Preliminary testing of the Cobb-Douglas production function and related inferential issues

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ABSTRACT

In this paper, we consider the multiple regression model in the presence of multicollinearity and study the performance of the preliminary test estimator (PTE) both analytically and computationally, when it is a priori suspected that some constraints may hold on the vector parameter space. The performance of the PTE is further analysed by comparing the risk of some well-known estimators of the ridge parameter through an extensive Monte Carlo simulation study under some bounded and or asymmetric loss functions. An application of the Cobb-Douglas production function is included and from these results as well as the simulation studies, it is clear that the bounded linear exponential loss function outperforms the other loss functions across all the proposed ridge parameters by comparing the risk values.

Key words: BLINEX loss; Cobb-Douglas production function; LINEX loss; Multicollinearity; Preliminary test estimator; Ridge regression

1 Introduction

In this section, we discuss the basic properties of the Cobb-Douglass production function, since this function forms the basis of the theoretical and simulation studies which follows. The Cobb-Douglas production function dates back to the work done by Paul Douglas and Charles Cobb in 1928, which relates production inputs and output for the U.S. manufacturing sector for 1899-1922. It is still the most universal functional form in both theoretical and empirical analyses of production growth and can be formulated as follows:

$$Y_i = \beta_0 X_{1i}^{\beta_1} X_{2i}^{\beta_2} e^{u_i},$$

where Y_i is the i^{th} production output, X_{1i} and X_{2i} are the i^{th} labour and capital input respectively, u_i is the i^{th} stochastic error term, $i = 1, 2, ..., n$, and e denotes the base of the natural log. By making use of the double log transformation, the model can be rewritten as

$$\ln Y_i = \ln \beta_0 + \beta_1 \ln X_{1i} + \beta_2 \ln X_{2i} + u_i \ , i = 1, 2, ..., n$$

The regression coefficients β_1 and β_2 of the Cobb-Douglas production function are interpreted as elasticities and the sum of the two coefficients represents the return to scale. The first coefficient β_1 measures the percentage change in production output for a percentage increase in labour input, by keeping capital input constant (partial elasticity) and β_2 can be interpreted in a similar way as the percentage change in production output for a percentage increase in capital input, by keeping labour input constant. The sum of the two coefficients, namely $\beta_1 + \beta_2$ measures the return to scale and can be interpreted as the typical response of production output to a proportionate change in the two inputs.

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When $\beta_1 + \beta_2 = 1$, it is an indication of constant return to scale which suggests that if the inputs double the production output will double. For $\beta_1 + \beta_2 > 1$, increasing return to scale is observed which suggests that when inputs double, production output will more than double and when $\beta_1 + \beta_2 < 1$, decreasing return to scale is observed which indicates that when inputs double, production output will less than double. It is therefore evident that in order to test for constant return to scale, the vector $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$ should be subjected to a linear restriction.

The use of asymmetric loss functions for econometric modelling has been discussed by Waud (1976), Kunstman (1984), Zellner (1986) and Horowitz (1987) to name a few. Ruge-Murcia (2002) developed and estimated a theoretical model of inflation targeting, similar to the Barro and Gordon model (1983) in which they suggested that the central banker's preferences are asymmetric around the targeted inflation rate. It was further shown that positive deviations from the inflation target must be weighed more heavily compared to negative deviations in the loss function used by the central banker and the linear exponential (LINEX) loss function was proposed. Christofferson and Diebold (1996) mentioned that in the field of volatility forecasting, using squared error loss is not the appropriate loss function to use and Christofferson and Diebold (1997) examined LINEX forecasts under the assumption that the process is normally distributed.

Granger (1999) also discussed the importance of using a bounded loss function, where he gives different examples of the advantage of penalising over or underestimation of estimators, depending on which one of the two is the most serious. In a paper by Wen and Levy (2001), a new parametric family of bounded and asymmetric loss functions, called the bounded linear exponential (BLINEX) loss function, was developed and the mathematical properties of the BLINEX loss function were discussed. Coetsee *et al.* (2012) motivated and discussed the use of theBLINEX loss and its impact regarding the use of the preliminary test estimator in normal models. They extensively studied the performance of the unrestricted maximum likelihood-, restricted maximum likelihood- and preliminary test estimators in the classical context as well as the Bayesian context.

In the example of the Cobb-Douglas production function it can be deemed that the loss/cost of overestimating production response to capital and labour input is generally more serious than underestimating production response and therefore overestimation should be penalised more heavily, therefore the use of the BLINEX loss function can be suggested. Furthermore, the two exploratory variables are deemed to be negatively correlated, which suggests that multicollinearity is a problem that needs to be addressed, by making use of the ridge regression approach suggested by Hoerl and Kennard (1970). Interested readers may refer to Akdeniz and Erol (2003), Akdeniz and Kaciranlar (1995), Akdeniz and Tabakan (2009), Akdeniz and Akdeniz (2012), Kibria (1996, 2003, 2012), Kibria and Saleh (2012), Kristofer, Kibria and Shukur (2012), Saleh and Kibria (2011), Groβ (2003) and Sarkar (1992) for more related contributions in this field. We are now considering the Cobb-Douglas production function and therefore there is a need to extend the work of Coetsee *et al.* (2012) to accommodate multicollinearity.

In the context of the above-mentioned scenario, it is apparent that in any inference regarding the Cobb-Douglas production function, the following aspects should be addressed, namely hyperspace restrictions, bounded asymmetric loss functions and multicollinearity. There is thus a need to propose a solution that fulfills all the above conditions. The main goal of this paper is to accommodate all the above-mentioned aspects in one model and to consider the performance of the related estimators. It will be shown that the BLINEX loss function outperforms both the reflected normal loss and LINEX loss functions in simulation studies and a practical application by comparing the risk values.

The layout of the paper is as follows: In section 2 the model and proposed estimators are given. In section 3 the BLINEX loss function is defined and the risk functions of the unrestricted ridge regression estimator (URRE), the restricted ridge regression estimator (RRRE) and the preliminary test ridge regression estimator (PTRRE) are derived under BLINEX loss and an alternative method is proposed to derive the risk function of the PTRRE. In section 4 the simulation results will be discussed which will illustrate the contribution of the BLINEX loss. Only the derivations of the risk functions for the BLINEX loss case are included, since the BLINEX loss outperforms all other considered loss functions in this paper, namely reflected normal loss and LINEX loss. In this section two other loss functions, namely reflected normal loss and LINEX loss are included in the simulation studies for comparison purposes. In section 5 we focus on a practical application where a linear constraint of constant return to scale is imposed on the Cobb-Douglas production function. In section 6 a general discussion follows, based on the simulation results obtained.

2 Model and Estimators

In this section, we briefly introduce the existing material in the literature in order to guide the reader to the questions which arised in the introduction and which will be addressed in the following sections. The derivation and details of these results can be found in Saleh (2006).

2.1 Multiple regression model

Consider a multiple regression model (MRM)

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where $\mathbf{Y}_{n\times 1}$ is the vector of response variables, $\mathbf{X}_{n\times p}$ is regarded as the ill-conditioned non-stochastic design matrix $(p < n)$ and $\boldsymbol{\beta} : p \times 1$ is the vector of unknown regression parameters. Also $\boldsymbol{\epsilon}$ is the vector of random error terms with the assumption that $\boldsymbol{\epsilon} \sim N_n\left(\mathbf{0}, \sigma^2\mathbf{I}_p\right)$, where $\sigma \in \mathbb{R}^+$ is unknown.
In the presence of perfect multicollinearity, $\mathbf{C} = \mathbf{X}'\mathbf{X}$ is not of full rank and therefore cannot be inverted which implies that the parameter estimates cannot be calculated. In the case of nearly perfect multicollinearity it is possible to calculate the inverse of $\mathbf{X}'\mathbf{X}$. The inverse will be ill-conditioned and therefore the inverse will be very sensitive to variation in the data and can provide very inaccurate results, which will lead to very large sampling variances.. Hoerl and Kennard (1970) suggested the following correction:

$$\mathbf{C}_{(k)} = \mathbf{X}'\mathbf{X} + k\mathbf{I}_p, \tag{1}$$

where $k > 0$ is known as the ridge or regularisation parameter.
The unrestricted ridge regression estimator (URRE) is defined as:

$$\widetilde{\boldsymbol{\beta}}_n(k) = \mathbf{C}_{(k)}^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n, \tag{2}$$

where $\widetilde{\boldsymbol{\beta}}_n = \mathbf{C}^{-1}\mathbf{X}'\mathbf{Y}$ and $\mathbf{R}(k) = \left(\mathbf{I}_p + k\left(\mathbf{X}'\mathbf{X}\right)^{-1}\right)^{-1}$.
The distribution of URRE is given by

$$\widetilde{\boldsymbol{\beta}}_n(k) \sim N_p\left(\mathbf{R}(k)\,\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\right),$$

where $\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)} = \sigma^2\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{R}'(k)$.

Suppose that the vector parameter $\boldsymbol{\beta}$ is restricted and lies in the sub-space restriction $\mathbf{H}\boldsymbol{\beta}=\mathbf{h}$, where $\mathbf{H}_{q\times p}$ is a known matrix of rank q $(q<p)$, and $\mathbf{h}_{q\times 1}$ is a pre-specified constant vector, where q represents the number of linear restrictions placed on the model. Then the restricted ridge regression estimator (RRRE) is defined as

$$\widehat{\boldsymbol{\beta}}_n(k)=\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n, \tag{3}$$

where $\widehat{\boldsymbol{\beta}}_n=\widetilde{\boldsymbol{\beta}}_n-\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n-\mathbf{h}\right)$
The distribution of RRRE is given by

$$\widehat{\boldsymbol{\beta}}_n(k)\sim N_p\left(\mathbf{R}(k)\boldsymbol{\beta},\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\right), \tag{4}$$

where $\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}=\sigma^2\mathbf{R}(k)\mathbf{A}\mathbf{R}'(k)$ and $\mathbf{A}=\mathbf{C}^{-1}-\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}$

When the researcher is uncertain about the applicability of the restriction $\mathbf{H}\boldsymbol{\beta}=\mathbf{h}$, the estimate of $\boldsymbol{\beta}$ can be obtained after preliminary testing the null hypothesis $H_0:\mathbf{H}\boldsymbol{\beta}=\mathbf{h}$. It is well-known that the preliminary test estimator (PTE), performs better than the unrestricted or restricted estimator in a specific region of the parameter space.
Preliminary test ridge regression estimator (PTRRE) is then defined as

$$\widehat{\boldsymbol{\beta}}_n^{PT}(k)=\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n^{PT}, \tag{5}$$

where $\widehat{\boldsymbol{\beta}}_n^{PT}=\widetilde{\boldsymbol{\beta}}_n-\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I\left(\pounds_n<F_{q,m}(\alpha)\right)$ and $\pounds_n$ is the likelihood ratio test statistic for testing H_0 against H_A given by

$$\pounds_n=\frac{\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n-\mathbf{h}\right)'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n-\mathbf{h}\right)}{qs_\varepsilon^2}, \tag{6}$$

$F_{q,m}(\alpha)$ is the critical value of the central F-distribution with q and m degrees of freedom and α is the level of significance. (In general, $I(A)$ is the indicator function of a set A.)

3 Risk Analysis Under BLINEX Loss Function

In order to consider the performance of an estimator, one needs to study the risk associated with the estimator In this section, the risk functions of all three estimators (URRE, RRRE and PTRRE) are derived under the BLINEX loss function. There is a need to investigate the performance of the preliminary test estimator when a linear restriction is places on the regression model, working under a bounded loss function and also taking into account the presence of multicollinearity. Assume $\boldsymbol{\beta}^*$ is a $p\times 1$ vector containing regression coefficient estimators and $\boldsymbol{\beta}$ is a $p\times 1$ vector of unknown regression coefficient parameters.
The multivariate BLINEX loss function is given by:

$$\mathcal{L}_{BLINEX}(\boldsymbol{\beta}^*,\boldsymbol{\beta})=\frac{1}{\lambda}\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'(\boldsymbol{\beta}^*-\boldsymbol{\beta})\right)-\mathbf{a}'(\boldsymbol{\beta}^*-\boldsymbol{\beta})-1\right]}\right], \tag{7}$$

with $b=\lambda d$ where $d=$the scale parameter of the LINEX loss function.
For an in depth discussion of the multivariate BLINEX loss function see Coetsee et al. (2012) and Kleyn (2014).

3.1 Risk Performance

The risk function of each of the proposed estimators (URRE, RRRE and PTRRE) of section 2 will be derived under the BLINEX loss function..
The risk function for $\boldsymbol{\beta}^*$ under the BLINEX loss function is therefore given by:

$$\Re_{BLINEX}\left(\boldsymbol{\beta}^*, \boldsymbol{\beta}\right) = \frac{1}{\lambda} E\left[1 - \frac{1}{1 + b\left[\exp\left(\mathbf{a}'\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right)\right) - \mathbf{a}'\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right) - 1\right]}\right]$$

where E indicates the expected value of the given function.

3.1.1 Risk function of the URRE

Theorem 1 *Ignoring the terms of order 3, the approximated risk function for the URRE under the BLINEX loss function is given as*

$$-\tfrac{1}{\lambda}\left[b\left[1 + \mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right] - \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right]\right.$$
$$+b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)} + \left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)'\right]\mathbf{a}$$
$$-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta} + \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) - \mathbf{a}'\boldsymbol{\beta} - 1\right]$$
$$\left.+2b^2\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right] + b^2\exp\left(-2\boldsymbol{\beta}'\mathbf{a} + 2\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a} + 4\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) + b^2\right]$$

Proof.
Let $M = b\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right) - \exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\right) + 1\right]$, then the risk function of $\widetilde{\boldsymbol{\beta}}_n(k)$ reduces to

$$\Re_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k), \boldsymbol{\beta}\right) = -\frac{1}{\lambda}\left[E[M] + E\left[M^2\right]\right] + O\left(M^3\right) \tag{8}$$

From (8) we have

$$E[M] = bE\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta} + \mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right) - \exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\right) + 1\right]$$
$$= -b\exp\left(-\boldsymbol{\beta}'\mathbf{a}\right)\left[MGF_{\widetilde{\boldsymbol{\beta}}'_n(k)}(\mathbf{a})\right] + b\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right] + b$$

where

$$MGF_{\widetilde{\boldsymbol{\beta}}'_n(k)}(\mathbf{a}) = E\left[\exp\left(\widetilde{\boldsymbol{\beta}}'_n(k)\mathbf{a}\right)\right] = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\left[\exp\left(\widetilde{\boldsymbol{\beta}}'_n(k)\mathbf{a}\right)\right.$$
$$\left.\times\frac{1}{(2\pi)^{\frac{p}{2}}\left|\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\right|^{\frac{1}{2}}}\exp\left(-\tfrac{1}{2}\left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\boldsymbol{\Sigma}^{-1}_{\widetilde{\boldsymbol{\beta}}_n(k)}\left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)\right)d\widetilde{\boldsymbol{\beta}}_n(k)\right]$$
$$= \exp\left(\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

Therefore,

$$E[M] = b\left[1 + \mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right] - \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right] \tag{9}$$

$E\left[M^2\right]$ can be written as

$$b^2E\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)'\mathbf{a}\right] - 2b^2\mathbf{a}'E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\right)\right] + 2b^2a'E\left[\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right]$$
$$-2b^2E\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\right)\right] + b^2E\left[\exp\left(2\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right)\right)\right] + b^2$$

Knowing that $E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\right] = 0$ and $E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\right] = \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}$, it follows that

$$E\left[M^2\right] = b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)} + \left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)'\right]\mathbf{a}$$

$$-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-\mathbf{a}'\boldsymbol{\beta}-1\right]$$
$$+2b^2\mathbf{a}'\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]+b^2\exp\left(-2\boldsymbol{\beta}'\mathbf{a}+2\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+4\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)+b^2 \tag{10}$$

By substituting (9) and (10) into (8) the proof is complete.■

3.1.2 Risk function of the RRRE

Theorem 2 *Ignoring the terms of order 3, the approximated risk function for the RRRE under the BLINEX loss function is given as*

$$-\frac{1}{\lambda}\Big[b\left[1+\mathbf{a}'\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]-\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right]$$
$$+b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}+\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\right]\mathbf{a}$$
$$-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-\mathbf{a}'\boldsymbol{\beta}-1\right]$$
$$+2b^2\mathbf{a}'\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]+b^2\exp\left(-2\boldsymbol{\beta}'\mathbf{a}+2\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+4\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)+b^2\Big]$$

Proof. The proof is similar to the proof of Theorem 1. ■

3.1.3 Risk function of the PTRRE

Theorem 3 *Ignoring the terms of order 2, the approximated risk function for the PTRRE under the BLINEX loss function is given as*

$$\frac{-b}{\lambda}\Big[\sigma^2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$
$$-2\sigma^2E\left[\phi\left(\chi^2_{(p+2,\delta^2)}\right)\right]\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$
$$+\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$
$$+2\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{(p+2,\delta^2)}\right)\right]$$
$$+\Big[\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2E\left[\varphi\left(\chi^2_{(p+2,\delta^2)}\right)\right]\mathbf{I}_p+E\left[\varphi\left(\chi^2_{(p+4,\delta^2)}\right)\right]\boldsymbol{\eta}\boldsymbol{\eta}'\right]$$
$$\times\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}\Big]\Big]$$

where $E\left[\varphi\left(\chi^2_{(p+2,\delta^2)}\right)\right]=E\left(\varphi\left(\frac{n-p}{q}\frac{\chi^2_{q+2,\delta^2}}{\chi^2_{n-p}}\right)\right)=E\left[I\left(\pounds_n<F_{q,m}(\alpha)\right)\right]=E\left\{I\left(F_{q+2,n-p,\delta^2}<\frac{q}{q+2}F_{q,m}(\alpha)\right)\right\}$, $\varphi(.)$ denotes any Borel measurable function, $\chi^2_{(p+2,\delta^2)}$ denotes the noncentral χ^2 distribution with $p+2$ degrees of freedom and noncentrality parameter $\delta^2=\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}$ and $F_{q+2,n-p,\delta^2}$ denotes the noncentral F distribution with $q+2$ and $n-p$ degrees of freedom and noncentrality parameter $\delta^2=\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}$.

Proof. The risk function of $\widehat{\boldsymbol{\beta}}^{PT}(k)$ under the BLINEX loss function , using Theorems 1 and 3 in Judge and Bock p 321 and 323, can be written as

$$\Re_{BLINEX}\left(\widehat{\boldsymbol{\beta}}^{PT}(k),\boldsymbol{\beta}\right)\approx-\frac{1}{\lambda}E[M] \tag{11}$$

due to the complexity of the BLINEX loss function, where

$$M=b\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}^{PT}(k)-\boldsymbol{\beta}\right)-\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}^{PT}(k)-\boldsymbol{\beta}\right)\right)+1\right]$$

$$\begin{aligned}E[M]&=-bE\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}^{PT}(k)-\boldsymbol{\beta}\right)-1\right]\\&=-b\left[\frac{E\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}^{PT}(k)-\boldsymbol{\beta}\right)\right]^2}{2!}\right]\end{aligned} \tag{12}$$

with

$$E\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}^{PT}(k)-\boldsymbol{\beta}\right)\right]^2$$

expanded as follows (see Kleyn (2014) and Saleh (2006) for the definition of $\widehat{\boldsymbol{\beta}}^{PT}(k)$ in terms of $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$):

$$\begin{aligned}&E\left[\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})+\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})-\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I(\pounds_n<F_{q,m}(\alpha))\right]^2\\&=E\left[\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)\right]+2E\left[\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'(\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}))\right]\\&-2E\left[\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I(\pounds_n<F_{q,m}(\alpha))\right)\right]\\&+E\left[(\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}))'(\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}))\right]\\&-2E\left[(\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}))'\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I(\pounds_n<F_{q,m}(\alpha))\right)\right]\\&+E\left[\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I(\pounds_n<F_{q,m}(\alpha))\right)'\right.\\&\left.\times\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I(\pounds_n<F_{q,m}(\alpha))\right)\right].\end{aligned}$$

By applying Theorems 1 and 3 in Judge and Bock p 321 and 323 we obtain

$$\begin{aligned}&\sigma^2\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}\\&-2\sigma^2E\left[\phi\left(\chi^2_{(p+2,\delta^2)}\right)\right]\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}\\&+\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\\&+2\mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\phi\left(\chi^2_{(p+2,\delta^2)}\right)\right]\\&+\left[\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2E\left[\phi\left(\chi^2_{(p+2,\delta^2)}\right)\right]\mathbf{I}_p+E\left[\phi\left(\chi^2_{(p+4,\delta^2)}\right)\right]\boldsymbol{\eta}\boldsymbol{\eta}'\right]\right.\\&\left.\times\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}\right].\end{aligned}$$

Substituting this result into (12) and then into (11) the proof is complete.■

Theorem 3 provides a weak approximation for the risk of the PTRRE due to its complex nature. For the purpose of obtaining a better result, we propose an alternative approach for the evaluatin of the risk of the PTRRE, which is based ont he double expectation concept, i.e., $E[L]\equiv E[E[L|T]]$,where $\pounds\equiv\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)$ (the BLINEX loss function) and $T\equiv$ Result of the hypothesis test based on the linear restriction $\mathbf{H}\boldsymbol{\beta}=\mathbf{h}$.

Therefore the risk of the PTRRE under the BLINEX loss function is given by:

$$E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\right]=\frac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\right]$$
$$=\left\{E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\middle|T\right]\right\}=\frac{1}{\lambda}E\left\{E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\middle|T\right]\right\}$$
$$=\frac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\middle|H_0\text{ not rejected}\right]P\left(H_0\text{ not rejected}\right)$$
$$+\frac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\middle|H_0\text{ rejected}\right]P\left(H_0\text{ rejected}\right)$$
$$=\frac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]}\right]P\left(H_0\text{ not rejected}\right)$$
$$+\frac{1}{\lambda}E\left[=1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]}\right]P\left(H_0\text{ rejected}\right)$$
$$=E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]P\left(H_0\text{ not rejected}\right)+E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]P\left(H_0\text{ rejected}\right)$$

where both $E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$ and $E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$ were calculated in Theorems 1 and 2.

4 Simulation study

In reality the ridge parameter k is unknown, therefore we need to consider some estimators which can be applied in practical situations. Kibria(2003), Gisela and Kibria (2009) and Najarian et al. (2013) extensively discussed the performance of ridge estimators for different ridge parameters. They only considered the squared error loss function as a measure of closeness to the target estimator. In this section, we also provide a numerical study of a number of ridge estimators under the BLINEX loss function. The simulation technique used is similar to the technique described in McDonald and Galarneau (1975) and Gibbons (1981), where the $p=4$ multicollinear explanatory variables in the simulation exercise are computed by

$$x_{ij}=\left(1-\gamma^2\right)^{\frac{1}{2}}z_{ij}+\gamma z_{i5,}\qquad i=1,2,....,n\text{ and }j=1,2,....4, \tag{13}$$

with $z_{ij}\sim N(0,1)$ random variables and γ chosen in such a way that the correlation between any two explanatory variables is given by γ^2. The explanatory variables are standardised in order to ensure that $\mathbf{X}'\mathbf{X}$ is in correlation form. In order to generate the dependent variable y_i, the true $\boldsymbol{\beta}$ vector is needed when using the following equation:

$$y_i=\beta_0+\beta_1x_{i1}+\beta_2x_{i2}+\beta_3x_{i3}+\beta_4x_{i4}+e_i\qquad i=1,2,....,n \tag{14}$$

with $e_i\sim N\left(0,\sigma^2\right)$ and $\beta_0=0$. The $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4)$ vector is determined as the normalised eigenvector corresponding to the largest eigenvalue of the $\mathbf{X}'\mathbf{X}$ matrix, which is subject to the constraint that $\boldsymbol{\beta}'\boldsymbol{\beta}=1$ as suggested by Newhouse and Oman (1971).

These regression coefficients are introduced into (14) in order to generate the dependent variable in the analysis. The simulated dependent (14) and explanatory variables (13) are then used to estimate the values of $\boldsymbol{\beta}$, for the three proposed estimators in the paper, namely URRE, RRRE and PTRRE. The performance of each estimator is then determined by evaluating the performance under the BLINEX loss function. In order to calculate the risk values the process is repeated 2000 times (2000 iterations).

The different ridge regression estimators (URRE, RRRE and PTRRE) are further evaluated by introducing seven different estimated ridge parameters (k) into the analysis:

1. **Hoerl and Kennard (1970)** ($\widehat{k}_{HK}$ or HK): $\widehat{k}_{HK} = \frac{\widehat{\sigma}^2}{\alpha_i^2}$

where $\widehat{\sigma}^2 = \dfrac{\sum \widehat{e}_i^2}{n-p}$ (the residual mean square error) , $\boldsymbol{\alpha} = \boldsymbol{\Gamma\beta} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\Gamma}$ represents the eigenvectors of $\mathbf{X}'\mathbf{X}$. α_i^2 will be replaced by α_{Max}^2, the maximum squared element of $\boldsymbol{\alpha}$.

2. **Hoerl, Kennard and Baldwin (1975)** ($\widehat{k}_{HKB}$ or HKB): $\widehat{k}_{HKB} = \frac{p\widehat{\sigma}^2}{\widehat{\boldsymbol{\alpha}}'\widehat{\boldsymbol{\alpha}}}$

3. **Lawless and Wang (1976)** ($\widehat{k}_{LW}$ or LW): $\widehat{k}_{LW} = \frac{p\widehat{\sigma}^2}{\widehat{\boldsymbol{\alpha}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\alpha}}}$

4. **Hocking, Speed and Lynn(1976)** ($\widehat{k}_{LW}$ or LW): $\widehat{k}_{HSL} = \widehat{\sigma}^2 \frac{\sum_{i=1}^{p} (\lambda_i \widehat{\alpha}_i)^2}{\sum_{i=1}^{p} \left(\lambda_i \widehat{\alpha}_i^2\right)^2}$

5. **Kibria (2003)** ($\widehat{k}_{AM}$ or AM): $\widehat{k}_{AM} = \frac{1}{p} \sum_{i=1}^{p} \frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}$. This represents the arithmetic mean of of k_{HK}.

6. **Kibria (2003)** ($\widehat{k}_{GM}$ or GM): $\widehat{k}_{GM} = \dfrac{\widehat{\sigma}^2}{\left(\prod_{i=1}^{p} \widehat{\alpha}_i^2\right)^{\frac{1}{p}}}$. This represents the geometric mean of k_{HK}.

7. **Kibria (2003)** ($\widehat{k}_{MED}$ or MED) for $p \geq 3$: $\widehat{k}_{MED} = Median \left(\dfrac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right)$, $i = 1, 2, ..., p$. This represents the median of k_{HK}.

For details, see Arashi and Valizadeh (2014) and very recently Kibria and Banik (2014) among others.

4.1 Evaluation of the performance of different ridge parameters under the BLINEX loss function

The simulation results in Tables 1 to 4 show the performance of the different proposed ridge parameters. By keeping specific parameters fixed in the simulation process and varying only one parameter at a time, the behaviour of the risk values can be evaluated. The ridge parameters were calculated as proposed at the beginning of this section and all the estimators (URRE, RRRE and PTRRE) under the BLINEX loss function were repeatedly calculated using all the proposed ridge parameters. In order to calculate the risk values, the average loss was calculated, for each estimator calculated with different ridge parameters. In Table1 in each section only the error variance varied, Table 2 represents the risk values over different levels of multicollinearity. Table 3 represents the effect of an increase in the number of parameters that needs to be estimated and Table 4 represents the effect of an increase in the sample size. In these tables, a subscript B represents BLINEX loss, L represents LINEX loss and R represents reflected normal loss. The only comparisons between the risk values calculated under the three different loss functions are indicated for the PTRRE. Similar results were obtained in Kleyn (2014) for all three estimators (URRE, RRRE and PTRRE), under all three loss functions (BLINEX, LINEX and reflected normal loss) and for all the conditions considered (increase in error variance, increase in the level of multicollinearity, increase in the number of parameters to be estimated and an increase in the sample size). It can be seen that the BLINEX loss function outperforms the LINEX loss and reflected normal loss functions by comparing the risk values (the values are highlighted in the tables).

Table 1: Effect of an increase in the variance on the risk of the three estimators under BLINEX loss and across different ridge parameters:

		$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	$Estimator$	HK	HKB	LW	HSL	AM	GM	MED
0.1	$URRE_B$	0.0002218	0.0002228	0.0002217	0.000222	0.0109315	0.000252	0.0003204
	$RRRE_B$	1.76E-06	1.21E-06	2.18E-06	1.65E-06	0.0106626	0.000024	0.0000899
	$PTRRE_B$	0.0000653	0.000065	0.0000657	0.0000653	0.0122171	0.000092	0.0001423
1	$URRE_B$	0.0215563	0.0240307	0.0222496	0.0214666	0.1615253	0.0528128	0.0560122
	$RRRE_B$	0.000544	0.0025669	4.75E-06	0.0010216	0.154075	0.0393545	0.0426766
	PTRRE$_B$	**0.0068635**	**0.0088707**	**0.0066511**	**0.0072474**	**0.1540832**	**0.0432259**	**0.0452642**
	PTRRE$_L$	**0.0067331**	**0.0088424**	**0.0066409**	**0.0070137**	**0.261523**	**0.0529673**	**0.049579**
	PTRRE$_R$	**0.1112168**	**0.0922945**	**0.1409591**	**0.1033731**	**0.1109183**	**0.05524**	**0.0555483**
20	$URRE_B$	0.4523026	0.5976154	0.6560864	0.4700296	0.5259865	0.5215678	0.5214459
	$RRRE_B$	0.4750527	0.0810547	0.0017288	0.4863195	0.5261745	0.5223268	0.5222596
	$PTRRE_B$	0.4704927	0.1365946	0.0561257	0.4820313	0.5257225	0.521726	0.5216334

Table 2: Effect of an increase in the level of multicollinearity on the risk of the three estimators under BLINEX loss and across different ridge parameters:

		$n = 30$ $p = 4$ and $\sigma = 1$						
γ	$Estimator$	HK	HKB	LW	HSL	AM	GM	MED
0.7	$URRE_B$	0.0254274	0.0284393	0.0266898	0.0254013	0.1777523	0.063313	0.0727785
	$RRRE_B$	0.0006403	0.0035879	0.0000252	0.0006256	0.1703246	0.0509422	0.0608166
	$PTRRE_B$	0.0083307	0.0114513	0.0082199	0.008318	0.1686848	0.0566555	0.0690406
0.8	$URRE_B$	0.0220932	0.0243859	0.0229981	0.0220907	0.164138	0.0530384	0.0524669
	$RRRE_B$	0.0005522	0.0025503	4.62E-06	0.0010226	0.1570133	0.0406039	0.0402203
	PTRRE$_B$	**0.0060882**	**0.0080786**	**0.0060507**	**0.0063681**	**0.152002**	**0.0424793**	**0.0453755**
	PTRRE$_L$	**0.0069031**	**0.0089518**	**0.0068472**	**0.0072279**	**0.2429004**	**0.0517412**	**0.0545445**
	PTRRE$_R$	**0.11061**	**0.0924855**	**0.1391345**	**0.1029873**	**0.1089375**	**0.056623**	**0.0586928**
0.99	$URRE_B$	0.0154786	0.0157537	0.0157828	0.0370948	0.1403219	0.0374711	0.0355819
	$RRRE_B$	0.0003626	0.0001871	0.0000451	0.0250294	0.1331218	0.0263292	0.0238209
	$PTRRE_B$	0.0039417	0.0038505	0.0037822	0.0272357	0.1411934	0.0306878	0.0282437

Table 3: Effect of an increase in the number of parameters to be estimated on the risk of the three estimators under BLINEX loss and across different ridge parameters:

		$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	$Estimator$	HK	HKB	LW	HSL	AM	GM	MED
4	$URRE_B$	0.0220932	0.0243859	0.0229981	0.0220907	0.164138	0.0530384	0.0524669
	$RRRE_B$	0.0005522	0.0025503	4.62E-06	0.0010226	0.1570133	0.0406039	0.0402203
	$PTRRE_B$	0.006565	0.0085628	0.0063321	0.0069623	0.1472063	0.0404024	0.0418997
10	$URRE_B$	0.0241123	0.0255239	0.0252804	0.0242387	0.4066395	0.0998795	0.0757541
	$RRRE_B$	0.3227751	0.3262358	0.3206902	0.3244085	0.5450151	0.3939551	0.3758208
	PTRRE$_B$	**0.0234876**	**0.0250226**	**0.0242882**	**0.0241134**	**0.4177921**	**0.1029818**	**0.0717673**
	PTRRE$_L$	**0.0242353**	**0.0256523**	**0.0253732**	**0.024257**	**0.9554551**	**0.1265353**	**0.0920304**
	PTRRE$_R$	**0.2607062**	**0.2138464**	**0.427464**	**0.231376**	**0.1719465**	**0.0471389**	**0.0617497**
20	$URRE_B$	0.0261123	0.0268236	0.0306112	0.0268788	0.5758074	0.1459182	0.0949289
	$RRRE_B$	0.5544572	0.559757	0.5979607	0.5525225	0.7042255	0.5831416	0.5688081
	$PTRRE_B$	0.0253094	0.0256043	0.0289253	0.0261733	0.6029361	0.1550401	0.0977558

Table 4: Effect on an increase in the sample size on the risk of the three estimators under BLINEX loss and across different ridge parameters:

		$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	$Estimator$	HK	HKB	LW	HSL	AM	GM	MED
15	$URRE_B$	0.0444846	0.050169	0.0476796	0.0452737	0.2148647	0.1061582	0.1155797
	$RRRE_B$	0.0022388	0.0075825	0.0001032	0.0044595	0.2058656	0.0881903	0.0990456
	$PTRRE_B$	0.0150589	0.0203113	0.0136204	0.0174938	0.221718	0.0957336	0.1030331
30	$URRE_B$	0.0220932	0.0243859	0.0229981	0.0220907	0.164138	0.0530384	0.0524669
	$RRRE_B$	0.0005522	0.0025503	4.62E-06	0.0010226	0.1570133	0.0406039	0.0402203
	PTRRE$_B$	**0.006565**	**0.0085628**	**0.0063321**	**0.0069623**	**0.1472063**	**0.0404024**	**0.0418997**
	PTRRE$_L$	**0.0069031**	**0.0089518**	**0.0068472**	**0.0072279**	**0.2429004**	**0.0517412**	**0.0545445**
	PTRRE$_R$	**0.11061**	**0.0924855**	**0.1391345**	**0.1029873**	**0.1089375**	**0.056623**	**0.0586928**
100	$URRE_B$	0.0067695	0.0071331	0.0068621	0.0067742	0.08703	0.0132359	0.0134058
	$RRRE_B$	0.0000536	0.0003122	7.93E-08	0.0000906	0.0831591	0.0074833	0.0078169
	$PTRRE_B$	0.0021742	0.0024503	0.0021468	0.0021968	0.0804973	0.0076744	0.0080379

According to Tables 1 to 4, it can be seen that for RRRE and PTRRE the LW ridge parameter performed the best, whereas for URRE the HK and HSL ridge parameters performed the best over different values of σ. In all cases the AM ridge parameter performed the worst. For all three estimators, namely URRE, RRRE and PTRRE the HK, LW and HKB ridge parameters lead to the lowest risk over different levels of multicollinearity. For the HSL ridge parameter the risk is also low for $\gamma < 0.95$. The AM ridge parameter performed the worst overall. For all three estimators, namely URRE, RRRE and PTRRE the HK, HKB and HSL performed the best over an increasing number of parameters to be estimated. For the LW ridge parameter also performs well, except for RRRE when $p > 20$ and the AM ridge parameter performs the worst overall. For all the estimators, namely URRE, RRRE and PTRRE the HK, HKB, LW and HSL ridge parameters performs the best over different sample sizes and the AM ridge parameter performs the worst overall. As expected for all estimators, it was found the risk increased when the variance, level of multicollinearity and number of parameters to be estimated increased and the risk of all the estimators decreased when the sample size increased.

4.2 Evaluation of the performance of the estimators relative to each other under BLINEX loss

The risk of RRRE, URRE and PTRRE calculated under BLINEX loss are computationally compared to each other making use of simulation studies in order to determine the performance of the estimators relative to each other. The panel of sketches in this section will be discussed in two ways, firstly by keeping σ constant and allowing the sample size, n, to increase and secondly by keeping n constant and allowing σ to increase. All the risk functions are represented as a function of the estimation error, namely $\boldsymbol{\delta} = \mathbf{H}\boldsymbol{\beta} - \mathbf{h}$ (in this application δ was a scalar), with $-1 \leq \delta \leq 1$ on the horizontal axis.

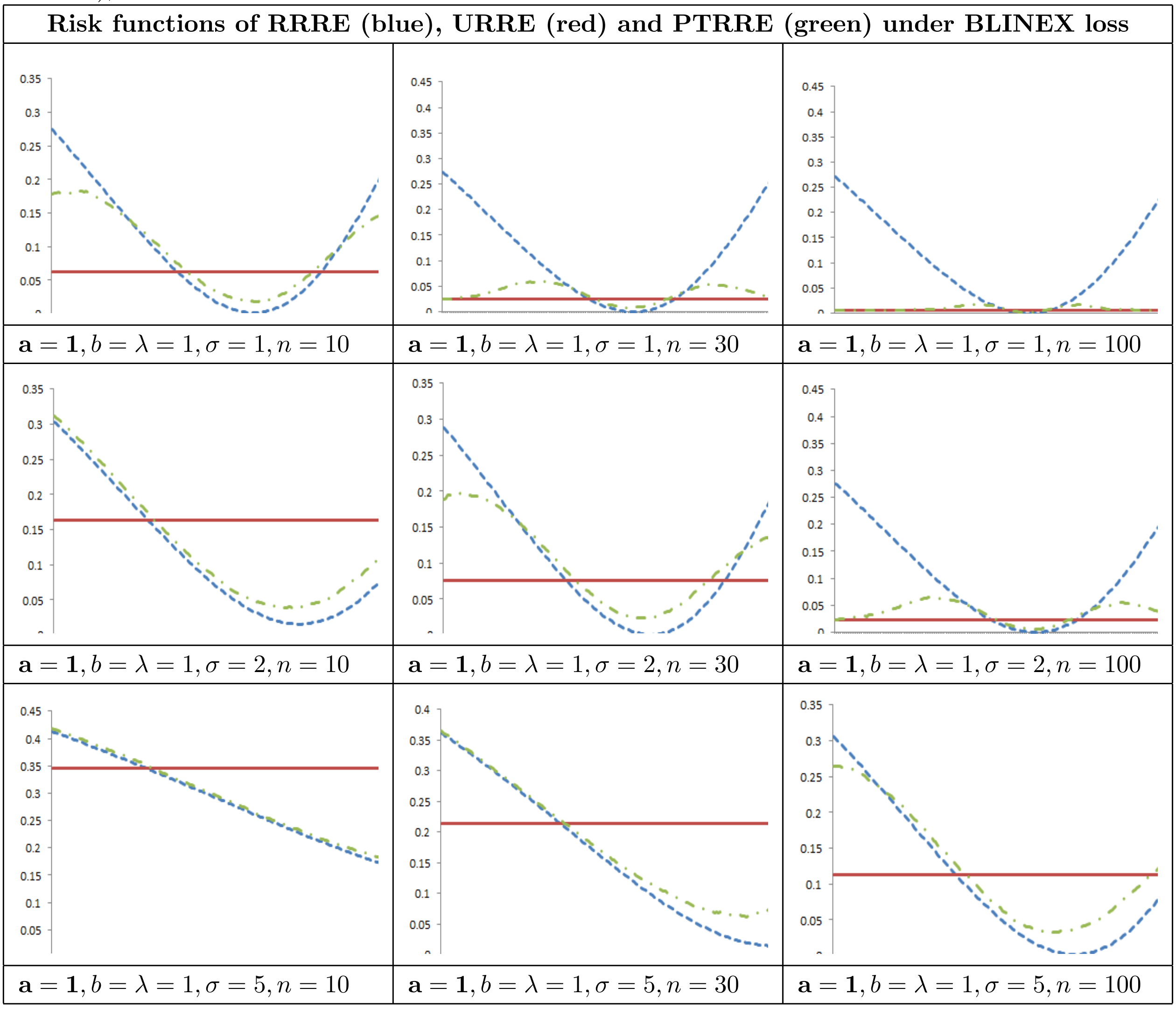


Figure 1 Comparison of URRE, RRRE and PTRRE across different values of σ and n under BLINEX loss.

It was found that for the case where $\sigma = 1$, as the sample size increases for this choice of σ, the risk of PTRRE tends towards the risk of URRE as $\delta \to \pm\infty$. It is well-known that URRE is the most efficient estimator as the sample size increases, since the variance of URRE, i.e. $\widetilde{\boldsymbol{\beta}}_n(k)$ is $\sigma^2 \mathbf{R}(k)\mathbf{C}^{-1}\mathbf{R}'(k)$ and $\sigma^2 = \frac{\sum e_i^2}{n-p}$, as the sample size increases the variance of URRE will tend to zero. Therefore as the sample size increases, PTRRE becomes equally as efficient as the URRE. It is clear that as the sample size increases the risk associated with all the estimators decreases. Similar results are obtained for different values of σ. When σ increases for a specific sample size, it can be seen that in the interval near the origin where the risk of the PTRRE is lower relative to the URRE becomes larger. Therefore when working with a specific sample size and if the variance of the sample is relatively high, the PTRRE is a more efficient estimator compared to the URRE.

5 Application

In this section, the Cobb-Douglas production function will be considered under the restriction of constant return to scale and the estimators which will be calculated for all the possible estimators of the ridge parameter. The real data set consists of following variables:

(1) The South African manufacturing production output as measured by the Gross Domestic product (GDP in real prices) measured in R million in the manufacturing sector.

(2) Labour input as measured by employment figures within the manufacturing sector.

(3) Capital input as measured by fixed capital stock (in real prices) measured in R million in the manufacturing sector.

The definitions of the variables are given below:

Gross Domestic Product: The total market value of all final goods and services produced in a country in a given year, equal to total consumer, investment and government spending, plus the value of exports, minus the value of imports.

Fixed capital stock: Fixed capital stock consists of buildings, installations, transmission devices, machinery, equipment, means of transport, tools, production and sales implements, draft animals, and commercial livestock.

Employment figures: The number of employees employed within the manufacturing sector.

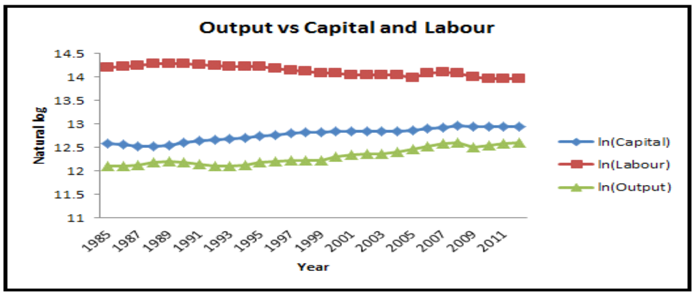


Figure 2: Relationship between production output, capital and labour.

It can be seen from Figure 2 that there has been a steady increase in production output over the period 1985 to 2012. The two explanatory variables in the analysis is capital input and labour input. From the graph it is clear that in the manufacturing sector there has been a steady increase in capital input and a steady decrease in labour input over the specified period. This is an indication of a capital intensive market, where more fixed capital stock is used in manufacturing and less labour is used in the production process.

The linear restriction of constant return to scale $\beta_1 + \beta_2 = 1$ suggests a sub-space formulation

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = 1,$$

where the restriction $\mathbf{H}\boldsymbol{\beta} = \mathbf{h}$ holds with $\mathbf{H}_{1\times 3} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{h}_{1\times 1} = 1$.

By using the ordinary least squares (OLS) method, the following estimated regression equation is obtained:

$$\begin{aligned} \widehat{\ln Y}_i &= 10.68 - 0.54 \ln X_{1i} + 0.72 \ln X_{2i} \\ & \quad\ (8.36) \quad (0.36) \qquad\quad (0.27) \\ Adj R^2 &= 0.77 \qquad r_{12} = -0.90 \qquad VIF = 5.52 \\ cov\left(\beta_1, \beta_2\right) &= 0.0885 \qquad CI = 9793.53 \end{aligned}$$

From the above model it follows that a degree of multicollinearity is present in the data, since the two explanatory variables are highly negatively correlated, the VIF (Variance inflating factor) and the condition index (CI) also indicates the presence of severe multicollinearity in the model. In this example the HK ridge parameter was used to adjust for multicollinearity in the data.
When estimating using OLS, variances and covariances of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are given by $var\left(\widehat{\beta}_1\right) = \frac{\sigma^2}{\sum x_{1i}^2(1-r_{12}^2)}$, $var\left(\widehat{\beta}_2\right) = \frac{\sigma^2}{\sum x_{2i}^2(1-r_{12}^2)}$ and $cov\left(\widehat{\beta}_1, \widehat{\beta}_2\right) = \frac{-r_{12}\sigma^2}{(1-r_{12}^2)\sqrt{\sum x_{1i}^2 \sum x_{2i}^2}}$ (see Gujarati ,2003, p 351).
Therefore, as $|r_{12}|$ tends to 1 the multicollinearity increases and the two variances and covariance will increase accordingly. The VIF measures the speed at which the variances and covariances increase and is given by $VIF = \frac{1}{(1-r_{12}^2)}$. This measure shows the extend to which the variance of an estimator is inflated due to the presence of multicollinearity, i.e. as $r_{12}^2 \to 1$, $VIF \to \infty$. In this sample, when estimating the parameter values with the OLS method, the variances of the estimated parameters are 5.5 times higher than it would have been if no multicollinearity was present.
Another condition that has not yet been taken into account is the fact that an economic restriction needs to be placed on the model in order to determine whether constant return to scale is present in the South African manufacturing sector for the sample period 1985 to 2012. In order to test the linear economic restriction of $\beta_1 + \beta_2 = 1$ we need to calculate the unrestricted, restricted and preliminary test ridge regression estimators as given by (2), (3) and (5).
The estimated regression equation for the unrestricted ridge regression estimators is given by:

$$\begin{aligned} \widehat{\ln Y_i} &= 0.11 - 0.078 \ln X_{1i} + 1.041 \ln X_{2i} \\ & \quad (0.065) \quad (0.063) \qquad (0.069) \\ AdjR^2 &= 0.73 \qquad cov\left(\beta_1, \beta_2\right) = -0.0043 \end{aligned}$$

The estimated regression equation for the restricted ridge regression estimator is given by:

$$\begin{aligned} \widehat{\ln Y_i} &= 0.027 - 0.075 \ln X_{1i} + 1.045 \ln X_{2i} \\ & \quad (< 0.001) \quad (0.063) \qquad (0.069) \\ AdjR^2 &= 0.75 \qquad cov\left(\beta_1, \beta_2\right) = -0.0043 \end{aligned}$$

Based on the likelihood ratio test given by (6) the estimated regression equation for the preliminary test ridge regression estimator is given by:

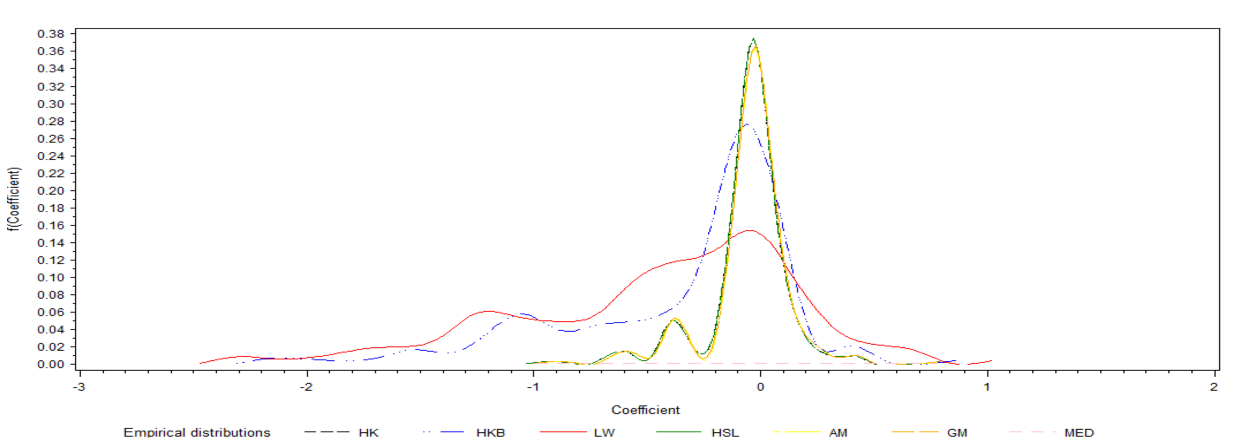
$$\begin{aligned} \widehat{\ln Y_i} &= 0.027 - 0.075 \ln X_{1i} + 1.045 \ln X_{2i} \\ & \quad (< 0.001) \quad (0.063) \qquad (0.069) \\ AdjR^2 &= 0.75 \qquad cov\left(\beta_1, \beta_2\right) = -0.0043 \end{aligned}$$

Therefore, the null hypothesis, namely $H_0 : \mathbf{H}\boldsymbol{\beta} = \mathbf{h}$ is not rejected at a 5% level of significance. It can therefore be concluded that for this sample period under consideration, considering the South African manufacturing sector from 1985 to 2012, the restriction of constant return to scale could not be rejected, which implies that as by doubling the input in the market will result in output doubling as well. Each of the slope coefficients can be interpreted as partial elasticities. The coefficient -0.075 measures the partial elasticity of production output to labour input, i.e. a 1% increase in the labour input, holding capital input constant, will result in a 0.075% decrease in production output. Whereas, the coefficient 1.045 measures the partial elasticity of production output to capital input, i.e., a 1% increase in capital input, holding labour input constant, will result in a 1.045% increase in production output.

The following results are based on a simulation study where block bootstrapping was employed to select samples from the original data set used in the above example. Block bootstrapping is the appropriate technique to use when time series data is considered, since the data is more likely to be time dependent (see Lahiri, 2003). Nonoverlapping block bootstrapping was used in this analysis. This procedure involves splitting the sample data into b blocks of size l (see Santana and Allison, 2013). The value of b is chosen in such a way that $lb \leq T$, where T is the sample size. In the case where $b = T$ this nonoverlapping block bootstrap method reduces to the standard bootstrapping procedure.

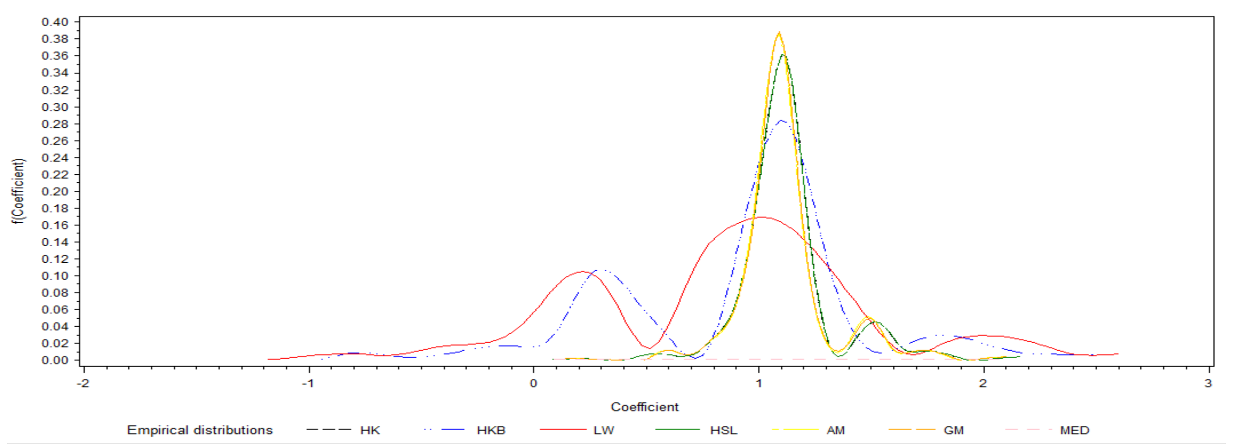
In Figure 3 the two coefficients associated with labour and capital input are estimated using the unrestricted ridge regression estimator and by using the different ridge parameters(k) as discussed in section 4. For the LW and HKB ridge parameters it can be seen that the variation in the estimates relative to the URRE's calculated under all the other ridge parameters is relatively high.

Unrestricted ridge regression estimator



Labour coefficient

Unrestricted ridge regression estimator



Capital coefficient

Figure 3 Empirical distributions of the slope coefficients of the unrestricted ridge regression estimator.

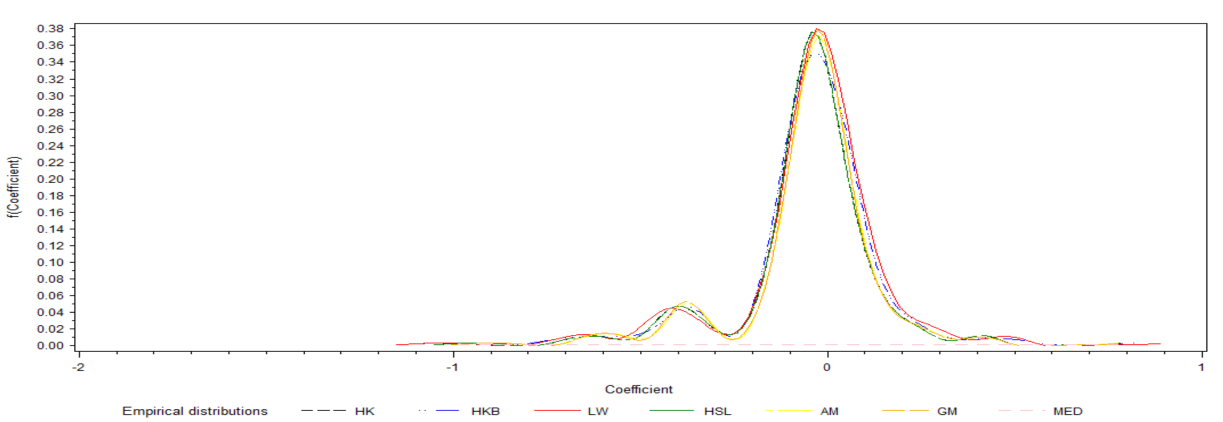
Table 5 Measures of risk performance for the unrestricted ridge regression estimator.

Ridge parameter	HK	HKB	LW	HSL	AM	GM	MED
Parameter estimate 1	**-0.10**	**-0.42**	**-0.53**	**-0.10**	**-0.09**	**-0.10**	**-0.09**
Standard error	0.060	0.181	0.347	0.060	0.059	0.059	0.059
Parameter estimate 2	**1.07**	**0.84**	**0.76**	**1.07**	**1.06**	**1.06**	**1.06**
Standard error	0.066	0.151	0.268	0.066	0.065	0.065	0.065
Cov(est1, est2)	-0.0039	0.0188	0.0833	-0.0039	-0.0038	-0.0038	-0.0038
Adj R-Square	0.73	0.78	0.79	0.73	0.71	0.72	0.71
Risk RNL	0.37	0.79	0.96	0.37	0.35	0.36	0.35
Risk LINEX	0.08	6.19E+22	1.8E+22	0.08	0.08	0.08	0.08
Risk BLINEX	0.07	0.63	0.87	0.07	0.08	0.08	0.08

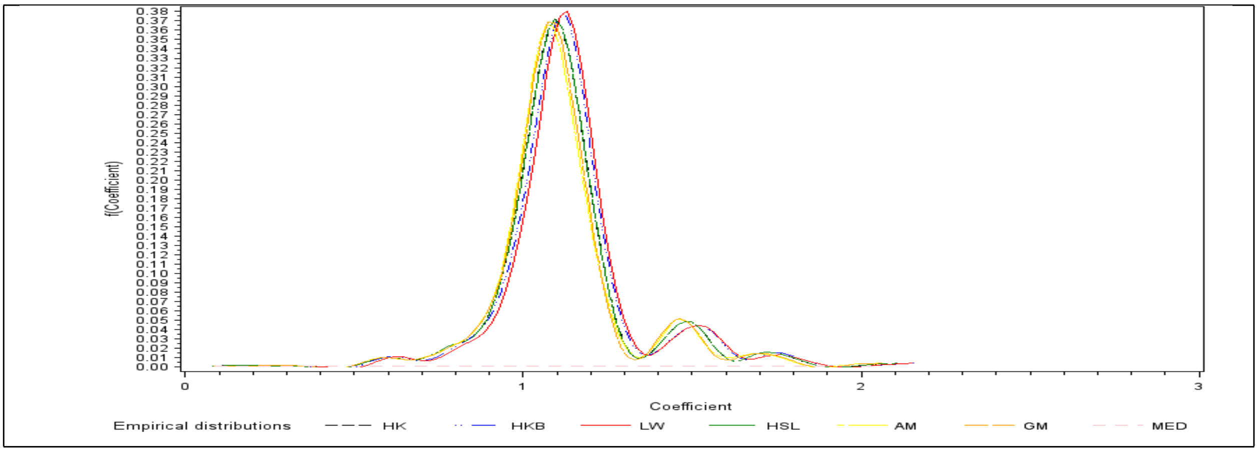
In Table 5 it is very obvious that the risk associated with the unrestricted ridge regression estimators using the HKB and LW ridge parameter is much higher compared to the risks associated with all the other ridge parameters. When the risk is evaluated under LINEX loss, it is very high due to the form of the LINEX loss function and the role of exponential term in the loss function. This is one of the disadvantages of working with the LINEX loss function and why a bounded loss function is more preferable.

In Figure 4 the two coefficients associated with labour and capital input were estimated using the restricted ridge regression estimator and by using the different ridge parameters(k) as discussed in Section 4. It can be seen that under all the ridge parameters the empirical distributions of the restricted ridge regression estimators are similar.

Restricted ridge regression estimator



Labour coefficient



Capital coefficient

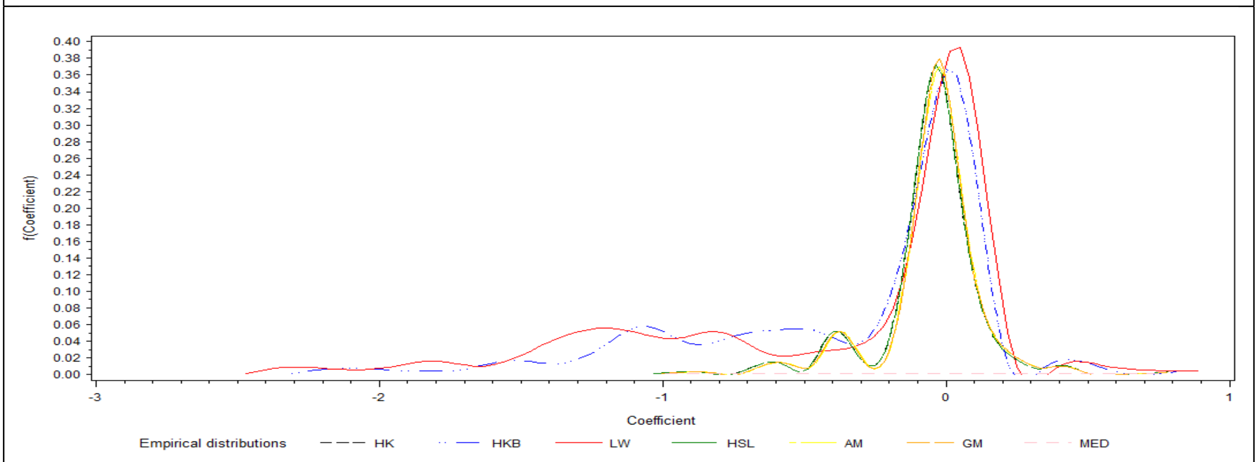
Figure 4 Empirical distributions of the slope coefficients of the restricted ridge regression estimator.

Table 6 Measures of risk performance for the restricted ridge regression estimator.

Ridge parameter	HK	HKB	LW	HSL	AM	GM	MED
Parameter estimate 1	**-0.10**	**-0.10**	**-0.09**	**-0.10**	**-0.09**	**-0.09**	**-0.09**
Standard error	0.060	0.062	0.065	0.060	0.059	0.059	0.059
Parameter estimate 2	**1.07**	**1.09**	**1.09**	**1.07**	**1.06**	**1.06**	**1.06**
Standard error	0.066	0.066	0.065	0.066	0.065	0.065	0.065
Cov(est1,est2)	-0.004	-0.004	-0.004	-0.0039	-0.0038	-0.0038	-0.0038
Adj R-Square	0.75	0.76	0.76	0.75	0.73	0.73	0.73
Risk RNL	0.36	0.39	0.43	0.36	0.35	0.35	0.35
Risk LINEX	0.09	0.17	0.27	0.09	0.09	0.09	0.09
Risk BLINEX	0.09	0.14	0.21	0.09	0.08	0.08	0.08

In Figure 6 the two coefficients associated with labour and capital input were estimated using the preliminary test ridge regression estimator and by using the different ridge parameters(k) as discussed in Section 4. The preliminary test estimator depends on the results of the hypothesis test, therefore the preliminary test estimator will be the appropriate combination of the two component estimators, namely the unrestricted ridge regression estimator and the restricted ridge regression estimator. In Figure 6 and Table 7 the effect of the variation in the LW and HKB ridge parameter is therefore again present in the preliminary test estimator, but not to the same extent as in Figure 3.

Preliminary test ridge regression estimator



Labour coefficient

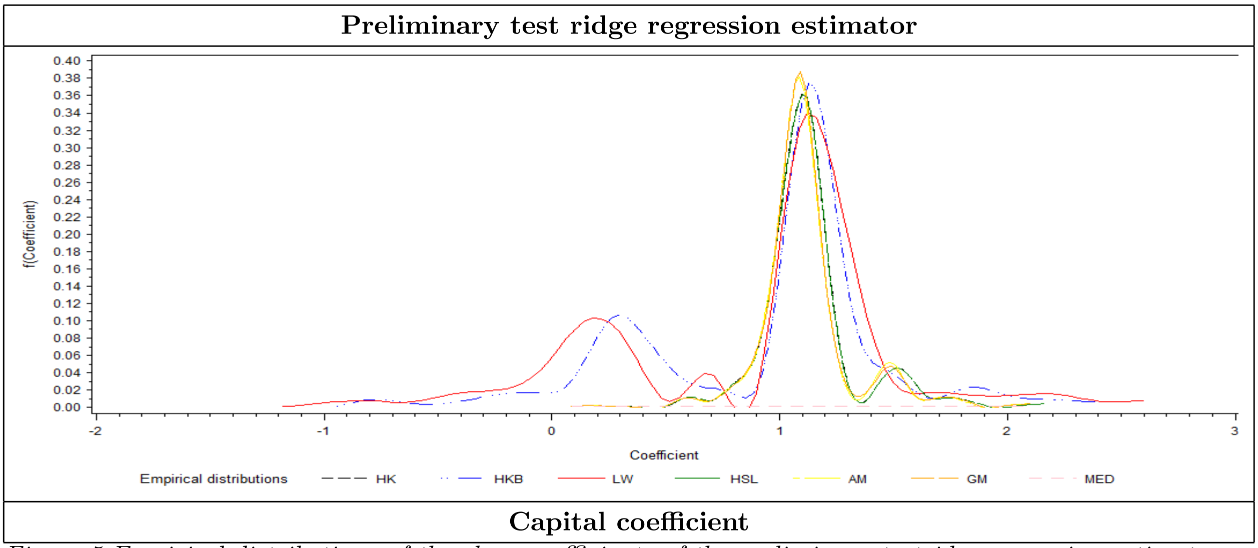


Figure 5 Empirical distributions of the slope coefficients of the preliminary testridge regression estimator.

Table 7 Measures of risk performance for the preliminary test ridge regression estimator.

Ridge parameter	*HK*	*HKB*	*LW*	*HSL*	*AM*	*GM*	*MED*
Parameter estimate 1	**-0.10**	**-0.40**	**-0.45**	**-0.10**	**-0.09**	**-0.10**	**-0.09**
Standard error	0.06	0.136	0.241	0.06	0.059	0.059	0.059
Parameter estimate 2	**1.07**	**0.86**	**0.82**	**1.07**	**1.06**	**1.06**	**1.06**
Standard error	0.066	0.119	0.192	0.066	0.065	0.065	0.065
Cov(est1, est2)	-0.004	0.010	0.051	-0.0039	-0.0038	-0.0038	-0.0038
Adj R-Square	0.73	0.78	0.79	0.73	0.72	0.72	0.71
Risk RNL	0.37	0.60	0.64	0.37	0.35	0.36	0.35
Risk LINEX	0.08	6.19E+19	1.84E+22	0.08	0.09	0.08	0.09
Risk BLINEX	0.07	0.43	0.49	0.07	0.08	0.08	0.08

In Table 7 it is clear that the risk associated with all the proposed estimators (regardless of ridge parameter) was consistently the lowest under BLINEX loss, which shows that the BLINEX loss function is the most preferable asymmetric loss function to work with.

Lastly, Figure 6 depicts the empirical distributions of the two parameter estimates of the Cobb-Douglas production function for the preliminary test estimator separately and also the empirical distribution of the restriction placed on the two slope parameters, namely $\beta_1 + \beta_2 = 1$. In this example the HK ridge parameter was used to control for multicollinearity, since this is one of the ridge parameters associated with lower risk.

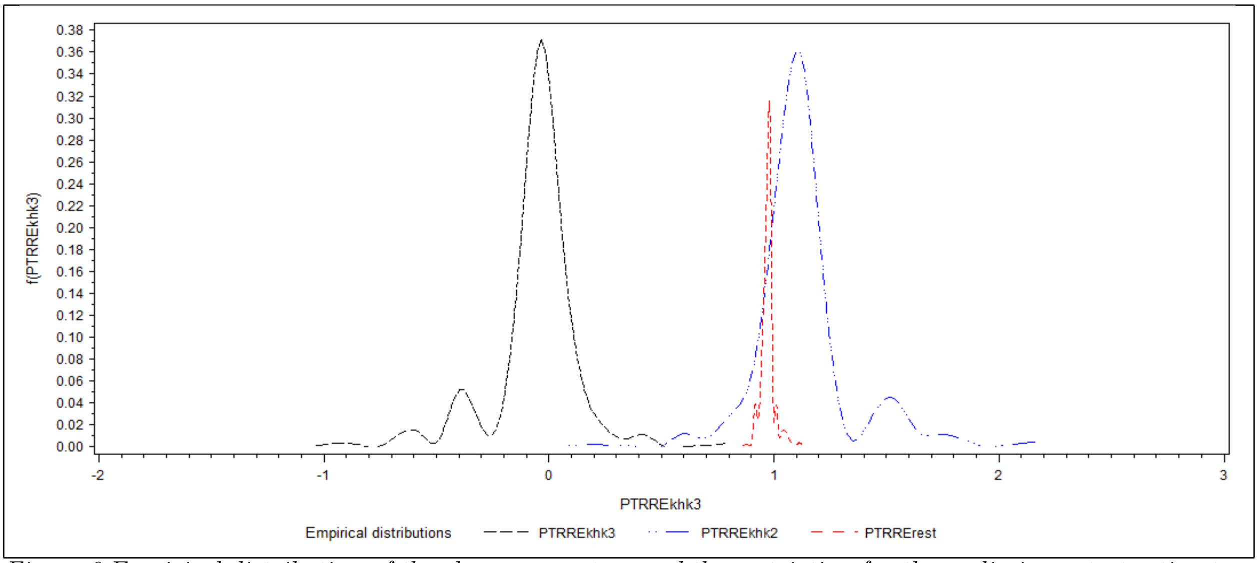


Figure 6 Empirical distribution of the slope parameters and the restriction for the preliminary test estimator.

The expected value of the labour and capital coefficients are -0.104 and 1.071 respectively and the expected value of the restricted coefficient, namely $\beta_1 + \beta_2 = 1$ is 0.97. By examining the empirical distribution of the PTRRE for the restriction $\beta_1 + \beta_2 = 1$, Figure 6 clearly indicates that the values are very closely distributed around the value of 1, with a standard error of 0.03.

6 Conclusion

In this paper, we examined the performance of the different ridge regression estimators, namely URRE, RRRE and PTRRE, under the BLINEX loss function. It was found that for the case where $\sigma = 1$, it can be seen that as the sample size increases for this choice of σ, the risk of PTRRE tends towards the risk of URRE as $\delta \rightarrow \pm\infty$, therefore as the sample size increases, PTRRE becomes just as efficient as the URRE. It was also shown that as the sample size increases the risk associated with all the estimators decreases. Similar results are obtained for different values of σ. When σ increases for a specific sample size, it was shown that in the interval near the origin, the interval where the risk of the PTRRE is lower relative to the URRE becomes larger. Therefore when working with a specific sample size and if the variance of the sample is relatively high, the PTRRE is a more efficient estimator compared to the URRE. Similar results are obtained under reflected normal loss and LINEX loss (see Kleyn,2014). In the practical example it can also be seen that the risk for all the proposed estimators, calculated under BLINEX loss versus the other considered loss functions namely reflected normal loss and LINEX loss, is consistently the lowest across all the proposed ridge parameters (k). Based on all the comparisons the significance of introducing estimation under the BLINEX loss function can clearly be seen. For an extensive comparison between the BLINEX loss function and LINEX loss function, as well as the reflected normal loss function, refer to Kleyn (2014).

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