On the bivariate Kummer-beta type IV distribution

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Abstract

In this paper the non-central bivariate Kummer-beta type IV distribution is introduced and derived via the Laplace transform of the non-central bivariate beta distribution by Gupta et al. (2009). We focus on and discuss the central bivariate Kummer-beta type IV distribution; this distribution is a special case of the non-central bivariate Kummer-beta type IV distribution and extends the popular Jones' bivariate beta distribution. The probability density functions of the product and the ratio of the components of the central bivariate Kummer-beta type IV distribution are also derived and we provide tabulations of the associated lower percentage points as well as some upper percentage points that are useful in reliability.

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1 Introduction

We explicitly derive the *non-central bivariate Kummer-beta type IV distribution* via the Laplace transform of the non-central bivariate beta distribution by Gupta et al. (2009); this latter distribution may be regarded as a generalization of the popular *Jones' bivariate beta distribution*, which was independently proposed by Jones (2001) and Olkin and Liu (2003) and is a special case of models proposed by Libby and Novick (1982) and Sarabia and Castillo (2006). The *central bivariate Kummer-beta type IV distribution* follows from the *non-central bivariate Kummer-beta type IV distribution* by setting the non-centrality parameter, δ, equal to zero (i.e. $\delta = 0$). The *central bivariate Kummer-beta type IV distribution* is a special case of the *bimatrix variate Kummer-beta type IV distribution* defined by Bekker et al. (2010).

Kummer-type distributions form an integral part of statistical distribution theory and a number of these distributions have been proposed. In the univariate case, for example, Armero and Bayarri (1997) introduced the *Kummer-gamma* distribution (which is an extension of the well-known gamma distribution). Ng and Kotz (1995) subsequently examined some properties of the *Kummer-gamma* distribution, introduced the *Kummer-beta* distribution (which is an extension of the familiar beta type I distribution) and also proposed and studied the *multivariate Kummer-gamma* and *multivariate Kummer-beta* families of distributions. In the matrix variate case, there is the work by Gupta et al (2001), Nagar and Gupta (2002) and Nagar and Cardeño (2001). These authors proposed and studied matrix variate generalizations of the multivariate Kummer-beta and the multivariate Kummer-gamma families of distributions, which are called the *matrix variate Kummer-Dirichlet* (or the *matrix variate Kummer-beta*) and the *matrix variate Kummer-gamma* distributions. It should be noted that these Kummer distributions get their name from the fact that their normalizing constants are all defined in terms of one of the two so-called Kummer functions (see e.g. Rainville, 1960, p. 124-126).

The distributions of the *product* and the *ratio* of the components of *independent* and *dependent* random variables arise in various applications (see e.g. Nagar et al. (2009), Gupta and Nadarajah (2008), Joarder (2007), Pham-Gia and Turkkan (2002) and Pham-Gia (2000)). In this paper, we also study the product and the ratio of the *central bivariate Kummer-beta type IV distribution.*

The benefits of introducing the *central bivariate Kummer-beta type IV distribution* and the distributions of the *product* and the *ratio* of its components will be discussed and demonstrated by graphical representations of their density functions.

The rest of this paper is organized as follows: In section 2 we derive the joint probability density function (pdf) $g(x_1, x_2; \delta, \psi)$ of the non-central bivariate Kummer-beta type IV distribution and its central counterpart with pdf $g(x_1, x_2; \delta = 0, \psi)$. Because $g(x_1, x_2; \delta, \psi)$ is a finite mixture of $g(x_1, x_2; \delta = 0, \psi)$, we focus on the central bivariate Kummer-beta type IV distribution and its properties. Then, the pdf's of the marginal distributions $m(x_i)$ for $i = 1, 2$, the pdf's of the conditional distributions $h(x_i|x_j)$ for $i, j = 1, 2$ and $i \neq j$ and the product moment $E(X_1^r X_2^s)$ of the central bivariate Kummer-beta type IV are derived. In section 3 we derive the distributions of the product $P = X_1X_2$ and the ratio $R = \frac{X_1}{X_2}$ of the correlated components of the central bivariate Kummer-beta type IV distribution. In section 4 we show the relationship between the central bivariate Kummer-beta type IV distribution and Jones' bivariate beta type I distribution (and their associated properties). In Section 5 we investigate the influence of the shape parameter, ψ, of the central bivariate Kummer-beta type IV distribution. Thus, its effect on the correlation between the components of this distribution as well as its effect on the pdf's of (X_1, X_2), the marginal of X_1 and the ratio R will be shown. We complete the paper with an application by looking at some percentage points.

2 The Bivariate Kummer-Beta type IV distribution

In this section we derive the pdf of the non-central bivariate Kummer-beta type IV distribution $g(x_1, x_2; \delta, \psi)$. The central bivariate Kummer-beta type IV distribution, which is obtained by setting the non-centrality parameter equal to zero (i.e. $\delta = 0$), will be used to derive the pdf's of the marginal distributions $m(x_i)$ for $i = 1, 2$, the pdf's of the conditional distributions $h(x_i|x_j)$ for $i, j = 1, 2$ and $i \neq j$ and the product moment $E(X_1^r X_2^s)$.

The pdf $g(x_1, x_2; \delta, \psi)$ is derived via the Laplace transform (a technique used in Marshall and Olkin 2007, p 260) of the non-central bivariate beta distribution by Gupta et al. (2009), which has pdf

$$f(x_1, x_2; \delta) = \frac{e^{-\delta} x_1^{a-1} x_2^{b-1} (1 - x_1)^{b+c-1} (1 - x_2)^{a+c-1}}{B(a, b, c)(1 - x_1x_2)^{a+b+c}} {}_1F_1\left(a + b + c; c; \frac{(1 - x_1)(1 - x_2)\delta}{1 - x_1x_2}\right) \quad (1)$$

or, equivalently,

$$f(x_1, x_2; \delta) = C \sum_{d=0}^{\infty} \frac{e^{-\delta}\delta^d}{d!} x_1^{a-1} x_2^{b-1} (1 - x_1)^{b+c+d-1} (1 - x_2)^{a+c+d-1} (1 - x_1x_2)^{-(a+b+c+d)} \quad (2)$$

for $0 \leq x_1, x_2 \leq 1$, $a, b, c > 0$ and where $C^{-1} = B(a, b, c+d) = \frac{\Gamma(a)\Gamma(b)\Gamma(c+d)}{\Gamma(a+b+c+d)}$ denotes the normalizing constant (Gupta et al, 2009) and $\delta \geq 0$ denotes the non-centrality parameter.

Note that, when $\delta = 0$ we obtain Jones' bivariate beta distribution, which has pdf

$$f(x_1, x_2; \delta = 0) = C x_1^{a-1} x_2^{b-1} (1 - x_1)^{b+c-1} (1 - x_2)^{a+c-1} (1 - x_1x_2)^{-(a+b+c)} \quad (3)$$

for $0 \leq x_1, x_2 \leq 1$, $a, b, c > 0$ and where $C^{-1} = B(a, b, c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$ denotes the normalizing constant (Jones, 2001). In this latter case (i.e. when $\delta = 0$), X_1 and X_2, each have a standard beta type I distribution, i.e. $X_1 \sim Beta(a, c)$ and $X_2 \sim Beta(b, c)$ over $0 \leq x_1, x_2 \leq 1$.

Also, note that, the pdf of the non-central bivariate beta distribution in (2) is expressed as an infinite mixture of central bivariate distributions (i.e. Jones' bivariate beta distribution); this alternative expression (see (2))simplifies the derivation of the non-central bivariate Kummer-beta type IV distribution.

Theorem 1

The pdf of the non-central bivariate Kummer-beta type IV distribution is given by

$$g(x_1, x_2; \delta, \psi) = K \sum_{d=0}^{\infty} \frac{\delta^d}{d!} x_1^{a-1} x_2^{b-1} (1 - x_1)^{b+c+d-1} (1 - x_2)^{a+c+d-1} (1 - x_1x_2)^{-(a+b+c+d)} e^{-\delta\psi(x_1+x_2)} \quad (4)$$

where $0 \leq x_1, x_2 \leq 1$, $a, b, c > 0$, $\delta \geq 0$, $-\infty < \psi < \infty$ *and the normalizing constant* K *is given by*

$$\begin{aligned} K^{-1} &= e^{-\delta} \sum_{d=0}^{\infty} \frac{\delta^d}{d!} \left(\sum_{k=0}^{\infty} \frac{(a+b+c+d)_k}{k! \left(B(a+c+d, b+k) B(b+c+d, a+k)\right)^{-1}} \right. \\ &\quad \left. \times {}_1F_1(b+k; a+b+c+d+k; -\psi) \, {}_1F_1(a+k; a+b+c+d+k; -\psi) \right) \end{aligned} \quad (5)$$

where $B(.)$ denotes the beta function, ${}_1F_1(.)$ denotes the confluent hypergeometric function (Gradshteyn, 2007, Section 9.2, p 1022) and $(n)_k$ is the Pochhammer symbol defined as $\frac{\Gamma(n+k)}{\Gamma(n)}$.

This distribution is denoted as $(X_1, X_2) \sim NCBKB^{IV}(a, b, c, \delta, \psi)$.

Proof

Apply the Laplace transform on the pdf $f(x_1, x_2; \delta)$ i.e. the non-central bivariate beta distribution by Gupta et al. (2009) as given in (2).

$$\begin{aligned}
L_f(\psi) &= \int_0^1 \int_0^1 e^{-\psi(x_1+x_2)} f(x_1, x_2; \delta) dx_1 dx_2 \\
&= e^{-\delta} \sum_{d=0}^{\infty} \frac{\delta^d}{d!} \left(\sum_{k=0}^{\infty} \frac{(a+b+c+d)_k}{k!} \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d-1} e^{-\psi x_2} \right. \\
&\quad \left. \times \int_0^1 x_1^{a+k-1} (1-x_1)^{b+c+d-1} e^{-\psi x_1} dx_1 dx_2 \right) \\
&= e^{-\delta} \sum_{d=0}^{\infty} \frac{\delta^d}{d!} \left(\sum_{k=0}^{\infty} \frac{(a+b+c+d)_k}{k!} \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d-1} e^{-\psi x_2} \right. \\
&\quad \left. \times \frac{{}_1F_1(a+k; a+b+c+d+k; -\psi)}{(B(b+c+d, a+k))^{-1}} dx_2 \right) \\
&= e^{-\delta} \sum_{d=0}^{\infty} \frac{\delta^d}{d!} \left(\sum_{k=0}^{\infty} \frac{(a+b+c+d)_k}{k! \left(B(a+c+d, b+k) B(b+c+d, a+k) \right)^{-1}} \right. \\
&\quad \left. \times {}_1F_1(b+k; a+b+c+d+k; -\psi)\, {}_1F_1(a+k; a+b+c+d+k; -\psi) \right)
\end{aligned}$$

The above result is obtained by expanding the term $(1-x_1x_2)^{-(a+b+c+d)}$ as a power series into $\sum_{k=0}^{\infty} \frac{(a+b+c+d)_k x_1^k x_2^k}{k!}$ and then using the integral representation of the confluent hypergeometric function, ${}_1F_1(.)$ (Gradshteyn, 2007, Eq 3.383, p 347).

Using the Laplace transform to obtain the normalizing constant, we define the non-central bivariate Kummer-beta type IV distribution with pdf as

$$g(x_1, x_2; \delta, \psi) = K \sum_{d=0}^{\infty} \frac{\delta^d}{d!} x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d-1} (1-x_1x_2)^{-(a+b+c+d)} e^{-\delta\psi(x_1+x_2)}$$

with

$$\begin{aligned}
K^{-1} &= L_f(\psi) \\
&= e^{-\delta} \sum_{d=0}^{\infty} \frac{\delta^d}{d!} \left(\sum_{k=0}^{\infty} \frac{(a+b+c+d)_k}{k! \left(B(a+c+d, b+k) B(b+c+d, a+k) \right)^{-1}} \right. \\
&\quad \left. \times {}_1F_1(b+k; a+b+c+d+k; -\psi)\, {}_1F_1(a+k; a+b+c+d+k; -\psi) \right).
\end{aligned}$$

■

Corollary 1

The pdf of the central bivariate Kummer-beta type IV distribution is obtained by substituting $\delta = 0$ in (4) and (5) and is given by

$$g(x_1, x_2; \delta = 0, \psi) = K x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c-1} (1-x_2)^{a+c-1} (1-x_1x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} \qquad (6)$$

where $0 \le x_1, x_2 \le 1$, $a, b, c > 0$, $-\infty < \psi < \infty$, and the normalizing constant K is given by

$$K^{-1}=\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1\left(a+k,a+b+c+k;-\psi\right){}_1F_1\left(b+k,a+b+c+k;-\psi\right)}{\left(B\left(a+c,b+k\right)B\left(b+c,a+k\right)\right)^{-1}} \tag{7}$$

where $B(.)$ denotes the beta function, ${}_1F_1(.)$ denotes the confluent hypergeometric function (Gradshteyn, 2007, Section 9.2, p 1022) and $(n)_k$ is the Pochhammer symbol defined as $\frac{\Gamma(n+k)}{\Gamma(n)}$.

This distribution is denoted as $(X_1, X_2) \sim BKB^{IV}(a,b,c,\psi)$.

Note the following:

1. The central bivariate Kummer-beta type IV distribution may also be obtained by substituting $p=1$ in the pdf of the bimatrix variate Kummer-beta type IV distribution defined by Bekker et al. (2010) (see Section 5.3 of their article). We, however, derived the bivariate case explicitly in the light of the work by Balakrishnan and Lai (2009) and more specifically as an extension of Jones' bivariate beta distribution also discussed in Balakrishnan and Lai (2009, p 379-381).
2. Balakrishnan and Lai (2009, p 377-378) also give various applications for the bivariate beta distribution which could possibly be extended to the Jones' bivariate beta distribution and the bivariate Kummer-beta type IV distribution. For example, the ratio of the components of bivariate beta distributions is often used in the context of reliability theory in the stress-strength model. The central bivariate Kummer-beta type IV distribution can easily be extended to this application as is done at the end of Section 5.
3. In the rest of the paper and in subsequent derivations of the pdf's of the product and the ratio we will focus on the central bivariate Kummer-beta type IV distribution, since the non-central bivariate Kummer-beta type IV distribution is given as an infinite mixture of central bivariate Kummer-beta type IV distributions (see (4) and (6)).
4. The infinite sums in (5) and (7) will converge even for relatively large values of ψ.

Theorem 2

If $(X_1, X_2) \sim BKB^{IV}(a,b,c,\psi)$, the marginal pdf of X_1 is given by

$$\begin{aligned} m(x_1) &= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}B\left(b,a+c\right)\Phi_1\left(b,a+b+c,a+b+c,\psi,x_1\right) && (8)\\ &= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(a+b+c)_k\, x_1^k}{k!}\frac{{}_1F_1\left(b+k;a+b+c+k;-\psi\right)}{\left(B\left(b+k,a+c\right)\right)^{-1}} && (9)\end{aligned}$$

where $0 \leq x_1 \leq 1$, $a,b,c>0$, $B(.)$ denotes the beta function, $\Phi_1(.)$ denotes the confluent hypergeometric series of two variables (Gradshteyn, 2007, Eq 9.261, p 1031) and K is defined in (7).

Proof

Using (6), the first representation of $m(x_1)$ given in (8), is obtained by using the integral representation of the confluent hypergeometric series of two variables, $\Phi_1(.)$ (Gradshteyn, 2007, Eq 3.385, p 349):

$$\begin{aligned} m(x_1) &= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\int_0^1 x_2^{b-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_2}dx_2\\ &= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}B\left(b,a+c\right)\Phi_1\left(b,a+b+c,a+b+c,\psi,x_1\right).\end{aligned}$$

The second representation of $m(x_1)$ given in (9), is obtained by expanding the term $(1-x_1x_2)^{-(a+b+c)}$ in (6) as a power series into $\sum_{k=0}^{\infty}\frac{(a+b+c)_k x_1^k x_2^k}{k!}$, hence

$$m(x_1)=Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(a+b+c)_k x_1^k}{k!}\int_0^1 x_2^{b+k-1}(1-x_2)^{a+c-1}e^{-\psi x_2}dx_2$$

and (9) follows directly by using the integral representation of the confluent hypergeometric function, ${}_1F_1(.)$. ∎

Equation (9) is more useful (in the sense that it is easier to implement and/or program) in computer packages such as Mathematica when we want to graph the pdf $m(x_i)$ for $i = 1, 2$ as the ${}_1F_1(.)$ is a built-in routine.

Note that, the marginal pdf of X_2 is obtained by substituting x_2 for x_1 in (8) and (9) and interchanging the parameters a and b.

Theorem 3

If $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$, *the conditional pdf of* $X_2|X_1$ *is given by*

$$h(x_2|x_1) = Dx_2^{b-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_2} \tag{10}$$

where $0 \leq x_2 \leq 1$, $a, b, c > 0$ *and the normalizing constant* D *is defined as*

$$D^{-1} = B(b, a+c)\,\Phi_1(b, a+b+c, a+b+c, \psi, x_1).$$

Proof

Using the joint pdf $g(x_1, x_2; \delta = 0, \psi)$ in (6) and the marginal pdf $m(x_1)$ in (8), expression (10) for the conditional pdf of $X_2|X_1$ follows directly, i.e. $h(x_2|x_1) = \frac{g(x_1,x_2;\delta=0,\psi)}{m(x_1)}$. ∎

Note that the conditional pdf of $X_1|X_2$ is obtained by interchanging the variables x_1 and x_2 and the parameters a and b in (10).

Theorem 4

If $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$, *the product moment i.e.* $E(X_1^r X_2^s)$, *equals*

$$\begin{aligned} & K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1(a+k+r, a+b+c+k+r; -\psi)\,{}_1F_1(b+k+s, a+b+c+k+s; -\psi)}{(B(a+c, b+k+s)\,B(b+c, a+k+r))^{-1}} \\ = \; & (A(a,b,c,0,0))^{-1} \times A(a,b,c,s,r) \end{aligned} \tag{11}$$

where

$$A(a,b,c,s,r) = \sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1(a+k+r, a+b+c+k+r; -\psi)\,{}_1F_1(b+k+s, a+b+c+k+s; -\psi)}{(B(a+c, b+k+s)\,B(b+c, a+k+r))^{-1}} \tag{12}$$

and $A(a,b,c,0,0)^{-1} = K$ *as defined in (7).*

Proof

From (6), expanding the term $(1-x_1x_2)^{-(a+b+c)}$ as a power series into $\sum\limits_{k=0}^{\infty}\frac{(a+b+c)_k x_1^k x_2^k}{k!}$ and using the integral representation of the confluent hypergeometric function, ${}_1F_1(.)$, we obtain the product moment as

$$\begin{aligned} & E(X_1^r X_2^s) \\ = \; & K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\int_0^1 x_2^{b+k+s-1}(1-x_2)^{a+c-1}e^{-\psi x_2}\int_0^1 x_1^{a+k+r-1}(1-x_1)^{b+c-1}e^{-\psi x_1}dx_1dx_2 \\ = \; & K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1(a+k+r; a+b+c+k+r; -\psi)}{B(b+c, a+k+r)^{-1}}\int_0^1 x_2^{b+k+s-1}(1-x_2)^{a+c-1}e^{-\psi x_2}dx_2 \\ = \; & K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1(a+k+r; a+b+c+k+r; -\psi)\,{}_1F_1(b+k+s; a+b+c+k+s; -\psi)}{(B(b+c, a+k+r)\,B(a+c, b+k+s))^{-1}} \\ = \; & (A(a,b,c,0,0))^{-1} \times A(a,b,c,s,r) \end{aligned}$$

with $A(a,b,c,s,r)$ as defined in (12). ■

The estimation of the parameters of the bivariate Kummer-beta type IV distribution (i.e. a, b, c and ψ) is an important issue. However, it is currently being investigated and will be reported elsewhere as it is beyond the scope of this paper.

3 Distribution of the ratio and product of the components

In this section we derive exact expressions for the pdf's of the product and ratio of the correlated components of the central bivariate Kummer-beta type IV distribution i.e. $P = X_1X_2$ and $R = \frac{X_1}{X_2}$, in terms of Meijer's G-function (see Mathai, 1993, Definition 2.1, p 60) using the Mellin transform and the inverse Mellin transform (see Mathai, 1993, Definition 1.8, p 23).

Theorem 5

If $(X_1, X_2) \sim BKB^{IV}(a,b,c,\psi)$ *and we let* $P = X_1X_2$ *and* $R = \dfrac{X_1}{X_2}$, *then the pdf's of* P *and* R *are given by*

1.

$$v(p) = K\Gamma(a+c)\Gamma(b+c)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(a+b+c)_k\frac{(-\psi)^{j+l}}{j!k!l!}G_{2,2}^{2,0}\left(p\,\middle|\,\begin{matrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix}\right)\ \text{for } 0\le p\le 1 \tag{13}$$

where

$$\begin{array}{ll}\alpha_1 = a+b+c+k+j-1 & \alpha_2 = a+b+c+k+l-1\\ \beta_1 = b+k+j-1 & \beta_2 = a+k+l-1\end{array}$$

2.

$$w(r) = K\Gamma(a+c)\Gamma(b+c)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(a+b+c)_k\frac{(-\psi)^{j+l}}{j!k!l!}G_{2,2}^{1,1}\left(r\,\middle|\,\begin{matrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix}\right)\ \text{for } r\ge 0 \tag{14}$$

where

$$\begin{array}{ll}-\alpha_1 = b+k+j & \alpha_2 = a+b+c+k+l-1\\ \beta_1 = a+k+l-1 & -\beta_2 = a+b+c+k+j\end{array}$$

with K *as defined in* (7).

Proof

1. Setting $r = s = h-1$ in (11) and using the series representation of the confluent hypergeometric function, ${}_1F_1(.)$ (Gradshteyn, 2007, Section 9.21, p 1023), we obtain an expression for the Mellin transform of $g(x_1,x_2;\delta=0,\psi)$ as

$$\begin{aligned}&M_g(h)\\ =\ &E(P^{h-1}) = E\left((X_1X_2)^{h-1}\right)\\ =\ &(A(a,b,c,0,0))^{-1}\times A(a,b,c,h-1,h-1)\\ =\ &K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Gamma(a+b+c+k)\frac{\Gamma(\beta_1+h)\Gamma(\beta_2+h)}{\Gamma(\alpha_1+h)\Gamma(\alpha_2+h)}\frac{(-\psi)^{j+l}}{j!k!l!}\end{aligned}$$

with

$$\begin{array}{ll}\alpha_1 = a+b+c+k+j-1 & \alpha_2 = a+b+c+k+l-1\\ \beta_1 = b+k+j-1 & \beta_2 = a+k+l-1.\end{array}$$

Using the inverse Mellin transform, the density of the product of the components of $g(x_1, x_2; \delta = 0, \psi)$ i.e. P in terms of the Meijer's G-function is given by:

$$\begin{aligned} v(p) &= K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Gamma(a+b+c+k)\frac{(-\psi)^{j+l}}{j!k!l!}\frac{1}{2\pi i}\int\frac{\prod_{j=1}^{2}\Gamma\left(\beta_j+h\right)}{\prod_{j=1}^{2}\Gamma\left(\alpha_j+h\right)}p^{-h}dh \\ &= K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Gamma(a+b+c+k)\frac{(-\psi)^{j+l}}{j!k!l!}G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix}\right) \end{aligned}$$

2. Setting $r = h - 1$ and $s = 1 - h$ in (11) and using the series representation of the confluent hypergeometric function, ${}_1F_1(.)$, we obtain an expression for the Mellin transform of $g(x_1, x_2; \delta = 0, \psi)$ as

$$\begin{aligned} & M_g(h) \\ =\; & E(R^{h-1}) = E\left(\left(\frac{X_1}{X_2}\right)^{h-1}\right) \\ =\; & \left(A(a,b,c,0,0)\right)^{-1}\times A(a,b,c,h-1,1-h) \\ =\; & K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Gamma(a+b+c+k)\frac{\Gamma\left(1-\alpha_1-h\right)\Gamma\left(\beta_1+h\right)}{\Gamma\left(1-\beta_2-h\right)\Gamma\left(\alpha_2+h\right)}\frac{(-\psi)^{j+l}}{j!k!l!} \end{aligned}$$

with

$$\begin{aligned} -\alpha_1 &= b+k+j & \alpha_2 &= a+b+c+k+l-1 \\ \beta_1 &= a+k+l-1 & -\beta_2 &= a+b+c+k+j. \end{aligned}$$

Similarly, using the inverse Mellin transform, the density of the ratio of the components of $g(x_1, x_2; \delta = 0, \psi)$ i.e. R in terms of the Meijer's G-function given by (14) follows. ■

4 Relationship with other distributions

In this section we show the relationship between the central bivariate Kummer-beta type IV distribution and Jones' bivariate beta distribution and the relationships between the associated properties of these distributions e.g. the distributions of the product and the ratio of their correlated components. These relationships follow when we set $\psi = 0$ i.e. the additional parameter that we have introduced in the case of the central bivariate Kummer-beta type IV distribution.

1. If we set $\psi = 0$ the marginal pdf $m(x_1)$ simplifies to the standard beta type I pdf i.e. $X_1 \sim Beta\,(a, c)$.

2. If we set $\psi = 0$ the pdf $g(x_1, x_2; \delta = 0, \psi)$ of the *central bivariate Kummer-beta type IV distribution* reduces to the pdf $f\,(x_1, x_2; \delta = 0)$ of *Jones' bivariate beta distribution.* This shows, as mentioned earlier, that $f\,(x_1, x_2; \delta = 0)$ may be regarded as a special case of $g(x_1, x_2; \delta = 0, \psi)$.

3. If we set $\psi = 0$ the pdf's of $v(p)$ and $w(r)$ (see (13) and (14)) simplify to the pdf's of the product and the ratio of the components of Jones' bivariate beta distribution derived by Nagar et al (2009).

5 Shape analysis and computations

In this section we illustrate the effect of the parameter ψ on the shape of the central bivariate Kummer-beta type IV density, the marginal density, the density of $R = \frac{X_1}{X_2}$ as well as on the correlation between X_1 and X_2. We provide tabulations of the percentage points of $R = \frac{X_1}{X_2}$. The programming was done by making use of built-in routines of the package Mathematica.

Figures 1 and 2 illustrate the effect of the parameter ψ for $\psi = -1.1, 0$ and 1.1 on the central bivariate Kummer-beta type IV pdf (see (6)) for different choices of the parameters a, b and c; these two figures may be compared to Figure 1 of Olkin and Liu (2003). Figure 2 in this paper also contains the contour plots for easy comparison. We note that the domain of the graphs in Figure 1 and 2 are all $\mathbb{R}^2 : [0,1] \times [0,1]$. Considering panels (i)(a), (i)(b) and (i)(c) in Figure 1 we see that the parameter ψ shifts the bell of the density. The correlations between X_1 and X_2 corresponding to the three values of ψ are 0.70399, 0.607467 and 0.506187, respectively. Looking at panels (ii)(a), (ii)(b) and (ii)(c) in Figure 1, we see that the parameter ψ heightens and lowers the graph. The maximum height is obtained for $\psi = 1.1$ at about 10, while the minimum is obtained for $\psi = 1.1$ at about 2.5. The correlations between X_1 and X_2 corresponding to the three values of ψ are 0.424531, 0.346873 and 0.263492, respectively.

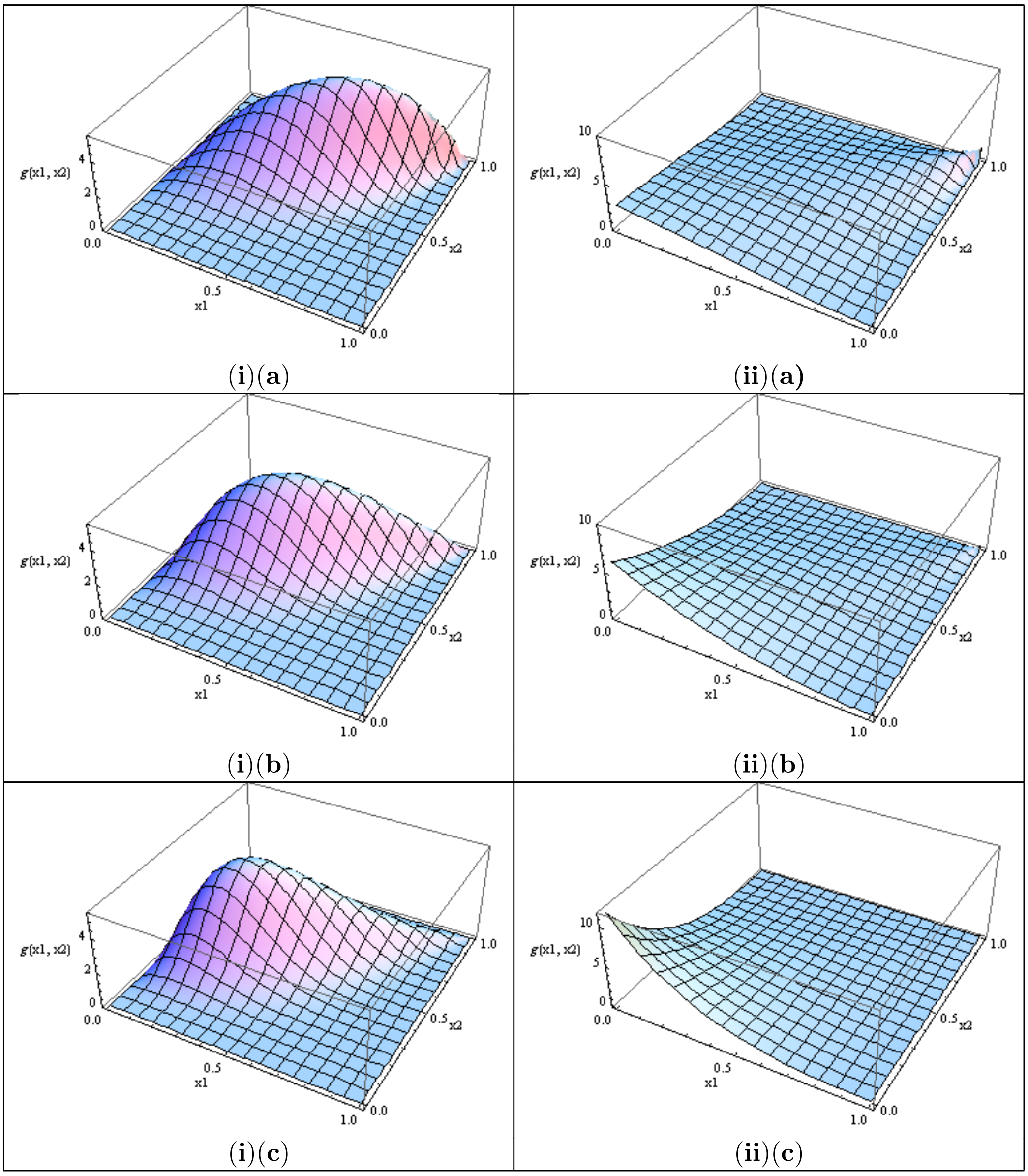

Figure 1: Bivariate Kummer beta type IV density function for (i) $a = 2$, $b = 5$, $c = 3$; (ii) $a = 1$, $b = 1$, $c = 2$. The 3 panels in each column are: (a) $\psi = -1.1$; (b) $\psi = 0$; (c) $\psi = 1.1$.

In Figure 2 we observe the same results as in panels (i)(a), (i)(b) and (i)(c) of Figure 1. The bell of the density moves towards the origin for positive values of ψ and away from the origin for negative values of ψ, as is also easily seen in the contour plots. The correlations between X_1 and X_2 corresponding to the three values of ψ are 0.535713, 0.490536 and 0.427197, respectively.

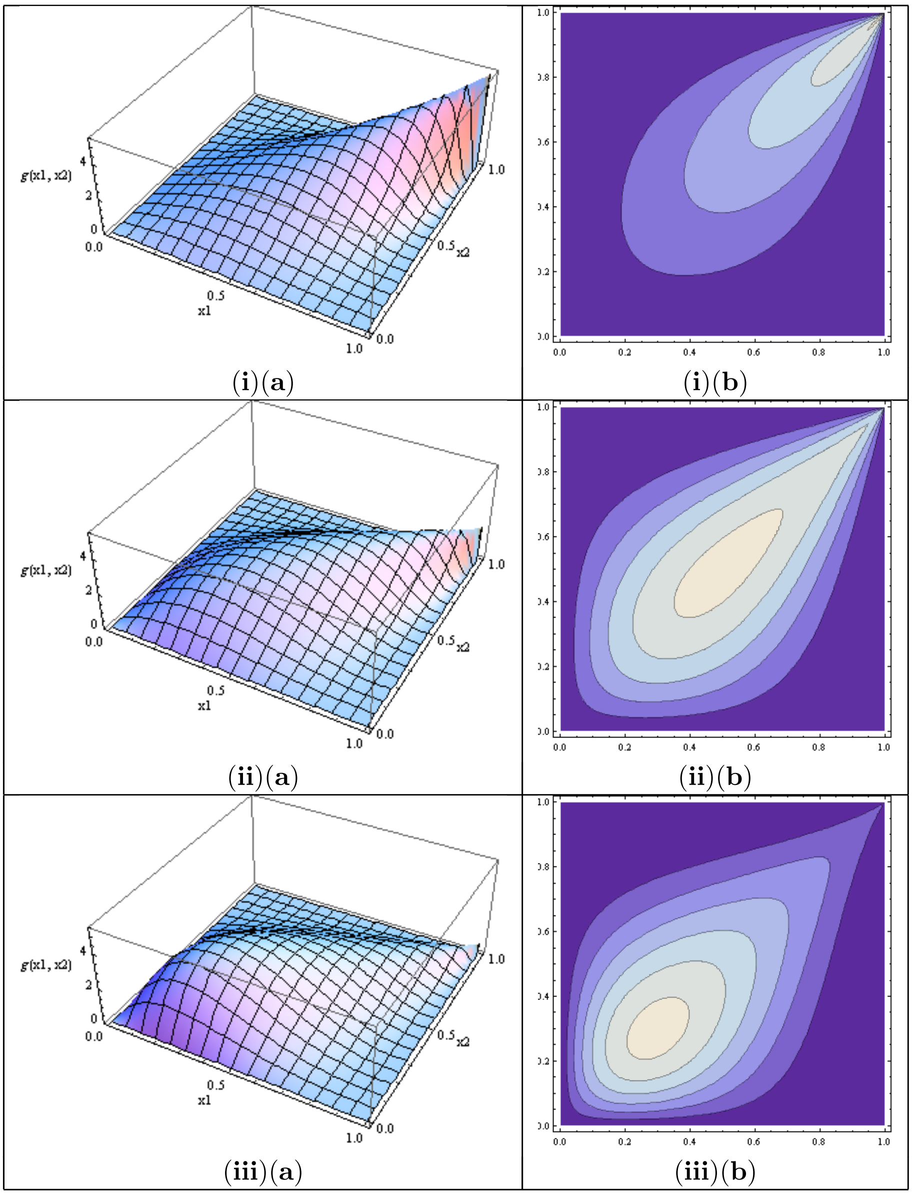

Figure 2: Bivariate Kummer beta type IV density function (a) and contour plots (b) for (i) $\psi = -1.1$; (ii) $\psi = 0$; (iii) $\psi = 1.1$. The parameter values are $a = b = c = 2$.

Figure 3 illustrates the effect of the parameter $\psi \in [-10, 10]$ on the correlation between X_1 and X_2 using Equation 11. We see that: (i) the parameter ψ can both increase and decrease the correlations for different values of a, b and c and (ii) we can obtain a wide range of correlations between 0 and 1 - depending on the values of a, b, c and ψ. The correlation cannot be negative because the central bivariate Kummer-beta type IV distribution is totally positive of order 2 (denoted TP_2). Balakrishnan and Lai (2009, p 115) define a bivariate distribution to be TP_2 if

$$h(x_1, y_1)h(x_2, y_2) \geq h(x_1, y_2)h(x_2, y_1) \tag{15}$$

for $x_1 < x_2$, $y_1 < y_2$ and where $h(x, y)$ denotes the pdf of the particular bivariate distribution.

In order to prove that the pdf $g(x_1, x_2; \delta = 0, \psi)$ is TP_2, we substitute $g(x_1, x_2; \delta = 0, \psi)$ in (15). We note that $g(x_1, y_1; \delta = 0, \psi)g(x_2, y_2; \delta = 0, \psi) \geq g(x_1, y_2; \delta = 0, \psi)g(x_2, y_1; \delta = 0, \psi)$ if and only if $(1 - x_1y_1)(1 - x_2y_2) \leq (1 - x_1y_2)(1 - x_2y_1)$, which always holds. We, therefore, conclude that the central bivariate Kummer-beta type IV distribution is TP_2 which implies that it is also positive quadrant dependent (denoted PQD), which in turn implies that the components X_1 and X_2 are always positively correlated (Balakrishnan and Lai, 2009, p 116).

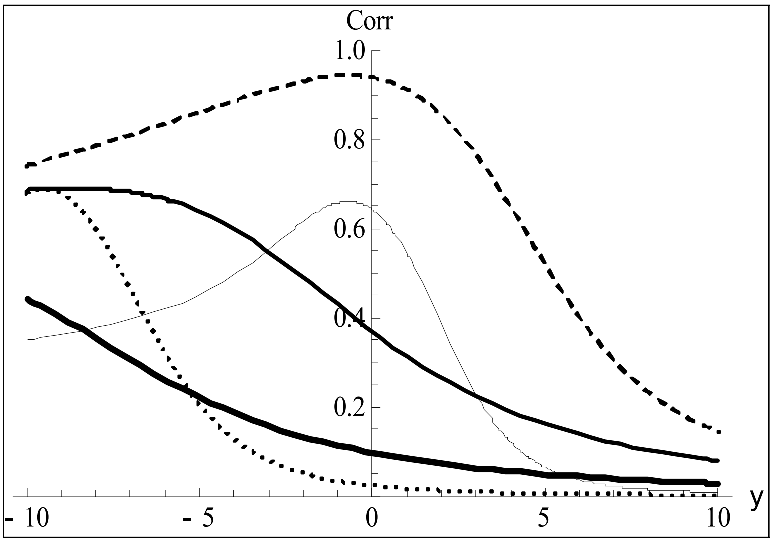

Figure 3: Correlation between X_1 and X_2. The five curves are: thick solid line $a = 1$, $b = 1$, $c = 10$; medium solid line $a = 2$, $b = 4$, $c = 5$; thin solid line $a = 0.5$, $b = 0.9$, $c = 0.1$; dashed line $a = 2.5$, $b = 4$, $c = 0.5$; dotted line $a = 0.1$, $b = 0.5$, $c = 5$.

The influence of the parameter ψ on the corresponding marginal density $m(x_1)$ (see Equation 9) is shown in Figure 4. The solid line in each panel is the pdf of the beta type I distribution. In all four these panels the domain is $\mathbb{R} : [0, 1]$. The four panels represent four of the more frequently found pdf shapes, namely symmetric, u-shaped, negatively skewed and positively skewed. In panel (i) the symmetric beta pdf is pushed a little off center by the parameter ψ i.e. to the left for negative ψ and to the right for positive ψ. In panel (ii) the symmetric u-shaped beta pdf is also pushed a little skew by the parameter ψ. In panel (iii) we see that the parameter ψ changes the kurtosis of the pdf i.e. positive ψ decreases the kurtosis while a negative ψ increases the kurtosis. In panel (iv), the effect of the parameter ψ is to make the pdf a little steeper or flatter.

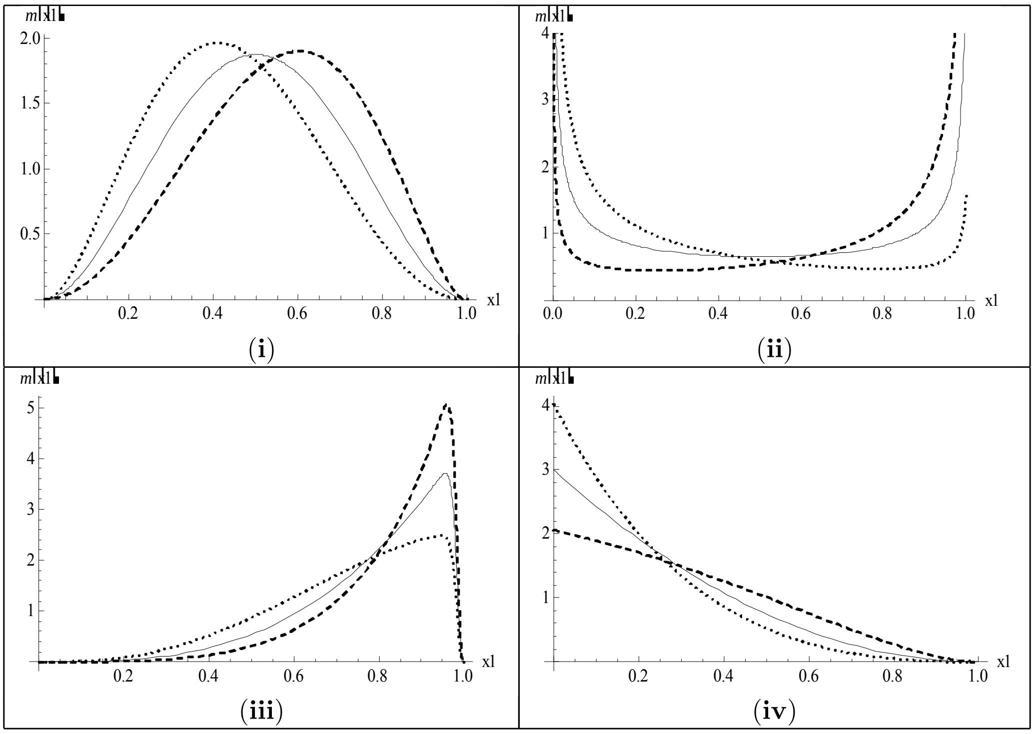

Figure 4: Marginal density of X_1 for (i) $a = 3$, $b = 3$, $c = 3$; (ii) $a = 0.5$, $b = 0.5$, $c = 0.5$; (iii) $a = 4$, $b = 4$, $c = 1$; (iv) $a = 1$, $b = 1$, $c = 3$. The three curves in each panel are: dashed line $\psi = -1.1$, solid line $\psi = 0$, dotted line $\psi = 1.1$.

Figure 5 illustrates the shape of the density of $R = \frac{X_1}{X_2}$ (see Equation 14) for the case $a = 1, b = c = 2$ and $a = b = c = 2$ for different values of ψ. The domain for these graphs is $\mathbb{R} : [0, \infty]$. In panel (i) we see that the value of ψ mostly affects the left tail of the pdf, while the right tail remains basically unchanged. It would seem that the value of ψ does actually have an effect on the shape of the pdf. In panel (ii) we see that the value of ψ only changes the kurtosis with a positive ψ decreasing kurtosis and a negative ψ increasing kurtosis.

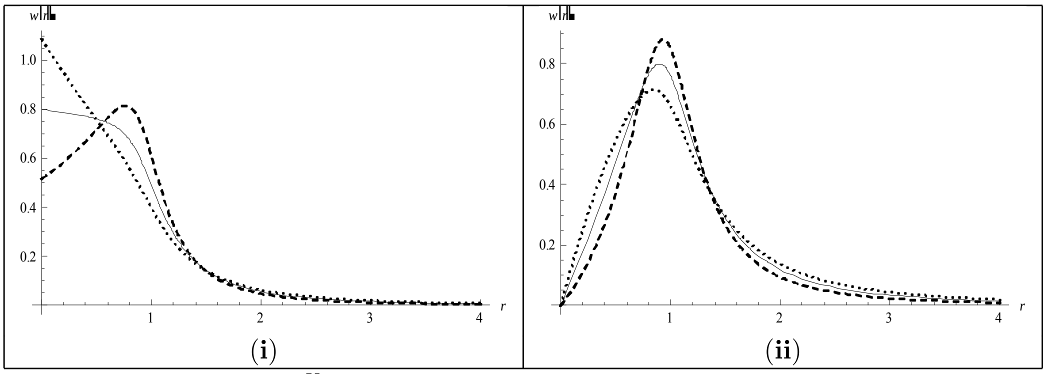

Figure 5: The pdf of the ratio $R = \frac{X_1}{X_2}$ for (i) $a = 1, b = c = 2$ and (ii) $a = b = c = 2$. The three curves in each panel are: dashed line $\psi = -1.1$, solid line $\psi = 0$, dotted line $\psi = 1.1$.

The percentage points r_α for $0 < \alpha < 1$ of R are obtained numerically by solving the equation

$$\int_0^{r_\alpha} w(r)dr = \alpha.$$

Evidently, solving the above integral involves the computation of Meijer's G-function. We used the build-in routines of the package Mathematica. Table 1 provides the numerical values of r_α for $\psi = -1.1$, 0, 1.1 and $\alpha = 0.01$, 0.025, 0.05, 0.1 for the cases $a = 1$, $b = c = 2$ and $a = b = c = 2$. Similar tabulations can be derived for the percentage points p_α for $0 < \alpha < 1$ of $P = X_1X_2$; this is also true for other values of the parameters as well as the upper percentiles. We see that the percentage points in Table 1 confirm the shapes of the pdf's in Figure 5. For example, when $a = 1, b = c = 2$, the first percentile (i.e. $\alpha = 0.01$) values decrease when ψ changes from negative to positive, which is confirmed in Figure 5 panel (i) where the dotted line ($\psi = 1.1$) is highest and the dashed line ($\psi = -1.1$) the lowest for small values of r.

a	**b**	**c**	ψ	$\alpha = \mathbf{0.01}$	**0.025**	**0.05**	**0.1**
1	2	2	−1.1	0.01926	0.04774	0.09413	0.18303
1	2	2	0	0.01250	0.03127	0.06261	0.12552
1	2	2	1.1	0.00925	0.02322	0.04681	0.09511
2	2	2	−1.1	0.19660	0.30078	0.41009	0.55136
2	2	2	0	0.15131	0.23798	0.33449	0.46875
2	2	2	1.1	0.12233	0.19555	0.28012	0.40378

Table 1: The lower percentage points r_α of R

The stress-strength model in the context of reliability is a well-known application of various bivariate beta distributions. This model describes the life of a component with a random strength X_2 subjected to a random stress X_1. The reliability of a component can be expressed as $P(X_1 < X_2)$ or $P(\frac{X_1}{X_2} < 1) = P(R < 1)$. Table 2 provides the reliability of the central bivariate Kummer type IV distribution for $\psi = -1.1$, 0, 1.1 with parameters $a = 1$, $b = c = 2$ and $a = b = c = 2$; these parameters are the same as those used in Table 1. For example, when $a = 1, b = c = 2$ and $\psi = -1.1$, we see that $P(R < 1) = 0.68471$; this implies that the probability that the component will function satisfactorily is 0.68471; or, in other words, the component will fail with probability 0.31529.

a	**b**	**c**	ψ	$\mathbf{P(R < 1)}$
1	2	2	−1.1	0.68471
1	2	2	0	0.74904
1	2	2	1.1	0.75217
2	2	2	−1.1	0.42325
2	2	2	0	0.49782
2	2	2	1.1	0.48213

Table 2: Some reliability values

6 Conclusion

In this paper we introduced and derived the new non-central bivariate Kummer-beta type IV distribution and studied the central bivariate Kummer-beta type IV distribution, which is a special case of the non-central version. We also obtained exact expressions for the density functions of the ratio and the product of the components of the central bivariate distribution. The effect of the shape parameter ψ on the shapes of the central bivariate Kummer-beta type IV density, the marginal density and the density of the ratio were also illustrated. Furthermore, lower percentage points of $R = \frac{X_1}{X_2}$ were given as well as some upper percentage points that are useful in reliability. It was shown that the densities can take different shapes and, therefore, the bivariate Kummer-beta type IV distribution can be used to analyse skewed bivariate data sets. The expressions derived in this paper are a valuable contribution to the existing literature on *Continuous Bivariate Distributions* as the comprehensive work by Balakrishnan and Lai (2009).

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