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Short Note

Extension of the Aki-Utsu *b*-Value Estimator for Incomplete Catalogs

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Abstract The Aki (1965) maximum likelihood estimate of the Gutenberg–Richter *b*-value is extended for use in the case of multiple catalogs with different levels of completeness. The most striking feature of this newly derived estimator is its simplicity—it is more manageable than the well-known and already easy to use Weichert (1980) solution to the analogs problem. In addition, confidence intervals for the newly derived estimator are provided.

Introduction

According to the Gutenberg–Richter frequency–magnitude relation (Gutenberg and Richter, 1944; 1954), the number of earthquakes n, having a magnitude equal to or larger than m, can be expressed by the equation

$$\log(n) = a - bm, \tag{1}$$

where parameter a is a measure of the level of seismicity and parameter b describes the ratio between the number of small and large events. The Gutenberg–Richter relation is of significant importance to seismic studies because it is used to describe both tectonic and induced seismicity, can be applied in different time scales, and holds true over a large interval of earthquake magnitudes.

Both parameters have been used in a variety of seismological studies, especially in seismicity simulation (Ogata and Zhuang 2006; Felzer, 2008), earthquake prediction (Kagan and Knopoff, 1987; Geller, 1997), and seismic hazard and risk assessment (Cornell, 1968; Beauval and Scotti, 2004). The accurate calculation of this parameter is therefore of critical importance. It can be shown that the *b*-value has a clear physical meaning (e.g., Scholz, 1968; Dieterich, 1994; Schorlemmer *et al.*, 2005; De Santis *et al.*, 2011). Some authors (e.g., Bak and Tang, 1989) believe in the universality of the *b*-value and its scale independence. Their model, which is known as the self-organized criticality, became a classic in the description of seismic activity. The *b*-value can be calculated in various ways (for a review on the different estimation techniques, see Marzocchi and Sandri, 2003). However, Aki's classic estimator (Aki, 1965), which considered earthquake magnitude as a continuous random variable, is still the preferred estimator. Aki (1965) derived the maximum likelihood estimate of the *b*-value, equivalent to

$$\beta = \frac{1}{\overline{m} - m_{\min}}, \tag{2}$$

where $\beta = b\ln(10)$. The parameters $\overline{m}$ and $m_{\min}$ are the average magnitude and the level of completeness of a given sample of earthquake magnitudes, respectively. Utsu (1965) was the first to derive this equation for the assessment of the *b*-value of Gutenberg–Richter by utilizing the method of moments; that is, by the comparison of the first population moment with an equivalent sample moment. It is interesting to note that estimator (2) was known to statisticians before its derivation by both Aki and Utsu (e.g., Kulldorff, 1961). In the following part of this work, estimator (2) will be called the Aki–Utsu estimator of the *b*-value.

A complete seismic event catalog, starting from a specified level of completeness $m_{\min}$, is needed in order to apply estimator (2). The question then arises as to how to calculate the *b*-value when the seismic event catalog is incomplete. In this work, an incomplete catalog is defined as a catalog that can be divided into subcatalogs, each of a different level of completeness (Fig. 1). The utilization of such incomplete catalogs has been discussed by Molchan *et al.* (1970), Kijko and Sellevoll (1989, 1992), Rosenblueth (1986), and Rosenblueth and Ordaz (1987), among others, but the most elegant, straightforward, and best-known is the procedure derived by Weichert (1980). In this short article, a new alternative estimator to the problem that is even more user friendly than Weichert's formula is suggested.

Theoretical Background

Assume that a given seismic event catalog can be divided into s subcatalogs, each with known, but different levels of completeness, $m^1_{\min}, m^2_{\min}, \ldots, m^s_{\min}$. Let each of these subcatalogs last t_i years and contain a record of n_i $(i = 1, 2, \ldots, s)$ number of events with known magnitudes. A schematic illustration of such a catalog is given in Figure 1.

An overall maximum likelihood estimate of the *b*-value can be obtained by the application of the additive property of likelihood functions (Rao, 1973). If applied to the current problem, the joint likelihood function, which utilizes

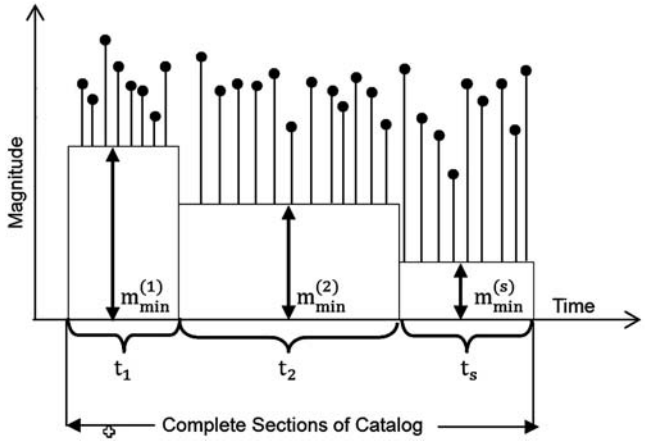

Figure 1. A schematic illustration of a seismic event catalog that can be used in the estimation of the Gutenberg–Richter b-value. (Modified after Kijko and Sellevoll, 1989.)

all earthquakes that occurred within the whole span of the catalog, is defined as

$$L = L_1 L_2 \ldots L_s, \tag{3}$$

where L_i represents the i-th likelihood function based on data observed within i-th subcatalog and $i = 1, 2, \ldots, s$.

If magnitudes of seismic events are assumed to be independent, identically distributed random variables following the frequency–magnitude Gutenberg–Richter relation in equation (1), the probability density function (PDF) of earthquake magnitude takes the form (Aki, 1965)

$$f(m;\beta) = \begin{cases} 0 & \text{for } m \le m_{\min} \\ \beta \exp[-\beta(m - m_{\min})] & \text{for } m \ge m_{\min} \end{cases}, \tag{4}$$

where magnitude m is considered as a continuous variable that may assume any value equal to or larger than the level of completeness $m_{\min}$.

The maximum likelihood estimator of the parameter β, $\hat{\beta}$, is defined as the value of β that maximizes the appropriate likelihood function.

If the magnitudes of seismic events are assumed to be independent, identically distributed random variables following the PDF in equation (4), the likelihood function for the i-h subcatalog takes the form

$$L_i(\beta) = \prod_{j=1}^{n_i} f(m^i_j;\beta) = \prod_{j=1}^{n_i} \beta \exp[-\beta(m^i_j - m^i_{\min})], \tag{5}$$

where m^i_j is the sample of n_i earthquake magnitudes recorded during the time span of the i-th subcatalog. Following equation (3), the joint likelihood function, which utilizes magnitudes of all the earthquakes that occurred within the entire span of the catalog, takes the form

$$L(\beta) = \prod_{i=1}^{s}\prod_{j=1}^{n_i} f(m^i_j,\beta) = \prod_{i=1}^{s}\prod_{j=1}^{n_i} \beta \exp[-\beta(m^i_j - m^i_{\min})]. \tag{6}$$

Maximization of the likelihood function in equation (6) provides the generalized Aki–Utsu $\hat{\beta}$-value estimator

$$\hat{\beta} = \left(\frac{r_1}{\hat{\beta}_1} + \frac{r_2}{\hat{\beta}_2} + \ldots + \frac{r_s}{\hat{\beta}_s}\right)^{-1}, \tag{7}$$

where $r_i = n_i/n$; $n = \sum_i^s n_i$ is the total number of events with magnitudes equal to or exceeding the relevant level of completeness, and $\hat{\beta}_i$ are the Aki–Utsu estimators of the β-values, calculated for individual subcatalogs i ($i = 1, \ldots, s$) according to the classic Aki–Utsu formulation in estimator (2). Details on the derivation of estimator (7) are provided in Appendix A. Equation (7) will be called the generalized Aki–Utsu β-value estimator, and it is applicable in the assessment of the β (or, equivalently, b-value) when the seismic event catalog can be divided into subcatalogs, each with a different level of completeness.

The advantage of applying the maximum likelihood procedure for the estimation of parameters lies in that it provides straightforward approximations for the standard errors and confidence intervals. Based on the central limit theorem, it can be shown (e.g., Mood *et al.*, 1974) that, under suitable regularity conditions and for a sufficiently large number of events, the newly derived estimator $\hat{\beta}$ is approximately normally distributed about its mean (see equation 7). The sample standard deviation is defined as

$$\hat{\sigma}_{\hat{\beta}} = \frac{\hat{\beta}}{\sqrt{n}} \tag{8}$$

and its confidence interval as

$$\hat{\beta} \pm z_{\alpha/2}\hat{\sigma}_{\hat{\beta}}. \tag{9}$$

In equation (9), $z_{\alpha/2}$ is the $(1 - \alpha/2)$ quintile of the standard normal distribution. The derivation of equation (8) is provided in Appendix B. The natural question is how many events are needed in order to be sure that the newly derived estimator of $\hat{\beta}$ is distributed normally. According to Jansson (1966), the sum of 12 uniformly distributed random numbers will create a set of random numbers with a bell-shaped distribution that is approximately Gaussian. Surprisingly, such an approximation by only 12 numbers is quite good, especially if only the central part (mean ± SD) of such a Gaussian-like distribution is explored. Obviously, if more inference is to be based on the tail of such a distribution (confidence intervals, significance levels, etc.) more observations are required.

The most striking feature of the newly derived β-value estimator is its simplicity. Its calculation is straightforward (it is a simple combination of the classic Aki–Utsu β-value estimators computed for each of s subcatalogs), and it is free of any kind of iterative process. Once the $\hat{\beta}$-value is known, the mean seismic activity rate can be calculated. Let $\lambda(m_{\min})$ denote the area-characteristic, seismic activity rate of events with magnitudes $m_{\min}$ and larger. It can be shown (Kijko and Sellevoll, 1989; 1992), that if the number of seismic events per unit of time is a Poisson random variable, the maximum likelihood estimator of $\lambda(m_{\min})$ takes the form

$$\hat{\lambda}(m_{\min}) = \frac{n}{\sum_{i=1}^{s} t_i \exp[-\hat{\beta}(m_{\min}^i - m_{\min})]}. \quad (10)$$

For a complete catalog (i.e., where $s = 1$; $m_{\min}^1 = m_{\min}^2 = \ldots = m_{\min}^s = m_{\min}$; $t = t_1$ with $t_2 = t_3 = \ldots = 0$ and $n = n_1$ with $n_2 = n_3 = \ldots = 0$), the generalized Aki–Utsu $\hat{\beta}$-value estimator (7) reduces to the classic Aki–Utsu formulation in estimator (2), and the maximum likelihood estimator of $\lambda(m_{\min})$ takes the standard form n/t.

Example

The performance of the newly derived estimator has been investigated by the Monte Carlo simulation technique. It was assumed that a hypothetic seismic catalog can be divided into two subcatalogs, each with the same time span (e.g., 38 years) and with a level of completeness of $m_{\min}^1 = 4.5$ and $m_{\min}^2 = 4.0$, respectively. The earthquake magnitudes were generated according to the distribution in equation (4), where $m_{\min} = 4.0$ and $\beta = 2.303$ or, equivalently, the Gutenberg–Richter b-value was equal to 1. In the first subcatalog, only magnitudes equal to or exceeding the value of $m_{\min}^1 = 4.5$ were considered. It can be shown that, after removing the magnitudes in the first subcatalog below the level of completeness, the second subcatalog contains on average $10^{-b(m_{\min}^2 - m_{\min}^1)} = 10^{0.5} \cong 3.16$ times more events than the first subcatalog. The simulation was repeated 1000 times for different number of events, ranging from 100 to 500. The number of events is defined as the total number of events in both subcatalogs.

Figure 2 displays the average of 1000 solutions of the b-value calculated according to the newly derived generalized Aki–Utsu estimator (7). From this figure it is clear that estimator (7) performs very well. The slight overestimation of the $\hat{b}$-value for a small number of events confirms the findings of Ogata and Yamashina (1986), that the Aki–Utsu formulation in estimator (2) is biased. If a more precise assessment of the b-value is required, the bias can be removed by applying the bias-reduction formula provided by Ogata and Yamashina (1986). One has to admit, that the Ogata–Yamashina bias is small, if not negligible. In the real world, the uncertainties surrounding the model, its parameters, and the errors in magnitude determination have a more significant effect on the estimated b-value than the Ogata–Yamashina bias. Figure 2 also indicates the confidence intervals of the estimated b-value. As expected, as the number of events increases, the width of the confidence intervals decreases.

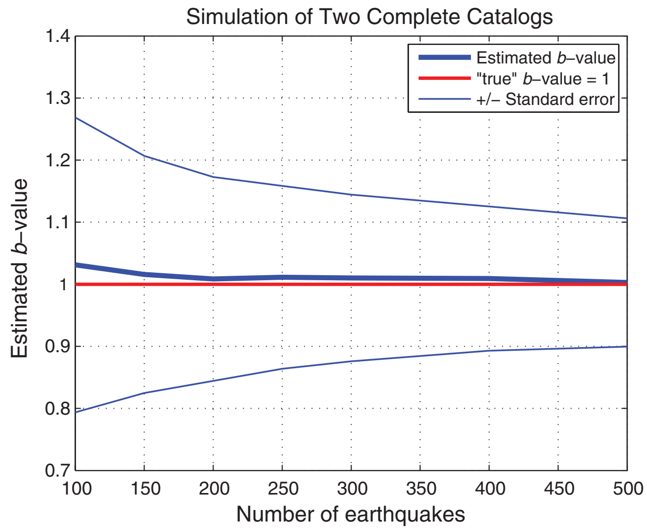

Figure 2. Performance of generalized Aki–Utsu b-value estimator (7). On thousand synthetic catalogs of seismic event magnitudes were generated. Each catalog was divided into two subcatalogs, each with a time span of 38 years, and levels of completeness $m_{\min}^1 = 4.5$ and $m_{\min}^2 = 4.0$, respectively.

Conclusions

A new maximum likelihood estimate of the Gutenberg–Richter b-value for multiple catalogs with different levels of completeness is derived. This estimator is easy to use and confidence intervals that estimate the margin of error are provided. The estimator is not based on an iterative process, and once the $\hat{\beta}$-value is known, the mean seismic activity rate can be calculated. The slight overestimation owing to bias for catalogs with a small number of events can be removed through the bias-reduction formula provided by Ogata and Yamashina (1986). This short note illustrates how to combine several catalogs of different levels of completeness in the case in which earthquake magnitudes are distributed according to PDF (see equation 4), which is based on the classic frequency–magnitude Gutenberg–Richter relation in equation (1). It is important to note that the applied formalism is not restricted to any model of earthquake magnitude distribution. It means, that our approach is open to any alternative model; for example, when earthquake magnitudes are binned (Bender, 1983; Guttorp and Hopkins, 1986; Tinti and Mulargia, 1987), when errors in earthquake magnitudes determination are taken into account (Tinti and Mulargia, 1985; Rhoades, 1996; Kijko and Sellevoll, 1992), when uncertainty of the model itself and its parameters are taken into account (Campbell, 1982), when new parameter such as the maximum possible earthquake magnitude $m_{\max}$ is introduced, etc. The only difference is that in some cases (such as in the case of application of Bender's model of magnitude binning), estimation of the b-value would require a recursive procedure.

Data and Resources

Only simulated data were used in this short note. Plots were made using the MATLAB package version 7.10 (R2010).

The MATLAB computer program, used for the calculation of the newly derived estimator of the b-value, is available from the authors. In addition to the b-value, the program can be used for the assessment of the mean activity rate λ and the area-characteristic, maximum possible earthquake magnitude $m_{\max}$.

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Appendix A

Derivation of the maximum likelihood estimator $\hat{\beta}$ for $s = 2$. The generalization for any number of subcatalogs follows intuitively.

From the condition $d \ln L(\beta)/d\beta = 0$, where $L(\beta) = \prod_{i=1}^{s} \prod_{j=1}^{n_i} \beta \exp[-\beta(m_j^i - m_{\min}^i)]$, it follows that

$$\sum_{i=1}^{2} \sum_{j=1}^{n_i} \frac{1}{\beta} + \sum_{i=1}^{2} \sum_{j=1}^{n_i} \left\{ \frac{\exp[-\beta(m_j^i - m_{\min}^i)]}{\exp[-\beta(m_j^i - m_{\min}^i)]} [-(m_j^i - m_{\min}^i)] \right\} = 0,$$

or, equivalently,

$$\sum_{i=1}^{2}\sum_{j=1}^{n_i}\frac{1}{\beta}+\sum_{i=1}^{2}\sum_{j=1}^{n_i}[-(m_j^i-m_{\min}^i)]=0. \quad \text{(A1)}$$

After the introduction of the following notation,

$$\frac{1}{\beta_1}=\frac{\sum_{j=1}^{n_1}m_j^1}{n_1}-m_{\min}^1, \qquad \frac{1}{\beta_2}=\frac{\sum_{j=1}^{n_2}m_j^2}{n_2}-m_{\min}^2,$$
$$r_1=\frac{n_1}{n_1+n_2}, \quad \text{and} \quad r_2=\frac{n_2}{n_1+n_2},$$

equation (A1) takes the form

$$\frac{1}{\hat{\beta}}=\frac{r_1}{\hat{\beta}_1}+\frac{r_2}{\hat{\beta}_2},$$

or, equivalently

$$\frac{1}{\hat{\beta}}=\left(\frac{r_1}{\hat{\beta}_1}+\frac{r_2}{\hat{\beta}_2}\right)^{-1}. \quad \text{(A2)}$$

Equation (A2) describes the generalized Aki–Utsu b-value estimator in the case of two subcatalogs ($s=2$). Extension of equation (A2) for any number of subcatalogs ($s=1$ or $s\geq 2$) is obvious and takes the form of equation (7).

Appendix B

Derivation of the standard error (sample standard deviation) $\hat{\sigma}_{\hat{\beta}}$ for $s=2$. The generalization for any number of subcatalogs follows intuitively.

Let $\mathcal{I}(\theta)$ denote the Fisher information and be defined as

$$\mathcal{I}(\theta)=-E\left[\frac{d^2}{d\theta^2}\ln\{L(\theta)\}\right],$$

where E denotes the operator of expectation and θ is the parameter to be estimated. The standard error of the maximum likelihood estimator $\hat{\theta}$ is then defined as

$$\hat{\sigma}_{\hat{\theta}}=\frac{1}{\sqrt{\mathcal{I}(\hat{\theta})}}.$$

For the problem in hand, $\theta\equiv\beta$. The Fisher information for the maximum likelihood estimator $\hat{\beta}$ is derived by differentiating equation (A1) for a second time in terms of β, which provides the equation

$$\mathcal{I}(\hat{\beta})=-E\left[-\frac{n_1}{\beta^2}-\frac{n_2}{\beta^2}\right] \quad \text{for } k=2.$$

The standard error for $\hat{\beta}$ is therefore

$$\hat{\sigma}_{\hat{\beta}}=\frac{1}{\sqrt{\left[\frac{n_1+n_2}{\hat{\beta}^2}\right]}},$$

or, equivalently,

$$\hat{\sigma}_{\hat{\beta}}=\frac{\hat{\beta}}{\sqrt{n_1+n_2}}.$$

For any number of subcatalogs $s\geq 1$, the standard error of the newly derived estimator for β takes the form

$$\hat{\sigma}_{\hat{\beta}}=\frac{\hat{\beta}}{\sqrt{\sum_{i=1}^{s}n_i}}.$$

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