{C}^\infty(\Omega) \ni u \mapsto \Delta_\Lambda^\infty(u) + \mathcal{I} \in \mathcal{A}(\Omega) \tag{1.10}$$

defines an injective algebra homomorphism, provided only that the subalgebra $\mathcal{S}$ of $\mathcal{C}^\infty(\Omega)^\Lambda$ and the ideal $\mathcal{I}$ in $\mathcal{S}$ satisfy the neutrix condition

$$\mathcal{U}_\Lambda^\infty(\Omega) \subseteq \mathcal{S}, \ \ \mathcal{U}_\Lambda^\infty(\Omega) \cap \mathcal{I} = \mathcal{O}, \tag{1.11}$$

where $\mathcal{O}$ denotes null ideal in $\mathcal{C}^\infty(\Omega)^\Lambda$. The neutrix condition (1.11) determines to a good extent the structure of ideals $\mathcal{I}$ which play a crucial role in the stability, generality and exactness properties of algebras of generalized functions, see [51, Chapters 1, 3 and 6].

Let us now consider the second requirement, namely, the extension of the partial differential operators

$$D^p : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega), \quad p \in \mathbb{N}^n \tag{1.12}$$

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to the algebras of generalised functions constructed in (1.8). In this regard, we note that each such partial differential operator extends to a linear mapping

$$D^p : \mathcal{C}^\infty(\Omega)^\Lambda \ni (u_\lambda) \mapsto (D^p u_\lambda) \in \mathcal{C}^\infty(\Omega)^\Lambda, \quad p \in \mathbb{N}^n.$$

An algebra (1.8) is called a *differential algebra of generalize functions* on Ω provided that the differential operators D^p, $p \in \mathbb{N}^n$, on $\mathcal{C}^\infty(\Omega)$ extend in a canonical way to linear mappings

$$D^p : \mathcal{A}(\Omega) \longrightarrow \mathcal{A}(\Omega)$$

that satisfy the Leibnitz rule

$$D^p(uv) = \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q} u D^q v \tag{1.13}$$

for all $u, v \in \mathcal{A}(\Omega)$.

For an algebra $\mathcal{A}(\Omega)$ given in (1.8), such an extension is possible whenever the subalgebra $\mathcal{S}$ and the ideal $\mathcal{I}$ satisfy

$$D^p(\mathcal{S}) \subseteq \mathcal{S} \quad \text{and} \quad D^p(\mathcal{I}) \subseteq \mathcal{I}, \quad p \in \mathbb{N}^n. \tag{1.14}$$

In this case, we define the mappings $D^p : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ as

$$D^p : \mathcal{A}(\Omega) \ni (u_\lambda) + \mathcal{I} \to (D^p u_\lambda) + \mathcal{I} \in \mathcal{A}(\Omega). \tag{1.15}$$

If in addition to (1.14), the algebra $\mathcal{S}$ and ideal $\mathcal{I}$ also satisfy (1.11) then the mappings (1.15) are extensions of the usual partial differential operators acting on $\mathcal{C}^\infty(\Omega)$, in the sense that the diagram

$$\begin{array}{ccc} \mathcal{A}(\Omega) & \xrightarrow{\quad D^p \quad} & \mathcal{A}(\Omega) \\ \uparrow & & \uparrow \\ \mathcal{C}^\infty(\Omega) & \xrightarrow[\quad D^p \quad]{} & \mathcal{C}^\infty(\Omega) \end{array} \tag{1.16}$$

commutes for all $p \in \mathbb{N}^n$.

As mentioned, differential algebras of generalized functions may provide a setting for the multiplication of distributions. In this context, multiplications of distributions is realized through a linear injection

$$E : \mathcal{D}'(\Omega) \ni T \mapsto E(T) \in \mathcal{A}(\Omega). \tag{1.17}$$

Although the product ST of given distributions S and T may not be defined as a distribution, due to the Schwartz impossibility result, the product may be formed in any differential algebra of generalized functions $\mathcal{A}(\Omega)$ that admits an embedding (1.17). In particular, the product ST of distributions S and T may be defined as the product of $E(S)$ and $E(T)$ in $\mathcal{A}(\Omega)$.

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1.4 Embedding $\mathcal{D}'(\Omega)$ into differential algebras

In this section we discuss briefly the details of the embedding (1.17) of distributions into a differential algebra of generalized functions. Recall [49, 51, 63] that there exists a vector space

$$\mathcal{V}_\Lambda^\infty \subseteq \mathcal{C}^\infty(\Omega)^\Lambda \tag{1.18}$$

and a linear surjection

$$\mathcal{L} : \mathcal{V}_\Lambda^\infty \ni s \ \mapsto \mathcal{L}(s) = T \in \mathcal{D}'(\Omega) \tag{1.19}$$

such that

$$\mathcal{U}_\Lambda^\infty \subseteq \mathcal{V}_\Lambda^\infty \tag{1.20}$$

with (1.19) an extension of the mapping

$$\mathcal{U}_\Lambda^\infty \ni (u_n)_\lambda = (u) \mapsto T_u \in \mathcal{D}'(\Omega) \tag{1.21}$$

where the distribution T_u is defined as

$$T_u : \mathcal{D}(\Omega) \ni \psi \mapsto \int_\Omega u(x)\psi(x)dx \in \mathbb{R}. \tag{1.22}$$

If we denote by

$$\mathcal{W}_\Lambda^\infty \tag{1.23}$$

the kernel of the mapping (1.19),

$$q_{\mathcal{D}'(\Omega)} : \mathcal{V}_\Lambda^\infty / \mathcal{W}_\Lambda^\infty \ni u + \mathcal{W}_\Lambda^\infty \mapsto T \in \mathcal{D}'(\Omega) \tag{1.24}$$

is a vector space isomorphism. In this case, the pair $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ is a *representation* of the vector space $\mathcal{D}'(\Omega)$ of distributions on Ω.

If, in addition, the vector subspaces $\mathcal{V}_\Lambda^\infty$ and $\mathcal{W}_\Lambda^\infty$ are such that

$$D^p(\mathcal{V}_\mathbb{N}^\infty) \subseteq \mathcal{V}_\mathbb{N}^\infty, \quad D^p(\mathcal{W}_\mathbb{N}^\infty) \subseteq \mathcal{W}_\mathbb{N}^\infty, \ p \in \mathbb{N}^n \tag{1.25}$$

and

$$D^p(\mathcal{L}(u)) = \mathcal{L}(D^p(u), \quad u \in \mathcal{V}_\mathbb{N}^\infty, \quad p \in \mathbb{N}^n \tag{1.26}$$

then the mappings

$$\mathcal{D}'(\Omega) \ni u + \mathcal{W}_\Lambda^\infty \mapsto D^p u + \mathcal{W}_\Lambda^\infty \in \mathcal{D}'(\Omega), \quad p \in \mathbb{N}^n \tag{1.27}$$

coincide with the distributional partial derivatives.

Definition 1.2. *The pair $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ is called a C^∞-smooth representation of distribution if (1.20) - (1.21) are satisfied. $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ is called a differentiable C^∞-smooth representation of distribution if, in addition, the condition (1.26) holds.*

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The following theorem gives a sufficient condition for embedding of distribution into an algebra of generalized functions.

Theorem 1.3. *[51]Let $\mathcal{S}$ be a subalgebra in $C^\infty(\Omega)^\Lambda$ and $\mathcal{I}$ an ideal in $\mathcal{S}$ such that the neutrix condition* (1.11) *is satisfied. Suppose that $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ is a C^∞-smooth representation of $\mathcal{D}'(\Omega)$, and $\mathcal{V}, \mathcal{W} \subseteq C^\infty(\Omega)^\Lambda$ are vector subspaces such that the diagram*

$$\begin{array}{ccccc} \mathcal{I} & \xrightarrow{\subset} & \mathcal{S} & \longrightarrow & C^\infty(\Omega)^\Lambda \\ \uparrow \subset & & \uparrow \subset & & \\ \mathcal{W} & \xrightarrow{\subset} & \mathcal{V} & & \\ \downarrow \subset & & \downarrow \subset & & \\ \mathcal{W}_\Lambda^\infty & \xrightarrow{\subset} & \mathcal{V}_\Lambda^\infty & & \end{array} \tag{1.28}$$

commutes. If

$$\mathcal{I} \cap \mathcal{V} = \mathcal{W} = \mathcal{W}_\Lambda^\infty \cap \mathcal{V} \tag{1.29}$$

and

$$\mathcal{W}_\Lambda^\infty + \mathcal{V} = \mathcal{V}_\Lambda^\infty, \tag{1.30}$$

then

$$\mathcal{D}'(\Omega) \ni s + \mathcal{W}_\Lambda^\infty \leftarrow\!\shortmid s + \mathcal{W} \in \mathcal{V}/\mathcal{W} \ni s + \mathcal{W} \mapsto s + \mathcal{I} \in \mathcal{A}(\Omega) \tag{1.31}$$

is a linear injection. If $\mathcal{U}_\Lambda^\infty \subseteq \mathcal{V}$, then the mapping E is an embedding of differential algebras, when restricted to $C^\infty(\Omega) \subset \mathcal{D}'(\Omega)$.

Remark 1.4. We note the following regarding Theorem 1.3.

1. In (1.31), the mapping $\mathcal{V}/\mathcal{W} \ni s + \mathcal{W} \mapsto s + \mathcal{W}_\Lambda^\infty \in \mathcal{D}'(\Omega)$ is an isomorphism of vector spaces, while $\mathcal{V}/\mathcal{W} \ni s + \mathcal{W} \mapsto s + \mathcal{I} \in \mathcal{A}(\Omega)$ is a linear injection. Therefore (1.31) does indeed define a linear injection $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{A}(\Omega)$.

2. It is important to note that the structure of the commutative diagram (1.28) is not only sufficient for the existence of the embedding (1.17) but also necessary in the following sense. Suppose $\mathcal{A}(\Omega) = \mathcal{S}/\mathcal{I}$ is an algebra of generalized functions, with $\mathcal{S}$ and $\mathcal{I}$ satisfying the neutrix condition (1.11), and $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ is a C^∞-smooth representation of $\mathcal{D}'(\Omega)$. Assume that $\mathcal{D}'(\Omega) \subset \mathcal{A}(\Omega)$ in the sense that the diagram

$$\begin{array}{ccc} \mathcal{V} \ni v & \longrightarrow & v + \mathcal{W}_\Lambda^\infty \in \mathcal{D}'(\Omega) \\ & \searrow & \downarrow \\ & & v + \mathcal{I} \in \mathcal{A}(\Omega) \end{array} \tag{1.32}$$

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commutes for some vector space

$$\mathcal{V} \subseteq \mathcal{S} \cap \mathcal{V}_\Lambda^\infty.$$

By setting $\mathcal{W} = \mathcal{I} \cap \mathcal{V}$, we obtain the commutative diagram (1.28).

In view of Theorem 1.3, and in particular the necessary and sufficient structure of the commutative diagram (1.28), the following definitions are introduced, see for instance [51].

Definition 1.5. *Let $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ be a $\mathcal{C}^\infty$-smooth representation of the distributions. Then $(\mathcal{S}, \mathcal{I}, \mathcal{V}, \mathcal{W})$ is called a* regularisation *of $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ if the following conditions are satisfied.*

1. *$\mathcal{S}$ is a subalgebra of $\mathcal{C}^\infty(\Omega)^\Lambda$ and $\mathcal{I}$ is an ideal in $\mathcal{S}$ that satisfy the neutrix condition (1.11).*

2. *$\mathcal{W} \subset \mathcal{V} \subseteq C^\infty(\Omega)^\Lambda$ are vector spaces such that $\mathcal{W} = \mathcal{I} \cap \mathcal{V} = \mathcal{W}_\Lambda^\infty \cap \mathcal{V}$ and $\mathcal{W}_\Lambda^\infty + \mathcal{V} = \mathcal{V}_\Lambda^\infty$.*

If, in addition, we have $\mathcal{U}_\Lambda^\infty \subseteq \mathcal{V}$, then $(\mathcal{S}, \mathcal{I}, \mathcal{V}, \mathcal{W})$ is called C^∞-smooth regular.

Definition 1.6. *An ideal $\mathcal{I}$ in a subalgebra $\mathcal{S}$ of $\mathcal{C}^\infty(\Omega)^\Lambda$ is called* regular *if there exists a $\mathcal{C}^\infty$-smooth representation $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ of $\mathcal{D}'(\Omega)$ and vector spaces $\mathcal{W} \subset \mathcal{V} \subseteq C^\infty(\Omega)^\Lambda$ so that $(\mathcal{S}, \mathcal{I}, \mathcal{V}, \mathcal{W})$ is a regularisation of $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$. If, in addition $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ is C^∞-smooth regular, then $\mathcal{I}$ is called C^∞-smooth regular.*

As mentioned, see also [51], the neutrix condition (1.11) determines, to a good extent, the structure of ideals $\mathcal{I}$ related to the stability, generality and exactness properties of algebras of generalized functions. In particular, (1.11) characterizes the regular ideals, among all *cofinal invariant* ideals in $\mathcal{I}$ in $C^\infty(\Omega)^\mathbb{N}$.

Definition 1.7. *[49, page 81] An ideal $\mathcal{I}$ in $C^\infty(\Omega)^\mathbb{N}$ is cofinal invariant if, for all $w \in C^\infty(\Omega)^\mathbb{N}$ we have*

$$\left(\begin{array}{ll} \exists & w' \in \mathcal{I},\ \ \mu \in \mathbb{N} \\ \forall & \nu \in \mathbb{N}, \nu \geq \mu : \\ & w_\nu = w'_\nu \end{array} \right) \Longrightarrow w \in \mathcal{I}.$$

A characterization of C^∞- regular ideals is given in the following

Theorem 1.8. *A cofinal ideal $\mathcal{I}$ in $C^\infty(\Omega)^\mathbb{N}$ is C^∞-regular if and only if $\mathcal{I} \cap \mathcal{U}_\mathbb{N}^\infty = \mathcal{O}$*

A large class of regular ideals is given by the so-called vanishing ideals, see for instance [51, Section 6.1]

Definition 1.9. *An ideal $\mathcal{I}$ in $\mathcal{C}^\infty(\Omega)^\mathbb{N}$ is* vanishing *if*

$$\begin{array}{lll} \forall & u = (u_\lambda) \in \mathcal{I},\ \ \mu \in \mathbb{N} & : \\ \exists & \nu \in \mathbb{N},\ \ \nu \geq \mu,\ \ x \in \Omega & : \\ & u_\nu(x) = 0. & \end{array}$$

Theorem 1.10. *Every vanishing ideal in $C^\infty(\Omega)^\mathbb{N}$ is regular.*

Combining Theorems 1.8 and 1.10, we have the following.

Theorem 1.11. *Every cofinal invariant, vanishing ideal in $\mathcal{C}^\infty(\Omega)^\mathbb{N}$ is off-diagonal, hence $\mathcal{C}^\infty$-smooth regular.*

According to [51, Proposition 2, p 238], an ideal in $\mathcal{C}^\infty(\Omega)^\mathbb{N}$ is vanishing if and only if it is a proper ideal. Therefore every proper ideal in $\mathcal{C}^\infty(\Omega)^\mathbb{N}$ is $\mathcal{C}^\infty$-smooth regular. Consequently, there is a lot of freedom in terms of the way in which the distributions can be embedded into differential algebras of generalised functions. However, it should be noted that the linear embedding

$$E : \mathcal{D}'(\Omega) \to \mathcal{A}(\Omega)$$

of distribution into a differential algebra $\mathcal{A}(\Omega)$ will in general not preserve the differential structure of $\mathcal{D}'(\Omega)$. That is, it may happen that for some $T \in \mathcal{D}'(\Omega)$ the identity

$$D^p(E(T)) = E(D^p(T)), \quad p \in \mathbb{N}^n \tag{1.33}$$

does not hold, where the derivative on the left of (1.33) is taken in the algebra $\mathcal{A}(\Omega)$, while that on the right of (1.33) is the distributional derivative. This is the case even if the C^∞-smooth representation $(\mathcal{V}_\mathbb{N}^\infty, \mathcal{L})$ is differentiable. Furthermore, different embeddings of $\mathcal{D}'(\Omega)$ into an algebra $\mathcal{A}(\Omega)$ may not determine the same differential structure on $\mathcal{D}'(\Omega)$. A sufficient condition for the identity (1.33) to hold is given in the following, see [51,].

Theorem 1.12. *Let $(\mathcal{S}, \mathcal{I}, \mathcal{V}, \mathcal{W})$ be a $\mathcal{C}^\infty$-smooth regular regularisation of a differentiable representation $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ of the distributions. If $\mathcal{S}$ and $\mathcal{I}$ satisfy (1.14), and*

$$D^p(\mathcal{V}) \subseteq \mathcal{V}, \;\; D^p(\mathcal{W}) \subseteq \mathcal{W}, \quad p \in \mathbb{N},$$

then $D^pE(T) = E(D^pT)$ for all $T \in \mathcal{D}'(\Omega)$ and $p \in \mathbb{N}$, where $E : \mathcal{D}'(\Omega) \to \mathcal{S}/\mathcal{I}$ is the linear injection given by Theorem 1.3.

As mentioned, there is a lot of freedom in the way in which distributions may be embedded into differential algebras of generalised functions. However, there is an essential limitation on such embeddings. Indeed, an embedding of $\mathcal{D}'(\Omega)$ into a differential algebra cannot, at the same time, preserve both the algebraic structure of $C(\Omega)$ and the differential structure of $\mathcal{D}'(\Omega)$. This limitation is due to a basic conflict between the trio of insufficient smoothness, multiplication and differentiability, see [51]. The basic result in this regard is given in the following

Theorem 1.13. *Let $\mathcal{A}(\Omega) = \mathcal{S}/\mathcal{I}$ be a differential algebra, and*

$$E : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}(\Omega)$$

a linear injection. Then one of the following holds

(a) $\exists \quad T \in \mathcal{D}'(\Omega), p \in \mathbb{N}^n$
$\quad D^p(E(T)) \neq E(D^pT)$

(b) $\exists \quad u, v \in C(\Omega)$
$\quad E(uv) \neq E(u)E(v)$

One way in which this limitation can be overcome is through the use of *chains of algebras of generalised functions*, see for instance [51, Chapter 6]. In the next section, we recall briefly the main points regarding such chains

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1.5 Chains of algebras of generalised functions

As mentioned in Section 1.4, in particular Theorem 1.3, it is possible to construct an algebra of generalised functions $\mathcal{A}(\Omega) = \mathcal{S}/\mathcal{I}$ such that the diagram

$$\begin{array}{ccc} C^\infty(\Omega) & \xrightarrow{\subset} & \mathcal{D}'(\Omega) \\ & \searrow^{\mathcal{E}_\infty} & \downarrow E \\ & & \mathcal{A}(\Omega) \end{array} \tag{1.34}$$

commutes, with $E : \mathcal{D}'(\Omega) \to \mathcal{A}(\Omega)$ a linear injection, and $\mathcal{E}_\infty$ the canonical injective algebra homomorphism (1.10). In particular, this is the case when the ideal $\mathcal{I}$ is $\mathcal{C}^\infty$-smooth regular. However, Theorem 1.13 shows that it is not possible to construct a differential algebra $\mathcal{A}(\Omega)$ admitting a linear embedding

$$E : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}(\Omega)$$

which is an algebra homomorphism when restricted to $\mathcal{C}^0(\Omega) \subset \mathcal{D}'(\Omega)$ and preserves distributional derivatives.

More generally, as will be explained at the end of this section, it is not possible to construct a differential algebra $\mathcal{A}(\Omega)$ admitting a linear embedding

$$E : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}(\Omega)$$

and the algebra homomorphism

$$q_m : C^m(\Omega) \longrightarrow \mathcal{A}(\Omega)$$

so that the diagrams

$$\begin{array}{ccc} \mathcal{A}(\Omega) & \xrightarrow{D^p} & \mathcal{A}(\Omega) \\ E \uparrow & & \uparrow E \\ \mathcal{D}'(\Omega) & \xrightarrow{D^p} & \mathcal{D}'(\Omega) \end{array} \tag{1.35}$$

and

$$\begin{array}{ccc} C^m(\Omega) & \xrightarrow{\subset} & \mathcal{D}'(\Omega) \\ & \searrow^{q_m} & \downarrow E \\ & & \mathcal{A}(\Omega) \end{array} \tag{1.36}$$

commute.

In order to overcome the mentioned limitation of the embedding of $\mathcal{D}'(\Omega)$ into a differential algebra $\mathcal{A}(\Omega)$, the concept of a chain of algebras of generalized functions were introduced, see [49, 50, 51].

Definition 1.14. *For $l \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, let $\mathcal{A}^l(\Omega)$ be a commutative and unital algebra. Assume that, for all $k, l \in \mathbb{N}$ such that $k \leq l$, there exists an algebra homomorphism*

$$\gamma_k^l : \mathcal{A}^l(\Omega) \to \mathcal{A}^k(\Omega).$$

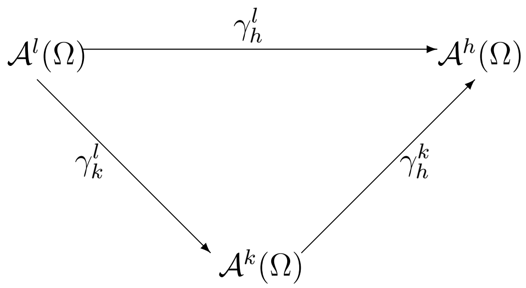
Assume that for $l > 0$ and $p \in \mathbb{N}^n$, with $|p| \leq l$ and $|p| + k \leq l$, there exists a linear differential operator

$$D^p : \mathcal{A}^l(\Omega) \to \mathcal{A}^k(\Omega)$$

that satisfies the Leibnitz rule

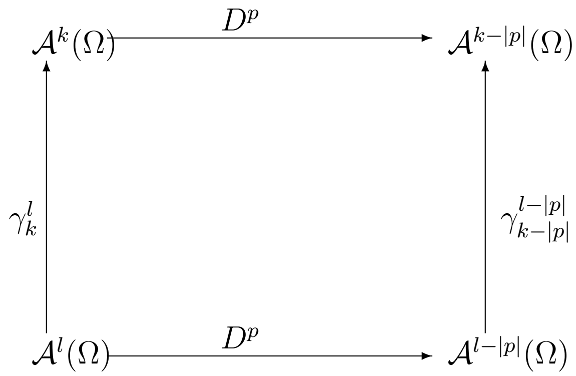
$$D^p(uv) = \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q} u D^q v \tag{1.37}$$

where $D^p, D^q, D^{p-q} : \mathcal{A}^l(\Omega) \longrightarrow \mathcal{A}^k(\Omega)$. If the diagram



(1.38)

commutes for all $h, k, l \in \overline{\mathbb{N}}$ such that $h \leq k \leq l$, then $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \ : \ k, l \in \overline{\mathbb{N}},\ k \leq l\}$ is called a chain of algebras of generalised functions. *If, in addition, the diagram*



(1.39)

commutes for all $l \geq k > 0$ and $p \in \mathbb{N}^n$, $|p| \leq k$, we call the chain $\mathbf{A}$ differential.

In this section we outline how such chains of algebras of generalised functions may be constructed. We give two possible ways in which such chains may be constructed, in particular chains that contain the distributions.

1.5.1 First method for constructing chains of algebras of generalised functions

The first is a simple extension of the construction of differential algebras of generalised functions presented in Section 1.2.

Suppose given $l \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let Λ be an infinite index set, and let

$$\mathcal{C}^l(\Omega)^\Lambda = \left\{ u = (u_\lambda)_{\lambda \in \Lambda} \;\middle|\; \begin{array}{l} \forall \quad \lambda \in \Lambda \;: \\ \quad u_\lambda \in \mathcal{C}^l(\Omega). \end{array} \right\} \tag{1.40}$$

Since the space $\mathcal{C}^l(\Omega)$ of l-times continuously differentiable functions on Ω is a commutative algebra with unit element, with respect to the usual pointwise operations on functions, the set $\mathcal{C}^l(\Omega)^\Lambda$ is also a commutative algebra with unit element, when considered with the termwise operations on sequences of functions. For a subalgebra $\mathcal{S}^l$ of $\mathcal{C}^l(\Omega)^\Lambda$, and a proper ideal $\mathcal{I}^l$ in $\mathcal{S}^l$, the quotient algebra

$$\mathcal{A}^l(\Omega) = \mathcal{S}^l / \mathcal{I}^l \tag{1.41}$$

is a unital and commutative algebra. In view of the construction of differential algebras of generalised functions discussed in Section 1.2, we call the algebra $\mathcal{A}^l(\Omega)$ an *algebra of generalized functions* on Ω.

If for $k \leq l$, the inclusions

$$\mathcal{S}^l \subseteq \mathcal{S}^k, \quad \mathcal{I}^l \subseteq \mathcal{I}^k \tag{1.42}$$

hold, then

$$\gamma_k^l : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}^l \mapsto u + \mathcal{I}^k \in \mathcal{A}^k(\Omega) \tag{1.43}$$

defines an algebra homomorphism. Clearly, in this case the diagram (1.38) commutes for $h \leq k \leq l$.

Suppose further that, for $l > 0$ and $p \in \mathbb{N}^n$ with $|p| \leq l$ and $k \leq |p| \leq l$ we have

$$D^p(\mathcal{S}^l) \subseteq \mathcal{S}^k, \quad D^p(\mathcal{I}^l) \subseteq \mathcal{I}^k. \tag{1.44}$$

Then

$$D^p : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}^l \mapsto D^p(u) + \mathcal{I}^k \in \mathcal{A}^k(\Omega)$$

defines a linear differential operator that satisfies the Leibnitz rule (1.37). For $k \leq l$ and $p \in \mathbb{N}^n$ such that $|p| \leq k$, the diagram (1.39) commutes. Hence we have the following.

Theorem 1.15. *For each $l \in \overline{\mathbb{N}}$, let $\mathcal{S}^l$ be a subalgebra of $\mathcal{C}^l(\Omega)^\Lambda$ and $\mathcal{I}^l$ an ideal in $\mathcal{S}^l$. If (1.42) and (1.44) are satisfied, then $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \; : \; k, l \in \overline{\mathbb{N}}, \; k \leq l\}$, with $\mathcal{A}^l(\Omega) = \mathcal{S}^l / \mathcal{I}^l$ and γ_k^l defined by (1.43), is a differential chain of algebras of generalised functions.*

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Remark 1.16. A more general construction of a chain of algebras of generalised functions is the following. Suppose given a subalgebra $\mathcal{S}^l$ of $\mathcal{C}^l(\Omega)^\Lambda$ and an ideal $\mathcal{I}^l$ in $\mathcal{S}^l$, for $l \in \overline{\mathbb{N}}$. Assume that

$$\alpha_k^l : \mathcal{S}^l \to \mathcal{S}^k \tag{1.45}$$

is an algebra homomorphism, for $k \leq l$, such that

$$\alpha_k^l(\mathcal{I}^l) \subseteq \mathcal{I}^k, \tag{1.46}$$

then

$$\gamma_k^l : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}^l \mapsto \alpha_k^l(u) + \mathcal{I}^k \in \mathcal{A}^k(\Omega)$$

defines an algebra homomorphism, where $\mathcal{A}^l(\Omega) = \mathcal{S}^l/\mathcal{I}^l$. If, in addition to (1.46), we have

$$\alpha_h^l = \alpha_h^k \circ \alpha_k^l$$

whenever $h \leq k \leq l$, then

$$\gamma_h^l = \gamma_h^k \circ \gamma_k^l, \quad h \leq k \leq l$$

so that the diagram (1.38) commutes. If the inclusions (1.44) holds, then

$$\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \ : \ k, l \in \overline{\mathbb{N}}, \ k \leq l\}$$

is a chain of algebras of generalised functions. If, in addition, the diagram

$$\begin{array}{ccc}
\mathcal{S}^k & \xrightarrow{\quad D^p \quad} & \mathcal{S}^{k-|p|} \\
\Big\uparrow \alpha_k^l & & \Big\uparrow \alpha_{k-|p|}^{l-|p|} \\
\mathcal{S}^l & \xrightarrow{\quad D^p \quad} & \mathcal{S}^{l-|p|}
\end{array}$$

commutes for all $k, l \in \overline{\mathbb{N}}$ with $k \leq l$ and $p \in \mathbb{N}^n$, $|p| \leq k$, then the chain

$$\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \ : \ k, l \in \overline{\mathbb{N}}, \ k \leq l\}$$

is differential.

Let us now consider the problem of embedding the distributions into differential chains of algebras of generalised functions given by Theorem 1.15. We therefore assume that $\mathcal{A}^l(\Omega) = \mathcal{S}^l/\mathcal{I}^l$, with $\mathcal{S}^l$ and $\mathcal{I}^l$ satisfying (1.42) and (1.44) for each $l \in \overline{\mathbb{N}}$, so that $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \ : \ k, l \in \overline{\mathbb{N}}, \ k \leq l\}$ is a differential chain of algebras of generalised functions. We proceed from the particular to the general, considering first the embedding of $\mathcal{C}^\infty(\Omega)$ into $\mathcal{A}^l(\Omega)$, for $l \in \overline{\mathbb{N}}$. For each $l \in \overline{\mathbb{N}}$, let

$$\mathcal{S}_\infty^l = \mathcal{S}^l \cap \mathcal{C}^\infty(\Omega)^\Lambda, \quad \mathcal{I}_\infty^l = \mathcal{I}^l \cap \mathcal{C}^\infty(\Omega)^\Lambda \tag{1.47}$$

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so that $\mathcal{S}^l_\infty$ is a subalgebra in $\mathcal{C}^\infty(\Omega)^\Lambda$, and $\mathcal{I}^l_\infty$ is an ideal in $\mathcal{S}^l_\infty$. Since $\mathcal{I}^l_\infty \subseteq \mathcal{I}^l$ and $\mathcal{S}^l_\infty \subseteq \mathcal{S}^l$, it follows that

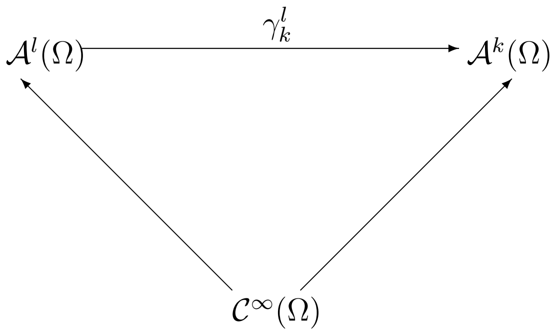
$$\mathcal{A}^l_\infty(\Omega) = \mathcal{S}^l_\infty/\mathcal{I}^l_\infty \ni s + \mathcal{I}^l_\infty \mapsto s + \mathcal{I}^l \in \mathcal{A}^l(\Omega) \tag{1.48}$$

defines an algebra homomorphism. Furthermore, it follows from (1.47) that (1.48) is injective. The following is therefore immediate.

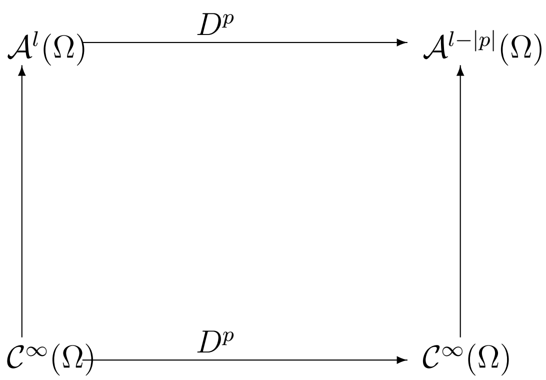
Theorem 1.17. *If $\mathcal{S}^l_\infty$ and $\mathcal{I}^l_\infty$ satisfy the neutrix condition (1.11), then*

$$\mathcal{C}^\infty(\Omega) \ni u \mapsto \Delta(u) + \mathcal{I}^l_\infty \in \mathcal{A}^l_\infty(\Omega) \ni \Delta(u) + \mathcal{I}^l_\infty \mapsto \Delta(u) + \mathcal{I}^l \in \mathcal{A}^l(\Omega)$$

defines an injective algebra homomorphism so that the diagram



commutes for all $k, l \in \overline{\mathbb{N}}$, $k \leq l$. If, in addition, $\mathcal{S}^l_\infty$ and $\mathcal{I}^l_\infty$ satisfy (1.14) for all $p \in \mathbb{N}^n$, then the diagram



commutes for all $l \in \overline{\mathbb{N}}$ and $p \in \mathbb{N}^n$, $|p| \leq l$.

We next consider the embedding of $\mathcal{C}^l(\Omega)$ into $\mathcal{A}^l(\Omega)$, for $l \in \overline{\mathbb{N}}$. As can be expected, the existence of an algebra embedding

$$\mathcal{C}^l(\Omega) \hookrightarrow \mathcal{A}^l(\Omega), \ l \in \overline{\mathbb{N}}$$

is determined by the neutrix condition

$$\mathcal{U}^l_\Lambda(\Omega) \subseteq \mathcal{S}^l, \qquad \mathcal{U}^l_\Lambda(\Omega) \cap \mathcal{I}^l = \{0\} \tag{1.49}$$

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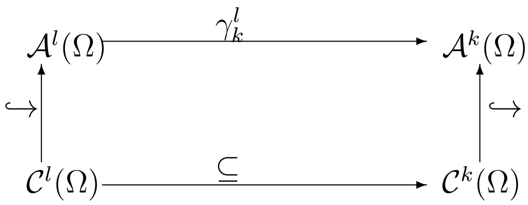
where

$$\mathcal{U}^l_\Lambda(\Omega) = \left\{ u = (u_\lambda)_{\lambda\in\Lambda} \;\middle|\; \begin{array}{ll} \exists & v \in \mathcal{C}^l(\Omega) : \\ \forall & \lambda \in \Lambda \;: \\ & u_\lambda = v \end{array} \right\}.$$

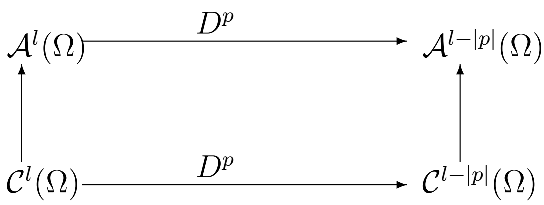
Theorem 1.18. *Suppose that (1.49) is satisfied for each $l \in \overline{\mathbb{N}}$. Then*

$$\mathcal{C}^l(\Omega) \ni u \mapsto \Delta(u) + \mathcal{I}^l \in \mathcal{A}^l(\Omega) \tag{1.50}$$

defines an injective algebra homomorphism for each $l \in \overline{\mathbb{N}}$. Furthermore, the diagrams

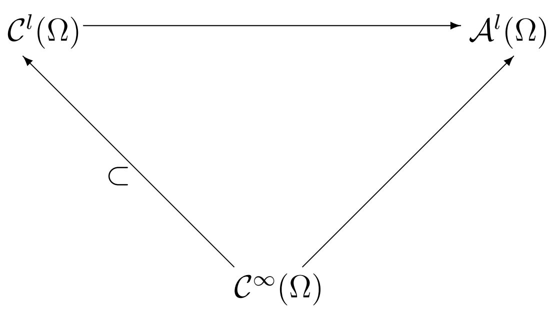


and



commutes for all $l, k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N}^n$, $|p| \leq l$.

Remark 1.19. It should be noted that condition (1.49) implies that $\mathcal{S}^l_\infty$ and $\mathcal{I}^l_\infty$ satisfy (1.11). Moreover, in this case the diagram



commutes for all $l \in \overline{\mathbb{N}}$.

Lastly, we consider the embedding of distributions into chains of algebras of generalised functions. The main result in this regard is a simple extension of Theorem 1.3. First we briefly recall the following main points concerning the embedding of $\mathcal{D}'(\Omega)$ into an algebra of generalised functions.

Given any $l \in \overline{\mathbb{N}}$, there exists, see [49, 51], a vector subspace

$$\mathcal{V}^l_\Lambda \subseteq \mathcal{C}^l(\Omega)^\Lambda \tag{1.51}$$

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and a linear surjection

$$\mathcal{L}_l : \mathcal{V}^l_\Lambda \ni u = (u_\lambda) \ \mapsto \mathcal{L}_l(u) = T \in \mathcal{D}'(\Omega) \tag{1.52}$$

such that $\mathcal{U}^l_\Lambda \subseteq \mathcal{V}^l_\Lambda$, with (1.52) an extension of the mapping

$$\mathcal{U}^l_\Lambda \ni (u_\lambda)_{\lambda\in\Lambda} \mapsto u \in \mathcal{C}^l(\Omega) \ni u \mapsto T_u \in \mathcal{D}'(\Omega)$$

where the distribution T_u is defined as

$$T_u : \mathcal{D}(\Omega) \ni \psi \mapsto \int_\Omega u(x)\psi(x)dx \in \mathbb{R}.$$

Let $\mathcal{W}^l_\Lambda$ denote the kernel of the mapping (1.52). Then we have a vector space isomorphism

$$q^l_{\mathcal{D}'(\Omega)} : \mathcal{V}^l_{\mathbb{N}}/\mathcal{W}^l_{\mathbb{N}} \ni (u) + \mathcal{W}^l_{\mathbb{N}} \mapsto T \in \mathcal{D}'(\Omega).$$

Thus we have the representation of distribution

$$\mathcal{D}'(\Omega) = \mathcal{V}^l_{\mathbb{N}}/\mathcal{W}^l_{\mathbb{N}}. \tag{1.53}$$

The pair $(\mathcal{V}^l_{\mathbb{N}}, \mathcal{L}_l)$ will be called a C^l-smooth representation of distribution if (1.51) - (1.53) are satisfied.

Theorem 1.20. *Let $\mathcal{S}^l$ be a subalgebra in $\mathcal{C}^l(\Omega)^\Lambda$ and $\mathcal{I}^l$ an ideal in $\mathcal{S}^l$ such that the neutrix condition (1.49) is satisfied, and let $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$ be a $\mathcal{C}^l$-smooth representation of $\mathcal{D}'(\Omega)$. Suppose further that $\mathcal{V}^l, \mathcal{W}^l \subseteq \mathcal{C}^l(\Omega)^{\mathbb{N}}$ are vector subspaces such that the diagram*

$$\begin{array}{ccccc}
\mathcal{I}^l & \xrightarrow{\subset} & \mathcal{S}^l & \longrightarrow & \mathcal{C}^l(\Omega)^\Lambda \\
\uparrow \subset & & \uparrow \subset & & \\
\mathcal{W}^l & \xrightarrow{\subset} & \mathcal{V}^l & & \\
\downarrow \subset & & \downarrow \subset & & \\
\mathcal{W}^l_\Lambda & \xrightarrow{\subset} & \mathcal{V}^l_\Lambda & &
\end{array} \tag{1.54}$$

commutes. If, in addition, the identities

$$\mathcal{I}^l \cap \mathcal{V}^l = \mathcal{W}^l = \mathcal{W}^l_\Lambda \cap \mathcal{V}^l \tag{1.55}$$

and

$$\mathcal{W}^l_\Lambda + \mathcal{V}^l = \mathcal{V}^l_\Lambda \tag{1.56}$$

hold, then

$$\mathcal{D}'(\Omega) \ni s + \mathcal{W}^l_\Lambda \leftarrow\!\shortmid s + \mathcal{W}^l \in \mathcal{V}^l/\mathcal{W}^l \ni s + \mathcal{W}^l \mapsto s + \mathcal{I}^l \in \mathcal{A}^l(\Omega) \tag{1.57}$$

defines a linear injection $E^l : \mathcal{D}'(\Omega) \to \mathcal{A}^l(\Omega)$. Moreover, if $\mathcal{U}^l_\Lambda \subseteq \mathcal{V}^l$, then the mapping E^l is an algebra homomorphism, when restricted to $\mathcal{C}^l(\Omega) \subset \mathcal{D}'(\Omega)$.

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The above theorem serves as motivation for the following definitions, see [51].

Definition 1.21. *Let* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$ *be a* $\mathcal{C}^l$*-smooth representation of* $\mathcal{D}'(\Omega)$*. The quadruple* $(\mathcal{W}^l, \mathcal{V}^l, \mathcal{I}^l, \mathcal{S}^l)$ *is called a regularization of* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$ *if and only if the following hold.*

(i) $\mathcal{S}^l$ *is a subalgebra of* $\mathcal{C}^l(\Omega)^\Lambda$ *and* $\mathcal{I}^l$ *an ideal in* $\mathcal{S}^l$ *so that the neutrix condition (1.49) holds.*

(ii) $\mathcal{W}^l, \mathcal{V}^l$ *are vector subspaces of* $\mathcal{C}^l(\Omega)^\Lambda$*.*

(iii) The diagram (1.54) *commutes.*

(iv) $\mathcal{I}^l \cap \mathcal{V}^l = \mathcal{W}^l = \mathcal{W}^l_\Lambda \cap \mathcal{V}^l$*.*

(v) $\mathcal{W}^l_\Lambda + \mathcal{V}^l = \mathcal{V}^l_\Lambda$*.*

If, in addition to (i) to (v), the inclusion $\mathcal{U}^l_\Lambda \subset \mathcal{V}^l$ *holds, then* $(\mathcal{W}^l, \mathcal{V}^l, \mathcal{I}^l, \mathcal{S}^l)$ *is said to be a* $\mathcal{C}^l$*-smooth regularization of* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$*.*

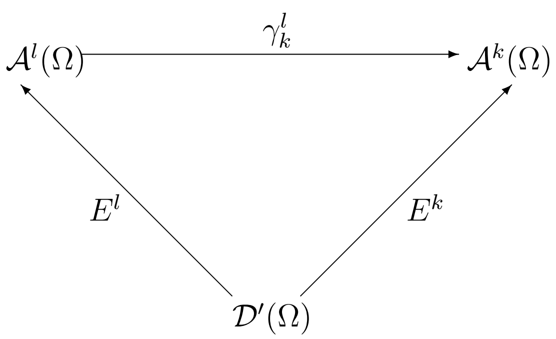
Definition 1.22. *An ideal* $\mathcal{I}^l$ *in* $\mathcal{S}^l$ *is regular if and only if there exist a* $\mathcal{C}^l$*-smooth representation* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$ *of* $\mathcal{D}'(\Omega)$*, vector subspaces* $\mathcal{W}^l,\ \mathcal{V}^l \subset \mathcal{C}^l(\Omega)^\Lambda$ *such that* $(\mathcal{W}^l, \mathcal{V}^l, \mathcal{I}^l, \mathcal{S}^l)$ *is a regularization. Furthermore, the ideal* $\mathcal{I}^l$ *is* $\mathcal{C}^l$*- smooth regular, if and only if* $(\mathcal{W}^l, \mathcal{V}^l, \mathcal{I}^l, \mathcal{S}^l)$ *is a* $\mathcal{C}^l$*-smooth regularization of* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$*.*

Next we consider the issue of consistency of the embedding of distributions in algebras of generalised functions within a differential chain of algebras.

Theorem 1.23. *Suppose that* $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma^l_k) \ :\ k, l \in \overline{\mathbb{N}},\ k \leq l\}$ *is a differential chain of algebras of generalised functions, as given by Theorem 1.15. Assume that, for each* $l \in \overline{\mathbb{N}}$*,* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$ *is a* $\mathcal{C}^l$*-smooth representation of* $\mathcal{D}'(\Omega)$*, and* $(\mathcal{W}^l, \mathcal{V}^l, \mathcal{I}^l, \mathcal{S}^l)$ *is a regularisation of* $(\mathcal{V}^l_\Lambda, \mathcal{L}_l)$*. If for each* $k \leq l$ *the inclusions*

$$\mathcal{V}^l_\Lambda \subseteq \mathcal{V}^k_\Lambda,\ \mathcal{W}^l_\Lambda \subseteq \mathcal{W}^k_\Lambda,\ \ \mathcal{V}^l \subseteq \mathcal{V}^k,\ \mathcal{W}^l \subseteq \mathcal{W}^k$$

hold, then the diagram



commutes, with E^l *and* E^k *given by* (1.57)*. If each of the ideals* $\mathcal{I}^l$ *is* $\mathcal{C}^l$*- smooth regular, then* E^l *is an algebra homomorphism, when restricted to* $\mathcal{C}^l(\Omega) \subset \mathcal{D}'(\Omega)$*. If the inclusions*

$$D^p(\mathcal{W}^l_\Lambda) \subseteq \mathcal{W}^{l-|p|}_\Lambda,\ D^p(\mathcal{V}^l_\Lambda) \subseteq \mathcal{V}^{l-|p|}_\Lambda,\ D^p(\mathcal{W}^l) \subseteq \mathcal{W}^{l-|p|},\ D^p(\mathcal{V}^l) \subseteq \mathcal{V}^{l-|p|}.$$

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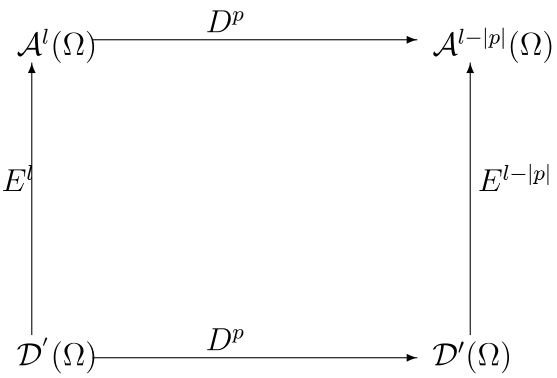
are satisfied for all $l \in \overline{\mathbb{N}}$ *and* $p \in \mathbb{N}^n$ *with* $|p| \leq l$, *then the diagram*



commutes.

1.5.2 An alternative way to construct chains of algebra containing the distributions

In this section we discuss an alternative method for constructing chains of algebra of generalized functions containing the distributions, see [49], [50], [51].

Definition 1.24. *A subset* $\mathcal{H} \subset \mathcal{C}^\infty(\Omega)^\Lambda$ *is said to be derivative invariant if* $D^p(\mathcal{H}) \subset \mathcal{H}$ *for all* $p \in \mathbb{N}$.

Clearly, $\mathcal{H} = \mathcal{C}^\infty(\Omega)^\Lambda$ is derivative invariant.

Suppose given a derivative invariant subalgebra $\mathcal{S}$ of $\mathcal{C}^\infty(\Omega)^\Lambda$, and a derivative invariant, $\mathcal{C}^\infty$-smooth regular ideal $\mathcal{I}$ in $\mathcal{S}$. In this case, there exists a representation $(\mathcal{V}_\Lambda^\infty, \mathcal{L})$ of $\mathcal{D}'(\Omega)$ and vector spaces $\mathcal{V}, \mathcal{W}$ of $\mathcal{C}^\infty(\Omega)^\Lambda$ so that $(\mathcal{W}, \mathcal{V}, \mathcal{I}, \mathcal{S})$ is a $\mathcal{C}^\infty$-smooth regularisation of $\mathcal{D}'(\Omega) = \mathcal{V}_\Lambda^\infty / \mathcal{W}_\Lambda^\infty$, where $\mathcal{W}_\Lambda^\infty$ is the kernel of $\mathcal{L}$. For each $l \in \mathbb{N}$, let

$$\mathcal{W}_l = \left\{ w \in \mathcal{W} \;\middle|\; \begin{array}{l} \forall \quad p \in \mathbb{N}^n, \ |p| \leq l \\ \quad D^p w \in \mathcal{W} \end{array} \right\}, \tag{1.58}$$

and let

$$\mathcal{S}_l(\mathcal{W}, \mathcal{V}) \tag{1.59}$$

denote the derivative invariant subalgebra in $\mathcal{S}$ generated by $\mathcal{W}^l + \mathcal{V}$. Furthermore, let

$$\mathcal{I}_l(\mathcal{W}, \mathcal{V}) \tag{1.60}$$

denote the ideal in $\mathcal{S}_l(\mathcal{W}, \mathcal{V})$ generated by $\mathcal{W}_l$. Lastly, we let

$$\mathcal{V}_l = \mathcal{W}_l \oplus \mathcal{V}. \tag{1.61}$$

That is,

$$\mathcal{V}_l = \mathcal{W}_l \oplus \mathcal{V} = \{(w, v) \ : \ w \in \mathcal{W}_l, \ v \in \mathcal{V}\}$$

with componentwise addition and scalar multiplication. With the above notations, we have the following, see [51, Section 6.4]

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Theorem 1.25. *If for each $l \in \overline{\mathbb{N}}$, we set $\mathcal{A}^l(\Omega) = \mathcal{S}_l(\mathcal{W},\mathcal{V})/\mathcal{I}_l(\mathcal{W},\mathcal{V})$, then the following hold.*

(i) For $k \leq l$

$$\gamma^l_k : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}_l(\mathcal{W},\mathcal{V}) \mapsto u + \mathcal{I}_k(\mathcal{W},\mathcal{V}) \in \mathcal{A}^k(\Omega)$$

defines an algebra homomorphism so that $\mathbf{A} = \{(\mathcal{A}^l, \mathcal{A}^k, \gamma^l_k) \ : \ k, l \in \overline{\mathbb{N}}, \ k \leq l\}$ is a differentiable chain of algebras of generalised functions.

(ii) For each $l \in \overline{\mathbb{N}}$ there exists a linear injection

$$E^l : \mathcal{D}'(\Omega) \to \mathcal{A}^l(\Omega).$$

When restricted to $\mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega)$, the map E^l is an algebra homomorphism.

(iii) For $h, k, l \in \overline{\mathbb{N}}$, with $h \leq k \leq l$, the diagram

$$\begin{array}{ccccc} & & \gamma^l_h & & \\ \mathcal{A}^l(\Omega) & \xrightarrow{\gamma^l_k} & \mathcal{A}^k(\Omega) & \xrightarrow{\gamma^k_h} & \mathcal{A}^h(\Omega) \\ \uparrow E^l & & \uparrow E^k & & \uparrow E^h \\ \mathcal{D}'(\Omega) & \xrightarrow{id} & \mathcal{D}'(\Omega) & \xrightarrow{id} & \mathcal{D}'(\Omega) \end{array} \tag{1.62}$$

commutes.

(iv) The differential operators $D^p : \mathcal{A}^l(\Omega) \to \mathcal{A}^k(\Omega)$, with $k + |p| \leq l$, extend the usual partial differential operators on $\mathcal{C}^\infty(\Omega)$.

1.5.3 Limitations of Embedding $\mathcal{D}'(\Omega)$ into chains of algebra of generalized functions

Let

$$\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma^l_k) | l, k \in \overline{\mathbb{N}}, \ k \leq l\}$$

be a differential chain of algebra of generalized functions. Based on [51, Theorem 9, pp. 68], we have the following

Theorem 1.26. *Assume that for each $l \in \overline{\mathbb{N}}$ there exists a linear injection*

$$E^l : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^l(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{A}^l(\Omega) & \xrightarrow{\gamma^l_k} & \mathcal{A}^k(\Omega) \\ {\scriptstyle E^l} \nwarrow & & \nearrow {\scriptstyle E^k} \\ & \mathcal{D}'(\Omega) & \end{array} \tag{1.63}$$

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commutes. If for some $l \geq 4$, the mapping $E^l : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^l(\Omega)$ is an algebra homomorphism when restricted to $C^{l-1}(\Omega) \subset \mathcal{D}'(\Omega)$, then the diagram

$$\begin{array}{ccc} \mathcal{A}^l(\Omega) & \xrightarrow{\quad D^p \quad} & \mathcal{A}^k(\Omega) \\ \uparrow E^l & & \uparrow E^k \\ \mathcal{D}'(\Omega) & \xrightarrow{\quad D^p \quad} & \mathcal{D}'(\Omega) \end{array} \tag{1.64}$$

does not commutes.

Let us now return to the issue of obtaining a differential algebra $\mathcal{A}(\Omega)$, a linear injection $E : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}(\Omega)$ and an algebra homomorphism $q_m : C^m(\Omega) \longrightarrow \mathcal{A}(\Omega)$ for some $m \in \mathbb{N}$, so that the diagrams (1.35) and (1.36) commute. Setting

$$\mathcal{A}^l(\Omega) = \mathcal{A}(\Omega), \quad E^l = E, \quad \gamma_k^l = id, \quad l, k \in \overline{\mathbb{N}}, \quad k \leq l, \tag{1.65}$$

we obtain a differential chain of algebra of generalized functions such that the diagrams (1.63) (1.64) commutes, and for each $l \in \mathbb{N}$, E^l restricted to $C^m(\Omega) \subset \mathcal{D}'(\Omega)$ is an algebra homomorphism. According to Theorem 1.26 this is impossible. Hence we have the following

Corollary 1.27. *If $\mathcal{A}(\Omega)$ is a differential algebra and $E : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}(\Omega)$ is a linear injection then one of the following holds*

1. *For some $p \in \mathbb{N}^n$, $T \in \mathcal{D}'(\Omega)$*

$$D^p(E(T)) \neq E(D^p(T)).$$

2. *For every $m \in \mathbb{N}$, there exists $u, v \in C^m(\Omega) \subset \mathcal{D}'(\Omega)$ so that*

$$E(uv) \neq E(u)E(v).$$

1.6 Nonlinear Partial Differential Operators

We now recall the way in which nonlinear partial differential operators may be defined on differential chains of algebras of generalised functions, see [51, Chapter 1, Sec. 13 & Chapter 6, Sec. 5]. In this regard, let $\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^l(\Omega), \gamma_k^l) \ : \ k, l \in \overline{\mathbb{N}}, \ k \leq l\}$ be a differential chain of algebras of generalised functions, given as in Theorem 1.15. That is, $\mathcal{A}^l(\Omega) = \mathcal{S}^l / \mathcal{I}^l$ where $\mathcal{S}^l$ is a subalgebra of $\mathcal{C}^l(\Omega)^\Lambda$ and $\mathcal{I}^l$ is an ideal in $\mathcal{S}^l$ so that (1.42) and (1.44) are satisfied.

Consider a nonlinear PDE

$$Tu(x) = f(x), \quad x \in \Omega \tag{1.66}$$

where $f \in \mathcal{C}^\infty(\Omega)$, and the nonlinear differential operator T is given by

$$Tu(x) = F(x, ..., D^p u(x), ...), \quad x \in \Omega, \quad |p| \leq m. \tag{1.67}$$

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Here $F : \Omega \times \mathbb{R}^M \to \mathbb{R}$, with M the cardinality of the set $\{p \in \mathbb{N}^n \ : \ |p| \leq m\}$, is $\mathcal{C}^\infty$-smooth. Clearly, the operator (1.67) defines mappings

$$T : \mathcal{C}^l(\Omega) \to \mathcal{C}^k(\Omega), \quad l, k \in \overline{\mathbb{N}}, \ k + m \leq l. \tag{1.68}$$

Applying each of the mappings in (1.68) termwise to elements of $\mathcal{C}^l(\Omega)^\Lambda$, we obtain extensions

$$T : \mathcal{C}^l(\Omega)^\Lambda \ni (u_\lambda) \mapsto (Tu_\lambda) \in \mathcal{C}^k(\Omega)^\Lambda, \quad l, k \in \overline{\mathbb{N}}, \ k + m \leq l. \tag{1.69}$$

of (1.68). Provided that the mappings (1.69) satisfy

$$T(\mathcal{S}^l) \subseteq \mathcal{S}^k, \quad l, k \in \overline{\mathbb{N}}, \ k + m \leq l$$

and

$$u, v \in \mathcal{S}^l, \ u - v \in \mathcal{I}^l \Rightarrow T(u) - T(v) \in \mathcal{I}^k, \quad l, k \in \overline{\mathbb{N}}, \ k + m \leq l,$$

it follows that

$$T : \mathcal{A}^l(\Omega) \ni u + \mathcal{I}^l \mapsto T(u) + \mathcal{I}^k \in \mathcal{A}^k(\Omega) \tag{1.70}$$

is well defined. If, in addition, the neutrix condition (1.49) is satisfied, then (1.70) is an extension of (1.68) in the sense that the diagram

$$\begin{array}{ccc} \mathcal{A}^l(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{A}^k(\Omega) \\ \uparrow & & \uparrow \\ \mathcal{C}^l(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{C}^k(\Omega) \end{array}$$

commutes for all $l, k \in \overline{\mathbb{N}}$ such that $k + m \leq l$.

Definition 1.28. *A generalised function $u + \mathcal{I}^\infty \in \mathcal{A}^\infty(\Omega)$ is a chain generalised solution of (1.66) in the chain* **A** *if*

$$T(\gamma_l^\infty(u + \mathcal{I}^\infty)) = \gamma_k^\infty(f + \mathcal{I}^\infty)$$

for all $k, l \in \overline{\mathbb{N}}$ so that $k + m \leq l$.

In [51, Chapter 7], chains of algebras of generalised functions, and chain generalised solutions in such chains, are applied to the resolution of nowhere dense singularities of weak solutions of certain polynomial nonlinear PDEs.

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Chapter 2

The order completion method

2.1 Solutions of continuous nonlinear PDEs through order completion

In the 1994 monograph [41], Oberguggenberger and Rosinger introduced a general and type-independent theory for the existence and basic regularity of solutions of large classes of linear and nonlinear PDEs. Their method is based essentially on the order structure of certain spaces of piecewise smooth functions, and the Dedekind order completion of these spaces. This theory was subsequently dramatically enriched through the introduction of convergence spaces, see [69, 70, 71, 72], resulting in a significant improvement of the regularity of generalised solutions of nonlinear PDEs, as well as a clarification of the structure of solutions obtained in [41].

In this section, we recall briefly the original results of Oberguggenberger and Rosinger [41]. In this regard, consider a nonlinear PDE

$$T(x,D)u(x) = h(x), x \in \Omega \tag{2.1}$$

of order m, where $\Omega \subseteq \mathbb{R}^n$ is an open set, $h : \Omega \to \mathbb{R}$ is continuous on Ω, and the differential operator $T(x,D)$ is defined through a jointly continuous function

$$F : \Omega \times \mathbb{R}^M \to \mathbb{R}$$

by the expression

$$T(x,D)u(x) = F(x, u(x), ..., D^p u(x), ...), \quad |p| \leq m, \tag{2.2}$$

where M is the cardinality of the set $\{p \in \mathbb{N}^n : |p| \leq m\}$. We assume that the mapping F and the right hand term h satisfy the simple condition

$$\begin{array}{l} \forall \quad x \in \Omega \ : \\ \quad h(x) \in \operatorname{int}\{F(x,\xi) \ : \ \xi = (\xi_p)_{|p|\leq m} \in \mathbb{R}^M\}. \end{array} \tag{2.3}$$

Under this condition, the following fundamental approximation result holds, [41].

Theorem 2.1. *Suppose that* (2.3) *holds. Then for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \subset \Omega$ closed and nowhere dense and $u^\varepsilon \in C^m(\Omega \setminus \Gamma_\varepsilon)$ such that*

$$h(x) - \varepsilon < T(x,D)u^\varepsilon(x) < h(x), \qquad x \in \Omega \setminus \Gamma_\varepsilon.$$

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The Order Completion Method consists of using Theorem 2.1, interpreted in appropriate function spaces, to construct generalised solutions of the PDE (2.1). This construction is summarized below, see [6, 7, 41, 50], for a detailed exposition.

In this regard, we consider the space $C^m_{nd}(\Omega)$ which is defined as follows: For any $m \in \overline{\mathbb{N}}$,

$$C^m_{nd}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \left| \begin{array}{ll} \exists & \Gamma_u \subset \Omega \text{ closed, nowhere dense :} \\ & u \in C^m(\Omega \backslash \Gamma_u) \end{array} \right. \right\}. \tag{2.4}$$

Clearly, $C^m(\Omega) \subseteq C^m_{nd}(\Omega)$, $m \in \overline{\mathbb{N}}$. Since the mapping F that defines $T(x,D)$ through (2.2) is continuous, it follows that if $u \in C^m(\Omega \backslash \Gamma_u)$ with $\Gamma_u \subset \Omega$ closed nowhere dense, then $T(x,D)u \in C^0(\Omega \backslash \Gamma_u)$. Hence the operator $T(x,D)$ induces a mapping

$$\overline{T}(x,D) : C^m_{nd}(\Omega) \longrightarrow C^0_{nd}(\Omega). \tag{2.5}$$

Define an equivalence relation on $C^0_{nd}(\Omega)$ as follows: For any $u, v \in C^0_{nd}(\Omega)$, we set

$$u \sim v \Longleftrightarrow \left\{ \begin{array}{lll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense :} & \\ & 1) & u, v \in C^0(\Omega \backslash \Gamma) \\ & 2) & u(x) = v(x),\ x \in \Omega \backslash \Gamma. \end{array} \right. \tag{2.6}$$

The quotient space $C^0_{nd}(\Omega)/\sim$ is denoted by $\mathcal{M}^0(\Omega)$.

On the space $C^m_{nd}(\Omega)$ define an equivalence relation in the following way: For any $u, v \in C^m_{nd}(\Omega)$, set

$$u \sim_{\overline{T}} v \Longleftrightarrow \overline{T}u \sim \overline{T}v. \tag{2.7}$$

The space $\mathcal{M}^m_T(\Omega)$ is defined as the quotient space $C^m_{nd}(\Omega)/\sim_{\overline{T}}$. The mapping (2.5) induces an injective mapping

$$\widehat{T} : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^0(\Omega) \tag{2.8}$$

in a canonical way, so that the diagram

$$\begin{array}{ccc} C^m_{nd}(\Omega) & \xrightarrow{\quad \overline{T} \quad} & C^0_{nd}(\Omega) \\ q_1 \Big\downarrow & & \Big\downarrow q_2 \\ \mathcal{M}^m_T(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{M}^0(\Omega) \end{array} \tag{2.9}$$

commutes, with q_1 and q_2 canonical quotient mappings associated with the equivalence relations (2.6) and (2.7), respectively. The mapping $\widehat{T}$ is defined as follows: If $U \in \mathcal{M}^m_T(\Omega)$ is the $\sim_{\overline{T}}$ equivalence class generated by $u \in C^m_{nd}(\Omega)$, then $\widehat{T}(U)$ is the $\sim$ equivalence class generated by $Tu \in C^0_{nd}(\Omega)$.

On the space $\mathcal{M}^0(\Omega)$, define a partial order as follows: For any $H, G \in \mathcal{M}^0(\Omega)$,

$$H \leq G \Longleftrightarrow \left\{ \begin{array}{lll} \exists & h \in H,\ g \in G,\ \Gamma \subset \Omega \text{ closed nowhere dense :} & \\ & (1) & h, g \in C^0(\Omega \backslash \Gamma) \\ & (2) & h \leq g \text{ on } \Omega \backslash \Gamma. \end{array} \right. \tag{2.10}$$

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On the space $\mathcal{M}^m_T(\Omega)$ define a partial order $\leq_{\widehat{T}}$ through the mapping $\widehat{T}$ as follows: For any $U, V \in \mathcal{M}^m_T(\Omega)$

$$U \leq_{\widehat{T}} V \Longleftrightarrow \widehat{T}U \leq \widehat{T}V \text{ in } \mathcal{M}^0(\Omega). \tag{2.11}$$

With respect to the partial orders (2.10) and (2.11) on $\mathcal{M}^0(\Omega)$ and $\mathcal{M}^m_T(\Omega)$, respectively, the mapping $\widehat{T}$ is an *order isomorphic embedding* [41]. That is, $\widehat{T}$ is injective and

$$\begin{array}{ll} \forall & U, V \in \mathcal{M}^m_T(\Omega) : \\ & U \leq_{\widehat{T}} V \Longleftrightarrow \widehat{T}U \leq \widehat{T}V. \end{array}$$

According to the McNeille Completion Theorem [37], see also [41, page 396], there exists unique Dedekind complete partially ordered sets $(\mathcal{M}^0(\Omega)^\sharp, \leq)$ and $(\mathcal{M}^m_T(\Omega)^\sharp, \leq_{\widehat{T}})$, and order isomorphic embeddings

$$i_{\mathcal{M}^m_T(\Omega)} : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^m_T(\Omega)^\sharp$$

and

$$i_{\mathcal{M}^0(\Omega)} : \mathcal{M}^0(\Omega) \longrightarrow \mathcal{M}^0(\Omega)^\sharp$$

so that the following *universal property* is satisfied: For every order isomorphic embedding

$$S : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$$

there exists a unique order isomorphic embedding $S^\sharp : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$ so that the diagram

$$\begin{array}{ccc} \mathcal{M}^m_T(\Omega) & \xrightarrow{\quad S \quad} & \mathcal{M}^0(\Omega) \\ \Big\downarrow i_{\mathcal{M}^m_T(\Omega)} & & \Big\downarrow i_{\mathcal{M}^0(\Omega)} \\ \mathcal{M}^m_T(\Omega)^\sharp & \xrightarrow{\quad S^\sharp \quad} & \mathcal{M}^0(\Omega)^\sharp \end{array} \tag{2.12}$$

commutes. In particular, there exists a unique order isomorphic embedding

$$\widehat{T}^\sharp : \mathcal{M}^m_T(\Omega)^\sharp \longrightarrow \mathcal{M}^0(\Omega)^\sharp,$$

which is an extension of the mapping $\widehat{T}$ in the sense that the diagram

$$\begin{array}{ccc} \mathcal{M}^m_T(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{M}^0(\Omega) \\ \Big\downarrow i_{\mathcal{M}^m_T(\Omega)} & & \Big\downarrow i_{\mathcal{M}^0(\Omega)} \\ \mathcal{M}^m_T(\Omega)^\sharp & \xrightarrow{\quad \widehat{T}^\sharp \quad} & \mathcal{M}^0(\Omega)^\sharp \end{array} \tag{2.13}$$

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commutes. In this way we arrive at an extension of the nonlinear PDE (2.1). Any solution $U^\sharp \in \mathcal{M}^m_T(\Omega)^\sharp$ of the equation

$$\widehat{T}^\sharp U^\sharp = f$$

is interpreted as a generalized solution of (2.1).

The main existence and uniqueness result for solutions of the PDE (2.1) is stated below, see [41, Theorem 5]

Theorem 2.2. *If the PDE* (2.1) *satisfies the condition* (2.3) *then there exists a unique solution* $U^\sharp \in \mathcal{M}^m_T(\Omega)^\sharp$ *of the equation*

$$\widehat{T}^\sharp U^\sharp = f.$$

As shown in [2], this generalized solution to the PDE (2.1) may be assimilated with so called Hausdorff continuous functions in $\mathbb{H}_{nf}(\Omega)$. Indeed, $\mathcal{M}^m_T(\Omega)^\sharp$ is order isomorphic to a subset of the space $\mathbb{H}_{nf}(\Omega)$ [4], [6]. A major deficiency of the OCM, is that the spaces of generalized functions containing solutions of a PDE may to a large extent depend on the particular nonlinear operator $T(x, D)$. Furthermore, there is no concept of generalized partial derivative for generalized functions. These issues were recently resolved by introducing suitable uniform convergence spaces, see [69, 68, 70, 72].

2.2 Structure and regularity of generalized solutions

In order to formulate the results obtained in [69, 70, 72] on the structure and regularity of generalised solution of nonlinear PDEs obtained through the OCM, we recall the necessary concepts from the theories of convergence spaces, order convergence on Riesz spaces and normal semi-continuous functions.

2.2.1 Convergence spaces

In this section we discuss some of the basic concepts related to convergence spaces, uniform convergence spaces and convergence vector spaces. For more details we refer the reader to [11], [13], [19], [23], [31], from where the concepts discussed here are taken.

Convergence spaces

A convergence space is a set together with a designated collection of filters. Recall that a filter $\mathcal{F}$ on a set X is a nonempty collection of subsets of X such that

(i) the empty set does not belong to $\mathcal{F}$,

(ii) for all $F \in \mathcal{F}$ and for all $G \subseteq X$, if $G \supseteq F$, then $G \in \mathcal{F}$,

(iii) if $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$.

A subset $\mathcal{B}$ of a filter $\mathcal{F}$ is a filter basis for $\mathcal{F}$ if each set in $\mathcal{F}$ contains a set in $\mathcal{B}$. The filter $\mathcal{F}$ is said to be generated by $\mathcal{B}$. We then write $\mathcal{F} = [\mathcal{B}]$. If $A \subseteq X$, the filter generated A is written as $[A]$. That is

$$[A] = \{F \subseteq X : F \supseteq A\}.$$

In particular for $x \in X$, $[x]$ is the filter generated by $\{x\}$. The filter $[x]$ is called the principal ultrafilter generated by x. Recall that a filter $\mathcal{G}$ on X is called an ultrafilter if $\mathcal{G} \not\subset \mathcal{F}$ for all filters $\mathcal{F}$ on X. The intersection of two filters $\mathcal{F}$ and $\mathcal{G}$ on X is defined as

$$\mathcal{F} \cap \mathcal{G} = [\{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}]$$

If $\mathcal{F}$ is a filter on X, and $\mathcal{G}$ is a filter on Y, then the product of the filters $\mathcal{F}$ and $\mathcal{G}$ is a filter on $X \times Y$ which is defined as

$$\mathcal{F} \times \mathcal{G} = [\{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}]$$

If filters $\mathcal{F}$ and $\mathcal{G}$ on X are such that $\mathcal{G} \subseteq \mathcal{F}$, then we say that $\mathcal{F}$ is finer than $\mathcal{G}$, or alternatively $\mathcal{G}$ is coarser than $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are filters on X, such that $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and all $G \in \mathcal{G}$ then

$$\mathcal{F} \vee \mathcal{G} = [\{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}]$$

is a filter on X. If (x_n) is a sequence in X, then we define the Frechét filter associated with (x_n) as

$$\langle (x_n) \rangle = [\{\{x_n : n \geq k\} : k \in \mathbb{N}\}].$$

If $f : X \longrightarrow Y$ is a mapping then we define the image of a filter $\mathcal{F}$ under f as

$$f(\mathcal{F}) = [\{f(F) : F \in \mathcal{F}\}].$$

A convergence structure on a set X is defined as follows:

Definition 2.3. *Let X be a nonempty set. A convergence structure on X is the mapping λ from X to the power set of the set of all filters on X that satisfies the following for all $x \in X$.*

(i) $[x] \in \lambda(x)$

(ii) If $\mathcal{F}, \mathcal{G} \in \lambda(x)$, then $\mathcal{F} \cap \mathcal{G} \in \lambda(x)$.

(iii) If $\mathcal{F} \in \lambda(x)$, then $\mathcal{G} \in \lambda(x)$, for all filters $\mathcal{G} \supseteq \mathcal{F}$.

The pair (X, λ) is called a convergence space. Whenever $\mathcal{F} \in \lambda(x)$ we say $\mathcal{F}$ converges to x and write "$\mathcal{F} \longrightarrow x$".

Definition 2.4. *Let λ and μ be two convergence structures on the same set X. Then λ is finer than μ (or μ is coarser than λ) if for every $x \in X$, $\lambda(x) \subseteq \mu(x)$.*

Example 2.5. Let X be a topological space. For each $x \in X$, denote by $\mathcal{V}_x$ the set of open neighbourhoods of x. Then

$$\mathcal{F} \in \lambda(x) \Leftrightarrow \mathcal{V}_x \subseteq \mathcal{F}, \quad x \in X$$

defines a convergence structure on X

Example 2.6. Let $\mathcal{M}(\mathbb{R})$ denote the set of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$, with the usual convention of identifying functions that are equal everywhere except possibly on a set of measure 0. Then

$$\mathcal{F} \in \lambda(f) \Leftrightarrow \left(\begin{array}{ll} \exists \quad E \subset \mathbb{R} \ : & \\ \quad 1) \quad E \text{ has measure } 0 & \\ \quad 2) \quad \mathcal{F}(x) \longrightarrow f(x) \text{ in } \mathbb{R}, \ x \in \mathbb{R} \setminus E & \end{array} \right.$$

defines a convergence structure on $\mathcal{M}(\mathbb{R})$, see [11]. This convergence structure is not induced by any topology on $\mathcal{M}(\mathbb{R})$ as in Example 2.5.

As Examples 2.5 and 2.6 show, the notion of a convergence space is a generalisation of that of a topological space. Most topological notions can be extended to convergence spaces. In particular, recall the following, see [11].

Definition 2.7. *Let X be a convergence space, and Y a subset of X. A filter $\mathcal{F}$ on Y converges to $y \in Y$ in the subspace convergence structure on Y if*

$$[\mathcal{F}]_X = \left\{ G \subseteq X \,\middle|\, \begin{array}{l} \exists \quad F \in \mathcal{F} \ : \\ \quad F \subseteq G \end{array} \right\}$$

converges to y in X.

Definition 2.8. *Let X be a convergence space. A subset Y of X is a dense subspace of X if*

$$a(Y) = \left\{ x \in X \,\middle|\, \begin{array}{l} \exists \quad \mathcal{F} \ \text{ a filter on } Y : \\ \quad [\mathcal{F}]_X \longrightarrow x \end{array} \right\} = X.$$

Definition 2.9. *Let $\{X_i \ : \ i \in I\}$ be a collection of convergence spaces. A filter $\mathcal{F}$ on $\prod_{i\in I} X_i$ converges to $x = (x_i)_{i\in I}$ with respect to the product convergence structure if for each $i \in I$ there exists a filter $\mathcal{F}_i \longrightarrow x_i$ so that $\prod_{i\in I} \mathcal{F}_i \subseteq \mathcal{F}$, where $\prod_{i\in I} \mathcal{F}_i$ is the filter with basis*

$$\left\{ \prod_{i\in I} F_i \,\middle|\, \begin{array}{ll} (1) & F_i \in \mathcal{F}_i, \ i \in I \\ (2) & \{i \in I \ : \ F_i \neq X_i\} \text{ is finite} \end{array} \right\}.$$

Definition 2.10. *A convergence space X is Hausdorff if every filter on X converges to at most one limit.*

Definition 2.11. *A convergence space X is first countable if for each $\mathcal{F} \longrightarrow x$ in X there exists a coarser filter $\mathcal{G}$ on X with countable basis that converges to x*

Definition 2.12. *Let X and Y be convergence spaces. A function $f : X \longrightarrow Y$ is continuous if for each $x \in X$,*

$$f(\mathcal{F}) \longrightarrow f(x) \text{ in } Y \quad \text{whenever} \quad \mathcal{F} \longrightarrow x \text{ in } X.$$

We call f an embedding if it is injective, and $f^{-1} : f(X) \to X$ is continuous. The function f is an isomorphism if it is a surjective embedding.

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Uniform Convergence Spaces

Recall [11] that a uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that the following conditions are satisfied.

(i) $\Delta \subseteq \mathcal{U}$ for each $U \in \mathcal{U}$.

(ii) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

(iii) For each $U \in \mathcal{U}$ there are some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

Here $\Delta = \{(x, x) : x \in X\}$ denotes the diagonal in $X \times X$. If U and V are subsets of $X \times X$ then

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\},$$

and the composition of subsets U and V of $X \times X$ is defined as

$$U \circ V = \left\{ (x, y) \in X \times X \;\middle|\; \begin{array}{ll} \exists & z \in X : \\ & (x, z) \in V \text{ and } (z, y) \in U \end{array} \right\}.$$

A uniformity $\mathcal{U}_X$ on X induces a topology on X in the following way: A set $A \subseteq X$ is open in X if

$$\begin{array}{ll} \forall & x \in A : \\ \exists & U \in \mathcal{U}_X : \\ & U[x] \subseteq A \end{array}$$

where $U[x] = \{y \in X | (x, y) \in U\}$. A filter $\mathcal{F}$ on X is a Cauchy filter if and only if

$$\mathcal{U}_X \subseteq \mathcal{F} \times \mathcal{F}.$$

The uniform space $X = (X, \mathcal{U})$ is called complete if every Cauchy filter on X is convergent, in the sense of Example 2.5.

A uniform convergence space generalises the notion of a uniform space to the broader context of convergence spaces. In order to formulate the definition of a uniform convergence space, we recall the following.

If $\mathcal{U}$ and $\mathcal{V}$ are filters on $X \times X$ then $\mathcal{U}^{-1}$ is defined as

$$\mathcal{U}^{-1} = [\{U^{-1} : U \in \mathcal{U}\}].$$

If $U \circ V \neq \emptyset$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$ then the filter $\mathcal{U} \circ \mathcal{V}$ exists and it is defined as

$$\mathcal{U} \circ \mathcal{V} = [\{U \circ V : U \in \mathcal{U}, V \in \mathcal{V}\}].$$

Definition 2.13. *Let X be a set. A family $\mathcal{J}_X$ of filters on $X \times X$ is called a uniform convergence structure on X if the following hold:*

(i) $[x] \times [x] \in \mathcal{J}_X$ for every $x \in X$

(ii) If $\mathcal{U} \in \mathcal{J}_X$ and $\mathcal{U} \subseteq \mathcal{V}$, then $\mathcal{V} \in \mathcal{J}_X$

(iii) If $\mathcal{U}, \mathcal{V} \in \mathcal{J}_X$, then $\mathcal{U} \cap \mathcal{V} \in \mathcal{J}_X$

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(iv) *If* $\mathcal{U} \in \mathcal{J}_X$*, then* $\mathcal{U}^{-1} \in \mathcal{J}_X$.

(v) *If* $\mathcal{U}, \mathcal{V} \in \mathcal{J}_X$*, then* $\mathcal{U} \circ \mathcal{V} \in \mathcal{J}_X$ *whenever* $\mathcal{U} \circ \mathcal{V}$ *exists.*

The pair $(X, \mathcal{J}_X)$ *is called a uniform convergence space.*

As mentioned, uniform convergence spaces generalize the concept of a uniform space in the sense that every uniformity $\mathcal{U}_X$ on X give rise to a uniform convergence structure $\mathcal{J}_{\mathcal{U}_X}$ defined through

$$\mathcal{U} \in \mathcal{J}_{\mathcal{U}_X} \Longrightarrow \mathcal{U}_X \subseteq \mathcal{U}.$$

Every uniform convergence structure $\mathcal{J}_X$ on X induces a convergence structure $\lambda_{\mathcal{J}_X}$ on X defined by

$$\begin{array}{ll} \forall & x \in X \ : \\ \forall & \mathcal{F} \text{ a filter on } X \ : \\ & \mathcal{F} \in \lambda_{\mathcal{J}_X}(x) \Longleftrightarrow \mathcal{F} \times [x] \in \mathcal{J}_X \end{array}$$

The convergence structure $\lambda_{\mathcal{J}_X}$ is called the induced convergence structure.

Definition 2.14. *A unform convergence space* X *is Hausdorff if the induced convergence structure on* X *is Hausdorff.*

Definition 2.15. *Let* X *be a uniform convergence space with uniform convergence structure* $\mathcal{J}_X$*. A filter* $\mathcal{U}$ *on* $Y \times Y$ *belongs to the subspace uniform convergence structure on* Y *if*

$$[\mathcal{U}]_{X \times X} \in \mathcal{J}_X.$$

The concepts of uniform continuity, Cauchy filters, completeness and completion extend to uniform convergence spaces in a natural way. In this regard let X and Y be uniform convergence spaces. A mapping $f : X \longrightarrow Y$ is *uniformly continuous* if

$$\begin{array}{ll} \forall & \mathcal{U} \in \mathcal{J}_X \\ & (f \times f)(\mathcal{U}) \in \mathcal{J}_Y. \end{array}$$

A uniformly continuous mapping f is called a uniformly continuous embedding if it is injective and f^{-1} is uniformly continuous on the subspace $f(X)$ of Y. A uniformly continuous embedding is a uniformly continuous isomorphism if it is also surjective.

A filter $\mathcal{F}$ on X is called a *Cauchy filter* if

$$\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X.$$

In particular, a sequence (x_n) in X is a Cauchy sequence if $\langle x_n \rangle$ is a Cauchy filter. A uniform convergence space X is complete if every Cauchy filter on X is convergent with respect to the induced convergence structure. Each Hausdorff uniform convergence space can be completed, in the following sense, see [75].

Theorem 2.16. *If* X *is a Hausdorff uniform convergence space, then there exists a complete, Hausdorff uniform convergence space* $X^\sharp$ *and a uniformly continuous embedding*

$$i_X : X \longrightarrow X^\sharp$$

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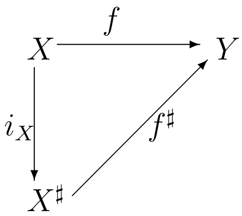
such that $i_X(X)$ *is dense in* $X^\sharp$*. Moreover, the completion* $X^\sharp$ *of* X *satisfies the* universal property*: If* Y *is a complete Hausdorff uniform convergence space and* $f : X \longrightarrow Y$ *is uniformly continuous, then there exists a uniformly continuous mapping*

$$f^\sharp : X^\sharp \longrightarrow Y$$

such that the diagram



(2.14)

commutes.

$X^\sharp$ is called the Wyler completion of X. This completion is unique up to uniformly continuous isomorphism.

Convergence vector spaces

Let V be a vector space over the scalar field $\mathbb{K}$ of real or complex numbers. A convergence structure λ_V on V is called a vector space convergence structure if the vector space operations

$$+ : (V, \lambda_V) \times (V, \lambda_V) \longrightarrow (V, \lambda_V)$$

and

$$\cdot : \mathbb{K} \times (V, \lambda_V) \longrightarrow (V, \lambda_V)$$

are continuous. In this case V is called a convergence vector space.

Example 2.17. We recall some important examples of convergence vector spaces.

1. Every topological vector space is a convergence vector space. Recall [43], [56] that a vector space V over the scalar field $\mathbb{K}$ of real or complex numbers is called a topological vector space if V is endowed with a topology τ_V such that addition

   $$+ : (V, \tau_V) \times (V, \tau_V) \longrightarrow (V, \tau_V)$$

   and scalar multiplication

   $$\cdot : \mathbb{K} \times (V, \tau_V) \longrightarrow (V, \tau_V)$$

   are (jointly) continuous.

2. For convergence spaces X and Y, the continuous convergence structure on $\mathcal{C}(X, Y)$ is defined as follows. A filter $\mathcal{F}$ on $\mathcal{C}(X, Y)$ converges to $f \in \mathcal{C}(X, Y)$ if for every $x \in X$ and every filter $\mathcal{G}$ on X that converges to x, the filter

   $$\mathcal{F}(\mathcal{G}) = [\{\{f(y) \ : f \in F, \ y \in G\} \ : \ F \in \mathcal{F}, \ G \in \mathcal{G}\}]$$

   converges to $f(x)$. We denote by $\mathcal{C}_c(X, Y)$ the set $\mathcal{C}(X, Y)$ equipped with the continuous convergence structure.

   If Y a convergence vector space, then $\mathcal{C}_c(X, Y)$ is a convergence vector space. In particular $\mathcal{C}_c(X) = \mathcal{C}_c(X, \mathbb{R})$ is a convergence vector space.

□

A convergence vector space is equipped with a natural uniform convergence structure, called the *induced uniform convergence structure*, which is denoted as $\mathcal{J}_V$. In this regard, let V be a convergence vector space with convergence structure λ_V, and let $\mathcal{U}$ be a filter on $V \times V$. Then

$$\mathcal{U} \in \mathcal{J}_V \Longleftrightarrow \left\{ \begin{array}{ll} \exists & \mathcal{F} \text{ a filter on } V : \\ & (1)\mathcal{F} \longrightarrow 0 \\ & (2)\Delta(\mathcal{F}) \subseteq \mathcal{U}. \end{array} \right. \tag{2.15}$$

Here $\Delta(\mathcal{F}) = [\{\Delta(F) : F \in \mathcal{F}\}]$ where for any set $F \subseteq V$

$$\Delta(F) = \{(x, y) \in V \times V : x - y \in F\}. \tag{2.16}$$

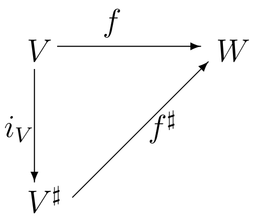
The convergence structure induced by the uniform convergence structure $\mathcal{J}_V$ agrees with the vector space convergence structure λ_V, that is, $\lambda_{\mathcal{J}_V} = \lambda_V$. If V and W are convergence vector space and a linear mapping $f : V \longrightarrow W$ is continuous then f is uniformly continuous, see [11, Proposition 2.5.3].

In a convergence vector space Cauchy filters are characterized as follows.

Proposition 2.18. *A filter $\mathcal{F}$ on a convergence vector space V is a Cauchy filter if and only if $\mathcal{F} - \mathcal{F}$ converges to* 0.

In general, the Wyler completion $V^\sharp$ of a convergence vector space V is not a convergence vector space. In particular, the convergence structure induced on $V^\sharp$ by the uniform convergence structure is not a vector space convergence structure, see [11, Section 2.3]. However, a suitable completion may be constructed for a large class of convergence vector spaces, see for instance [24].

Theorem 2.19. *Let V be a Hausdorff convergence vector space. If every Cauchy filter $\mathcal{F}$ in V is bounded, that is, there is some $F \in \mathcal{F}$ so that $\mathcal{V}(0)F \longrightarrow 0$ where $\mathcal{V}(0)$ denotes the neighbourhood filter at* 0 *in $\mathbb{K}$, then there is a complete, Hausdorff convergence vector space $V^\sharp$ and a linear embedding $i_V : V \longrightarrow V^\sharp$ such that $i_V(V)$ is dense in $V^\sharp$. Furthermore, for every complete Hausdorff convergence vector space W and every continuous linear mapping $f : V \longrightarrow W$ there exists a continuous linear mapping $f^\sharp : V^\sharp \longrightarrow W$ so that the diagram*



(2.17)

commutes.

The Structure of Completions

Of particular interest to us in this work is the structure of the underlying set of the completion of a convergence vector space. In this regard, let $X = (X, \mathcal{J}_X)$ be a Hausdorff

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uniform convergence space, and let $X^\sharp = (X^\sharp, \mathcal{J}_{X^\sharp})$ denote its completion, in the sense of Theorem 2.16, and $\iota_X : X \to X^\sharp$ the canonical uniformly continuous embedding. We recall how the *set* $X^\sharp$ and the mapping ι_X may be constructed. Denote by $C[X]$ the set of all Cauchy filters on X. Then

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \mathcal{F} \cap \mathcal{G} \in C[X] \tag{2.18}$$

defines an equivalence relation on $C[X]$, and

$$X^\sharp = C[X]/ \sim_C . \tag{2.19}$$

That is, $X^\sharp$ is the set of $\sim_C$-equivalence classes in $C[X]$. Furthermore, for each $x \in X$, the set $\lambda_X(x) \subseteq C[X]$ is a $\sim_C$-equivalence class in $C[X]$. Indeed, if $\mathcal{F}, \mathcal{G} \in \lambda_X(x)$, then $\mathcal{F} \cap \mathcal{G} \in \lambda_X(x) \subseteq C[X]$ so that $\mathcal{F} \sim_C \mathcal{G}$. Moreover, if $\mathcal{F} \in \lambda_X(x)$ and $\mathcal{F} \sim_C \mathcal{G}$ for some $\mathcal{G} \in C[X]$, then $\mathcal{F} \cap \mathcal{G} \in C[X]$. Since $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{F} \in \lambda_X(x)$, it follows [11, Proposition 2.3.2] that $\mathcal{F} \cap \mathcal{G} \in \lambda_X(x)$, hence $\mathcal{G} \in \lambda_X(x)$. Therefore

$$\iota_X : X \ni x \mapsto \lambda_X(x) \in X^\sharp$$

defines an injective mapping. According to Theorem 2.16, for each complete Hausdorff uniform convergence space Y and every uniformly continuous map $f : X \longrightarrow Y$, there exists a unique uniformly continuous map $f^\sharp : X^\sharp \longrightarrow Y$ such that $f = f^\sharp \circ i_X$. This extension $f^\sharp$ of f is defined in the following way: Since f is uniformly continuous, $(f \times f)(\mathcal{U}) \in \mathcal{J}_Y$ whenever $\mathcal{U} \in \mathcal{J}_X$ so that $f(\mathcal{F})$ is Cauchy in Y whenever $\mathcal{F}$ is Cauchy in X. As Y is complete and Hausdorff, there exist a unique $y \in Y$ so that $f(\mathcal{F}) \longrightarrow y$ in Y. We therefore have a map

$$f' : C[X] \longrightarrow Y.$$

If $\mathcal{F} \sim_C \mathcal{G}$ then $\mathcal{F} \cap \mathcal{G} \in C[X]$ so that

$$(f \times f)(\mathcal{F} \cap \mathcal{G} \times \mathcal{F} \cap \mathcal{G}) \in \mathcal{J}_Y.$$

But

$$(f \times f)(\mathcal{F} \cap \mathcal{G} \times \mathcal{F} \cap \mathcal{G}) = f(\mathcal{F} \cap \mathcal{G}) \times f(\mathcal{F} \cap \mathcal{G}) \tag{2.20}$$
$$= f(\mathcal{F}) \cap f(\mathcal{G}) \times f(\mathcal{F}) \cap f(\mathcal{G}) \tag{2.21}$$

so that $f(\mathcal{F}) \cap f(\mathcal{G})$ is Cauchy in Y hence convergent. Since Y is complete and Hausdorff , it follows that $f'(\mathcal{F}) = f'(\mathcal{G})$. Thus

$$f^\sharp : X^\sharp = C[X]/ \sim_C \ni [\mathcal{F}] \mapsto f'(\mathcal{F}) \in Y$$

is well defined. It remains to verify that $f^\sharp$ is indeed an extension of f. That is,

$$f^\sharp \circ i_X = f.$$

For $x \in X$, $i_X(x) = \lambda(x) \in X^\sharp$. Since f is continuous, $f'(\mathcal{F}) = f(x)$ for all $\mathcal{F} \in i_X(x) = \lambda(x)$. Hence $f^\sharp \circ i_X(x) = f(x)$, as desired.

If X is a convergence vector space, and $\mathcal{J}_X$ is the induced uniform convergence structure, then $X^\sharp$ is, in a natural way, a vector space, and the mapping ι_X is a linear injection.

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Indeed, it follows from Proposition 2.18 that if $\mathcal{F},\mathcal{G} \in C[X]$, then $\mathcal{F}+\mathcal{G} \in C[X]$ and $\alpha\mathcal{F} \in C[X]$ for all $\alpha \in \mathbb{K}$.

Furthermore, $\mathcal{F} \sim_C \mathcal{G}$ if and only if $\mathcal{F}-\mathcal{G} \longrightarrow 0$ in X. Indeed, if $\mathcal{F} \sim_C \mathcal{G}$, then $\mathcal{F}\cap\mathcal{G} \in C[X]$ so that $\mathcal{F}\cap\mathcal{G} - \mathcal{F}\cap\mathcal{G} \longrightarrow 0$ in X. But $\mathcal{F}\cap\mathcal{G} - \mathcal{F}\cap\mathcal{G} \subseteq \mathcal{F}-\mathcal{G}$, so that $\mathcal{F}-\mathcal{G} \longrightarrow 0$ in X.

Conversely, if $\mathcal{F}-\mathcal{G} \longrightarrow 0$ in X for some $\mathcal{F},\mathcal{G} \in C[X]$ then $\Delta(\mathcal{F}-\mathcal{G}) \in \mathcal{J}_X$. But for $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $\Delta(F-G) = \{(u,v) | u-v \in F-G\} \supseteq (F\cap G)\times(F\cap G)$ so that $\Delta(\mathcal{F}-\mathcal{G}) \subseteq (\mathcal{F}\cap\mathcal{G})\times(\mathcal{F}\cap\mathcal{G})$ Thus $(\mathcal{F}\cap\mathcal{G})\times(\mathcal{F}\cap\mathcal{G}) \in \mathcal{J}_X$ so that $\mathcal{F}\cap\mathcal{G} \in C[X]$. Hence $\mathcal{F} \sim_C \mathcal{G}$. It therefore follows that

$$[\mathcal{F}]+[\mathcal{G}] = [\mathcal{F}+\mathcal{G}] \tag{2.22}$$

$$\alpha[\mathcal{F}] = [\alpha\mathcal{F}],\ \ \alpha \in \mathbb{R} \tag{2.23}$$

are well defined in $X^\sharp = C[X]/\sim_C$. An elementary argument shows that $X^\sharp$ is a vector space over $\mathbb{R}$ with respect to the operations (2.22) and (2.23). The linearity of $i_X : X \longrightarrow X^\sharp$ follows from the continuity of addition and scalar multiplication on X.

2.2.2 Order convergence

Most of the important spaces encountered in analysis, and in particular functional analysis, are equipped with partial order in a natural way. For example, the space $\mathcal{C}^0(X)$ of all real valued continuous functions defined on a topological space X is equipped with the usual point-wise order

$$\begin{array}{l}\forall\quad u,v \in \mathcal{C}^0(X)\\ u \leq v \Longleftrightarrow \left(\begin{array}{ll}\forall & x \in X\\ & u(x) \leq v(x).\end{array}\right)\end{array}$$

Likewise, the space $M(X)$ of all real valued measurable functions on a measure space (X,Γ,μ) is equipped with the almost everywhere point-wise order

$$\begin{array}{l}\forall\quad u,v \in M(X)\\ u \leq v \Longleftrightarrow \left(\begin{array}{ll}\exists & E \subset X,\ \mu(E)=0\\ & u(x) \leq v(x),\ x \in X\backslash E\end{array}\right)\end{array}$$

Definition 2.20. *A real ordered vector space L is a vector space equipped with a partial order such that the following hold for all $f,g,h \in L$.*

(i) If $f \leq g$ then $f+h \leq g+h$.

(ii) If $f \geq 0$ then $\alpha f \geq 0$ for all real numbers $\alpha \geq 0$.

Definition 2.21 (Riesz space)**.** *A Riesz space is an ordered vector space in which every pair of elements (f,g) has a supremum* $\sup(f,g)$ *and an infimum* $\inf(f,g)$ *in L.*

Definition 2.22. *Let L be a Riesz space. For any element $f \in L$, set*

$$f^+ = \sup(f,0)\quad f^- = \sup(-f,0) = -\inf(f,0),\qquad |f| = f^+ + f^- = \sup(f,-f).$$

Below are some examples of Riesz spaces

Example 2.23. (i) The space $\mathbb{R}^n$, equipped with the partial ordering

$$x \leq y \Longrightarrow x_k \leq y_k \text{ for } x, y \in \mathbb{R}^n,\ 1 \leq k \leq n,$$

is a Riesz space.

(ii) Let X be a topological space. The space $C(X)$ of all continuous real valued function on X is a Riesz space with respect to the point-wise partial ordering.

Riesz spaces were introduced, independently, by F. Riesz [45, 46], L. V. Kantorovitch [28, 29], and H Freudenthal [22]. The theory has been extensively developed, see for instance [25, 32, 77].

Definition 2.24 (Archimedean Riesz space)**.** *A Riesz space L, is called* Archimedean *if for all $u \in L, u \geq 0$, the decreasing sequence $(n^{-1}u : n = 1, 2, \cdots)$ has infimum* 0.

Definition 2.25 (Distributive lattice)**.** *A lattice L is said to be distributive if*

$$\inf(u, \sup(v, w)) = \sup(\inf(u, v), \inf(u, w))$$

and

$$\sup(u, \inf(v, w)) = \inf(\sup(u, v), \sup(u, w))$$

hold for all $u, v, w \in L$.

Definition 2.26 (σ-distributive lattice)**.** *A lattice L is* σ-distributive*, if*

$$\inf(u, \sup A) = \sup\{\inf(u, a) : a \in A\}$$

and

$$\sup(u, \inf B) = \inf\{\sup(u, b) : b \in B\}$$

hold for all $u \in L$ and all countable sets $A, B \subseteq L$ for which the supremum and infimum exist.

Proposition 2.27. *Any Riesz space L is a σ-distributive lattice, see [36].*

On a partially ordered set L, and in particular on a Riesz L, one may define a notion of convergence of sequences in terms of the partial order on the set X. Several such notions of convergence of sequences on partially ordered sets have been introduced in the literature, see for instance [12, 18, 35, 36, 40]. It often turns out that these notions of convergence of sequences cannot be associated with any topology. One such notion of convergence of sequences defined through a partial order that is, in general, not topological, is order convergence of sequences, [64, 67] .

Definition 2.28. *A sequence (u_n) in a partially ordered set X order converges to $u \in X$ whenever*

$$\left\{\begin{array}{ll} \exists & (\alpha_n), (\beta_n) \subset X : \\ & (i)\ \alpha_n \leq \alpha_{n+1} \leq u_n \leq \beta_{n+1} \leq \beta_n,\ \ n \in \mathbb{N} \\ & (ii)\ \sup\{\alpha_n : n \in \mathbb{N}\} = u = \inf\{\beta_n : n \in \mathbb{N}\}. \end{array}\right. \quad (2.24)$$

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For a Riesz space L, the condition (2.24) is equivalent to the following:

$$\begin{array}{ll}\exists & (\lambda_n) \subset L : \\ & (i)\ \lambda_{n+1} \leq \lambda_n,\ \ n \in \mathbb{N} : \\ & (ii)\ \inf\{\lambda_n : n \in \mathbb{N}\} = 0 \\ & (iii)\ |u - u_n| \leq \lambda_n, \quad n \in \mathbb{N}.\end{array}$$

where definition of $|\cdot|$ is given in Definition 2.22. In general, order convergence sequences is not topological as shown in the following

Example 2.29. Consider the Archimedean Riesz space $C(\mathbb{R})$, and the sequence $(u_n) \subset C(\mathbb{R})$ given by

$$u_n(x) = \begin{cases} 1 - n|x - q_n| & \text{if} \quad |x - q_n| < \frac{1}{n} \\ \\ 0 & \text{if} \quad |x - q_n| \geq \frac{1}{n} \end{cases} \tag{2.25}$$

where $\{q_n \mid n \in \mathbb{N}\} = [0,1] \cap \mathbb{Q}$. The sequence (u_n) does not order converge to 0. Indeed,the complement of any finite subset of $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$. For any $N_0 \in \mathbb{N}$ we therefore have

$$\sup\{u_n : n \geq N_0\} = 1.$$

This means that a sequence $(\beta_n) \subseteq C(\mathbb{R})$ such that $u_n \leq \beta_n$ for all $n \in \mathbb{N}$ cannot decrease to 0.

Thus if there is a topology τ on $C^0(\mathbb{R})$ that induces order convergence, then there is some τ-neighborhood V of 0 and a subsequence (u_{n_k}) of (u_n) which is always outside of V. Let (q_{n_k}) denote the sequence of rational numbers associated with the subsequence (u_{n_k}) according to (2.25). Since the sequence (q_{n_k}) is bounded, there exist a subsequence $(q_{n_{k_i}})$ of (q_{n_k}) that converges to some $q \in [0,1]$. Let $(u_{n_{k_i}})$ be the sequence associated with the sequence of rational numbers $(q_{n_{k_i}})$. Then

$$\begin{array}{l}\forall\ \varepsilon > 0 : \\ \exists\ N_\varepsilon \in \mathbb{N} : \\ \ u_{n_{k_i}}(x) = 0,\ \text{whenever} |x - q| > \varepsilon \text{ and } n_{k_i} \geq N_\varepsilon.\end{array}$$

For each $j \in \mathbb{N}$ set $\varepsilon_j = \frac{1}{j}$ and let the sequence $(\mu_{n_{k_i}}) \subseteq C^0(\mathbb{R})$ be defined as

$$\mu_{n_{k_i}}(x) = \begin{cases} 0 & if|x - q| \geq 2\varepsilon_j \\ 1 & if|x - q| \leq \varepsilon_j \\ \frac{|x-q|}{\varepsilon_j} + 2 & if\varepsilon_j < |x - q| < 2\varepsilon_j \end{cases} \tag{2.26}$$

whenever $N_{\varepsilon_j} \leq n_{k_i} < N_{\varepsilon_{j+1}}$. The sequence $(\mu_{n_{k_i}})$ decreases to 0, and $0 \leq u_{n_{k_i}} \leq \mu_{n_{k_i}}$, for all i. This means that the sequence $(u_{n_{k_i}})$ order converges to 0. Therefore it must eventually be in V, a contradiction. Thus the topology τ cannot exist.

In [8], see also [62], a convergence structure, called the *order convergence structure*, was defined on a Riesz space L which induces the order convergence of sequences.

Definition 2.30. *Let L be a Riesz space. A filter $\mathcal{F}$ on L converges to u in L with respect to the order convergence structure, denoted as λ_0, if and only if*

$$\begin{array}{ll}\exists & (\alpha_n), (\beta_n) \subset L : \\ & (i)\ \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n,\ n \in \mathbb{N} \\ & (ii)\ \sup\{\alpha_n : n \in \mathbb{N}\} = u = \inf\{\beta_n : n \in \mathbb{N}\} \\ & (iii)\ [\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}.\end{array}$$

In [64], it was shown that any Riesz space equipped with the order convergence structure is a Hausdorff and first countable convergence space.

Theorem 2.31. *[64] If L is a Riesz space, then the order convergence structure defines a Hausdorff and first countable convergence structure on L. A sequence (u_n) on L converges to $u \in L$ if and only if it order converges to $u \in L$. Furthermore, If L is an Archimedean Riesz space, then the order convergence structure is a vector space convergence structure.*

In what follows, we consider the case when a Riesz space L is also an algebra, [27, 42, 64, 76].

Definition 2.32. *A Riesz algebra is a Riesz space that is also an associative algebra such that*

$$L^+ \cdot L^+ \subseteq L^+, \tag{2.27}$$

where L^+ denote the positive cone of L, defined as $L^+ = \{f \in L : f \geq 0\}$. We note that the inclusion in (2.27) is equivalent to

$$f \leq g \Longrightarrow fh \leq gh$$

where $f, g, \in L$ and $h \in L^+$.

Definition 2.33. *The multiplication in a Riesz algebra is called σ-order continuous if*

$$\sup\{ab : a \in A, b \in B\} = a_0 b_0 \tag{2.28}$$

holds for all countable sets $A, B \subseteq L^+$ such that $a_0 = \sup A$ and $b_0 = \sup B$.

Note that in any Riesz algebra L, the identity

$$fg = f^+g^+ + f^-g^- - f^+g^- - f^-g^+$$

holds for all $f, g \in L$, where f^+, f^- denote positive part and negative part of f respectively, given by $f^+ = \sup\{f, 0\}$ and $f^- = \sup\{-f, 0\}$.

Theorem 2.34. *Let L be an Archimedean Riesz algebra with σ-order continuous multiplication. Then the order convergence structure is an algebra convergence structure.*

2.2.3 Normal lower semi-continuous functions

In this section we discuss some of the properties of the spaces of nearly finite normal lower semi-continuous functions. In this regard, let X be a topological space, and denote by $\mathbb{R}^*$ the extended real line $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$. Denote by $\mathcal{A}(X)$ the set of all extended real valued function on X. That is

$$\mathcal{A}(X) = \{u : X \longrightarrow \mathbb{R}^*\}.$$

Definition 2.35. *A function $u : X \longrightarrow \mathbb{R}^*$ is called lower semi-continuous at $x \in X$ if $u(x) = -\infty$ or*

$$\begin{array}{ll} \forall & M < u(x): \\ \exists & V \in \mathcal{V}_x : \\ & y \in V \Longrightarrow M < u(y). \end{array}$$

Definition 2.36. *A function $u : X \longrightarrow \mathbb{R}^*$ is called upper semi-continuous at $x \in X$ if $u(x) = +\infty$ or*

$$\begin{array}{ll} \forall & M > u(x): \\ \exists & V \in \mathcal{V}_x : \\ & y \in V \Longrightarrow M > u(y). \end{array}$$

Definition 2.37. *A function*

$$u : X \longrightarrow \mathbb{R}^*$$

is called lower(upper) semi-continuous on X if it is lower(upper) semi-continuous at every point of X.

Example 2.38. (i) A real valued function is continuous if and only if it is both upper and lower semi-continuous.

(ii) The characteristic function χ_A defined on a set $A \subseteq X$, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

is lower semi-continuous if A is open and upper semi-continuous if A is closed.

(iii) Let the function f be defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0. \end{cases}$$

Then the function f is upper semi-continuous but not lower semi-continuous, at $x = 0$.

We recall [71] that the point-wise supremum of any collection of lower semi-continuous functions is a lower semi-continuous function. That is, if A is a set of lower semi-continuous functions on X, then the function

$$u : X \ni x \mapsto \sup\{v(x) : v \in A\} \in \mathbb{R}^*$$

is lower semi-continuous. Similarly, the infimum of any collection of upper semi-continuous functions is an upper semi-continuous function. That is, if B is a set of upper semi-continuous functions on X, then the function

$$w : X \ni x \mapsto \inf\{v(x) : v \in B\} \in \mathbb{R}^*$$

is upper semi-continuous. In particular,

$$\begin{array}{ll} \forall & A \subseteq C(X) : \\ & (1) u : X \ni x \mapsto \sup\{v(x) : v \in A\} \in \mathbb{R}^* \text{ is lower semi} - \text{continuous}, \\ & (2) w : X \ni x \mapsto \inf\{v(x) : v \in A\} \in \mathbb{R}^* \text{ is upper semi} - \text{continuous}. \end{array}$$

Conversely, if X is a metric space, then for each lower semi-continuous function $u : X \longrightarrow \mathbb{R}$ we have that

$$\begin{array}{ll} \exists & A \subseteq C(X) : \\ & u(x) = \sup\{v(x) : v \in A\}, x \in X \end{array}$$

while for every upper semi-continuous function $w : X \longrightarrow \mathbb{R}$ we have that

$$\begin{array}{ll} \exists & A \subseteq C(X) : \\ & w(x) = \inf\{v(x) : v \in A\}, x \in X. \end{array}$$

We remark that the pointwise infimum of a set of continuous functions need not be the infimum of such a set with respect to the pointwise order on $C(X)$. Indeed, consider the sequence (u_n) in $C(\mathbb{R})$ which is defined by

$$u_n(x) = \left\{ \begin{array}{lll} 1 - n|x| & \text{if} & |x| < \frac{1}{n} \\ \\ 0 & \text{if} & |x| \geq \frac{1}{n}. \end{array} \right.$$

The pointwise infimum of the set $\{u_n : n \in \mathbb{N}\}$ is the function

$$u(x) = \left\{ \begin{array}{lll} 1 & \text{if} & x = 0 \\ 0 & \text{if} & x \neq 0 \end{array} \right.$$

while the infimum of the set $\{u_n : n \in \mathbb{N}\}$ in $C(\mathbb{R})$ is the function $u(x) = 0,\ x \in \mathbb{R}$. The same is true of the pointwise supremum of a set of continuous functions.

The concept of normal lower semi-continuous function is formulated in terms of two fundamental operators which are associated with semi-continuous functions, and extended real valued functions in general, namely, the Baire operators. The Lower and Upper Baire Operators, [3], [9], are mappings

$$I : \mathcal{A}(X) \rightarrow \mathcal{A}(X)$$

and

$$S : \mathcal{A}(X) \rightarrow \mathcal{A}(X)$$

defined by

$$I(u)(x) = \sup\{\inf\{u(y) \ : \ y \in V\} \ : \ V \in \mathcal{V}_x\} \tag{2.29}$$

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and

$$S(u)(x) = \inf\{\sup\{u(y) \ : \ y \in V\} \ : \ V \in \mathcal{V}(x)\}, \tag{2.30}$$

respectively. The Baire operators I and S, as well as their composition $I \circ S$, are idempotent and monotone with respect to the usual pointwise order on $\mathcal{A}(X)$, given by

$$u \leq v \Longleftrightarrow \left(\begin{array}{ll} \forall & x \in X : \\ & u(x) \leq v(x) \end{array} \right). \tag{2.31}$$

That is, for all $u, v \in \mathcal{A}(X)$ we have

$$I(I(u)) = I(u), \quad S(S(u)) = S(u), \quad I(S(I(S(u)))) = I(S(u)) \tag{2.32}$$

and

$$u \leq v \Rightarrow \left(\begin{array}{l} I(u) \leq I(v) \\ S(u) \leq S(v) \\ I(S(u)) \leq I(S(v)) \end{array} \right) \tag{2.33}$$

Furthermore, the inequalities

$$I(u) \leq u \leq S(u) \tag{2.34}$$

are satisfied.

An easy computation shows that a function $u \in \mathcal{A}(X)$ is lower semi-continuous if and only if $I(u) = u$, while u is upper semi-continuous if and only if $S(u) = u$, see[5], [17].

Definition 2.39. *A function u is normal lower semi-continuous whenever*

$$I(S(u)) = u. \tag{2.35}$$

A normal lower semi-continuous function is called *nearly finite* whenever the set

$$\{x \in X \ : \ u(x) \in \mathbb{R}\} \tag{2.36}$$

is dense in X. In fact, due to the lower semi-continuity of a normal lower semi-continuous function, one may assume that the set (2.36) is open and dense. We denote by $\mathcal{NL}(X)$ the set of all nearly finite normal lower semi-continuous functions on X. That is,

$$\mathcal{NL}(X) = \left\{ u \in \mathcal{A}(X) \left| \begin{array}{l} (1)(I \circ S)u(x) = u(x), \ x \in X \\ (2) \ \{x \in X : u(x) \in \mathbb{R}\} \text{ is open and dense in } X. \end{array} \right. \right\}$$

Note that every continuous, real valued function is nearly finite normal lower semi-continuous. Thus we have $C(X) \subseteq \mathcal{NL}(X)$. Conversely, every $u \in \mathcal{NL}(X)$ is nearly continuous in a topologically large set in the following sense.

Theorem 2.40. *For every $u \in \mathcal{NL}^0(X)$ and $\epsilon > 0$ the set*

$$D_\epsilon = \{x \in X : \omega(u, x) < \epsilon\}$$

contains an open and dense subset of X, where

$$\omega(u, x) = \inf\{\sup\{|u(x) - u(y)| : y \in V\} : V \in \mathcal{V}\}$$

is the modulus of continuity of u at $x \in X$

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The following are useful properties of the set $\mathcal{NL}(X)$, see [69].

Proposition 2.41. *(P_1) For all $u, v \in \mathcal{NL}(X)$ and for all dense sets $D \subseteq X$, we have*

$$\left(\begin{array}{ll} \forall & x \in D \\ & u(x) \leq v(x) \end{array} \right) \Longrightarrow u \leq v.$$

(P_2) The set $\mathcal{NL}(X)$ is a Dedekind order complete lattice with respect to the pointwise order. That is,

(i) every $B \subseteq \mathcal{NL}(X)$ satisfying

$$\begin{array}{ll} \exists & u_0 \in \mathcal{NL}(X) : \\ \forall & u \in B \\ & u \leq u_0, \end{array}$$

has a supremum, which is given by

$$\sup B = (I \circ S)(\psi), \text{ where } \psi(x) = \sup\{u(x) : u \in B\}, \ x \in X \tag{2.37}$$

(ii) every $A \subseteq \mathcal{NL}(X)$ satisfying

$$\begin{array}{ll} \exists & u_0 \in \mathcal{NL}(X) : \\ \forall & u \in A \\ & u_0 \leq u, \end{array}$$

has an infimum, which is given by

$$\inf A = (I \circ S)(\varphi), \text{ where } \varphi(x) = \inf\{u(x) : u \in A\}, \ x \in A \tag{2.38}$$

(P_3) The lattice $\mathcal{NL}(X)$ is fully distributive. That is

$$\begin{array}{ll} \forall & v \in \mathcal{NL}(X) : \\ \forall & B \subset \mathcal{NL}(X) : \\ & u_0 = \sup B \Longrightarrow \sup\{\inf\{u, v\} : u \in B\} = \inf\{u_0, v\}. \end{array}$$

Corollary 2.42. *If X is Baire space, then a function $u \in \mathcal{NL}(X)$ is real valued and continuous on a residual set, that is a set with complement of first Baire category*

On the space $\mathcal{NL}(X)$ we introduce the algebraic operations $\oplus$, $\odot$ and $\otimes$ as the usual point-wise operations on real functions, with understanding that the result of any operation involving $\pm\infty$ is again $\pm\infty$, with the appropriate sign determined as usual [63]. Note that, for $u, v \in \mathcal{NL}(X)$, the function $u \oplus v$ may fail to be normal lower semi-continuous. Indeed, if

$$u(x) = \left\{ \begin{array}{ll} 1 & if x > 0 \\ 0 & if x \leq 0 \end{array} \right.$$

and

$$v(x) = \left\{ \begin{array}{ll} -1 & if x \geq 0 \\ 0 & if x < 0 \end{array} \right.$$

then

$$(u \oplus v)(x) = \begin{cases} -1 & if x = 0 \\ 0 & if x \neq 0 \end{cases}$$

so that $u \oplus v \notin \mathcal{NL}(\mathbb{R})$. We therefore define the algebraic oprations on $\mathcal{NL}(X)$ in the following way. For $u, v \in \mathcal{NL}(X)$ and $\alpha \in \mathbb{R}$ we set

$$u + v = I(S(u \oplus v)), \tag{2.39}$$
$$\alpha u = I(S(\alpha \odot u)), \tag{2.40}$$
$$uv = I(S(u \otimes v)) \tag{2.41}$$

Theorem 2.43. *Let space the $\mathcal{NL}(X)$ be equipped with the algebraic operations defined in* (2.39) - (2.41). *Then the following statements are true.*

(i) $\mathcal{NL}(X)$ is a unital Archimedean f-algebra and hence a commutative algebra.

(ii) The multiplication on $\mathcal{NL}(X)$ is σ-order continuous.

2.2.4 Structure and regularity results

In this section, we recall briefly the main ideas underlying the reformulation of the OCM in terms of convergence spaces. For $l \in \overline{\mathbb{N}}$, consider the space

$$\mathcal{ML}^m(\Omega) = \{u \in \mathcal{NL}(\Omega) : u \in C^m_{nd}(\Omega)\}. \tag{2.42}$$

The space $\mathcal{ML}^m(\Omega)$ is a sublattice and a subalgebra of $\mathcal{NL}(\Omega)$. In particular, the space

$$\mathcal{ML}^0(\Omega) = \{u \in \mathcal{NL}(\Omega) : u \in C^0_{nd}(\Omega)\}, \tag{2.43}$$

is σ-order dense subalgebra of $\mathcal{NL}(\Omega)$, see [69]. That is, for each $u \in \mathcal{NL}(\Omega)$

$$\begin{array}{ll} \exists & (\lambda_n), (\mu_n) \subset \mathcal{ML}^0(\Omega) : \\ & \begin{array}{ll} (i) & \lambda_n \leq \lambda_{n+1} \leq u \leq \mu_{n+1} \leq \mu_n \quad n \in \mathbb{N}, \\ (ii) & \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\}. \end{array} \end{array} \tag{2.44}$$

The partial derivatives

$$D^p : C^m(\Omega) \longrightarrow C^0(\Omega), \quad p \in \mathbb{N}^n, \ |p| \leq m$$

extends to the mappings

$$\mathcal{D}^p : \mathcal{ML}^m(\Omega) \ni u \mapsto (I \circ S)(D^p u) \in \mathcal{ML}^0(\Omega), \quad p \in \mathbb{N}^n, \ |p| \leq m.$$

The partial differential operator (2.2) induces a mapping

$$T : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega) \tag{2.45}$$

defined as follows

$$Tu = (I \circ S)(F(\cdot, u, \cdots \mathcal{D}^p u \cdots)). \tag{2.46}$$

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Within the current content, this gives rise to a first generalization of the PDE (2.1), namely,

$$Tu = f \tag{2.47}$$

where the unknown function u is not restricted to belong to $\mathcal{C}^l(\Omega)$, but may belong to the larger space $\mathcal{ML}^m(\Omega)$. Proceeding in much the same way as in (2.7) - (2.9), we consider the equivalence relation $\sim_T$ induced by T through

$$\begin{array}{l} \forall \quad u, v \in \mathcal{ML}^m(\Omega) : \\ \quad u \sim_T v \Longleftrightarrow Tu = Tv \end{array} \tag{2.48}$$

Denote by $\mathcal{ML}^m_T(\Omega)$ the quotient space $\mathcal{ML}^m(\Omega) \backslash \sim_T$. With the mapping (2.45) one may associate in a canonical way an injective mapping

$$\widehat{T} : \mathcal{ML}^m_T(\Omega) \longrightarrow \mathcal{ML}^0(\Omega) \tag{2.49}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{ML}^0(\Omega) \\ q_T \downarrow & & \downarrow id \\ \mathcal{ML}^m_T(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{ML}^0(\Omega) \end{array} \tag{2.50}$$

commutes. Here, q_T denotes the canonical quotient map associated with the equivalence relation (2.48), and id is the identity map on $\mathcal{ML}^0(\Omega)$. Note that $U \in \mathcal{ML}^m_T(\Omega)$ satisfies the equation

$$\widehat{T}U = f \tag{2.51}$$

if and only if each $u \in U$ satisfies (2.47). Hence a solution $U \in \mathcal{ML}^m_T(\Omega)$ of (2.51) is the $\sim_T$-equivalence class of solutions to (2.47). In order to obtain a further extension of (2.1), the following is introduced on $\mathcal{ML}^0(\Omega)$.

Definition 2.44. Let Λ consists of all nonempty order intervals in $\mathcal{ML}^0(\Omega)$. Let $\mathcal{J}_0$ denote the family of filters on $\mathcal{ML}^0(\Omega) \times \mathcal{ML}^0(\Omega)$ defined as follows:

$$\mathcal{U} \in \mathcal{J}_0 \Longleftrightarrow \left\{ \begin{array}{ll} \exists & k \in \mathbb{N} : \\ \forall & j = 1, \cdots, k : \\ \exists & \Lambda_j = \{I^j_n\} \subseteq \Lambda : \\ \exists & u_j \in \mathcal{NL}(\Omega) : \\ & \quad (i) \quad I^j_{n+1} \subseteq I^j_n, \quad n \in \mathbb{N} \\ & \quad (ii) \quad \sup_n \{\inf_j \{I^j_n\}\} = u_j = \inf_n \{\sup_j \{I^j_n\}\} \\ & \quad (iii) \quad ([\Lambda_1] \times [\Lambda_1]) \cap \cdots \cap ([\Lambda_k] \times [\Lambda_k]) \subseteq \mathcal{U}. \end{array} \right. \tag{2.52}$$

Since $\mathcal{ML}^0(\Omega)$ is an Archimedean Riesz algebra with σ-order continuous multiplication, we have the following as a consequence of Theorem 2.31 and 2.34, see [63].

Proposition 2.45. *The convergence structure $\lambda_{\mathcal{J}_0}$ is a first countable, Hausdorff algebra convergence structure.*

It was shown in [63] that the induced uniform convergence structure on $\mathcal{ML}^0(\Omega)$ agrees with the uniform convergence structure $\mathcal{J}_0$.

Proposition 2.46. *The uniform convergence spaces* $(\mathcal{ML}^0(\Omega), \mathcal{J}_0)$ *and* $(\mathcal{ML}^0(\Omega), \mathcal{J}_{\lambda_0})$ *are uniformly isomorphic. In particular, a filter* $\mathcal{F}$ *on* $\mathcal{ML}^0(\Omega)$ *is Cauchy with respect to* $\mathcal{J}_0$ *if and only if* $\mathcal{F} - \mathcal{F} \longrightarrow 0$ *with respect to* λ_0.

Cauchy sequences on $\mathcal{ML}^0(\Omega)$ are characterized in the following way, see [63]

Proposition 2.47. *A sequence in* $\mathcal{ML}^0(\Omega)$ *is Cauchy with respect to* $\mathcal{J}_0$ *if and only if there exists a set* $B \subset \Omega$ *of first Baire category such that* $(u_n(x))$ *is convergent in* $\mathbb{R}$ *for all* $x \in \Omega \backslash B$.

On the space $\mathcal{ML}^m_T(\Omega)$ we consider the initial uniform convergence structure $\mathcal{J}_T$ with respect to the mapping $\widehat{T}$: For any filter $\mathcal{U} \in \mathcal{ML}^m_T(\Omega) \times \mathcal{ML}^m_T(\Omega)$

$$\mathcal{U} \in \mathcal{J}_T \Longleftrightarrow (\widehat{T} \times \widehat{T})(\mathcal{U}) \in \mathcal{J}_0 \tag{2.53}$$

Since the mapping $\widehat{T}$ is injective, it follows that the space $\mathcal{ML}^m_T(\Omega)$ is uniformly isomorphic to the subspace $\widehat{T}(\mathcal{ML}^m_T(\Omega))$ of $\mathcal{ML}^0(\Omega)$, see [72]. Thus the mapping $\widehat{T}$ is a uniformly continuous embedding.

The Wyler completion of the space $(\mathcal{ML}^0(\Omega), \mathcal{J}_0)$ is constructed as the space $\mathcal{NL}(\Omega)$ of nearly finite normal lower semi-continuous functions equipped with the uniform convergence structure $\mathcal{J}_0^\sharp$ defined as follows, see [72].

Definition 2.48. *Let* Λ *consists of all nonempty order intervals in* $\mathcal{ML}^0(\Omega)$. *Let* $\mathcal{J}_0^\sharp$ *denote the family of filters on* $\mathcal{NL}(\Omega) \times \mathcal{NL}(\Omega)$ *defined as follows*

$$\mathcal{U} \in \mathcal{J}_0^\sharp \Longleftrightarrow \left\{ \begin{array}{ll} \exists & k \in \mathbb{N} : \\ \forall & i = 1, \cdots, k \\ \exists & \Lambda_i = \{I_n^i : n \in \mathbb{N}\} \subseteq \Lambda : \\ \exists & u_i \in \mathcal{NL}(\Omega) : \\ & \begin{array}{ll} (i) & I_{n+1}^i \subseteq I_n^i \quad n \in \mathbb{N} \\ (ii) & \sup_N\{\inf_i\{I_n^i\}\} = u_i = \inf_n\{\sup_i\{I_n^i\}\} \\ (iii) & \bigcap_{i=1}^k (([\Lambda_i] \times [\Lambda_i]) \cap ([u_i] \times [u_i])) \subseteq \mathcal{U}. \end{array} \end{array} \right. \tag{2.54}$$

The completion of the space $\mathcal{ML}^m_T(\Omega)$ is denoted by $\mathcal{NL}^m_T(\Omega)$, and is realized as a subspace of $\mathcal{NL}^0(\Omega)$. In particular, the mapping $\widehat{T}$ extends uniquely to an injective uniformly continuous mapping

$$\widehat{T}^\sharp : \mathcal{NL}^m_T(\Omega) \longrightarrow \mathcal{NL}^0(\Omega).$$

This is summarized in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{ML}^m_T(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{ML}^0(\Omega) \\ \Big\downarrow \phi & & \Big\downarrow \psi \\ \mathcal{NL}^m_T(\Omega) & \xrightarrow{\quad \widehat{T}^\sharp \quad} & \mathcal{NL}^0(\Omega) \end{array} \tag{2.55}$$

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Here ϕ and ψ are the canonical uniformly continuous embeddings associated with the completions $\mathcal{NL}_T^m(\Omega)$ and $\mathcal{NL}(\Omega)$, of $\mathcal{ML}_T^m(\Omega)$ and $\mathcal{ML}^0(\Omega)$ respectively. A first existence and uniqueness result for generalized solutions of the PDE (2.1) is given below.

Theorem 2.49. *For every $f \in C^0(\Omega)$ satisfying (2.3), there exists a unique $U^\sharp \in \mathcal{NL}_T^m(\Omega)$ such that*

$$\widehat{T}^\sharp U^\sharp = f.$$

Theorem 2.49 is essentially a reformulation of Theorem 2.2 in the context of uniform convergence spaces. Thus the mentioned deficiencies of the OCM also applies to Theorem 2.49. However, by introducing a parallel construction of spaces of generalized functions, which is independent of the particular nonlinear operator T, we may resolve these difficulties. In this regard, consider the space $C_{nd}^m(\Omega)$ defined in (2.4).

Equip the space $\mathcal{ML}^m(\Omega)$ with the initial uniform convergence structure $\mathcal{J}_m$ with respect to the mappings

$$\mathcal{D}^p : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega),\ |p| \leq m \tag{2.56}$$

Definition 2.50. *A filter on $\mathcal{ML}^m(\Omega)$ belongs to $\mathcal{J}_m$ if and only if*

$$\begin{array}{l} \forall \quad p \in \mathbb{N}^n,\ |p| \leq m : \\ \quad (\mathcal{D}^p \times \mathcal{D}^p)(\mathcal{U}) \in \mathcal{J}_0. \end{array}$$

Proposition 2.51. *A filter $\mathcal{F}$ on $\mathcal{ML}^m(\Omega)$ converges to $u \in \mathcal{ML}^m(\Omega)$ with respect to the induced convergence structure λ_m if and only if $\mathcal{D}^p(\mathcal{F})$ converges to $\mathcal{D}^p u$ in $\mathcal{ML}^0(\Omega)$ for every $p \in \mathbb{N}$, $|p| \leq m$. In particular, a sequence (u_n) converges to $u \in \mathcal{ML}^m(\Omega)$ if and only if*

$$\begin{array}{l} \forall \quad p \in \mathbb{N}^n,\ |p| \leq m : \\ \quad \mathcal{D}^p(u_n) \text{ order converges to } \mathcal{D}^p(u) \in \mathcal{ML}^0(\Omega). \end{array}$$

It is clear from Definition 2.50 that each of the mappings in (2.56) is uniformly continuous with respect to the uniform convergence structure, $\mathcal{J}_m$ and $\mathcal{J}_0$ of $\mathcal{ML}^m(\Omega)$ and $\mathcal{ML}^0(\Omega)$, respectively. In fact, see [70, 72], the mapping

$$\mathbf{D} : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega)^M$$

defined through

$$\mathbf{D}(u) = (\mathcal{D}^p u)_{|p| \leq m}.$$

is a uniformly continuous embedding. Therefore, see [72], the mapping $\mathbf{D}$ extends uniquely to an injective, uniformly continuous mapping

$$\mathbf{D}^\sharp : \mathcal{NL}^m(\Omega) \longrightarrow \mathcal{NL}^0(\Omega)^M. \tag{2.57}$$

where $\mathcal{NL}^m(\Omega)$ denotes the completion of $\mathcal{ML}^m(\Omega)$. This gives a first and basic regularity result: The generalized functions in $\mathcal{NL}^0(\Omega)$ may be represented, through their generalized partial derivatives, as normal lower semi-continuous functions. Indeed, the mapping (2.57) may be represented as

$$\mathbf{D}^\sharp(u) = (\mathcal{D}^{p^\sharp} u^\sharp)_{|p| \leq m}$$

where, for $|p| \leq m$, $(\mathcal{D}^{p^\sharp})$ denotes the unique uniformly continuous extension of $\mathcal{D}^p$ to $\mathcal{NL}^m(\Omega)$.

In order to formulate the concept of a generalized solution of (2.1) in the space $\mathcal{NL}^m(\Omega)$, the operator

$$T : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega)$$

must be extended to $\mathcal{NL}^m(\Omega)$ in a meaningful way. In this regard, we have the following

Theorem 2.52. *The mapping*

$$T : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega)$$

defined in (2.45) - (2.46) *is uniformly continuous.*

In view of Theorem 2.52, the mapping T extends uniquely to a uniformly continuous mapping

$$T^\sharp : \mathcal{NL}^m(\Omega) \longrightarrow \mathcal{NL}(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{ML}^0(\Omega) \\ \varphi \downarrow & & \downarrow \psi \\ \mathcal{NL}^m(\Omega) & \xrightarrow{\quad T^\sharp \quad} & \mathcal{NL}(\Omega) \end{array} \tag{2.58}$$

commutes. Here φ and ψ are the uniformly continuous embeddings associated with the completion $\mathcal{NL}^m(\Omega)$ and $\mathcal{NL}(\Omega)$ of $\mathcal{ML}^m(\Omega)$ and $\mathcal{ML}^0(\Omega)$, respectively. The main existence result for generalised solutions of (2.1) in $\mathcal{NL}^m(\Omega)$ is the following

Theorem 2.53. *If for each $x \in \Omega$ there is some $\zeta \in \mathbb{R}^M$ and neighborhoods V and W of x and ζ so that*

$$F(x, \zeta) = h(x)$$

and

$$F : V \times W :\longrightarrow \mathbb{R}$$

is open, then there exists $u^\sharp \in \mathcal{NL}^l(\Omega)$ such that

$$T^\sharp u^\sharp = h.$$

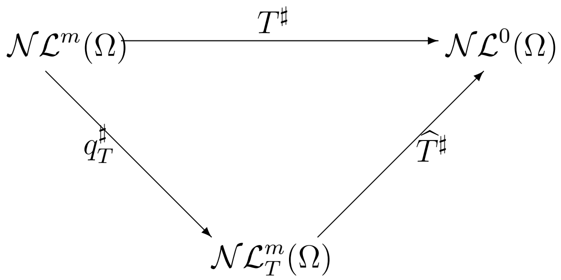
Theorem 2.53 provides some insight into the meaning and structure of the unique generalized solution of (2.1) obtained Theorem 2.49. In this regard, observe that the diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{ML}^0(\Omega) \\ & {\scriptstyle q_T} \searrow \quad \nearrow {\scriptstyle \widehat{T}} & \\ & \mathcal{ML}^m_T(\Omega) & \end{array} \tag{2.59}$$

commutes. Since T is uniformly continuous and $\widehat{T}$ is a uniformly continuous embedding, it follows that q_T is uniformly continuous. Therefore there exists a unique uniformly continuous extension

$$q_T^\sharp : \mathcal{NL}_T^m(\Omega) \longrightarrow \mathcal{NL}(\Omega)$$

of q_T. Since the diagram (2.59), the diagram



(2.60)

also commutes. Since $\widehat{T}^\sharp$ is injective, it follows that for $u^\sharp, v^\sharp \in \mathcal{NL}^m(\Omega)$,

$$q_T^\sharp u^\sharp = q_T^\sharp v^\sharp \Longleftrightarrow T^\sharp u^\sharp = T^\sharp v^\sharp.$$

That is, $q_T^\sharp$ is the canonical quotient map associated with the equivalence relation

$$u^\sharp \sim_{T^\sharp} v^\sharp \Longleftrightarrow T^\sharp u^\sharp = T^\sharp v^\sharp$$

on $\mathcal{NL}^l(\Omega)$. Therefore, for a PDE (2.1) that satisfies the conditions of Theorem 2.53, and hence also (2.3), we may interpret the generalized solution $U^\sharp \in \mathcal{NL}_T^m(\Omega)$ as the $\sim_T^\sharp$-equivalence class

$$\{u^\sharp \in \mathcal{NL}^m(\Omega) | T^\sharp u^\sharp = h\}.$$

Existence of C^∞-smooth generalized solutions

In the previous section we discussed the existence of generalised solution to nonlinear PDEs of order at most m and obtained such solution in the space $\mathcal{NL}^m(\Omega)$ of generalized functions which admits only generalized partial derivatives of arbitrary but fixed and finite order m. In this section we discuss the existence of generalised solution in the space $\mathcal{NL}^\infty(\Omega)$ of generalised functions which admit generalised partial derivatives of all orders. Details of the result discussed here are found in [66].

We consider the PDE,

$$T(x, D)u(x) = h(x), \ x \in \Omega. \tag{2.61}$$

The differential operator $T(x, D)$ is defined by a C^∞-smooth mapping

$$F : \Omega \times \mathbb{R}^M \longrightarrow \mathbb{R} \tag{2.62}$$

through

$$T(x, D)u(x) = F(x, u(x), \cdots, D^p u(x), \cdots), \ x \in \Omega, \ |p| \leq m \tag{2.63}$$

for sufficiently smooth $u : \Omega \longrightarrow \mathbb{R}$. The right-hand term h is a $\mathcal{C}^\infty$-smooth function. Assume that the PDE (2.61) satisfies

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & \xi(x) \in \mathbb{R}^{\mathbb{N}^n} : \\ \exists & V \in \mathcal{V}_x,\ W \in \mathcal{V}_{\xi(x)} : \\ & 1)\ F^\infty : V \times W \longrightarrow \mathbb{R}^{\mathbb{N}^n} \text{ open}, \\ & 2)\ F^\infty(x, \xi(x)) = (D^\beta f(x))_{\beta \in \mathbb{N}^n} \end{array} \tag{2.64}$$

where $\mathbb{R}^{\mathbb{N}^n}$ is equipped with the product topology τ, the mapping

$$F^\infty : \Omega \times \mathbb{R}^{\mathbb{N}^n} \longrightarrow \mathbb{R}^{\mathbb{N}^n}$$

is defined by setting

$$F^\infty(x, (\xi_\alpha)_{\alpha \in \mathbb{N}^n}) = (F^\beta(x, \cdots, \xi_\alpha, \cdots,))_{\beta \in \mathbb{N}^n}, \tag{2.65}$$

where, for each $\beta \in \mathbb{N}^n$, the mapping

$$F^\beta : \Omega \times \mathbb{R}^{\mathbb{N}^n} \longrightarrow \mathbb{R}$$

is defined by setting

$$D^\beta(T(x, D)u(x)) = F^\beta(x, \cdots, D^\alpha u(x), \cdots), \qquad |\alpha| \leq m + |\beta| \tag{2.66}$$

for all $u \in C^\infty(\Omega)$.

The nonlinear operator $T(x, D)$, which is a mapping

$$T : C^\infty(\Omega) \longrightarrow C^\infty(\Omega) \tag{2.67}$$

may be extended to the mapping

$$T : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{ML}^\infty(\Omega)$$

defined by setting

$$Tu = (I \circ S)(F(., u, \cdots, \mathcal{D}^p u, \cdots)),\ |p| \leq l. \tag{2.68}$$

Thus we have the first extension of the nonlinear PDE (2.61) given as

$$Tu = h \tag{2.69}$$

where u is in $\mathcal{ML}^\infty(\Omega)$.

Theorem 2.54. *The mapping $T : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{ML}^\infty(\Omega)$ defined through* (2.68) *is uniformly continuous.*

As a consequence of Theorem 2.54, there exists a unique uniformly continuous extension

$$T^\sharp : \mathcal{NL}^\infty(\Omega) \longrightarrow \mathcal{NL}^\infty(\Omega)$$

of T. This give rise to the concept of generalised solution of (2.61) as a solution $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ of the extended equation

$$T^\sharp u^\sharp = h. \tag{2.70}$$

The main existence result for the C^∞-smooth PDE (2.61) is the following, see [66].

Theorem 2.55. *Consider the nonlinear PDE of the form* (2.61)*. If the nonlinear operator T satisfies* (2.64)*, then there exists some $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ that satisfies* (2.70).

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Part II

Differential algebraic interpretation of the order completion method

Chapter 3

The spaces $\mathcal{NL}^l(\Omega)$ as a chain of algebras

In this chapter we show how the spaces of generalised functions introduced in [70, 72] may be interpreted as a chain of algebras of generalised functions, as discussed in Section 1.5. In this regard, it will be shown that $\mathcal{NL}^l(\Omega)$ is, for each $l \in \overline{\mathbb{N}}$, an algebra of generalised functions admitting an embedding of $\mathcal{C}^l(\Omega)$ as a subalgebra. We then proceed to study the chain structure

$$\mathcal{NL}^\infty(\Omega) \to \cdots \to \mathcal{NL}^l(\Omega) \to \mathcal{NL}^{l-1}(\Omega) \to \cdots \to \mathcal{NL}^0(\Omega).$$

The existence result for generalised solutions of $\mathcal{C}^\infty$-smooth PDEs, Theorem 2.55, is interpreted in the differential-algebraic frame work.

3.1 $\mathcal{NL}^l(\Omega)$ as an algebra of generalised functions

Recall that, for $l \in \overline{\mathbb{N}}$, $\mathcal{NL}^l(\Omega)$ is the completion of $\mathcal{ML}^l(\Omega)$ with respect to the Hausdorff uniform order convergence structure given in Definition 2.54. Therefore, applying the abstract construction of the completion of a uniform convergence space outlined in Section 2.2.4, we may express the *set* $\mathcal{NL}^l(\Omega)$ as

$$\mathcal{NL}^l(\Omega) = C[\mathcal{ML}^l(\Omega)]/\sim_C .$$

The structure of the set $\mathcal{NL}^l(\Omega)$ is therefore determined, to a good extent, by the structure of the Cauchy filters on $\mathcal{ML}^l(\Omega)$. We thus turn first to an investigation of the structure of such filters.

Proposition 3.1. *The space $\mathcal{ML}^l(\Omega)$ is a subalgebra of $\mathcal{NL}(\Omega)$. Furthermore, the differential operators*

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}, \quad |p| \le l$$

are linear and satisfy the Leibnitz rule

$$\mathcal{D}^p(uv) = \sum \binom{p}{q} \mathcal{D}^{p-q} u \mathcal{D}^q v$$

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Proof. For $u, v \in \mathcal{ML}^l(\Omega)$ and $\alpha \in \mathbb{R}$, it follows from the definition of $\mathcal{ML}^l(\Omega)$ and (2.39) - (2.41) that there exists a closed, nowhere dense subset Γ of Ω so that

$$(u+v)(x) = u(x) + v(x),\ (uv)(x) = u(x)v(x) \text{ and} (\alpha u)(x) = \alpha u(x), \quad x \in \Omega\backslash\Gamma$$

The set Γ may be chosen in such a way that $u, v \in C^l(\Omega\backslash\Gamma)$. Then $u{+}v,\ uv,\ \alpha u \in C^l(\Omega\backslash\Gamma)$. Hence $u+v,\ uv,\ \alpha u \in \mathcal{ML}^l(\Omega)$.

Furthermore, for $|p| \leq l$,

$$\begin{aligned}\mathcal{D}^p(\alpha u + \beta v)(x) &= \alpha D^p u(x) + \beta D^p v(x)\\ &= \alpha \mathcal{D}^p u(x) + \beta \mathcal{D}^p v(x), \quad x \in \Omega\backslash\Gamma\end{aligned}$$

It follows from Proposition 2.41(P1) that $\mathcal{D}^p(\alpha u + \beta v) = \alpha \mathcal{D}^p u + \beta \mathcal{D}^p v$, so that $\mathcal{D}^p$ is linear. That $\mathcal{D}^p$ satisfies the Leibnitz rule follows in the same way. □

Proposition 3.2. *The induced convergence structure λ_l on $\mathcal{ML}_l(\Omega)$ is a Hausdorff and first countable algebra convergence structure.*

Proof. According to Proposition 2.51, a filter $\mathcal{F}$ on $\mathcal{ML}^l(\Omega)$ converges to $u \in \mathcal{ML}^l(\Omega)$ if and only if $\mathcal{D}^p(\mathcal{F})$ converges to $\mathcal{D}^p(u)$ in $\mathcal{ML}^0(\Omega)$ with respect to λ_0 for every $|p| \leq l$. Assume that $\mathcal{F} \longrightarrow u$ and $\mathcal{G} \longrightarrow v$ in $\mathcal{ML}^l(\Omega)$. Then $\mathcal{D}^p(\mathcal{F}) \longrightarrow \mathcal{D}^p(u)$ and $\mathcal{D}^p(\mathcal{G}) \longrightarrow \mathcal{D}^p(v)$ in $\mathcal{ML}^0(\Omega)$, $|p| \leq l$. By Proposition 2.45,

$$\mathcal{D}^p(\mathcal{F} + \mathcal{G}) = \mathcal{D}^p(\mathcal{F}) + \mathcal{D}^p(\mathcal{G}) \longrightarrow \mathcal{D}^p(u) + \mathcal{D}^p(u)$$

in $\mathcal{ML}^0(\Omega)$, $|p| \leq l$. Hence $\mathcal{F} + \mathcal{G} \longrightarrow u + v$ in $\mathcal{ML}^l(\Omega)$.

In the same way it follows that, for $\alpha \in \mathbb{R}$, $(\mathcal{V} + \alpha)(\mathcal{F}) \longrightarrow \alpha u$, with $\mathcal{V}$ denoting the neighbourhood filter at $0 \in \mathbb{R}$. Lastly,

$$\mathcal{D}^p(\mathcal{F}\mathcal{G}) \supseteq = \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} \mathcal{D}^{p-q} u \mathcal{D}^q v, \quad |p| \leq l.$$

By Proposition 2.45,

$$\sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} \mathcal{D}^{p-q} \mathcal{F} \mathcal{D}^q \mathcal{G} \longrightarrow \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} \mathcal{D}^{p-q} u \mathcal{D}^q v = \mathcal{D}^p(uv) \text{ in } \mathcal{ML}^0(\Omega).$$

Hence $\mathcal{F}\mathcal{G} \longrightarrow uv$ in $\mathcal{ML}^l(\Omega)$.

To see that λ_l is Hausdorff, consider a filter $\mathcal{F}$ on $\mathcal{ML}^l(\Omega)$ so that $\mathcal{F} \longrightarrow u$ and $\mathcal{F} \longrightarrow v$. Then $[\mathcal{F}]_{\mathcal{ML}^0(\Omega)} \longrightarrow u$ and $[\mathcal{F}]_{\mathcal{ML}^0(\Omega)} \longrightarrow v$ in $\mathcal{ML}^0(\Omega)$. Since λ_0 is Hausdorff, it follows that u = v so that λ_l is Hausdorff. It remains to show that λ_l is first countable. Consider a filter $\mathcal{F} \in \lambda_l(0)$. Then for each $|p| \leq l$ there exists a sequence (μ_n^p) in $\mathcal{ML}^0(\Omega)$ that decreases to 0, and

$$\mathcal{D}^p(\mathcal{F}) \supseteq [\{[-\mu_n^p, \mu_n^p] | n \in \mathbb{N}\}]$$

Let $\mu_n = \sup\{\mu_n^p \mid |p| \leq l\}$, $n \in \mathbb{N}$. Then (μ_n) decreases to 0 in $\mathcal{ML}^0(\Omega)$, and

$$\mathcal{D}^p(\mathcal{F}) \supseteq [\{[-\mu_n, \mu_n] \mid n \in \mathbb{N}\}], \quad |p| \leq l.$$

For each $n \in \mathbb{N}$ let $G_n = \{u \in \mathcal{ML}^l(\Omega) \mid \mathcal{D}^p u \in [-\mu_n, \mu_n]\}$. Clearly, $g = [\{G_n \mid n \in \mathbb{N}\}] \subseteq \mathcal{F}$. Furthermore, $\mathcal{D}^p(g\mathcal{G}) \supseteq [\{[-\mu_n, \mu_n] | n \in \mathbb{N}\}]$, $|p| \leq l$. So that $\mathcal{D}^p(g) \longrightarrow 0$ in $\mathcal{ML}(\Omega)$, $|p| \leq l$. Hence $\mathcal{G} \longrightarrow 0$ in $\mathcal{ML}^l(\Omega)$. Since $\mathcal{ML}^l(\Omega) \ni u \mapsto u + v \in \mathcal{ML}^l(\Omega)$ is continuous for every $v \in \mathcal{ML}^l(\Omega)$ it follows that $\mathcal{ML}(\Omega)$ is first countable. □

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Proposition 3.3. *The uniform convergence structure* $\mathcal{J}_l$ *on* $\mathcal{ML}^l(\Omega)$ *is the uniform convergence structure induced by the convergence structure* λ_l.

Proof. According to Definition 2.50,

$$\mathcal{U} \in \mathcal{J}_l \Longleftrightarrow (\mathcal{D}^p \times \mathcal{D}^p)(\mathcal{U}) \in \mathcal{J}_0, \quad |p| \leq l. \tag{3.1}$$

It follows from (2.15) and Proposition 2.46that

$$\mathcal{U} \in \mathcal{J}_l \Longleftrightarrow \left(\begin{array}{ll} \forall & p \in \mathbb{N}^n, \ |p| \leq l : \\ \exists & \mathcal{F}_p \in \lambda_0(0) : \\ & \Delta(\mathcal{F}_p) \subseteq (\mathcal{D}^p \times \mathcal{D}^p)(\mathcal{U}). \end{array} \right. \tag{3.2}$$

In particular, upon setting

$$\mathcal{F} = \bigcap_{|p| \leq l} \mathcal{F}_p,$$

in (3.2), we find that

$$\mathcal{U} \Longleftrightarrow \left(\begin{array}{ll} \exists & \mathcal{F} \in \lambda_0(0) : \\ \forall & p \in \mathbb{N}^n, \ |p| \leq l : \\ & \Delta(\mathcal{F}) \subseteq (\mathcal{D}^p \times \mathcal{D}^p)(\mathcal{U}). \end{array} \right. \tag{3.3}$$

Let

$$\mathcal{G} = [\{\{u - v \,|(u,v) \in U \,|U \in \mathcal{U}\,\}\}].$$

It follows from the fact that $\mathcal{U}$ is a filter on $\mathcal{ML}^l(\Omega) \times \mathcal{ML}^l(\Omega)$ that $\mathcal{G}$ is a filter on $\mathcal{ML}^l(\Omega)$. Without loss of generality we may assume that $\mathcal{F} \subseteq [0]$. Then

$$\mathcal{F} = \{\{u - v|(u,v) \in \Delta(F)\} : \quad F \in \mathcal{F}\}.$$

Hence it follows from (3.3) that

$$\mathcal{F} \subseteq \mathcal{D}^p(\mathcal{G}), \quad |p| \leq l.$$

Thus $\mathcal{G} \longrightarrow 0$ in $\mathcal{ML}^l(\Omega)$. But $\Delta(g) \subseteq \mathcal{U}$ so that (3.3) implies that

$$\mathcal{U} \in \mathcal{J}_l \Longleftrightarrow \left(\begin{array}{ll} \exists & \mathcal{G} \in \lambda_l(0) : \\ & \Delta(\mathcal{G}) \subseteq (\mathcal{U}). \end{array} \right. \tag{3.4}$$

This completes the proof. □

Basd on the abstract construction of the completion of a uniform converence space, as discussed in Section 2.2.1, we may represent the set $\mathcal{NL}^l(\Omega)$ as

$$\mathcal{NL}^l(\Omega) = C[\mathcal{ML}^l(\Omega)]/ \sim_C, \tag{3.5}$$

where, due to Proposition 3.3,

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \mathcal{F} - \mathcal{G} \in \lambda_l(0).$$

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The representation of $\mathcal{NL}^l(\Omega)$ can be further particularised. By Propostition 3.2, λ_l is first countable. Hence for $\mathcal{F} \in C[\mathcal{ML}^l(\Omega)]$ there exists $\mathcal{G} = [\{G_n | n \in \mathbb{N}\}] \longrightarrow 0$ in $\mathcal{ML}^l(\Omega)$ so that $\mathcal{G} \subseteq \mathcal{F} - \mathcal{F}$. Thus

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ \exists & F_n \in \mathcal{F} : \\ & F_n - F_n \subseteq G_n. \end{array} \tag{3.6}$$

For each $n \in \mathbb{N}$, select $u \in F_1 \cap \cdots \cap F_n$. Then $\langle u_n \rangle - \langle u_n \rangle \supseteq \mathcal{G}$ so that (u_n) is a Cauchy sequence in $\mathcal{ML}^l(\Omega)$. Furthermore, $\langle u_n \rangle \sim_C \mathcal{F}$ so that each $\sim_C$-equivalence class contains a Cauchy sequence. Therefore we may represent $\mathcal{NL}^l(\Omega)$ as

$$\mathcal{NL}^l(\Omega) = C_s[\mathcal{ML}^l(\Omega)]/\sim_{C_s}$$

where $C_s[\mathcal{ML}^l(\Omega)]$ denotes the set of Cauchy sequences in $\mathcal{ML}^l(\Omega)$, and for $(u_n), (v_n) \in C_s[\mathcal{ML}^l(\Omega)$,

$$(u_n) \sim_C (v_n) \Longleftrightarrow \langle u_n - v_n \rangle \in \lambda_l(0).$$

In view of (3.5), the structure of $\mathcal{NL}^l(\Omega)$ depends only on the properties of the Cauchy sequences in $\mathcal{ML}^l(\Omega)$. In this regard, we have the following

Proposition 3.4. *A sequence (u_n) in $\mathcal{ML}^l(\Omega)$ is Cauchy sequence with respect to the uniform convergence structure on $\mathcal{J}_l$ if and only if there exists a residual set $R \subset \Omega$ such that $(\mathcal{D}^p u_n(x))$ is a convergent sequence in $\mathbb{R}$ for each $x \in R$ and $p \in \mathbb{N}^n$ with $|p| \leq l$.*

Proof. It follows from Proposition 3.2 and 3.3 that a sequence (u_n) in $\mathcal{ML}^l(\Omega)$ is Cauchy if and only if $(\mathcal{D}^p u_n)$ is Cauchy in $\mathcal{ML}^0(\Omega)$ for every $|p| \leq l$. The result now follows from Proposition 2.47. □

By Proposition 3.4, we have that

$$(u_n) \sim_{C_s} (v_n) \Longleftrightarrow \left(\begin{array}{ll} \exists & R \subseteq \Omega, \text{ a residual set} : \\ \forall & p \in \mathbb{N}^n, \ |p| \leq l, \ x \in R : \\ & \mathcal{D}^p u_n(x) - \mathcal{D}^p v_n(x) \to 0 \text{ in } \mathbb{R}. \end{array} \right) \tag{3.7}$$

In order to represent the space $\mathcal{NL}^l(\Omega)$ as an algebra of generalized functions, we show that each $\sim_{C_s}$-equivalence class contains a sequence of C^l-smooth functions. To do this, we make use of the Principle of Partition of Unity, see [60].

Theorem 3.5. *Let O be a locally finite open cover of an open subset Ω of $\mathbb{R}^n$. Then there is a collection*

$$\{\phi_U : \Omega \longrightarrow [0,1] : U \in O\}$$

of C^l-smooth functions ϕ_U such that the following hold:

(i) For each $U \in O$, the support of ϕ_U is contained in U.

(ii) $\sum_{U \in O} \phi_U(x) = 1$, for each $x \in M$.

A consequence of Theorem 3.5 is that disjoint, closed sets in Ω are separated by C^l-smooth, real valued functions. In this regard, let A and B be disjoint, nonempty, closed subsets of Ω. Then it follows from Theorem 3.5 that

$$\begin{array}{ll} \exists & \phi \in C^l(\Omega,[0,1]) : \\ & (1)\ x \in A \Longrightarrow \phi(x) = 1 \\ & (2)\ x \in B \Longrightarrow \phi(x) = 0 \end{array} \tag{3.8}$$

Lemma 3.6. *Let (u_n) be a Cauchy sequence in $\mathcal{ML}^l(\Omega)$ with respect to $\mathcal{J}_l$. Then $C^l(\Omega)^{\mathbb{N}} \cap [(u_n)]_{C_s} \neq \emptyset$, where $[(u_n)]_{C_s}$ denotes the $\sim_{C_s}$-equivalence class generated by (u_n).*

Proof. Let (u_n) be a Cauchy sequence in $\mathcal{ML}^l(\Omega)$. Then for each $n \in \mathbb{N}$, there exists a closed, nowhere dense set $\Gamma_n \subset \Omega$ so that $u_n \in C^l(\Omega \backslash \Gamma_n)$. For each $k \in \mathbb{N}$, let

$$B_n^k = cl\left(\left\{ x \in \Omega \left| \begin{array}{ll} \exists & y \in \Gamma_n : \\ & \|x - y\| \leq \frac{1}{k} \end{array} \right.\right\}\right)$$

and

$$A_n^k = \left\{ x \in \Omega \left| \begin{array}{ll} \exists & y \in \Gamma_n : \\ & \|x - y\| \geq \frac{2}{k} \end{array} \right.\right\}.$$

Then for fixed $n, k \in \mathbb{N}$, B_n^k and A_n^k are disjoint, closed subsets of Ω, and

$$\Gamma_n \subset B_n^k.$$

By Theorem 3.5, there exist for all $n, k \in \mathbb{N}$ a function $\phi_k^n \in C^l(\Omega, [0,1])$ so that

$$x \in A_n^k \Longrightarrow \phi_k^n(x) = 1$$

and

$$x \in B_n^k \Longrightarrow \phi_k^n(x) = 0.$$

Since $u_n \in C^l(\Omega \backslash \Gamma_n)$, it follows that

$$v_{n,k} = u_n \phi_k^n \in C^l(\Omega).$$

Furthermore,

$$v_{n,k}(x) = u_n(x), \quad x \in A_n^k.$$

Since $A_n^k \subset A_n^{k+1}$ for all $n, k \in \mathbb{N}$ and

$$\bigcup_{k \in \mathbb{N}} A_n^k = \Omega \backslash \Gamma_n,$$

it follows that

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma_n : \\ \exists & V \subset \Omega \backslash \Gamma_n \text{ open},\ x \in V : \\ \exists & K_V \in \mathbb{N} : \\ \forall & k \geq K_V,\ y \in V : \\ & u_n(x) = v_{n,k}(x). \end{array}$$

It therefore follows that $(\mathcal{D}^p v_{n,k}(x)) \longrightarrow \mathcal{D}^p u_n(x)$ in $\mathbb{R}$ for every $x \in \Omega \backslash \Gamma_n$.

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Let $R_0 = \Omega \setminus \bigcup \Gamma_n$. It follows that for each $n \in \mathbb{N}$ there exists a sequence $(v_{n,k})$ of C^l-smooth functions on Ω so that

$$(\mathcal{D}^p v_{n,k}(x)) \longrightarrow \mathcal{D}^p u_n(x), \ x \in R_0, \ |p| \leq l.$$

But (u_n) is Cauchy in $\mathcal{ML}^l(\Omega)$ so there exists a residual set $R_1 \subseteq \Omega$ so that $(\mathcal{D}^p u_n(x))$ is convergent in $\mathbb{R}$ to same $\alpha(x) \in \mathbb{R}, \ |p| \leq l$. Let $R = R_0 \cap R_1$. Then R is a residual subset of Ω, and

$$(\mathcal{D}^p v_{n,k}(x)) \longrightarrow \mathcal{D}^p(u_n)(x), \ x \in \mathbb{R}, \ |p| \leq l,$$

and

$$(\mathcal{D}^p u_n(x)) \longrightarrow \alpha(x), \quad x \in R, \ |p| \leq l$$

Thus there exists a strictly increasing sequence (k_n) of natural numbers so that

$$(\mathcal{D}^p v_{n,k_n}(x)) \longrightarrow \alpha(x), \ x \in R, \ |p| \leq l \qquad \text{(see Proposition3.4).}$$

It now follows from Proposition 3.4 that $(v_{n,k_n}) \subset C^l(\Omega)$ is Cauchy in $\mathcal{ML}^l(\Omega)$. Furthermore,

$$(\mathcal{D}^p u_n(x) - \mathcal{D}^p v_{n,k_n}(x)) \longrightarrow 0, \ x \in R, \ |p| \leq l$$

so that $(u_n) \sim_{C_s} (v_{n,k_n})$ by (3.7). This completes the proof. □

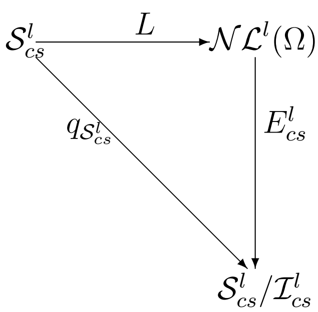
The main result of this section is the following.

Theorem 3.7. *Let $\mathcal{S}^l_{cs} = C_s[\mathcal{ML}^l(\Omega)] \cap C^l(\Omega)^{\mathbb{N}}$ and $\mathcal{I}^l_{cs} = \lambda_l(0) \cap C^l(\Omega)^{\mathbb{N}}$. Then*

(i) $\mathcal{S}^l_{cs}$ *is a subalgebra of* $C^l(\Omega)^{\mathbb{N}}$ *and* $\mathcal{I}^l_{cs}$ *is an ideal in* $\mathcal{S}^l_{cs}$.

(ii) $\Delta(C^l(\Omega)) \subseteq \mathcal{S}^l_{cs}$ *and* $\Delta(C^l(\Omega)) \cap \mathcal{I}^l_{cs} = \{0\}$.

(iii) There exists a bijective mapping $E^l_{cs} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$ *such that the diagram*



(3.9)

commutes. Here, $q_{\mathcal{S}^l_{cs}}$ *is the canonical mapping associated with the quotient algebra* $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$*, and the mapping L is defined as*

$$L : \mathcal{S}^l_{cs} \ni u = (u_n) \mapsto u^\sharp \in \mathcal{NL}^l(\Omega), \tag{3.10}$$

where $u^\sharp$ *is the limit of* (u_n) *in* $\mathcal{NL}^l(\Omega)$

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Proof. The result in (i) follows from Proposition 3.4. Indeed, by Proposition 3.4, $(a_n), (b_n) \in \mathcal{S}^l_{cs}$ if and only if there exist residual sets R_a and R_b such that $(D^p a_n(x))$ and $(D^p b_n(x))$ converge in $\mathbb{R}$ for each $x \in R_a$ and $x \in R_b$ respectively and for all $|p| \leq l$. It follows, by the linearity of D^p and the Leibnitz rule that $\mathcal{S}^l_{cs}$ is closed under addition and multiplication. Since any $(u_n) \in \mathcal{I}^l_{cs}$ is Cauchy, it follows that $\mathcal{I}^l_{cs} \subseteq \mathcal{S}^l_{cs}$. Moreover, for any $(a_n) \in \mathcal{S}^l_{cs}$, there exists residual set R_a such that $(D^p a_n(x))$ converges in $\mathbb{R}$, for every $x \in R_a$ and $|p| \leq l$. For any $(u_n) \in \mathcal{I}^l_{cs}$ there exists residual set R_u such that $(D^p u_n(x))$ converges to 0 for each $x \in R_u$ and $|p| \leq l$. Thus for $x \in R = R_a \cap R_u$ and $|p| \leq l$ we have that $(D^p(a_n u_n)(x))$ converge to 0 in $\mathbb{R}$ by the Leibnitz rule. This implies that $(a_n)(u_n) \in \mathcal{I}^l_{cs}$. Thus $\mathcal{I}^l_{cs}$ is an ideal in $\mathcal{S}^l_{cs}$.
To proof the result in (ii), let $u \in C^l(\Omega)$. Then the sequence $\Delta(u)$ converges to u in $\mathcal{NL}^l(\Omega)$ and hence it is Cauchy. Thus $\Delta(C^l(\Omega)) \subseteq \mathcal{S}^l_{cs}$. Moreover, $\Delta(u) \in \Delta(C^l(\Omega)) \cap \mathcal{I}^l_{cs}$, implies that $\Delta(u) \in \mathcal{I}^l_{cs}$. Since λ_l is Hausdorff, it follows that $\Delta(u)$ converges to $u = 0$. Thus $\Delta(C^l(\Omega)) \cap \mathcal{I}^l_{cs} = \{0\}$.
We now proof the result in (iii). According to Lemma 3.6, the mapping L is surjective. The quotient mapping $q_{\mathcal{S}^l_{cs}}$ is also a surjection. For $u, v \in \mathcal{S}^l_{cs}$ we have that

$$\begin{aligned} L(u) = L(v) &\Longleftrightarrow u^\sharp = v^\sharp \\ &\Longleftrightarrow [u] = [v] \\ &\Longleftrightarrow u \sim_{C_s} v \\ &\Longleftrightarrow (u - v) \in \lambda_l(0) \\ &\Longleftrightarrow u - v \in \mathcal{I}^l_{cs} \\ &\Longleftrightarrow u + \mathcal{I}^l_{cs} = v + \mathcal{I}^l_{cs} \\ &\Longleftrightarrow q_{\mathcal{S}^l_{cs}}(u) = q_{\mathcal{S}^l_{cs}}(v). \end{aligned}$$

Hence $L(u) = L(v)$ if and only if $q_{\mathcal{S}^l_{cs}}(u) = q_{\mathcal{S}^l_{cs}}(v)$, so that the mapping

$$E^l_{cs} : \mathcal{NL}^l(\Omega) \ni u^\sharp \mapsto q_{\mathcal{S}^l_{cs}}(L^{-1}(u^\sharp)) \in \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \tag{3.11}$$

is well defined on $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$.

Furthermore, since L is surjective, the mapping (3.11) is defined at each $u^\sharp \in \mathcal{NL}^l(\Omega)$ so that $q_{\mathcal{S}^l_{cs}}$ being surjective implies that is E^l_{cs} is surjective. Moreover, the mapping E^l_{cs} is injective. Indeed, for $u^\sharp, v^\sharp \in \mathcal{NL}^l(\Omega)$ we have that

$$E^l_{cs}(u^\sharp) = E^l_{cs}(v^\sharp) \Longleftrightarrow u - v \in \mathcal{I}^l_{cs}$$

for some $u \in L^{-1}(u^\sharp)$ and $v \in L^{-1}(v^\sharp)$, so that $u \sim_{C_s} v$. This implies $u^\sharp = v^\sharp$ by (3.5). Thus E^l_{cs} is a bijection. □

3.2 Chain Structure of $\{\mathcal{NL}^l(\Omega) \ : \ l \in \overline{\mathbb{N}}\}$

In this section we show how the spaces of generalized functions $\mathcal{NL}^l(\Omega)$, $l \in \overline{\mathbb{N}}$, may be represented as a chain of algebras of generalized functions. By virtue of the definition of the uniform convergence structure on $\mathcal{ML}^l(\Omega)$, the partial derivative operators

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^k(\Omega), \quad k + |p| \leq l \tag{3.12}$$

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are uniformly continuous. Hence there exists unique uniformly continuous extension

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega), \quad k + |p| \leq l \tag{3.13}$$

of the mappings in (3.12). On the other hand, since $\mathcal{S}^l_{cs} \subset C_s[\mathcal{ML}^l(\Omega)]$ and $\mathcal{I}^l_{cs}$ consists of null sequences in $\mathcal{ML}^l(\Omega)$, it follows by uniform continuity of the mapping in (3.12) that

$$D^p(\mathcal{I}^l_{cs}) \subseteq \mathcal{I}^k_{cs}, \quad \text{and} \quad D^p(\mathcal{S}^l_{cs}) \subseteq \mathcal{S}^k_{cs}, \quad p \in \mathbb{N}^n, \quad |p| \leq l - k, \tag{3.14}$$

so that

$$D^p : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \ni (u) + \mathcal{I}^l_{cs} \mapsto D^p(u) + \mathcal{I}^k_{cs} \in \mathcal{S}^k_{cs}/\mathcal{I}^k_{cs} \tag{3.15}$$

define linear mappings that satisfy the Leibnitz rule.

Proposition 3.8. *The diagram*

$$\begin{array}{ccc} \mathcal{NL}^l(\Omega) & \xrightarrow{\quad \mathcal{D}^{p\sharp} \quad} & \mathcal{NL}^k(\Omega) \\ E^l_{cs} \big\downarrow & & \big\downarrow E^l_{cs} \\ \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} & \xrightarrow{\quad D^p \quad} & \mathcal{S}^k_{cs}/\mathcal{I}^k_{cs} \end{array} \tag{3.16}$$

commutes for all $p \in \mathbb{N}^n$, $l, k \in \overline{\mathbb{N}}$, *so that* $k + |p| \leq l$ *with*

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega), \quad k + |p| \leq l.$$

given by (3.13) *and*

$$D^p : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \longrightarrow \mathcal{S}^k_{cs}/\mathcal{I}^k_{cs} \tag{3.17}$$

given by (3.15).

Proof. Fix $u^\sharp \in \mathcal{NL}^l(\Omega)$. According to Theorem 3.7, $u^\sharp \in L(u)$ for some $u \in \mathcal{S}^l_{cs}$, and $E^l_{cs}(u^\sharp) = u + \mathcal{I}^l_{cs}$. So $\mathcal{D}^p(E^l_{cs}(u^\sharp)) = \mathcal{D}^p(u) + \mathcal{I}^k_{cs}$. But $\mathcal{D}^{p\sharp}u^\sharp = L(\mathcal{D}^p u)$ so that, by Theorem 3.7, $E^k_{cs}(\mathcal{D}^{p\sharp}u^\sharp) = \mathcal{D}^p u + \mathcal{I}^k_{cs} = \mathcal{D}^p(E^l_{cs}(u^\sharp))$. Thus the diagram (3.16) commutes. □

Observe that

$$\mathcal{S}^l_{cs} \subseteq \mathcal{S}^k_{cs} \quad \text{and} \quad \mathcal{I}^l_{cs} \subseteq \mathcal{I}^k_{cs} \tag{3.18}$$

for all $l, k \in \overline{\mathbb{N}}$ such that $k \leq l$. Indeed, it follows directly from the definition of the uniform convergence structure on $\mathcal{ML}^l(\Omega)$ and $\mathcal{ML}^k(\Omega)$, respectively, that the inclusion map

$$\mathcal{ML}^l(\Omega) \ni u \mapsto u \in \mathcal{ML}^k(\Omega)$$

is uniformly continuous. Thus (3.18) follows immediately from the definition of $\mathcal{I}^l_{cs}$ and $\mathcal{S}^l_{cs}$. Thus

$$\gamma^l_k : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \ni (u) + \mathcal{I}^l_{cs} \mapsto u + \mathcal{I}^k_{cs} \in \mathcal{S}^k_{cs})/\mathcal{I}^k_{cs} \tag{3.19}$$

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defines an algebra homomorphism, see (1.42) and (1.43).

In view of Theorem 3.7 and Proposition 3.8, the spaces $\mathcal{NL}^l(\Omega)$, and the differential operators

$$\mathcal{D}^{p\sharp} : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega)$$

with $|p| + k \leq l$, may be identified with the algebras $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$, with differential operators $D^p : \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} \longrightarrow \mathcal{S}^k/\mathcal{I}^k_{cs}$ defined in (3.15). Therefore we denote the algebras $\mathcal{S}^l_{cs}/\mathcal{I}^l_{cs}$ by $\mathcal{NL}^l(\Omega)$.

As a direct application of Theorem 1.15 we now have the following

Theorem 3.9. *With the algebra homomorphism*

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega)$$

define as in (3.19) *and the differential operator*

$$\mathcal{D}^p : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega),$$

with $k + |p| \leq l$ define as in (3.17),

$$\mathbf{A}_{oc} = \{(\mathcal{NL}^l(\Omega), \mathcal{NL}^k(\Omega), \gamma^l_k) \mid k, l \in \overline{\mathbb{N}},\ k \leq l\}$$

is a differential chain of algebra of generalized functions.

Proof. The result follows from (3.14), (3.18) and Theorem 1.15. □

Next we address the issue of embedding smooth functions into the chain $\mathbf{A}_{oc}$ of algebras of generalised functions.

Theorem 3.10. *For each $l \in \mathbb{N}$, there exists an injective algebra homomorphism*

$$\mathcal{E}^l_{cs} : C^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

so that the diagram

$$\begin{array}{ccccc} & & \xrightarrow{\qquad\qquad \gamma^l_h \qquad\qquad} & & \\ \mathcal{NL}^l(\Omega) & \xrightarrow{\gamma^l_k} & \mathcal{NL}^k(\Omega) & \xrightarrow{\gamma^k_h} & \mathcal{NL}^h \\ \uparrow \mathcal{E}^l_{cs} & & \uparrow \mathcal{E}^k_{cs} & & \uparrow \mathcal{E}^h_{cs} \\ C^l(\Omega) & \xrightarrow{\subseteq} & C^k(\Omega) & \xrightarrow{\subseteq} & C^h(\Omega) \end{array} \qquad (3.20)$$

commutes. Here γ^l_k, γ^k_h, γ^l_h, are injective algebra homomorphisms defined by (3.19), *while $\mathcal{E}^l_{cs}$, $\mathcal{E}^h_{cs}$, $\mathcal{E}^k_{cs}$ are linear injective algebra homomorphisms define as in* (1.50).

Proof. Since $\mathcal{S}^l_{cs}$ is contained in the set $C_s[\mathcal{ML}^l(\Omega)]$ of $\mathcal{J}_l$ - Cauchy sequences in $\mathcal{ML}^l(\Omega)$, it follows that $\mathcal{U}^l_{\mathbb{N}}(\Omega) \subseteq \mathcal{S}^l_{cs}$. Furthermore, $\mathcal{I}^l_{cs} \subset \lambda_l(0)$, so that, since λ_l is Hausdorff $\mathcal{I}^l_{cs} \cap \mathcal{U}^l_{\mathbb{N}}(\Omega) = \{0\}$. The result now follows from Theorem 1.18. □

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3.3 Embedding of $\mathcal{ML}^l(\Omega)$ into $\mathcal{NL}^l(\Omega)$

The embedding $\mathcal{C}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$ extends in a natural way to an embedding

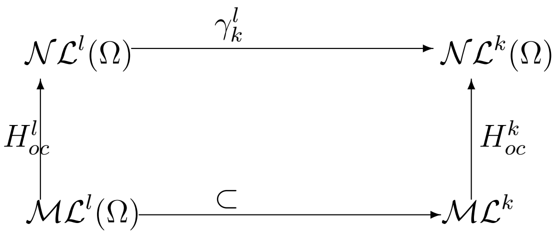
$$H^l_{oc} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega). \tag{3.21}$$

Theorem 3.11. *For each $l \in \overline{\mathbb{N}}$ there exists an injective homomorphism*

$$H^l_{oc} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega).$$

so that the following hold.

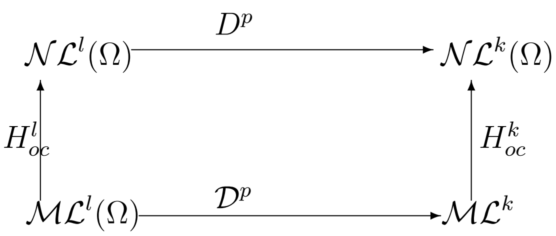
(i) The diagram



(3.22)

commutes whenever $k \leq l$.

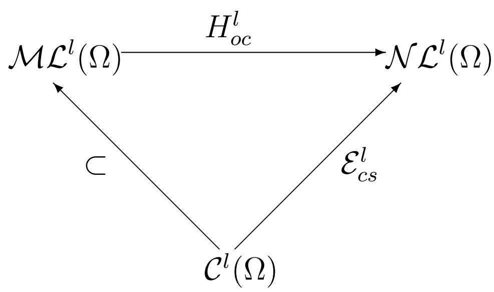
ii The diagram



(3.23)

commutes whenever $k + |p| \leq l$.

(iii) The diagram



(3.24)

commutes for all $l \in \overline{\mathbb{N}}$.

Proof. Consider the map

$$H^l_{oc} : \mathcal{ML}^l(\Omega) \ni u \mapsto (u_n) + \mathcal{I}^l_{cs} \in \mathcal{NL}^l(\Omega) \tag{3.25}$$

where $(u_n) \in \mathcal{S}^l_{cs}$ converges to u with respect to λ_l. The existence of such a sequence follows from Lemma 3.6. To see that Γ^l is well defined, let $(u_n), (v_n)$ be two sequences in

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$\mathcal{S}^l_{cs}$ converging to u with respect to λ_l. Based on Proposition 3.2 we conclude that $(u_n - v_n)$ converges to 0 with respect to λ_l, so that $(u_n - v_n) \in \mathcal{I}^l_{cs}$. It follows from Proposition 3.2 that H^l_{oc} is an injective algebra homomorphism. Indeed, if $H^l_{oc}(u) = H^l_{oc}(v)$ for some $u, v \in \mathcal{ML}(\Omega)$, then there exists $(u_n) \in \mathcal{S}^l_{cs}$ that converges to u and v with respect to λ_l. Since λ_l is Hausdorff, it follows that $u = v$. If $(u_n), (v_n) \in \mathcal{S}^l_{cs}$ converge to $u, v \in \mathcal{ML}^l(\Omega)$ with respect to λ_l, respectively, then $(u_n v_n)$ converges to uv with respect to λ_l. Hence

$$\begin{aligned} H^l_{oc}(u) H^l_{oc}(v) &= ((u_n) + \mathcal{I}^l_{cs})((v_n) + \mathcal{I}^l_{cs}) \\ &= (u_n v_n) + \mathcal{I}^l_{cs} \\ &= H^l_{oc}(uv). \end{aligned}$$

Linearity of H^l_{oc} follows the same way.

(i) The commutativity of the diagram (3.22), follows immediately from th definitions of the homomorphisms H^l_{oc}, H^k_{oc} and γ^l_k.

(ii Recall that, for $k + |p| \leq l$, the partial differential operator

$$\mathcal{D}^p : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^k(\Omega)$$

is uniformly continuous, thus continuous with respect to the convergence structure λ_l and λ_k. Thus if

$$H^l_{oc}(u) = (u_n) + \mathcal{I}^l_{oc}$$

for some $u \in \mathcal{ML}^l(\Omega)$, then $D^p(u_n) = (\mathcal{D}^p u_n)$ converges to $\mathcal{D}^p u$ in $\mathcal{ML}^k(\Omega)$ with respect to λ_k. Hence

$$H^k_{oc}(\mathcal{D}^p u) = D^p(u_n) + \mathcal{I}^k_{cs}.$$

By definition,

$$D^p(H^l_{oc} u) = D^p(u_n) + \mathcal{I}^k_{cs}.$$

Thus (3.23) is commutative.

(iii) The embedding $\mathcal{E}^l_{cs} : \mathcal{C}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$ is given by

$$\mathcal{E}^l_{cs}(u) = \Delta(u) + \mathcal{I}^l_{cs}$$

where $\Delta : \mathcal{C}^l(\Omega) \longrightarrow \mathcal{S}^l_{cs}$ maps each $u \in \mathcal{C}^l(\Omega)$ to the constant sequence with all terms equal to u. Since this sequence converges to u with respect to λ_l, the result follows immediately from the definition of the map H^l_{oc}.

□

3.4 Existence of Chain Generalised Solutions

In this section, we give an interpretation of the existence result for smooth PDEs, Theorem 2.55, in the context of the chain

$$\mathbf{A}_{oc} = \{(\mathcal{NL}^l_{\cdot}, \mathcal{NL}^k_{\cdot}, \gamma^l_k) : k, l \in \mathbb{N},\ k \leq l\}$$

of algebra of generalized functions. In particular, we show that the generalized solution $u^\sharp \in \mathcal{NL}^\infty(\Omega)$ obtained through the Theorem 2.55 is a chain generalized solution, see Definition 1.28 in Section 1.6. In this regard, consider the nonlinear partial differential operator

$$T : C^l(\Omega) \longrightarrow C^k(\Omega), \quad k + m \leq l \tag{3.26}$$

of order at most m, defined through a C^∞-smooth mapping

$$F : \Omega \times \mathbb{R}^M \longrightarrow \mathbb{R}$$

by setting

$$Tu(x) = F(x, u(x), \cdots, D^p u(x), \cdots), \ |p| \leq m \tag{3.27}$$

for each $x \in \Omega$. Since

$$T(C^l(\Omega)) \subseteq C^k(\Omega),$$

it follows that

$$T(C^l(\Omega)^\mathbb{N}) \subseteq C^k(\Omega)^\mathbb{N}.$$

Using (2.42), and owing to F being C^∞-smooth, the mapping (3.26) may be extended to a map

$$T : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{ML}^k(\Omega) \quad k + m \leq l. \tag{3.28}$$

It follows from Theorem 2.52 and the uniform continuity of the embedding

$$\mathcal{ML}^l(\Omega) \ni u \mapsto u \in \mathcal{ML}^k(\Omega) \quad k \leq l.$$

that (3.28) is uniformly continuous for all $l, k \in \overline{\mathbb{N}}$ such that $k + m \leq l$. Hence there exists unique uniformly continuous extensions

$$T^\sharp : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega), \quad k + m \leq l. \tag{3.29}$$

of (3.28).

On the other hand, in view of the construction of the extension of a uniformly continuous map to the completion of its domain, see Section 2.2.1, the map

$$T : C^l(\Omega)^\mathbb{N} \ni (u_n) \mapsto (Tu_n) \in C^k(\Omega)^\mathbb{N} \quad k + m \leq l.$$

satisfies

$$T(\mathcal{S}^l_{cs}) \subseteq \mathcal{S}^k_{cs}, \quad k + m \leq l$$

and

$$(u_n) - (v_n) \in \mathcal{I}^l_{cs} \Longrightarrow T(u_n) - T(v_n) \in \mathcal{I}^k_{cs}, \quad k + m \leq l$$

Thus in view of (1.66) - (1.70), and since $\mathcal{S}^l_{cs}$ and $\mathcal{I}^l_{cs}$ satisfy the neutrix condition (1.49), it follows that

$$T : \mathcal{NL}^l(\Omega) \ni u + \mathcal{I}^l_{cs} \mapsto Tu + \mathcal{I}^k_{cs} \in \mathcal{NL}^k(\Omega) \quad k + m \leq l. \tag{3.30}$$

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defines an extension of (3.26). Using the same argument as in the proof of Theorem 3.9, it follows that (3.29) and (3.30) are equal, in the sense that the diagram

$$\begin{array}{ccc} \mathcal{NL}^l(\Omega) & \xrightarrow{\quad T^\sharp \quad} & \mathcal{NL}^k(\Omega) \\ \downarrow E^l_{cs} & & \downarrow E^l_{cs} \\ \mathcal{S}^l_{cs}/\mathcal{I}^l_{cs} & \xrightarrow{\quad T \quad} & \mathcal{S}^k_{cs}/\mathcal{I}^l_{cs} \end{array} \tag{3.31}$$

commutes for all $l, k \in \overline{\mathbb{N}}, \quad k + m \leq l$

Our main result is the following

Theorem 3.12. *Assume that the PDE*

$$T(x, D)u(x) = f(x), \quad x \in \Omega \tag{3.32}$$

with $f \in C^\infty(\Omega)$ *and* T *defined as in* (3.27) *satisfies* (2.64)*. Then* (3.32) *admits a chain generalized solution* $u + \mathcal{I}^\infty_{cs} \in \mathcal{NL}^\infty(\Omega)$.

Proof. According to Theorem 2.55, there exists a generalized solution $u \in \mathcal{NL}^\infty(\Omega)$ of the PDE (3.32). Thus there exists a sequence $(u_n) \in \mathcal{S}^l_{cs}$ so that

$$u = (u_n) + \mathcal{I}^\infty_{cs}$$

satisfies $Tu = f$ in $\mathcal{NL}^\infty(\Omega)$. That is,

$$(Tu_n) - f \in \mathcal{I}^\infty_{cs} \subseteq \mathcal{I}^k_{cs}, \quad k \in \overline{\mathbb{N}}. \tag{3.33}$$

By definition of the algebra homomorphism

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega) \tag{3.34}$$

we have

$$T(\gamma^l_k(u)) = T(u_n) + \mathcal{I}^k_{cs}, \quad k + m \leq l \tag{3.35}$$

and

$$\gamma^\infty_k(f) = f + \mathcal{I}^l_{cs} \tag{3.36}$$

Thus (3.33), (3.35) and (3.36) imply that

$$T(\gamma^l_k(u)) = \gamma^\infty_{cs}(f + \mathcal{I}^\infty), \quad k + m \leq l$$

□

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Chapter 4

Nowhere Dense Algebras

In this chapter we considered so-called nowhere dense algebra of generalized functions. These are quotient algebras $\mathcal{A} = \mathcal{S}/\mathcal{I}$, where the ideal $\mathcal{I}$ in $\mathcal{S} \subseteq C^\infty(\Omega)^\Lambda$ is defined in terms of a dense vanishing condition. In particular, we consider Rosinger's nowhere dense algebra $\mathcal{A}_{nd}^\infty(\Omega)$, see [49, 50, 51], and the associated chain

$$\mathbf{A}_{nd} = \{(A_{nd}^l(\Omega), A_{nd}^k(\Omega), \gamma_k^l) : k, l \in \mathbb{N},\ k \leq l\}$$

of algebras of generalized functions, as well as Verneave's almost everywhere algebra $\mathcal{A}_{ae}^\infty(\Omega)$, see [73, 74]. Using Verneave's construction of the algebra $\mathcal{A}_{ae}^\infty$, we obtain a differential chain

$$\mathbf{A}_{ae} = \{(A_{ae}^l(\Omega), A_{ae}^k(\Omega), \gamma_k^l) : k, l \in \mathbb{N},\ k \leq l\}$$

of algebras of generalized functions.

We considered the extent to which the distributions may be embedded into the chain $\mathbf{A}_{ae}$, as well as the way in which the chain $\mathbf{A}_{oc}$ relates to the chain $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$, respectively. This leads to an interpretation of Theorem 3.12 within the chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$, giving an existence result of a large class of nonlinear PDEs within the mentioned chains of algebras of generalized functions. An application to so-called space-time foam differential algebras of generalised functions.

4.1 Two Constructions of Nowhere Dense Algebras

4.1.1 Rosinger's Nowhere Dense Algebra

In this section we discuss the construction of nowhere dense algebra of generalized functions introduced by Rosinger, see [49], [50], [51]. In particular, we recall how the nowhere dense chain of algebras of generalised functions is constructed, and discuss the embedding of distributions into this chain. In this regard, let $l \in \overline{\mathbb{N}}$ and denote by $\mathcal{I}_{nd}^l$ the set of all sequences of functions in $C^l(\Omega)$ satisfying the following asymptotic vanishing condition:

$$u = (u_n)_{n\in\mathbb{N}} \in \mathcal{I}_{nd}^l \Longleftrightarrow \left\{ \begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense :} \\ \forall & x \in \Omega \backslash \Gamma : \\ \exists & V \subset \Omega\backslash\Gamma, \text{ neighbourhood of } x,\ N_V \in \mathbb{N} : \\ \forall & y \in V,\ n \geq N_V : \\ & u_n(y) = 0 \end{array} \right. \qquad (4.1)$$

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In other words, the terms of the sequence (u_n) vanish at each point of the open and dense subset $\Omega\backslash\Gamma$, provided $n \in \mathbb{N}$ is sufficiently large. The set $\mathcal{I}^l_{nd}$ is an ideal in $C^l(\Omega)^{\mathbb{N}}$ as stated in [49, Chapter 1 Sction 7]. The ideal $\mathcal{I}^l_{nd} \subseteq C^l(\Omega)^{\mathbb{N}}$ is called the *nowhere dense ideals* on Ω.

Furthermore, $\mathcal{I}^l_{nd}$ satisfied the neutrix condition, and the inclusions

$$\mathcal{I}^l_{nd} \subset \mathcal{I}^k_{nd}, \qquad k \leq l$$

and

$$D^p(\mathcal{I}^l_{nd}) \subset \mathcal{I}^k_{nd}, \qquad |p| + k \leq l$$

hold. In view of Theorem 1.15, we have

Theorem 4.1.

$$\mathbf{A}_{nd} = \{(\mathcal{A}^l_{nd}(\Omega), \mathcal{A}^k_{nd}(\Omega), \gamma^l_k) : k, l \in \mathbb{N},\ k \leq l\}$$

is a differential chain of algebras of generalized functions.

The algebra homomorphism

$$\gamma^l_k : \mathcal{A}^l_{nd}(\Omega) \longrightarrow \mathcal{A}^k_{nd}(\Omega), \quad k \leq l$$

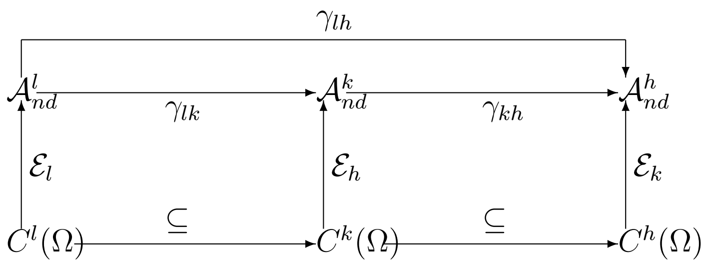
is defined as (1.43). That is,

$$\gamma^l_k : \mathcal{A}^l_{nd}(\Omega) \ni u + \mathcal{I}^l_{nd} \mapsto u + \mathcal{I}^k_{nd} \in \mathcal{A}^k_{nd}(\Omega), \quad k \leq l \tag{4.2}$$

Since the neutrix condition (1.49) is satisfied, it follows from Theorem 1.18 that

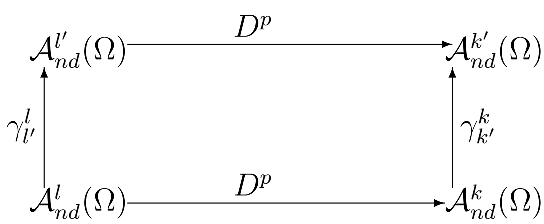
$$\mathcal{E}_l : C^l(\Omega) \ni u \mapsto \Delta(u) + \mathcal{I}^l_{nd} \in \mathcal{A}^k_{nd}(\Omega).$$

defines an injective algebra homomorphism for each $l \in \overline{\mathbb{N}}$. Furthermore, the diagrams



(4.3)

and



(4.4)

commute, for all $h \leq k \leq l$ and all $k + |p| \leq l$.

We next discuss briefly the details of embedding the distributions $\mathcal{D}'(\Omega)$ into the chain $\mathbf{A}_{nd}$. It follows from a modification of Theorem 1.11 that $\mathcal{I}^l_{nd}$ is C^l-regular in $C^l(\Omega)^{\mathbb{N}}$ for every $l \in \overline{\mathbb{N}}$. Therefore each of the algebras $\mathcal{A}^l_{nd}(\Omega)$ admits a linear embedding of $\mathcal{D}'(\Omega)$

which is an algebra homomorphism when restricted to $C^l(\Omega) \subset \mathcal{D}'(\Omega)$. In what follows we give an explicit construction of such a linear embedding. In this regard, we consider suitable representations of the distributions. For $l \in \overline{\mathbb{N}}$, let

$$\mathcal{V}^l_{\mathbb{N}} = \{(u_n) \in C^l(\Omega)^{\mathbb{N}} \ : \ u_n \text{ converges weakly to } u \in \mathcal{D}'(\Omega)\} \tag{4.5}$$

and

$$\mathcal{W}^l_{\mathbb{N}} = \{(u_n) \in C^l(\Omega)^{\mathbb{N}} \ : \ u_n \text{ converges weakly to } 0 \in \mathcal{D}'(\Omega)\}. \tag{4.6}$$

Clearly,

$$\mathcal{L}_l : \mathcal{V}^l_{\mathbb{N}} \ni (u_n) \mapsto \lim_{n\to\infty} u_n \in \mathcal{D}'(\Omega) \tag{4.7}$$

is a linear surjection, where $\lim_{n\to\infty} u_n$ denotes the weak limit of $(u_n) \in \mathcal{V}^l_{\mathbb{N}}$. Furthermore, $\mathcal{U}^l_{\mathbb{N}} \subset \mathcal{V}^l_{\mathbb{N}}$, and (4.7) is an extension of

$$\mathcal{U}^l_{\mathbb{N}} \ni (u_n) \mapsto u \in C^l(\Omega) \ni u \mapsto T_u \in \mathcal{D}'(\Omega)$$

where for $u \in C^l(\Omega)$, the distribution T_u is defined by

$$T_u : \mathcal{D}(\Omega) \ni \psi \mapsto \int_\Omega u(x)\psi(x)dx \in \mathbb{R}.$$

With, $\mathcal{W}^l_{\mathbb{N}}$ the kernel of the mapping (4.7), we have a vector space isomorphism

$$q_{\mathcal{D}'(\Omega)} : \mathcal{V}^l_{\mathbb{N}}/\mathcal{W}^l_{\mathbb{N}} \ni (u) + \mathcal{W}^l_{\mathbb{N}} \mapsto T \in \mathcal{D}'(\Omega), \tag{4.8}$$

and thus a representation of distributions given as

$$\mathcal{D}'(\Omega) = \mathcal{V}^l_{\mathbb{N}}/\mathcal{W}^l_{\mathbb{N}}. \tag{4.9}$$

Due to the weak continuity of the differential operators D^p, the pair $(\mathcal{V}^l_{\mathbb{N}}, \mathcal{L}_l)$ is a C^l-smooth representation of the distributions.

In order to obtain a regularization for the representation (4.5) - (4.7) of the distribution, we set

$$\mathcal{W}^l_{nd} = \mathcal{I}^l_{nd} \cap \mathcal{W}^l_{\mathbb{N}}$$

and

$$\mathcal{V}^l_{nd} = \mathcal{W}^l_{nd} \oplus \mathcal{V}^l_{\mathbb{N}} \oplus \mathcal{L}^l_{nd}$$

where $\mathcal{L}^l_{nd}$ is defined as follows. According to [51, Section 6.2] there exists algebraic bases $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ for $\mathcal{W}^l_{\mathbb{N}}$ and $\mathcal{I}^l_{nd} \cap \mathcal{V}^l_{\mathbb{N}}$, respectively, such that $\{c_k : k \in K\}$ is a basis for $\mathcal{W}^l_{\mathbb{N}} \cap \mathcal{I}^l_{nd}$, where

$$K = I \cap J, \quad c_k = a_k = b_k, \quad k \in K.$$

Furthermore, there exists an injection $\alpha : J \setminus K \longrightarrow I \setminus K$. The linear space $\mathcal{L}^l_{nd}$ is defined as

$$\mathcal{L}^l_{nd} = \text{span}\{a_{\alpha(i)} + b_i | i \in J \setminus K\}.$$

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It follows from [51, Theorem 1 p 228] that

$$(\mathcal{W}^l_{nd}, \mathcal{V}^l_{nd}, \mathcal{I}^l_{nd}, C^l(\Omega)^{\mathbb{N}}) \tag{4.10}$$

is a regularization of the representation $(\mathcal{V}^l_{\mathbb{N}}, \mathcal{L}_l)$ of $\mathcal{D}'(\Omega)$. In particular, since

$$\Delta(C^l(\Omega)^{\mathbb{N}}) = \mathcal{U}^l_{\mathbb{N}}(\Omega) \subseteq \mathcal{V}^l_{nd}$$

it follows that the regularization (4.10) is C^l-regular. That is, there exists a linear injection

$$E_l : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega) \tag{4.11}$$

so that the diagram

$$\begin{array}{ccc} C^l(\Omega) & \xrightarrow{\subset} & \mathcal{D}'(\Omega) \\ & \searrow^{\mathcal{E}_l} & \downarrow E_l \\ & & \mathcal{A}^l_{nd}(\Omega) \end{array} \tag{4.12}$$

commutes.

Clearly the inclusions

$$\mathcal{V}^l_{\mathbb{N}} \subseteq \mathcal{V}^k_{\mathbb{N}}, \quad \mathcal{V}^l_{\mathbb{N}} \subseteq \mathcal{W}^k_{\mathbb{N}}, \quad \mathcal{W}^l_{nd} \subseteq \mathcal{W}^k_{nd}$$

hold for all $k \leq l$. Furthermore, the vector spaces $\mathcal{L}^l_{nd}$ may be constructed in such a way that

$$\mathcal{L}^l_{nd} \subseteq \mathcal{L}^k_{nd}, \quad k \leq l$$

It then follows that

$$\mathcal{V}^l_{nd} \subseteq \mathcal{V}^k_{nd}, \quad k \leq l.$$

Applying Theorem 1.23, we therefore have the following

Theorem 4.2. *For all $h, k, l \in \overline{\mathbb{N}}$ with $h \leq k \leq l$, the diagram*

$$\begin{array}{ccccc} \mathcal{A}^l_{nd} & \xrightarrow{\gamma_{lk}} & \mathcal{A}^k_{nd} & \xrightarrow{\gamma_{kh}} & \mathcal{A}^h_{nd} \\ \uparrow E_l & & \uparrow E_k & & \uparrow E_h \\ \mathcal{D}'(\Omega) & \xrightarrow{id} & \mathcal{D}'(\Omega) & \xrightarrow{id} & \mathcal{D}'(\Omega) \end{array} \qquad \gamma_{lh} : \mathcal{A}^l_{nd} \to \mathcal{A}^h_{nd} \tag{4.13}$$

commutes. Here γ^l_k, γ^k_h, γ^l_h, are algebra homomorphisms defined by (1.43) *while the linear injections E_l, E_h, E_k are defined by* (4.11).

Furthermore, for $l \in \overline{\mathbb{N}}$, the map E_l restricted to $C^l(\Omega) \subset \mathcal{D}'(\Omega)$ is an algebra homomorphism. In particular, the diagram

$$\begin{array}{ccc} C^l(\Omega) & \xrightarrow{\subset} & \mathcal{D}'(\Omega) \\ & \searrow^{\mathcal{E}_l} & \downarrow E_l \\ & & \mathcal{A}^l_{nd}(\Omega) \end{array} \tag{4.14}$$

commutes.

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4.1.2 Verneave's Almost Everywhere Algebra

It is unknown whether or not a linear embedding

$$\mathcal{D}'(\Omega) \hookrightarrow \mathcal{A}^{\infty}_{nd}(\Omega)$$

exists which commutes with the partial differential operators D^p. However, the algebra $\mathcal{A}^{\infty}_{nd}(\Omega)$ and the chain $\mathbf{A}_{nd}$ have some desirable properties. In particular, as shown in [51, Theorem 1, Chapter 2], $\mathcal{A}^{\infty}_{nd}(\Omega)$ admits a global Cauchy - Kovalevskaia Theorem. An analytic PDE with analytic initial conditions specified on a non-characteristic hyperplane has a global generalized solution in $\mathcal{A}^{\infty}_{nd}$ which is analytic everywhere except possibly in a closed nowhere dense set Γ.

In order to address the issue of embedding $\mathcal{D}'(\Omega)$ into an algebra of generalised functions which admits global generalised solutions of analytic nonlinear PDEs such that distributional derivatives are preserved, Vernaeve [73] introduced the so called almost everywhere algebras, the construction of which is now recalled.

Let $\mathcal{M}_0$ be a subset of $\{\Gamma \subset \Omega\ :\ \Gamma \text{ is closed and nowhere dense}\}$ that is closed under the formation of finite unions. Consider the set

$$\mathcal{E}^{\infty}_{ae}(\Omega) = \left\{ (u_n) \,\middle|\, \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & (1)\ u_n : \Omega \longrightarrow \mathbb{R} : \\ & (2)\ u_n \in C^{\infty}(\Omega \backslash \Gamma) \end{array} \right\} \tag{4.15}$$

With respect to componentwise operations on sequences of real valued functions, $\mathcal{E}^{\infty}_{ae}$ is an algebra over $\mathbb{R}$. Let

$$\mathcal{I}_E = \left\{ (u_n) \in \mathcal{E}^{\infty}_{ae} \,\middle|\, \begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & y \in V,\ n \geq N : \\ & u_n(y) = 0 \end{array} \right\} \tag{4.16}$$

and

$$\mathcal{I}_{ae} = \left\{ (u_n) \in \mathcal{E}^{\infty}_{ae} \,\middle|\, \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N}, x \in \Omega \setminus \Gamma : \\ & u_n(x) = 0 \end{array} \right\}. \tag{4.17}$$

Both $\mathcal{I}_E$ and $\mathcal{I}_{ae}$ are ideals in $\mathcal{E}^{\infty}_{ae}(\Omega)$, hence $\mathcal{I}_E + \mathcal{I}_{ae}$ is an ideal as well. The almost everywhere algebra $\mathcal{A}^{\infty}_{ae}(\Omega)$ is defined as

$$\mathcal{A}^{\infty}_{ae}(\Omega) = \mathcal{E}^{\infty}_{ae}/(\mathcal{I}_E + \mathcal{I}_{ae}) \tag{4.18}$$

We note that the ideal $\mathcal{I}_E + \mathcal{I}_{ae}$ may be expressed as

$$\mathcal{I}_E + \mathcal{I}_{ae} = \left\{ (u_n) \in \mathcal{E}^{\infty}_{ae} \,\middle|\, \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & y \in V \setminus \Gamma,\ n \geq N : \\ & u_n(y) = 0 \end{array} \right\}. \tag{4.19}$$

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With each partial differential operator

$$D^p : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)$$

we may associate a relation

$$D^p : \mathcal{E}^\infty_{ae}(\Omega) \longrightarrow \mathcal{E}^\infty_{ae}(\Omega)$$

by setting

$$D^p(u_n) = \left\{ v_n \in \mathcal{E}^\infty_{ae} \;\middle|\; \begin{array}{ll} \forall & n \in \mathbb{N} : \\ & (1) v_n : \Omega \longrightarrow \mathbb{R} : \\ & (2) D^p u_n(x) = v_n(x),\ x \in \Omega \setminus \Gamma. \end{array} \right\} \tag{4.20}$$

where for $(u_n) \in \mathcal{E}^\infty_{ae}(\Omega)$, $\Gamma \in \mathcal{M}_0$ is the set associated with (u_n) through (4.15).

It is easy to see, using (4.19), that $D^p(u_n)$ is an $(\mathcal{I}_E + \mathcal{I}_{ae}$-equivalence class in $\mathcal{A}^\infty_{ae}(\Omega)$ for each $(u_n) \in \mathcal{E}^\infty_{ae}(\Omega)$ and $p \in \mathbb{N}^n$. Moreover, if $(u_n) - (v_n) \in \mathcal{I}_E + \mathcal{I}_{ae}$, then $D^p(u_n) = D^p(v_n)$ for all $p \in \mathbb{N}$. Thus we have a mapping

$$D^p : \mathcal{A}^\infty_{ae}(\Omega) \ni (u_n) + (\mathcal{I}_E + \mathcal{I}_{ae}) \mapsto D^p(u_n) + (\mathcal{I}_E + \mathcal{I}_{ae}) \in \mathcal{A}^\infty_{ae}, \tag{4.21}$$

for each $p \in \mathbb{N}^n$. The mappings (4.21) are linear and satisfy the Leibnitz rule for derivative of products. Hence $\mathcal{A}^\infty_{ae}(\Omega)$ is a differential algebra.

Since

$$\mathcal{U}^\infty_{\mathbb{N}}(\Omega) \subseteq \mathcal{E}^\infty_{ae}(\Omega), \quad \mathcal{U}^\infty_{\mathbb{N}}(\Omega) \cap (\mathcal{I}_E + \mathcal{I}_{ae}) = \{0\},$$

it follows that

$$C^\infty(\Omega) \ni u \mapsto \Delta^\infty_{\mathbb{N}}(u) + (\mathcal{I}_E + \mathcal{I}_{ae}) \in \mathcal{A}^\infty_{ae}(\Omega) \tag{4.22}$$

defines an injective algebra homomorphism, with

$$\Delta^\infty_{\mathbb{N}} : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)^{\mathbb{N}} \subset \mathcal{E}^\infty_{ae}(\Omega)$$

defined as in (1.9). The homomorphism (4.22) commutes with the differential operator (4.21), hence (4.21) defines extensions of the classical differential operators $D^p : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)$.

We now discuss the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{A}^\infty_{ae}(\Omega)$. In this regards, recall that a sequence (χ_n) in $\mathcal{D}(\Omega) \subset C^\infty(\Omega)$ is a unit sequence on Ω if

$$\begin{array}{ll} \forall & K \subset \Omega \text{ compact} : \\ \exists & N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & \chi_n(x) = 1,\ x \in K \end{array} \tag{4.23}$$

A sequence (ψ_n) in $\mathcal{D}(\Omega)$ is a strict delta sequence if

$$\begin{array}{ll} \forall & V \in \mathcal{V}_0 : \\ \exists & N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & \mathrm{supp}\psi_n \subset V \end{array}$$

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$$\int_{\Omega} \psi_n = 1, \ n \in \mathbb{N}$$

and there exists $\mu > 0$ so that

$$\int_{\Omega} |\psi_n(x)| dx \leq \mu, \ n \in \mathbb{N}.$$

For $T \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, the convolution $T \star \phi$ of T and ϕ is defined as

$$T \star \phi : \Omega \ni x \mapsto T(\tau_x \check{\phi}) \in \mathbb{R}$$

where

$$\tau_x \check{\phi}(y) = \check{\phi}(y - x), \quad y \in \Omega \tag{4.24}$$

and

$$\check{\phi}(y) = \phi(-y), \ y \in \Omega. \tag{4.25}$$

Note that, in (4.24) and (4.25), we are implicitly extending $\phi \in \mathcal{D}(\Omega)$ to a function in $\mathcal{D}(\mathbb{R}^n)$ by setting $\phi(x) = 0, \quad x \in \mathbb{R} \setminus \Omega$. It is well known, see for instance [55, Chapter 6], that $T \star \phi \in C^\infty(\Omega)$, for all $T \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$.

Finally, a smooth-part map is a linear surjection

$$F : \mathcal{D}'(\Omega) \longrightarrow C^\infty(\Omega)$$

such that

$$F(u) = u, \quad u \in C^\infty(\Omega)$$

and

$$D^p(F(T)) = F(D^p T), \quad T \in \mathcal{D}'(\Omega), \quad p \in \mathbb{N}^n.$$

The existence of a smooth - path map is guaranteed whenever Ω is convex, [73, 74]

Theorem 4.3. *Let (ψ_n) be a strict delta sequence, (χ_n) a unit sequence, and $F : \mathcal{D}'(\Omega) \longrightarrow C^\infty(\Omega)$ a smooth - part map. Then*

$$\mathcal{D}'(\Omega) \ni T \mapsto ((F(T) + [(T - F(T))\chi_n] \star \psi_n) + (\mathcal{I}_{ae} + \mathcal{I}_E)) \in \mathcal{A}^\infty_{ae}(\Omega)$$

defines a linear injection which commutes with partial derivatives and is an algebra homomorphism when restricted to $C^\infty(\Omega) \subset \mathcal{D}'(\Omega)$.

The algebra $\mathcal{A}^\infty_{ae}(\Omega)$ and $\mathcal{A}^\infty_{nd}(\Omega)$ are related to each other in the following way.

Theorem 4.4. *There exists a surjective algebra homomorphism*

$$\mathcal{A}^\infty_{ae}(\Omega) \longrightarrow \mathcal{A}^\infty_{nd}(\Omega)$$

that commutes with partial derivatives

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4.2 The chain of almost everywhere algebras

In this section we introduce, following the construction discussed in Section 4.1.2, the almost everywhere chain $\mathbf{A}_{ae}$ of algebras of generalized functions. We also consider the embedding of distribution into this chain. Let $\mathcal{M}_0$ be a set of closed nowhere dense subset of Ω that is closed under the formation of finite unions of its elements. For $l \in \overline{\mathbb{N}}$, let

$$\mathcal{E}^l_{ae}(\Omega) = \left\{ (u_n) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & (1)\ u_n : \Omega \longrightarrow \mathbb{R} : \\ & (2)\ u_n \in C^l(\Omega \backslash \Gamma) \end{array} \right\} \tag{4.26}$$

Clearly, $\mathcal{E}^l_{ae}(\Omega)$ is an algebra over $\mathbb{R}$ with respect to the termwise operations on sequences of functions. Following (4.16) and (4.17), we introduce the ideals

$$\mathcal{I}^l_E := \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \end{array} \right\} \tag{4.27}$$

and

$$\mathcal{I}^l_{ae} := \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & n \in \mathbb{N} : \\ & u_n(x) = 0, \quad x \in \Omega \setminus \Gamma \end{array} \right\} \tag{4.28}$$

Lemma 4.5. *For each $l \in \overline{\mathbb{N}}$,*

$$\mathcal{I}^l_E + \mathcal{I}^l_{ae} = \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \setminus \Gamma \end{array} \right\}. \tag{4.29}$$

Proof. Let (u_n) be a sequence in $\mathcal{E}^l_{ae}$. Then

$$\begin{array}{rl} (u_n) \in \mathcal{I}^l_E + \mathcal{I}^l_{ae} & \Longleftrightarrow (u_n) = (a_n) + (b_n) \ \text{ for some } \ (a_n) \in \mathcal{I}^l_E \ \text{ and } \ (b_n) \in \mathcal{I}^l_{ae} \\ & \Longleftrightarrow \left(\begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \cap (\Omega \setminus \Gamma) = V \setminus \Gamma \end{array} \right. \end{array}$$

Hence

$$\mathcal{I}^l_E + \mathcal{I}^l_{ae} = \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \setminus \Gamma \end{array} \right\}.$$

□

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Define the algebra $\mathcal{A}^l_{ae}(\Omega)$ as

$$\mathcal{A}^l_{ae}(\Omega) = \mathcal{E}^l_{ae}/(\mathcal{I}^l_E + \mathcal{I}^l_{ae}). \tag{4.30}$$

Since

$$\mathcal{E}^l_{ae}(\Omega) \subseteq \mathcal{E}^k_{ae}(\Omega) \quad \text{and} \quad \mathcal{I}^l_E + \mathcal{I}^l_{ae} \subseteq \mathcal{I}^k_E + \mathcal{I}^k_{ae} \tag{4.31}$$

whenever $l \geq k$, it follows that

$$\gamma^l_k : \mathcal{A}^l_{ae}(\Omega) \ni u + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \mapsto u + (\mathcal{I}^k_E + \mathcal{I}^k_{ae}) \in \mathcal{A}^k_{ae}(\Omega) \tag{4.32}$$

defines an algebra homomorphism. By setting

$$\overline{D^p}(u_n) = \left\{ (v_n) \in \mathcal{E}^k_{ae}(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma_0 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x, \;\; N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \;\; n \geq N : \\ & v_n(y) = D^p u_n(y), \quad y \in V \setminus (\Gamma \cup \Gamma_0) \end{array} \right\} \tag{4.33}$$

for each $(u_n) \in \mathcal{E}^l_{ae}(\Omega)$ and $|p|, k \leq l$, where $\Gamma \in \mathcal{M}_0$ is the set associated with (u_n) through (4.26), we obtain a relation

$$\overline{D^p} : \mathcal{E}^l_{ae}(\Omega) \rightrightarrows \mathcal{E}^k_{ae}(\Omega), \quad k + |p| \leq l.$$

Proposition 4.6. *For (u_n), $(u'_n) \in \mathcal{E}^l_{ae}(\Omega)$ and $|p| + k \leq l$, the following are true.*

(i) $\overline{D^p}(u_n) - \overline{D^p}(u_n) \subseteq \mathcal{I}^k_E + \mathcal{I}^k_{ae}$.

(ii) If $(v_n) - (w_n) \in \mathcal{I}^k_E + \mathcal{I}^k_{ae}$, *for some* $(v_n) \in \overline{D^p}(u_n)$ *and* $(w_n) \in \mathcal{E}^k_{ae}(\Omega)$, *then* $(w_n) \in \overline{D^p}(u_n)$.

(iii) $\alpha\overline{D^p}(u_n) + \beta\overline{D^p}(u'_n) \subseteq \overline{D^p}(\alpha u_n + \beta u'_n)$

Proof. (i) Let (v_n) and (w_n) be sequences in $\overline{D^p}(u_n)$. Then according to (4.33),

$$\begin{array}{ll} \exists & \Gamma_1 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V_1 \in \mathcal{V}_x, \;\; N_1 \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \;\; n \geq N_1 : \\ & v_n(y) = D^p u_n(y), \quad y \in V_1 \setminus (\Gamma \cup \Gamma_1) \end{array}$$

and

$$\begin{array}{ll} \exists & \Gamma_2 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V_2 \in \mathcal{V}_x, \;\; N_2 \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \;\; n \geq N_2 : \\ & w_n(y) = D^p u_n(y), \quad y \in V_2 \setminus (\Gamma \cup \Gamma_2). \end{array}$$

where $\Gamma \in \mathcal{M}_0$ is the set associated with (u_n) through (4.26). Let $\Gamma_0 = \Gamma_1 \cup \Gamma_2$ and $v = v_1 + v_2$. Then

$$\begin{aligned}&\forall \quad x \in \Omega : \\ &\forall \quad n \in \mathbb{N}, n \geq \max\{N_1,\ N_2\} : \\ &\quad\ v_n(y) - w_n(y) = D^p u_n(y) - D^p u_n(y) = 0, \\ &\quad\ y \in V \setminus (\Gamma \cup \Gamma_0).\end{aligned}$$

Thus $(v_n) - (w_n) \in \mathcal{I}_E^k + \mathcal{I}_{ae}^k$ which implies that $\overline{D^p}(u_n) - \overline{D^p}(u_n) \subseteq \mathcal{I}_E^k + \mathcal{I}_{ae}^k$.

(ii) Let $(v_n - w_n) \in \mathcal{I}_E^l + \mathcal{I}_{ae}^l$ for some $(v_n) \in \overline{D^p}(u_n)$ and $(w_n) \in \mathcal{E}_{ae}^k(\Omega)$. Let $\Gamma_1 \in \mathcal{M}_0$ be the closed nowhere dense set associated with $(v_n - w_n)$ through Lemma 4.5, and let $\Gamma_2 \in \mathcal{M}_0$ be the closed nowhere dense set associated with (v_n) through (4.33). Let $\Gamma_0 = \Gamma_1 \cup \Gamma_2$. Fix $x \in \Omega$. Then by Lemma 4.5, there exists $V_1 \in \mathcal{V}_x$ and $N_1 \in \mathbb{N}$ so that

$$v_n(y) - w_n(y) = 0, \quad y \in V \setminus \Gamma_0, \quad n \geq N_1.$$

By (4.33) there exists $V_2 \in \mathcal{V}_x$ and $N_2 \in \mathbb{N}$ so that

$$v_n(y) = D^p u_n(y), \quad y \in V_2 \setminus (\Gamma \cup \Gamma_2) \quad n \geq N_2,$$

where $\Gamma \in \mathcal{M}_0$ is the closed nowhere dense se associated with (u_n) through (4.26). If $N = \max\{N_1, N_2\}$ and $V = V_1 \cap V_2$, then

$$D^p u_n(y) - w_n(y) = v_n(y) - w_n(y) = 0,\ y \in V \setminus (\Gamma \cup \Gamma_0), \quad n \geq N,$$

so that $w_n(y) = D^p u_n(y),\ y \in V \setminus (\Gamma \cup \Gamma_0), \quad n \geq N$. Hence $(w_n) \in \overline{D^p}(u_n)$.

(iii) Let $(v_n) \in \overline{D^p}(u_n)$ and $(v_n') \in \overline{D^p}(u_n')$. Then there exists $\Gamma_1,\ \Gamma_2, \in \mathcal{M}_0$, so that for every $x \in \Omega$ there exists $N_1, N_2 \in \mathbb{N}$ and $V_1, V_2 \in \mathcal{V}_x$, so that

$$v_n(y) = D^p(u_n)(y),\ y \in V_1 \setminus (\Gamma \cup \Gamma_1), \quad n \geq N_1$$

and

$$v_n'(y) = D^p(u_n')(y),\ y \in V_2 \setminus (\Gamma' \cup \Gamma_2), \quad n \geq N_2,$$

where $\Gamma,\ \Gamma' \in \mathcal{M}_0$ are closed nowhere dense sets associated with (v_n) and (v_n'), respectively, through (4.26). Hence, for $\alpha,\ \beta \in \mathbb{R}$,

$$\alpha v_n(y) = \alpha D^p(u_n)(y),\ y \in V_1 \setminus (\Gamma \cup \Gamma_1), \quad n \geq N_1,$$

and

$$\beta v_n'(y) = \beta D^p(u_n')(y),\ y \in V_2 \setminus (\Gamma' \cup \Gamma_2), \quad n \geq N_2.$$

Thus

$$\begin{aligned}\alpha v_n(y) + \beta v_n'(y) = \alpha D^p(u_n)(y) + \beta D^p(u_n')(y) = D^p(\alpha u_n + \beta u_n')(y),\\ y \in (V_1 \cap V_2) \setminus ((\Gamma \cup \Gamma') \cup (\Gamma_1 \cup \Gamma_2)), \quad n \geq \max\{N_1,\ N_2\}\end{aligned}$$

which implies that

$$\alpha(v_n) + \beta(v_n') \in D^p(\alpha u_n + \beta u_n').$$

Therefore

$$\alpha\overline{D^p}(u_n) + \beta\overline{D^p}(u_n') \subseteq \overline{D^p}(\alpha u_n + \beta u_n').$$

□

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In view of Proposition 4.6 above, we have the following.

Proposition 4.7. *For all $l,k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N}^n$ so that $|p|+k \leq l$*

$$D^p : \mathcal{A}^l_{ae}(\Omega) \ni (u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \mapsto \overline{D^p}(u_n) \in \mathcal{A}^k_{ae}(\Omega) \tag{4.34}$$

is well defined and linear. Furthermore, (4.34) *satisfies the Leibnitz rule*

$$D^p\left(((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))\right)$$
$$= \sum_{q\leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q}((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))D^q((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))$$

Proof. That (4.34) is well defined and linear follows immediately from Proposition 4.6. To see that the Leibnitz rule holds, consider

$$(u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}), (v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \in \mathcal{A}^l_{ae}(\Omega).$$

Then by (4.33) and (4.34),

$$D^p\left(((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))\right) = D^p((u_nv_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) = \overline{D^p}(u_nv_n).$$

Let (w_n) be a sequence in $\overline{D^p}(u_nv_n)$. According to (4.33),

$$\begin{array}{ll} \exists & \Gamma_1 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & w_n(y) = D^p(u_nv_n)(y) = \sum_{q\leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q}u_n(y)D^qv_n(y), \quad y \in V \setminus (\Gamma \cup \Gamma_1) \end{array}$$

where $\Gamma \in \mathcal{M}_0$ is the set associated with $(u_nv_n) \in \mathcal{E}^l_{ae}(\Omega)$ through (4.26). This implies, since D^p satisfies Leibnitz rule, that

$$\begin{array}{ll} \exists & \Gamma_1 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & w_n(y) = \sum_{q\leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q}u_n(y)D^qv_n(y), \quad y \in V \setminus (\Gamma \cup \Gamma_1), \end{array}$$

so that for $p,q \in \mathbb{N}^n$ the sequences $D^{p-q}u_n = w'_n$ and $D^qv_n = w''_n$ for some sequences (w'_n) and (w''_n) in $\overline{D^{p-q}}(u_n)$ and $\overline{D^q}(v_n)$ respectively. Thus we have that

$$\overline{D^p}(u_nv_n) = \sum_{q\leq p} \begin{pmatrix} p \\ q \end{pmatrix} \overline{D^{p-q}}u_n\overline{D^q}v_n.$$

Hence according to (4.34) we have

$$D^p\left(((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))\right)$$
$$= \sum_{q\leq p} \begin{pmatrix} p \\ q \end{pmatrix} \overline{D^{p-q}}u_n(y)\overline{D^q}v_n(y)$$
$$= \sum_{q\leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q}((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}))D^q((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})).$$

This completes the proof. □

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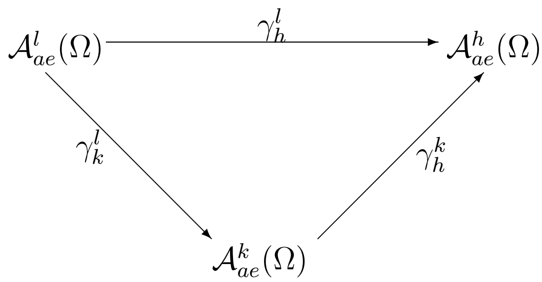
Theorem 4.8.

$$\mathbf{A}_{ae} = \{\mathcal{A}^l_{ae}(\Omega), \mathcal{A}^k_{ae}(\Omega), \gamma^l_k\} \mid l,\ k \in \overline{\mathbb{N}},\ k \leq l\}$$

is a differential chain of algebras of generalized functions with γ^l_k *defined in* (4.32)*.*

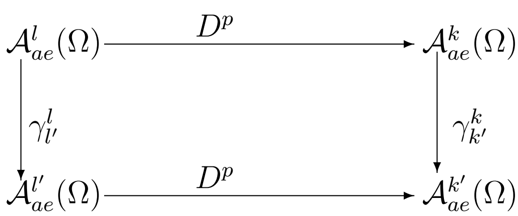
Proof. The commutativity of the diagram



(4.35)

with $h \leq k \leq l$ follows immediately from (4.32).

Now consider the diagram



(4.36)

with $l' \leq l,\ k' \leq l,\ |p| + k \leq l$ and $|p| + k' \leq l'$. The commutativity of the diagram follows immediately upon noting that for $(u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \in \mathcal{A}^l_{ae}(\Omega)$,

$$D^p((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) = (v_n) + (\mathcal{I}^k_E + \mathcal{I}^k_{ae})$$

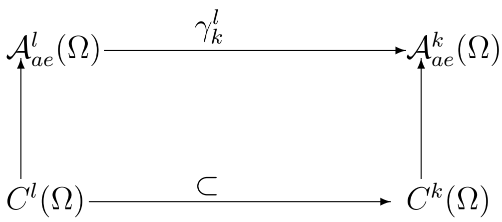
for any member (v_n) of the set $\overline{D^p}(u_n)$. □

We now consider the embedding of the distributions into the chain $\mathbf{A}_{ae}$. A first result in this regard is the following

Proposition 4.9. *For each* $l \in \overline{\mathbb{N}}$*, there exists an injective algebra homomorphism*

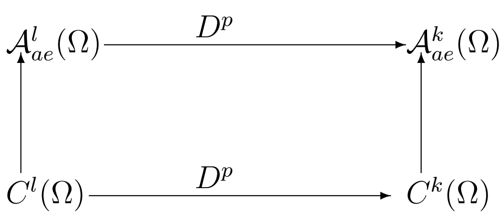
$$C^l(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega) \tag{4.37}$$

such that the diagrams



(4.38)

and



(4.39)

commutes, for all $k \leq l$ *and* $|p| + k \leq l$*, respectively.*

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Proof. The existence of the injective algebra homomorphism (4.37) follows immediately from

$$\mathcal{U}^l_{\mathbb{N}} \subset \mathcal{E}^l_{ae}(\Omega), \ \ \mathcal{U}^l_{\mathbb{N}} \cap (\mathcal{I}^l_E + \mathcal{I}^l_{ae} = \{0\}.$$

In particular, we obtain (4.37) by setting

$$C^l(\Omega) \ni u \mapsto \Delta^\infty_{\mathbb{N}}(u) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \in \mathcal{A}^l_{ae}(\Omega)$$

where $\Delta^\infty_{\mathbb{N}}(u)$ denotes the sequence in $C^l(\Omega)^{\mathbb{N}} \subseteq \mathcal{E}^l_{ae}(\Omega)$ with all terms equal to u. The commutativity of the diagram (4.38) now follows immediately from (4.32).

Since for $u \in C^l(\Omega)$, and $|p| + k \leq l$, $\overline{D^p}\Delta^\infty_{\mathbb{N}}(u) = \Delta^\infty_{\mathbb{N}}(\overline{D^p}(u))$, it follows that the diagram (4.39) commutes. □

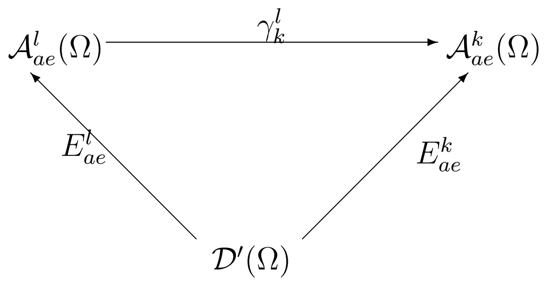
Next we consider the embedding of $\mathcal{D}'(\Omega)$ into $\mathbf{A}_{ae}$. In this regard we have the following.

Theorem 4.10. *Assume that Ω is convex. For each $l \in \overline{\mathbb{N}}$, there exists a linear injection*

$$E^l_{ae} : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$$

with the following properties.

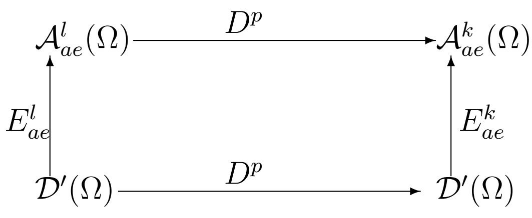
(i) The diagram



(4.40)

commutes, for all $k \leq l$.

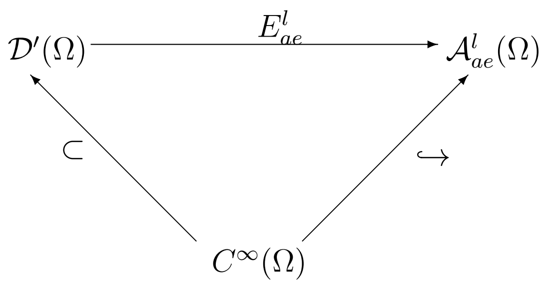
(ii) The diagram



(4.41)

commutes whenever $k + |p| \leq l$.

(iii) E^l_{ae} is an algebra homomorphism when restricted to $C^\infty(\Omega) \subseteq \mathcal{D}'(\Omega)$. In particular, the diagram



(4.42)

commutes for each $l \in \overline{\mathbb{N}}$, with the algebra homomorphism $\mathcal{C}^\infty(\Omega) \hookrightarrow \mathcal{A}^l_{ae}(\Omega)$ the restriction of $E^k_{ae} : \mathcal{C}^l(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$ to $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^l(\Omega)$.

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(iv) E^l_{ae} *is not an algebra homomorphism when restricted to* $C^l(\Omega) \subset \mathcal{D}'(\Omega)$*. In particular, the diagram*

$$\begin{array}{ccc} \mathcal{D}'(\Omega) & \xrightarrow{E^l_{ae}} & \mathcal{A}^l_{ae}(\Omega) \\ & \nwarrow \subset \quad \hookrightarrow \nearrow & \\ & C^l(\Omega) & \end{array} \tag{4.43}$$

does not commute, with the embedding $C^l(\Omega) \hookrightarrow \mathcal{A}^l_{ae}(\Omega)$ *given by Proposition 4.9.*

Proof. According to Theorem 4.3, there exists a linear injection

$$E^\infty_{ae} : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^\infty_{ae}(\Omega)$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{D}'(\Omega) & \xrightarrow{E^\infty_{ae}} & \mathcal{A}^\infty_{ae}(\Omega) \\ & \nwarrow \subset \quad \hookrightarrow \nearrow & \\ & C^\infty(\Omega) & \end{array} \tag{4.44}$$

with $C^\infty \hookrightarrow \mathcal{A}^\infty_{ae}(\Omega)$. given in (4.22), and

$$\begin{array}{ccc} \mathcal{A}^\infty_{ae}(\Omega) & \xrightarrow{D^p} & \mathcal{A}^\infty_{ae}(\Omega) \\ \uparrow E^\infty_{ae} & & \uparrow E^\infty_{ae} \\ \mathcal{D}'(\Omega) & \xrightarrow{D^p} & \mathcal{D}'(\Omega) \end{array} \tag{4.45}$$

commute for all $p \in \mathbb{N}^n$.

For each $l \in \overline{\mathbb{N}}$, let

$$E^l_{ae} : \mathcal{D}'(\Omega) \ni T \mapsto \gamma^\infty_l(E^\infty_{ae}(T)) \in \mathcal{A}^\infty_{ae}(\Omega).$$

Note that $\gamma^\infty_l : \mathcal{A}^\infty_{ae}(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$ is injective if and only if $(\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \bigcap \mathcal{E}^\infty_{ae}(\Omega) = (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})$. To see this, let $(\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \bigcap \mathcal{E}^\infty_{ae}(\Omega) = (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})$. Then for any $(u_n), (v_n) \in \mathcal{E}^\infty_{ae}$, we have that

$$\begin{aligned} & \gamma^l_\infty((u_n) + (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})) = \gamma^l_\infty((v_n) + (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})) \\ \Longleftrightarrow\ & (u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) = (v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \\ \Longleftrightarrow\ & (u_n) - (v_n) \in (\mathcal{I}^l_E + \mathcal{I}^l_{ae}). \end{aligned}$$

Since $(u_n) - (v_n) \in \mathcal{E}^\infty_{ae}$, it follows that $(u_n) - (v_n) \in (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \bigcap \mathcal{E}^\infty_{ae}(\Omega) = (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})$. Thus $(u_n) + (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae}) = (v_n) + (\mathcal{I}^\infty_E + \mathcal{I}^\infty_{ae})$. Hence γ^l_∞ is injective. Conversely, let γ^l_∞

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be injective. Take $(u_n) \in (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty) \cap \mathcal{E}_{ae}^\infty$, and since γ_∞^l is injective, the inverse image of $(u_n) + (\mathcal{I}_E^l + \mathcal{I}_{ae}^l)$ in $\mathcal{A}_{ae}^l(\Omega)$ is $(u_n) + (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty)$ in $\mathcal{A}_{ae}^\infty(\Omega)$ so that $(u_n) \in (\mathcal{I}_E^l + \mathcal{I}_{ae}^l)$ implies $(u_n) \in (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty)$ Thus

$$(\mathcal{I}_E^l + \mathcal{I}_{ae}^l) \cap \mathcal{S}_{ae}^\infty \subseteq (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty). \tag{4.46}$$

From (4.31) we deduce that

$$\mathcal{E}_{ae}^\infty(\Omega) \subseteq \mathcal{E}_{ae}^l(\Omega) \ \text{ and } \ \mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty \subseteq \mathcal{I}_E^l + \mathcal{I}_{ae}^l \tag{4.47}$$

Therefore, from (4.46) and (4.47) we have that

$$(\mathcal{I}_E^l + \mathcal{I}_{ae}^l) \cap \mathcal{S}_{ae}^\infty = (\mathcal{I}_E^\infty + \mathcal{I}_{ae}^\infty).$$

We now show that the results in (i) - (iv) hold.

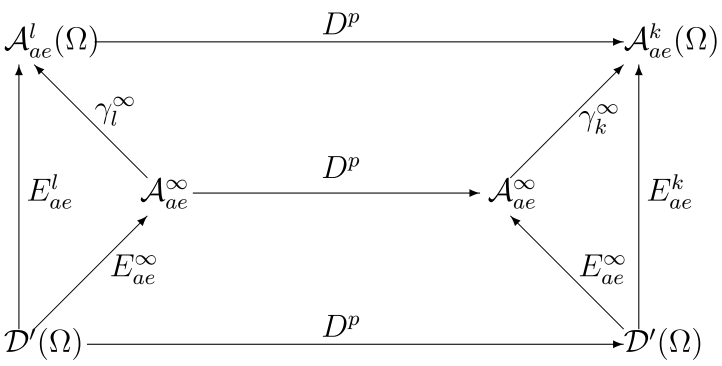
(i) It follows from Theorem 4.8 that

$$\gamma_k^l \circ \gamma_l^\infty = \gamma_k^\infty \ \text{ whenever } \ k \leq l.$$

Hence for $k \leq l$

$$\gamma_k^l \circ E_{ae}^l = E_{ae}^k$$

(ii) It follows from Theorems 4.3 and 4.8 that the diagram



(4.48)

commutes whenever $|p| + k \leq l$. Thus the result follows.

(iii) This follows immediately from Theorem 4.3.

(iv) Suppose that, for some $l \in \mathbb{N}$, E_{ae}^l is an algebra homomorphism when restricted to $C^l(\Omega) \subset \mathcal{D}'(\Omega)$.4 Since $\gamma_l^\infty : \mathcal{A}_l^\infty(\Omega) \longrightarrow \mathcal{A}_{ae}^l(\Omega)$ is injective, it follows from (i) that E_{ae}^∞ is an algebra homomorphism when restricted to $C^l(\Omega) \ \subset \mathcal{D}'(\Omega)$ This is impossible by Corollary 1.27. Therefore E_{ae}^l is not an algebra homomorphism when restricted to $C^l(\Omega) \subset \mathcal{D}'(\Omega)$. The fact that the diagram (4.43) does not commute follows immediately.

□

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4.3 Functions with Nowhere Dense Singularities

In this Section, we consider the embedding of the spaces $\mathcal{ML}^l(\Omega)$ into the chain of algebras of generalized functions considered in Section 4.1. We also consider the questions of wether or not the embedding introduced here is compatible with the embedding of distribution into the chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$ as discussed in Section 4.1.

4.3.1 Embedding $\mathcal{ML}^l(\Omega)$ into $\mathbf{A}_{nd}$

For each $l \in \overline{\mathbb{N}}$ and $u \in \mathcal{ML}^l(\Omega)$, there exists $\Gamma \subset \Omega$ closed and nowhere dense so that

$$u \in C^l(\Omega \setminus \Gamma) \tag{4.49}$$

Applying Theorem 3.5, we find for each $n \in \mathbb{N}$ a function $\phi_n \in C^\infty(\Omega, [0,1])$ so that , for each $x \in \Omega$,

$$\left(\begin{array}{l} \exists \quad \overline{y} \in \Gamma : \\ \quad \|x-y\| \leq \frac{1}{2n} \end{array} \right) \Longrightarrow \phi_n(x) = 0 \tag{4.50}$$

and

$$\left(\begin{array}{l} \forall \quad y \in \Gamma : \\ \quad \|x-y\| \geq \frac{1}{n} \end{array} \right) \Longrightarrow \phi_n(x) = 1 \tag{4.51}$$

Thus $u_n = \phi_n u \in C^l(\Omega)$ satisfies

$$\left(\begin{array}{l} \exists \quad y \in \Gamma : \\ \quad \|x-y\| \leq \frac{1}{2n} \end{array} \right) \Longrightarrow u_n(x) = 0 \tag{4.52}$$

and

$$\left(\begin{array}{l} \forall \quad y \in \Gamma : \\ \quad \|x-y\| \geq \frac{1}{n} \end{array} \right) \Longrightarrow u_n(x) = u(x). \tag{4.53}$$

for each $x \in \Omega$.

For each $u \in \mathcal{ML}^l(\Omega)$, let

$$\mathcal{I}_u = \left\{ (u_n) \in C^l(\Omega)^{\mathbb{N}} \;\middle|\; \begin{array}{l} \exists \quad \Gamma \subset \Omega \text{ closed nowhere dense :} \\ \quad (1) \quad u \in C^l(\Omega \setminus \Gamma) : \\ \quad (2) \;\forall\; x \in \Omega \backslash \Gamma : \\ \quad \exists\; V \in \mathcal{V}_x, \;\; N \in \mathbb{N} : \\ \quad \forall\; y \in V, \;\; n \geq N : \\ \qquad\qquad u_n(y) = u(y) \end{array} \right\}. \tag{4.54}$$

In view of the construction (4.50) - (4.53), $\mathcal{I}_u$ is non-empty for each $u \in \mathcal{ML}^l(\Omega)$. We will use the following properties of the set $\mathcal{I}_u$ to construct the desired embedding of $\mathcal{ML}^l(\Omega)$ into $\mathcal{A}^l_{nd}(\Omega)$.

Proposition 4.11. *The following is true for all $u, v \in \mathcal{ML}^l(\Omega)$.*

(i) If $(u_n) \in \mathcal{I}_u$ and $(v_n) \in \mathcal{I}_v$, then $(u_n v_n) \in \mathcal{I}_{uv}$

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(ii) *If* $(u_n) \in \mathcal{I}_u$ *and* $(v_n) \in \mathcal{I}_v$*, then* $(\alpha u_n + \beta v_n) \in \mathcal{I}_{\alpha u+\beta v}$*, for all* $\alpha,\ \beta \in \mathbb{R}$.

(iii) *If* $(u_n), (u'_n) \in \mathcal{I}_u$*, then* $(u_n - u'_n) \in \mathcal{I}_{nd}$.

(iv) *If* $(u_n) \in \mathcal{I}_u$ *and* $(v_n - v_n) \in \mathcal{I}_{nd}$ *for some* $(v_n) \in C^l(\Omega)^{\mathbb{N}}$*, then* $(v_n) \in \mathcal{I}_u$.

(v) $D^p(\mathcal{I}_u) \subseteq \mathcal{I}_{D^p(u)}$ *for all* $|p| \leq l$.

(vi) *If* $\mathcal{I}_u \cap \mathcal{I}_v \neq \emptyset$*, then* $u = v$.

Proof. (i) Let $\Gamma_0, \Gamma_1 \subset \Omega$ be the closed, nowhere dense set associated with (u_n) and (v_n) respectively, through (4.54). Then $\Gamma = \Gamma_0 \cup \Gamma_1$ is closed and nowhere dense and $u, v \in C^l(\Omega \setminus \Gamma)$. Fix $x \in \Omega \setminus \Gamma$. Then according to (4.54), there exists $V_0,\ V_1 \in \mathcal{V}_x$ and $N_0, N_1 \in \mathbb{N}$ so that

$$u_n(y) = u(y), \quad y \in V_0, \quad n \geq N_0$$

and

$$v_n(y) = v(y), \quad y \in V_1, \quad n \geq N_1.$$

Hence the result follows upon setting $V = V_0 \cap V_1$ and $N = \max\{N_0,\ N_1\}$.

(ii) The proof of (2) follows in the same way, and is therefore omitted.

(iii) Let $\Gamma_0, \Gamma_1 \subset \Omega$ be the closed nowhere dense set associated with (u_n) and (u'_n) respectively, through (4.54). Let $\Gamma = \Gamma_0 \cup \Gamma_1$. For $x \in \Omega \setminus \Gamma$ there exists, according to (4.54), neighbourhoods V_0 and V_1 of x and natural numbers N_0 and N_1 so that

$$u_n(y) = u(y), \quad y \in V_0, \quad n \geq N_0$$

and

$$v_n(y) = v(y), \quad y \in V_1, \quad n \geq N_1.$$

Setting $V = V_0 \cap V_1$ and $N = \max\{N_0,\ N_1\}$, we find that

$$u_n(y) - v_n(y) = 0, \quad y \in V, \ n \geq N.$$

Since V is open, it follows that

$$D^p(u_n(y) - v_n(y)) = 0, \quad y \in V, \ n \geq N, |p| \leq l.$$

Hence $(u_n - v_n) \in \mathcal{I}^l_{nd}$.

(iv) Since $(u_n - v_n) \in \mathcal{I}^l_{nd}$, it follows from (4.1) that there exists $\Gamma_0 \subset \Omega$ closed and nowhere dense so that

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma_0 : \\ \exists & V_0 \in \mathcal{V}_x, N_0 \in \mathbb{N} : \\ \forall & y \in V_0,\ n \geq N_0 : \\ & u_n(y) - v_n(y) = 0. \end{array} \tag{4.55}$$

Let $\Gamma_1 \subset \Omega$ be the closed nowhere dense set associated with (u_n) through (4.54). Let $\Gamma = \Gamma_0 \cup \Gamma_1$. Then $u \in C^l(\Omega \setminus \Gamma)$ and according to (4.54), we have for $x \in \Omega \setminus \Gamma$

an open neighbourhood V_1 of x and a natural number N_1 so that Setting $V = V_0 \cap V_1$ and $N = \max\{N_0,\ N_1\}$, we find that

$$u_n(y) = u(y), \quad y \in V,\ n \geq N.$$

Since $x \in \Omega \setminus \Gamma \subset \Omega \setminus \Gamma_0$, (4.55) applies so that

$$v_n(y) = u(y), \quad y \in V,\ n \geq N.$$

Hence $(v_n) \in \mathcal{I}_u$.

(v) Let $\Gamma \subset \Omega$ be the closed nowhere dense set associated with (u_n) through (4.54). Then $\mathcal{D}^p u \in C^{l-|p|}(\Omega \setminus \Gamma)$ and for each $x \in \Omega \setminus \Gamma$ there exists $V \in \mathcal{V}_x$ and $N \in \mathbb{N}$ so that

$$u_n(y) = u(y), \quad y \in V,\ n \geq N.$$

Since V is open,

$$D^p u_n(y) = D^p u(y) = \mathcal{D}^p u(y), \quad y \in V,\ n \geq N.$$

Hence $(D^p u_n) \in \mathcal{I}_{\mathcal{D}^p(u)}$.

(vi) If $\mathcal{I}_u \cap \mathcal{I}_v \neq \emptyset$, then it follows from (4.54) that $u(x) = v(x)$ for all x in some open and dense set $D \subset \Omega$. Proposition 2.41 now implies that $u = v$.

□

Theorem 4.12. *There exists, for each $l \in \overline{\mathbb{N}}$, an injective algebra homomorphism*

$$H^l_{nd} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega)$$

so that the diagrams

$$\begin{array}{ccc} \mathcal{ML}^l(\Omega) & \xrightarrow{\ \mathcal{D}^p\ } & \mathcal{ML}^k(\Omega) \\ {\scriptstyle H^l_{nd}}\downarrow & & \downarrow{\scriptstyle H^k_{nd}} \\ \mathcal{A}^l_{nd}(\Omega) & \xrightarrow{\ D^p\ } & \mathcal{A}^k_{nd}(\Omega) \end{array} \tag{4.56}$$

and

$$\begin{array}{ccc} \mathcal{ML}^l(\Omega) & \xrightarrow{\ \subset\ } & \mathcal{ML}^k(\Omega) \\ {\scriptstyle H^l_{nd}}\downarrow & & \downarrow{\scriptstyle H^k_{nd}} \\ \mathcal{A}^l_{nd}(\Omega) & \xrightarrow{\ \gamma^l_k\ } & \mathcal{A}^k_{nd}(\Omega) \end{array} \tag{4.57}$$

commute for $k + |p| \leq l$ and $k \leq l$, respectively.

Proof. It follows from Proposition 4.11 (iii) and (iv) that $\mathcal{I}_u$ is an $\mathcal{I}^l_{nd}$-equivalence class for each $u \in \mathcal{ML}^l(\Omega)$, so that

$$H^l_{nd} : \mathcal{ML}^l(\Omega) \ni u \mapsto \mathcal{I}_u \in \mathcal{A}^l_{nd}(\Omega)$$

is well-defined. It follows from Proposition 4.11(vi) that H^l_{nd} is injective,while properties (i) and (ii) of the same proposition imply that H^l_{nd} is an algebra homomorphism. Proposition 4.11 (5) implies the commutativity of the diagram (4.56), while the commutativity of diagram (4.57) follows immediately from the definitions of the respective algebra homomorphisms. □

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4.3.2 Embedding $\mathcal{ML}^l(\Omega)$ into $\mathbf{A}_{ae}$

We now deal with the embedding of $\mathcal{ML}^l(\Omega)$ into the chain $\mathbf{A}_{ae}$ introduced in Section 4.2. We consider the particular case where $\mathcal{M}_0 = \{\Gamma \subseteq \Omega \ : \ \Gamma \text{ closed nowhere dense }\}$. In this regard, we have the following

Theorem 4.13. *For each $l \in \overline{\mathbb{N}}$ there exists an injective algebra homomorphism*

$$H^l_{ae} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega) \tag{4.58}$$

so that the diagrams

$$\begin{array}{ccc} \mathcal{ML}^l(\Omega) & \xrightarrow{\ \subset\ } & \mathcal{ML}^k(\Omega) \\ \Big\downarrow H^l_{ae} & & \Big\downarrow H^k_{ae} \\ \mathcal{A}^l_{ae}(\Omega) & \xrightarrow{\ \gamma^l_k\ } & \mathcal{A}^k_{ae}(\Omega) \end{array} \tag{4.59}$$

and

$$\begin{array}{ccc} \mathcal{ML}^l(\Omega) & \xrightarrow{\ \mathcal{D}^p\ } & \mathcal{ML}^k(\Omega) \\ \Big\downarrow H^l_{ae} & & \Big\downarrow H^k_{ae} \\ \mathcal{A}^l_{ae}(\Omega) & \xrightarrow{\ D^p\ } & \mathcal{A}^k_{ae}(\Omega) \end{array} \tag{4.60}$$

commute whenever $k \leq l$ and $k + |p| \leq l$ respectively.

Proof. Let $u \in \mathcal{ML}^l(\Omega)$ for some $l \in \overline{\mathbb{N}}$. Then there exists $\Gamma \in \mathcal{M}_0$ so that $u \in C^l(\Omega \setminus \Gamma)$. Let

$$\Delta_u = \left\{ (u_n) \in \mathcal{E}^l_{ae}(\Omega) \left| \begin{array}{ll} \exists & \Gamma_0 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x, \ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \ n \geq N : \\ & u_n(y) = u(y), \quad y \in V \setminus (\Gamma \cup \Gamma_0) \end{array} \right. \right\} \tag{4.61}$$

It follows from Lemma 4.5 that $(u_n - v_n) \in \mathcal{I}^l_E + \mathcal{I}^l_{ae}$ for all $(u_n), (v_n) \in \Delta_u$.

Assume that $(u_n - v_n) \in \mathcal{I}^l_E + \mathcal{I}^l_{ae}$ for some $(u_n) \in \Delta_u$ and $(v_n) \in \mathcal{E}^l_{ae}$. Let Γ_1 be the closed nowhere dense set associated with $(u_n - v_n)$ through Lemma 4.5, and let Γ_2 be the closed nowhere dense set associated with (u_n) through (4.61). Let $\Gamma = \Gamma_1 \cup \Gamma_2$. Fix $x \in \Omega$. Then by Lemma 4.5, there exists $V_1 \in \mathcal{V}_x$ and $N_1 \in \mathbb{N}$ so that

$$u_n(y) - v_n(y) = 0, \quad y \in V \setminus \Gamma_1, \quad n \geq N_1.$$

By (4.61) there exists $V_2 \in \mathcal{V}_x$ and $N_2 \in \mathbb{N}$ so that

$$u_n(y) - u(y) = 0, \quad y \in V_2 \setminus (\Gamma \cup \Gamma_2), \quad n \geq N_2.$$

If $N = \max\{N_1, N_2\}$ and $V = V_1 \cap V_2$, then

$$v_n(y) = u(y), \quad y \in V \setminus (\Gamma \cup \Gamma_2), \quad n \geq N,$$

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so that $(v_n) \in \Delta_u$. It therefore follows that Δ_u is an $\mathcal{I}^l_E + \mathcal{I}^l_{ae}$-equivalence class so that

$$H^l_{ae} : \mathcal{ML}^l(\Omega) \ni u \mapsto \Delta_u \in \mathcal{A}^l_{ae} \tag{4.62}$$

is well defined. That H^l_{ae} is an algebra homomorphism follows immediately upon noting that

$$\Delta_u \Delta_v \subseteq \Delta_{uv}, \qquad \Delta_{\alpha u + \beta v} \subseteq \alpha \Delta_u + \beta \Delta_v$$

for all $u, v \in \mathcal{ML}^l(\Omega)$ and $\alpha, \beta \in \mathbb{R}$. Indeed, if $(u_n) \in \Delta_u$ and $(v_n) \in \Delta_v$, then there exists $\Gamma_1, \Gamma_2 \in \mathcal{M}_0$ so that for every $x \in \Omega$ there exist $N_2 \in \mathbb{N}$ and $V_1, V_2 \in \mathcal{V}_x$, so that

$$u_n(y) = u(y),\ \ y \in V_1 \setminus (\Gamma \cup \Gamma_1),\ \ \ n \geq N_1,$$

and

$$v_n(y) = v(y),\ \ y \in V_2 \setminus (\Gamma \cup \Gamma_2),\ \ \ n \geq N_2$$

where $u \in \mathcal{C}^l(\Omega \setminus \Gamma)$ and $v \in \mathcal{C}^l(\Omega \setminus \Gamma')$. Hence

$$u_n(y)v_n(y) = u(y)v(y),\ \ y \in (V_1 \cap V_2) \setminus ((\Gamma \cup \Gamma') \cup (\Gamma_1 \cup \Gamma_2)),\ \ \ n \geq \max\{N_1,\ N_2\}.$$

Thus $(u_n v_n) \in \Delta_{uv}$. The second inclusion follows in the same way. In order to show that H^l_{ae} is injective, it is sufficient to show that, for $u, v \in \mathcal{ML}^l(\Omega)$, $u = v$ whenever $\Delta_u \cap \Delta_v \neq \emptyset$. Suppose that $u \in \mathcal{C}^l(\Omega \setminus \Gamma)$ and $v \in \mathcal{C}^l(\Omega \setminus \Gamma')$ for some $\Gamma,\ \Gamma' \in \mathcal{M}_0$. According to (4.61) thre exists $\Gamma_0, \Gamma_1 \in \mathcal{M}_0$ so that for each $x \in \Omega$ there exists $V_1, V_2 \in \mathcal{V}_x$ and $N_1, N_2 \in \mathbb{N}$ so that

$$w_n(y) = u(y),\ \ y \in V_1 \setminus (\Gamma \cup \Gamma_0),\ \ n \geq N_1$$

and

$$w_n(y) = v(y),\ \ y \in V_2 \setminus (\Gamma' \cup \Gamma_1),\ \ n \geq N_2.$$

Hence $u(y) = u_n(y) = v(y)$ whenever $n \geq \max\{N_1, N_2\}$ and $y \in (V_1 \cap V_2) \setminus (\Gamma \cup \Gamma') \cup (\Gamma_0 \cup \Gamma_1)$. Thus $u = v$ on an open and dense subset of Ωso that $u = v$ on Ω by Proposition 2.41 The commutativity of the diagram (4.59) follows immediately from (4.32) and (4.62). To see that (4.60) commutes, fix $k, l \in \overline{\mathbb{N}}$, $l \geq k$ and $p \in \mathbb{N}^n$ so that $|p| + k \leq l$. Then for $u \in \mathcal{ML}^l(\Omega)$, it follows from (4.34) that

$$D^p(H^l_{ae}(u)) = \overline{D^p}(u_n) = D^p((u_n) + \mathcal{I}^l_E + \mathcal{I}^l_{ae})$$

for any $(u_n) \in \Delta_u$. But from (4.61), (4.33) and (4.62) that

$$H^k_{ae}(\mathcal{D}^p u) = \Delta_{\mathcal{D}^p u} = \overline{D^p}(u_n) = D^p((u_n) + \mathcal{I}^l_E + \mathcal{I}^l_{ae})$$

for any $(u_n) \in \Delta_u$. Therefore the diagram (4.60) commutes. □

We now consider the compatibility of the embedding of $\mathcal{D}'(\Omega)$ into $\mathbf{A}_{ae}$ obtained in Theorem 4.8 with that of $\mathcal{ML}^l(\Omega)$ discussed in Theorem 4.13 above. In this regard, let

$$\mathcal{ML}^l_0(\Omega) = \mathcal{ML}^l(\Omega) \cap L^l_{loc}(\Omega).$$

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So that $\mathcal{ML}^l(\Omega) \subset \mathcal{D}'(\Omega)$ and $\mathcal{ML}_0^l(\Omega) \subset \mathcal{ML}^l(\Omega)$. As we show next, the commutativity of diagram

$$\begin{array}{ccc} \mathcal{D}'(\Omega) & \xrightarrow{\quad E_{ae}^l \quad} & \mathcal{A}_{ae}^l(\Omega) \\ \subset \nwarrow & & \nearrow H_{ae}^l \\ & \mathcal{ML}_0^l(\Omega) & \end{array} \tag{4.63}$$

fails in a dramatic way.

Theorem 4.14. *For each $l \in \overline{\mathbb{N}}$, the following is true.*

(i) The diagram (4.63) does not commute.

(ii) If $l \geq 1$, then there exists $u \in \mathcal{ML}_0^l(\Omega)$ so that

$$D^p(E_{ae}^l(u)) \neq D^p(H_{ae}^l(u)) \text{ in } \mathcal{A}_{ae}^k(\Omega)$$

for some $p \in \mathbb{N}^n$ with $|p| = 1$ and all $k \in \overline{\mathbb{N}}$ such that $k < l$.

Proof. (i) Since $C^l(\Omega) \subset \mathcal{ML}_0^l(\Omega)$ is a subalgebra of $\mathcal{ML}^l(\Omega)$, the result follows immediately from Theorem 4.10 (iv).

(ii) Fix $a = (a_1, \cdots, a_n) \in \Omega$. Let $\Gamma = \{x \in \Omega | x_1 = a_1\}$. Then $\Gamma \in \mathcal{M}_0$, so that $u : \Omega \longrightarrow \mathbb{R}$ defined by

$$u(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x_1 \leq a_1 \\ 1 & \text{if} & x_1 > a_1 \end{array} \right\}$$

belongs to$\mathcal{ML}_0^l(\Omega)$ for every $l \in \overline{\mathbb{N}}$. Furthermore, $\mathcal{D}^p u = 0$ for every $p \in \mathbb{N}^n$. Thus $D^p(H_{ae}^l(u)) = 0$ in $\mathcal{A}_{ae}^k(\Omega)$, for all $|p| = 1$ and $k < l$. But for $k < l$ and $|p| = 1$,

$$D^p(E_{ae}^l u) = E_{ae}^k(D^p u) \neq 0 \text{ in } \mathcal{A}_{ae}^k(\Omega),$$

since $D^p u \neq 0$ in $\mathcal{D}'(\Omega)$ and E_{ae}^l is a linear injection.

□

In view of Theorem 4.14, a function $u \in \mathcal{ML}_0^l(\Omega)$ may have at least two distinct representations in the algebra $\mathcal{A}_{ae}^l(\Omega)$, namely, $E_{ae}^l(u)$ and $H_{ae}^l(u)$. These representations are different in the sense that

$$E_{ae}^l(u) \neq H_{ae}^l(u)$$

and more generally,

$$D^p(E_{ae}^l(u)) \neq D^p(H_{ae}^l(u))$$

for some $p \in \mathbb{N}^n$ and all k so that $k + |p| \leq l$. In particular, the heaviside function $u \in \mathcal{ML}_0^l(\mathbb{R})$ given by

$$u(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x \leq 0 \\ 1 & \text{if} & x > 0 \end{array} \right\}$$

has representations $H^l_{ae}(u),\ E^l_{ae}(u) \in \mathcal{A}^l_{ae}(\mathbb{R}),\ l \in \overline{\mathbb{N}}$, for which

$$\frac{\partial}{\partial x} H^l_{ae}(u) = 0, \quad \frac{\partial}{\partial x} E^l_{ae}(u) = \delta \neq 0 \ \text{ in } \mathcal{A}^k_{ae}(\mathbb{R})$$

for all $l \in \overline{\mathbb{N}}$ and $k + 1 \leq l$. Therefore u is a chain generalized solution of the equation

$$\frac{\partial u}{\partial x} = 0 \tag{4.64}$$

in the chain $\mathbf{A}_{ae}$. In particular, u is a generalized solution of (4.64) in the algebra $\mathcal{A}^\infty_{ae}(\Omega)$. Clearly, u is not a generalized solution of (4.64) in the sense of distributions. It follows that a generalized solution $u \in \mathcal{ML}^\infty_0(\Omega)$ of a linear or nonlinear PDE

$$T(x, D)u(x) = f(x)$$

in $\mathcal{A}^\infty_{ae}(\Omega)$ such as those given by [73, Theorem 10], may fail to be a solution in the sense of distributions.

4.4 $\mathcal{NL}^l(\Omega)$ and Nowhere Dense Algebras

In this section we show how the chain $\mathbf{A}_{oc}$ is related to the chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$ considered in Section 4.1. It is shown how the existence result for chain generalized solutions of nonlinear PDEs in $\mathbf{A}_{oc}$, Theorem 3.10, leads to corresponding existence result in the chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$.

4.4.1 The chain $\mathbf{A}_{oc}$ and $\mathbf{A}_{nd}$

In order to establish the relationship between the chains $\mathbf{A}_{oc}$ and $\mathbf{A}_{nd}$, we introduce an auxiliary chain $\mathbf{A}^0_{nd}$. In this regard, we note that

$$\mathcal{I}^l_{nd} \subset \mathcal{I}^l_{cs} \subset \mathcal{S}^l_{cs}, \quad l \in \overline{\mathbb{N}}. \tag{4.65}$$

Indeed, for each $(u_n) \in \mathcal{I}^l_{nd}$ there exists $\Gamma \subset \Omega$ closed and nowhere dense such that

$$\begin{array}{ll} \forall & x \in \Omega \setminus \Gamma : \\ \exists & N \in \mathbb{N} : \\ \forall & n \geq N, \quad |p| \leq l : \\ & D^p u_n(x) = 0 \end{array}$$

Thus (u_n) converges to 0 pointwise on an open and dense, hence residual, subset of Ω. It follows from Proposition 3.4 that $(u_n) \in \mathcal{I}^l_{cs}$. Since $\mathcal{I}^l_{nd}$ is an ideal in $C^l(\Omega)^{\mathbb{N}}$, it is also an ideal in $\mathcal{S}^l_{cs}$. Furthermore, the inclusions,

$$\mathcal{I}^l_{nd} \subseteq \mathcal{I}^k_{nd}, \quad \mathcal{S}^l_{cs} \subseteq \mathcal{S}^k_{cs}, \quad k \leq l$$

and

$$D^p(\mathcal{I}^l_{nd}) \subseteq \mathcal{I}^k_{nd}, \quad D^p(\mathcal{S}^l_{cs}) \subseteq \mathcal{S}^k_{cs}, \quad |p| + k \leq l$$

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imply that

$$\mathbf{A}^0_{nd} = \{\mathcal{A}^l_0(\Omega), \mathcal{A}^k_0(\Omega), \gamma^l_k)|l, k \in \mathbb{N}, \quad k \leq l\}$$

with $\mathcal{A}^l_0(\Omega) = \mathcal{S}^l_{cs}/\mathcal{I}^l_{nd}$ and γ^l_k defined as

$$\gamma^l_k : \mathcal{A}^l_0(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto (u_n) + \mathcal{I}^k_{nd} \in \mathcal{A}^k_0(\Omega) \quad k \leq l \tag{4.66}$$

is a differential chain of algebras of generalized functions. The way in which $\mathbf{A}_{oc}$ is related to $\mathbf{A}_{nd}$ is given in the following

Theorem 4.15. *For each $l \in \overline{\mathbb{N}}$ then there exists an injective algebra homomorphism*

$$H^l : \mathcal{A}^l_0(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega)$$

and a surjective algebra homomorphism

$$G^l : \mathcal{A}^l_0(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

such that the following hold.

(i) The diagrams

$$\begin{CD} \mathcal{A}^l_0(\Omega) @>H^l>> \mathcal{A}^l_{nd}(\Omega) \\ @V{\gamma^l_k}VV @VV{\gamma^l_k}V \\ \mathcal{A}^k_0(\Omega) @>H^k>> \mathcal{A}^k_{nd}(\Omega) \end{CD} \tag{4.67}$$

and

$$\begin{CD} \mathcal{A}^l_0(\Omega) @>G^l>> \mathcal{NL}^l(\Omega) \\ @V{\gamma^l_k}VV @VV{\gamma^l_k}V \\ \mathcal{A}^k_0(\Omega) @>G^k>> \mathcal{NL}^k(\Omega) \end{CD} \tag{4.68}$$

commute for all $k \leq l$.

(ii) The diagrams

$$\begin{CD} \mathcal{A}^l_0(\Omega) @>D^p>> \mathcal{A}^k_0(\Omega) \\ @V{H^l}VV @VV{H^k}V \\ \mathcal{A}^l_{nd}(\Omega) @>D^p>> \mathcal{A}^k_{nd}(\Omega) \end{CD} \tag{4.69}$$

and

$$\begin{CD} \mathcal{A}^l_0(\Omega) @>D^p>> \mathcal{A}^k_0(\Omega) \\ @V{G^l}VV @VV{G^k}V \\ \mathcal{NL}^l(\Omega) @>D^p>> \mathcal{NL}^k(\Omega) \end{CD} \tag{4.70}$$

commute whenever $k + |p| \leq l$.

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Proof. For each $l \in \overline{\mathbb{N}}$ define H^l and G^l as

$$H^l : \mathcal{A}_0^l(\Omega) \ni (u_n) + \mathcal{I}_{nd}^l \mapsto (u_n) + \mathcal{I}_{nd}^l \in \mathcal{A}_{nd}^l(\Omega) \tag{4.71}$$

and

$$G^l : \mathcal{A}_0^l(\Omega) \ni (u_n) + \mathcal{I}_{nd}^l \mapsto (u_n) + \mathcal{I}_{cs}^l \in \mathcal{NL}^l(\Omega) \tag{4.72}$$

H^l is well defined since $\mathcal{S}_{cs}^l \subseteq C^l(\Omega)^{\mathbb{N}}$, while G^l also well defined since $\mathcal{I}_{nd}^l \subseteq \mathcal{I}_{cs}$. Clearly H^l is injective, and G^l is surjective.

The commutativity of the diagrams in (i) follows immediately from (4.66), (4.71) and (4.72) as well as the definition of the algebra homomorphisms

$$\gamma_k^l : \mathcal{A}_{nd}^l(\Omega) \longrightarrow \mathcal{A}_{nd}^k(\Omega) \quad k \leq l \tag{4.73}$$

and

$$\gamma_k^l : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega) \tag{4.74}$$

see (4.2) and (3.19).

The commutativity of the diagrams in (ii) follows in a similar way taking into account the definitions of the differential operators in the algebras $\mathcal{A}_{nd}^l(\Omega)$, $\mathcal{A}_0^l(\Omega)$ and $\mathcal{NL}^l(\Omega)$, respectively. □

As shown in Sections 3.3 and 4.3.1, each of the algebras $\mathcal{A}_{nd}^l$ and $\mathcal{NL}^l(\Omega)$ contain $\mathcal{ML}^l(\Omega)$ as a subalgebra. In particular, there exists injective algebra homomorphisms

$$H_{oc}^l : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega) \tag{4.75}$$

and

$$H_{nd}^l : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}_{nd}^l(\Omega) \tag{4.76}$$

so that the diagrams

$$\begin{array}{ccc}
\mathcal{NL}^l(\Omega) & \xrightarrow{\gamma_k^l} & \mathcal{NL}^k(\Omega) \\
H_{oc}^l \uparrow & & \uparrow H_{oc}^k \\
\mathcal{ML}^l(\Omega) & \xrightarrow{\subset} & \mathcal{ML}^k(\Omega) \\
H_{nd}^l \downarrow & & \downarrow H_{nd}^k \\
\mathcal{A}_{nd}^l(\Omega) & \xrightarrow{\gamma_k^l} & \mathcal{A}_{nd}^k(\Omega)
\end{array} \tag{4.77}$$

and

$$\begin{array}{ccc} \mathcal{NL}^l(\Omega) & \xrightarrow{D^p} & \mathcal{NL}^k(\Omega) \\ \uparrow H^l_{oc} & & \uparrow H^k_{oc} \\ \mathcal{ML}^l(\Omega) & \xrightarrow{\mathcal{D}^p} & \mathcal{ML}^k(\Omega) \\ \downarrow H^l_{nd} & & \downarrow H^k_{nd} \\ \mathcal{A}^l_{nd}(\Omega) & \xrightarrow{D^p} & \mathcal{A}^k_{nd}(\Omega) \end{array} \tag{4.78}$$

commute whenever $k \leq l$ and $k + |p| \leq l$, respectively.

A trivial modification of Theorem 4.12 yields the following

Proposition 4.16. *For each $l \in \overline{\mathbb{N}}$, there exists an injective algebra homomorphism*

$$\Gamma^l_0 : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_0(\Omega)$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{A}^l_0(\Omega) & \xrightarrow{\gamma^l_k} & \mathcal{A}^k_0(\Omega) \\ \uparrow \Gamma^l_0 & & \uparrow \Gamma^k_0 \\ \mathcal{ML}^l(\Omega) & \xrightarrow{\subset} & \mathcal{ML}^k(\Omega) \end{array} \tag{4.79}$$

and

$$\begin{array}{ccc} \mathcal{A}^l_0(\Omega) & \xrightarrow{D^p} & \mathcal{A}^k_0(\Omega) \\ \uparrow \Gamma^l_0 & & \uparrow \Gamma^k_0 \\ \mathcal{ML}^l(\Omega) & \xrightarrow{\mathcal{D}^p} & \mathcal{ML}^k(\Omega) \end{array} \tag{4.80}$$

commute, whenever $k \leq l$ and $|p| + k \leq l$, respectively.

As we show next, the homomorphism

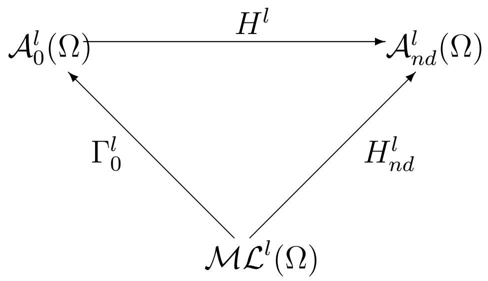
$$H^l : \mathcal{A}^l_0(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega)$$

and

$$G^l : \mathcal{A}^l_0(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

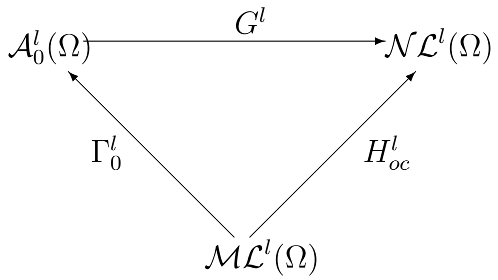
leave $\mathcal{ML}^l(\Omega)$ invariant.

Theorem 4.17. *The following diagrams*



(4.81)

and



(4.82)

commute for all $l \in \overline{\mathbb{N}}$.

Proof. For each $u \in \mathcal{ML}^l(\Omega)$,

$$\Gamma_0^l(u) = (u_n) + \mathcal{I}_{nd}^l$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}} \subset \mathcal{S}_{cs}^l$ satisfies

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma : \\ \exists & V \in \mathcal{V}_X, \ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \ n \geq \mathbb{N} : \\ & u_n(y) = u(y), \ y \in V. \end{array} \tag{4.83}$$

where $\Gamma \subset \Omega$ is closed, nowhere dense set so that $u \in \mathcal{C}^l(\Omega \backslash \Gamma)$. Likewise, the map $H_{nd}^l(u)$ may be expressed as

$$H_{nd}^l(u) = (u_n) + \mathcal{I}_{nd}^l$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ satisfies (4.83). Clearly, $(u_n) \in \mathcal{S}_{cs}^l$ for any $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (4.83). Thus the commutativity of (4.81) follows from the Definition 4.71 of H^l.

Since any sequence $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (4.83) converges to $u \in \mathcal{ML}^l(\Omega)$ with respect to λ_l, the commutativity of (4.82) follows the same way as that of (4.81), taking into account the definition (3.25) of H_{oc}^l. □

4.4.2 The chain $\mathbf{A}_{oc}$ and $\mathbf{A}_{ae}$

We now consider the relationship between the chain $\mathbf{A}_{oc}$ and $\mathbf{A}_{ae}$. In this regards, we note that

$$\mathcal{I}_0^l = (\mathcal{I}_E^l + \mathcal{I}_{ae}^l) \cap C^l(\Omega)^{\mathbb{N}} \subseteq \mathcal{I}_{cs}^l \subset \mathcal{S}_{cs}^l, \quad l \in \overline{\mathbb{N}}$$

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Indeed, for $(u_n) \in \mathcal{I}_0^l$ there exists, by Lemma 4.5, a closed nowhere dense set $\Gamma \in \mathcal{M}_0$ so that

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\ n \geq N : \\ & u_n(y) = 0, \quad y \in V \setminus \Gamma. \end{array}$$

Hence (u_n) converges point-wise to 0 on the open and dense set $\Omega \setminus \Gamma$ so that $(u_n) \in \mathcal{I}_{cs}^l$.

Furthermore, the inclusions

$$\mathcal{I}_0^l \subset \mathcal{I}_0^k, \quad \mathcal{S}_{cs}^l \subset \mathcal{S}_{cs}^k, \quad l,\ k \in \mathbb{N} \tag{4.84}$$

and

$$D^p(\mathcal{I}_0^l) \subset \mathcal{I}_0^k, \quad D^p(\mathcal{S}_{cs}^l) \subset \mathcal{S}_{cs}^k, \quad k + |p| \leq l \tag{4.85}$$

hold. Therefore

$$\mathbf{A}_{ae}^0 = \{(\mathcal{B}_{ae}^l, \mathcal{B}_{ae}^k, \gamma_k^l) | l, k \in \overline{\mathbb{N}}, k \leq l\}$$

is a differential chain of algebra of generalized functions, where

$$\mathcal{B}_{ae}^l(\Omega) = \mathcal{S}_{cs}^l / \mathcal{I}_0^l$$

and

$$\gamma_k^l : \mathcal{B}_{ae}^l(\Omega) \ni (u_n) + \mathcal{I}_0^l \mapsto (u_n) + \mathcal{I}_0^k \in \mathcal{B}_{ae}^k(\Omega) \tag{4.86}$$

for all $l, k \in \overline{\mathbb{N}}$ with $k \leq l$. The differential operators are defined in the usual way, that is

$$D^p : \mathcal{B}_{ae}^l(\Omega) \ni (u_n) + \mathcal{I}_0^l \mapsto D^p(u_n) + \mathcal{I}_0^k \in \mathcal{B}_{ae}^k(\Omega), \quad |p| + k \leq l.$$

Theorem 4.18. *For every $l \in \overline{\mathbb{N}}$ there exists an injective algebra homomorphism*

$$F_{ae}^l : \mathcal{B}_{ae}^l(\Omega) \longrightarrow \mathcal{A}_{ae}^l(\Omega)$$

and a surjective algebra homomorphism

$$G_{ae}^l : \mathcal{B}_{ae}^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

so that the following hold.

(i) The diagrams

$$\begin{array}{ccc} \mathcal{B}_{ae}^l(\Omega) & \xrightarrow{\ F_{ae}^l\ } & \mathcal{A}_{ae}^l(\Omega) \\ \gamma_k^l \downarrow & & \downarrow \gamma_k^l \\ \mathcal{B}_{ae}^k(\Omega) & \xrightarrow{\ F_{ae}^k\ } & \mathcal{A}_{ae}^k(\Omega) \end{array} \tag{4.87}$$

and

$$\begin{array}{ccc} \mathcal{B}_{ae}^l(\Omega) & \xrightarrow{\ G_{ae}^l\ } & \mathcal{NL}^l(\Omega) \\ \gamma_k^l \downarrow & & \downarrow \gamma_k^l \\ \mathcal{B}_{ae}^k(\Omega) & \xrightarrow{\ G_{ae}^k\ } & \mathcal{NL}^k(\Omega) \end{array} \tag{4.88}$$

commute for all $k \leq l$.

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(ii) The diagrams

$$\begin{array}{ccc} \mathcal{B}^l_{ae}(\Omega) & \xrightarrow{D^p} & \mathcal{B}^k_{ae}(\Omega) \\ {\scriptstyle F^l_{ae}}\downarrow & & \downarrow{\scriptstyle F^k_{ae}} \\ \mathcal{A}^l_{ae}(\Omega) & \xrightarrow{D^p} & \mathcal{A}^k_{ae}(\Omega) \end{array} \tag{4.89}$$

and

$$\begin{array}{ccc} \mathcal{B}^l_{ae}(\Omega) & \xrightarrow{D^p} & \mathcal{B}^k_{ae}(\Omega) \\ {\scriptstyle G^l_{ae}}\downarrow & & \downarrow{\scriptstyle G^k_{ae}} \\ \mathcal{NL}^l(\Omega) & \xrightarrow{D^p} & \mathcal{NL}^k(\Omega) \end{array} \tag{4.90}$$

commute whenever $k + |p| \leq l$.

Proof. For each $l \in \overline{\mathbb{N}}$ define algebra homomorphisms F^l_{ae} and G^l_{ae} as

$$F^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \ni (u_n) + \mathcal{I}^l_0 \mapsto (u_n) + (\mathcal{I}^l_E + \mathcal{I}_{ae}) \in \mathcal{A}^l_{ae}(\Omega) \tag{4.91}$$

and

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \ni (u_n) + \mathcal{I}^l_0 \mapsto (u_n) + \mathcal{I}^l_{cs} \in \mathcal{NL}^l(\Omega). \tag{4.92}$$

Since $\mathcal{S}^l_{cs} \subseteq C^l(\Omega)^{\mathbb{N}} \subseteq \mathcal{E}^l_{ae}$ and $\mathcal{I}^l_0 \subseteq (\mathcal{I}^l_E + \mathcal{I}_{ae})$ it follows that F^l_{ae} is well defined. Also G^l_{ae} is well defined since $\mathcal{I}^l_0 \subseteq \mathcal{I}_{cs}$. The mapping F^l_{ae} is injective since $\mathcal{I}^l_0 = (\mathcal{I}^l_E + \mathcal{I}^l_{oc}) \cap \mathcal{S}^l_{cs}$ which implies that $\{(u_n) + \mathcal{I}^l_0 \in \mathcal{B}^l_{ae} \mid F^l_{ae}((u_n) + \mathcal{I}^l_0) = 0\} = \{0\}$. G^l_{ae} is surjective since

$$\mathcal{I}^l_0 \subseteq \mathcal{I}^l_{cs}.$$

The commutativity of the diagrams in (i) follows immediately from (4.86), (4.91) and (4.92) as well as the definition of the algebra homomorphisms

$$\gamma^l_k : \mathcal{A}^l_{ae}(\Omega) \longrightarrow \mathcal{A}^k_{ae}(\Omega) \quad k \leq l \tag{4.93}$$

and

$$\gamma^l_k : \mathcal{NL}^l(\Omega) \longrightarrow \mathcal{NL}^k(\Omega) \tag{4.94}$$

given by (4.32) and (3.19) respectively.

The commutativity of the diagrams in (ii) follows in a similar way taking into account the definitions of the differential operators in the algebras $\mathcal{A}^l_{ae}(\Omega)$, $\mathcal{B}^l_{ae}(\Omega)$ and $\mathcal{NL}^l(\Omega)$ respectively. □

As shown in Section 4.3.2, if $\mathcal{M}_0$ consists of all closed nowhere dense subsets of Ω, then each of the algebras $\mathcal{A}^l_{ae}(\Omega)$ contain $\mathcal{ML}^l(\Omega)$ as a subalgebra. In particular, there exists for each $l \in \overline{\mathbb{N}}$ an injective algebra homomorphism

$$H^l_{ae} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega) \tag{4.95}$$

so that the diagrams

$$\begin{array}{ccc}
\mathcal{NL}^l(\Omega) & \xrightarrow{\gamma_k^l} & \mathcal{NL}^k(\Omega) \\
\uparrow H_{oc}^l & & \uparrow H_{oc}^k \\
\mathcal{ML}^l(\Omega) & \xrightarrow{\subset} & \mathcal{ML}^k(\Omega) \\
\downarrow H_{ae}^l & & \downarrow H_{ae}^k \\
\mathcal{A}_{ae}^l(\Omega) & \xrightarrow{\gamma_k^l} & \mathcal{A}_{ae}^k(\Omega)
\end{array} \tag{4.96}$$

and

$$\begin{array}{ccc}
\mathcal{NL}^l(\Omega) & \xrightarrow{D^p} & \mathcal{NL}^k(\Omega) \\
\uparrow H_{oc}^l & & \uparrow H_{oc}^k \\
\mathcal{ML}^l(\Omega) & \xrightarrow{\mathcal{D}^p} & \mathcal{ML}^k(\Omega) \\
\downarrow H_{ae}^l & & \downarrow H_{ae}^k \\
\mathcal{A}_{ae}^l(\Omega) & \xrightarrow{D^p} & \mathcal{A}_{ae}^k(\Omega)
\end{array} \tag{4.97}$$

commute whenever $k \leq l$ and $k + |p| \leq l$ respectively, where H_{oc}^l is defined by (4.75).

In view of Theorem 4.13 we have the following

Proposition 4.19. *Assume that* $\mathcal{M}_0 = \{\Gamma \subset \Omega | \Gamma$ is closed nowhere dense$\}$. *Then for each* $l \in \overline{\mathbb{N}}$, *there exists an injective algebra homomorphism*

$$H_{ae}^l : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{B}_{ae}^l(\Omega)$$

such that the diagrams

$$\begin{array}{ccc}
\mathcal{B}_{ae}^l(\Omega) & \xrightarrow{\gamma_k^l} & \mathcal{B}_{ae}^k(\Omega) \\
\uparrow H_{ae}^l & & \uparrow H_{ae}^k \\
\mathcal{ML}^l(\Omega) & \xrightarrow{\subset} & \mathcal{ML}^k(\Omega)
\end{array} \tag{4.98}$$

and

$$\begin{array}{ccc}
\mathcal{B}_{ae}^l(\Omega) & \xrightarrow{D^p} & \mathcal{B}_{ae}^k(\Omega) \\
\uparrow H_{ae}^l & & \uparrow H_{ae}^k \\
\mathcal{ML}^l(\Omega) & \xrightarrow{\mathcal{D}^p} & \mathcal{ML}^k(\Omega)
\end{array} \tag{4.99}$$

commute, whenever $k \leq l$ *and* $|p| + k \leq l$ *respectively*

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We note that the algebra homomorphism

$$H^l_{oc}(u) : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{B}^l_{ae}(\Omega)$$

is obtained by setting

$$H^l_{oc}(u)(u) = (u_n) + \mathcal{I}^l_0$$

where $(u_n) \in \mathcal{S}^l_{cs}$ is any sequence satisfying (4.61). The existence of such a sequence is guaranteed by Lemma 3.6.

We now show that the homomorphism

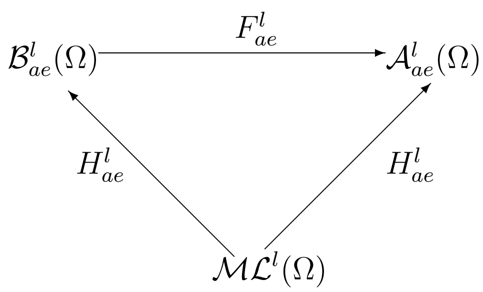
$$F^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$$

and

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

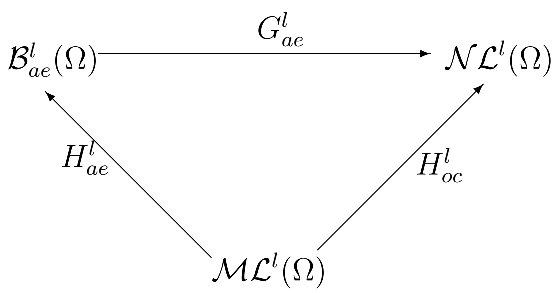
leave the subalgebra $\mathcal{ML}^l(\Omega)$ of $\mathcal{B}^l_{ae}(\Omega)$ invariant.

Theorem 4.20. *Assume that* $\mathcal{M}_0 = \{\Gamma \subset \Omega | \Gamma$ is closed nowhere dense . *Then the diagrams*



(4.100)

and



(4.101)

commute for all $l \in \overline{\mathbb{N}}$.

Proof. The proof is similar to that of Theorem 4.17 which we outline below.

For each $u \in \mathcal{ML}^l(\Omega)$,

$$\Gamma^l_0(u) = (u_n) + \mathcal{I}^l_{nd}$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}} \subset \mathcal{S}^l_{cs}$ satisfies

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma : \\ \exists & V \in \mathcal{V}_X, \ N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \ n \geq \mathbb{N} : \\ & u_n(y) = u(y), \ y \in V. \end{array} \tag{4.102}$$

where $\Gamma \subset \Omega$ is closed, nowhere dense set so that $u \in \mathcal{C}^l(\Omega \backslash \Gamma)$. Likewise, the map $H^l_{ae}(u)$ may be expressed as

$$H^l_{ae}(u) = (u_n) + \mathcal{I}^l_{nd}$$

where $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ satisfies (4.102). Clearly, $(u_n) \in \mathcal{S}^l_{cs}$ for any $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (4.102). Thus the commutativity of (4.100) follows from the definition (4.91) of F^l_{ae}.

Since any sequence $(u_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ that satisfies (4.102) converges to $u \in \mathcal{ML}^l(\Omega)$ with respect to λ_l, the commutativity of (4.101) follows the same way as that of (4.100), taking into account the definition (3.25) of H^l_{oc}. □

4.4.3 Chain generalized solutions in $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$

In this section we show how the existence result for chain generalized solutions of nonlinear PDEs in $\mathbf{A}_{oc}$ given in Theorem 3.10 leads to corresponding existence results in the chains $\mathbf{A}_{ae}$ and $\mathbf{A}_{nd}$, respectively. In this regards, consider a polynomial nonlinear differential operator

$$T = \sum_{1 \leq i \leq h} c_i(x) \prod_{1 \leq j \leq k_i} D^{p_{ij}}, \quad x \in \Omega \tag{4.103}$$

where $h, k_i \in \mathbb{N}$, $c_i \in C^\infty(\Omega)$ and $p_{ij} \in \mathbb{N}^n$ satisfies $|p_{ij}| \leq m$ for all $i = 1, \cdots, h$ and $j = 1, \cdots, k_i$. For $f \in C^\infty(\Omega)$ we show that, under a mild assumption on the operator T, the polynomial PDE,

$$Tu = f. \tag{4.104}$$

admits a chain generalized solutions in $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$ respectively.

We deal first with the case of solutions in $\mathbf{A}_{nd}$. In this regard, it is clear that

$$T(\mathcal{I}^l_{nd}) \subset \mathcal{I}^k_{nd}$$

whenever $k + m \leq l$ and, obviously,

$$T(C^l(\Omega)^{\mathbb{N}}) \subset C^k(\Omega)^{\mathbb{N}}, \quad k + m \leq l$$

Therefore, since $\mathcal{I}^l_{nd}$ is off diagonal, $(u_n) - (v_n) \in \mathcal{I}^l_{nd}$ which implies $(Tu_n) - (tv_n) \in \mathcal{I}^k_{nd}$, see Section 1.6, so that

$$T_{nd} : \mathcal{A}^l_{nd}(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto T(u_n) + \mathcal{I}^k_{nd} \in \mathcal{A}^k_{nd}(\Omega) \quad k + m \leq l$$

defines an extension of

$$T : C^l(\Omega) \longrightarrow C^k(\Omega),$$

for $k + m \leq l$. In the same way,

$$T_{oc} : \mathcal{NL}^l(\Omega) \ni (u_n) + \mathcal{I}^l_{cs} \mapsto T(u_n) + \mathcal{I}^k_{cs} \in \mathcal{NL}^k(\Omega), \quad k + m \leq l$$

and

$$T_0 : \mathcal{A}^l_0(\Omega) \ni (u_n) + \mathcal{I}^l_{nd} \mapsto T(u_n) + \mathcal{I}^k_{nd} \in \mathcal{A}^k_0(\Omega) \quad k + m \leq l$$

defines an extension of $T : C^l(\Omega) \longrightarrow C^k(\Omega)$

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Proposition 4.21. *The diagrams*

$$\begin{array}{ccc} \mathcal{A}_0^l(\Omega) & \xrightarrow{\quad T_0 \quad} & \mathcal{A}_0^k(\Omega) \\ {\scriptstyle F_{nd}^l}\downarrow & & \downarrow{\scriptstyle F_{nd}^k} \\ \mathcal{A}_{nd}^l(\Omega) & \xrightarrow{\quad T_{nd} \quad} & \mathcal{A}_{nd}^k(\Omega) \end{array} \tag{4.105}$$

and

$$\begin{array}{ccc} \mathcal{A}_0^l(\Omega) & \xrightarrow{\quad T_0 \quad} & \mathcal{A}_0^k(\Omega) \\ {\scriptstyle G_{nd}^l}\downarrow & & \downarrow{\scriptstyle G_{nd}^k} \\ \mathcal{NL}^l(\Omega) & \xrightarrow{\quad T_{oc} \quad} & \mathcal{NL}^k(\Omega) \end{array} \tag{4.106}$$

commute whenever $k+m \leq l$, with F_{nd}^l and G_{nd}^l the algebra homomorphisms obtain in Theorem 4.12

Proof. For $u = (u_n) + \mathcal{I}_{nd}^l \in \mathcal{A}_0^l$ with $k+m \leq l$,

$$\begin{aligned} T_{nd}(F_{nd}^l(u)) &= T_{nd}((u_n) + \mathcal{I}_{nd}^l) \\ &= T(u_n) + \mathcal{I}_{nd}^k \end{aligned}$$

and

$$\begin{aligned} F_{nd}^k(T_0(u)) &= F_{nd}^k(T(u_n) + \mathcal{I}_{nd}^k) \\ &= T(u_n) + \mathcal{I}_{nd}^k \end{aligned}$$

Hence (4.105) commutes. The commutativity of the diagram (4.106) follows in the same way. □

Theorem 4.22. *If $f \in C^\infty(\Omega)$, and the operator T defined in* (4.103) *satisfies* (2.63) *to* (2.64) *then the PDE*

$$Tu = f \tag{4.107}$$

admits a chain generalized solution in $\mathbf{A}_{nd}$.

Proof. According to Theorem 3.12, there exists a chain generalized solution of (4.107) in $\mathbf{A}_{oc}$. That is, there exists $(u_n) \in \mathcal{S}_{cs}^\infty$ so that $u = (u_n) + \mathcal{I}_{cs}^l$ satisfies

$$Tu = T(u_n) + \mathcal{I}_{cs}^k = f + \mathcal{I}_{cs}^k$$

for all $l, k \in \overline{\mathbb{N}}$ with $k+m \leq l$. Since

$$G_{nd}^l : \mathcal{A}_0^l(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

is surjective for each $l \in \overline{\mathbb{N}}$, it follows that there exists $v = (v_n) + \mathcal{I}_{nd}^l \in \mathcal{A}_0^\infty(\Omega)$ so that $G_{nd}^\infty(v) = u$. It follows from Theorem 4.12 and Proposition 4.21 that

$$T_o((v_n) + \mathcal{I}_{nd}^l) = f + \mathcal{I}_{nd}^k$$

for all $k, l \in \overline{\mathbb{N}}$ so that $k+m \leq l$. In the same way, it follows that

$$F_{nd}^\infty(v) = (v_n) + \mathcal{I}_{nd}^\infty \in \mathcal{A}_{nd}^\infty(\Omega)$$

is a chain generalized solution of (4.107) in $\mathbf{A}_{nd}$. □

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Let us now consider the existence of chain generalized solutions of the PDE (4.107) in the chain $\mathbf{A}_{ae}$. It is clear that

$$(u_n) - (v_n) \in \mathcal{I}_0^l \Longrightarrow T(u_n) - T(v_n) \in \mathcal{I}_0^k$$

for all $(u_n), (v_n) \in C^\infty(\Omega)^{\mathbb{N}}$ and $k + m \leq l$. Thus

$$T_{\mathcal{B}} : \mathcal{B}_{ae}^l(\Omega) \in (u_n) + \mathcal{I}_0^l \mapsto T(u_n) + \mathcal{I}_0^k \in \mathcal{B}_{ae}^k(\Omega)$$

is a well-defined extension of $T : C^l(\Omega) \longrightarrow C^k(\Omega)$, for all $l, k \in \overline{\mathbb{N}}$ such that $m + k \leq l$. With each $(u_n) \in \mathcal{E}_{ae}^l(\Omega)$ and $k \in \overline{\mathbb{N}}$, we associate the set

$$\overline{T}_{ae}(u_n) = \left\{ (v_n) \in \mathcal{E}_{ae}^k(\Omega) \;\middle|\; \begin{array}{ll} \exists & \Gamma_0 \in \mathcal{M}_0 : \\ \forall & x \in \Omega : \\ \exists & V \in \mathcal{V}_x,\;\; N \in \mathbb{N} : \\ \forall & n \in \mathbb{N},\;\; n \geq N : \\ & v_n(y) = Tu_n(y), \quad y \in V \setminus \Gamma \end{array} \right\}. \tag{4.108}$$

This gives rise to a relation

$$\mathcal{E}_{ae}^l(\Omega) \ni (u_n) \mapsto \overline{T}_{ae}(u_n) \subset \mathcal{E}_{ae}^k(\Omega).$$

It follows from Proposition 4.6 that

$$\overline{T}_{ae}(u_n) - \overline{T}_{ae}(u_n) \subseteq \mathcal{I}_E^k + \mathcal{I}_{ae}^k$$

and

$$(v_n) \in \overline{T}_{ae}(u_n), \quad ((v_n) - (w_n)) \in \mathcal{I}_E^k + \mathcal{I}_{ae}^k \quad \Longrightarrow (w_n) \in \overline{T}_{ae}(u_n)$$

for all $(u_n) \in \mathcal{E}_{ae}^l(\Omega)$ and $l, k \in \overline{\mathbb{N}}$ such that $k + m \leq l$. Therefore,

$$T_{ae} : \mathcal{A}_{ae}^l(\Omega) \ni (u_n) + (\mathcal{I}_E^l + \mathcal{I}_{ae}^l) \mapsto \overline{T}_{ae}(u_n) \in \mathcal{A}_{ae}^k(\Omega) \tag{4.109}$$

is well-defined for all $k, l \in \overline{\mathbb{N}}$ such that $m + k \leq l$. Note that

$$T_{ae}((u_n) + (\mathcal{I}_E^l + \mathcal{I}_{ae}^l)) = (v_n) + (\mathcal{I}_E^k + \mathcal{I}_{ae}^l)$$

where (v_n) is any member of the set $\overline{T}_{ae}(u_n)$. Since the ideal $\mathcal{I}_E^l + \mathcal{I}_{ae}^l$ is off diagonal, it follows that (4.109) is an extension of

$$T : C^l(\Omega) \longrightarrow C^k(\Omega), \quad k + m \leq l.$$

Proposition 4.23. *For all $k, l \in \overline{\mathbb{N}}$ so that $m + k \leq l$, the diagrams*

$$\begin{array}{ccc} \mathcal{B}_{ae}^l(\Omega) & \xrightarrow{\;\;T_{\mathcal{B}}\;\;} & \mathcal{B}_{ae}^k(\Omega) \\ {\scriptstyle G_{ae}^l}\downarrow & & \downarrow{\scriptstyle G_{ae}^k} \\ \mathcal{NL}^l(\Omega) & \xrightarrow{\;\;T_{oc}\;\;} & \mathcal{NL}^k(\Omega) \end{array} \tag{4.110}$$

and

$$
\begin{array}{ccc}
\mathcal{B}^l_{ae}(\Omega) & \xrightarrow{\quad T_{\mathcal{B}} \quad} & \mathcal{B}^k_{ae}(\Omega) \\
{\scriptstyle F^l_{ae}}\downarrow & & \downarrow{\scriptstyle F^k_{ae}} \\
\mathcal{A}^l_{ae}(\Omega) & \xrightarrow{\quad T_{ae} \quad} & \mathcal{A}^k_{ae}(\Omega)
\end{array}
\tag{4.111}
$$

commute.

Proof. For $u = (u_n) + \mathcal{I}^l_0 \in \mathcal{B}^l_{ae}$ with $k + m \leq l$,

$$
\begin{aligned}
T_{oc}(G^l_{ae}(u)) &= T_{oc}((u_n) + \mathcal{I}^l_{cs}) \\
&= T(u_n) + \mathcal{I}^k_{cs}
\end{aligned}
$$

and

$$
\begin{aligned}
G^k_{ae}(T_{\mathcal{B}}(u)) &= G^k_{ae}(T(u_n) + \mathcal{I}^k_0) \\
&= T(u_n) + (\mathcal{I}^k_{cs})
\end{aligned}
$$

Hence (4.110) commutes. The commutativity of the diagram (4.111) follows in the same way. □

Theorem 4.24. *If* $f \in C^\infty(\Omega)$, *and the operator* T *defined in* (4.103) *satisfies* (2.63) *to* (2.64) *then the PDE*

$$Tu = f \tag{4.112}$$

admits a chain generalized solution in $\mathbf{A}_{ae}$.

Proof. According to Theorem 3.12, there exists a chain generalized solution of (4.107) in $\mathbf{A}_{oc}$. That is, there exists $(u_n) \in \mathcal{S}^\infty_{cs}$ so that $u = (u_n) + \mathcal{I}^l_{cs}$ satisfies

$$Tu = T(u_n) + \mathcal{I}^k_{cs} = f + \mathcal{I}^k_{cs}$$

for all $l, k \in \overline{\mathbb{N}}$ with $k + m \leq l$. Since

$$G^l_{ae} : \mathcal{B}^l_{ae}(\Omega) \longrightarrow \mathcal{NL}^l(\Omega)$$

is surjective for each $l \in \overline{\mathbb{N}}$, there exists $v = (v_n) + \mathcal{I}^l_0 \in \mathcal{B}^\infty_{ae}(\Omega)$ so that $G^\infty_{ae}(v) = (u_n) + \mathcal{I}^l_0$. It follows from Theorem 4.12 and Proposition 4.23 that

$$T_{\mathcal{B}}((v_n) + \mathcal{I}^l_0) = f + \mathcal{I}^k_0$$

for all $k, l \in \overline{\mathbb{N}}$ so that $k + m \leq l$. In the same way, it follows that

$$F^\infty_{ae}(v) = (v_n) + \mathcal{I}^\infty_0 \in \mathcal{A}^\infty_{nd}(\Omega)$$

is a chain generalized solution of (4.107) in $\mathbf{A}_{ae}$. □

Theorem 4.24 establishes the existence of chain generalized solution in $\mathbf{A}_{ae}$ for a large class of PDEs, as demonstrated in the following

Example 4.25. Consider the PDE

$$D_t u(x,t) = \sum_{1\leq i\leq h} c_i(x) \prod_{1\leq j\leq k_i} D_x^{p_{ij}} u(x,t), \quad (x,t) \in \Omega = \Omega' \times \mathbb{R} \tag{4.113}$$

where $\Omega' \subset \mathbb{R}^{n-1}$ is open, $h, k_i \in \mathbb{N}$, $c_i \in C^\infty(\Omega)$ and $p_{ij} \in \mathbb{N}^n$ satisfies $|p_{ij}| \leq m$ for all $i = 1, \cdots, h$ and $j = 1, \cdots, k_i$. The PDE (4.113) can be written in the form

$$T(x,t,D)u(x,t) = 0, \quad (x,t) \in \Omega$$

where 0 denotes the zero function on Ω. The operator $T(x,t,D)$ is defined through a jointly continuous, C^∞-smooth mapping

$$F : \Omega \times \mathbb{R}^{M+1} \longrightarrow \mathbb{R} \tag{4.114}$$

as

$$T(x,t,D) = F(x,t,u(x,t),\cdots, D_x^{p_{ij}} u(x,t), \cdots, D_t u(x,t)).$$

where M is the cardinality of $\{p_{ij} \mid i = 1\cdots h,\ j = 1 \cdots k_i\}$. In particular,

$$F(x,t,\xi_1 \cdots, \xi_{M+1}) = \xi_{M+1} - \sum_{1\leq i\leq h} c_i(x) \prod_{1\leq j\leq k_i} \xi_{p_{ij}}, \quad (x,t) \in \Omega = \Omega' \times \mathbb{R}.$$

Since the PDE in (4.113) is a linear ξ_{M+1}, it follows that the range of F in $\mathbb{R}$ is given by

$$R_F = \{F(x,t,\xi_1 \cdots, \xi_{M+1}) | (x,t) \in \Omega, \quad (x,t,\xi_1 \cdots, \xi_{M+1}) \in \mathbb{R}^{M+1}\} = \mathbb{R}.$$

Hence R_F is open and F is surjective. Furthermore, $R_F = intR_F = \mathbb{R}$ so that $0 \in intR_F$.

Now define the mapping

$$F^\infty : \Omega \times \mathbb{R}^{\mathbb{N}^{n+1}} \longrightarrow \mathbb{R}^{\mathbb{N}^{n+1}}$$

by setting

$$F^\infty(x,t,(\xi_{M+1})_{M\in\mathbb{N}^n}) = (F^\beta(x,t,\cdots,\xi_M,\xi_{M+1})),\ \beta \in \mathbb{N}^{n+1}$$

where, for each $\beta \in \mathbb{N}^{n+1}$, the mapping

$$F^\beta : \Omega \times \mathbb{R}^{\mathbb{N}^{n+1}} \longrightarrow \mathbb{R}^{\mathbb{N}^{N+1}}$$

is defined by setting

$$D^\beta(T(x,t,D)u(x,t)) = F^\beta(x,t,\cdots, D^{p_{ij}}u(x,t),\cdots, D_t u(x,t)), \quad |p_{ij}| \leq m + |\beta|$$

for all $u \in C^\infty(\Omega)$. Note that for each $\beta \in \mathbb{N}^{n+1}$, F^β is linear in at least one factor of $\mathbb{R}^{\mathbb{N}^{n+1}}$, so that, for $\beta' = ne\beta$, F^β is independent of this factor. Hence

$$\begin{array}{ll} \forall & (x,t) \in \Omega \\ \exists & \xi(x,t) \in \mathbb{R}^{\mathbb{N}^{n+1}}, \quad F^\infty(x,t,\xi(x,t)) = \mathbf{0} \\ \exists & V \in \mathcal{V}_{(x,t)}, W \in \mathcal{V}_{\xi(x,t)} : \\ & F^\infty : V \times W \in \mathbb{R}^{\mathbb{N}^{n+1}} \text{ is open} \end{array}$$

Thus the PDE (4.113) satisfies (2.64). Therefore by Theorems 4.24, the PDE (4.113) has a chain generalized solution in $\mathbf{A}$

4.5 Space-time Foam Algebras

Recently, see [39, 47, 52, 53, 54, 65], Rosinger introduced so-called space-time foam (STF) algebras. These are differential algebras of generalized functions which can deal with singularities that occur on a dense subset of Ω, as opposed to the closed, nowhere dense singularity set used in the nowhere dense algebra $\mathcal{A}_{nd}^{\infty}(\Omega)$. The main motivation for this work is to provide a mathematical model for space-time foam singularities in general relativity, proposed by physicists in order to deal with quantum phenomena. Let us now recall briefly the construction of such space-time foam algebras, see [39, 47, 52, 53]. Let $\mathfrak{S}$ be a collection of subsets of Ω such that

$$\Sigma \in \mathfrak{S} \Longrightarrow \Omega \setminus \Sigma \text{ is dense in } \Omega \tag{4.115}$$

and

$$\begin{array}{ll} \forall & \Sigma,\ \Sigma' \in \mathfrak{S} : \\ \exists & \Sigma'' \in \mathfrak{S} : \\ & \Sigma \cup \Sigma' \subseteq \Sigma'' \end{array} \tag{4.116}$$

Let $L = (\Lambda, \leq)$ be a right directed partially ordered set. That is,

$$\begin{array}{ll} \forall & \lambda, \lambda' \in \Lambda \\ \exists & \lambda'' \in \Lambda \\ & \lambda, \lambda' \leq \lambda'' \end{array}$$

For $\Sigma \in \mathfrak{S}$, the ideal $\mathcal{I}_{L,\Sigma} \subseteq C^{\infty}(\Omega)^{\Lambda}$ is defined as the set of Λ-sequences $(u_{\lambda})_{\lambda \in \Lambda} \in C^{\infty}(\Omega)^{\Lambda}$ that satisfy the asymptotic vanishing condition

$$\begin{array}{ll} \forall & x \in \Omega \setminus \Sigma : \\ \exists & \lambda \in \Lambda : \\ \forall & \mu \in \Lambda, \mu \geq \lambda : \\ \forall & p \in \mathbb{N}^p : \\ & D^p u_{\mu}(x) = 0. \end{array} \tag{4.117}$$

Clearly, for all $\Sigma, \Sigma' \in \mathfrak{S}$, we have

$$\Sigma \subseteq \Sigma' \Longrightarrow \mathcal{I}_{L,\Sigma} \subseteq \mathcal{I}_{L,\Sigma}$$

so that

$$\mathcal{I}_{L,\mathfrak{S}} = \bigcup_{\Sigma \in \mathfrak{S}} \mathcal{I}_{L,\Sigma} \tag{4.118}$$

is an ideal in $C^{\infty}(\Omega)^{\Lambda}$. Based on (4.118), we associate with the collection of singularity sets the multi-foam algebra

$$\mathcal{B}_{L,\mathfrak{S}}(\Omega) = C^{\infty}(\Omega)^{\Lambda} / \mathcal{I}_{L,\mathfrak{S}}. \tag{4.119}$$

Note that, due to the denseness of $\Omega \setminus \Sigma$ in Ω for each $\Sigma \in \mathfrak{S}$, it follows that the ideal $\mathcal{I}_{L,\mathfrak{S}}$ satisfies the neutrix condition (1.11). Hence

$$C^{\infty}(\Omega) \ni u \mapsto \Lambda_{\infty}(u) + \mathcal{I}_{L,\mathfrak{S}} \in \mathcal{B}_{L,\mathfrak{S}}(\Omega) \tag{4.120}$$

defines an injective algebra homomorphism where Δ_∞ is defined by. Furthermore, since

$$D^p(\mathcal{I}_{L,\mathfrak{S}}) \subseteq \mathcal{I}_{L,\mathfrak{S}}, \quad p \in \mathbb{N}^n,$$

it follows that for each $p \in \mathbb{N}^n$

$$D^p : \mathcal{B}_{L,\mathfrak{S}}(\Omega) \ni (u_\lambda) + \mathcal{I}_{L,\mathfrak{S}}(\Omega) \mapsto D^p(u_\lambda) + \mathcal{I}_{L,\mathfrak{S}} \in \mathcal{B}_{L,\mathfrak{S}}(\Omega) \tag{4.121}$$

defines an extension of the differential operator

$$D^p : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$$

which is linear and satisfy the Leibnitz rule.

If we set

$$\mathfrak{S} = \mathfrak{S}_{nd} = \{\Gamma \subseteq \Omega \mid \Gamma \text{ is closed nowhere dense}\}$$

then the construction (4.115) to (4.121) above reduces to that of the nowhere dense algebra $\mathcal{A}^\infty_{nd}(\Omega)$ discussed in Section 4.1.1.

In order to deal with singularities that occur on a dense subset of Ω, using the construction (4.115) to (4.121), the collection $\mathfrak{S}$ of singularity sets must satisfy

$$\begin{array}{ll} \exists & \Sigma \in \mathfrak{S} : \\ & \Sigma \text{ is dense in} \Omega. \end{array} \tag{4.122}$$

Condition (4.122) is clearly not satisfied by the collection $\mathfrak{S}_{nd}$, which gives rise to the nowhere dense algebra $\mathcal{A}^\infty_{nd}(\Omega)$. Given the utility of the algebra $\mathcal{A}^\infty_{nd}(\Omega)$, in particular when it comes to the solution of nonlinear PDEs, solutions which may exhibit singularities in closed nowhere dense subsets of Ω, the following is a natural choice.

Let

$$\mathfrak{S}_{Baire-I} = \{\Sigma \subset \Omega \ \mid \Sigma \text{ is of first Baire category in } \Omega\}.$$

Clearly, $\mathfrak{S}_{Baire-I}$ satisfies (4.115) and (4.116) so that

$$\mathcal{B}_{Baire-I}(\Omega) = C^\infty(\Omega)^\Lambda / \mathcal{I}_{L,\mathfrak{S}_{Baire-I}} \tag{4.123}$$

is a differential algebra of generalized functions admitting a canonical embedding of $C^\infty(\Omega)$. Since $\mathfrak{S}_{BaireI}$ satisfies (4.122), it would seem that the algebra $\mathcal{B}_{Baire-I}(\Omega)$ can deal with functions admitting singularities on dense subset of Ω, in particular, on arbitrary sets of first Baire category. However, in [65] it is shown that if $L = (\Lambda, \leq)$ is countably co-final, that is

$$\begin{array}{ll} \exists & \Lambda_0 \subset \Lambda \text{ countable} : \\ \forall & \lambda \in \Lambda : \\ \exists & \lambda_0 \in \Lambda_0 : \\ & \lambda \leq \lambda_0, \end{array}$$

then $\mathcal{I}_{L'\mathfrak{S}_{Baire-I}} = \mathcal{I}^\infty_{nd}$, so that

$$\mathcal{B}_{Baire-I}(\Omega) = \mathcal{A}^\infty_{nd}(\Omega).$$

In this section, we introduced an alternative construction of algebras admitting dense singularities of a particular form based on the theory of chains of algebra of generalized functions.

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4.5.1 Constructing differential algebras from differential chains of algebras

Let

$$\mathbf{A} = \{(\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l) \mid l, k \in \overline{\mathbb{N}}, k \leq l\}$$

be a differential chain of algebras of generalized functions . Among the algebras $\mathcal{A}^l(\Omega)$ in $\mathbf{A}$, only $\mathcal{A}^\infty(\Omega)$ is a differential algebra. Our aim in this section is to construct, using the algebras $\mathcal{A}^l(\Omega)$, $l \in \mathbb{N}$, a differential algebra that is typically larger than the algebra $\mathcal{A}^\infty(\Omega)$. In this regard, let

$$\mathcal{A}_0^\infty(\Omega) = \bigcap_{l \in \mathbb{N}} \gamma_0^l(\mathcal{A}^l(\Omega)) \subset \mathcal{A}^0(\Omega). \tag{4.124}$$

Since each

$$\gamma_0^l : \mathcal{A}^l(\Omega) \longrightarrow \mathcal{A}^0(\Omega)$$

is an algebra homomorphism, it follows that $\mathcal{A}_0^\infty(\Omega)$ is a subalgebra of $\mathcal{A}^0(\Omega)$. Note that $\mathcal{A}_0^\infty(\Omega)$ is not the trivial algebra $\{0\}$, except when $\gamma_0^\infty(\mathcal{A}^\infty(\Omega)) = \{0\}$. Indeed, due to the commutative diagram (1.38), it follows that

$$\gamma_0^\infty(\mathcal{A}^\infty(\Omega)) \subset \mathcal{A}_0^\infty(\Omega).$$

Theorem 4.26. *Let* $\mathbf{A}$ *be a differential chain of algebra of generalized functions. Let* $\mathcal{A}_0^\infty(\infty)(\Omega)$ *be defined as in* (4.124), *and assume that* γ_0^l *is injective for all* $l \in \mathbb{N}$. *Then for each* $p \in \mathbb{N}^n$ *there exists a map*

$$D^p : \mathcal{A}_0^\infty(\Omega) \longrightarrow \mathcal{A}_0^\infty(\Omega)$$

so that the following hold

(i) For each $u \in \mathcal{A}_0^\infty(\Omega)$, $D^p(u) = \gamma_0^l(u)$ *for all* $l \in \mathbb{N}$ *so that* $|p| \leq l$.

(ii) D^p *is linear and satisfies the Leibnitz rule.*

Proof. Since each γ_0^l is injective and the diagram (1.38) commutes, it follows that

$$\begin{array}{ll} \forall & u \in \mathcal{A}_0^\infty(\Omega),\ l \in \mathbb{N} : \\ \exists! & u_l \in \mathcal{A}^l(\Omega) : \\ & (1)\ \gamma_0^l(u_l) = u \\ & (2)\ \gamma_k^l(u_l) = u_k, \quad k \leq l. \end{array} \tag{4.125}$$

In view of the commutativity of the diagrams (1.38) and (1.39), the diagram

$$\begin{array}{ccccc} \mathcal{A}^l(\Omega) & \xrightarrow{\ D^p\ } & \mathcal{A}^k(\Omega) & \overset{\gamma_0^k}{\searrow} & \\ \gamma_{l'}^l \downarrow & & \downarrow \gamma_{k'}^k & & \mathcal{A}^0(\Omega) \\ \mathcal{A}^{l'}(\Omega) & \xrightarrow{\ D^p\ } & \mathcal{A}^{k'}(\Omega) & \underset{\gamma_0^{k'}}{\nearrow} & \end{array} \tag{4.126}$$

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commutes for all $p \in \mathbb{N}^n$, $l \geq l' \geq |p|$ and $k, k' \in \mathbb{N}$ so that $k + |p| \leq l$ and $k' + |p| \leq l'$. Hence for each $u \in \mathcal{A}_0^\infty(\Omega), p \in \mathbb{N}^n, l \geq l' \geq |p|$ and $k, k' \in \mathbb{N}$ so that $k + |p| \leq l$ and $k' + |p| \leq l'$, we have

$$\gamma_0^k(D^p u_l) = \gamma_0^{k'}(D^p(u_{l'})) \text{ in } \mathcal{A}^0(\Omega) \tag{4.127}$$

Thus for each $u \in \mathcal{A}_0^\infty(\Omega)$ and $p \in \mathbb{N}^n$ there exists a unique $w \in \mathcal{A}_0^\infty(\Omega)$ so that

$$\gamma_0^k(D^p u_l) = w, \quad l \in \mathbb{N}, \quad k + |p| \leq l. \tag{4.128}$$

This proves the existence of a map

$$D^p : \mathcal{A}_0^\infty(\Omega) \longrightarrow \mathcal{A}_0^\infty(\Omega),$$

for each $p \in \mathbb{N}$, so that Theorem 4.26(i) holds.

For $u, v \in \mathcal{A}_0^\infty(\Omega)$ and $\alpha, \beta \in \mathbb{R}$,

$$\gamma_0^l(\alpha u_l + \beta v_l) = \alpha u + \beta v$$

so that

$$(\alpha u + \beta v)_l = \alpha u_l + \beta v_l$$

for all $l \in \mathbb{N}$. Thus

$$\begin{aligned} D^p(\alpha u + \beta v) &= \gamma_0^k(D^p(\alpha u_l + \beta v_l)) \\ &= \alpha \gamma_0^k(D^p u_l) + \beta \gamma_0^k(D^p v_l) \end{aligned}$$

for all $l, k \in \mathbb{N}$ so that $k + |p| \leq l$. Hence

$$D^p(\alpha u + \beta v) = \alpha D^p(u) + \beta D^p(v).$$

In the same way, $(uv)_l = u_l v_l$ for all $l \in \mathbb{N}$. Hence

$$\begin{aligned} D^p(uv) &= \gamma_0^k(D^p(u_l v_l)) \\ &= \gamma_0^k\left(\sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q} u_l D^q v_l\right) \\ &= \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} \gamma_0^k(D^{p-q} u_l) \gamma_0^k(D^q v_l) \end{aligned}$$

whenever $k + |p| \leq l$. In fact, due to the commutativity of the diagrams (1.38) and (1.39) we have

$$D^p(uv) = \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} \gamma_0^{k_{p-q}}(D^{p-q} u_l) \gamma_0^{k_q}(D^q v_l)$$

whenever $k_{p-q} + |p - q| \leq l$ and $k_q + |q| \leq l$. It therefore follows that

$$D^p(uv) = \sum_{q \leq p} \begin{pmatrix} p \\ q \end{pmatrix} D^{p-q} u D^q v$$

for all $u, v \in \mathcal{A}_0^\infty(\Omega)$. □

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Assuming that γ_0^l is injective for each $l \in \mathbb{N}$, we have obtained a differential algebra $\mathcal{A}_0^\infty(\Omega)$ from the algebras $\mathcal{A}^l(\Omega)$, $l \in \mathbb{N}$, in the chain $\mathbf{A}$. Due to the commutativity of the diagram (1.38), we have

$$\gamma_0^\infty(\mathcal{A}^\infty(\Omega)) \subseteq \mathcal{A}_0^\infty(\Omega) \subset \mathcal{A}^0(\Omega).$$

Thus, if $\gamma_0^\infty : \mathcal{A}^\infty(\Omega) \longrightarrow \mathcal{A}^0(\Omega)$ is also injective, it follows that $\mathcal{A}_0^\infty(\Omega)$ contains $\mathcal{A}^\infty(\Omega)$ as a subalgebra. Furthermore, due to (1.39), the diagram

$$\begin{array}{ccc} \mathcal{A}_0^\infty(\Omega) & \xrightarrow{D^p} & \mathcal{A}_0^\infty(\Omega) \\ \gamma_0^\infty \uparrow & & \uparrow \gamma_0^\infty \\ \mathcal{A}^\infty(\Omega) & \xrightarrow{D^p} & \mathcal{A}^\infty(\Omega) \end{array} \qquad (4.129)$$

commutes for all $p \in \mathbb{N}^n$, so that we may view $\mathcal{A}_0^\infty(\Omega)$ as an extension of $\mathcal{A}^\infty(\Omega)$. It can be shown that

$$\gamma_0^\infty : \mathcal{A}^\infty(\Omega) \longrightarrow \mathcal{A}_0^\infty(\Omega)$$

is typically not an isomorphism, so that $\mathcal{A}_0^\infty(\Omega)$ is a proper extension of $\mathcal{A}^\infty(\Omega)$.

We now investigate the extent the which properties of the chain $\mathbf{A}$ carry over to the algebra $\mathcal{A}_0^\infty(\Omega)$. In particular, we consider the embedding of smooth functions and distributions into the $\mathcal{A}_0^\infty(\Omega)$ as well as the existence of generalized solutions to nonlinear PDEs.

Let us consider first the embedding of smooth functions into $\mathcal{A}_0^\infty(\Omega)$.

Theorem 4.27. *Assume that γ^l is injective for each $l \in \mathbb{N}$. Further assume that there exists for each $l \in \overline{\mathbb{N}}$ an injective algebra homomorphism*

$$C^\infty(\Omega) \hookrightarrow \mathcal{A}^l(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{A}^l(\Omega) & \xrightarrow{\gamma_k^l} & \mathcal{A}^k(\Omega) \\ & \nwarrow \hookrightarrow \quad \hookrightarrow \nearrow & \\ & C^\infty(\Omega) & \end{array} \qquad (4.130)$$

commutes for all $l, k \in \overline{\mathbb{N}}$ with $k \leq l$. Then there exists an injective algebra homomorphism

$$C^\infty(\Omega) \hookrightarrow \mathcal{A}_0^\infty(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{A}^\infty(\Omega) & \xrightarrow{\gamma_0^\infty} & \mathcal{A}^\infty(\Omega) \\ & \nwarrow \hookrightarrow \quad \hookrightarrow \nearrow & \\ & C^\infty(\Omega) & \end{array} \qquad (4.131)$$

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commutes.

If in addition, the diagram

$$\begin{array}{ccc} \mathcal{A}^l(\Omega) & \xrightarrow{D^p} & \mathcal{A}^k(\Omega) \\ \hookrightarrow \uparrow & & \uparrow \hookrightarrow \\ C^\infty(\Omega) & \xrightarrow{D^p} & C^\infty(\Omega) \end{array} \tag{4.132}$$

commutes for all $p \in \mathbb{N}^n$ *and* $l, k \in \mathbb{N}$ *so that* $k + |p| \leq l$ *then the diagram*

$$\begin{array}{ccc} \mathcal{A}_0^l(\Omega) & \xrightarrow{D^p} & \mathcal{A}_0^k(\Omega) \\ \hookrightarrow \uparrow & & \uparrow \hookrightarrow \\ C^\infty(\Omega) & \xrightarrow{D^p} & C^\infty(\Omega) \end{array} \tag{4.133}$$

commutes.

Proof. If $u \in C^\infty(\Omega)$, then $\gamma_0^l(u) = \gamma_0^k(u)$ for all $l, k \in \mathbb{N}$. Hence there exists a unique $w_u \in \mathcal{A}^0(\Omega)$ so that

$$\gamma_0^l(u) = w_u \quad l \in \mathbb{N}.$$

Consider the map

$$E_0^\infty : C^\infty(\Omega) \ni u \mapsto w_u \in \mathcal{A}_0^\infty(\Omega).$$

Then $E_0^\infty(u) = \gamma_0^l(u)$ for all $u \in C^\infty(\Omega)$ and $l \in \mathbb{N}$, and if $\gamma_0^l(u) = w_u$ for all $l \in \mathbb{N}$, then $w_u = E_0^\infty(u)$. The injectivity of E_0^∞ follows from that of γ_0^l. For $u, v \in C^\infty(\Omega)$ and $\alpha,\ \beta \in \mathbb{R}$ we have , for all $l \in \mathbb{N}$,

$$E_0^\infty(uv) = \gamma_0^l(uv) = \gamma_0^l(u)\gamma_0^l(v) = E_0^\infty(u)E_0^\infty(v)$$

and

$$E_0^\infty(\alpha u + \beta v) = \gamma_0^l(\alpha u + \beta v) = \alpha\gamma_0^l(u) + \beta\gamma_0^l(v) = \alpha E_0^\infty(u) + \beta E_0^\infty(v)$$

so that E_0^∞ is an algebra homomorphism.

The commutativity of the diagram (4.131) follows immediately from (4.130).

Now assume that (4.132) commutes for all $p \in \mathbb{N}^n$ and $l, k \in \mathbb{N}$ such that $k + |p| \leq l$. Then for $u \in C^\infty(\Omega)$ we have, for $l \geq |p|$,

$$\begin{aligned} D^p(E_0^\infty(u)) &= D^p(\gamma_0^l(u)) \\ &= \gamma_0^l(D^p(u)) \\ &= E_0^\infty(D^p(u)). \end{aligned}$$

Hence the diagram (4.133) commutes. □

In the same way as above we obtain the following

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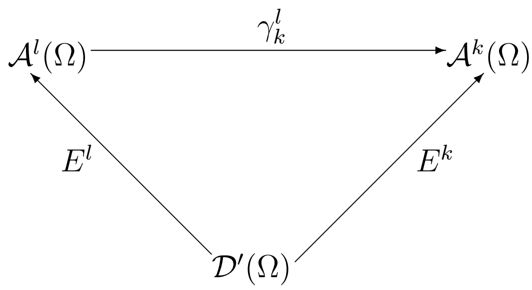
Theorem 4.28. *Assume that γ_0^l is injective for all $l \in \mathbb{N}$, and there exists a linear injection*

$$E^l : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^l(\Omega)$$

so that the diagram

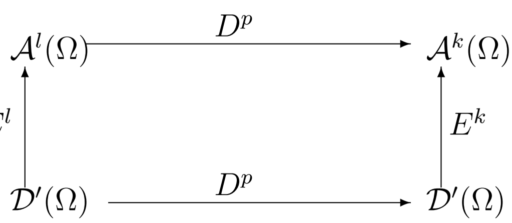


(4.134)

commutes for all $l, k \in \mathbb{N}$ so that $k \leq l$. Then there exists a linear injection

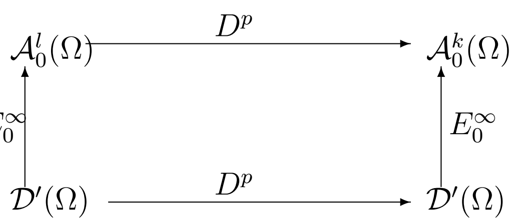
$$E_0^\infty : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}_0^\infty(\Omega).$$

If, in addition, the diagram



(4.135)

commutes whenever $|p| + k \leq l$, then the diagram



(4.136)

commutes for all $p \in \mathbb{N}^n$.

Proof. For all $l, k \in \mathbb{N}$ and $T \in \mathcal{D}'(\Omega)$, we have

$$\gamma_0^l(E^l(T)) = \gamma_0^k(E^k(T)).$$

Hence for each $T \in \mathcal{D}'(\Omega)$ there exists a unique $w_T \in \mathcal{A}_0^\infty(\Omega) \subset \mathcal{A}^0(\Omega)$ so that

$$\gamma_0^l(E^l(T)) = w_T \text{ in } \mathcal{A}^0(\Omega),\ l \in \mathbb{N}.$$

Consider the map

$$E_0^\infty : \mathcal{D}'(\Omega) \ni T \mapsto w_T \in \mathcal{A}_0^\infty(\Omega).$$

Since each E^l is a linear injection, it follows by arguments essentially simillar to those employed in the proof of Theorem 4.27 that E_0^∞ is a linear injection. The commutativity of (4.136), subject to that of (4.135), follows likewise by arguments similar to those in the proof of Theorem 4.27. □

Let us now consider a polynomial nonlinear PDE

$$Tu = f \tag{4.137}$$

of order at most m with $f \in C^\infty(\Omega)$, and T defined as in (4.103). We observe that, due to the polynomial nature of T, it admits an extension to any differential algebra containing $\mathcal{C}^\infty(\Omega)$. Thus in view of Theorem (4.27), T may be extended to $\mathcal{A}_0^\infty(\Omega)$.

Theorem 4.29. *Assume that γ_0^l is injective for each $l \in \mathbb{N}$ and that the chain* $\mathbf{A}$ *admits an embedding of $\mathcal{C}^\infty(\Omega)$. If there exists a chain generalized solution $u \in \mathcal{A}^\infty(\Omega)$ of* (4.137) *in* $\mathbf{A}$, *then there exists a solution of* (4.137) *in the algebra $\mathcal{A}_0^\infty(\Omega)$.*

Proof. If $u \in \mathcal{A}^\infty(\Omega)$ is a chain generalized solution of (4.137), then

$$T(\gamma_l^\infty(u)) = f \;\; in \;\; \mathcal{A}^k(\Omega) \tag{4.138}$$

whenever $m + k \leq l$. In view of the fact that the diagram

$$\begin{array}{ccccc} \mathcal{A}^\infty(\Omega) & \xrightarrow{\gamma_0^\infty} & \mathcal{A}_0^\infty(\Omega) & \overset{\subset}{\searrow} & \\ \downarrow D^p & & \downarrow D^p & & \mathcal{A}^0(\Omega) \\ \mathcal{A}^\infty(\Omega) & \xrightarrow{\gamma_0^\infty} & \mathcal{A}_0^\infty(\Omega) & \underset{\subset}{\nearrow} & \end{array} \tag{4.139}$$

commutes for all $p \in \mathbb{N}$, it follows that the diagram

$$\begin{array}{ccccc} \mathcal{A}^\infty(\Omega) & \xrightarrow{\gamma_0^\infty} & \mathcal{A}_0^\infty(\Omega) & \overset{\subset}{\searrow} & \\ \downarrow T & & \downarrow T & & \mathcal{A}^0(\Omega) \\ \mathcal{A}^\infty(\Omega) & \xrightarrow{\gamma_0^\infty} & \mathcal{A}_0^\infty(\Omega) & \underset{\subset}{\nearrow} & \end{array} \tag{4.140}$$

commutes. It therefore follows from (4.138) that $\gamma_0^\infty(u) \in \mathcal{A}_0^\infty(\Omega) \subset \mathcal{A}^0(\Omega)$ is a solution of (4.137) in $\mathcal{A}_0^\infty(\Omega)$. □

4.5.2 Differential algebras with dense singularities

In this section we give two examples of differential algebras of generalized functions obtained through the construction in Section 4.5.1, which are able to deal with dense singularities of a certain type. As a first illustrative example, we consider the following.

Example 4.30. Recall from Section 2.1 that, for $l \in \overline{\mathbb{N}}$,

$$C_{nd}^l(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \exists \quad \Gamma \subset \Omega, \text{ closed and nowhere dense :} \\ \quad u \in C^l(\Omega \setminus \Gamma) \end{array} \right\}.$$

We claim that, in general,

$$\bigcap C_{nd}^l(\Omega) \neq C_{nd}^\infty(\Omega).$$

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In this regard, let $\Omega = (0,1)$ and let $\{q_n \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q} \cap (0,1)$. For each $l \in \mathbb{N}$ and $x \in \Omega$, let

$$u_l(x) = \begin{cases} 0 & if\ x \leq q_l \\ \frac{x^{l+1}}{(l+1)!} & if\ x > q_l. \end{cases} \tag{4.141}$$

Then $u_l \in (C^l(\Omega) \setminus C^{l+1}(\Omega)) \cap C^\infty_{nd}(\Omega)$ for all $l \in \mathbb{N}$. Let

$$u(x) = \sum_{l \in \mathbb{N}} \frac{1}{2^l} u_l(x). \tag{4.142}$$

Since the series is uniformly convergent on $(0,1)$, it follows that $u \in C^0(\Omega)$. More generally, the series

$$u_p(x) = \sum_{l \in \mathbb{N}} \frac{1}{2^l} u_l^{(p}(x) \tag{4.143}$$

is uniformly convergent on $\Omega \setminus \{q_1, \cdots, q_{p-1}\}$ for all $p \in \mathbb{N}$, so that $u \in \mathcal{C}^l_{nd}(\Omega)$ for every $l \in \mathbb{N}$. Since $\{q_l \mid l \in \mathbb{N}\}$ is dense in Ω, it follows that u is not $\mathcal{C}^\infty$-smooth on any non-empty open subset of Ω. That is, u is singular on a dense subset of Ω. In particular, $u \notin \mathcal{C}^\infty_{nd}(\Omega)$.

Example 4.30 may be extrapolated in a straight forward way to an arbitrary open set $\Omega \subset \mathbb{R}^n$. It follows that the set

$$(\bigcap_{l \in \mathbb{N}} \mathcal{ML}^l(\Omega)) \setminus \mathcal{ML}^\infty(\Omega) \tag{4.144}$$

is nonempty. In particular, there exists $u \in \mathcal{ML}^\infty_0(\Omega) = \bigcap_{l \in \mathbb{N}} \mathcal{ML}^l(\Omega)$ such that $u \notin C^\infty(U)$ for every open subsets U of Ω. As shown in Theorem 4.12 and 4.13, the algebras $\mathcal{A}^\infty_{nd}(\Omega)$ and $\mathcal{A}^\infty_{ae}(\Omega)$ can handle singularities of functions in $\mathcal{ML}^\infty(\Omega)$. In particular, there exist injective algebra homomorphisms

$$H^\infty_{nd} : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{A}^\infty_{nd}(\Omega) \tag{4.145}$$

and

$$H^\infty_{ae} : \mathcal{ML}^\infty(\Omega) \longrightarrow \mathcal{A}^\infty_{ae}(\Omega) \tag{4.146}$$

so that the diagrams

$$\begin{array}{ccc} \mathcal{A}^\infty_{nd}(\Omega) & \xrightarrow{D^p} & \mathcal{A}^\infty_{nd}(\Omega) \\ H^\infty_{nd} \uparrow & & \uparrow H^\infty_{nd} \\ \mathcal{ML}^\infty(\Omega) & \xrightarrow{D^p} & \mathcal{ML}^\infty(\Omega) \end{array} \tag{4.147}$$

and

$$\begin{array}{ccc} \mathcal{A}^\infty_{ae}(\Omega) & \xrightarrow{D^p} & \mathcal{A}^\infty_{ae}(\Omega) \\ H^\infty_{ae} \uparrow & & \uparrow H^\infty_{ae} \\ \mathcal{ML}^\infty(\Omega) & \xrightarrow{D^p} & \mathcal{ML}^\infty(\Omega) \end{array} \tag{4.148}$$

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commute for all $p \in \mathbb{N}^n$. The method by which the embeddings (4.145) and (4.146) are obtained does not yield an embedding

$$\mathcal{ML}_0^\infty(\Omega) = \bigcap_{l \in \mathbb{N}} \mathcal{ML}^l(\Omega) \hookrightarrow \mathcal{A}^\infty_{nd}(\Omega)$$

or

$$\mathcal{ML}_0^\infty(\Omega) = \bigcap_{l \in \mathbb{N}} \mathcal{ML}^l(\Omega) \hookrightarrow \mathcal{A}^\infty_{ae}(\Omega).$$

In this regard, let us recall briefly the main points involved in the construction of the embedding (4.145) and (4.146), respectively. We deal first with the embedding (4.145). For $u \in \mathcal{ML}^\infty(\Omega)$ there exists $\Gamma \subset \Omega$ closed and nowhere dense so that $u \in C^\infty(\Omega \setminus \Gamma)$. An application of the Principle of Partition of Unity , see Theorem 3.5, yields a sequence (u_n) in $C^\infty(\Omega, [0,1])$ so that

$$\begin{array}{ll} \forall & x \in \Gamma : \\ \forall & n \in \mathbb{N} : \\ \exists & V \in \mathcal{V}_x : \\ & \phi_n(y) = 0, \ y \in V, \end{array}$$

$$\begin{array}{ll} \forall & x \in \Omega \setminus \Gamma : \\ \exists & N \in \mathbb{N} : \\ \exists & V \in \mathcal{V}_x : \\ & \phi_n(y) = 1, \ y \in V, n \geq N. \end{array}$$

The embedding (4.145) is obtained by setting

$$H^\infty_{nd}(u) = (u\phi_n) + \mathcal{I}^\infty_{nd}.$$

Clearly this strategy will not deliver an embedding

$$\mathcal{ML}_0^\infty(\Omega) \hookrightarrow \mathcal{A}^\infty_{nd}(\Omega)$$

since , as mentioned, there exists $u \in \mathcal{ML}_0^\infty(\Omega)$ so that

$$u \notin C^\infty(U), \ U \subseteq \Omega \text{ open.}$$

Let us now consider the construction of algebras of generalized functions that admit embeddings of $\mathcal{ML}_0^\infty(\Omega)$. In this regard, we have the following

Theorem 4.31. *The set*

$$\mathcal{A}^\infty_{nd,0}(\Omega) = \bigcap_{l \in \mathbb{N}} \gamma_0^l(\mathcal{A}^l_{nd}(\Omega))$$

is a differential algebra of generalized functions. Furthermore, the following is true

(i) There exists an injective algebra homomorphism

$$C^\infty(\Omega) \hookrightarrow \mathcal{A}^\infty_{nd,0}(\Omega) \tag{4.149}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{A}^{\infty}_{nd,0}(\Omega) & \xrightarrow{\;D^p\;} & \mathcal{A}^{\infty}_{nd,0}(\Omega) \\ \uparrow \hookrightarrow & & \uparrow \hookrightarrow \\ C^{\infty}(\Omega) & \xrightarrow{\;D^p\;} & C^{\infty}(\Omega) \end{array} \tag{4.150}$$

commutes for all $p \in \mathbb{N}^n$.

(ii) There exists a linear injection

$$\Gamma^{\infty}_{nd,0} : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^{\infty}_{nd,0}(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{D}'(\Omega) & \xrightarrow{\;\Gamma^{\infty}_{nd,0}\;} & \mathcal{A}^{\infty}_{nd,0}(\Omega) \\ \nwarrow \subset & & \hookrightarrow \nearrow \\ & C^{\infty}(\Omega) & \end{array} \tag{4.151}$$

commutes.

Proof. Let

$$\mathcal{A}^{\infty}_{nd,0}(\Omega) = \bigcap_{l \in \mathbb{N}} \gamma^l_0(\mathcal{A}^l_{nd}(\Omega))$$

with $\gamma^l_0 : \mathcal{A}^l_{nd} \longrightarrow \mathcal{A}^0_{nd}$ the algebra homomorphism defined by $\gamma^l_0((u_n) + \mathcal{I}^l_{nd}) = (u_n) + \mathcal{I}^0_{nd}$. Since each γ^l_0 is an algebra homomorphism, $\mathcal{A}^0_{nd}$ is a subalgebra of $\mathcal{A}^0_{nd}$. Each γ^l_0 is injective. To see that this is so, observe that

$$\mathcal{I}^l_{nd} = \mathcal{C}^l(\Omega)^{\mathbb{N}} \bigcap \mathcal{I}^0_{nd}$$

for each $l \in \overline{\mathbb{N}}$. Indeed, $\mathcal{I}^l_{nd} \subset \mathcal{I}^0_{nd}$ and $\mathcal{I}^l_{nd} \subset \mathcal{C}^l(\Omega)^{\mathbb{N}}$ so that $\mathcal{I}^l_{nd} \subseteq \mathcal{C}^l(\Omega)^{\mathbb{N}}$, then there exists $\Gamma \in \Omega$ closed nowhere dense so that

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma : \\ \exists & V \in \mathcal{V}_x, \;\; N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \;\; n \geq N : \\ & u_n(y) = 0, \;\; y \in V. \end{array} \tag{4.152}$$

Since $V \in \mathcal{V}_x$ is open for each x, it follows that

$$\begin{array}{ll} \forall & x \in \Omega \backslash \Gamma : \\ \exists & V \in \mathcal{V}_x, \;\; N \in \mathbb{N} : \\ \forall & n \in \mathbb{N}, \;\; n \geq N : \\ \forall & p \in \mathbb{N}^n, \;\; |p| \leq l : \quad D^p u_n(y) = 0, \;\; y \in V. \end{array} \tag{4.153}$$

Hence $(u_n) \in \mathcal{I}^l_{nd}$ so that $\mathcal{I}^0_{nd} \bigcap \mathcal{C}^l(\Omega)^{\mathbb{N}} \subseteq \mathcal{I}^l_{nd}$. If for $(v_n) \in \mathcal{I}^l_{nd}$,

$$\gamma^l_0((u_n) + \mathcal{I}^l_{nd}) = \gamma^l_0((v_n) + \mathcal{I}^l_{nd}),$$

then according to the definition of γ^l_0, $(u_n) - (v_n) \in \mathcal{I}^0_{nd}$. But $(u_n) - (v_n) \in \mathcal{C}^l(\Omega)^{\mathbb{N}}$ so that $(u_n) - (v_n) \in \mathcal{I}^l_{nd}$. Hence $(u_n) + \mathcal{I}^l_{nd} = (v_n) + \mathcal{I}^l_{nd}$ so that γ^l_0 is injective.

It follows from Theorem 4.26 that there exists for each $p \in \mathbb{N}^n$ a linear map

$$D^p : \mathcal{A}^\infty_{nd,0}(\Omega) \longrightarrow \mathcal{A}^\infty_{nd,0}(\Omega)$$

that satisfies the Leibnitz rule. Furthermore, $D^p(u) = \gamma^k_0(D^p\gamma^-_0(u))$ for all $u \in \mathcal{A}^\infty_{nd,0}(\Omega)$, $p \in \mathbb{N}^n$ and $l, k \in \mathbb{N}$ so tha $k|p| \leq l$.

Item (i) follows immediately from (4.3), (4.4) and Theorem 4.27. The assertion in (ii) follows directly from Theorem 4.2 and 4.28.

□

The following is an immediate consequence of Theorem 4.22 and Theorem 4.31.

Corollary 4.32. *If a polynomial nonlinear PDE* (4.104) *satisfies* (2.63) *to* (2.64), *then there exists a generalized solution* $u \in \mathcal{A}^\infty_{nd,0}(\Omega)$ *of* (4.104).

We now show that $\mathcal{A}^\infty_{nd,0}(\Omega)$ admits an embedding of $\mathcal{ML}^\infty_0(\Omega)$.

Theorem 4.33. *There exists an injective algebra homomorphism*

$$H^\infty_{nd,0} : \mathcal{ML}^\infty_0(\Omega) \longrightarrow \mathcal{A}^\infty_{nd,0}(\Omega)$$

so that the diagram

$$\begin{array}{ccc}
\mathcal{A}^\infty_{nd,0}(\Omega) & \xrightarrow{\quad D^p \quad} & \mathcal{A}^\infty_{nd,0}(\Omega) \\
\uparrow H^\infty_{nd,0} & & \uparrow H^\infty_{nd,0} \\
\mathcal{ML}^\infty_0(\Omega) & \xrightarrow{\quad D^p \quad} & \mathcal{ML}^\infty_0(\Omega)
\end{array} \tag{4.154}$$

commutes for all $p \in \mathbb{N}^n$.

Proof. It follows from Theorem 4.12 that there exists, for each $l \in \mathbb{N}$, an injective algebra homomorphism

$$H^l_{nd} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{nd}(\Omega)$$

so that the diagram

$$\begin{array}{ccccccc}
 & \overset{\subset}{\nearrow} & \mathcal{ML}^l(\Omega) & \xrightarrow{\quad H^l_{nd} \quad} & \mathcal{A}^l_{nd}(\Omega) & \overset{\gamma^l_0}{\searrow} & \\
\mathcal{ML}^\infty_0(\Omega) & & \subset \downarrow & & \downarrow \gamma^l_k & & \mathcal{A}^0_{nd}(\Omega) \\
 & \underset{\subset}{\searrow} & \mathcal{ML}^k(\Omega) & \xrightarrow{\quad H^k_{nd} \quad} & \mathcal{A}^k_{nd}(\Omega) & \underset{\gamma^k_0}{\nearrow} &
\end{array} \tag{4.155}$$

commutes whenever $k \leq l$. Hence for each $u \in \mathcal{ML}^\infty_0(\Omega)$ there exists a unique $w_u \in \mathcal{A}^0_{nd}(\Omega)$ so that

$$(\gamma^k_0 \circ H^k_{nd})(u) = w_u = (\gamma^l_0 \circ H^l_{nd})(u), \quad k \leq l.$$

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Hence the map

$$H^{\infty}_{nd,0} : \mathcal{ML}^{\infty}_{0}(\Omega) \ni u \mapsto w_u \in \mathcal{A}^{\infty}_{nd,0}(\Omega) \subset \mathcal{A}^{0}_{nd}(\Omega)$$

is well defined. The injectivity of $H^{\infty}_{nd,0}$ follows from the fact that the algebra homomorphisms H^{l}_{nd} and γ^{l}_{0} are injective.

Since the diagram

$$\begin{array}{ccccc} \mathcal{ML}^{\infty}_{0}(\Omega) & \xrightarrow{H^{\infty}_{nd,0}} & \mathcal{A}^{\infty}_{nd,0}(\Omega) & \xrightarrow{\subset} & \mathcal{A}^{0}_{nd}(\Omega) \\ \subset \downarrow & & & & \uparrow \gamma^{l}_{0} \\ \mathcal{ML}^{l}(\Omega) & & \xrightarrow{H^{l}_{nd}} & & \mathcal{A}^{l}_{nd}(\Omega) \end{array} \tag{4.156}$$

commutes for all $l \in \mathbb{N}$, it follows that $H^{\infty}_{nd,0}$ is an algebra homomorphism. It follows from Theorems 4.1 and 4.12 and the definition of $H^{\infty}_{nd,0}$ that

$$\begin{aligned} D^{p}(H^{\infty}_{nd,0}(u)) &= D^{p}(\gamma^{l}_{0}(H^{l}_{nd}(u))) \\ &= \gamma^{l-|p|}_{0}(D^{p}(H^{l}_{nd}(u))) \\ &= \gamma^{l-|p|}_{0}(H^{l-|p|}_{nd}(\mathcal{D}^{p}(u))) \\ &= H^{\infty}_{nd,0}(\mathcal{D}^{p}(u)) \end{aligned}$$

for all $u \in \mathcal{ML}^{\infty}_{0}(\Omega)$, $p \in \mathbb{N}^n$ and $l \geq |p|$. Therefore the diagram (4.154) commutes. □

We now consider the construction of an algebra of generalized functions from the almost everywhere algebra of generalized functions that admit embedding of $\mathcal{ML}^{\infty}_{0}(\Omega)$. In this regard, we have the following

Theorem 4.34. *Assume Ω is convex. Then*

$$\mathcal{A}^{\infty}_{ae,0}(\Omega) = \bigcap_{l \in \mathbb{N}} \gamma^{l}_{0}(\mathcal{A}^{l}_{ae}(\Omega))$$

is a differential algebra of generalized functions. Furthermore, the following hold.

(i) There exists an injective algebra homomorphism

$$C^{\infty}(\Omega) \hookrightarrow \mathcal{A}^{\infty}_{ae,0}(\Omega)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{A}^{\infty}_{ae,0}(\Omega) & \xrightarrow{D^{p}} & \mathcal{A}^{\infty}_{ae,0}(\Omega) \\ \hookrightarrow \uparrow & & \uparrow \hookrightarrow \\ \mathcal{C}^{\infty}(\Omega) & \xrightarrow{D^{p}} & \mathcal{C}^{\infty}(\Omega) \end{array} \tag{4.157}$$

commutes for all $p \in \mathbb{N}^n$.

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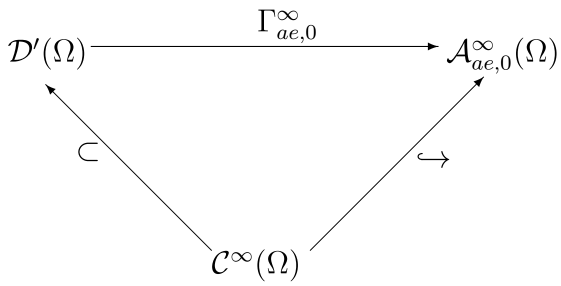
(ii) There exists a linear injection

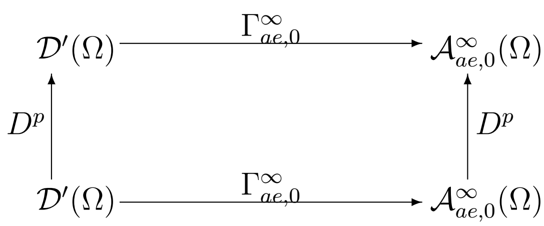
$$\Gamma^{\infty}_{ae,0} : \mathcal{D}'(\Omega) \longrightarrow \mathcal{A}^{\infty}_{ae,0}(\Omega)$$

so that the diagrams



(4.158)

and



(4.159)

commute.

Proof. Let

$$\mathcal{A}^{\infty}_{ae,0}(\Omega) = \bigcap_{l \in \mathbb{N}} \gamma^l_0(\mathcal{A}^l_{ae}(\Omega)),$$

where $\gamma^l_0 : \mathcal{A}^l_{ae} \longrightarrow \mathcal{A}^0_{ae}$ is an algebra homomorphism defined by $\gamma^l_0((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) = (u_n) + (\mathcal{I}^0_E + \mathcal{I}^0_{ae})$. Note that γ^l_0 is injective for all $l \in \mathbb{N}$. Indeed, for $(u_n), (v_n) \in \mathcal{E}^l_{ae}(\Omega)$, we have

$$\begin{aligned} &\gamma^l_0((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) = \gamma^l_0((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) \\ &\Longleftrightarrow (u_n) + (\mathcal{I}^0_E + \mathcal{I}^0_{ae}) = (v_n) + (\mathcal{I}^0_E + \mathcal{I}^0_{ae}) \\ &\Longleftrightarrow (u_n) - (v_n) \in (\mathcal{I}^0_E + \mathcal{I}^0_{ae}). \end{aligned}$$

But $(\mathcal{I}^l_E + \mathcal{I}^l_{ae}) = \mathcal{E}^l_{ae}(\Omega) \bigcap (\mathcal{I}^0_E + \mathcal{I}^0_{ae})$. The inclusion $(\mathcal{I}^l_E + \mathcal{I}^l_{ae}) \subseteq \mathcal{E}^l_{ae}(\Omega) \bigcap (\mathcal{I}^0_E + \mathcal{I}^0_{ae})$ holds trivially. the opposite inclusion follows immediately from Lemma 4.5. Hence

$$\begin{aligned} \gamma^l_0((u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) &= \gamma^l_0((v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae})) \\ &\Longleftrightarrow (u_n) - (v_n) \in \mathcal{I}^l_E + \mathcal{I}^l_{ae} \end{aligned}$$

so that

$$(u_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}) = (v_n) + (\mathcal{I}^l_E + \mathcal{I}^l_{ae}).$$

Item (i) follows immediately from Theorems4.10 and 4.27. The assertion in (ii) follows immediately from Theorems 4.10 and 4.28.

□

An immediate consequence of Theorem 4.24 and Theorem 4.34 is the following.

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Corollary 4.35. *Assume that $\mathcal{M}_0$ consists of all closed, nowhere dense subsets of Ω. If a polynomial nonlinear PDE* (4.104) *satisfies* (2.63) *to* (2.64)*, then there exists a generalized solution $u \in \mathcal{A}^{\infty}_{ae,0}(\Omega)$ of* (4.104).

Finally we establish the existence of an embedding of $\mathcal{ML}^{\infty}_0(\Omega)$ into $\mathcal{A}^{\infty}_{ae,0}$.

Theorem 4.36. *Assume that $\mathcal{M}_0$ consists of all closed, nowhere dense subsets of Ω. There exists an injective algebra homomorphism*

$$H^{\infty}_{ae,0} : \mathcal{ML}^{\infty}_0(\Omega) \longrightarrow \mathcal{A}^{\infty}_{ae,0}(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{A}^{\infty}_{ae,0}(\Omega) & \xrightarrow{\;D^p\;} & \mathcal{A}^{\infty}_{ae,0}(\Omega) \\ \uparrow H^{\infty}_{ae,0} & & \uparrow H^{\infty}_{ae,0} \\ \mathcal{ML}^{\infty}_0(\Omega) & \xrightarrow{\;D^p\;} & \mathcal{ML}^{\infty}_0(\Omega) \end{array} \qquad (4.160)$$

commutes for all $p \in \mathbb{N}^n$

Proof. It follows from Theorem 4.13 that there exists, for each $l \in \mathbb{N}$, an injective algebra homomorphism

$$H^l_{ae} : \mathcal{ML}^l(\Omega) \longrightarrow \mathcal{A}^l_{ae}(\Omega)$$

so that the diagram

$$\begin{array}{ccccccc} & & \mathcal{ML}^l(\Omega) & \xrightarrow{\;H^l_{ae}\;} & \mathcal{A}^l_{ae}(\Omega) & & \\ & \nearrow \subset & & & & \searrow \gamma^l_0 & \\ \mathcal{ML}^{\infty}_0(\Omega) & \subset & \downarrow & & \downarrow \gamma^l_k & & \mathcal{A}^0_{ae}(\Omega) \\ & \searrow \subset & & & & \nearrow \gamma^k_0 & \\ & & \mathcal{ML}^k(\Omega) & \xrightarrow{\;H^k_{ae}\;} & \mathcal{A}^k_{ae}(\Omega) & & \end{array} \qquad (4.161)$$

commutes whenever $k \leq l$. Hence for each $u \in \mathcal{ML}^{\infty}_0(\Omega)$ there exists a unique $w_u \in \mathcal{A}^0_{ae}(\Omega)$ so that

$$(\gamma^k_0 \circ H^k_{ae})(u) = w_u = (\gamma^l_0 \circ H^l_{ae})(u), \quad k, \in \mathbb{N}.$$

Hence the map

$$H^{\infty}_{ae,0} : \mathcal{ML}^{\infty}_0(\Omega) \ni u \mapsto w_u \in \mathcal{A}^{\infty}_{ae,0}(\Omega) \subset \mathcal{A}^0_{ae}(\Omega)$$

is well defined. The injectivity of $H^{\infty}_{ae,0}$ follows from the fact that the algebra homomorphisms H^l_{ae} and γ^l_0 are injective.

Since the diagram

$$\begin{array}{ccccc} \mathcal{ML}^{\infty}_0(\Omega) & \xrightarrow{\;H^{\infty}_{ae,0}\;} & \mathcal{A}^{\infty}_{ae,0}(\Omega) & \xrightarrow{\;\subset\;} & \mathcal{A}^0_{ae}(\Omega) \\ \subset \downarrow & & & & \uparrow \gamma^l_0 \\ \mathcal{ML}^l(\Omega) & & \xrightarrow{\;H^l_{ae}\;} & & \mathcal{A}^l_{ae}(\Omega) \end{array} \qquad (4.162)$$

commutes for all $l \in \mathbb{N}$, it follows that $H^{\infty}_{ae,0}$ is an algebra homomorphism. It follows from Theorems 4.8 and 4.13 and the definition of $H^{\infty}_{ae,0}$ that

$$\begin{aligned} D^p(H^{\infty}_{ae,0}(u)) &= D^p(\gamma_0^l(H^l_{ae}(u)) \\ &= \gamma_0^{l-|p|}(D^p(H^l_{ae}(u))) \\ &= \gamma_0^{l-|p|}(H^{l-|p|}_{ae}(\mathcal{D}^p(u))) \\ &= H^{\infty}_{ae,0}(\mathcal{D}^p(u)) \end{aligned}$$

for all $u \in \mathcal{ML}^{\infty}_0(\Omega)$, $p \in \mathbb{N}^n$ and $l \geq |p|$. Therefore the diagram (4.160) commutes. $\square$

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Chapter 5

Concluding Remarks

5.1 Main results

We have shown that the underlying spaces of generalized functions, $\mathcal{NL}^l(\Omega)$, involve in the Order Completion Method as formulated in the setting of convergence spaces, may be represented as algebras of generalized functions. These algebra of generalized functions are shown to form a differential chain $\mathbf{A}_{oc}$ of algebras of generalized functions. Any generalized solution in the underlying space may be interpreted as a chain generalized solution.

We also considered chains of nowhere dense algebras, and established the way in which such chains are related to the chain $\mathbf{A}_{oc}$. In particular, we considered the Rosinger's nowhere dense algebras, which constitutes the chain $\mathbf{A}_{nd}$, and, based on a construction introduced by Verneave [73, 74], see also [20], the chain $\mathbf{A}_{ae}$ of almost-everywhere algebras was introduced. It was shown that the existence results for chain generalized solution of nonlinear PDEs lead to corresponding existence results in $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$, respectively. The embedding of $\mathcal{D}'(\Omega)$ and the spaces of smooth functions into the chains $\mathbf{A}_{ae}$ was also obtained. It was shown that chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$ admits embeddings of the spaces $\mathcal{NL}^l(\Omega)$ which preserve both the algebraic and differential structure of $\mathcal{NL}^l(\Omega)$. These results demonstrates the extent to which these chains of are able to handle singularities occurring on closed, nowhere dense set. The embedding of $\mathcal{ML}^l(\Omega)$ into $\mathbf{A}_{ae}$ was shown not to be compatible with the embedding of $\mathcal{D}'(\Omega)$ into the chain $\mathbf{A}_{ae}$. Thus a locally integrable function in $\mathcal{ML}^l(\Omega)$ may have more than one representation in the algebra $\mathcal{A}^l(\Omega)$.

This leads naturally to the consideration of the questions of whether or not these chains can deal also with singularities occurring on more general sets. In this regards, we considered the problem of embedding the algebra

$$\mathcal{ML}_0^\infty(\Omega) = \bigcap_{l\in\mathbb{N}} \mathcal{ML}^l(\Omega)$$

into a differential algebra $\mathcal{A}(\Omega)$. The question is motivated by the problem of constructing so-called space-time foam algebras [52, 53].

A general method was introduced by which a differential algebra $\mathcal{A}_0^\infty(\Omega)$ may be constructed from the algebras in a differential chain $\mathbf{A} = \{\mathcal{A}^l(\Omega), \mathcal{A}^k(\Omega), \gamma_k^l \mid k \leq l\}$. In general, the differential algebra $\mathcal{A}_0^\infty(\Omega)$ is larger than $\mathcal{A}^\infty(\Omega)$, and may therefore be able

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to deal with a larger class of singularities than the algebra $\mathcal{A}^\infty(\Omega)$. It was shown how properties of the chain $\mathbf{A}$ induces the corresponding properties of the algebra $\mathcal{A}_0^\infty(\Omega)$. Applying the general method to the chains $\mathbf{A}_{nd}$ and $\mathbf{A}_{ae}$, we constructed algebras of generalized function admitting embedding of $\mathcal{ML}_0^\infty(\Omega)$. The embedding of $\mathcal{D}'(\Omega)$ into these algebras, as well as the existence of generalized solutions of a large class of nonlinear PDEs in these algebras was also established.

5.2 Further research

As we have shown, there is a close connection between the spaces of generalised functions upon which the Order Completion Method is based and, on the other hand, the nowhere dense and almost everywhere algebras. The extent to which this connection can be exploited in order to improve the regularity results of generalised solutions of nonlinear PDEs obtained through the Order Completion Method is a possibly fruitful avenue for future research. A further possibility for future research is the way in which the chain $\mathbf{A}_{oc}$ relates to other differential algebras of generalized functions, such as the Colombeau algebra [14, 15, 16].

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