Onderstepoort Journal of Veterinary Science and Animal Industry, Volume 18, Numbers 1 and 2, July and October, 1943.

Printed in the Union of South Africa by the Government Printer, Pretoria.

Curve Fitting by the Orthogonal Polynomials of Least Squares.

By D. VAN DER REYDEN, Section Statistics, Onderstepoort.

CONTENTS.

1. INTRODUCTION.

THE use of the orthogonal polynomials of least squares as a statistical procedure of curve fitting has become widespread in recent years. It has found applications in many fields of research; its ultimate success in Biology and related subjects will largely depend upon the ability of the biologist to realize the advantages and, at the same time, the limitations that are inherent in this procedure. Unfortunately most of the theory of polynomial fitting is available only in a highly technical form.

It is therefore intended to present in a series of papers, of which this is the first one, the implications of orthogonal polynomial fitting. At the same time, the best practical procedures will be given.

In this paper the systems of orthogonal polynomials mainly used in practice are derived from a common general formula, which is established by the principle of least squares, utilizing results from the Finite Calculus. The Aitken–Chebyshev orthogonal polynomial is recommended for practical use, and, due to a simplification, the fitting process, by means of an extensive set of appended tables, becomes very easy in practice.

At the same time these papers must be regarded as forming the background for a series of papers, appearing in the near future, on the problem of Bio-climatology.

Since it is attempted to present a more or less selfsufficient paper, the non-statistical reader, for whom it is actually intended, should have a fair amount of success, if it be kept in mind that a statistical paper should be *worked* through.

2. Mathematical Introduction.

(a) *Polynomial Interpolation.*

The name *polynomial* (Gr. *polys*, many, L. *nomen*, a name) is given to an algebraical function to express the fact that it is constituted of a number of terms containing different powers of x, connected by the signs $+$ or $-$, i.e., an expression of the form

$$C_0 + C_1x + C_2x^2 + \ldots + C_r x^r.$$

The general problem of interpolation consists in representing a function, known or unknown, in a form chosen in advance with the aid of given values, which this function takes for definite values of the independent variable.

Weierstrass first enunciated the theorem that an arbitrary function can be represented by a polynomial with any assigned degree of accuracy. For a mathematical discussion of the theory of approximation *see* ([5]).*

(b) *Results from the Finite Calculus* ([1], [2], [3], [4]).*

(1) *Operations.*

Whereas, if $u(x)$ be some function of x, the Differential Calculus is concerned with the properties of

$$\frac{d}{dx}\,u(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

the Difference Calculus deals with

$$\underset{h}{\triangle}\, u(x) = \frac{u(x+h) - u(x)}{h} \quad \text{(i)}$$

i.e., it deals with discrete quantities which can be displayed in a table; h is the length of the interval between two consecutive values of x. This quantity is called the *First Advancing Difference* of $u(x)$; the *n–th* Advancing Difference is defined as

$$\triangle^n u(x) = \triangle\,[\triangle^{n-1}u(x)].$$

If $u(x)$ is a function whose values are given for the values $x_0, x_1, \ldots, x_n$ of the variable x, the Divided Difference of $u(x)$ for the arguments x_0, x_1, is denoted by $[x_0x_1]$ and is defined by

$$[x_0x_1] = \frac{u(x_0) - u(x_1)}{x_0 - x_1}$$

* Raised figures in parentheses refer to list of References at the end of the paper.

while the divided difference of the three arguments x_0, x_1, x_2, is defined by

$$[x_0x_1x_2] = \frac{[x_0x_1] - [x_1x_2]}{x_0 - x_2}, \ etc.$$

Since all our work is based on unit interval (i) becomes

$$\triangle u(x) = (E - 1)\, u(x)$$

where the operator E is defined as $E^r u(x) = u(x + r)$.

If the Central Difference Operator, δ, defined by

$$\delta^{2n} u(k) = \triangle^{2n} u\ (k-n)$$
$$\delta^{2n+1} u\ (k + \tfrac{1}{2}) = \triangle^{2n+1} u\ (k-n)$$

is introduced, we find

$$\begin{aligned}\delta\, u(x) &= u(x + \tfrac{1}{2}) - u(x - \tfrac{1}{2}) \\ &= (E^{\frac{1}{2}} - E^{-\frac{1}{2}})\, u(x) \\ &= \triangle\, E^{-\frac{1}{2}}\, u(x).\end{aligned}$$

If two adjacent entries are *averaged*, the operation is denoted by

$$\begin{aligned}\mu\, u(x) &= \tfrac{1}{2}\, [u\ (x + \tfrac{1}{2}) + u(x - \tfrac{1}{2})] \\ &= \tfrac{1}{2}\, [E^{\frac{1}{2}} + E^{-\frac{1}{2}}]\, u(x).\end{aligned}$$

Defining the *indefinite sum* as

$$\Sigma\ u(x) = u_{x-1} + u_{x-2} + \ldots$$

it follows that $\triangle \Sigma\ u(x) = u(x)$, so that in this sense Σ is an operation inverse to $\triangle$. Therefore, the *n–th* sum is expressed by $\Sigma^n u(x) = \Sigma[\Sigma^{n-1} u(x)]$. Furthermore, the problem of finding the *definite sum*

$$\sum_a^b u(x) = u\ (a) + u\ (a + 1) + \ldots + u\ (b)$$

is evidently solved if we know a function $U(x)$ such that $\triangle U(x) = u(x)$.

Leibnitz' well-known theorem in the Differential Calculus

$$D^n\ (u_x v_x) = \sum_{s=0}^{n} \binom{n}{s}\ D^{n-s}\, u(x)\ D^s v(x)$$

has an analogue in the Finite Calculus

$$\triangle^n u(x)\ v(x) = \sum_{s=0}^{n} \binom{n}{s}\ \triangle^{n-s}\, u(x + s)\ \triangle^s\, v(x) \qquad \text{(ii)}$$

with, as special case

$$\triangle\ u(x)\ v(x) = u\ (x + 1)\ \triangle\ v(x) + v(x)\ \triangle\ u(x).$$

Corresponding formulae for summation can be derived :—

$$\Sigma\ u(x)\ v(x) = u(x)\ \Sigma v(x) - \Sigma[\triangle u(x)\ \Sigma\ v\ (x + 1)] \qquad \text{(iii)}$$

and, in the case where $u(x)$ is a polynomial of the *n–th* degree :

$$\begin{aligned}\Sigma\ u(x)\ v(x) &= u(x)\ \Sigma\ v(x) - \triangle\ u(x)\ \Sigma^2\ v(x + 1) + \triangle^2\ u(x)\ \Sigma^3\ v(x + 2) \\ &\quad - \ldots (-)^n\ \triangle^n u(x)\ \Sigma^{n+1}\, v(x + n) \qquad \text{(iv)}\end{aligned}$$

2) *Factorial Notation and Properties.*

The Binomial Theorem is well-known :—

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$
$$a^{n-r}b^r + \ldots + b^n$$

Now, $x^{(r)} = x(x-1)(x-2)\ldots(x-r+1)$ is defined as a *descending factorial,* and by dividing both sides by $r!$ the *reduced* descending factorial

$$x_{(r)} = \frac{x^{(r)}}{r!} = \binom{x}{r}$$

is obtained. The analogy with the binomial coefficients is clear.

It can be easily established that

$$\triangle x^{(r)} = rx^{(r-1)}$$
$$\triangle x_{(r)} = x_{(r-1)}$$

and, continuing the process, we find

$$\triangle^s x^{(r)} = r(r-1)(r-2)\ldots(r-s+1)\,x^{(r-s)}$$
$$\triangle^s x_{(r)} = x_{(r-s)} \quad \text{(v)}$$

From the above it follows that

$$\Sigma x^{(r)} = x^{(r+1)}/_{(r+1)}$$
$$\Sigma x_{(r)} = x_{(r+1)}.$$

Describing *central factorials* as that product of factors, where the factors are in arithmetical progression of common difference unity, and centred at x, e.g.,

$$x^{[2]} = (x+\tfrac{1}{2})(x-\tfrac{1}{2})$$
$$x^{[3]} = (x+1)\,x\,(x-1)$$

we find, in general,

$$x^{[r]} = [x+\tfrac{1}{2}(r-1)]^{(r)}$$
$$= [x+\tfrac{1}{2}(r-1)][x+\tfrac{1}{2}(r-3)][x+\tfrac{1}{2}(r-5)]\ldots$$
$$[x-\tfrac{1}{2}(r-5)][x-\tfrac{1}{2}(r-3)][x-\tfrac{1}{2}(r-1)]$$

with the special cases, where the exponent is even and odd respectively

$$x^{[2s]} = (x^2-\tfrac{1}{4})(x^2-\tfrac{9}{4})\ldots(x^2-\overline{s-\tfrac{1}{2}}^2) \quad \text{(vi)}$$
$$x^{[2s+1]} = x(x^2-1)(x^2-4)\ldots(x^2-s^2) \quad \text{(vii)}$$

If the averaging operation μ is applied to the central factorials, we derive the *mean central factorials,* defined as

$$\mu x^{[r]} = x^{[r-1]+1}$$

with, as special cases

$$x^{[2s]+1} = x(x^2-\tfrac{1}{4})\ldots(x^2-\overline{s-\tfrac{1}{2}}^2) \quad \text{(viii)}$$
$$x^{[2s+1]+1} = x^2(x^2-1)\ldots(x^2-s^2) \quad \text{(ix)}$$

By dividing both classes of factorials by $r!$ the reduced cases are formed.

In exactly the same way as with the descending factorials, we find the following *properties* :

$$\delta x^{[r]} = r x^{[r-1]}$$

$$\delta^s x^{[r]} = r\,(r-1)\,(r-2)\,\ldots\,(r-s+1)\;x^{[r-s]} \ldots\ldots\ldots\ldots \quad \text{(x)}$$

$$\delta x^{]r-1[+1} = r\; x^{[r-2]+1}$$

$$\delta^s x^{[r-1]+1} = r\,(r-1)\,(r-2)\,\ldots\,(r-s+1)\;x^{[[r-s]-1]+1} \ldots\ldots \quad \text{(xi)}$$

(3) *Interpolation Formulae.*

Since it can be shown that the *n–th* difference of a polynomial of degree n is constant, the following formulae are exact if applied to a polynomial function, and consequently, the remainder term is not shown.

Newton's Divided Difference Formula :

$$u(x) = u(x_1) + \sum_{s=1}^{n-1} (x - x_1)\,(x - x_2)\ldots(x - x_s)\,[x_1 x_2 \ldots x_{s+1}] \ldots\ldots \quad \text{(xii)}$$

Gregory–Newton Formula :

$$u_x = u_0 + x\Delta u_0 + x_{(2)}\Delta^2 u_0 + x_{(3)}\Delta^3 u_0 + \ldots \ldots\ldots \quad \text{(xiii)}$$

Newton–Stirling formula (odd No. of values of $u(x)$ with central value $u(o)$:

$$u_x = u_0 + x\cdot\mu\delta u_0 + \mu x^{[2]}\delta^2 u_0/2! + x^{[3]}\cdot\mu\delta^3 u_0/3! + \ldots \ldots\ldots \quad \text{(xiv)}$$

Newton–Bessel formula (even No. of values with two central values $u\,(-\tfrac{1}{2})$ and $u\,(\tfrac{1}{2})$:

$$u_x = \mu u_0 + \mu x\cdot\delta u_0 + x^{[2]}\cdot\mu\delta^2 u_0/2! + \mu x^{[3]}\cdot\delta^3 u_0/3! + \ldots \ldots\ldots \quad \text{(xv)}$$

(4) *Identities in Central and Mean Central Factorials.*

The elegantness of Aitken's derivation depends on certain less familiar factorial identities, of which the limiting case is $(x^2 - q^2)^r$, allowing a perfect analogy with the continuous Legendre polynomial.

We have :

(*a*) $x^2 - q^2 = (x^2 - \tfrac{1}{4}) - (q^2 - \tfrac{1}{4})$

(*b*) $$(x^2 - \overline{q+\tfrac{1}{2}}^2)(x^2 - \overline{q-\tfrac{1}{2}}^2) = x^2(x^2-1) - 2(x^2-1)(q^2-\tfrac{1}{4}) + (q^2-\tfrac{1}{4})(q^2-9/4)$$
$$= (x^2-\tfrac{1}{4})(x^2-9/4) - 2(x^2-\tfrac{1}{4})(q^2-1) + q^2(q^2-1)$$

(*c*) $$(x^2 - \overline{q+1}^2)(x^2-q^2)(x^2 - \overline{q-1}^2) = x^2(x^2-1)(x^2-4) - 3x^2(x^2-1)(q^2-1) + 3(x^2-1)q^2(q^2-1) - q^2(q^2-1)(q^2-4)$$
$$= (x^2-\tfrac{1}{4})(x^2-9/4)(x^2-25/4) - 3(x^2-\tfrac{1}{4})(x^2-9/4)(q^2-9/4) + 3(x^2-9/4)(q^2-\tfrac{1}{4})(q^2-9/4) - (q^2-\tfrac{1}{4})(q^2-9/4)(q^2-25/4)$$

(*d*) $$(x^2 - \overline{q+{}^3/_2}^2)(x^2 - \overline{q+\tfrac{1}{2}}^2)(x^2 - \overline{q-\tfrac{1}{2}}^2)(x^2 - \overline{q-{}^3/_2}^2) =$$
$$a\,b\,c\,d - 4\,a\,b\,c\,B + 6\,bc\,AB - 4\,c\,A\,B\,C + A\,B\,C\,D$$
$$= a'b'c'd' - 4\,a'b'c'C' + 6\,a'b'B'C' - 4b'A'B'C' + A'B'C'D'$$

where $abcd$ denotes $x^2(x^2-1)(x^2-4)(x^2-9)$, $ABCD$ denotes $(q^2-\tfrac{1}{4})(q^2-9/4)(q^2-25/4)\;q^2-49/4)$

and $a'b'c'd'$ denotes $(x^2-\frac{1}{4})(x^2-9/4)(x^2-25/4)(x^2-49/4)$ $A'B'C'D'$ denotes $q^2(q^2-1)(q^2-4)(q^2-9)$

From the above it will be seen :—

(i) Although expansions are special cases of the Newton Dividend Difference formula, in practice a combinatorial scheme can be set up.

(ii) For an odd number of factors : $f(x^2,q^2) = -f(q^2,x^2)$ and $f(x^2-\frac{1}{4}, q^2-\frac{1}{4}) = -f(q^2-\frac{1}{4}, x^2-\frac{1}{4})$.

For an even number of factors : $f(x^2-\frac{1}{4}, q^2) = f(q^2-\frac{1}{4}, x^2)$ and $f(x^2, q^2-\frac{1}{4}) = f(q^2, x^2-\frac{1}{4})$

(iii) Denoting expressions on left by $I(r)$, we have as special cases :—

$$I(2s) = R(o) + \sum_{t=0}^{s} (-)^{s-t}\binom{2s}{s-t} (q^2-s^2)(q^2-\overline{s-1}^2) \ldots (q^2-\overline{t+1}^2)\, x^{[2s+2t]} \qquad \text{(xvi)}$$

and

$$I(2s+1) = R(1) + \sum_{t=0}^{s} (-)^{s-t}\binom{2s+1}{s\ \ t} (q^2-s^2)(q^2-\overline{s-1}^2 \ldots (q^2-\overline{t+1}^2)\, x^{[2s+2t+1]} \qquad \text{(xvii)}$$

where $R(o)$ and $R(1)$ are aggregates of terms falling away with the $2s-th$ and $(2s+1)\ th$ differences respectively.

(5) *Examples of Tables of Differences.*

			u_0			
		Σu_1		Δu_0		
	$\Sigma^2 u_2$		u_1		$\Delta^2 u_0$	
$\Sigma^3 u_3$		Σu_2		Δu_1		$\Delta^3 u_0$
	$\Sigma^2 u_3$		u_2		$\Delta^2 u_1$	
$\Sigma^3 u_4$		Σu_3		Δu_2		$\Delta^3 u_1$
	$\Sigma^2 u_4$		u_3		$\Delta^2 u_2$	
$\Sigma^3 u_5$		Σu_4		Δu_3		
	$\Sigma^2 u_5$		u_4			
		Σu_5				

Thus the above elaborated for x^3 becomes for $x = 0, 1, 2, 3, 4$:

Sums.			*Function.*	*Differences.*		
			0			
		0		1		
	0		1		6	
0		1		7		6
	1		8		12	
1		9		19		6
	10		27		18	
11		36		37		
	46		64			
		100				

Central Differences: *n odd*

u_{-2}				
	$\delta u_{-3/2}$			
u_{-1}		$\delta^2 u_{-1}$		
	$\delta u_{-1/2}$		$\delta^3 u_{-1/2}$	
u_0		$\delta^2 u_0$		$\delta^3 u_0$
	$\delta u_{1/2}$		$\delta^3 u_{1/2}$	
u_1		$\delta^2 u_1$		
	$\delta u_{3/2}$			
u_2				

n even

$u_{-5/2}$			
	δu_{-2}		
$u_{-3/2}$		$\delta^2 u_{-3/2}$	
	δu_{-1}		$\delta^3 u_{-1}$
$u_{-1/2}$		$\delta^2 u_{-1/2}$	
	δu_0		$\delta^3 u_0$
$u_{1/2}$		$\delta^2 u_{1/2}$	
	δu_1		$\delta^3 u_1$
$u_{3/2}$		$\delta^2 u_{3/2}$	
	δu_2		
$u_{5/2}$			

3. The Approximation Problem.

Suppose we have statistical data as displayed in Table 1. By means of the Method of Least Squares we wish to fit a curve of the polynomial type, i.e.,

$$Y' = C_0 + C_1 X + C_2 X^2 + \ldots + C_r X^r \qquad (1)$$

Considering only the simplest case, viz., n independent observations Y_x of equal weight, corresponding to n equi-spaced values of X, the polynomial of best fit is given by the minimum of the sum of squared residuals:

$$\sum_x (Y - Y')^2 = \sum_x [Y - C_0 - C_1 X - C_2 X^2 - \ldots - C_r X^r]^2 \qquad (2)$$

which give $(r + 1)$ normal equations by means of which the C values are determined, expressible as

$$\sum_x X^i (Y - Y') = 0 \quad (i = 0, 1, 2, \ldots, r) \qquad (3)$$

Table 1.

X	x	Y
a	$-\frac{1}{2}(n-1)$	Y_0
$a+1$	$-\frac{1}{2}(n-3)$	Y_1
$a+2$	$-\frac{1}{2}(n-5)$	Y_2
.	.	.
.	.	.
.	.	.
$b-2$	$\frac{1}{2}(n-3)$	Y_{n-2}
$b-1$	$\frac{1}{2}(n-1)$	Y_{n-1}

Apart from the fact that the method becomes extremely laborious with large n and r, since the sums of powers of X up to the $2r$–*th* are required, another disadvantage attaches itself to this approach. Since we do not know in advance which degree will give a satisfactory fit, this implies that if the degree of the polynomial is changed, then the previous coefficients have to be recalculated. The advantage of using some system, which will allow the raising of the degree of the polynomial, while the coefficients already calculated retain their value, is apparent.

By transforming the power polynomial into an aggregate of special components having the property of being *uncorrelated* or *orthogonal* (Gr. *orthos*, right, *gonia*, angle) our object is attained.

If $F_r = F_r(X)$ is defined as an orthogonal polynomial of degree r, it has the properties

$$\left.\begin{aligned} &\sum_a^{b-1} F_r F_s = 0 \text{ if } r \neq s \ldots\ldots \\ &\sum_a^{b-1} F_r^2 \neq 0 \text{ if } r = s \ldots\ldots \end{aligned}\right\} \quad (4)$$

Y, the dependent variable, is now expressed, not in terms of *powers* of X, the determining variable, but in terms of *orthogonal polynomials* of X. Thus

$$Y' = a_0 + a_1 F_1 + a_2 F_2 + \ldots + a_r F_r$$

Utilizing properties (4), the minimizing condition (2) becomes

$$a_r \sum_a^{b-1} F_r^2 - \sum_a^{b-1} YF_r = 0$$

$$\text{i.e.,} \qquad a_r = \frac{\sum_a^{b-1} Y F_r}{\sum_a^{b-1} F_r^2} \ldots\ldots \quad (5)$$

Thus each coefficient a_r in the regression equation is found independently from the others, without the labour of solving simultaneous equations.

The minimum sum of squared residuals (2) becomes by (5)

$$\begin{aligned} \sum_a^{b-1} (Y - Y')^2 &= \sum_a^{b-1} [Y^2 - a_0^2 - a_1^2 F_1^2 - \ldots - a_r F_r^2] \\ &= \Sigma Y^2 - a_0 \Sigma Y - a_1 \Sigma Y F_1 - \ldots - a_r \Sigma Y F_r \ldots \quad (6) \end{aligned}$$

enabling the evaluation beforehand of what value of r will give the best polynomial Y'.

$$\text{Also } t = \frac{(a_r' - a_r)\sqrt{(N - r - 1)\,\Sigma F_r^2}}{\sqrt{\Sigma (Y - Y')^2}}$$

will be distributed in the t–distribution with $(N - r - 1)$ degrees of freedom, and can be used to test the significance of the deviation $a_i - a_i$ of any of the coefficients from a hypothetical value a_i. In practice this hypothetical value is usually taken to be zero.

4. Derivation of $F_r(x)$.

Let $f_{r-1} = f_{r-1}(x)$ be an arbitrary polynomial of degree $(r - 1)$. Since f_{r-1} can be expressed as F_s and since $(r - l) \neq r$, we have from (4)

$$\sum_a^{b-1} f_{r-1} F_r = 0.$$

Applying formula (iv) of the Mathematical Introduction

$$\sum_a^{b-1} f_{r-1} F_r = f_{r-1} \Sigma F_r(X) - \triangle f_{r-1} \Sigma^2 F_r (X + 1) + \triangle^2 f_{r-1} \Sigma^3 F_r (X + 2) - \ldots$$

$(-)^{r-1} \triangle^{r-1} f_{r-1} \Sigma^r F_r (X + r - 1) \cdot \Sigma F_r (X)$ contains an arbitrary constant which can be chosen so that $\Sigma F_r (a)$ becomes zero. In the same way $\Sigma^2 F_r (a+1)$, . . . , $\Sigma_r F_r (a + r - 1)$ vanish. In other words $[\Sigma f_{r-1} F_r]_{x=a} = O$ i.e., the definite sum vanishes for $X = a$. For $X = b$, the expression also vanishes. Since the polynomial F_{r-1} is arbitrary for all values of r, it can be deduced that $(X-a)$ and $X-b)$ are factors of the orthogonal polynomial F_r, and by successive summation it can be shown that $(X-a)_{(r)}$ and $(X-b)_{(r)}$ are multiplying factors of $\Sigma^r F_r(X)$. That is,

$$\sum_a^{b-1} F_r (X) = C. (X-a)_{(r)} (X-b)_{(r)} \ldots\ldots \quad (8)$$

where, since both sides are of degree $2r$, C is an arbitrary constant.

Taking r-th differences of both sides, the general expression for the orthogonal polynomials *w.r.t.* $X = a, a+1, \ldots, b-1$. becomes

$$F_r(X) = C. \triangle^r (X-a)_{(r)} (X-b)_{(r)} \quad \ldots \ldots \ldots (9)$$

Applying formula (ii) it can be written

$$F_r(X) = C. \sum_{t=0}^{r} \binom{r}{t} \binom{x-a+t}{t} \binom{x-b}{r-t} \quad \ldots \ldots \ldots (10)$$

or, utilizing formula (xiii) differenced r times with a as origin, that is

$$\triangle^r u_x = \sum_{s=0}^{n-r} (X-a)_{(s)} \triangle^{r+s} u_a$$

and expanding (8) into a Newton-series of binomial coefficients $(X-b)_{(t)}$ we have

$$\Sigma^r F_r(X) = C. \sum_{t=0}^{2r} (X-b)_{(t)} \triangle^t [(X-a)_{(r)} (X-b)_{(r)}]_{X=b}.$$

According to formula (ii)

$$\triangle^t [(X-a)_{(r)} (X-b)_{(r)}]_{x=b} = [\sum_{s=0}^{t} t_{(s)} \triangle^{t-s} (X-a+s)_{(r)}$$
$$\triangle^s (X-b)_{(r)}]_{X=b}$$
$$= t_{(r)} (b-a+r)_{(2r-t)}$$

all other terms vanishing.

Thus

$$\Sigma^r F_r(X) = C. \sum_{t=r}^{2r} t_{(r)} (b-a+r)_{(2r-t)} (X-b)_{(t)} \quad \ldots \ldots \ldots (11)$$

Since $t > r$, let $t = r + s$. Substituting and determining the r-th difference, we find

$$F_r(X) = C. \sum_{s=0}^{r} (X-b)_{(s)} (r+s)_{(r)} (b-a+r)_{(r-s)} \quad \ldots \ldots \ldots (12)$$

Now, since $\Sigma^r F_r(X)$ is symmetric *w.r.t.* a and b, expression (12) can also be expressed in terms of the initial value of the determining variable, i.e., a. Thus, noting that $b - a = n$, and $(a-b+r)_{(r-s)} = (-)^{r-s} (n-s-1)_{(r-s)}$ we have from (12).

$$F_r(X) = C. \sum_{s=0}^{r} (-)^{r-s} (r+s)_{(r)} (n-s-1)_{(r-s)} (X-a)_{(s)} \quad \ldots \ldots \ldots (12^1)$$

This is the general explicit form of the orthogonal polynomial, and the various systems proposed differ only in the value assigned to the arbitrary constant C, determined in each case by the criterion that the numerical work involved in the particular system should be a minimum.

5. Derivation of $\sum_{a}^{b-1} F_r^2(X)$.

Applying formula (iv) to $\Sigma[F_r F_r]$, we find

$$\Sigma [F_r F_r] = F_r \Sigma F_r(X) - \triangle F_r \Sigma^2 F_r(X+1) + \triangle^2 F_r \Sigma^3 F_r$$
$$(X+2) - \ldots (-)^{s-1} \triangle^{s-1} F_r \Sigma^s F_r (X+s-1)$$
$$\ldots (-)^r \triangle^r F_r \Sigma^{r+1} F_r (X+r) \quad \ldots \ldots \ldots (13)$$

If $s < r+1$, the quantities $\Sigma^s F_r(X+s-1)$ are easily obtained. Applying expression (11), replacing X by $X+s-1$, it will be seen that at both limits $X = a$, $X = b$ all the terms of (11), excepting the last, vanish. Now

$$\Sigma^{r+1} F_r(X+r) = C \cdot \sum_{s=r}^{2r} (X+r-b)_{(s+1)} s_{(r)} (b-a+r)_{(2r-s)},$$

which is the expression for the indefinite sum of $\Sigma^r F_r$ vanishing for $X = b$, since $r < s + 1$.

Its value for $X = a$ will be

$$\Sigma^{r+1} F_r (a + r) = C \cdot \sum_{s=r}^{2r} s_{(r)} (a - b + r)_{(s+1)} (b - a + r)_{(2r-s)}$$

and, utilizing the fact that $b - a = n$, it becomes

$$\Sigma^{r+1} F_r (a + r) = C \cdot (n + r)_{(2r+1)} \sum_{s=r}^{2r} (-)^{s+1} s_{(r)} (2r + 1)_{(s+1)}$$

By combinatory analysis the sum on the righthand-side is equal to $(-1)^{r+1}$, giving

$$\Sigma^{r-1} F_r (a + r) = (-)^{r+1} C \cdot (n + r)_{(2r+1)}$$

From (12), $\triangle^r F_r (X) = C (2r)_{(r)}$, and, on combining these quantities into the definite sum of (13), the general expression for the sums of squares of the orthogonal polynomials is derived

$$\sum_a^{b-1} F_r^2 = C^2 \cdot (2r)_{(r)} (n + r)_{(2r+1)} \qquad (14)$$

6. Determination of $\sum_a^{b-1} YF_r (X)$.

If $Y' = a_0 + a_1 F_1 + a_2 F_2 + \ldots + a_r F_r$ it was shown in section 3 that the coefficients a_r can be determined from the data, by means of the formula

$$a_r = \frac{\Sigma YF_r}{\Sigma F_r^2} \qquad (5)$$

Since the denominator is independent of the origin of X, and only depends upon the values of n, the number of data, and r, the degree of the polynomial, it only becomes necessary to evaluate the product.

It can be shown that, if the origin of X is chosen suitably, this product can be evaluated by a process of consecutive summation. Since F_r is expressed either in the form of a Gregory–Newton series (taking the initial datum as origin) or in the form of a Newton–Stirling or Newton–Bessel series (*see* Section 8), depending upon whether the number of data are odd or even (taking the origin at the centre of the data), consecutive summation will respectively yield the reduced factorial moments and reduced central factorial moments, which, when combined according to the formula in question, will yield the numerator of the expression in (5).

For practical purposes, however, if n and r are not too large, the direct multiplication method according to expression (5), with the aid of the standard tables of F_r (Appendix 1), and a calculating machine, is far superior.

For large n and r (usually not met with in practice), some summation method will probably be preferred, and the reader is referred to ([38]) and ([41]) for a succint description.

7. Compilation of the Standard Tables.

If, in the expression for $F_r (x)$, we omit the arbitrary constant C for the moment, we have

$$\begin{aligned} F_r(X) = & \sum_{s=0}^{r} (-)^{r-s} (r + s)_{(r)} (n - s - 1)_{(r-s)} (X - a)_{(s)} \\ = & (2r)_{(r)} (X - a)_{(r)} - (2r - 1)_{(r)} (n - r) (X - a)_{(r-1)} + \\ & (2r - 2)_{(r)} (n - r + 1)_{(2)} (X - a)_{(r-2)} - \ldots \\ & \ldots (-)^{r-1} (r + 1)_{(r)} (n - 2)_{(r-1)} (X - a) + (-)^r \\ & (n - 1)_{(r)} \end{aligned} \qquad (15)$$

Differencing this expression t times, utilizing formula (v)

$$\triangle^t F_r(X) = (2r)_{(r)} (X - a)_{(r-t)} - (2r-1)_{(r)} (n-r) (X-a)_{(r-t+1)} + (2r-2)_{(r)} (n-r+1)_{(2)} (X-a)_{(r-t-2)} - \dots \dots (-)^{r-4} (r+1)_{(r)} (n-2)_{(r-1)} (X-a)_{(1-t)} \dots \quad (16)$$

Putting $X = a$, these two expressions can be combined into one formula, viz.,

$$\triangle^t F_r(a) = (-)^{r-t} (r+t)_{(r)} (n-t-1)_{(r-t)} \dots\dots \quad (17)$$

giving the initial difference of $F_r(x)$ for $X = a$.

As special cases we find

$$F_r(a) = (-)^r (n-1)_{(r)}.$$
$$\triangle F_r(a) = (-)^{r-1} (r+1)_{(r)} (n-2)_{(r-1)}.$$
$$\triangle^2 F_r(a) = (-)^{r-2} (r+2)_{(r)} (n-3)_{(r-2)}.$$

. . .

. . .

. . .

$$\triangle^{r-2} F_r(a) = (2r-2)_{(r)} (n-r+1)_{(2)}.$$
$$\triangle^{r-1} F_r(a) = -(2r-1)_{(r)} (n-r).$$
$$\triangle r F_r(a) = (2r)_{(r)}.$$

yielding, e.g., for $r = 5$, $n = 7$.

$$F_5(a) = -5_{(5)} 6_{(5)}.$$
$$\triangle F_5(a) = 6_{(5)} 5_{(4)}.$$
$$\triangle^2 F_5(a) = -7_{(5)} 4_{(3)}.$$
$$\triangle^3 F_5(a) = 8_{(5)} 3_{(2)}.$$
$$\triangle^4 F_5(a) = -9_{(5)} 2_{(1)}.$$
$$\triangle^5 F_5(a) = 10_{(5)} 1_{(0)}.$$

Removing common factor 6, we find consecutively -1, 5, -14, 28, -42, 42 for the the leading term and differences, enabling us by summation to build up the table of values for $F_5(X)$ with $n = 7$.

It will be noticed that each number in the above scheme consists of the product of two factors, both of which can be easily derived from the *figurate number* properties (modification of the well-known Pascal triangle algorithm) of binomial coefficients in the following way: write down $r + 1$ columns of figurate numbers from right to left. Multiply the values in each column by the first $r + 1$ values in the $(r + 1)$*–th* column with alternate + and — signs from the right. Remove common factors within each row; the rows in the table of products give the leading term and initial differences for values of n from $(r + 1)$ upwards.

Example: $r = 5$, $n = 6, 7, 8, \dots$

First Step: *Figurate Numbers*.

n						
6	1	1	1	1	1	1
7	6	5	4	3	2	1
8	21	15	10	6	3	1
9	56	35	20	10	4	1
10	126	70	35	15	5	1
11	252	126	56	21	6	1
12	462	210	84	28	7	1

Each number in any column after the first from the right is the progressive sum of the numbers in the column to the right up to the row, in which it stands, e.g., 35 (in the 5–*th* column) = 1 + 4 + 10 + 20.

Second Step: Multiplication.

n	$F_r(a)$	$\triangle F_r(a)$	$\triangle^2 F_r(a)$	$\triangle^3 F_r(a)$	$\triangle^4 F_r(a)$	$\triangle^5 F_r(a)$
6	— 1	6	— 21	56	—126	252
7	— 6	30	— 84	168	—252	252
8	— 21	90	—210	336	—378	252
9	— 56	210	—420	560	—504	252
10	—126	420	—735	840	—630	252
11	—252	756	—1176	1176	—756	252
12	—462	1260	—1764	1568	—882	252

Third Step: Removing Common Factor.

n							*Common Factor*
6	— 1	6	— 21	56	—126	252	1
7	— 1	5	— 14	28	— 42	42	6
8	— 7	30	— 70	112	—126	84	3
9	— 4	15	— 30	40	— 36	18	14
10	— 6	20	— 35	40	— 30	12	21
11	— 3	9	— 14	14	— 9	3	84
12	—33	90	—126	112	— 63	18	14

Fourth Step: Derivation of F_r (X) by Multiplication.

By summation from the leading term and initial differences the values of $F_r(X)$ for the different values of X can be found. In compiling our standard tables (Appendix I) it was found easier to substitute a process of multiplication, as follows :—

Since $F_r(X)$ is nothing else than a Gregory–Newton interpolation formula, it can be written

$$F_r(X) = F_r(a) + (X - a) \triangle F_r(a) + (X - a)_{(2)} \triangle^2 F_r(a) + \dots + (X - a)_{(r)} \triangle^r F_r(a) \qquad (18)$$

Substituting respectively $X = a, a + 1, \ldots$ in this expression, we derive

$$F_r(a) = F_r(a)$$
$$F_r(a+1) = F_r(a) + \triangle F_r(a)$$
$$F_r(a+2) = F_r(a) + 2\triangle F_r(a) + \triangle^2 F_r(a)$$
$$F_r(a+3) = F_r(a) + 3\triangle F_r(a) + 3\triangle^2 F_r(a) + \triangle^3 F_r(a), \text{ etc.}$$

Denoting $\triangle^t F_r(a)$ by $(-)^{r-t} A_r$, where A_r is the absolute value of $\triangle^t F_r(a)$ we have respectively for $r = even$ and $r = odd$.

X	F_{2s}	F_{2s+1}
a	A_0	$-A_0$
$a+1$	$A_0 - A_1$	$-A_0 + A_1$
$a+2$	$A_0 - 2A_1 + A_2$	$-A_0 + 2A_1 - A_2$
$a+3$	$A_0 - 3A_1 + 3A_2 - A_3$	$-A_0 + 3A_1 - 3A_2 + A_3$
$a+4$	$A_0 - 4A_1 + 6A_2 - 4A_3 + A_4$	$-A_0 + 4A_1 - 6A_2 + 4A_3 - A_4$

It is observed that the coefficients of the A's form a Pascal triangle, i.e., they correspond to the diagonals of the figurate numbers in Step 1. In this way the table of figurate numbers is used twice, viz., first to find the leading differences of an orthogonal polynomial, and then showing how to combine these differences to obtain the values for different X.

It will now be recalled that the arbitrary constant C was temporarily omitted for this discussion. Since we have used the Aitken-criterion (Section 11), viz., $C = 1$, the above development is immediately applicable. In the case of the other systems, all results above must be multiplied by the value C assumes in that particular development. In practice this amounts to multiplying the common factor removed by C.

As can be seen from the derivation in Section 4, $F_r(X)$ is symmetrical in absolute value about the centre of the data, the same signs thus occuring in the upper and lower portions of the table when r is even, opposite signs when r is odd. Thus it is only necessary to apply the derivation methods to one half of the table.

8. The Derivation of the Central Case: $\hat{F}_r(x)$.

In section 4 we derived

$$F_r(X) = C \triangle^r (X-a)_{(r)} (X-b)_{(r)} \ldots \quad (8)$$

where X took the values a to $b-1$, or, the same thing, a to $a+n-1$. Measure X from the centre of the data, thus $x = X - \bar{X}$, and transform to central difference notation by the relation $\triangle^r u(x) = \delta^r u(x + r/_2)$.

Using Aitken's criterion, let $C = 1$ and put $n = 2q$, thus finding

$$\hat{T}_r(x) = \delta^r [(x + q + \tfrac{1}{2}\overline{r-1})_{(r)} (x - q + \tfrac{1}{2}\overline{r-1})_{(r)}]$$
$$= \frac{\delta^r}{(r!)^2} [(x + q + \tfrac{1}{2}\overline{r-1})(x + q + \tfrac{1}{2}\overline{r-3}) \ldots (x + q - \tfrac{1}{2}\overline{r-3})$$
$$(x + q - \tfrac{1}{2}\overline{r-1}) \cdot (x - q + \tfrac{1}{2}\overline{r-1})(x - q + \tfrac{1}{2}\overline{r-3})$$
$$\ldots (x - q - \tfrac{1}{2}\overline{r-3})(x - q - \tfrac{1}{2}\overline{r-1})]$$

where $\hat{T}_r$ now denotes the orthogonal polynomial. By appropriate multiplication within the brackets we readily find

$$\hat{T}_r(x) = \frac{\delta^r}{(r!)^2}[(x^2 - \overline{q + \tfrac{1}{2}(r-1)}^2)(x^2 - \overline{q + \tfrac{1}{2}(r-3)}^2 \ldots$$
$$(x^2 - \overline{q - \tfrac{1}{2}(r-3)}^2)(x^2 - \overline{q - \tfrac{1}{2}(r-1)}^2 \ldots\ldots\ldots \quad (19)$$

By means of the factorial identities, Section 2, the general case can be solved, but it is more instructive to treat the even and odd cases separately.

If r is even, i.e., $r = 2s$, it follows that, using formulae (xvi) and (x)

$$\hat{T}_{2s}(x) = \frac{\delta^{2s}}{(2s)!\,(2s)!} I_{2s}$$
$$= \sum_{t=o}^{s} (-)^{s-t}(2s)_{s-t} \frac{(2s+2t)(2s+2t-1)\ldots(2t+1)}{(2s)!\ (2s)!}$$
$$(q^2 - s^2)\ldots(q^2 - \overline{t+1}^2)\, x^{[2t]}$$
$$= \sum_{t=o}^{s} (-)^{s-t}(2s+2t)_{(2t)} \frac{(q^2 - s^2)\ldots(q^2 - \overline{t+1}^2)}{(s+t)!\ (s-t)!} x^{[2t]} \ldots\ldots \quad (20)$$

Using formulae (xvii) and (xi) we find

$$\hat{T}_{2s+2}(x) = \sum_{t=o}^{s} (-)^{s-t}(2s+2t+2)_{(2t+2)} \frac{(q^2-s^2)\ldots(q^2-\overline{t+1}^2)}{(s+t+1)!\ (s-t)!} \mu x^{[2t+1]} \quad (21)$$

It will be noted that the above formulae are suitable for application to cases with an *even* number of data. By utilizing the alternative forms of the factorial identities, two corresponding expressions are derived for an odd number of data, viz.,

$$\hat{T}_{2s}(x) = \sum_{t=o}^{s} (-)^{s-t}(2s+2t)_{(2t)} \frac{(q^2-\overline{s-{}^1/_2}^2)(q^2-\overline{s-{}^3/_2}^2)\ldots(q^2-\overline{t+{}^1/_2}^2)}{(s+t)!\ (s-t)!}$$
$$\mu x^{[2t]} \ldots\ldots\ldots\ldots \quad (22)$$

and

$$\hat{T}_{2s+1}(x) = \sum_{t=o}^{s} (-)^{s-t}(2s+2t+2)_{(2t+1)} \frac{(q^2-\overline{s+{}^1/_2}^2)(q^2-\overline{s-{}^1/_2}^2)\ldots(q^2-\overline{t+{}^3/_2}^2)}{(s+t+1)!\ (s-t)!}$$
$$x^{[2t+1]} \ldots\ldots\ldots \quad (23)$$

In Table 2 the explicit expressions for the two cases are given up to the ninth degree. It will be noted that the polynomials are but special cases of the Newton–Stirling (if n is odd) and the Newton–Bessel (if n is even) formulae of interpolation, thus revealing very lucidly the interpolatory character of this method of curve fitting. This fact also enables us to draw up tables, giving the central and mean central differences, from which standard tables (which are of course the same as those obtained previously) may be computed. From these central and mean central differences, the reduced factorial and reduced central factorial moments can also be obtained, and, as pointed out in Section 6, the fitting-process may then be carried out by a process of consecutive summation ([38]).

Figure 1, giving the graphical representation to these orthogonal polynomials for $n = 39$, is intended to bring out the extreme flexibility inherent in this method of curve fitting. Note that with increasing r, there is a proportionate increase of turning-points in the graphs.

TABLE 2.

n = even.

$$\hat{T}_0 = 1$$

$$\hat{T}_1 = \binom{2}{1}\frac{\mu x^{[1]}}{1!}$$

$$\hat{T}_2 = \binom{4}{2}\frac{x^{[2]}}{2!} - \frac{(q^2-1)}{1!}$$

$$\hat{T}_3 = \binom{6}{3}\frac{\mu x^{[3]}}{3!} - \binom{4}{1}\frac{(q^2-1)}{2!}\mu x^{[1]}$$

$$\hat{T}_4 = \binom{8}{4}\frac{x^{[4]}}{4!} - \binom{6}{2}\frac{(q^2-4)}{3!}x^{[2]} + \frac{(q^2-4)(q^2-1)}{2!\,2!}$$

$$\hat{T}_5 = \binom{10}{5}\frac{\mu x^{[5]}}{5!} - \binom{8}{3}\frac{(q^2-4)}{4!}\mu x^{[3]} + \binom{6}{1}\frac{(q^2-4)(q^2-1)}{3!\,2!}\mu x^{[1]}$$

$$\hat{T}_6 = \binom{12}{6}\frac{x^{[6]}}{6!} - \binom{10}{4}\frac{(q^2-9)}{5!}x^{[4]} + \binom{8}{2}\frac{(q^2-9)(q^2-4)}{4!\,2!}x^{[2]} - \frac{(q^2-9)(q^2-4)(q^2-1)}{3!\,3!}$$

$$\hat{T}_7 = \binom{14}{7}\frac{\mu x^{[7]}}{7!} - \binom{12}{5}\frac{(q^2-9)}{6!}\mu x^{[5]} + \binom{10}{3}\frac{(q^2-9)(q^2-4)}{5!\,2!}\mu x^{[3]} - \binom{8}{1}\frac{(q^2-9)(q^2-4)(q^2-1)}{4!\,3!}\mu x^{[1]}$$

$$\hat{T}_8 = \binom{16}{8}\frac{x^{[8]}}{8!} - \binom{14}{6}\frac{(q^2-16)}{7!}x^{[6]} + \binom{12}{4}\frac{(q^2-16)(q^2-9)}{6!\,2!}x^{[4]} - \binom{10}{2}\frac{(q^2-16)(q^2-9)(q^2-4)}{5!\,3!}x^{[2]} + \frac{(q^2-16)(q^2-9)(q^2-4)(q^2-1)}{4!\,4!}$$

$$\hat{T}_9 = \binom{18}{9}\frac{\mu x^{[9]}}{9!} - \binom{16}{7}\frac{(q^2-16)}{8!}\mu x^{[7]} + \binom{14}{5}\frac{(q^2-16)(q^2-9)}{7!\,2!}\mu x^{[5]} - \binom{12}{3}\frac{(q^2-16)(q^2-9)(q^2-4)}{6!\,3!}\mu x^{[3]} + \binom{10}{1}\frac{(q^2-16)(q^2-9)(q^2-4)(q^2-1)}{5!\,4!}\mu x^{[1]}$$

n = uneven.

$$\hat{T}_0 = 1$$

$$\hat{T}_1 = \binom{2}{1}\frac{x^{[1]}}{1!}$$

$$\hat{T}_2 = \binom{4}{2}\frac{\mu x^{[2]}}{2!} - \frac{(q^2-{}^1/_4)}{1!}$$

$$\hat{T}_3 = \binom{6}{3}\frac{x^{[3]}}{3!} - \binom{4}{1}\frac{(q^2-{}^9/_4)}{2!}x^{[1]}$$

$$\hat{T}_4 = \binom{8}{4}\frac{\mu x^{[4]}}{4!} - \binom{6}{2}\frac{(q^2-{}^9/_4)}{3!}\mu x^{[2]} + \frac{(q^2-{}^9/_4)(q^2-{}^1/_4)}{2!\,2!}$$

$$\hat{T}_5 = \binom{10}{5}\frac{x^{[5]}}{5!} - \binom{8}{3}\frac{(q^2-{}^{25}/_4)}{4!}x^{[3]} + \binom{6}{1}\frac{(q^2-{}^{25}/_4)(q^2-{}^9/_4)}{3!\,2!}x^{[1]}$$

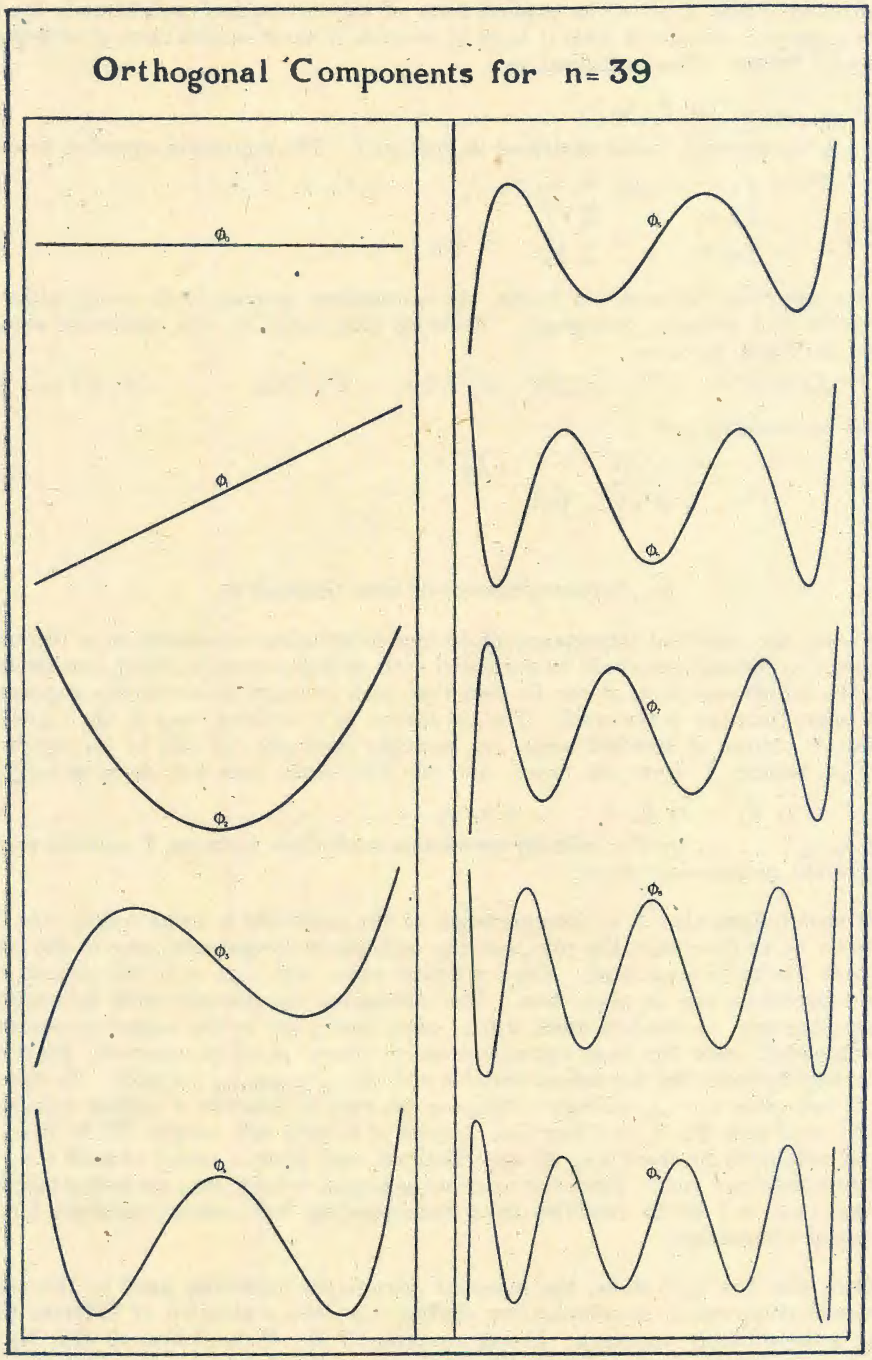
$$\hat{T}_6 = \binom{12}{6}\frac{\mu x^{[6]}}{6!} - \binom{10}{4}\frac{(q^2-{}^{25}/_4)}{5!}\mu x^{[4]} + \binom{8}{2}\frac{(q^2-{}^{25}/_4)(q^2-{}^9/_4)}{4!\,2!}\mu x^{[2]} - \frac{(q^2-{}^{25}/_4)(q^2-{}^9/_4)(q^2-{}^1/_4)}{3!\,3!}$$

$$\hat{T}_7 = \binom{14}{7}\frac{x^{[7]}}{7!} - \binom{12}{5}\frac{(q^2-{}^{49}/_4)}{6!}x^{[5]} + \binom{10}{3}\frac{(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)}{5!\,2!}x^{[3]} - \binom{8}{1}\frac{(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)(q^2-{}^9/_4)}{4!\,3!}x^{[1]}$$

$$\hat{T}_8 = \binom{16}{8}\frac{\mu x^{[8]}}{8!} - \binom{14}{6}\frac{(q^2-{}^{49}/_4)}{7!}\mu x^{[6]} + \binom{12}{4}\frac{(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)}{6!\,2!}\mu x^{[4]} - \binom{10}{2}\frac{(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)(q^2-{}^9/_4)}{5!\,3!}\mu x^{[2]} + \frac{(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)(q^2-{}^9/_4)(q^2-{}^1/_4)}{4!\,4!}$$

$$\hat{T}_9 = \binom{18}{9}\frac{x^{[9]}}{9!} - \binom{16}{7}\frac{(q^2-{}^{81}/_4)}{8!}x^{[7]} + \binom{14}{5}\frac{(q^2-{}^{81}/_4)(q^2-{}^{49}/_4)}{7!\,2!}x^{[5]} - \binom{12}{3}\frac{(q^2-{}^{81}/_4)(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)}{6!\,3!}x^{[3]} + \binom{10}{1}\frac{(q^2-{}^{81}/_4)(q^2-{}^{49}/_4)(q^2-{}^{25}/_4)(q^2-{}^9/_4)}{5!\,4!}x^{[1]}$$

FIGURE 1.



9. Simplified Polynomials.

Although Table 2 gives the explicit form of the orthogonal polynomials for the various degrees, a simplified form is used in practice in most applications of orthogonal polynomial fitting. This is defined as

$$\varphi_r\,(x) = {}^1/\lambda\ \hat{T}_r\,(x) \qquad (24)$$

where λ is the common factor explained in Section 7. The regression equation becomes

$$Y' = a'_0 + a'_1\,\varphi_1 + a'_2\,\varphi_2 + \ldots + a'_r\,\varphi_r \qquad (25)$$

$$a'_r = \frac{\Sigma Y\varphi_r}{\Sigma\,\varphi_r{}^2} = \lambda\,\frac{\Sigma Y\hat{T}_r}{\Sigma\,\hat{T}_r{}^2} = \lambda a_r \qquad (26)$$

Thus by removing the common factor, the normalized property falls away, although the system still remains orthogonal. Utilizing (24) and (26), the minimum sum of squared residuals becomes

$$\Sigma\,(Y\text{–}Y')^2 = \Sigma Y^2 - a'_0\Sigma Y - a'_1\,\Sigma Y\varphi_1 - Y'_2\Sigma Y\varphi_2 - \ldots - a'_r\,\Sigma Y\varphi_r \qquad (27)$$

and the significance test

$$t = \frac{(a'_r - a_r)\sqrt{(N - r - 1)\,\Sigma\varphi_r{}^2}}{\sqrt{\Sigma\,(Y - Y')^2}} \qquad (28)$$

10. Interpretation of the Constants.

Above, the practical importance of having independent constants in a regression equation, i.e., that the one could be evaluated without influencing the other, was stressed. From the significance tests it can be seen that each constant measures the importance of its approximating polynomial. This is shown in a striking way if the regression equation is written in *standard units,* i.e., multiply both sides of (25) by the reciprocal of $\sqrt{\Sigma y^2}$, taking Y from its mean, and at the same time put $\psi_r = \varphi_r/\sqrt{\Sigma\varphi_r{}^2}$. Thus

$$y' = r_1\,\psi_1 + r_2\,\psi_2 + \ldots + r_r\,\psi_r \qquad (29)$$

where $r_1, r_2, \ldots, r_r$ are the ordinary correlation coefficients between Y and the various uncorrelated polynomial values.

It thus follows that if an interpretation of the constants is being sought, the first step must be to determine the rôle that the orthogonal components play in the set of data that has to be evaluated. Since a future paper will deal with this aspect, only a few suggestions can be made here. The orthogonal components must be regarded, in a certain sense, as *standard units,* which, when multiplied by the respective constants and summated, give the best *approximation* to some, possibly unknown, functional relationship between the dependent variable and the determining variable. To take an easy, if not quite correct, analogy: Suppose we wish to describe a certain individual. We will state that Mr. Y. is 6 feet (i.e., 6 units of length) tall, weighs 170 lb. (i.e., 170 units of weight) is 30 years (i.e., 30 age-units) old, and draws a salary of £400 (i.e., 400 monetary units) per year. Since the concepts of height, weight, etc., are well-established our first task will be to establish in a corresponding way certain concepts for the orthogonal components.

Once this has been done, the manifest advantages attaching itself to the use of orthogonal polynomials in comparative studies, e.g., the evaluation of different time-series, is immediately apparent. In our example, if Mr. Y. is compared with Mr. Z., we only compare the various multiplier-constants of the standard units of height, etc.

However, a word of caution must be given when interpretations are being sought. It has been stressed throughout that the orthogonal polynomial method of curve fitting is an interpolatory, i.e., approximative, one. Thus, it is the best representation of the dependent variable in terms of *integral* values, or combinations of values, of the determining variable.

The validity of the process will depend on how far that underlying, in most cases unknown, functional relationship as influenced by " random error ", *can* be approximated by a polynomial or combination of polynomials. In other words, are our criteria, by which the set of independent polynomials were derived adequate? Research on this point seems necessary. It appears, however, that if the data conform to such conditions as to give validity to the use of the arithmetic mean, then the use of the higher constants are also permissible. In this case, the constants can be regarded either as parameters, with which the symmetry of the data can be investigated, or, as certain measures of the rates of increase. In other words, in certain cases an analogy can be set up between the parameters characterizing the one-dimensional frequency-distribution and the parameters, by means of which a two-dimensional distribution, not necessarily of a frequency-nature, can be evaluated.

Figure 2, which is a special adaptation of Figure 1, emphasizes the transformatory character of the various orthogonal components. It must be kept in mind that this transformation is of an *integral* nature, i.e., all the exponents of the determining variable are integers. That is, the straight line, parabola and third degree polynomial which are so often met with in biological curve fitting are all special cases of the procedure evaluated in the preceding pages.

11. Historical Notes.

(a) Continuous Case.

The orthogonal polynomials treated above are all discrete cases of the well-known *Legendre-polynomials,* particulars of which can be found in any standard textbook on Analysis.

If the limiting case of (19) is taken between the limits $-q$, q, Rodrigues' formula is derived

$$P_r\,(x) = \left(\frac{d}{dx}\right)^r \frac{(x^2 - q^2)^r}{(r!)^2}.$$

The explicit values of the first few polynomials become :—

$$P_0 = 1$$

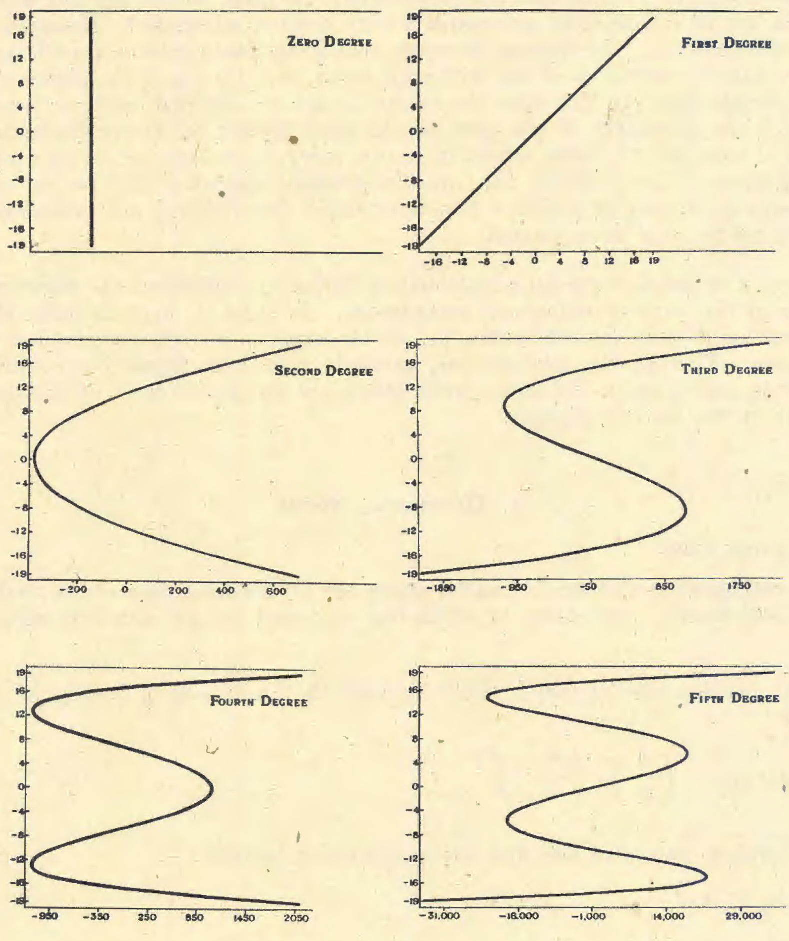
$$P_1 = 2x$$

$$P_2 = \binom{4}{2}\frac{x^2}{2!} - q^2$$

$$P_3 = \binom{6}{3}\frac{x^3}{3!} - \binom{4}{1}\frac{q^2x}{2!}$$

$$P_4 = \binom{8}{4}\frac{x^4}{4!} - \binom{6}{2}\frac{q^2x^2}{3!} + \frac{q^4}{2!\,2!}$$

FIGURE 2.



The orthogonal polynomials as transformation functions. The ordinate represents the equi-spaced values of the determining variable, while the abscissa represents the values of the orthogonal components.

(b) P. L. Chebyshev.

The problem of interpolation by means of orthogonal functions was first introduced by the famous Russian mathematician, P. L. Chebyshev, in a series of papers on orthogonal representation ([8], [9], [10], [11], [12], [13]). His researches are of a very general nature, since he treats the non-equidistant case ([8], [9]), as well as the equidistant case ([10], [11], [12], [13]) but the application in practice is extremely complicated. Using his reduction formula([11]).

$$\psi_r = 2\,(2r - 1)\,x\,\psi_{r-1} - (r-1)^2\,[n^2 - (r-1)^2]\,\psi_{r-2}$$

or, putting $C = (r!)^2$ and $a = -\frac{n-1}{2}$ in our (10′) the first five explicit values are

$$\psi_0 = 1$$
$$\psi_1 = 2x$$
$$\psi_2 = 12x^2 + (n^2 - 1)$$
$$\psi_3 = 120\,x^3 - 6\,(3n^2 - 7)\,x$$
$$\psi_4 = 1680\,x^4 - 120\,(3n^2 - 13)\,x^2 + 9\,(n^2 - 1)\,(n^2 - 9)$$

(c) Intermediate Period.

In his "Calcul des Probabilités"([14]), Poincaré, independently from Chebyshev, develops the interpolatory function by means of a continuous function identity for non-equidistant values. These functions are proportional to those of Chebyshev. A. Quiquet ([15]) applies this development to practical cases. J. P. Gram([17]) suggests in a general way a step-wise derivation of the general approximating function as a convergent sequence of polynomials for various orthogonal functions and applies it to the smoothing of empirical curves.

(d) Charl Jordan.

Although the theoretical basis of the orthogonal polynomials of least squares was given by Chebyshev as early as 1855, it was not until 1920, with the publication of Jordan's methods, that this approach became really practicable. In his first paper([18]) Jordan treats the mathematical theory of orthogonal polynomials for equidistant values in the general case. C was chosen as $r!/2^r\,h^r$; where h is the length of the interval, and the coefficients were derived by multiplication of the Y-values with standard values, which were calculated for the different degrees. In ([19]) the coefficients were obtained by the product of the binomial moments, obtained by successive summation, and certain standard numbers; in ([20]) mean orthogonal moments are introduced, which eliminates the calculation of ΣU_r^2, where U_r denotes the Jordan-polynomial. Our derivation of F_r is based to a certain extent upon ([22]), which paper is the final presentation of the Jordan system. Practical applications of his work will be found in ([23], [24]).

(e) Fredrik Esscher.

Denoting the orthogonal polynomial by P_r, taking X from the centre of the data, Esscher determines the value of C by the convention([25]) that the coefficient of x^r shall be unity, i.e., in our notation $C = r!/\binom{2r}{r}$. In his second paper ([26]) using polynomials $X_r(x)$, x taking the values $1, 2, 3, \ldots, n$, he chooses C in such a way that ΣX_r^2 becomes equal to n, i.e., in our notation

$$C = \sqrt{n \Big/ \binom{2r}{r}\binom{n+r}{2r+1}}$$

thus simplifying the expression ΣX_r^2, but complicating the polynomials.

(*f*) *P. Lorenz.*

Using the determinantal approach, Lorenz ([27], [28]) derives orthogonal polynomials $X_r(x)$, distinguishing between even and odd n, so that $\Sigma X_r^2 = n$. Thus, in our notation the value of C is found to be

$$C = \sqrt{\frac{n}{\binom{2r}{r}\binom{n+r}{2r+1}}} \quad \text{in the even case, and}$$

$$C = \frac{1}{2^{2r}}\sqrt{\frac{n}{\binom{2r}{r}\binom{n+r}{2r+1}}} \quad \text{in the odd case.}$$

For an interesting application of the Lorentz-system *see* ([30]).

(*g*) *R. A. Fisher.*

Independently from Esscher, Fisher derived his system approximately at the same time ([31] [32]). He derives his polynomials T_r, where X is measured from the mean value, so that $\Sigma T_r^2 = 1$, i.e., if the arbitrary constant in (10′) is chosen equal to $1/\sqrt{\binom{2r}{r}\binom{n+r}{2r+1}}$ and a is put equal to $-\frac{1}{2}(n-1)$, we have, e.g., for the first four explicit values

$$T_0 = \frac{1}{\sqrt{n}}$$

$$T_1 = \sqrt{\frac{12}{n(n^2-1)}}\; x$$

$$T_2 = \sqrt{\frac{180}{n(n^2-1)(n^2-4)}}\; [x^2 - \tfrac{1}{12}(n^2-1)]$$

$$T_3 = \sqrt{\frac{2800}{n(n^2-1)(n^2-4)(n^2-9)}}\; [x^3 - \tfrac{1}{20}x(3n^2-7)]$$

Later ([33]) by utilizing the convention that the coefficient of x^r shall be unity, i.e., in our notation C becoming equal to $r!/\binom{2r}{r}$, he presents his polynomials in a new form, the general expression of which is derived by Allan ([36]) as

$$\xi r = \frac{r!}{(r-\frac{1}{2})!}\left[\frac{n}{2}\right]^r \xi_1 \sum_{q=o} \frac{(-)^q(r-q-\frac{1}{2})!}{(r-2q)!\; q!\; 2^{2q}}\; \frac{[\xi_1]^{r-2q-1}}{[\frac{n}{2}]^{r-2q}}$$

where $[x]^n$ corresponds to $x^{[n]}$ in our notation, and where the series terminate in $\frac{1}{2}(r+1)$ or $\frac{1}{2}(r+2)$ terms. The first few polynomials become

$$\xi_0 = 1$$
$$\xi_1 = x$$
$$\xi_2 = \xi_1^2 - {}^1/_{12}(n^2-1)$$
$$\xi_3 = \xi_1^3 - {}^1/_{20}(3n^2-7)\,\xi_1$$
$$\xi_4 = \xi_1^4 - {}^1/_{14}(3n^2-13)\,\xi_1^2 + {}^3/_{560}(n^2-1)(n^2-9)$$

and $$\Sigma\xi_r^2 = \frac{(r!)^4}{(2r)!\;(2r+1)!}\; n(n^2-1)(n^2-4)\ldots(n^2-r^2)$$

In ([35]) standard tables of the polynomial values for n, from 4 to 52, and r, from 1 to 5, with common factors λ removed, are presented, while in ([34]) this method is elucidated. Thus $\xi'_r(x) = \frac{1}{\lambda}\xi_r(x)$. The application of the last procedure is well illustrated in ([37]).

(*h*) *A. C. Aitken.*

With the appearance of the work of Aitken ([38], [39], [40], [41]), nearly a century's search after suitable practical methods can be said to have been completed. By deriving new forms for the orthogonal polynomials in terms of central and mean central factorials the relationship between the theory of interpolation and the theory of orthogonal polynomial curve fitting is shown in a remarkably lucid way. The choice of C equal to 1 allows the double utilization of his tables of the central and mean central, and terminal values and differences, and gives integers throughout the standard tables.

Practical Example.

To illustrate the method of curve fitting, enunciated in the previous sections, we choose the average diurnal variation of atmospheric temperature during December, 1940, at Armoedsvlakte, Bechuanaland. Our aim is to express the temperature (Y) in terms of a linear combination of orthogonal polynomials (φ_r) of the time of the day (X, or x if measured from the midpoint of the data). This operation will yield—

(i) the regression equation between Y and X.

(ii) the smoothed values of the atmospheric temperature;

(iii) statistics, independently summarizing certain aspects of the diurnal variation of air temperature.

Table 3.

Example of Curve Fitting: Air Temperature.

Hour.	X.	x.	Observation.	Combinations.		φ_1.	φ_3.	φ_5.	Check. $\varphi_1+\varphi_3+\varphi_5$.
6 a.m.	1	−11·5	65·25	1−24	− 0·63	− 23	−1771	−4807	−6601
7	2	−10·5	69·37	2−23	2·66	− 21	− 847	1463	595
8	3	− 9·5	74·44	3−22	7·35	− 19	− 133	3743	3591
9	4	− 8·5	79·00	4−21	10·88	− 17	391	3553	3927
10	5	− 7·5	82·72	5−20	13·64	− 15	745	2071	2801
11	6	− 6·5	85·97	7−19	15·40	− 13	949	169	1105
12	7	− 5·5	88·11	7−18	16·21	− 11	1023	−1551	− 539
1	8	− 4·5	89·67	8−17	16·54	− 9	987	−2721	−1743
2	9	− 3·5	90·36	9−16	16·29	− 7	861	−3171	−2317
3	10	− 2·5	90·35	10−15	14·20	− 5	665	−2893	−2233
4	11	− 1·5	88·92	11−14	8·51	− 3	419	−2005	−1589
5	12	− 0·5	87·81	12−13	2·86	− 1	143	− 715	− 573
6	13	0·5	84·95	Total.	123·91	φ_2	φ_4	φ_6	$\varphi_2+\varphi_4+\varphi_6$
7	14	1·5	80·41	1+24	131·13	253	253	4807	5313
8	15	2·5	76·15	2+23	136·08	187	33	−3971	−3751
9	16	3·5	74·07	3+22	141·53	127	− 97	−4769	−4739
10	17	4·5	73·13	4+21	147·12	73	− 157	−2147	−2231
11	18	5·5	71·90	5+20	151·80	25	− 165	1045	905
12	19	6·5	70·57	7+19	156·54	− 17	− 137	3271	3117
1 a.m.	20	7·5	69·08	7+18	160·01	− 53	− 87	3957	3817
2	21	8·5	68·12	8+17	162·80	− 83	− 27	3183	3073
3	22	9·5	67·09	9+16	164·43	− 107	33	1419	1345
4	23	10·5	66·71	10+15	166·50	− 125	85	− 695	− 735
5	24	11·5	65·88	11+14	169·33	− 137	123	−2525	−2539
				12+13	172·76	− 143	143	−3575	−3575
		Total.	1,860·03	Total.	1,860·03				

Step 1.—Combine the observations as shown in column 5, Table 3. These can be checked by the relation—

Sums + Differences = 2 (Sum of first half of original observations), i.e., 123·91 + 1,860·03 = 2 (919·97).

Step 2.—Multiply these sums and differences respectively by the values of the even and odd degrees of φ_r, taken from the standard tables (Appendix I), and enter the sums of products in Table 4. The sums of products can be checked by the relations

$$\Sigma Y\varphi_1 + \Sigma Y\varphi_3 + \Sigma Y\varphi_5 = \Sigma Y\ (\varphi_1 + \varphi_3 + \varphi_5)$$

and $$\Sigma Y\varphi_2 + \Sigma Y\varphi_4 + \Sigma Y\varphi_6 = \Sigma Y\ (\varphi_2 + \varphi_4 + \varphi_6)$$

TABLE 4.

Evaluation of the Constants.

s	$\Sigma Y\ \varphi_r$	$\Sigma\ \varphi_r^2$	a_r'	Sums of Squares.
0	1,860·03	24	77·501250	144,154·650
1	— 1,311·37	4,600	— 0·285080	373·846
2	— 19,831·59	394,680	— 0·050247	996·483
3	87,266·81	17,760,600	0·004914	428·786
4	1,141·79	394,680	0·002893	3·303
5	— 78,051·97	177,928,920	— 0·000439	34·239
6	— 6,396·25	250,925,400	— 0·000003	0·163

Step 3.—Divide the sums of products $\Sigma Y\varphi_r$ by $\Sigma\varphi_r^2$, taken from standard tables, giving the a'_r values, column 4. Divide the square of $\Sigma Y\varphi_r$ by $\Sigma\varphi_r^2$, giving column 5, the sums of squares due to each term fitted.

TABLE 5.

Analysis of Variance.

Variance due to.	D.F.	S.S.	M.S.	F.	5 Per Cent.	1 Per Cent.
Total........................	23	1,849·395	—	—	—	—
Degree 1....................	1	373·846	373·846	5·57	4·30	7·94
Residual 1..................	22	1,475·549	67·070	S.	—	—
Degree 2....................	1	996·483	996·483	43·68	4·32	8·02
Residual 2..................	21	479·066	22·813	S.S.	—	—
Degree 3....................	1	428·786	428·786	170·56	4·35	8·10
Residual 3..................	20	50·280	2·514	S.S.	—	—
Degree 4....................	1	3·303	3·303	1·34	4·38	8·18
Residual 4..................	19	46·977	2·473	—	—	—
Degree 5....................	1	34·239	34·239	48·38	4·41	8·28
Residual 5..................	18	12·738	0·708	S.S.	—	—
Degree 6....................	1	0·163	0·163	0·22	4·45	8·40
Residual 6..................	17	12·575	0·740	—	—	—

Step 4.—From the sums of squares in Table 4 it appears that a 5–*th* degree polynomial will give a satisfactory fit, and thus, before calculating any further coefficients, a significance test by means of the analysis of variance is applied, so as to establish whether the 5–*th* degree is appropriate. This procedure is set out in Table 5. Squaring and summing the original values and subtracting the first entry in column 5, Table 4, the total S.S. value in Table 5, 1,849·395, with $(N - 1) = 23$ degrees of freedom is obtained. Subtraction of the S.S. value for the first degree from this value yields the first residual. For the second residual subtract the S.S. for the second degree from the first residual, and so on. The mean square (M.S.) column is obtained by dividing the S.S. column by the D.F. column, and the F-column is the result of dividing the M.S. for residual into the M.S. for degree. These F-values are compared with the theoretical values [35] in order to test significance. *S* denotes significance at 5 per cent. theoretical, while S.S. denotes significance at the 1 per cent. level. In this way significance is established for the 1st, 2nd, 3rd and 5th degrees, while the 4th degree does not attain the significance level.

This can be seen at a glance when expression (29) is utilized, giving

$$y^1 = -\,0{\cdot}450\,\psi_1 - 0{\cdot}734\,\psi_2 + 0{\cdot}482\psi_3 + 0{\cdot}042\,\psi_4 - 0{\cdot}136\,\psi_5.$$

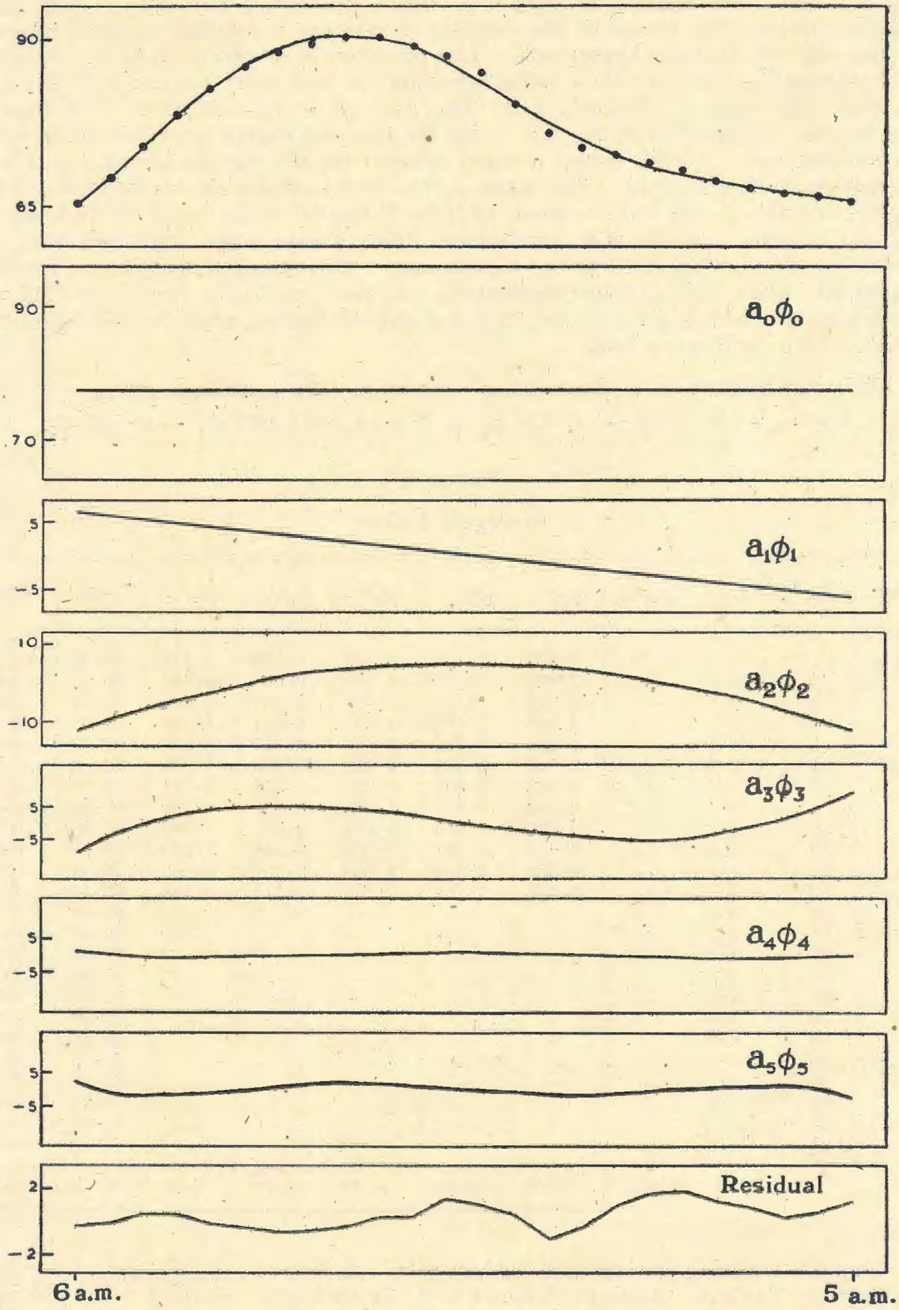
Table 6.

Graduated Values.

Hour.	X.	x.	$\varphi_0 a'_0$.	$\varphi_1 a'_1$.	$\varphi_2 a'_2$.	$\varphi_3 a'_3$.	$\varphi_4 a'_4$.	$\varphi_5 a'_5$.	Total.	Diff.
6 a.m.	1	−11·5	77·501	6·557	−12·712	−8·703	0·732	2·110	65·49	−0·24
7	2	−10·5	—	5·987	− 9·396	−4·162	0·095	−0·642	69·38	−0·01
8	3	− 9·5	—	5·417	− 6·381	−0·654	−0·281	−1·643	73·96	0·48
9	4	− 8·5	—	4·846	− 3·668	1·921	−0·454	−1·560	78·59	0·41
10	5	− 7·5	—	4·276	−1·256	3·661	−0·477	−0·909	82·80	−0·08
11	6	− 6·5	—	3·706	0·854	4·663	−0·396	−0·074	86·25	−0·28
12	7	− 5·5	—	3·136	2·663	5·027	−0·252	0·681	88·76	−0·65
1	8	− 4·5	—	2·566	4·171	4·850	−0·078	1·195	90·21	−0·54
2	9	− 3·5	—	1·996	5·376	4·231	0·095	1·392	90·60	−0·23
3	10	− 2·5	—	1·425	6·281	3·268	0·246	1·270	89·99	0·36
4	11	− 1·5	—	0·855	6·884	2·059	0·356	0·880	88·54	0·38
5	12	− 0·5	—	0·285	7·185	0·703	0·414	0·314	86·40	1·41
6	13	0·5	—	—	—	—	—	—	83·80	1·15
7	14	1·5	—	—	—	—	—	—	80·95	−0·54
8	15	2·5	—	—	—	—	—	—	78·06	−1·91
9	16	3·5	—	—	—	—	—	—	75·34	−1·28
10	17	4·5	—	—	—	—	—	—	72·98	0·15
11	18	5·5	—	—	—	—	—	—	71·07	0·83
12	19	6·5	—	—	—	—	—	—	69·66	0·91
1 a.m.	20	7·5	—	—	—	—	—	—	68·75	0·34
2	21	8·5	—	—	—	—	—	—	68·17	−0·05
3	22	9·5	—	—	—	—	—	—	67·71	−0·62
4	23	10·5	—	—	—	—	—	—	67·01	−0·31
5	24	11·5	—	—	—	—	—	—	65·56	0·32
			TOTAL.	0·00	0·00	0·00	0·00	0·00	1,860·03	0·00

Step 5.—Utilizing the standard tables again, calculate $a'_0\ \varphi_0$, $a'_1\ \varphi_1, \ldots, a'_5\,\varphi_5$ as shown in Table 6. Adding columns 4 to 9, the graduated values of Y are obtained. Because of symmetry, only the first half of the products is calculated. The graduated values of the second half is obtained by simply changing the sign of the odd degrees, leaving the signs of the even degrees unaltered, and adding. The total of the graduated

FIGURE 3.



Diurnal Variation of Air Temperature, December 1940 at Armoedsvlakte, analyzed into its orthogonal components. The circles represent the actual observations, while the smooth curve results from the addition of the components.

values should be equal to the total of the original observations. Column 11 gives the differences between the original and graduated values, the sums of squares of which should be equal to the S.S. of the 5–*th* residual in Table 5. Figure 3 shows the original values with the 5–*th* degree curve, as well as the different components.

Step 6.—The regression equation can now be written

$$Y' = 77 \cdot 501 - 0 \cdot 2851\, \varphi_1 - 0 \cdot 0502\, \varphi_2 + 0 \cdot 0049\, \varphi_3 + 0 \cdot 0029\, \varphi_4 - 0 \cdot 0004\, \varphi_5$$

or, utilizing (24) and (26), i.e., dividing each term by the common factor of the standard tables, it can be written in terms of Aitken's polynomials.

$$Y' = 77 \cdot 501 - 0 \cdot 2851\, \hat{T}_1 - 0 \cdot 0502\, \hat{T}_2 + 0 \cdot 0049\, \hat{T}_3 + 0 \cdot 00014\, \hat{T}_4 - 0 \cdot 00008\, \hat{T}_5$$

Using Table 2 and substituting for $\hat{T}_r$, we find

$$Y' = 85 \cdot 145 - 2 \cdot 613\, x - 0 \cdot 180\, x^2 + 0 \cdot 037\, x^3 + 0 \cdot 00024\, x^4 - 0 \cdot 00013\, x^5$$

and, remembering that $x = X - \overline{X}$, i.e., $x = X - 12 \cdot 5$, we finally have

$$Y' = 63 \cdot 062 + 1 \cdot 370\, X + 1 \cdot 223\, X^2 - 0 \cdot 181\, X^3 + 0 \cdot 0085\, X^4 - 0 \cdot 00013\, X^5$$

giving the relationship between air temperature and time as an ordinary power polynomial of the 5–*th* degree.

Remarks.—From this application it follows that the use of this method not only yields the graduated values, but at the same time, certain statistics, the coefficients, each of which represents some aspect of the temperature distribution in time. For instance, a'_0 is the average temperature, a'_1 is a measure of the average linear rate of increase, a'_2 is a function of the parabolic rate of increase, etc., or, a'_1 is the regression coefficient between air temperature and a certain combination of first degree time values, etc. In a forthcoming article this interpretational aspect will be fully treated.

13. Summary.

The systems of orthogonal polynomials mainly used in practice are derived from a common general formula, which is established by the principle of least squares, utilizing results from the Finite Calculus. A simplified method of utilizing the Aitken–Chebyshev polynomials, by means of an extensive set of appended standard tables, is presented.

14. Acknowledgments.

I wish to acknowledge :—

(1) the encouragement of this type of research by the Director, Dr. P. J. du Toit, and the Deputy-Director of Veterinary Services, Dr. Gilles de Kock;

(2) the friendly co-operation of Prof. J. H. R. Bisschop, of Onderstepoort, who kindly placed his data at my disposal;

(3) the willing assistance of Prof. B. de Loor, Professor of Statistics at the University of Pretoria, especially in regard to the supply of literature.

(4) the excellent services rendered by the Statistical Assistant, Mr. D. F. I. van Heerden, who was solely responsible for the calculation and checking of the appended standard tables.

REFERENCES.

(1) WALTHER, A. " Einführung in die mathematische Behandlung naturwissenschaftlicher Fragen ", Methodik der Wissenschaftlichen Biologie, Vol. I (1928), pp. 1–202.

(2) MILNE-THOMSON, L. M. The Calculus of Finite Differences. London : Macmillan & Co., 1933.

(3) STEFFENSEN, J. F. Interpolation. Baltimore : Williams & Wilkins Co., 1927.

(4) WHITTAKER, E. T. and ROBINSON, G. The Calculus of Observations. London : Blackie & Son, Ltd.

(5) JACKSON, D. The Theory of Approximation. Amer. Math. Soc. Colloq. Publ., Vol. 23 (1939), pp. v + 178.

(6) SZEGO, GABOR. Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ., Vol. 23 (1939), pp. vii + 401.

(7) SHOHAT, J. Théorie générale des polynomes orthogonaux de Tchebicheff. Mem. des Sci. Math., Fasc. 66 (1936), Paris : Gauthier-Villars.

(8) CHEBYSHEV, P. L. (1855). " Sur les fractions continues ", Oeuvres, Vol. 1, pp. 201–230.

(9) CHEBYSHEV, P. L. (1854). " Sur une formule d'analyse ", Oeuvres, Vol. 1, pp. 701–770.

(10) CHEBYSHEV, P. L. (1858). " Sur une nouvelle série ", Oeuvres, Vol. 1, pp. 381–384.

(11) CHEBYSHEV, P. L. (1859). " Sur l'interpolation par la méthode des moindres carrés ", Oeuvres, Vol. 1, pp. 471–498.

(12) CHEBYSHEV, P. L. (1864). " Sur l'interpolation ", Oeuvres, Vol. 1, pp. 539–560.

(13) CHEBYSHEV, P. L. (1875). " Sur l'interpolation des valeurs equidistantes ", Oeuvres, Vol. 2, pp. 219–242.

(14) POINCARE, H. (1896). Calcul des Probabilités. Paris : 1912.

(15) QUIQUET, A. (1912). " Sur une méthode d'interpolation ", Proc. Intern. Congr. Math., Cambridge (Engl.), Vol. 2 (1913), pp. 385–388.

(16) GRAM, J. P. " Uber partielle Ausgleichung mittelst Orthogonalfunktionen ", Mitt. Ver. schweiz. Versich. Math., Heft 10 (1915).

(17) GRAM, J. P. " Uber die Entwicklung reeler Funktionen in Reihen ", Jour. für Math., Vol. 94 (1883), pp. 41–73.

(18) JORDAN, C. " Sur une série de polynomes, etc.", Proc. London Math. Soc., 2nd Ser., Vol. 20 (1921), pp. 297–325.

(19) JORDAN, C. " Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate ", Mitt. ungarischen Landeskom. für Wirtschaftsstat. und Konjunkturforsch., Budapest, 1930.

(20) JORDAN, C. " Sur la détermination de la tendence séculaire des grandeurs statistiques par la méthode des moindres carrés ", Jour. de la Soc. Hongr. de Stat., Vol. 7 (1929), pp. 567.

(22) JORDAN, C. " Statistique Mathématique, Paris : 1927.

(22) JORDAN, C. " Approximation and graduation according to the Principle of least Squares by orthogonal polynomials ". Ann. Math. Stat., Vol. 3 (1932), pp. 257–333.

(23) SIPOS, A. " Praktische Anwendung der Trendberechnungsmethode von Jordan ", Mitt. ungarischen Landeskom., No. II (1930).

([24]) SIPOS, A. "Practical application of Jordan's method for trend measurement", Report Hungarian Nat. Comm., 1930.

([25]) ESSCHER, M. F. "Uber die Sterblichkeit in Schweden (1886–1914)", Medd. fran Lunds Astron. Obser., Ser. 2, No. 23 (1920).

([26]) ESSCHER, M. F. "On graduation according to the Method of Least Squares by means of certain polynomials", Försakrings-aktiebolaget Skand., Festskrift, 1930, Stockholm.

([27]) LORENZ, P. Der Trend. Vierteljahrshefte zur Konj., Sonderheft 9 (1928).

([28]) LORENZ, P. Der Trend. Vierteljahrshefte zur Konj., Sonderheft 21 (1931).

([29]) HENNIG, H. Die Analyse von Wirtschaftskurven. Vierteljahrshefte zur Konj., Sonderheft 4 (1927).

([30]) SCHUMACHER, F. X. and MEYER, H. A. "Effect of Climate on Timber-Growth Fluctuations", Jour. Agr. Res., Vol. 54 (1937), pp. 79–108.

([31]) FISHER, R. A. "Studies in Crop Variation I. An examination of the yield of dressed grain from Broadbalk", Jour. Agr. Sci., Vol. 11 (1921), pp. 107–135.

([32]) FISHER, R. A. "The Influence of Rainfall on the yield of wheat at Rothamsted". Phil. Trans. Roy. Soc., London, Ser. B., Vol. 213 (1924), pp. 96–113.

([33]) FISHER, R. A. "Statistical Methods for Research Workers. Edinburgh: Oliver & Boyd, Editions 1 to 7.

([34]) FISHER, R. A. "Statistical Methods for Research Workers. Edinburgh: Oliver & Boyd, 8th edition, 1941.

([35]) FISHER, R. A. and YATES, F. Statistical Tables for Biol., Agric., and Med., Research. Edinburgh: Oliver & Boyd, 1938.

([36]) ALLAN, F. E. "The general form of the orthogonal polynomials for Simple Series with proofs of their Simple Properties". Proc. Roy. Soc. Edinb., Vol. 50 (1929–30), pp. 310–320.

([37]) DAVIS, F. E. and PALLESEN, J. E. "Effect of the amount and Distribution of Rainfall and Evaporation during the growing season on Yields of Corn and Spring Wheat". Jour. Agr. Res., Vol. 60 (1940), pp. 1–24.

([38]) AITKEN, A. C. "On the Graduation of Data by the Orthogonal Polynomials of Least Squares". Proc. Roy. Soc., Edinb., Vol. 53 (1932–33), pp. 54–78.

([39]) AITKEN, A. C. "On fitting polynomials to weighted data by Least Squares", Proc. Roy. Soc., Edinb., Vol. 54 (1933–34), pp. 1–11.

([40]) AITKEN, A. C. "On fitting polynomials to data with Weighted and Correlated Errors", Proc. Roy. Soc., Edinb., Vol. 54 (1933–34), pp. 12–16.

([41]) AITKEN, A. C. Statistical Mathematics. Edinburgh: Oliver & Boyd, 1939.

([42]) WISHART, J. "Growth-rate Determinations in Nutrition Studies with the Bacon Pig, and their Analysis", Biometrika, Vol. 30 (1938), pp. 16–28.

Appendix I.

Standard Tables.

In the following pages the values of the orthogonal polynomials are given for n, from 5 to 52, and for r, from 1 to 9. Because φ_r is symmetrical in absolute value about the centre of the date, the same signs occurring in the upper and lower portions of the table when r is even, opposite signs when r is odd, *only the first half* of the tables is given. In the first row at the base of the tables the sums of squares are given. Since these figures tend to become very large with large n and r, they have been abbreviated to 10 significant figures, e.g., for $n = 42$, $r = 8$, the sums of squares is 563, 270, 101, 173, 780; this is abbreviated to 10 figures and is denoted as 563, 270, 101, 2 (5), the 5 between brackets denoting that 5 figures have been omitted. In the second row the common factor eliminated, is given. Below follows an example of the standard tables. For completeness sake, all the values are given in this example :—

$n = 11.$

X	x	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7
1	− 5	− 5	15	− 30	6	− 3	15	− 5
2	− 4	− 4	6	6	− 6	6	− 48	23
3	− 3	− 3	− 1	22	− 6	1	29	− 33
4	− 2	− 2	− 6	23	− 1	− 4	36	2
5	− 1	− 1	− 9	14	4	− 4	− 12	28
6	0	0	− 10	0	6	0	− 40	0
7	1	1	− 9	− 14	4	4	− 12	− 28
8	2	2	− 6	− 23	− 1	4	36	− 2
9	3	3	− 1	− 22	− 6	1	29	33
10	4	4	6	− 6	− 6	− 6	− 48	− 23
11	5	5	15	30	6	3	15	5
	$\Sigma \varphi r^2$	110	858	4,290	286	156	11,220	4,862
	λ	2	3	4	35	84	14	24

5			6				7				
φ_1	φ_2	φ_3	φ_1	φ_2	φ_3	φ_4	φ_1	φ_2	φ_3	φ_4	φ_5
−2	2	−1	−5	5	−5	1	−3	5	−1	3	−1
−1	−1	2	−3	−1	7	−3	−2	0	1	−7	4
0	−2	0	−1	−4	4	2	−1	−3	1	1	−5
							0	−4	0	6	0
10	14	10	70	84	180	28	28	84	6	154	84
2	3	4	1	2	2	5	2	3	20	5	6

8						9						
φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7
−7	7	−7	7	−7	1	− 4	28	−14	14	− 4	4	− 1
−5	1	5	−13	23	−5	− 3	7	7	−21	11	− 17	6
−3	−3	7	−3	−17	9	− 2	− 8	13	−11	− 4	22	−14
−1	−5	3	9	−15	−5	− 1	−17	9	9	− 9	1	14
						0	−20	0	18	0	−20	0
168	168	264	616	2,184	264	60	2,772	990	2,002	468	1,980	858
1	3	5	5	3	7	2	1	4	5	14	7	8

10

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7
−9	6	−42	18	−6	3	−9
−7	2	14	−22	14	−11	47
−5	−1	35	−17	−1	10	−86
−3	−3	31	3	−11	6	42
−1	−4	12	18	−6	−8	56
330	132	8,580	2,860	780	660	29,172
1	6	2	7	21	28	4

11

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7
−5	15	−30	6	−3	15	−5
−4	6	6	−6	6	−48	23
−3	−1	22	−6	1	29	−33
−2	−6	23	−1	−4	36	2
−1	−9	14	4	−4	−12	28
0	−10	0	6	0	−40	0
110	858	4,290	286	156	11,220	4,862
2	3	4	35	84	14	24

12

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
−11	55	−33	33	−33	11	−55	11
−9	25	3	−27	57	−31	225	−61
−7	1	21	−33	21	11	−251	119
−5	−17	25	−13	−29	25	−83	−65
−3	−29	19	12	−44	4	204	−74
−1	−35	7	28	−20	−20	140	70
572	12,012	5,148	8,008	15,912	4,488	369,512	65,208
1	1	5	10	14	42	6	15

13

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
−6	22	−11	99	−22	22	−33	11
−5	11	0	−66	33	−55	121	−55
−4	2	6	−96	18	8	−103	89
−3	−5	8	−54	−11	43	−75	−19
−2	−10	7	11	−26	22	65	−71
−1	−13	4	64	−20	−20	100	10
0	−14	0	84	0	−40	0	70
182	2,002	572	68,068	6,188	14,212	92,378	38,038
2	3	20	5	36	42	24	45

14

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−13	13	− 143	143	− 143	143	− 143	13	− 13
−11	7	− 11	− 77	187	− 319	473	− 59	77
− 9	2	66	− 132	132	− 11	− 297	79	− 163
− 7	− 2	98	− 92	− 28	227	− 353	7	107
− 5	− 5	95	− 13	− 139	185	95	− 65	89
− 3	− 7	67	63	− 145	− 25	375	− 25	− 105
− 1	− 8	24	108	− 60	− 200	200	50	− 90
910	728	97,240	136,136	235,144	497,420	1,293,292	34,580	142,324
1	6	2	5	9	12	12	99	55

15

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 7	91	− 91	1,001	− 1,001	143	− 13	91	− 91
− 6	52	− 13	− 429	1,144	− 286	39	− 377	494
− 5	19	35	− 869	979	− 55	− 17	415	− 901
− 4	− 8	58	− 704	44	176	− 31	157	344
− 3	−29	61	− 249	− 751	197	− 3	− 311	659
− 2	−44	49	251	− 1,000	50	25	− 275	− 250
− 1	−53	27	621	− 675	− 125	25	125	− 675
0	−56	0	756	0	− 200	0	350	0
280	37,128	39,780	6,466,460	10,581,480	426,360	8,398	1,193,010	4,269,720
2	1	4	1	2	21	264	33	22

16

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−15	35	− 455	273	− 143	65	− 195	65	− 91
−13	21	− 91	− 91	143	− 117	533	− 247	455
−11	9	143	− 221	143	− 39	− 143	221	− 715
− 9	− 1	267	− 201	33	59	− 423	149	95
− 7	− 9	301	− 101	− 77	87	− 157	− 133	575
− 5	−15	265	23	− 131	45	235	− 205	53
− 3	−19	179	129	− 115	− 25	375	− 25	− 505
− 1	−21	63	189	− 45	− 75	175	175	− 315
1,360	5,712	1,007,760	470,288	201,552	77,520	1,545,232	454,480	2,846,480
1	3	1	5	21	77	33	99	55

17

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−8	40	−28	52	−104	104	−130	26	−8
−7	25	−7	−13	91	−169	325	−91	37
−6	12	7	−39	104	−78	−39	65	−50
−5	1	15	−39	39	65	−247	65	−5
−4	−8	18	−24	−36	128	−149	−25	40
−3	−15	17	−3	83	93	75	−73	19
−2	−20	13	17	−88	2	215	−37	−26
−1	−23	7	31	−55	−85	175	35	−35
0	−24	0	36	0	−120	0	70	0
408	7,752	3,876	16,796	100,776	178,296	579,462	56,810	15,640
2	3	20	35	42	77	88	495	1,430

18

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−17	68	−68	68	−884	442	−442	34	−34
−15	44	−20	−12	676	−650	1,014	−110	146
−13	23	13	−47	871	−377	13	61	−169
−11	5	33	−51	429	169	−715	85	−55
−9	−10	42	−36	−156	481	−585	−5	125
−7	−22	42	−12	−588	439	31	−77	107
−5	−31	35	13	−733	145	563	−65	−43
−3	−37	23	33	−583	−209	651	7	−133
−1	−40	8	44	−22[illegible]	−440	280	70	−70
1,938	23,256	23,256	28,424	6,953,544	2,941,884	5,794,620	78,660	211,140
1	2	10	35	7	28	44	715	715

19

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−9	51	−204	612	−102	1,326	−306	34	−34
−8	34	−68	−68	68	−1,768	646	−102	136
−7	19	28	−388	98	−1,222	86	42	−134
−6	6	89	−453	58	234	−411	81	−74
−5	−5	120	−354	−3	1,235	−425	15	85
−4	−14	126	−168	−54	1,352	−97	−57	112
−3	−21	112	42	−79	729	267	−69	7
−2	−26	83	227	−74	−214	427	−21	−98
−1	−29	44	352	−44	−1,012	308	42	−98
0	−30	0	396	[illegible]	−1,320	0	70	0
570	13,566	213,180	2,288,132	89,148	24,515,700	2,451,570	65,550	164,220
2	3	4	5	84	14	104	1,287	1,430

20

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−19	57	− 969	1,938	− 1,938	1,938	− 646	646	− 646
−17	39	− 357	− 102	1,122	− 2,346	1,258	− 1,802	2,414
−15	23	85	− 1,122	1,802	− 1,870	306	510	− 2,006
−13	9	377	− 1,402	1,222	6	− 702	1,422	− 1,586
−11	− 3	539	− 1,187	187	1,497	− 891	549	979
− 9	−13	591	− 687	− 771	1,931	− 387	− 723	1,993
− 7	−21	553	− 77	− 1,351	1,353	321	− 1,239	763
− 5	−27	445	503	− 1,441	195	777	− 735	− 1,127
− 3	−31	287	948	− 1,076	− 988	756	294	− 1,862
− 1	−33	99	1,188	− 396	− 1,716	308	1,078	− 882
2,660	17,556	4,903,140	22,881,320	31,201,800	49,031,400	9,806,280	20,189,400	47,623,800
1	3	1	2	6	14	78	117	143

21

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 10	190	− 285	969	− 3,876	6,460	− 3,230	3,230	− 1,292
− 9	133	− 114	0	1,938	− 7,106	5,814	− 8,398	4,522
− 8	82	12	− 510	3,468	− 6,392	2,006	1,394	− 3,128
− 7	37	98	− 680	2,618	− 918	− 2,754	6,426	− 3,298
− 6	− 2	149	− 615	788	3,996	− 4,266	3,618	932
− 5	− 35	170	− 406	− 1,063	6,075	− 2,565	− 2,025	3,479
− 4	− 62	166	− 130	− 2,654	5,088	543	− 5,421	2,264
− 3	− 83	142	150	− 2,819	2,001	3,087	− 4,557	− 931
− 2	− 98	103	385	− 2,444	− 1,716	3,822	− 588	− 3,136
− 1	−107	54	540	− 1,404	− 4,628	2,548	3,626	− 2,646
0	−110	0	594	0	− 5,720	0	5,390	0
770	201,894	432,630	5,720,330	121,687,020	514,829,700	223,092,870	439,119,450	157,158,540
2	1	4	5	4	6	24	39	130

22

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 21	35	− 133	1,197	− 2,261	646	− 9,690	4,522	− 4,522
− 19	25	− 57	57	969	− 646	16,150	− 10,982	14,858
− 17	16	0	− 570	1,938	− 646	7,106	646	− 8,398
− 15	8	40	− 810	1,598	− 170	− 6,222	7,990	− 11,458
− 13	1	65	− 775	663	306	− 11,934	5,814	442
− 11	− 5	77	− 563	− 363	558	− 8,910	− 954	10,054
− 9	− 10	78	− 258	− 1,158	537	− 1,035	− 6,231	9,139
− 7	− 14	70	70	− 1,554	303	6,717	− 6,783	469
− 5	− 17	55	365	− 1,509	− 30	10,626	− 2,940	− 8,036
− 3	− 19	35	585	− 1,079	− 338	9,282	2,548	− 9,996
− 1	− 20	12	702	− 390	− 520	3,640	6,370	− 4,410
3,542	7,084	96,140	8,748,740	40,562,340	4,903,140	1,848,483,780	761,140,380	1,623,971,580
1	6	10	5	9	84	12	45	65

23

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 11	77	− 77	1,463	− 209	3,553	− 7,106	7,106	− 24,871
− 10	56	− 35	133	76	− 3,230	10,982	− 16,150	76,874
− 9	37	− 3	− 627	171	− 3,553	5,814	− 646	− 34,561
− 8	20	20	− 950	152	− 1,292	− 3,230	10,982	− 60,724
− 7	5	35	− 955	77	1,207	− 7,990	9,758	− 9,469
− 6	− 8	43	− 747	− 12	2,754	− 7,038	918	43,282
− 5	− 19	45	− 417	− 87	2,985	− 2,334	− 7,530	51,637
− 4	− 28	42	− 42	− 132	2,076	3,117	− 10,383	17,752
− 3	− 35	35	315	− 141	501	6,750	− 6,816	− 27,188
− 2	− 40	25	605	− 116	− 1,166	7,210	476	− 50,568
− 1	− 43	13	793	− 65	− 2,405	4,550	7,280	− 38,220
0	− 44	0	858	0	− 2,860	0	10,010	0
1,012	35,420	32,890	13,123,110	340,860	142,191,060	924,241,890	1,685,382,270	42,223,261,08 (1)
2	3	20	5	126	21	24	45	20

24

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 23	253	− 1,771	253	− 4,807	4,807	− 81,719	81,719	− 7,429
− 21	187	− 847	33	1,463	− 3,971	117,249	− 174,097	21,641
− 19	127	− 133	− 97	3,743	− 4,769	72,029	− 22,933	− 7,429
− 17	73	391	− 157	3,553	− 2,147	− 23,579	108,851	− 17,119
− 15	25	745	− 165	2,071	1,045	− 82,365	114,665	− 5,491
− 13	− 17	949	− 137	169	3,271	− 83,317	32,657	9,503
− 11	− 53	1,023	− 87	− 1,551	3,957	− 40,953	− 61,251	14,773
− 9	− 83	987	− 27	− 2,721	3,183	16,767	− 109,419	8,383
− 7	−107	861	33	− 3,171	1,419	63,093	− 92,652	− 3,452
− 5	−125	665	85	− 2,893	− 695	80,845	− 27,340	− 12,372
− 3	−137	419	123	− 2,005	− 2,525	65,625	49,700	− 13,020
− 1	−143	143	143	− 715	− 3,575	25,025	100,100	− 5,460
4,600	394,680	17,760,600	394,680	177,928,920	250,925,400	114,605,994,4 (2)	202,245,872,4 (2)	3,290,124,240
1	1	1	35	7	21	3	6	110

25

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 12	92	− 506	1,518	− 1,012	19,228	− 14,421	22,287	− 1,748
− 11	69	− 253	253	253	− 14,421	19,228	− 44,574	4,807
− 10	48	− 55	− 517	748	− 18,810	13,376	− 9,690	− 1,178
− 9	29	93	− 897	753	− 9,899	− 2,052	25,194	− 3,743
− 8	12	196	− 982	488	2,052	− 12,806	31,008	− 1,748
− 7	− 3	259	− 857	119	11,229	− 14,668	13,566	1,501
− 6	− 16	287	− 597	− 236	15,142	− 9,096	− 10,302	3,166
− 5	− 27	285	− 267	− 501	13,635	− 12	− 26,010	2,411
− 4	− 36	258	78	− 636	8,028	8,409	− 26,793	116
− 3	− 43	211	393	− 631	391	13,092	− 14,136	− 2,124
− 2	− 48	149	643	− 500	− 7,050	12,700	4,800	− 3,000
− 1	− 51	77	803	− 275	− 12,375	7,700	21,000	− 2,100
0	− 52	0	858	0	− 14,300	0	27,300	0
1,300	53,820	1,480,050	14,307,150	7,803,900	3,889,343,700	3,370,764,540	13,789,491,30 (1)	161,280,600
2	3	4	7	42	7	24	33	748

26

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 25	50	− 1,150	2,530	− 2,530	6,325	− 10,925	10,925	− 2,185
− 23	38	− 598	506	506	− 4,301	13,547	− 20,539	5,681
− 21	27	− 161	− 759	1,771	− 6,072	10,488	− 6,118	− 874
− 19	17	171	− 1,419	1,881	− 3,608	− 152	10,298	− 4,294
− 17	8	408	− 1,614	1,326	46	− 8,398	14,744	− 2,584
− 15	0	560	− 1,470	482	3,090	− 10,830	8,360	1,064
− 13	− 7	637	− 1,099	− 377	4,672	− 7,904	− 2,242	3,458
− 11	− 13	649	− 599	− 1,067	4,624	− 1,936	− 10,594	3,278
− 9	− 18	606	− 54	− 1,482	3,231	4,329	− 13,011	1,063
− 7	− 22	518	466	− 1,582	1,033	8,641	− 9,163	− 1,647
− 5	− 25	395	905	− 1,381	− 1,340	9,740	− 1,360	− 3,312
− 3	− 27	247	1,221	− 935	− 3,300	7,500	6,800	− 3,120
− 1	− 28	84	1,386	− 330	− 4,400	2,800	11,900	− 1,260
5,850	16,380	7,803,900	40,060,020	48,384,180	409,404,600	1,838,598,840	3,064,331,400	225,792,840
1	6	2	5	21	28	44	99	935

27

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 13	325	− 130	2,990	− 16,445	1,495	− 7,475	28,405	− 28,405
− 12	250	− 70	690	2,530	− 920	8,625	− 50,255	69,920
− 11	181	− 22	− 782	10,879	− 1,403	7,337	− 18,791	− 4,807
− 10	118	15	− 1,587	12,144	− 920	736	21,850	− 50,692
− 9	61	42	− 1,872	9,174	− 122	− 4,878	36,556	− 37,012
− 8	10	60	− 1,770	4,188	592	− 7,090	24,928	4,712
− 7	− 35	70	− 1,400	− 1,162	1,018	− 5,870	532	37,772
− 6	− 74	73	− 867	− 5,728	1,096	− 2,424	− 21,698	43,244
− 5	−107	70	− 262	− 8,803	865	1,643	− 31,895	22,679
− 4	−134	62	338	− 10,058	424	4,891	− 27,287	− 9,216
− 3	−155	50	870	− 9,479	− 101	6,375	− 11,339	− 34,731
− 2	−170	35	1,285	− 7,304	− 584	5,780	8,704	− 41,616
− 1	−179	18	1,548	− 3,960	− 920	3,400	24,820	− 27,540
0	−182	0	1,638	0	− 1,040	0	30,940	0
1,638	712,530	101,790	56,448,210	2,032,135,560	22,331,160	822,531,060	19,305,287,82 (1)	34,546,304,52 (1)
2	1	20	5	4	154	88	55	110

28

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 27	117	− 585	1,755	− 13,455	13,455	− 40,365	4,485	− 85,215
− 25	91	− 325	455	1,495	− 7,475	43,355	− 7,475	198,835
− 23	67	− 115	− 395	8,395	− 12,305	40,135	− 3,335	2,185
− 21	45	49	− 879	9,821	− 8,763	7,935	2,737	− 136,781
− 19	25	171	− 1,074	7,866	− 2,162	− 21,850	5,428	− 117,116
− 17	7	255	− 1,050	4,182	4,138	− 36,074	4,276	− 9,044
− 15	− 9	305	− 870	22	8,310	− 33,162	940	91,276
− 13	− 23	325	− 590	− 3,718	9,682	− 17,914	− 2,516	125,476
− 11	− 35	319	− 259	− 6,457	8,401	2,365	− 4,559	86,317
− 9	− 45	291	81	− 7,887	5,139	20,565	− 4,551	4,247
− 7	− 53	245	395	− 7,931	841	31,457	− 2,723	− 75,843
− 5	− 59	185	655	− 6,701	− 3,485	32,521	85	− 116,433
− 3	− 63	115	840	− 4,456	− 6,936	24,072	2,788	− 101,388
− 1	− 65	39	936	− 1,560	− 8,840	8,840	4,420	− 39,780
7,308	95,004	2,103,660	19,634,160	1,354,757,040	1,771,605,360	23,030,869,68 (1)	451,585,680	284,047,392,7 (2)
1	3	5	10	6	22	22	495	55

29

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 14	126	— 819	4,095	— 8,190	26,910	— 4,485	31,395	— 62,790
— 13	99	— 468	1,170	585	— 13,455	4,485	— 49,335	139,035
— 12	74	— 182	— 780	4,810	— 23,920	4,485	— 25,415	11,960
— 11	51	44	— 1,930	5,885	— 18,285	1,265	14,605	— 89,815
— 10	30	215	— 2,441	4,958	— 6,210	— 1,955	35,305	— 88,366
— 9	11	336	— 2,460	2,946	6,026	— 3,726	31,372	— 20,884
— 8	— 6	412	— 2,120	556	14,832	— 3,754	11,584	51,416
— 7	— 21	448	— 1,540	— 1,694	18,678	— 2,414	— 11,564	86,156
— 6	— 34	449	— 825	— 3,454	17,534	— 381	— 27,827	71,846
— 5	— 45	420	— 66	— 4,521	12,375	1,635	— 31,875	22,567
— 4	— 54	366	660	— 4,818	4,752	3,063	— 23,631	— 34,808
— 3	— 61	292	1,290	— 4,373	— 3,571	3,567	— 7,127	— 74,043
— 2	— 66	203	1,775	— 3,298	— 10,914	3,077	11,407	— 79,458
— 1	— 69	104	2,080	— 1,768	— 15,912	1,768	25,636	— 50,388
0	— 70	0	2,184	0	— 17,680	0	30,940	0
2,030		4,207,320		500,671,080		274,177,020		142,023,696,4 (2)
	113,274		107,987,880		6,959,878,200		20,885,837,70 (1)	
2	3	4	5	12	14	264	99	110

30

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 29	203	— 1,827	23,751	— 16,965	5,655	— 130,065	130,065	— 70,035
— 27	161	— 1,071	7,371	585	— 2,535	121,095	— 192,855	147,315
— 25	122	— 450	— 3,744	9,360	— 4,875	130,065	— 112,125	23,115
— 23	86	46	— 10,504	11,960	— 3,965	46,345	43,355	— 88,435
— 21	53	427	— 13,749	10,535	— 1,655	— 43,815	134,435	— 98,785
— 19	23	703	— 14,249	6,821	823	— 99,199	132,779	— 36,271
— 17	— 4	884	— 12,704	2,176	2,734	— 108,698	64,952	40,664
— 15	— 28	980	— 9,744	— 2,384	3,730	— 79,386	— 24,760	86,744
— 13	— 49	1,001	— 5,929	— 6,149	3,751	— 27,313	— 96,697	84,539
— 11	— 67	957	— 1,749	— 8,679	2,937	29,271	— 126,849	42,229
— 9	— 82	858	2,376	— 9,768	1,551	74,547	— 109,677	— 16,757
— 7	— 94	714	6,096	— 9,408	— 87	97,899	— 55,461	— 65,987
— 5	—103	535	9,131	— 7,753	— 1,655	95,113	15,485	— 86,127
— 3	—109	331	11,271	— 5,083	— 2,873	68,289	79,781	— 70,737
— 1	—112	112	12,376	— 1,768	— 3,536	24,752	117,572	— 21,132
8,990		21,360,240		2,145,733,200		223,180,094,3 (2)		163,873,495,8 (2)
	302,064		3,671,587,920		302,603,400		340,140,785,4 (2)	
1	2	2	1	7	84	12	33	143

31

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 15	145	− 1,015	783	− 1,131	28,275	− 84,825	50,025	− 10,005
− 14	116	− 609	261	0	− 11,310	73,515	− 70,035	20,010
− 13	89	− 273	− 99	585	− 23,595	84,435	− 45,195	4,485
− 12	64	− 2	− 324	780	− 20,280	35,685	10,695	− 11,040
− 11	41	209	− 439	715	− 9,815	− 20,735	47,035	− 13,915
− 10	20	365	− 467	496	2,050	− 58,675	51,175	− 6,670
− 9	1	471	− 429	207	11,759	− 69,651	30,199	3,611
− 8	− 16	532	− 344	− 88	17,488	− 56,234	− 1,472	10,856
− 7	− 31	553	− 229	− 343	18,727	− 26,869	− 29,813	12,161
− 6	− 44	539	− 99	− 528	15,906	8,019	− 45,309	7,846
− 5	− 55	495	33	− 627	10,065	38,775	− 44,385	415
− 4	− 64	426	156	− 636	2,568	58,263	− 28,977	− 6,848
− 3	− 71	337	261	− 561	− 5,139	62,757	− 4,959	− 11,153
− 2	− 76	233	341	− 416	− 11,726	52,117	19,969	− 11,058
− 1	− 79	119	391	− 221	− 16,133	29,393	38,437	− 6,783
0	− 80	0	408	0	− 17,680	0	45,220	0
2,480	158,224	6,724,520	4,034,712	9,536,592	3,888,399,800	92,183,032,42 (1)	47,968,572,30 (1)	3,121,399,920
2	3	4	35	123	21	24	117	1,430

32

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 31	155	− 899	899	− 2,697	35,061	− 67,425	13,485	− 310,155
− 29	125	− 551	319	− 87	− 12,441	54,375	− 17,835	590,295
− 27	97	− 261	− 87	1,305	− 28,275	66,555	− 12,615	170,085
− 25	71	− 25	− 347	1,815	− 25,545	32,115	1,425	− 295,665
− 23	47	161	− 487	1,725	− 13,845	− 10,695	11,415	− 419,865
− 21	25	301	− 531	1,267	169	− 41,775	13,615	− 212,995
− 19	5	399	− 501	627	12,251	− 53,675	9,235	50,255
− 17	− 13	459	− 417	− 51	20,081	− 47,107	1,531	285,821
− 15	− 29	485	− 297	− 661	22,825	− 27,381	− 6,065	366,551
− 13	− 43	481	− 157	− 1,131	20,739	− 1,677	− 11,001	282,061
− 11	− 55	451	− 11	− 1,419	14,817	22,935	− 12,045	87,175
− 9	− 65	399	129	− 1,509	6,483	40,815	− 9,285	− 132,395
− 7	− 73	329	253	− 1,407	− 2,673	48,501	− 3,843	− 294,083
− 5	− 79	245	353	− 1,137	− 11,115	44,973	2,565	− 344,033
− 3	− 83	151	423	− 737	− 17,537	31,521	8,113	− 270,123
− 1	− 85	51	459	− 255	− 20,995	11,305	11,305	− 101,745
10,912	185,504	5,379,616	5,379,616	54,285,216	11,345,610,14 (1)	56,728,050,72 (1)	3,336,944,160	2,815,502,728 (3)
1	3	5	35	63	21	39	585	65

33

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 16	496	— 248	7,192	— 14,384	43,152	— 10,788	53,940	— 215,760
— 15	403	— 155	2,697	— 899	— 13,485	8,091	— 67,425	391,065
— 14	316	— 77	— 493	6,496	— 33,582	10,527	— 51,765	136,590
— 13	235	— 13	— 2,581	9,425	— 31,755	5,655	435	— 175,305
— 12	160	38	— 3,756	9,260	— 18,840	— 855	40,665	— 281,280
— 11	91	77	— 4,193	7,139	— 2,487	— 5,907	53,085	— 188,265
— 10	28	105	— 4,053	3,984	12,290	— 8,231	40,225	— 3,290
— 9	— 29	123	— 3,483	519	22,607	— 7,767	12,865	162,985
— 8	— 80	132	— 2,616	— 2,712	27,248	— 5,170	— 16,664	240,952
— 7	—125	133	— 1,571	— 5,327	26,247	— 1,425	— 38,451	— 212,747
— 6	—164	127	— 453	— 7,088	20,514	2,439	— 46,875	103,010
— 5	—197	115	647	— 7,883	11,505	5,547	— 40,935	— 39,595
— 4	—224	98	1,652	— 7,708	936	7,299	— 23,535	— 162,400
— 3	—245	77	2,499	— 6,649	— 9,459	7,425	— 153	— 225,211
— 2	—260	53	3,139	— 4,864	— 18,126	5,985	22,743	— 210,406
— 1	—269	27	3,537	— 2,565	— 23,845	3,325	39,235	— 125,685
— 0	—272	0	3,672	0	— 25,840	0	45,220	0
2,992	1,947,792	417,384	348,330,136	1,547,128,656	17,018,415,22 (1)	1,418,201,268	51,305,516,46 (1)	1,285,338,202 (3)
2	1	20	5	14	21	312	195	130

34

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 33	88	— 2,728	8,184	— 79,112	39,556	— 356,004	118,668	— 593,340
— 31	72	— 1,736	3,224	— 7,192	— 10,788	248,124	— 140,244	1,024,860
— 29	57	— 899	— 341	33,263	— 29,667	342,519	— 115,971	418,035
— 27	43	— 207	— 2,721	50,373	— 29,261	201,231	— 9,483	— 404,115
— 25	30	350	— 4,112	51,040	— 18,705	— 2,175	78,735	— 738,195
— 23	18	782	— 4,696	41,032	— 4,551	— 169,671	112,647	— 556,485
— 21	7	1,099	— 4,641	25,037	8,803	— 257,679	93,639	— 101,535
— 19	— 3	1,311	— 4,101	6,897	18,717	— 259,559	41,071	351,215
— 17	— 12	1,428	— 3,216	— 10,608	23,946	— 191,386	— 20,912	610,640
— 15	— 20	1,460	— 2,112	— 25,376	24,310	— 80,730	— 71,520	606,208
— 13	— 27	1,417	— 901	— 36,049	20,397	41,847	— 97,509	374,725
— 11	— 33	1,309	319	— 41,899	13,299	148,863	— 94,413	21,835
— 9	— 38	1,146	1,464	— 42,744	4,381	219,843	— 65,697	— 323,795
— 7	— 42	938	2,464	— 38,864	— 4,917	243,387	— 20,601	— 551,285
— 5	— 45	695	3,263	— 30,917	— 13,245	217,665	28,665	— 595,889
— 3	— 47	427	3,819	— 19,855	— 19,475	149,625	69,825	— 451,335
— 1	— 48	144	4,104	— 6,840	— 22,800	53,200	93,100	— 167,580
13,090	62,832	51,477,360	456,432,592	46,929,569,23 (1)	14,182,012,68 (1)	1,511,802,552 (3)	239,425,743,5 (2)	9,211,590,447 (3)
1	6	2	5	3	28	12	117	65

35

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−17	187	−1 496	46 376	−23 188	672 452	−672 452	2 017 356	−183 396
−16	154	−968	19 096	−2 728	−158 224	435 116	−2,254,692	302,064
−15	123	−520	−744	9,052	−485,460	636,492	−1,995,780	140,244
−14	94	−147	−14,229	14,322	−498,046	403,651	−320,943	−102,486
−13	67	156	−22,374	14,937	−339,097	41,093	1,162,059	−216,021
−12	42	394	−26,124	12,458	−112,752	−273,963	1,830,219	−180,264
−11	19	572	−26,354	8,173	109,589	−457,391	1,648,887	−56,199
−10	−2	695	−23,869	3,118	283,490	−489,547	882,195	79,746
−9	−21	768	−19,404	−1,902	386,166	−391,806	−107,572	169,876
−8	−38	796	−13,624	−6,292	411,632	−207,818	−988,104	187,736
−7	−53	784	−7,124	−9,646	366,314	10,946	−1,527,708	136,868
−5	−66	737	−429	−11,726	265,122	215,787	−1,622,049	42,038
−5	−77	660	6,006	12,441	127,985	367,939	−1,289,145	−62,377
−4	−86	558	11,796	−11,826	−23,152	442,727	−643,341	−143,512
−3	−93	436	16,626	−10,021	−166,869	431,151	142,443	−178,507
−2	−98	299	20,251	−7,250	−284,350	339,325	878,325	−159,250
−1	−101	152	22,496	−3,800	−361,000	186,200	1,396,500	−93,100
0	−102	0	23,256	0	−387,600	0	1,582,700	0
3,570		15,775,320		4,045,652,520		5,291,308,931 (3)		837,417,313,3 (2)
	290,598		14,834,059,24 (1)		4,070,237,639 (3)		66,919,495,30 (4)	
2	3	4	1	12	2	8	9	286

36

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−35	595	−6,545	5,236	−162,316	115,940	−3,362,260	2,139,620	−1,283,772
−33	493	−4,301	2,244	−23,188	−23,188	2,017,356	−2,261,884	2,017,356
−31	397	−2,387	44	58,652	−80,476	3,124,924	−2,132,428	1,046,436
−29	307	−783	−1,476	97,092	−85,684	2,118,044	−492,652	−571,764
−27	223	531	−2,421	104,067	−61,597	412,641	1,052,729	−1,421,319
−25	145	1,575	−2,889	89,685	−25,015	−1,144,775	1,839,905	−1,297,605
−23	73	2,369	−2,971	62,353	12,323	−2,129,731	1,781,789	−542,271
−21	7	2,933	−2,751	28,903	42,881	−2,415,579	1,098,293	367,899
−19	−53	3,287	−2,306	−5,282	62,534	−2,070,734	125,852	1,044,924
−17	−107	3,451	−1,706	−36,142	69,842	−1,275,646	−807,844	1,282,004
−15	−155	3,445	−1,014	−60,814	65,390	−257,790	−1,456,780	1,059,812
−13	−197	3,289	−286	−77,506	51,194	757,042	−1,688,908	502,892
−11	−233	3,003	429	−85,371	30,173	1,578,863	−1,488,331	−186,043
−9	−263	2,607	1,089	−84,381	5,687	2,074,743	−936,739	−793,633
−7	−287	2,121	1,659	−75,201	−18,859	2,178,907	−182,287	−1,151,563
−5	−305	1,565	2,111	−59,063	−40,345	1,893,395	597,065	−1,172,521
−3	−317	959	2,424	−37,640	−56,200	1,281,000	1,229,900	−863,380
−1	−323	323	2,584	−12,920	−64,600	452,200	1,582,700	−316,540
15,540		307,618,740		199,046,104,0 (2)		130,015,019,4 (5)		39,191,130,26 (4)
	3,011,652		191,407,216		120,302,590,3 (2)		73,003,085,78 (4)	
1	1	1	10	2	14	2	11	55

37

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 18	210	— 357	11,781	— 4,488	— 139,128	— 94,860	61,132	— 855,848
— 17	175	— 238	5,236	— 748	— 23,188	52,700	— 61,132	1,283,772
— 16	142	— 136	374	1,496	— 92,752	86,428	— 61,132	733,584
— 15	111	— 50	— 3,036	2,596	— 102,300	62,124	— 17,980	— 291,276
— 14	82	21	— 5,211	2,856	— 77,128	16,988	25,172	— 884,616
— 13	55	78	— 6,354	2,535	— 36,115	— 26,195	49,445	— 876,525
— 12	30	122	— 6,654	1,850	7,320	— 55,455	51,185	— 445,440
— 11	7	154	— 6,286	979	44,327	— 66,583	35,177	127,281
— 10	— 14	175	— 5,411	64	69,800	— 60,584	9,770	596,316
— 9	— 33	186	— 4,176	— 786	81,618	— 41,598	— 16,412	815,156
— 8	— 50	188	— 2,714	— 1,492	79,952	— 15,250	— 36,472	747,688
— 7	— 65	182	— 1,144	— 2,002	66,638	12,610	— 46,228	448,708
— 6	— 78	169	429	— 2,288	44,616	36,816	— 44,434	29,068
— 5	— 89	150	1,914	— 2,343	17,435	53,483	— 32,455	— 382,817
— 4	— 98	126	3,234	— 2,178	— 11,176	60,381	— 13,591	— 674,912
— 3	—105	98	4,326	— 1,819	— 37,671	56,775	7,781	— 775,859
— 2	—110	67	5,141	— 1,304	— 58,984	43,820	27,104	— 667,184
— 1	—113	34	5,644	— 680	— 72,760	23,800	40,460	— 383,180
0	—114	0	5,814	0	— 77,520	0	45,220	0
4,218		932,178		152,877,192		101,892,648,5 (2)		16,692,518,45 (4)
	383,838		980,961,982		172,433,712,8 (2)		57,938,956,97 (1)	
2	3	20	5	84	14	88	495	110

38

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 37	111	— 777	1,887	— 20,757	7,548	— 77,996	77,996	—2,261,884
— 35	93	— 525	867	— 3,927	— 1,020	40,052	— 73,780	3,239,996
— 33	76	— 308	102	6,358	— 4,828	69,564	— 77,996	2,017,356
— 31	60	— 124	— 442	11,594	— 5,508	52,700	— 27,404	— 550,188
— 29	45	29	— 797	13,079	— 4,332	17,980	26,164	—2,168,388
— 27	31	153	— 993	11,925	— 2,260	— 16,740	58,900	—2,319,420
— 25	18	250	— 1,058	9,070	15	— 41,695	64,945	—1,361,985
— 23	6	322	— 1,018	5,290	2,025	— 53,015	48,745	50,025
— 21	— 5	371	— 897	1,211	3,488	— 50,880	19,474	1,307,958
— 19	— 15	399	— 717	— 2,679	4,272	— 38,000	— 12,814	2,013,658
— 17	— 24	408	— 498	— 6,018	4,362	— 18,394	— 39,616	2,024,224
— 15	— 32	400	— 258	— 8,558	3,830	3,558	— 55,280	1,420,784
— 13	— 39	377	— 13	— 10,153	2,808	23,816	— 57,434	436,514
— 11	— 45	341	223	— 10,747	1,464	39,160	— 46,778	— 629,266
— 9	— 50	294	438	— 10,362	— 19	47,475	— 26,447	—1,491,601
— 7	— 54	238	622	— 9,086	— 1,461	47,867	— 1,127	—1,942,327
— 5	— 57	175	767	— 7,061	— 2,700	40,636	23,920	—1,887,472
— 3	— 59	107	867	— 4,471	— 3,604	27,132	43,792	—1,357,552
— 1	— 60	36	918	— 1,530	— 4,080	9,520	54,740	— 492,660
18,278		4,496,388		3,286,859,628		67,928,432,31 (1)		112,078,338,1 (5)
	109,668		25,479,532		505,670,712		91,903,173,13 (1)	
1	6	10	35	21	308	132	495	55

39

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−19	703	−2,109	2,109	−35,853	11,951	−47,804	1,481,924	−740,962
−18	592	−1,443	999	−7,548	−1,258	22,644	−1,325,932	1,013,948
−17	487	−867	159	10,047	−7,327	41,684	−1,477,708	681,938
−16	388	−376	−446	19,312	−8,636	33,116	−596,564	−113,832
−15	295	35	−849	22,321	−7,055	13,260	389,980	−654,534
−14	208	371	−1,081	20,860	−4,010	−7,460	1,037,260	−753,300
−13	127	637	−1,171	16,445	−545	−23,140	1,215,820	−495,690
−12	52	838	−1,146	10,340	2,620	−31,215	982,855	−68,820
−11	−17	979	−1,031	3,575	5,035	−31,460	487,090	341,715
−10	−80	1,065	−849	−3,036	6,470	−25,160	−98,170	604,894
−9	−137	1,101	−621	−8,877	6,869	−14,436	−618,694	662,197
−8	−188	1,092	−366	−13,512	6,308	−1,714	−964,168	522,652
−7	−233	1,043	−101	−16,667	4,957	10,676	−1,078,322	244,817
−6	−272	959	159	−18,212	3,046	20,784	−958,046	−86,698
−5	−305	845	401	−18,143	835	27,220	−644,810	−384,553
−4	−332	706	614	−16,564	−1,412	29,243	−211,313	−577,996
−3	−353	547	789	−13,669	−3,449	26,772	254,104	−625,876
−2	−368	373	919	−9,724	−5,066	20,332	663,136	−522,376
−1	−377	189	999	−5,049	−6,103	10,948	941,528	−295,596
0	−380	0	1,026	0	−6,460	0	1,040,060	0
4,940		33,722,910		9,860,578,884		25,199,257,15 (1)		11,594,310,84 (4)
	4,496,388		32,224,114		1,264,176,780		32,395,868,53 (4)	
2	1	4	35	14	231	264	33	220

40

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−39	247	−9,139	82,251	−9,139	155,363	−466,089	310,726	−9,632,506
−37	209	−6,327	40,071	−2,109	−11,951	203,167	−262,922	12,596,354
−35	173	−3,885	7,881	2,331	−91,205	396,899	−308,210	9,086,534
−33	139	−1,793	−15,579	4,741	−110,959	329,307	−139,298	−684,046
−31	107	−31	−31,499	5,611	−93,823	149,141	60,554	−7,788,006
−29	77	1,421	−40,999	5,365	−57,205	−46,835	200,090	−9,628,290
−27	49	2,583	−45,129	4,365	−14,015	−202,095	249,170	−6,959,190
−25	23	3,475	−44,869	2,915	26,675	−290,215	214,850	−1,890,690
−23	−1	4,117	−41,129	1,265	59,015	−306,245	123,455	3,347,535
−21	−23	4,529	−34,749	−385	79,805	−259,485	7,735	7,074,085
−19	−43	4,731	−26,499	−1,881	87,967	−167,561	−101,741	8,417,551
−17	−61	4,743	−17,079	−3,111	84,061	−51,697	−181,733	7,306,141
−15	−77	4,585	−7,119	−4,001	69,845	66,921	−218,735	4,293,961
−13	−91	4,277	2,821	−4,511	47,879	169,793	−209,287	309,491
−11	−103	3,839	12,251	−4,631	21,173	242,869	−158,779	−3,609,199
−9	−113	3,291	20,751	−4,377	−7,121	277,533	−79,267	−6,546,733
−7	−121	2,653	27,971	−3,787	−33,973	270,911	13,244	−7,881,028
−5	−127	1,945	33,631	−2,917	−56,695	225,607	101,660	−7,377,388
−3	−131	1,187	37,521	−1,837	−73,117	148,971	170,476	−5,203,428
−1	−133	399	39,501	−627	−81,719	52,003	208,012	−1,872,108
21,320		644,482,280		644,482,280		2,368,730,172 (3)		1,893,737,437 (6)
	567,112		49,625,135,56 (1)		213,224,483,6 (2)		1,393,370,689 (3)	
1	3	1	1	63	21	33	198	22

41

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 20	260	− 2,470	18,278	− 36,556	182,780	− 776,815	776,815	−1,242,904
− 19	221	− 1,729	9,139	− 9,139	− 9,139	310,726	− 621,452	1,553,630
− 18	184	− 1,083	2,109	8,436	− 102,638	645,354	− 764,864	1,195,100
− 17	149	− 527	− 3,071	18,241	− 128,797	557,294	− 379,916	6,290
− 16	116	− 56	− 6,646	22,096	− 112,396	278,596	101,422	− 913,240
− 15	85	335	− 8,847	21,583	− 72,685	− 37,230	456,620	−1,212,678
− 14	56	651	− 9,891	18,060	− 24,110	− 298,010	604,520	− 949,620
− 13	29	897	− 9,981	12,675	23,005	− 457,990	552,140	− 358,410
− 12	4	1,078	− 9,306	6,380	61,820	− 505,065	354,335	296,760
− 11	− 19	1,199	− 8,041	− 55	88,385	− 450,010	84,430	804,595
10	− 40	1,265	− 6,347	− 6,028	101,090	− 317,570	− 185,930	1,042,798
− 9	− 59	1,281	− 4,371	− 11,091	100,159	− 139,266	− 399,338	982,165
− 8	− 76	1,252	− 2,246	− 14,936	87,188	52,226	− 519,086	670,660
− 7	− 91	1,183	− 91	− 17,381	64,727	227,266	− 530,894	207,025
− 6	−104	1,079	1,989	− 18,356	35,906	362,154	− 441,142	− 288,730
− 5	−115	945	3,903	− 17,889	4,105	440,890	− 272,710	− 702,601
− 4	−124	786	5,574	− 16,092	− 27,332	455,933	− 59,401	− 948,040
− 3	−131	607	6,939	− 13,147	− 55,339	408,102	160,208	− 979,540
− 2	−136	413	7,949	− 9,292	− 77,326	305,762	348,772	− 797,640
− 1	−139	209	8,569	− 4,807	− 91,333	163,438	475,456	− 445,740
0	−140	0	8,778	0	− 96,140	0	520,030	0
5,740	641,732	47,900,710	2,481,256,778	10,376,164,71 (1)	294,751,492,0 (2)	6,514,007,973 (3)	8,534,395,472 (3)	30,544,152,22 (4)
2	3	4	5	18	21	24	99	220

42

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 41	410	− 1,066	20,254	− 749,398	374,699	−1,873,495	6,369,883	− 374,699
− 39	350	− 754	10,374	− 201,058	− 9,139	685,425	−4,816,253	447,811
− 37	293	− 481	2,717	155,363	− 201,058	1,517,074	−6,214,520	365,560
− 35	239	− 245	− 2,983	359,233	− 260,110	1,359,602	−3,346,280	28,120
− 33	188	− 44	− 6,978	445,258	− 233,692	737,484	446,590	− 248,270
− 31	140	124	− 9,506	443,734	− 158,932	3,100	3,389,086	− 354,578
− 29	95	261	− 10,791	380,799	− 63,998	− 626,690	4,779,584	− 297,888
− 27	53	369	− 11,043	278,685	30,670	−1,038,690	4,600,880	− 138,480
− 25	14	450	− 10,458	155,970	111,205	−1,195,945	3,222,035	50,925
− 23	− 22	506	− 9,218	27,830	169,235	−1,114,465	1,178,155	208,955
− 21	− 55	539	− 7,491	− 93,709	200,882	− 843,810	− 983,906	296,858
− 19	− 85	551	− 5,431	− 199,519	205,838	− 451,250	−2,805,074	300,998
− 17	−112	544	− 3,178	− 283,118	186,518	− 9,214	−3,969,110	229,670
− 15	−136	520	− 858	− 340,418	147,290	414,258	−4,322,630	106,730
− 13	−157	481	1,417	− 369,473	93,782	762,346	−3,868,670	− 36,010
− 11	−175	429	3,549	− 370,227	32,266	993,850	−2,741,518	− 166,166
− 9	−190	366	5,454	− 344,262	− 30,881	1,085,445	−1,169,677	− 256,931
− 7	−202	294	7,062	− 294,546	− 89,639	1,032,077	567,035	− 291,365
− 5	−211	215	8,317	− 225,181	− 138,730	845,786	2,181,100	− 264,540
− 3	−217	131	9,177	− 141,151	− 173,926	553,242	3,417,340	− 183,540
− 1	−220	44	9,614	− 48,070	− 192,280	192,280	4,085,950	− 65,550
24,682	1,629,012	9,075,924	3,084,805,724	4,389,117,671 (3)	1,237,956,266 (3)	37,551,340,08 (4)	563,270,101,2 (5)	2,695,072,254 (3)
1	2	10	5	1	12	12	15	935

43

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 21	287	− 574	22,386	− 70,889	374,699	−1,124,097	2,622,893	− 201,761
− 20	246	− 410	11,726	− 20,254	0	374,699	−1,873,495	230,584
− 19	207	− 266	3,406	13,091	− 191,919	886,483	−2,531,503	198,949
− 18	170	− 141	− 2,847	32,604	− 255,892	822,510	−1,462,240	28,120
− 17	135	− 34	− 7,292	41,344	− 236,208	478,040	35,150	− 119,510
− 16	102	56	− 10,174	41,992	− 167,832	54,316	1,251,266	− 184,112
− 15	71	130	− 11,724	36,872	− 77,560	− 322,032	1,886,270	− 164,762
− 14	42	189	− 12,159	27,972	14,892	− 582,014	1,906,744	− 88,872
− 13	15	234	− 11,682	16,965	95,865	− 699,595	1,434,865	7,995
− 12	− 10	266	− 10,482	5,230	156,800	− 679,155	665,465	94,280
− 11	− 33	286	− 8,734	− 6,127	193,347	− 544,951	− 193,171	148,577
− 10	− 54	295	− 6,599	− 16,252	204,540	− 332,438	− 958,630	161,492
− 9	− 73	294	− 4,224	− 24,522	192,038	− 81,306	−1,497,394	134,642
− 8	− 90	284	− 1,742	− 30,524	159,432	169,910	−1,734,850	77,960
− 7	−105	266	728	− 34,034	111,618	387,790	−1,655,290	6,230
− 6	−118	241	3,081	− 34,996	54,236	546,702	−1,294,618	− 64,444
− 5	−129	210	5,226	− 33,501	− 6,825	630,331	− 728,195	− 119,729
− 4	−138	174	7,086	− 29,766	− 65,856	632,207	− 55,967	− 149,384
− 3	−145	134	8,598	− 24,113	− 117,691	555,375	613,265	− 148,635
− 2	−150	91	9,713	− 16,948	− 158,004	411,350	1,176,860	− 118,560
− 1	−153	46	10,396	− 8,740	− 183,540	218,500	1,551,350	− 65,550
0	−154	0	10,626	0	− 192,280	0	1,682,450	0
6,622		2,676,234		39,541,600,64 (1)		13,411,192,89 (4)		760,148,584,6 (2)
	814,506		3,815,417,606		1,237,956,266 (3)		93,878,350,20 (4)	
2	3	20	5	12	14	24	45	2,210

44

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 43	301	− 12,341	12,341	− 22,919	435,461	−16,112,057	1,239,389	−8,675,723
− 41	259	− 8,897	6,601	− 6,929	10,127	4,871,087	− 835,867	9,482,767
− 39	219	− 5,863	2,091	3,731	− 212,667	12,365,067	−1,181,743	8,618,077
− 37	181	− 3,219	− 1,329	10,101	− 292,201	11,853,283	− 726,199	1,721,647
− 35	145	− 945	− 3,792	13,104	− 276,640	7,311,200	− 49,210	−4,548,410
− 33	111	979	− 5,424	13,552	− 204,288	1,484,736	524,438	−7,593,806
− 31	79	2,573	− 6,344	12,152	− 104,776	−3,863,096	849,446	−7,200,866
− 29	49	3,857	− 6,664	9,512	− 184	−7,735,576	899,414	−4,352,726
− 27	21	4,851	− 6,489	6,147	93,903	−9,713,979	719,497	− 431,151
− 25	− 5	5,575	− 5,917	2,485	167,405	−9,795,875	390,545	3.277,475
− 23	− 29	6,049	− 5,039	− 1,127	214,823	−8,258,311	3,077	5.845,841
− 21	− 51	6,293	− 3,939	− 4,417	234,381	−5,544,159	− 360,451	6,802,091
− 19	− 71	6,327	− 2,694	− 7,182	227,234	−2,169,914	− 636,514	6,111,806
− 17	− 89	6,171	− 1,374	− 9,282	196,742	1,346,774	− 785,842	4,089,146
− 15	−105	5,845	− 42	− 10,634	147,810	4,541,550	− 794,410	1,272,110
− 13	−119	5,369	1,246	− 11,206	86,294	7,041,502	− 671,194	−1,710,982
− 11	−131	4,763	2,441	− 11,011	18,473	8,588,723	− 443,683	−4,263,787
− 9	−141	4,047	3,501	− 10,101	− 49,413	9,051,003	− 152,027	−5,919,377
− 7	−149	3,241	4,391	− 8,561	− 111,559	8,421,367	157,409	−6,404,787
− 5	−155	2,365	5,083	− 6,503	− 162,925	6,808,175	438,425	−5,669,505
− 3	−159	1,439	5,556	− 4,060	− 199,500	4,417,500	650,750	−3,878,850
− 1	−161	483	5,796	− 1,380	− 218,500	1,529,500	764,750	−1,376,550
28,380		1,257,829,980		4,162,273,752		2,735,883,349 (6)		1,369,787,749 (6)
	913,836		1,173,974,648		1,672,913,873 (3)		20,632,604,44 (4)	
1	3	1	10	42	14	2	117	65

45

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 22	946	— 3,311	19,393	— 38,786	504,218	— 368,467	13,633,279	— 27,266,558
— 21	817	— 2,408	10,578	— 12,341	22,919	100,491	— 8,675,723	28,505,947
— 20	694	— 1,610	3,608	5,494	— 234,520	274,987	— 12,826,235	27,208,912
— 19	577	— 912	— 1,722	16,359	— 332,059	271,871	— 8,329,847	6,888,697
— 18	466	— 309	— 5,607	21,714	— 321,958	176,643	— 1,218,299	— 12,503,558
— 17	361	204	— 8,232	22,848	— 246,064	49,096	5,051,758	— 22,778,606
— 16	262	632	— 9,772	20,888	— 137,032	— 71,668	8,856,394	— 22,799,696
— 15	169	980	— 10,392	16,808	— 19,480	— 162,816	9,806,110	— 15,104,066
— 14	82	1,253	— 10,247	11,438	88,922	— 213,923	8,274,301	— 3,595,946
— 13	1	1,456	— 9,482	5,473	176,429	— 223,639	5,036,137	7,916,519
— 12	— 74	1,594	— 8,232	— 518	236,264	— 196,851	1,002,517	16,538,096
— 11	—143	1,672	— 6,622	— 6,083	265,727	— 142,307	— 2,965,259	20,626,661
— 10	—206	1,695	— 4,767	— 10,878	265,370	— 70,669	— 6,173,465	19,800,206
— 9	—263	1,668	— 2,772	— 14,658	238,238	7,038	— 8,158,946	14,729,786
— 8	—314	1,596	— 732	— 17,268	189,176	80,614	— 8,708,522	6,815,096
— 7	—359	1,484	1,268	— 18,634	124,202	141,542	— 7,845,194	— 2,181,202
— 6	—398	1,337	3,153	— 18,754	49,946	183,549	— 5,790,557	— 10,482,982
— 5	—431	1,160	4,858	— 17,689	— 26,845	202,903	— 2,911,835	— 16,593,397
— 4	—458	958	6,328	— 15,554	— 99,736	198,479	339,037	— 19,505,232
— 3	—479	736	7,518	— 12,509	— 162,967	171,627	3,487,849	— 18,816,327
— 2	—494	499	8,393	— 8,750	— 211,750	125,875	6,096,625	14,748,750
— 1	—503	252	8,928	— 4,500	— 242,500	66,500	7,813,750	— 8,079,750
0	—506	0	9,108	0	— 253,000	0	8,412,250	0
7,590	9,203,634	92,036,340	2,934,936,620	12,006,558,90 (1)	2,245,226,514 (3)	1,421,976,792 (3)	2,460,438,079 (6)	13,208,667,58 (7)
2	1	4	7	28	14	104	13	26

46

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 45	165	— 7,095	4,257	— 58,179	290,895	— 872,685	1,842,335	— 1,239,389
— 43	143	— 5,203	2,365	— 19,393	19,393	213,323	— 1,105,401	1,239,389
— 41	122	— 3,526	860	7,052	— 128,699	632,917	— 1,708,347	1,239,389
— 39	102	— 2,054	— 300	23,452	— 187,821	644,397	— 1,166,163	374,699
— 37	83	— 777	— 1,155	31,857	— 186,263	438,413	— 2 50,059	— 489,991
— 35	65	315	— 1,743	34,083	— 146,825	149,845	588,525	— 983,497
— 33	48	1,232	— 2,100	31,724	— 87,444	— 131,604	1,128,258	— 1,036,222
— 31	32	1,984	— 2,260	26,164	— 21,788	— 352,036	1,305,186	— 740,962
— 29	17	2,581	— 2,255	18,589	40,183	— 485,141	1,154,949	— 256,447
— 27	3	3,033	— 2,115	9,999	91,689	— 525,069	768,077	255,183
— 25	— 10	3,350	— 1,868	1,220	128,635	— 480,325	256,605	664,645
— 23	— 22	3,542	— 1,540	— 7,084	149,149	— 368,621	— 269,643	892,147
— 21	— 33	3,619	— 1,155	— 14,399	153,153	— 212,619	— 718,641	910,027
— 19	— 43	3,591	— 735	— 20,349	141,967	— 36,499	— 1,025,049	736,117
— 17	— 52	3,468	— 300	— 24,684	117,946	136,714	— 1,153,722	421,702
— 15	— 60	3,260	132	— 27,268	84,150	287,130	— 1,099,050	37,162
— 13	— 67	2,977	545	— 28,067	44,047	399,347	— 881,229	— 342,329
— 11	— 73	2,629	925	— 27,137	1,249	463,243	— 540,453	— 649,319
— 9	— 78	2,226	1,260	— 24,612	— 40,719	474,273	— 129,907	— 833,659
— 7	— 82	1,778	1,540	— 20,692	— 78,617	433,337	291,669	— 868,509
— 5	— 85	1,295	1,757	— 15,631	— 109,655	346,285	667,185	— 752,571
— 3	— 87	787	1,905	— 9,725	— 131,625	223,125	947,625	— 508,725
— 1	— 88	264	1,980	3,300	— 143,000	77,000	1,097,250	— 179,550
32,430	285,384	429,502,920	143,167,640	27,214,866,84 (1)	748,408,838,1 (2)	7,933,133,684 (3)	44,332,217,65 (4)	26,684,176,94 (4)
1	6	2	35	21	28	52	117	715

47

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−23	345	−759	32,637	−32,637	1,338,117	−2,230,195	40,549	−770,431
−22	300	−561	18,447	−11,352	116,358	484,825	−22,919	736,934
−21	257	−385	7,095	3,311	−562,397	1,570,833	−37,023	770,431
−20	216	−230	−1,720	12,556	−846,240	1,644,879	−26,445	267,976
−19	177	−95	−8,285	17,461	−857,433	1,166,163	−7,257	−257,849
−18	140	21	−12,873	18,984	−695,114	463,095	10,947	−578,018
−17	105	119	−15,743	17,969	−437,871	−242,505	23,283	−639,863
−16	72	200	−17,140	15,152	−146,184	−813,924	28,134	−488,528
−15	41	265	−17,295	11,167	135,265	−1,180,647	25,995	−211,793
−14	12	315	−16,425	6,552	375,414	−1,321,879	18,571	96,402
−13	−15	351	−14,733	1,755	554,775	−1,252,355	8,095	357,825
−12	−40	374	−12,408	−2,860	663,520	−1,010,295	−3,165	520,760
−11	−63	385	−9,625	−7,007	699,699	−647,361	−13,239	562,793
−10	−84	385	−6,545	−10,472	667,590	−220,473	−20,655	488,138
−9	−103	375	−3,315	−13,107	576,181	214,659	−24,531	321,623
−8	−120	356	−68	−14,824	437,784	608,430	−24,582	101,048
−7	−135	329	3,077	−15,589	266,781	920,805	−21,063	−130,747
−6	−148	295	6,015	−15,416	78,502	1,123,497	−14,667	−333,106
−5	−159	255	8,655	−14,361	−111,765	1,201,001	−6,395	−473,381
−4	−168	210	10,920	−12,516	−289,632	1,150,627	2,587	−530,936
−3	−175	161	12,747	−10,003	−442,337	981,675	11,091	−499,149
−2	−180	109	14,087	−6,968	−559,338	713,895	18,039	−385,434
−1	−183	55	14,905	−3,575	−632,775	375,375	22,575	−209,475
0	−184	0	15,180	0	−657,800	0	24,150	0
8,648	1,271,256	4,994,220	8,518,474,580	8,629,104,120	15,866,267,37 (4)	51,565,368,95 (4)	21,211,587,39 (1)	10,096,715,60 (4)
2	3	20	5	42	7	24	6,435	1,430

48

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
−47	1,081	−3,243	35,673	−1,533,939	511,313	−1,905,803	1,905,803	−1,905,803
−45	943	−2,415	20,493	−554,829	54,395	364,941	−1,013,725	1,743,607
−43	811	−1,677	8,283	126,291	−203,863	,1302,857	−1,711,873	1,902,277
−41	685	−1,025	−1,265	562,397	−316,437	1,401,585	−1,274,649	742,223
−39	565	−455	−8,445	801,047	−327,273	1,031,355	−417,831	−527,137
−37	451	37	−13,537	884,633	−272,211	459,651	423,489	−1,345,579
−35	343	455	−16,807	850,633	−179,865	−130,257	1,020,285	−1,563,289
−33	241	803	−18,507	731,863	−72,459	−622,809	1,288,341	−1,264,219
−31	145	1,085	−18,875	556,729	33,381	−955,575	1,238,877	−640,429
−29	55	1,305	−18,135	349,479	125,879	−1,106,495	939,653	92,481
−27	−29	1,467	−16,497	130,455	197,405	−1,082,835	485,225	747,555
−25	−107	1,575	−14,157	−83,655	243,815	−911,755	−24,895	1,193,865
−23	−179	1,633	−11,297	−279,565	263,835	−632,385	−501,345	1,365,625
−21	−245	1,645	−8,085	−447,139	258,489	−289,305	−874,377	1,258,579
−19	−305	1,615	−4,675	−579,139	230,571	72,675	−1,098,693	918,289
−17	−359	1,547	−1,207	−670,973	184,161	412,539	−1,154,589	424,099
−15	−407	1,445	2,193	−720,443	124,185	695,997	−1,046,265	−128,231
−13	−449	1,313	5,413	−727,493	56,019	897,429	−798,081	−642,941
−11	−485	1,155	8,355	−693,957	−14,863	1,000,945	−449,461	−1,038,103
−9	−515	975	10,935	−623,307	−83,197	1,000,665	−49,069	−1,255,813
−7	−539	777	13,083	−520,401	−144,193	900,323	351,197	−1,268,197
−5	−557	565	14,743	−391,231	−193,755	712,299	701,925	−1,079,127
−3	−569	343	15,873	−242,671	−228,657	456,183	961,233	−721,917
−1	−575	115	16,445	−82,225	−246,675	156,975	1,098,825	−253,575
36,848	12,712,560	92,620,080	10,301,411,12(1)	19,208,385,77 (4)	2,321,892,786 (3)	37,502,086,51 (4)	46,326,106,86 (4)	60,580,293,59 (4)
1	1	5	5	1	21	33	165	715

49

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 24	376	— 4,324	38,916	— 95,128	371,864	— 278,898	3,811,606	— 15,246,424
— 23	329	— 3,243	22,701	— 35,673	46,483	46,483	— 1,905,803	13,340,621
— 22	284	— 2,277	9,591	6,072	— 140,438	184,943	— 3,365,567	15,165,326
— 21	241	— 1,421	— 729	33,187	— 225,019	204,207	— 2,603,951	6,517,811
— 20	200	— 670	— 8,560	48,444	— 237,360	155,445	— 978,465	— 3,370,856
— 19	161	— 19	— 14,189	54,321	— 202,143	75,981	671,703	— 10,082,597
— 18	124	537	— 17,889	53,016	— 139,206	— 8,271	1,891,371	— 12,288,602
— 17	89	1,003	— 19,919	46,461	— 64,089	— 80,631	2,498,499	— 10,462,667
— 16	56	1,384	— 20,524	36,336	11,448	— 131,724	2,494,242	— 5,955,152
— 15	25	1,685	— 19,935	24,083	78,935	— 157,785	1,991,515	— 359,517
— 14	— 4	1,911	— 18,369	10,920	132,730	— 159,185	1,159,795	4,888,650
— 13	— 31	2,067	— 16,029	— 2,145	169,585	— 139,165	184,015	8,732,685
— 12	— 56	2,158	— 13,104	— 14,300	188,240	— 102,765	— 764,465	10,581,480
— 11	— 79	2,189	— 9,769	— 24,915	189,045	— 55,935	— 1,546,695	10,300,565
— 10	— 100	2,165	— 6,185	— 33,528	173,610	— 4,815	— 2,066,535	8,140,322
— 9	— 119	2,091	— 2,499	— 39,831	144,483	44,829	— 2,274,243	4,628,267
— 8	— 136	1,972	1,156	— 43,656	104,856	88,026	— 2,164,746	447,032
— 7	— 151	1,813	4,661	— 44,961	58,299	120,891	— 1,771,959	— 3,685,183
— 6	— 164	1,619	7,911	— 43,816	8,522	140,799	— 1,160,387	— 7,119,338
— 5	— 175	1,395	10,815	— 40,389	— 40,835	146,455	— 415,115	— 9,357,089
— 4	— 184	1,146	13,296	— 34,932	— 86,384	137,873	368,839	— 10,104,296
— 3	— 191	877	15,291	— 27,767	— 125,143	116,277	1,096,963	— 9,294,761
— 2	— 196	593	16,751	— 19,272	— 154,662	83,937	1,684,767	— 7,085,106
— 1	— 199	299	17,641	— 9,867	— 173,121	43,953	2,065,791	— 3,823,911
0	— 200	0	17,940	0	— 179,400	0	2,197,650	0

9,800 167,230,700 74,451,107,64 (1) 800,349,407,1 (2) 3,806,461,780 (6)
1,566,040 12,408,517,94(1) 1,231,306,780 (3) 18 3,374,173,0(5)

2	3	4	5	18	33	264	99	110

50

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
— 49	196	— 9,212	211,876	— 211,876	15,134	— 650,762	4,555,334	— 186,768,694
— 47	172	— 6,956	125,396	— 82,156	2,162	92,966	— 2,138,218	156,275,846
— 45	149	— 4,935	55,131	9,729	— 5,405	418,347	— 3,951,055	184,862,891
— 43	127	— 3,139	— 529	70,219	— 8,947	473,731	— 3,167,767	86,328,821
— 41	106	— 1,558	— 43,124	105,124	— 9,619	371,993	— 1,327,969	— 31,404,319
— 39	86	— 182	— 74,124	119,652	— 8,373	196,209	601,527	— 115,122,137
— 37	67	999	— 94,929	118,437	— 5,979	4,773	2,081,931	— 147,226,367
— 35	49	1,995	— 106,869	105,567	— 3,045	— 164,019	2,882,355	— 131,363,057
— 33	32	2,816	— 111,204	84,612	— 36	— 287,892	2,981,754	— 81,965,642
— 31	16	3,472	— 109,124	58,652	2,708	— 356,996	2,490,458	— 16,994,262
— 29	1	3,973	— 101,749	30,305	4,955	— 370,765	1,588,955	46,765,545
— 27	— 13	4,329	— 90,129	1,755	6,565	— 335,205	481,715	96,404,295
— 25	— 26	4,550	— 75,244	— 25,220	7,475	— 260,585	— 636,025	124,076,745
— 23	— 38	4,646	— 58,004	— 49,220	7,685	— 159,505	— 1,600,465	127,094,895
— 21	— 49	4,627	— 39,249	— 69,195	7,245	— 45,315	— 2,292,255	107,279,795
— 19	— 59	4,503	— 19,749	— 84,417	6,243	69,141	— 2,642,007	69,866,477
— 17	— 68	4,284	— 204	— 94,452	4,794	172,482	— 2,629,866	22,201,082
— 15	— 76	3,980	18,756	— 99,132	3,030	255,474	— 2,280,570	— 27,581,578
— 13	— 83	3,601	36,571	— 98,527	1,091	311,467	— 1,655,299	— 71,758,193
— 11	— 89	3,157	52,751	— 92,917	— 883	336,611	— 841,483	— 103,995,023
— 9	— 94	2,658	66,876	— 82,764	— 2,759	329,877	58,391	— 120,041,591
— 7	— 98	2,114	78,596	— 68,684	— 4,417	292,909	938,063	— 118,110,881
— 5	— 101	1,535	87,631	— 51,419	— 5,755	229,733	1,698,095	— 98,945,651
— 3	— 103	931	93,771	— 31,809	— 6,693	146,349	2,255,127	— 65,594,781
— 1	— 104	312	96,876	— 10,764	— 7,176	50,232	2,549,274	— 22,943,466

41,650 770,715,400 372,255,538,2 (2) 4,344,753,924 (3) 56 1,453,112,6(8)
433,160 372,255,588,2 (2) 2,045,360,100 259,407,366,7(5)

1	6	2	1	9	924	132	99	11

11

51

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 25	1,225	− 4,900	46,060	− 75,670	378,350	− 378,350	16,269,050	− 22,776,670
− 24	1,078	− 3,724	27,636	− 30,268	60,536	45,402	− 7,158,382	18,221,336
− 23	937	− 2,668	12,596	2,162	− 127,558	235,658	− 13,851,934	22,404,806
− 22	802	− 1,727	611	23,782	− 218,362	273,493	− 11,481,301	11,248,886
− 21	673	− 896	− 8,634	36,547	− 239,131	221,007	− 5,268,403	− 2,681,179
− 20	550	− 170	− 15,440	42,214	− 212,440	124,295	1,459,205	− 13,018,336
− 19	433	456	− 20,094	42,351	− 156,657	16,131	6,801,009	− 17,484,617
− 18	322	987	− 22,869	38,346	− 86,394	− 81,621	9,893,913	− 16,290,722
− 17	217	1,428	− 24,024	31,416	− 12,936	− 155,856	10,592,322	− 10,959,662
− 16	118	1,784	− 23,804	22,616	55,352	− 200,324	9,210,326	− 3,498,672
− 15	25	2,060	− 22,440	12,848	112,640	− 213,960	6,318,770	4,147,998
− 14	− 62	2,261	− 20,149	2,870	155,270	− 199,435	2,590,385	10,424,850
− 13	− 143	2,392	− 17,134	− 6,695	181,415	− 161,915	− 1,313,455	14,321,385
− 12	− 218	2,458	− 13,584	− 15,350	190,760	− 108,015	− 4,820,395	− 15,403,680
− 11	− 287	2,464	− 9,674	− 22,715	184,205	− 44,935	− 7,494,835	13,759,515
− 10	− 350	2,415	− 5,565	− 28,518	163,590	20,235	− 9,061,005	9,889,082
− 9	− 407	2,316	− 1,404	− 32,586	131,442	81,054	− 9,406,254	4,567,562
− 8	− 458	2,172	2,676	− 34,836	90,744	132,126	− 8,569,038	− 1,299,448
− 7	− 503	1,988	6,556	− 35,266	44,726	169,366	− 6,715,702	− 6,811,538
− 6	− 542	1,769	10,131	− 33,946	− 3,322	190,149	− 4,109,761	− 11,188,418
− 5	− 575	1,520	13,310	− 31,009	− 50,215	193,355	− 1,076,995	− 13,856,459
− 4	− 602	1,246	16,016	− 26,642	− 93,016	179,323	2,030,717	− 14,502,656
− 3	− 623	952	18,186	− 21,077	− 129,157	149,727	4,870,289	− 13,094,921
− 2	− 638	643	19,771	− 14,582	− 156,538	107,387	7,138,901	− 9,870,266
− 1	− 647	324	20,736	− 7,452	− 173,604	56,028	8,600,298	− 5,294,646
0	− 650	0	21,060	− 0	− 179,400	0	9,104,550	0
11,050	17,218,110	221,375,700	17,803,525,74(1)	47,861,426,34 (1)	1,282,440,783 (3)	1,465,091,440 (3)	3,279,650,279 (6)	8,216,387,014 (6)
2	1	4	5	28	42	264	33	110

52

φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
− 51	425	− 4,165	3,570	− 55,930	1,286,390	− 19,295,850	1,286,390	− 55,314,770
− 49	375	− 3,185	2,170	− 23,030	227,010	1,891,750	− 529,690	42,299,530
− 47	327	− 2,303	1,022	658	− 408,618	11,638,046	− 1,074,514	54,013,246
− 45	281	− 1,515	102	16,638	− 724,270	13,834,638	− 918,850	28,912,426
− 43	237	− 817	− 613	26,273	− 807,507	11,481,301	− 455,101	− 3,858,089
− 41	195	− 205	− 1,145	30,791	− 731,193	6,822,605	63,227	− 29,154,731
− 39	155	325	− 1,515	31,291	− 554,947	1,471,665	488,063	− 41,186,561
− 37	117	777	− 1,743	28,749	− 326,529	− 3,478,407	748,743	− 39,961,147
− 35	81	1,155	− 1,848	24,024	− 83,160	− 7,354,776	828,870	− 28,647,202
− 33	47	1,463	− 1,848	17,864	147,224	− 9,816,312	747,542	− 11,694,342
− 31	15	1,705	− 1,760	10,912	344,784	− 10,775,600	544,454	6,432,438
− 29	− 15	1,885	− 1,600	3,712	496,656	− 10,330,960	268,406	22,029,618
− 27	− 43	2,007	− 1,383	− 3,285	595,895	− 8,707,905	− 31,225	32,553,915
− 25	− 69	2,075	− 1,123	− 9,715	640,485	− 6,209,465	− 310,465	36,738,945
− 23	− 93	2,093	− 833	− 15,295	632,415	− 3,174,805	− 534,565	34,512,075
− 21	− 115	2,065	− 525	− 19,817	576,821	54,435	− 680,029	26,780,937
− 19	− 135	1,995	− 210	− 23,142	481,194	3,160,650	− 735,186	15,147,142
− 17	− 153	1,887	102	− 25,194	354,654	5,870,202	− 699,618	1,593,682
− 15	− 169	1,745	402	− 25,954	207,290	7,967,346	− 582,730	− 11,817,658
− 13	− 183	1,573	682	− 25,454	49,566	9,302,722	− 401,722	− 23,211,838
− 11	− 195	1,375	935	− 23,771	− 108,207	9,796,985	− 179,197	− 31,110,409
− 9	− 205	1,155	1,155	− 21,021	− 256,333	9,440,145	59,387	− 34,574,539
− 7	− 213	917	1,337	− 17,353	− 386,127	8,287,189	288,239	− 33,276,721
− 5	− 219	665	1,477	− 12,943	− 490,245	6,450,557	483,575	− 27,502,411
− 3	− 223	403	1,572	− 7,988	− 562,948	4,090,044	625,646	− 18,087,706
− 1	− 225	135	1,620	− 2,700	− 600,300	1,400,700	700,350	− 6,303,150
46,852	2,108,340	162,342,180	108,228,120	26,358,466,68 (1)	14,876,313,08 (4)	3,803,377,377 (6)	20,338,916,46 (4)	47,733,295,98 (7)
1	3	5	70	42	14	6	495	55

Appendix II.

Some Useful Relations.

$$x^{(1)} = x$$
$$x^{(2)} = x(x-1)$$
$$x^{(3)} = x(x-1)(x-2)$$
$$x^{(4)} = x(x-1)(x-2)(x-3)$$
$$x^{(5)} = x(x-1)(x-2)(x-3)(x-4)$$
$$x^{(6)} = x(x-1)(x-2)(x-3)(x-4)(x-5)$$

$$x^{[1]} = x$$
$$x^{[2]} = (x^2 - \tfrac{1}{4})$$
$$x^{[3]} = x(x^2 - 1)$$
$$x^{[4]} = (x^2 - \tfrac{1}{4})(x^2 - \tfrac{9}{4})$$
$$x^{[5]} = x(x^2 - 1)(x^2 - 4)$$
$$x^{[6]} = (x^2 - \tfrac{1}{4})(x^2 - \tfrac{9}{4})(x^2 - \tfrac{25}{4})$$

$$\mu x^{[1]} = x$$
$$\mu x^{[2]} = x^2,$$
$$\mu x^{[3]} = x(x^2 - \tfrac{1}{4})$$
$$\mu x^{[4]} = x^2(x^2 - 1)$$
$$\mu x^{[5]} = x(x^2 - \tfrac{1}{4})(x^2 - \tfrac{9}{4})$$
$$\mu x^{[6]} = x^2(x^2 - 1)(x^2 - 4)$$

$$x = \mu x^{[1]} = x^{[1]}$$
$$x^2 = \mu x^{[2]}$$
$$x^3 = x^{[3]} + x^{[1]}$$
$$x^4 = \mu x^{[4]} + \mu x^{[2]}$$
$$x^5 = x^{[5]} + 5x^{[3]} + x^{[1]}$$
$$x^6 = \mu x^{[6]} + 5\mu x^{[4]} + \mu x^{[2]}$$
$$x^7 = x^{[7]} + 14x^{[5]} + 21x^{[3]} + x^{[1]}$$
$$x^8 = \mu x^{[8]} + 14\mu x^{[6]} + 21\mu x^{[4]} + \mu x^{[2]}$$

$$x^{(1)} = \mu x^{[1]} = x^{[1]}$$
$$x^{(2)} = \mu x^{[2]} - x^{[1]}$$
$$x^{(3)} = x^{[3]} - 3\mu x^{[2]} + 3x^{[1]}$$
$$x^{(4)} = \mu x^{[4]} - 6x^{[3]} + 12\mu x^{[2]} - 12x^{[1]}$$
$$x^{(5)} = x^{[5]} - 10\mu x^{[4]} + 40x^{[3]} - 60\mu x^{[2]} + 60x^{[1]}$$
$$x^{(6)} = \mu x^{[6]} - 15x^{[5]} + 90\mu x^{[4]} - 300x^{[3]} + 360\mu x^{[2]} - 360x^{[1]}$$

Appendix III.

Binomial Coefficients.

n	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
2	1								
3	3	1							
4	6	4	1						
5	10	10	5	1					
6	15	20	15	6	1				
7	21	35	35	21	7	1			
8	28	56	70	56	28	8	1		
9	36	84	126	126	84	36	9	1	
10	45	120	210	252	210	120	45	10	1
11	55	165	330	462	462	330	165	55	11
12	66	220	495	792	924	792	495	220	66
13	78	286	715	1287	1716	1716	1287	715	286
14	91	364	1001	2002	3003	3432	3003	2002	1001
15	105	455	1365	3003	5005	6435	6435	5005	3003
16	120	560	1820	4368	8008	11440	12870	11440	8008
17	136	680	2380	6188	12376	19448	24310	24310	19448
18	153	816	3060	8568	18564	31824	43758	48620	43758
19	171	969	3876	11628	27132	50388	75582	92378	92378
20	190	1140	4845	15504	38760	77520	1 25970	1 67960	1 84756
21	210	1330	5985	20349	54264	1 16280	2 03490	2 93930	3 52716
22	231	1540	7315	26334	74613	1 70544	3 19770	4 97420	6 46646
23	253	1771	8855	33649	1 00947	2 45157	4 90314	8 17190	11 44066
24	276	2024	10626	42504	1 34596	3 46104	7 35471	13 07504	19 61256