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Generalized Geometry and Hopf Twists

by

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ABSTRACT

The Leigh-Strassler theories are marginal deformations of the $\mathcal{N} = 4$ SYM theory preserving $\mathcal{N} = 1$ Supersymmetry. As such they admit a Hopf algebra structure which is a quantum group deformation of the SU(3) structure of the R-symmetry of $\mathcal{N} = 4$ SYM. We reproduce the β-deformed theory, a subset of the Leigh-Strassler theories, from the Hopf twist approach and investigate how the twist manifests itself on the gravity dual by defining a star product between chiral superfields of the β-deformed field theory. The treatment on the gravity side is done in the Generalized Geometry framework. This star product is then used to deform the pure spinors of six-dimensional flat space and from the deformed spinors we obtain an $\mathcal{N} = 2$ solution of Supergravity. The Lunin-Maldacena background dual to the β-deformed theory is recovered when a stack of D3-branes is introduced in this $\mathcal{N} = 2$ solution. Alongside the β-deformed theory we consider a unitarily equivalent theory, which we refer to as a w-deformed theory. In this approach the role of the twist is transparent from the field theory to the gravity dual, making it useful in constructing backgrounds dual to the full Leigh-Strassler family of theories.

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"For who distinguishes you? And what do you have that you did not receive? ..."

- 1 Cor. 4:7, *The Bible*

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By now at least two things are evident:

1. this work belongs to more than one person
2. this page cannot contain them all.

Hector Dlamini

DECLARATION

I, Siphesihle Hector Dlamini, declare that the thesis/dissertation, which I hereby submit for the degree Master of Science in Physics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature:...............................

Date:..

CONTENTS

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LIST OF FIGURES

LIST OF TABLES

ACRONYMS

e.o.m Equation(s) of Motion

h.c. Hermitian conjugate

c.c. complex conjugate

SUSY Supersymmetry

SUGRA Supergravity

QFT or CFT Quantum or Conformal Field Theory

HA Hopf Algebra

QYBE Quantum Yang-Baxter Equation

GCG Generalized Complex Geometry

GCS Generalized Complex Structure

Part I

THE GAUGE FIELD THEORY SIDE

1

SUPERSYMMETRY

It is the aim of science to explain phenomena observed in nature, mainly by proposing plausible models whose predictions are tested against experimental data. Theoretical physics is not an exception to this. Typically and historically, models have been invented to help understand specific phenomena and usually they would be limited to that problem [52]. The need to invent a model for every phenomenon may attest to a lack of understanding the underlying principles of nature. So, a model which can explain many phenomena and do so in a simple [and elegant] way, is most sought after. In no other science is this search more prevalent than in particle physics. A concept of import to particle physicists in their quest for such models[1] is one of *symmetry*.

1.1 SYMMETRY

Symmetry can simply be described as the attribute of an 'object' to remain unchanged or invariant under a certain transformation. A rotation of a uniform sphere leaves it unchanged and so does a reflection. Some symmetries are continuous, like the rotation of the sphere, and others are discrete, the reflection of the sphere. In physics, symmetries of a system with respect to a given transformation usually manifest themselves as invariances in the *action* of that particular system. Our interests shall be on continuous symmetries because these kinds of symmetries are related to conserved quantities or charges by the elegant *Noether theorem* which we state here and prove in Appendix [A.2] [63]:

Theorem. *Every continuous symmetry of the Lagrangian, $\mathcal{L}$, gives rise to a conserved current $j^{\mu}(x)$, that is*

$$\partial_{\mu} j^{\mu}(x) = 0 \tag{1–1}$$

1.2 EXTERNAL AND INTERNAL SYMMETRY

Another way to classify symmetries which will simplify our introduction to Supersymmetry [SUSY] is whether the symmetry is *internal* or *external*. External symmetries are those that result from transformations which operate on spacetime coordinates, they are sometimes called geometric symmetries. Internal symmetries, on the other hand, are associated with transformations that affect the objects defined on the spacetime and not on the spacetime itself. In the case of field theory internal symmetries are observed in theories with more than one real field such that these real fields appear symmetrically in the Lagrangian [2]. These symmetries have a

1 Or "The Model" in the case of Grand Unified Theory [GUT].
2 There need not be multiple fields in the case where the fields are complex.

property that they commute with Lorentz transformations and an outcome of this is that their multiplets always contain particles of the same spin, either of bosonic kind or fermionic, not both[10].

1.3 THE LORENTZ GROUP AND POINCARÉ GROUP

1.3.1 *Lorentz group*

Now we turn to the *Lorentz group*, a group of linear transformations acting on four-vectors x^μ

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

such that the quadratic form

$$x^2 = x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1 x^1 + x^2 x^2 + x^3 x^3)$$

is invariant. Here we understand $\eta_{\mu\nu}$ to be the "mostly negative" Minkowski metric, $\eta_{\mu\nu} = (1, -1, -1, -1)$, (x^0) to be the time component and (x^1, x^2, x^3) the spatial components of space-time respectively. From a geometric view point the elements $\Lambda^\mu{}_\nu$ form a group of rotations and boosts, which mathematically is $SO(1,3|\mathbb{R})$ [3]. From Lie algebra theory, we know that an element, $\Lambda^\mu{}_\nu$, of the group, $SO(1,3|\mathbb{R})$, can be written in the form

$$\Lambda = exp(iT^i)$$

where T^i is a group generator, belonging to the Lie algebra $\mathfrak{so}(1,3|\mathbb{R})$. We will refer to J^i and K^i as generators for rotations and boosts respectively. These generators satisfy the following commutation relations

$$\begin{aligned} [J_i, J_k] &= i\epsilon_{ikl} J_l \\ [J_i, K_l] &= i\epsilon_{ilm} K_m \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \\ \text{where } & i, j, k = 1, 2, 3 \text{ and } \epsilon_{123} = 1 \end{aligned} \tag{1–2}$$

The boosts and rotations can be combined into a single anti-symmetric two-indexed tensor, $\mathcal{J}_{\mu\nu}$ by choosing that

$$\mathcal{J}_{i0} = -\mathcal{J}_{0i} = -K_i \ , \ \mathcal{J}_{ij} = -\mathcal{J}_{ji} = \epsilon_{ijk} J_k$$

In matrix form $\mathcal{J}$ looks like this

$$\mathcal{J} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

The commutation relations of the generators of the Lorentz group, (1–2), become

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = i(\eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}). \tag{1–3}$$

3 This is because the Minkowski treats time and space differently. In the Euclidean metric the appropriate group is $SO(4)$.

1.3.2 *Poincaré group*

If we include spacetime translations, $x^\mu \longrightarrow x^\mu + a^\mu$ to the Lorentz group, we obtain the *Poincaré group*. This is the full group of isometries of the Minkowski spacetime. An isometry is a transformation that preserves distances between points; rigid rotations, translations and boosts are examples of isometries. This group extension implies an algebra extension with momenta, P_μ - the generator of translations, and the resultant algebra is called the *Poincaré algebra* and is presented below:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i(\eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}) \\ [P_\mu, \mathcal{J}_{\nu\sigma}] &= i(\eta_{\mu\nu}P_\sigma - \eta_{\mu\sigma}P_\nu) \end{aligned} \tag{1–4}$$

1.3.3 *Conformal group*

Another yet important extension of the Poincaré group useful in field theory is the inclusion of conformal transformations. Conformal transformations are coordinate transformations that preserve the quantity

$$\frac{x \cdot y}{\sqrt{(x \cdot x)\ (y \cdot y)}} \tag{1–5}$$

Just as isometric transformations preserves distances between points, conformal transformations preserve angles. Their effect on the metric is

$$g_{\mu\nu}(x^a) \to \Omega^2(x^a) g_{\mu\nu}(x^a) \tag{1–6}$$

and field theories that display this symmetry are called Conformal Field Theories [CFT]. Infinitesimally these transformations are determined by vector fields ξ_μ that induce a change in the metric by a scale factor. These vector fields must satisfy the conformal Killing equation

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{d} g_{\mu\nu} \partial_\rho \xi^\rho = 0 \tag{1–7}$$

d is the spacetime dimension[4]. At the algebra level this implies the introduction of two generators : the dilation $x'^\mu \to \alpha x^\mu$ with generator $D = -i x^\mu \partial_\mu$ and the special conformal transformation [SCT]

$$x'^\mu \to \frac{x^\mu - (x \cdot x) b^\mu}{1 - 2(b \cdot x) + b^2 x^2} \tag{1–8}$$

which is generated by $K_\mu = -i(2x_\mu x^\mu \partial_\nu - (x \cdot)\partial_\mu)$ and b^μ is a translation. So the conformal algebra is the Poincaré algebra with the inclusion of following relations:

$$[K_\mu, \mathcal{J}_{\nu\rho}] = 2g_{\mu[\nu}K_{\rho]} \qquad [P_\mu, K_\nu] = 2(g_{\mu\nu}D + \mathcal{J}_{\mu\nu}) \tag{1–9}$$

$$[D, P_\mu] = P_\mu \qquad [D, K_\mu] = -K_\mu \tag{1–10}$$

Eugene Wigner's work [59] on representation theory ascertains that in (1–4) is all the information needed to build representations and states. This is because in representation theory particles are defined as elements of the representation space for a given representation of the Poincaré representation. Put in another way, the object that transforms in a given representation of the Poincare group is a particle. Now we need only to find a good filing system which we can use to order the various representations of the Poincaré group. A fruitful detour is a turn to Casimir operators.

4 The case d=2 is very special because the conformal group is infinite.

1.3.4 *Casimirs*

For a given group G, a *Casimir* operator is $c \in G$ such that $xc = cx \; \forall \; x \in G$. A Casimir operator is much like a central element of a group, which by definition is a member of the group which commutes with all the other elements of that group. For the Poincaré group, the first Casimir is $C_1 = P_\mu P^\mu$, momentum squared and here is why.

$$\begin{aligned} [P_\nu, P_\mu P^\mu] &= P_\nu P_\mu P^\mu - P_\mu P^\mu P_\nu \\ &= P_\nu (P_\mu P^\mu - P_\mu P^\mu) \\ &= 0 \end{aligned} \tag{1–11}$$

This is simply because P_μ commutes with itself according to the first relation in (1–4). Now for $\mathcal{J}$, also with the help of (1–4), we have

$$\begin{aligned} [\mathcal{J}_{\nu\rho}, P_\mu P^\mu] &= \mathcal{J}_{\nu\rho} P_\mu P^\mu - P_\mu P^\mu \mathcal{J}_{\nu\rho} \\ &= \mathcal{J}_{\nu\rho} P_\mu P^\mu - P_\mu \mathcal{J}_{\nu\rho} P^\mu + P_\mu \mathcal{J}_{\nu\rho} P^\mu - P_\mu P^\mu \mathcal{J}_{\nu\rho} \\ &= [\mathcal{J}_{\nu\rho}, P_\mu] P^\mu + P_\mu [\mathcal{J}_{\nu\rho}, P^\mu] \\ &= -i(\eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu) P^\mu - i P^\tau (\eta_{\tau\nu} P_\rho - \eta_{\tau\rho} P_\nu) \\ &= 0 \end{aligned} \tag{1–12}$$

The purpose here is to obtain a good label to use in classifying the representations of the Poincaré group. C_1 helps us obtain this label. To demonstrate this, let us consider a massive particle, mass m, with four-momentum, k^μ, and boost to its rest frame. In this rest frame its momentum becomes

$$k^\mu = (m, 0, 0, 0)$$

and applying C_1, we have

$$C_1 \left| k^\mu \right\rangle = P_\mu P^\mu \left| k^\mu \right\rangle = k_\mu k^\mu \left| k^\mu \right\rangle = m^2 \left| k^\mu \right\rangle$$

and m^2 is an invariant of the state $\left| k^\mu \right\rangle$ because it is an eigenvalue of C_1, thence a good label for ordering [5]. For massless particles, we have to adhere to the cosmic speed limit and cannot boost to the rest frame, but we can boost to a frame where

$$k^\mu = (E, 0, 0, E)$$

and here the action of C_1 is still

$$C_1 \left| k^\mu \right\rangle = k_\mu k^\mu \left| k^\mu \right\rangle$$

but the Minkowski metric implies that the covariant momentum of the state is

$$k_\mu = (E, 0, 0, -E)$$

and we conclude that for massless particles the eigenvalue of C_1 is 0. The second Casimir requires that a definition of the *Pauli-Lubanski pseudovector*

$$W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{J}^{\nu\rho} P^\sigma \tag{1–13}$$

Facts about this pseudovector which will prove to be useful later are that it is orthogonal to and commutes with translation generators P^μ, that is

$$[W_\mu, P^\nu] = 0 \text{ and } W_\mu P^\mu = 0. \tag{1–14}$$

5 Since C_1 is a Casimir, m^2 will not change under any Poincaré transformation.

The second Casimir is given by $C_2 = W_\mu W^\mu$ and this will give us another label with which to classify the representations of the Poincaré group. Consider again a particle of mass m with momentum k^μ. In the particle's rest frame the momentum is $k^\mu = (m, 0, 0, 0)$. The spatial components of momentum in this frame are zero, this then implies that the time component of the pseudovector is also zero

$$W_0 = -\frac{1}{2}\epsilon_{0ijk}\mathcal{J}^{ij}P^k = 0$$

so that the spatial components of W_μ give

$$\begin{aligned} W_i &= -\frac{1}{2}\epsilon_{ijk0}\mathcal{J}^{jk}P^0 \\ &= \frac{m}{2}\epsilon_{0ijk}\mathcal{J}^{jk} \\ &= -\frac{m}{2}\epsilon_{ijk}\epsilon^{jkl}J_l \\ &= -mJ_i \end{aligned} \tag{1–15}$$

Here J_i is the rotation generator, thus spatial components of W_μ are proportional to the spin matrices, J_i. So the action of C_2 on a rest frame state $k^\mu = (m, 0, 0, 0)$ is

$$\begin{aligned} W_\mu W^\mu |k^\mu\rangle &= \eta^{\mu\nu}W_\mu W_\nu |k^\mu\rangle \\ &= -W_i W_i |k^\mu\rangle \\ &= -m^2 J^2 |k^\mu\rangle \end{aligned} \tag{1–16}$$

From quantum mechanics we know that for a state with spin s, the eigenvalue of J^2 is $s(s+1)$. Spin is therefore another label with which to order our representations of the Poincaré group. This argument fails for massless particles so we recall from (1–14) that W_μ and P^μ are orthogonal and for massless particles both W and P are light-like, thus they are linearly dependent

$$W_\mu = \lambda P_\mu. \tag{1–17}$$

The constant of proportionality λ is called *helicity*. We define helicity as the projection of the spin of a particle along the direction of its momentum

$$\lambda = \frac{P \cdot J}{P_0}. \tag{1–18}$$

Spin and mass, for massive particles, and helicity, for massless ones, are the useful label in classifying representations. This is enough to embark into SUSY.

1.4 $\mathcal{N} = 1$ SUSY ALGEBRA

In section 1.2 we pointed out that symmetries are either of the internal kind or of the external kind. It is thus only natural to wonder if there exists another kind of symmetry which closes the chasm, one that mixes both internal and external symmetries. To the Coleman- Mandula "no-go theorem" [6] credit is due because it helped shed light on exactly how to search for this new kind of symmetry. By listing the axioms needed for a plausible physical theory in their proof, Coleman and Mandula exposed a silent assumption, namely that all continuous symmetries are Lie algebraic. Haag, Lopuszanski and Sohnius [19] demonstrated that by generalizing Lie Algebras to include algebraic systems whose defining relations incorporates both commutators and anti-commutators, the new symmetry could exist and it would be exempt from the no-go theorem. SUSY is simply then the symmetry that arises when we extend the Poincaré group by adding an anti-commuting spin $-\frac{1}{2}$ operator. We refer to the generators of SUSY as *supercharges*, denote them by Q_α and treat them as Weyl spinors. If parity invariance is imperative then we include the conjugate supercharges, $\bar{Q}_{\dot{\alpha}}$, also. In the latter setting it is meaningful to speak of the four-component Dirac spinors, Q_D, composed in the following fashion,

$$Q_D = \begin{pmatrix} Q_\alpha \\ \bar{Q}_{\dot{\alpha}} \end{pmatrix} \tag{1–19}$$

We use $\mathcal{N}$ to denote the amount of SUSY, the number of supercharges, by which the Poincaré algebra is extended. $\mathcal{N}=1$ SUSY means there is only one Q_D and cases where $\mathcal{N}>1$ are said to have *extended SUSY*. To avoid unnecessary complexities we consider the $\mathcal{N}=1$ case and specifically we look at the algebraic relations that the supercharges, Q_α and $\bar{Q}_{\dot{\alpha}}$, have with the Poincaré generators.

In 1965 O'Raifeartaigh, [46], demonstrated that translations commute with all generators beyond those of the Lorentz group. From this we conclude that

$$[P_\mu, Q_\alpha] = 0\ ,\ [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0 \tag{1–20}$$

The spinorial nature of the supercharges imposes the following commutation relations between the supercharges and $\mathcal{J}_{\mu\nu}$:

$$[Q_\alpha, \mathcal{J}_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \text{ and } [\bar{Q}_{\dot{\alpha}}, \mathcal{J}_{\mu\nu}] = -\bar{Q}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \tag{1–21}$$

$\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are 2x2 matrices, belonging to the group $SL(2,\mathbb{C})$ and they give a representation of the Lorentz group in the space of two-component left-moving and right moving spinors respectively. Their components are given by

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu{}_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}\beta} - \sigma^\nu{}_{\alpha\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta}) \tag{1–22}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}(\bar{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^\nu{}_{\gamma\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\gamma}\sigma^\mu{}_{\gamma\dot{\beta}}) \tag{1–23}$$

We now consider how the supercharges relate to one another by using anticommutators[6], because they are of fermionic nature. Thus in the left-moving representation

$$\{Q_\alpha, Q_\beta\} = 0 \tag{1–24}$$

The same is true of the conjugates in the right-moving representation. The mixed case is quiet interesting because supercharge, Q, is in the $(\frac{1}{2},0)$ representation while its conjugate, $\bar{Q}$ is in the $(0,\frac{1}{2})$ representation. Their product is therefore in the $(\frac{1}{2},\frac{1}{2})$, which is nothing but a spacetime vector. So their anticommutator must result in an object that transforms like a spacetime vector; the only candidate is P_μ and by imposing consistency in the indices their relation becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu{}_{\alpha\dot{\alpha}}P_\mu = 2P_{\alpha\dot{\alpha}} \tag{1–25}$$

It turns out that by extending the Poincaré group to include the supercharges there is an extra symmetry called the R-symmetry. To see it, let the Q's to be charged under an internal symmetry generated by R so that

$$[Q_\alpha, R] = Q_\alpha \text{ and } [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}} \tag{1–26}$$

6 $\{A,B\} = AB + BA$.

It follows that $[R, P_\mu] = 0$ and $[R, \mathcal{J}_{\mu\nu}] = 0$ because such a symmetry is internal to the Poincaré group. R-symmetry is scalar in nature and is simply the symmetry of the supercharges, Q, under phase rotations. We will refer to it as $U(1)_R$ symmetry. In exponential form it is

$$Q_\alpha \longrightarrow e^{-i\alpha} Q_\alpha \ , \ \bar{Q}_{\dot{\alpha}} \longrightarrow e^{i\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \ . \tag{1–27}$$

In summary, below is a collection of the relations of the $\mathcal{N} = 1$ super-Poincaré algebra [56]:

$$\begin{aligned}
[P_\mu, Q_\alpha] &= 0 & [P_\mu, \bar{Q}_{\dot{\alpha}}] &= 0, \\
[P_\mu, P_\nu] &= 0 & [P_\mu, \mathcal{J}_{\rho\sigma}] &= i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho), \\
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu & \{Q_\alpha, Q_\beta\} &= 0, \\
\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0 & [Q_\alpha, R] &= Q_\alpha, \\
[\bar{Q}_{\dot{\alpha}}, R] &= -\bar{Q}_{\dot{\alpha}} & [\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} + \eta_{\nu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}), \\
[Q_\alpha, \mathcal{J}_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta & [\bar{Q}_{\dot{\alpha}}, \mathcal{J}_{\mu\nu}] &= -\bar{Q}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}
\end{aligned} \tag{1–28}$$

It is to be noted that SUSY was obtained by extending the idea of Lie algebras to include fermionic generators as well as anti-commutators, this algebra then is a *graded Lie Algebra* and for consistency it is required to satisfy the graded Jacobi identities:

$$\begin{aligned}
[b_1, [b_2, b_3]] + [b_2, [b_2, b_1]] + [b_3, [b_1, b_2]] &= 0 \\
[b_1, [b_2, f]] + [b_2, [f, b_1]] + [f, [b_1, b_2]] &= 0 \\
[b, \{f_1, f_2\}] + \{f_1, [f_2, b]\} - \{f_2, [b, f_1]\} &= 0 \\
\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} &= 0
\end{aligned} \tag{1–29}$$

b_i and f_i represent bosons and fermions respectively. Any three elements of the super-Poincaré algebra satisfy these identities.

1.4.1 *SUSY Multiplets*

The two-fold question: What ”things” can live in a supersymmetric theory and how are these ”things” related to one another? Note that the s supercharges are fermionic, thus their action on a bosonic (fermionic) state will change it to a fermionic (bosonic) state.

$$Q\,|boson\rangle = |fermion\rangle \text{ and } Q\,|fermion\rangle = |boson\rangle \tag{1–30}$$

This observation demands that the theory have a fermionic state for every bosonic state [56]. So the idea of a ”particle” is adjusted to one that contains both the bosonic component and its fermionic counterpart and this generalized particle will be called a *superparticle*[7]. The components of a sparticle belong to the same supermultiplet. This takes care of the first part of our question. Sparticles, having boson components and fermionic components, are the “things” that live in a supersymmetric theory. To answer the second part there’s need to consider the Casimirs of the superalgebra. The first Casimir is

$$C_1 = P_\mu P^\mu$$

as it was for the Poincaré algebra and this follows from $[P_\mu, Q_\alpha] = 0$ in (1–28). The second Casimir C_2 is given by

$$C_2 = C_{\mu\nu}C^{\mu\nu}$$

7 or sparticle.

where

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \text{ and } B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\beta}Q_\beta\,.$$

The irreducible representations of the $\mathcal{N}=1$ super-Poincaré algebra are characterized by eigenvalues of these Casimirs.

1.4.2 $\mathcal{N}=1$ *massless supermultiplets*

In order to confirm the demand in (1–30) we consider the action of SUSY on a massless particle and observe the changes in helicity. From (1–17) we have that

$$W_0 = \lambda P_0$$

Let us now consider a state $|E,\lambda_1\rangle$ of specific helicity λ_1 in the frame where $P_\mu = (E,0,0,E)$. We observe that W_0

$$W_0\,|E,\lambda_1\rangle = \lambda_1 E\,|E,\lambda_1\rangle \tag{1–31}$$

gives the helicity of the state. In this setting the anticommutation relation $\{Q_\alpha,\bar{Q}_{\dot{\alpha}}\}$ becomes

$$\{Q_\alpha,\bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu = 2E(\sigma^0+\sigma^3)_{\alpha\dot{\alpha}} = 4E\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_{\alpha\dot{\alpha}} \tag{1–32}$$

so that

$$\{Q_1,\bar{Q}_{\dot{1}}\} = 4E \text{ and } \{Q_2,\bar{Q}_{\dot{2}}\} = 0 \tag{1–33}$$

Finding the helicity of the state $Q_\alpha\,|E,\lambda_1\rangle$ is what we are interested in, so we act on it with W_0

$$\begin{aligned}
W_0 Q_\alpha\,|E,\lambda_1\rangle &= Q_\alpha W_0\,|E,\lambda_1\rangle + [W_0,Q_\alpha]\,|E,\lambda_1\rangle\\
&= \lambda_1 E Q_\alpha\,|E,\lambda_1\rangle + E[\mathcal{J}^{12},Q_\alpha]\,|E,\lambda_1\rangle\\
&= \lambda_1 E Q_\alpha\,|E,\lambda_1\rangle + E(-\sigma^{12})_\alpha{}^\beta Q_\beta\,|E,\lambda_1\rangle\\
&= E\left(\lambda_1\delta_\alpha{}^\beta - \frac{1}{2}(\sigma^3)_\alpha{}^\beta\right)Q_\beta\,|E,\lambda_1\rangle\\
&= E\begin{bmatrix}\lambda_1-\frac{1}{2} & 0\\ 0 & \lambda_1+\frac{1}{2}\end{bmatrix}_\alpha^{\ \beta} Q_\beta\,|E,\lambda_1\rangle
\end{aligned}$$

We therefore have that

$$W_0 Q_1\,|E,\lambda_1\rangle = E(\lambda_1-\frac{1}{2})Q_1\,|E,\lambda_1\rangle \text{ and } W_0 Q_2\,|E,\lambda_1\rangle = E(\lambda_1+\frac{1}{2})Q_2\,|E,\lambda_1\rangle$$

and conclude that Q_1 lowers the helicity by $\frac{1}{2}$ while Q_2 increases it by $\frac{1}{2}$. From (1–33) it follows that $\bar{Q}_{\dot{1}}$ increases helicity by $\frac{1}{2}$ and $\bar{Q}_{\dot{2}}$ lowers it by $\frac{1}{2}$. We then define a state of helicity λ_0 to be the highest weight state if the following is true

$$Q_1\,|E,\lambda_0\rangle = 0 \tag{1–34}$$

Thus we can build our multiplet by applying $\bar{Q}_{\dot{1}}$ to such a state and of course this action can only be done once because the SUSY generators anticommute thus $\bar{Q}_{\dot{1}}\bar{Q}_{\dot{1}}\,|E,\lambda_0\rangle = 0$. And the second relation in (1–33) is not an option since it implies that the norm of every state $|\psi\rangle$ in Hilbert space vanishes [8].

$$\langle\psi|\,Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2\,|\psi\rangle = \langle\psi|\,Q_2\bar{Q}_{\dot{2}} - Q_2\bar{Q}_{\dot{2}}\,|\psi\rangle = 0 \tag{1–35}$$

In a nutshell, multiplets of $\mathcal{N} = 1$ supersymmetric theories only have two helicities $\{\lambda_0, \lambda_0 + \frac{1}{2}\}$ unless we are considering a CPT invariant theory then the allowed helicities are $\{-(\lambda_0 + \frac{1}{2}), -\lambda_0, \lambda_0, (\lambda_0 + \frac{1}{2})\}$. To make all these findings concrete we first note that theories without gravity have fields whose spin is not larger than one, thus our sample space of helicities for the highest weight state is narrowed down to just, $\lambda_0 = 0, \frac{1}{2}$.

1.4.2.1 *Example 1: Chiral Multiplet*

If we begin with a highest weight state,$|E, \lambda_0\rangle$ with helicity $\lambda_0 = 0$, then the multiplet is given by $\{-\frac{1}{2}, 0, 0, \frac{1}{2}\}$. This multiplet has room for two scalars and two spinors and is referred to as a *chiral multiplet* [8]. The on-shell field content is then a complex scalar field ϕ and a Weyl spinor ψ_α.

1.4.2.2 *Example 2: Vector Multiplet*

Now if our highest weight state has helicity $\lambda_0 = \frac{1}{2}$ then the resultant multiplet is called a *vector multiplet* and looks like this $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$. There are two vector and two spinor degrees of freedom. The on-shell field content for this multiplet is a vector, A_μ, and a Weyl spinor, ψ_α.

1.5 EXTENDED SUSY

At this point, we take a detour and insert a brief discussion on extended SUSY after which attention will be placed on $\mathcal{N} = 4$ SUSY specifically . Extended supersymmetry is nothing but an extension of $\mathcal{N} = 1$ SUSY in the following way:

$$Q_\alpha, \bar{Q}^{\dot{\alpha}} \longrightarrow Q^A_\alpha, \bar{Q}^{\dot{\alpha}}{}_A$$

with $A = 1, ..., \mathcal{N}$. This extension has little effect on the SUSY algebra (1–28) except for the following generalization and new relations:

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^A{}_B \\ \{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB} \\ \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} &= -\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{AB} \end{aligned} \tag{1–36}$$

Z^{AB}, an antisymmetric matrix, has been introduced in order to preserve the symmetry properties on both sides of the relation. Actually the algebra places another requirement on the elements, Z^{AB} and $(Z^{AB})^*$, and that is, they must commute amongst themselves and also with every element of the algebra in order for the algebra to be closed and consistent; algebra elements exhibiting such a property are called *central charges*.

8 or scalar.

1.5.1 *Massless Multiplets*

The massless multiplets in the extended SUSY case are similar to the $\mathcal{N} = 1$ case in that, if we boost to the frame $P_\mu = (E, 0, 0, E)$, we recover exactly what we found in (1–32)

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 4E\delta^A_B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \tag{1–37}$$

We again can use Q^A_1 to define the highest weight state $|E, \lambda\rangle$

$$Q^A_1 |E, \lambda\rangle = 0 \tag{1–38}$$

and by applying $\bar{Q}^A_1$ to such a state we can build the multiplets. By the same token as before, the case where A = B and $\alpha = \dot{\beta} = 2$ will introduce zero norm states in the theory so we lay it aside.

1.5.2 *Massive Multiplets*

Massive multiplets in the case of extended SUSY are slightly different, especially with the central charges switched on. In the rest frame, $P_\mu = (m, 0, 0, 0)$, the relation

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}} \tag{1–39}$$

already points to new features. Previously the case where $A = B$ and $\alpha = \dot{\beta} = 2$ issued in zero-norm states and in building the multiplets we excluded them. Here however they must be included and this means massive multiplets will contain more states than the massless ones. In order to handle the full SUSY algebra it is convenient to diagonalize the antisymmetric matrix Z^{AB} into blocks of 2×2 and we do so by splitting the index A into two $A = (a, i)$ where $a = 1, 2$ and $i = 1, ..., r$ with $\mathcal{N} = 2r$ [9]. The outcome is

$$Z = diag(\epsilon Z_1, ..., \epsilon Z_r, \#) \quad \text{where} \quad \epsilon^{12} = -\epsilon^{21} = 1 \tag{1–40}$$

In this basis, the only non-vanishing SUSY anticommutators are

$$\{a^i_{\alpha\pm}, (a^j_{\beta\pm})^\dagger\} = \delta^i_j \delta^\beta_\alpha (m \pm Z_i) \tag{1–41}$$

where a and $a^\dagger$ are linear combinations givens by

$$a^i_{\alpha\pm} = \frac{1}{2}\left(Q^{1i}_\alpha \pm \bar{Q}^{2i\dot{\beta}} \sigma^0_{\alpha\dot{\beta}}\right) \tag{1–42}$$

$$a^\dagger_{i\dot{\alpha}\pm} = \frac{1}{2}\left(\bar{Q}_{1i\dot{\alpha}} \pm \bar{\sigma}^0_{\dot{\alpha}\beta} Q^{\dot{\beta}}_{2i}\right) \tag{1–43}$$

Requiring that $m \geq Z_i$,$\forall$ m, drives away any ghost states. If $m > Z_i$ then we're in the typical massive multiplet case but if $m = Z_i$ then we have a *multiplet shortening*, because a^i_- will give zero-norm states hence a^i_+ are to be employed in creating states. This means a lesser number of states compared to the typical massive multiplet case. These special states have a special name: *BPS-saturated states*.

9 Assuming $\mathcal{N}$ is even. In the case where it is odd we would append a zero in the # .

1.5.2.1 *Example 3: $\mathcal{N} = 4$ SUSY and its massless multiplet*

Let us now consider a special SUSY case, $\mathcal{N} = 4$, where we introduce four generators and construct the massless multiplets of such a theory. As per custom in the massless multiplet case, we exclude the generators carrying indices $2, \dot{2}$ since they rise to zero-norm states and build with generators with indices $1, \dot{1}$ by applying $\bar{Q}^A_{\dot{1}}$ to the highest weight state ,$|\lambda\rangle$, of helicity λ defined by:

$$Q^A_1 \, |\lambda\rangle = 0 \tag{1–44}$$

And since A runs to 4, it is clear that the helicity can be raised to a maximum value of $\lambda + 2$. If we steer clear from theories with gravity, then we need to take $\lambda = -1$. We then have the following states in table (1)

λ	State
1	$\bar{Q}^1_{\dot{1}}\bar{Q}^2_{\dot{1}}\bar{Q}^3_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$
$\frac{1}{2}$	$\bar{Q}^1_{\dot{1}}\bar{Q}^2_{\dot{1}}\bar{Q}^3_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^1_{\dot{1}}\bar{Q}^3_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^2_{\dot{1}}\bar{Q}^3_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^1_{\dot{1}}\bar{Q}^2_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$
0	$\bar{Q}^1_{\dot{1}}\bar{Q}^2_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^1_{\dot{1}}\bar{Q}^3_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^1_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^2_{\dot{1}}\bar{Q}^3_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^2_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^3_{\dot{1}}\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$
$-\frac{1}{2}$	$\bar{Q}^1_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^2_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^3_{\dot{1}}\,|\lambda\rangle$ $\bar{Q}^4_{\dot{1}}\,|\lambda\rangle$
-1	$|\lambda\rangle$

Table 1.: $\mathcal{N} = 4$ SUSY massless multiplet

This multiplet consists of four Weyl fermions, $\psi^{A,a}_\alpha$, six real scalars, $\phi^{I,a}$, which can be combined to give 3 complex ones. It is the presence of a massless vector boson, A^a_μ, that makes this multiplet really special; this theory is called the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. Its field content in a compact form is

$$(A^a_\mu, \psi^{A,a}_\alpha, \phi^{I,a}) \tag{1–45}$$

where a is the gauge index, $A = 1, ..., 4$ and $I = 1, ..., 6$. The presence $\mathcal{N} = 4$ supercharges means the $U(1)_R$ symmetry observed in $\mathcal{N} = 1$ SUSY is enhanced in $\mathcal{N} = 4$ SYM to $SU(4)_R$. The scalar fields can be combined into

$$\Phi_j = \frac{1}{\sqrt{2}}[\phi^j + i\phi^j] \text{ where } i = 1,2,3 \tag{1–46}$$

The scalar fields transform in the **6** of SU(4) which is equivalent as SO(6). This fact becomes useful later when we match global symmetries between two theories. Below is the action of the theory where the SU(4) symmetry is evident [29].

$$S = \int d^4x \Big[(D_\mu \phi^{AB})(D^\mu \bar{\phi}_{AB}) - \frac{1}{2} i (\psi^{\alpha A}) \gamma_\mu D^\mu_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}_A - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ - g\psi^{\alpha A}[\psi^B_\alpha, \bar{\phi}_{AB}] - g\bar{\psi}_{\dot{\alpha}A}[\bar{\psi}^{\dot{\alpha}}_B, \phi^{AB}] + 2g^2[\phi^{AB}, \phi^{CD}][\bar{\phi}_{AB}, \bar{\phi}_{CD}] \Big] \tag{1–47}$$

Notice that the field content of $\mathcal{N} = 4$ SYM can be decomposed to that of one $\mathcal{N} = 1$ vector multiplet and three $\mathcal{N} = 1$ chiral multiplets:

$$V = (A_\mu, \psi^4_\alpha) \, , \quad \Phi = (\psi^i_\alpha, \phi^i) \tag{1–48}$$

1.5.3 *Superspace and Lagrangians*

The most logical way to proceed after obtaining the field content of a field theory is to construct a Lagrangian and obtain the e.o.m. Historically, this procedure entailed much trial and error simply because of the amount of freedom available to start with and it is thus best to construct the Lagrangian by components. A well-known such construction is the *Wess-Zumino* lagrangrian presented below

$$\mathcal{L}_{WZ} = \mathcal{L}_{kinetic} + \mathcal{L}_{mass} + \mathcal{L}_{int} \tag{1–49}$$

where

$$\begin{aligned} \mathcal{L}_{kin} &= \partial_\mu A^* \partial^\mu A + i\partial_\mu \psi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + F^* F \\ \mathcal{L}_{mass} &= -mAF + \frac{1}{2} m\psi^\alpha \psi_\alpha - mA^* F^* + \frac{1}{2} m\bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \\ \mathcal{L}_{int} &= -gAAF + gA\psi^\alpha \psi_\alpha - gA^* A^* F^* + gA^* \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \end{aligned} \tag{1–50}$$

with A and ψ being the scalar field and the Weyl spinor of the multiplet respectively, g a dimensionless coupling constant and F an auxiliary field. It is auxiliary because it needs to be supplied in order to have the right number of degrees of freedom because SUSY is not closed off-shell. Thus F guarantees closure of the SUSY algebra off-shell and has zero on-shell degrees of freedom, so it does not propagate [58]. Such a Lagrangian is invariant under the following SUSY transformations

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\xi^\alpha \psi_\alpha, & \delta_{\bar{\xi}} A^* &= \sqrt{2}\bar{\xi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}, \\ \delta_\xi \psi_\alpha &= \sqrt{2}\xi_\alpha F + i\sqrt{2}\sigma^\mu{}_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \partial_\mu A, & \delta_{\bar{\xi}} \bar{\psi}_{\dot{\alpha}} &= \sqrt{2}\bar{\xi}_{\dot{\alpha}} F^* - i\sqrt{2}\xi^\alpha \bar{\sigma}^\mu{}_{\alpha\dot{\alpha}} \partial_\mu A^*, \\ \delta_\xi F &= -i\sqrt{2}\partial_\mu \psi^\alpha \sigma^\mu{}_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}, & \delta_{\bar{\xi}} F^* &= i\sqrt{2}\xi^\alpha \bar{\sigma}^\mu{}_{\alpha\dot{\alpha}} \partial_\mu \bar{\psi}^{\dot{\alpha}} \end{aligned} \tag{1–51}$$

ξ and its conjugate $\bar{\xi}$ are anti-commuting variation parameters which ascertain consistency in the statistics.

Computational technology has since been developed and allows us to not only construct Lagrangians with ease but also to handle both the bosonic and fermionic part simultaneously. *Superspace* is what we are referring to and it is an extension of the spacetime coordinates, x^μ, by including Grassmann coordinates as follows:

$$x^\mu \to (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$$

This is thus called $\mathcal{N} = 1$ superspace because the extension is by the addition of only one Grassmann coordinate with its conjugate. In principle we can append any number of Grassmann coordinates but this formalism becomes cumbersome. We will thus consider the simple case of $\mathcal{N} = 1$. While this extension may seem arbitrary, a closer look at the algebra in (1–28) shows signs that which may suggest otherwise. The relation

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$$

implies that the action of two supercharges is as good as spacetime translation and furthermore indicates that superspace is not flat. In superspace, the supercharges,Q_α and $\bar{Q}_{\dot{\alpha}}$, can be represented as differential operators

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \ , \ \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu$$

And since superspace is not flat, we define the covariant derivatives as

$$D_\mu = \partial_\mu \ , \ D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \ \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{1–52}$$

These derivatives act on superfields, $\chi(x,\theta,\bar{\theta})$, the superspace version of spacetime fields. Superfields are functions of superspace and they are to be understood in terms of their power expansion in the Grassmann coordinates, θ and $\bar{\theta}$. Thus, generally a superfield in $\mathcal{N} = 1$ superspace will have the form

$$\begin{aligned}\chi(x,\theta,\bar{\theta}) = A(x) + \theta\psi(x) + \bar{\theta}\bar{\xi}(x) + \theta\theta B(x) + \bar{\theta}\bar{\theta}C(x) + \\ \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\ \bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\ \theta\eta(x) + \theta\theta\ \bar{\theta}\bar{\theta}D(x)\end{aligned} \tag{1–53}$$

and all other higher order terms vanish because of the anticommuting nature of Grassmann coordinates. We now have to impose some restriction on the superfield because its degrees of freedom do not match those obtained in section (1.4.2), neither the chiral multiplet nor the vector multiplet. The appropriate constraint for the chiral multiplet is

$$\bar{D}_{\dot{\alpha}}\chi(x,\theta,\bar{\theta}) = 0 \tag{1–54}$$

and we shall use $\Phi(x,\theta,\bar{\theta})$ to designate a superfield that satisfies (1–54), thus referring to it as a *chiral superfield*. To see the reason behind this nomenclature, we use a coordinate y^μ defined as

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{1–55}$$

which satisfies the following condition

$$\bar{D}_{\dot{\alpha}}y^\mu = 0 \tag{1–56}$$

It is clear that $\bar{D}_{\dot{\alpha}}\theta^\alpha = 0$. Any function $\Phi(y^\mu,\theta^\alpha)$ will satisfy the constraint (1–54) and solving it gives

$$\Phi(y^\mu,\theta^\alpha) = A(y) + \theta^\alpha\psi_\alpha(y) + \theta\theta F(y) \tag{1–57}$$

There are two complex scalar fields, $A(y)$ and $F(y)$, and one Weyl spinor,$\psi(y)$. In one superfield is the entire field content of the chiral multiplet; the name is thus fitting. We can regain the chiral superfield in terms of the original coordinate x by expanding the fermionic part of y. The result is

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = A(x) + \sqrt{2}\theta\psi(x) + \theta\theta\ F(x) + i\theta\sigma^\mu\bar{\theta}\ \partial_\mu A(x) \\ - \frac{i}{\sqrt{2}}\theta\theta\ \partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta\theta\ \bar{\theta}\bar{\theta}\ \Box A(x)\end{aligned} \tag{1–58}$$

The mixed product, $\Phi^\dagger\Phi$, constitutes the kinetic term of the superspace *action* from which we can recover the Wess-Zumino lagrangian in (1–49). Let us first turn to dimensional analysis and recall that the action must be dimensionless, so we require that the mass dimension $[\mathcal{L}] = 4$. The component fields, A, F, and ψ have the usual canonical mass dimensions $[A] = 1$,$[F] = 2$ and $[\psi] = \frac{3}{2}$ respectively. The mass dimension of the Grassmann coordinate is $[\theta] = -\frac{1}{2}$ and corresponds to that of the anti-commuting parameter, ξ, introduced in the transformation rules (1–51). The chiral superfield then has mass dimension $[\Phi] = 1$. We conclude that the measure over all of superspace must have mass dimension $[d\theta^2 d\bar{\theta}^2] = \frac{1}{2}$ since $[d\theta] = [\theta]^{-1} = \frac{1}{2}$ as observed from the Grassmann integral

$$\int d\theta\ \theta = 1 \tag{1–59}$$

For a free massless theory the action is thus given by

$$S = \int d\theta^2 \, d\bar{\theta}^2 \, \Phi\bar{\Phi} = \int d^4x \left[\partial^\mu A \, \partial_\mu A^* - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + FF^* \right] \tag{1–60}$$

A useful fact in extending to a massive and interacting theory is that a product of chiral superfields, Φ and Γ is also a chiral superfield. The SUSY transformation rules when expressed in superspace language will impose restrictions on the auxiliary F-term (the θ^2 term) of the product chiral field $\Phi\Gamma$ and the only way to extract it is to integrate over half of superspace. Since the mass dimension, $[\mathcal{L}]$, must be 4 and $[d\theta^2] = 1$, it follows that the mass term and interaction term of the theory must each be of mass dimension 3. A simple candidate is

$$W(\Phi) = -\frac{1}{2} m\Phi^2 - \frac{1}{3}\lambda\Phi^3 \tag{1–61}$$

λ is a coupling constant and m the mass. By putting all the results together we recover the Wess-Zumino lagrangian/action

$$\mathcal{S}_{WZ} = \int d^4\theta \; \Phi^\dagger \Phi - \left[\int d^2\theta \left(\frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3 \right) + \text{ h.c.} \right] \tag{1–62}$$

From (1–62) we see that part of the mass and interacting term is a holomorphic function of Φ and the other part, denoted by h.c., an antiholomorphic function of $\Phi^\dagger$. A feature which persists even in general cases is that the kinetic term of a supersymmetric theory composed only of fields from the chiral multiplet is integrated over all of superspace but the interacting term over half. In general the *action* of such a theory will have the form

$$\mathcal{S} = \int d^4\theta \; \mathcal{K}(\Phi^\dagger{}_i, \Phi^i) - \left[\, d^2\theta \; \mathcal{W}(\Phi^i) + d^2\bar{\theta} \; \bar{\mathcal{W}}(\Phi_i^\dagger) \right] \tag{1–63}$$

here $\mathcal{K}$ is a general kinetic term, called the Kähler potential and $\mathcal{W}$ is the superpotential of the theory, a general interacting term. This superpotential will be of import to us in the following chapters. Imposing a reality constraint

$$V = V^\dagger$$

on the general superfield (1–53) gives rise to a different superfield whose field content matches that of the vector multiplet we saw in section (1.4.2.2). Since this multiplet includes a vector v_μ and the reality constraint is not enough to adequately reduce the degrees of freedom either on-shell or off-shell, gauge symmetry will feature. The procedure involves the promotion of the gauge transformation parameter ($\alpha(x)$) to a chiral field (Λ) and realizing that the superspace version of the regular gauge transformation is

$$V \to V + i(\Lambda - \Lambda^\dagger) \tag{1–64}$$

Under this new gauge symmetry, the gauge field transforms as $A_\mu \to A_\mu + \partial_\mu a(x)$, where $a(x)$ is a scalar. We capture this by covariantizing the derivative as follows:

$$D_\mu = \partial_\mu + iA_\mu \tag{1–65}$$

The different ways we can choose Λ is the superspace "gauge fixing". It turns out that the proper way to define a superfield which will function as a field strength, $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$, which ever present in gauge theories is

$$W_\alpha = -\frac{1}{4} \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \; D_\alpha V(x, \theta, \bar{\theta}) \tag{1–66}$$

And in the case of what is known as the vector superfield in the Wess-Zumino gauge

$$V_{WZ} = -\theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta^2\bar{\theta}\,\bar{\Lambda}(x) - i\bar{\theta}^2\theta\,\lambda(x) + \frac{1}{2}\bar{\theta}^2\theta^2\,D(x) \tag{1–67}$$

In this gauge the prescribed superfield W_{α} in terms of the fields of the vector multiplet, v_{μ}, λ and $D(x)$ takes the form

$$W_{\alpha} = -i\lambda_{\alpha} + \theta_{\alpha}D(y) - \frac{i}{2}(\sigma^{\mu}\bar{\sigma}^{\nu})_{\alpha}F_{\mu\nu}(y) + \theta^2\sigma^{\mu}_{\alpha\dot{\beta}}\partial_{\mu}\bar{\lambda}^{\dot{\beta}}(y) \tag{1–68}$$

Here D is an auxiliary term supplied in order to balance the degrees of freedom on- and off-shell. y is familiar coordinate used in (1–55). We are on the verge of obtaining the Lagrangian because

$$\int d^2\theta W^{\alpha}W_{\alpha} = -2i\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{1–69}$$

Now for the Lagrangian of the abelian theory in superspace we have:

$$\mathcal{L} = \frac{1}{4g^2}\left\{\int d^2\theta W^{\alpha}W_{\alpha} + \int d^2\bar{\theta}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right\} \tag{1–70}$$

For the non-abelian theory we first convert the fields in the vector multiplet to matrices, that is place them in the adjoint representation with the help of the non-abelian gauge group generators T^a

$$A_{\mu} = A^a_{\mu}T^a,\ \lambda_{\alpha} = \lambda^a_{\alpha}T^a \text{ and } D = D^aT^a \tag{1–71}$$

and then promote all derivatives to covariant ones. The end result is

$$\mathcal{L} = \frac{1}{g^2}\,\mathrm{Tr}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + iD_{\mu}\lambda\sigma^{\mu}\bar{\lambda} + \frac{1}{2}D^2\right\} \tag{1–72}$$

which in the superspace formalism is given by

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left\{\tau\int d^2\theta\ \mathrm{Tr}\ W^{\alpha}W_{\alpha}\right\} \tag{1–73}$$

where

$$\tau = \frac{\theta}{2\pi} + \frac{4i\pi}{g^2}.$$

The interacting Lagrangian for general gauge invariance in the non-abelian theory is

$$\mathcal{L} = \mathrm{Tr}\int d^4\theta\Phi^{\dagger}e^{2V}\Phi - \mathrm{Tr}\int d^2\theta\left\{\frac{1}{2}m_{ij}\Phi^i\Phi^j + \frac{1}{3}\lambda_{ijk}\Phi^i\Phi^j\Phi^k\ + h.c.\right\} \tag{1–74}$$

and precise generalization of gauge invariance is

$$\Phi \to e^{-i2\Lambda}\Phi,\ \Phi^{\dagger} \to \Phi^{\dagger}e^{i2\Lambda^{\dagger}},\ W_{\alpha} \to e^{-i2\Lambda}W_{\alpha}e^{i2\Lambda}$$

Finally we present the superpotential of the $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ superspace notation

$$\mathcal{W} = g\ tr(\Phi^1[\Phi^2, \Phi^3]) \tag{1–75}$$

The Φ fields are complex and they are constructed using the 6 scalar fields X^i of $\mathcal{N} = 4$ SYM as $\Phi^j = \frac{1}{\sqrt{2}}(X^j + iX^{j+1})$ so that $j = 1, 2, 3$. In this guise the superpotential exhibits an $SU(3) \times U(1)_R$ symmetry concerning which there will be much discussion in the light of deformations.

2

ADS/CFT CORRESPONDENCE AND DEFORMATIONS

2.1 ADS/CFT CORRESPONDENCE

An idea that has proved useful in physics research, especially since the advent of string theory, is that of dualities or correspondences between theories. To have a duality that relates theories of the same kind, (i.e. QFT's $\rightarrow$ QFT's etc.), is sensible and is somewhat expected. In [39] a new kind of duality was conjectured and this duality relates field theories without gravity to theories of gravity with a particular background geometry. In the finding example which inspired this conjecture, on the field theory side was the maximally supersymmetric 4-dimensional $\mathcal{N} = 4$ SYM in Minkowski spacetime and on the gravity side was the 10-dimensional Type IIB string theory in the $AdS_5 \otimes S^5$ background geometry. More information was however obtained by considering the low energy limit where Type IIB string theory is well approximated by the 10-dimensional Type IIB Supergravity [SUGRA] [13]. The CFT is said to live in the boundary of AdS where AdS is the bulk.

Let us briefly discuss how this conjecture can be arrived at and for simplicity we shall take the D3-brane perspective. A detailed exposition on Dp-branes can be found in [48] [25]. Dp-branes are massive objects found in Type IIB string theories, they serve as higher-dimensional surfaces extending in (p+1) spacetime coordinates on which the ends of open strings attach. When a Dp-brane is introduced in flat space the BPS bound (as discussed in chapter 1[1].) is saturated. This implies that there will be a shortening of the supermultiplet of IIB SUGRA (a low energy approximation of IIB string theory), thus Dp-branes are appropriately called BPS objects. Only charged odd-dimensional (where p is odd) Dp-branes are found in Type IIB string theory. An open string trapped on a D3-brane represents matter and its dynamics are governed by a 4-dimensional U(1)-gauge theory. Closed strings on the other hand can travel through the bulk; their massless excitations are spin-2 gravitons. Since a brane is a BPS object it preserves only 16 of the 32 supersymmetries present in 10-dimensional SUGRA and this means in the *near-horizon* limit (details of which are specified below) the induced gauge theory must have $\mathcal{N} = 4$ Poincaré supersymmetry at the low energy limit. In the case where we have N D3-branes with the two string ends attached to different branes the induced gauge theory has $U(1)^N$ symmetry. At the limit when the D3-branes coincide we end up with a full $U(N) = U(1) \times SU(N)$ gauge theory. The U(1) describes the center of mass of the branes and can be ignored since focus is toward on-brane dynamics [61]. The final outcome is an $\mathcal{N} = 4$ supersymmetric gauge theory in 4 dimensions with gauge group $SU(N)$ [57] [62].

1 see 1–41

The spacetime metric of N coincident D3-branes is given by

$$ds^2 = H(y)^{-\frac{1}{2}}\eta_{ij}dx^i dx^j + H(y)^{\frac{1}{2}}\left(dy^2 + y^2 d\Omega_5{}^2\right) \tag{2–1}$$
$$H(y) = \left(1 + \frac{L^4}{y^4}\right) \quad \text{and } L^4 = 4\pi g_s N(\alpha')^2$$

where L is the radius of the D3-brane, η_{ij} – 4-dimensional Minkowski metric with *mostly plus* signature, g_s – the string coupling and α' is related to the string length. In the regime where $y \gg L$, $H(y) \to 1$ and the 10 dimensional flat spacetime is recovered. The regime where $y \ll L$ has a geometry whose one part is a 5-sphere with radius L and the other is the hyperbolic space AdS$_5$ also with radius L. A redefinition of coordinate to $u \equiv \frac{L^2}{y}$ makes this apparent. In the new coordinate $H(u) = (1 + \frac{u^4}{L^4})$. The regime $y \ll L$ corresponds to 'large' u and in this limit $H(u) \to \frac{u^2}{L^2}$. This transforms the metric to

$$ds^2 = L^2\left[\frac{1}{u^2}\eta_{ij}dx^i dx^j + \frac{du^2}{u^2}\right] + L^2\left[d\Omega_5{}^2\right] \tag{2–2}$$

The near-horizon limit corresponds to $y \to 0$ (or $u \to \infty$) with g_s and N fixed. Now reducing the 'string nature', $\alpha' \to 0$, Maldacena noticed that only the AdS$_5\otimes$S^5 part of the D3-brane geometry survives and the dynamics of the asymptotically flat region decouple from the theory. In this way was the conjecture born. Explicit comparisons have been made in view of the conjecture and these include matching the spectra of the theories [60] [18]. At present focus will be on comparing the symmetries of the theories for this is a necessary condition,though not sufficient.

2.1.1 *Global Symmetries*

2.1.1.1 *Symmetries of $\mathcal{N} = 4$ SYM*

A D3-brane in 10-dimensional spacetime has $SO(1,3) \times SO(6)$ global symmetry. The $SO(1,3)$ part is associate with Lorentz invariance and while the $SO(6) \sim SU(4)$ part is associate to the R-symmetry that relates the 6 scalar fields of $\mathcal{N} = 4$ SYM (1–45); these 6 scalar fields parametrize the 6 coordinates which are transverse to the D3-brane. The vanishing of the β-function of the theory means it is a CFT. The presence of conformal symmetry extends the superalgebra to a superconformal algebra and the Lorentz invariance $SO(1,3)$ is amplified to $SO(2,4)$ which is homomorphic to $SU(2,2)$. The global symmetry of $\mathcal{N} = 4$ SYM is thus given by $SO(2,4) \times SO(6) \sim SU(2,2) \times SU(4)$.

2.1.1.2 *Type IIB SUGRA on $AdS_5 \otimes S^5$*

Anti-de Sitter space is a vacuum solution to Einstein's field equations with positive cosmological constant. By definition then AdS has constant negative curvature [47]. In a flat (d+2)-dimensional manifold having coordinates X_i where $i = 0, 1, ..., d+1$ with a pseudo-Minkowski signature[2] metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{j=1}^{d} dX_j^2 \tag{2–3}$$

2 or *mostly plus* = (-,+,...,+,-).

AdS_{d+1} is defined as a solution to the constraint

$$-X_0^2 - X_{d+1}^2 + \sum_{j=1}^{d} X_j^2 = -L^2 \,. \tag{2–4}$$

L is refered to as the radius of AdS. Our particular case has $d+1=5$ and constraint becomes

$$-X_0^2 - X_5^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -L^2 \iff X_5^2 = L^2 - g_{\mu\nu}X^\mu X^\nu \tag{2–5}$$

$g_{\mu\nu} = diag(-,+,+,+,+)$. Contained in (2–5) is the fact that AdS_5 has an isometry group $SO(2,4)$. By solving the constraint (2–5) in what is usually referred to as *global coordinates* given via [9]:

$$\begin{aligned}
X_0 &= L\cosh\rho\ \cos\tau \\
X_5 &= L\cosh\rho\ \sin\tau \\
X_i &= \hat{X}_i\ L\sinh\rho,\ \text{where the } \hat{X}_i \text{ obey } \sum_{i=1}^{4}\hat{X}_i^2 = 1
\end{aligned}$$

one obtains the metric

$$ds^2 = L^2\left[-\cosh^2\rho\ d\tau^2 + d\rho^2 + \sinh^2\rho\ d\Omega_3{}^2\right] \tag{2–6}$$

where Ω_3 is a 3-sphere. However in order to make conformal invariance apparent we ought to rewrite the metric in Poincaré coordinates (u, x^a) which are defined by

$$\begin{aligned}
X_0 &= \frac{1}{2u}\left(1+u^2\left(L^2+\eta_{ab}x^a x^b\right)\right) = \frac{1}{2u}\left(1+u^2\left(L^2+\vec{x}^2-t^2\right)\right) \\
X_4 &= \frac{1}{2u}\left(1+u^2\left(-L^2+\eta_{ab}x^a x^b\right)\right) = \frac{1}{2u}\left(1+u^2\left(-L^2+\vec{x}^2-t^2\right)\right) \\
X_5 &= uLt \\
X_i &= uLx_i \ \ i=1,2,3.
\end{aligned} \tag{2–7}$$

x^a is a 4-vector which means η_{ab} is the Minkowski metric $diag(-,+,+,+)$ and the condition on the fifth coordinate is $u > 0$. This reduces the form of the AdS_5 metric to

$$ds^2 = L^2\left(\frac{du^2}{u^2} + u^2 dx_a dx^a\right) = \frac{L^2}{u^2}\left(u^2du^2 + \eta_{ab}dx^a dx^b\right). \tag{2–8}$$

AdS_5 is conformal manifold with an isometry group $SO(2,4)$ which happens to be equivalent to $SU(2,2)$, the conformal group of a manifold with one dimension less. The isometry group of S^5 is $SO(6) \sim SU(4)$. The global symmetry group of $\text{AdS}_5\times \text{S}^5$ is $SU(2,2)\times SU(4)$ and matches that of $\mathcal{N}=4$ SYM .

$\mathcal{N}=4$ **SYM**	**Type IIB SUGRA on** $\text{AdS}_5\times \text{S}^5$
Conformal group : $SO(2,4) \sim SU(2,2)$ R-Symmetry group : $SU(4)$	Isometry group of AdS_5 : $SO(2,4) \sim SU(2,2)$ Isometry group of S^5 : $SO(6) \sim SU(4)$

Table 2.: Global symmetry match

2.2 DEFORMATIONS

The $\mathcal{N} = 4$ SYM is undeniably a remarkable theory possessing interesting features, one such feature is finiteness. It however is removed from most theories which are of interest to physicists in that it is too ideal. It is only natural to wonder if there are other theories that are similar to $\mathcal{N} = 4$ SYM but are closer to reality. A logical starting point in searching for such theories is with the $\mathcal{N} = 4$ SYM itself and make changes to it. These changes are called *deformations* and the changes that preserve conformal invariance are referred to as *marginal*. It has been known that $\mathcal{N} = 4$ SYM is a member of a larger family of four-dimensional CFTs that preserve only $\mathcal{N} = 1$ supersymmetry and these $\mathcal{N} = 1$ theories can be arrived at by a suitable marginal deformation of the $\mathcal{N} = 4$ superpotential as follows

$$\mathcal{W}_{\mathcal{N}=4} = g\text{Tr}\left(\Phi^1[\Phi^2,\Phi^3]\right) \to \mathcal{W}_{LS} = \kappa\text{Tr}\left\{\Phi^1[\Phi^2,\Phi^3]_q + \frac{h}{3}\left[(\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right]\right\} \quad (2\text{–}9)$$

This deformation of the superpotential is in the formalism of $\mathcal{N} = 1$ superspace with chiral superfields $\Phi^i$and $[\Phi^i,\Phi^j]_q = \Phi^i\Phi^j - q\Phi^j\Phi^i$ is a q-deformed commutator. Finiteness at the quantum level of the $\mathcal{N} = 4$ theory only extends to $\mathcal{N} = 1$ theories whose superpotential is parameterized by q and h [33]. The parameter choice $(q,h) = (1,0)$ restores the $\mathcal{N} = 4$ theory whose superpotential is invariant under $SU(3) \times U(1)_R$ in terms of $\mathcal{N} = 1$ superspace but switching on both parameters breaks the $SU(3)$ part to a discrete $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry so that only the $U(1)_R$-symmetry is continuous [41]. Having begun with a CFT whose gravity dual is known, how then does the (q,h)-deformation manifest on the dual side? At the heart of this paper is an attempt to answer this question. We know that this question is valid because (q,h)-deformations are marginal and thus conformal symmetry is still present. With the presence of conformal symmetry in the deformed field theory it is reasonable to anticipate that in dual description the AdS_5 part of the geometry will not be affected and thus expect the deformation will manifest on the S^5.

2.2.1 *β-deformed theory and non-commutativity*

In [36], a class of deformed field theories corresponding to the parameter choice $(q,h) = (e^{i\pi\beta},0)$ was studied and a solution-generating technique of obtaining the gravity duals was devised. These field theories are referred to as $\beta-$deformed and their superpotential is given by

$$\mathcal{W}_{LS} = \kappa\text{Tr}\left[e^{i\pi\beta}\Phi^1\Phi^2\Phi^3 - e^{-i\pi\beta}\Phi^1\Phi^3\Phi^2\right] \quad (2\text{–}10)$$

An important consequence of the β-deformations is that the $SU(3)$ part of the global symmetry is broken to a subgroup $U(1) \times U(1)$. This is key because the solution-generating technique developed requires that the field theory possess a global $U(1) \times U(1)$ symmetry and that there be a 2-torus in the geometry of the parent gravity dual description, in other words, two isometries associated with the two $U(1)$ in the field theory. Then according to [36] the gravity dual of the β-deformed field theory can be obtained by transforming the complexified Kähler modulus, $\tau = B + i\sqrt{g}$, of the original theory as follows

$$\tau \longrightarrow \tau_\beta = \frac{\tau}{1+\beta\tau}. \quad (2\text{–}11)$$

The Kähler modulus is associated with the 2-torus whose volume is $\sqrt{g}$ and B is the B-field. β is the deformation parameter which in this work we restrict to real values only. The custom is to use γ as a parameter when $\beta \in \mathbb{R}$ but we will retain β, keeping in mind that it must be real. The deformation can alternatively be viewed as a result of promoting the product between the fields to a new one, called a *star product*. In this case we have

$$f \star g \equiv e^{i\pi\beta(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} fg \tag{2–12}$$

here f and g are fields belonging to the chiral/anti-chiral multiplets of the theory and the Q's are the $U(1)$ charges associated with them. This prescription is not foreign in the context of non-commutative gauge field theories where the field theory lives in a non-commutative space. In order to maintain the validity of the usual differential geometric methods, the non-commutativity of the spacetime is encoded by defining a star product on the spacetime coordinates (which now commute) [50][7].

Thus while the star product in (2–12) might seem to appear purely from inspiration it actually has a good basis from the field theory. In [30] there is a demonstration of how it can be obtained. Returning to the D3-brane picture in 10-dimensions with the 6 transverse coordinates, $\mathbb{C}^3$, we now can introduce the deformation by making these transverse coordinates non-commutative. Thus the D3-brane is surrounded by a six-dimensional non-commutative space and this non-commutativity of these coordinates is taken to be

$$z^I z^J = q z^J z^I \ , \ \bar{z}^I \bar{z}^J = q \bar{z}^J \bar{z}^I \tag{2–13}$$

I and J are cyclically ordered and run from 1 to 3. The relations (2–13) are nothing more than the constrains that the deformed superpotential (2–10) imposes on the F-term[3] of the Lagrangian, namely that

$$\Phi^I \Phi^J = q \Phi^J \Phi^I \ , \ \bar{\Phi}^{\bar{I}} \bar{\Phi}^{\bar{J}} = q \bar{\Phi}^{\bar{J}} \bar{\Phi}^{\bar{I}}. \tag{2–14}$$

This allows for the construction of an anti-symmetric contravariant 2-tensor in which the non-commutativity of the coordinates is captured in the following way

$$\begin{aligned} [z^I, z^J]_* &= z^I * z^J - z^J * z^I = i\Theta^{IJ} \\ [\bar{z}^{\bar{I}}, \bar{z}^{\bar{J}}]_* &= \bar{z}^{\bar{I}} * \bar{z}^{\bar{J}} - \bar{z}^{\bar{J}} * \bar{z}^{\bar{I}} = i\Theta^{\bar{I}\bar{J}} \\ [z^I, \bar{z}^{\bar{J}}]_* &= z^I * \bar{z}^{\bar{J}} - \bar{z}^{\bar{J}} * z^I = i\Theta^{I\bar{J}} \end{aligned} \tag{2–15}$$

The holomorphic components are $\Theta^{IJ} = 2sin\beta z^I z^J$ and by the same token the anti-holomorphic components are $\Theta^{\bar{I}\bar{J}} = 2sin\beta \bar{z}^{\bar{I}} \bar{z}^{\bar{J}}$. The mixed components are $\Theta^{I\bar{J}} = -2sin\beta z^I \bar{z}^{\bar{J}}$. The non-commutativity 2-tensor for the β-deformed theory in matrix form Θ_β is

$$\Theta_\beta = a \left(\begin{array}{ccc|ccc} 0 & z_1 z_2 & -z_1 z_3 & 0 & -z_1 \bar{z}_2 & z_1 \bar{z}_3 \\ -z_1 z_2 & 0 & z_2 z_3 & \bar{z}_1 z_2 & 0 & -z_2 \bar{z}_3 \\ z_1 z_3 & -z_2 z_3 & 0 & -\bar{z}_1 z_3 & \bar{z}_2 z_3 & 0 \\ \hline 0 & -\bar{z}_1 z_2 & \bar{z}_1 z_3 & 0 & \bar{z}_1 \bar{z}_2 & -\bar{z}_1 \bar{z}_3 \\ z_1 \bar{z}_2 & 0 & -\bar{z}_2 z_1 & -\bar{z}_1 \bar{z}_2 & 0 & \bar{z}_2 \bar{z}_3 \\ -\bar{z}_3 z_1 & \bar{z}_3 z_2 & 0 & \bar{z}_1 \bar{z}_3 & -\bar{z}_2 \bar{z}_3 & 0 \end{array} \right) \ , \ a \equiv 2sin\beta \tag{2–16}$$

Θ_β^{IJ} is coordinate independent; this fact evident when we change to a spherical coordinate system [30]. The choice

$$z^1 = r\cos(\alpha) e^{i\phi_1} \ , \ z^2 = r\sin(\alpha)\sin(\theta) e^{i\phi_2} \ , \ z^3 = r\cos(\theta)\sin(\alpha) e^{i\phi_3} \text{ including } c.c. \tag{2–17}$$

3 an auxillary term introduced into the supersymmetric lagrangian to ascertain the closure of the super-algebra, both on-shell and off-shell.

results in

$$\Theta_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a \\ 0 & 0 & 0 & a & 0 & -a \\ 0 & 0 & 0 & -a & 0 & a \end{pmatrix}, \text{where } a = 2\sin(\beta) \tag{2–18}$$

It must be emphasized that the holomorphic (top left block) and anti-holomorphic (bottom right block) parts are determined from the F-term constrains which means as long as we have the deformed superpotential we can always obtain them. The mixed components (top right and bottom left blocks) however were obtained from the star product definition (2–12) and in the absence of a star product prescription/definition one will have to rely on other constraints (i.e. the reality of the entries of Θ and symmetries of the lagrangian) in order to narrow down the possibilities for the mixed sectors.

2.3 QUANTUM ALGEBRA

From the view point of quantum algebra the non-commutativity of coordinates follows naturally. Quantum algebras are to classical (Lie) algebras what quantum mechanics is to classical mechanics in the sense that classical quantities are promoted to operators (which do not necessarily commute). So we can consider a non-commutative space V from which quantum vectors $\mathbf{x} = (x^i)$ and co-vectors $\mathbf{u} = (u_j)$ source their components. Commutation relations between quantum vector elements and quantum co-vector elements are given in terms of a complex-valued matrix R belonging to the vector space $V \otimes V$ as follows

$$\lambda x^b x^a = R^{a\ b}_{\ j\ l} x^j x^l \quad , \quad \lambda u_a u_b = u_j u_l R^{j\ l}_{\ b\ a} \tag{2–19}$$

where λ is an eigenvalue of the matrix $\hat{R}^{i\ k}_{\ j\ l} = R^{k\ i}_{\ l\ j}$. In quantum algebra linear transformations are governed by quantum matrices $\mathbf{t} = (t^i{}_j)$ where the entries t^i_j are operators and linear transformations that preserve the relations (2–19) will be associated with a quantum symmetry [41] [27].

Proposition 1. *The commutation relations in (2–19) are preserved by quantum linear transformations of the form*

$$x'^i = t^i{}_j x^j \quad \text{and} \quad u'_i = u_j (t^{-1})^j{}_i \tag{2–20}$$

if the elements t^i_j satisfy

$$R^{i\ k}_{\ a\ b} t^a_j t^b_l = t^k_b t^i_a R^{a\ b}_{\ j\ l} \tag{2–21}$$

Proof. Let t^i_j commute with x^i and u_i

$$\begin{aligned} x'^i x'^j &= R^{j\ i}_{\ k\ l} x'^k x'^l \\ \left(t^i_m x^m\right)\left(t^j_n x^n\right) &= R^{j\ i}_{\ k\ l} \left(t^k_p x^p\right)\left(t^l_q x^q\right) \\ t^i_m t^j_n\ R^{n\ m}_{\ p\ q}\ x^p x^q &= R^{j\ i}_{\ k\ l}\ t^k_p t^l_q\ x^p x^q \\ t^i_m t^j_n\ R^{n\ m}_{\ p\ q}\ x^p x^q &= R^{j\ i}_{\ k\ l}\ t^k_p t^l_q\ x^p x^q \\ \therefore t^i_k t^j_l\ R^{k\ l}_{\ p\ q} &= R^{j\ i}_{\ k\ l}\ t^k_p t^l_q \end{aligned}$$

□

The last equality is obtained by renaming the dummy indices $(m, n) \to (k, l)$. The equations (2–21) are called RTT relations [12]. For definiteness we rewrite the commutation relations (2–15) in the language of quantum algebra[4].

$$[z^i, z^j]_* = i\beta\tilde{\Theta}^{i\ j}_{\ k\ l} z^k z^l \ , \ [z^i, \bar{z}^{\bar{j}}]_* = i\beta\tilde{\Theta}^{i\ \bar{j}}_{\ k\ \bar{l}} z^k z^{\bar{l}} \ , \ [\bar{z}^{\bar{i}}, \bar{z}^{\bar{j}}]_* = i\beta\tilde{\Theta}^{\bar{i}\ \bar{j}}_{\ \bar{k}\ \bar{l}} \bar{z}^{\bar{k}} \bar{z}^{\bar{l}} \tag{2–22}$$

In this light, the commutation relations for the mixed coordinates are easily obtained from the mixed quantum planes defined by [27]

$$x^j u_l = u_i R^{j\ i}_{\ k\ l} x^k \text{ and } u_j x^i = x^k \tilde{R}^{\ k\ j}_{i\ l} u_l \tag{2–23}$$

where u_i is a quantum co-vector to which the antiholomorphic coordinates are mapped $\bar{x}^{\bar{i}} = u_i$ and $\tilde{R}$ is a second inverse matrix defined such that

$$\tilde{R}^{i\ n}_{\ m\ n} R^{m\ k}_{\ l\ n} = \delta^i_l \delta^k_j = R^{i\ n}_{\ m\ j} \tilde{R}^{m\ k}_{\ l\ n}. \tag{2–24}$$

Now by choosing $\tilde{\Theta}^{i\ j}_{\ k\ l}$ to be $\mathrm{diag}(0, -1, 1, 1, 0, -1, -1, 1, 0)$ we fully recover the holomorphic part of Θ_β (and in principle the anti-holomorphic part also), $i, j, k, l = 1, 2, 3$ [32]. $\tilde{\Theta}$ is not just chosen to make things work but rather comes from the matrix $R \in V \otimes V$ used to define the commutation relations in quantum algebra (2–19). The matrix R for the β-deformed theory is given by $diag(1, q, \frac{1}{q}, q, q, 1, \frac{1}{q}, \frac{1}{q}, q, 1)$ and its first order expansion around a small β, called a classical R-matrix, is equal to $\tilde{\Theta}$. The general R-matrix for a (q, h)-deformed theory is given by [41]

$$R = \frac{1}{2d^2}\begin{pmatrix}
1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\
0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q} \\
0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0 \\
0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\
0 & 0 & 2\bar{h}q & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0 \\
2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 \\
0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\
-2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 \\
0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h}
\end{pmatrix} \tag{2–25}$$

$$\text{where } d^2 = \frac{1 + \bar{q}q + \bar{h}h}{2} \tag{2–26}$$

This matrix is arrived at when the $SU(3)$-invariant tensor ϵ_{ijk}, appearing in the superpotential, is deformed to E_{ijk}. E_{ijk} is invariant under a quantum deformed SU(3) [41]. This point about the quantum deformation is important because the deformations which are of interest to us deform the $SU(3)$, much like the β-deformation. Whilst one might "naively" expect the general deformations to break the $SU(3)$ symmetry to $U(1)$'s on the contrary they actually deform to a Hopf algebra. This is true of the β-deformation. Notice that the parameter choice $(q, h) = (0, 0)$ gives the R-matrix for the undeformed $\mathcal{N} = 4$ SYM while the choice $(q, h) = (e^{i\beta}, 0)$ gives the R matrix of the β-deformed theory. Admittedly our introduction of quantum algebra was informal, necessity is laid upon us to take a more thorough and somewhat in-depth look at the theory of Quantum Algebras.

4 Repeated indices imply summation.

3

THE LANGUAGE OF HOPF ALGEBRAS

We devote this section to the formal discussion of the theory underlying Hopf (Quantum) [1] Algebras which we began to use in chapter 2. We shall also look at their properties pertaining to our scope by using mainly the conventions and notations of [37].

3.1 DEFINITIONS

3.1.0.1 *Algebra*

Definition 3.1.1. An *algebra* A is a vector space defined over a field k together with a compatible associative map, $m : A \otimes A \to A$, and a unit element, 1_A.

The map m, which sometimes is referred to as a "product" or "multiplication", is associative in the sense that $(ab)c = a(bc)\ \forall\ a, b, c \in A$ where $ab = m(a \otimes b)$. And if

$$ab = m(a \otimes b) = m(b \otimes a) = ba\ \forall\ a, b \in\ A$$

then A is commutative. The unit element 1_A can be defined in terms of a map as follows $\eta : k \to A$ as $\eta(1) = 1_A$. It is customary to both compactly write all of the above as $(A, +, m, \eta; k)$ and also summarize with the help of commutative diagrams as we do below

$$\begin{tikzcd} & A \otimes A \otimes A \arrow[dl, "m\otimes id"'] \arrow[dr, "id\otimes m"] & \\ A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

(a) Associativity of Algebra

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes id"] \arrow[d, "\sim"'] & A \otimes A \arrow[dl, "m"] \arrow[dr, "m"'] & A \otimes k \arrow[l, "id\otimes\eta"'] \arrow[d, "\sim"] \\ A & & A \end{tikzcd}$$

(b) The unit map η

Figure 1.: Commutative diagram for algebra A

3.1.0.2 *Co-algebra*

An unusual and somewhat novel structure is the *coalgebra*. Its formal definition is below.

1 In this section Hopf will be used instead of Quantum.

Definition 3.1.2. A *co-algebra* C is a vector space defined over a field k coupled with a *co-product*, a map $\Delta : C \to C \otimes C$, which is *co-associative* and a *co-unit*, a map $\epsilon : C \to k$.

The co-product Δ acts on an element $c \in C$ so as to share it over $C \otimes C$,

$$\Delta(c) = \sum_i c_{(1)i} \otimes c_{(2)i} \equiv c_{(1)} \otimes c_{(2)} \quad where \quad c_{(1),(2)} \in C \tag{3–1}$$

The last equality exhibits the use of the *Sweedler notation* in which the summation and indices are suppressed to help simplify complicated expressions [26]. *Co-associativity* means that

$$(\Delta \otimes id) \circ \Delta(c) = (id \otimes \Delta) \circ \Delta(c) \tag{3–2}$$

$$(\Delta \otimes id)(\sum c_{(1)} \otimes c_{(2)}) = (id \otimes \Delta)(\sum c_{(1)} \otimes c_{(2)}) \tag{3–3}$$

$$\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}, \quad \forall\, c \in C \tag{3–4}$$

hence it is of little import which vector space we share out first [2]. The condition on the *co-unit* map is that it obey

$$(\epsilon \otimes id) \circ \Delta(c) = c = (id \otimes \epsilon) \circ \Delta(c) \tag{3–5}$$

The coalgebra parallel of algebra commutativity is co-commutativity and is defined via a transposition map $\tau : C \otimes C \to C \otimes C$ which reverses the order of the vector spaces, that is $\tau(a \otimes b) = b \otimes a$. Thus a co-algebra is co-commutative

$$\tau \circ \Delta(c) = \Delta(c) \ \forall\, c \in C \tag{3–6}$$

As we did for the algebra so also we compactly write $(C, +, \Delta, \epsilon; k)$ and summarize by means of commutative diagrams for the coalgebra.

$$\begin{tikzcd}
 & C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "id \otimes \Delta"] & & C \otimes C \arrow[ul, "\Delta \otimes id"'] \\
 & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(a) Coassociativity of coalgebra C

$$\begin{tikzcd}
k \otimes C \arrow[d, "\sim"'] & C \otimes C \arrow[l, "\epsilon \otimes id"'] \arrow[r, "id \otimes \epsilon"] & C \otimes k \arrow[d, "\sim"] \\
C \arrow[ur, "\Delta"'] & & C \arrow[ul, "\Delta"]
\end{tikzcd}$$

(b) Co-unit map ϵ

Figure 2.: Commutative diagram for coalgebra C

The notion of a co-algebra parallels that of an algebra and is achieved by reversing the arrows of a commutative diagram of an algebra. What the algebra takes away is brought back by the co-algebra. One can then define extended structures which contain both the algebra and coalgebra properties, that is they are invariant under an arrow-reversal operation of their commutativity diagrams; such structures are called *bi-algebras* and they have built-in an "input-output" symmetry.

3.1.0.3 *Bialgebra and Hopf Algebra*

Definition 3.1.3. A *bialgebra* $(H, +, m, \eta, \Delta, \epsilon; k)$ over a field k is a vector space $(H, +; k)$ defined over a field k which is both an algebra and a co-algebra in a compatible way, where the maps m and Δ are co-algebra homomorphic and algebra homomorphic respectively.

2 $\circ$ found in 3–2 means composition i.e. $(f \circ g)(x) = f(g(x))$. So throughout this work.

Compatibility means the product and unit maps m and η preserve the coalgebra structure on the vector space and so do the co-product and co-unit maps Δ and ϵ the algebra structure:

$$\Delta(hg) = \Delta(h)\Delta(g), \ \ \Delta(1_H) = 1_H \otimes 1_H, \ \ \epsilon(hg) = \epsilon(h)\epsilon(g) \text{ and } \epsilon(1_H) = 1 \ \forall\, h, g, 1_H \in H$$

We will be concerned with a particular class of bialgebras called *Hopf Algebras*. They have an additional axiom, that is they have a linear map $S : H \to H$, called an *antipode*. This map must obey

$$m\,(S \otimes id) \circ \Delta = m\,(id \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{3–7}$$

Condition (3–7), at the level of elements, translates to [49]: $S(h_{(1)})h_{(2)} = \epsilon(a)1 = h_{(1)}S(h_{(2)})$ and for this reason many view it as encapsulating the idea of an inverse[3]. The commutative diagram of a Hopf Algebra will be a composition of (1) and (2) but will also include two additional diagrams for the antipode S

$$\begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & k \arrow[r, "\eta"] & H \\ H \otimes H \arrow[r, "id\otimes S"] & H \otimes H \arrow[r, "id\otimes id"] & H \otimes H \arrow[u, "m"'] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & k \arrow[r, "\eta"] & H \\ H \otimes H \arrow[r, "S\otimes id"] & H \otimes H \arrow[r, "id\otimes id"] & H \otimes H \arrow[u, "m"'] \end{tikzcd}$$

Figure 3.: The antipode axioms of HA H

3.2 HOPF TWISTS

The existence of HA's is certain but the task of constructing an one is not simple. One way of constructing HA's called Twisting [4] was presented in [11]. The core idea is: given an HA $(A, m, \eta, \Delta, \epsilon, S; k)$ and a counital 2-cocycle element $\chi \in A \otimes A$ then a new HA A_χ can be constructed by *twisting* the coproduct and the antipode as follows

$$\Delta_\chi a = \chi\Delta(a)\chi^{-1}\ ,\ S_\chi a = U(S(a))U^{-1}\ , \forall\ a \in A \tag{3–8}$$

and using the product map m and unit map η from A. U is given by $m \circ (id \otimes S)(\chi) = \sum \chi^{(1)}(S\chi^{(2)})$ and must be invertible. To say that χ ought to be counital and 2-cocyclic respectively means

$$\begin{gathered}(\epsilon \otimes id)\chi = 1 \\ (1 \otimes \chi)(id \otimes \Delta)\chi = (\chi \otimes 1)(\Delta \otimes id)\chi.\end{gathered} \tag{3–9}$$

The benefit of the twist approach is that it preserves properties of the original HA which depend on the coproduct and counit. This is useful because can one relax certain HA axioms to obtain special kinds of HA's with richer structure. The method of twisting will preserve the extra structure on these special HA's. A special HA that will be important in our case is the *Quasitriangular Hopf Algebra* which arises when the co-commutativity of the HA is relaxed up to conjugation with an invertible element upon which there are certain restrictions.

3 The antipode need not be an involution nor invertible.
4 Or twist construction, sometimes deformation.

3.2.0.1 *Quasitriangular Hopf Algebra*

Definition 3.2.1. A Quasitriangular Hopf Algebra is the pair (H, R) where H is an HA and R is an invertible element of $H \otimes H$ that satisfies

$$(\Delta \otimes id)R = R_{13}R_{23} \text{ and } (id \otimes \Delta)R_{13}R_{12} \tag{3–10}$$

$$\tau \circ \Delta(h) = R\Delta(h)R^{-1} \; \forall \; h \in H \tag{3–11}$$

In this notation the expression R_{ij} means $1 \otimes \cdots \otimes R_i \otimes \cdots \otimes R_j \otimes \cdots \otimes 1$ where R_i is in the i-th position of the chain and R_j in the j-th position. The quasitriangularity of HA H is preserved by a twist χ, so the twisted HA (H_χ, R_χ) is also quasitriangular. The twisted co-product and antipode are as in (3–8) and the twisted quasitriangular structure is

$$R_\chi = \chi_{21} R \chi^{-1} \tag{3–12}$$

Notice that the quasitriangular structure actually satisfies the Quantum Yang-Baxter Equation (QYBE), $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, a consistency equation. This property will ascertain associativity when the non-commutativity of the coordinates is encoded via a star product.

3.2.0.2 *Representation of Hopf Algebra*

Hopf algebras can act on other sets, be they algebras, coalgebras or even other HA's and generally the side from which this action is applied matters. If the action of HA H is from the left then A is called a left *H-module algebra* and the same can be said about the right action. The left action[5] is then a linear map $\alpha : H \otimes A \to A$.

$$\alpha(h \otimes a) \equiv h \triangleright a \;\; \in A, \quad \text{where } h \in H \text{ and } a \in A \tag{3–13}$$

The *module* is then a representation of the Hopf algebra. A desired property of the action of HA's on algebras is one of compatibility with structures already existing within the algebra in question (i.e. multiplication and unit maps). So given an algebra A with product map $\mu : A \otimes A \to A$, the elements $g, h \in H$ act on elements $x, y \in A$ such that

$$g \triangleright (\mu([x \otimes y])) = \mu(\Delta(g) \triangleright [x \otimes y]) \tag{3–14}$$

This means twisting an HA (which twists the co-product Δ) will translate to a twist in the product μ of the module. To maintain the Leibniz property of products one observes that for an F-twisted co-product $\Delta_F = F\Delta F^{-1}$ the F-twisted module product ought to be [2]

$$\mu_F(x \otimes y) = \mu(F^{-1} \triangleright [x \otimes y]) \tag{3–15}$$

5 The same definitions can be made for the right action and for coalgebras.

3.3 TWIST CONSTRUCTION OF β-DEFORMATION

In section (2.3) we saw that the deformation parameter choice $(q,h) = (1,0)$ in the general (q,h)-deformed R-matrix (2–25) restores the $\mathcal{N} = 4$ SYM with R-matrix, $R_0 = \mathbb{1}_{9\times 9}$, and the β-deformed theory arises by setting $(q,h) = (e^{i\beta}, 0)$ with the corresponding R-matrix

$$R_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3–16}$$

We observe that R_β can be constructed from R_0 with a suitable twist F_β which is given by

$$F_\beta = \mathrm{diag}(1, \sqrt{q}, \frac{1}{\sqrt{q}}, \sqrt{q}, \sqrt{q}, 1, \frac{1}{\sqrt{q}}, \frac{1}{\sqrt{q}}, \sqrt{q}, 1) \tag{3–17}$$

using the twist construction $R_\beta = F_\beta R_0 F^{-1}{}_\beta$. To see that F_β is 2-cocyclic we first note that is can be written in terms of the Cartan sub-algebra generators H_i of $SU(3)$ as

$$F_\beta = e^{i\frac{\beta}{2} H_1 \wedge H_2}, \text{ where } H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3–18}$$

and this means F_β is a twist of a Abelian type [53]. Since the co-product acts on algebra elements as

$$\Delta(H_i) = H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i \tag{3–19}$$

its effect on F_β exponentiates[6] so that

$$\begin{aligned}(\Delta \otimes id)F_\beta &= (\Delta \otimes id) e^{i\frac{\beta}{2} H_1 \wedge H_2} \\ &= (\Delta \otimes id) e^{i\frac{\beta}{2}(H_1 \otimes H_2 - H_2 \otimes H_1)} \\ &= e^{i\frac{\beta}{2}(\Delta \otimes id)(H_1 \otimes H_2 - H_2 \otimes H_1)} \\ &= e^{i\frac{\beta}{2}(H_1 \otimes \mathbb{1} \otimes H_2 + \mathbb{1} \otimes H_1 \otimes H_2 - H_2 \otimes \mathbb{1} \otimes H_1 - \mathbb{1} \otimes H_2 \otimes H_1)} \\ &= e^{i\frac{\beta}{2}(H_1 \otimes \mathbb{1} \otimes H_2 - H_2 \otimes \mathbb{1} \otimes H_1)} e^{i\frac{\beta}{2}(\mathbb{1} \otimes H_1 \otimes H_2 - \mathbb{1} \otimes H_2 \otimes H_1)} \\ &= F_{\beta,13} F_{\beta,23}\end{aligned}$$

By the exact same argument we arrive at $(id \otimes \Delta)F_\beta = F_{\beta,13}F_{\beta,12}$. This means then 2-cocycle condition reduces to

$$F_{\beta,12} F_{\beta,13} F_{\beta,23} = F_{\beta,23} F_{\beta,13} F_{\beta,12}. \tag{3–20}$$

6 much like Lie group elements are obtained by the exponentiation of Lie algebra generators.

Let us first define the co-unit map ϵ such that $\epsilon(1) = 1$ and $\epsilon(X) = 0$ for any other algebra element X [53]. It becomes clear that F_β is counital because we can expand in the deformation parameter β as

$$\sqrt{q} = e^{i\frac{\beta}{2}} = 1 + i\frac{\beta}{2} - \frac{\beta^2}{8} - i\frac{\beta^3}{48} + \ldots \tag{3–21}$$

$$\frac{1}{\sqrt{q}} = e^{-i\frac{\beta}{2}} = 1 - i\frac{\beta}{2} - \frac{\beta^2}{8} + i\frac{\beta^3}{48} + \ldots \tag{3–22}$$

so that F_β can be written as

$$F_\beta = \mathbb{1}_{3\times3} \otimes \mathbb{1}_{3\times3} + \beta F^{(1)} + \beta^2 F^{(2)} + \ldots \tag{3–23}$$

where $F^{(1)}$ is $\frac{i}{2} \times \mathrm{diag}(0, 1, -1, 1, 1, 0, -1, -1, 1, 0)$ and also for the higher order terms. Note, however, that only the first term survives the action of the co-unit map and this implies that $(\epsilon \otimes id)F_\beta = \mathbb{1}_{3\times3}$. Thus F_β is a legitimate twist.

It is easy to see that R_0 satisfies the axioms in (3.2.1) and thus is actually a quasitriangular structure of the $SU(3)$ symmetry of $\mathcal{N} = 4$ SYM. Twisting by F_β produces a quasitriangular structure R_β and a twisted $SU(3)$ symmetry $\mathcal{SU}(3)_q$. This is what motivates our view which is that the β-deformation does not break the $SU(3)$ symmetry, it only deforms it. And, as we mentioned previously,the price: we must describe symmetry in terms of Hopf algebras rather than Lie algebras.

By a Taylor expansion of R_β in the β we obtain

$$R_\beta = \mathbb{1} + \beta R^{(1)} + \frac{\beta^2}{2} R^{(2)} + \ldots \tag{3–24}$$

from which we isolate $R^{(1)} = r_\beta^{(1)} = \mathrm{diag}\,(0, 1, -1, 1, 1, 0, -1, -1, 1, 0)$ — the β classical R-matrix— employed to define the antisymmetric 2-tensor Θ^{ij} which carries the information of non-commutativity of the coordinates. Note that $r_\beta^{(1)}$ is exactly the $\tilde{\Theta}^{i\,j}_{\;k\,l}$ which appears in (2–22).

3.4 W-DEFORMATION AND THE TWIST

In this work we considered a deformation which corresponds to parameter choice $(q, h) = (1 + w, w)$ where $w \in \mathbb{R}$. We call it a *w-deformation*[7]. Since the q and h are not independent in the w-deformation it follows that the w-deformed theory is a proper subset of Leigh-Strassler theories. This means it possesses $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry which all Leigh-Strassler theories possess. We will extensively employ the $\mathbb{Z}_3$ symmetry which rotates the fields as $\Phi^i \to \Phi^{i+1}$. It should be noted that the w-deformed superpotential, can be obtained from the β-deformed one (3–16) by a field redefinition $\Phi^i \to (T^\dagger)^i_j \Phi^j$ where T is the $SU(3)$ matrix

$$T = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}} \\ 1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \end{pmatrix}. \tag{3–25}$$

Although the w-deformed theory is the same as the β-deformed it obscures the familiar symmetries of the β-deformed theory, making it a good candidate on which to test Hopf algebraic

7 and hope this name choice has not been used before.

approach to deformations. So the gravity background dual the w-deformed theory can be compared against the known *Lunnin-Maldacena* background [36] in order to build confidence in this approach for future work. The w-deformed theory has an R-matrix equal to

$$R_w = \frac{1+w}{1+w+w^2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{w}{w+1} & 0 & w & 0 \\ 0 & 1 & 0 & -\frac{w}{w+1} & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 1 & 0 & -\frac{w}{w+1} & 0 & w & 0 & 0 \\ 0 & w & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{w}{w+1} \\ 0 & 0 & w & 0 & 1 & 0 & -\frac{w}{w+1} & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{w}{w+1} & 0 \\ 0 & 0 & -\frac{w}{w+1} & 0 & w & 0 & 1 & 0 & 0 \\ -\frac{w}{w+1} & 0 & 0 & 0 & 0 & w & 0 & 1 & 0 \\ 0 & -\frac{w}{w+1} & 0 & w & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3–26}$$

and it can be viewed as a result of twisting R_0. We first dismantle R_w in terms of the shift matrix, its square and the identity matrix

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; V = U^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } U^3 = \mathbb{1}_{3\times 3}. \tag{3–27}$$

Thus R_w becomes

$$R_w = \frac{1+w}{1+w+w^2}\Big[\mathbb{1}\otimes\mathbb{1} + w\, U\otimes V - \frac{w}{1+w}\, V\otimes U\Big]. \tag{3–28}$$

From this angle, the twist corresponding to the w-deformation is

$$F_w = \tilde{C}\begin{pmatrix} 1+w & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 \\ 0 & 1+w & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+w & 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+w & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 1+w & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+w & 0 & w & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 1+w & 0 & 0 \\ w & 0 & 0 & 0 & 0 & 0 & 0 & 1+w & 0 \\ 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 1+w \end{pmatrix} \tag{3–29}$$

The coefficient $\tilde{C}$ serves as a normalization constant. Note that we have freedom to choose its form. This is evident from the definition of the R_w-matrix ($R_w = F_{w_{21}} R_0 F_{w_{12}}^{-1} = F_{w_{21}} F_{w_{12}}^{-1}$) that the effect of $\tilde{C}$ is cancelled out since the twist and its inverse are

$$F_w = \tilde{C}\Big[(1+w)\mathbb{1}\otimes\mathbb{1} + w\, V\otimes U\Big] \tag{3–30}$$

$$F_w^{-1} = \frac{(1+w)^2}{\tilde{C}(1+2w)(1+w+w^2)}\Big[\mathbb{1}\otimes\mathbb{1} - \frac{w}{(1+w)}\, V\otimes U + \frac{w^2}{(1+w)^2} U\otimes V\Big] \tag{3–31}$$

In order to choose the form of $\tilde{C}$ we first note that F_w satisfies the YBE, as can be confirmed by explicit calculation. From this we know therefore that F_w is a 2-cocycle and thus the deformed R_w-matrix is quasitriangular, as is the undeformed R_0-matrix. The co-unitality of F_w is however not guaranteed, so we will choose the normalization constant $\tilde{C}$ so as to guarantee it. The shift matrices possess properties

$$(V\otimes U)^2 = (U\otimes V) \text{ and } (V\otimes U)^3 = \mathbb{1}\otimes\mathbb{1} \tag{3–32}$$

which allow for F_w to be written in terms of the exponential function

$$F_w = \exp[a(w) V \otimes U + b(w) U \otimes V] \tag{3–33}$$

where

$$a_w := a(w) = \frac{1}{6} \ln \left[\frac{(1+2w)^2}{1+w+w^2} \right] + \frac{1}{\sqrt{3}} \left[\tan^{-1} \left(\frac{1+2w}{\sqrt{3}} \right) - \frac{\pi}{6} \right] \tag{3–34}$$

and

$$b_w := b(w) = \frac{1}{6} \ln \left[\frac{(1+2w)^2}{1+w+w^2} \right] - \frac{1}{\sqrt{3}} \left[\tan^{-1} \left(\frac{1+2w}{\sqrt{3}} \right) - \frac{\pi}{6} \right]. \tag{3–35}$$

Using hypergoniometric cosine and sine functions $C(x)$ and $S(x)$ we fix the normalization constant to be [35] [34]:

$$\tilde{C} = \frac{C(a_w)}{1+w}. \tag{3–36}$$

The result of substituting (3–34) into the hypergoniometric sine and cosine functions is

$$C(a_w) = \frac{1+w}{[(1+2w)(1+w+w^2)]^{\frac{1}{3}}} \tag{3–37}$$

and

$$S(a_w) = \frac{w}{[(1+2w)(1+w+w^2)]^{\frac{1}{3}}}. \tag{3–38}$$

This means we can safely expand (3–33) to obtain

$$\begin{aligned} F_w &= C(a_w) [\mathbb{1} \otimes \mathbb{1}] + S(a_w) [V \otimes U] && (3–39) \\ &= C(a_w) \left[\mathbb{1} \otimes \mathbb{1} + \frac{S(a_w)}{C(a_w)} V \otimes U \right] && (3–40) \\ &= C(a_w) \left[\mathbb{1} \otimes \mathbb{1} + \frac{w}{1+w} V \otimes U \right] && (3–41) \end{aligned}$$

because the properties (3–32) mean that the higher-order terms in $U \otimes V$ will recollect and cancel out. Thus (3–30) and (3–33) are equivalent. The exponential form, however, best highlights the counital property. F_w is therefore a proper twist and the twisted R-matrix R_w is a quasitriangular structure of the twisted algebra.

The classical R-matrix, r_w, associated with the w-deformed theory is given

$$r_w = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3–42}$$

and from it we obtain holomorphic part of the non-commutativity 2-tensor Θ_w via the definition

$$\Theta_w^{ij} = (r_w)^{i\ j}_{k\ l} z^k z^l = \begin{pmatrix} 0 & z_3^2 - z_1 z_2 & z_1 z_3 - z_2^2 \\ z_1 z_2 - z_3^2 & 0 & z_1^2 - z_2 z_3 \\ z_2^2 - z_1 z_3 & z_2 z_3 - z_1^2 & 0 \end{pmatrix} \tag{3–43}$$

Using the map of antiholomorphic coordinates to co-vectors, $\bar{x}^{\bar{i}} = u_i$, in conjunction with the mixed plane relations in (2–23) we find that the full non-commutativity matrix is

$$\Theta_w^{IJ} = w \begin{pmatrix} 0 & (z^3)^2 - z^1z^2 & z^1z^3 - (z^2)^2 & z^2\bar{z}^2 - z^3\bar{z}^3 & z^2\bar{z}^3 - \bar{z}^1z^3 & \bar{z}^1z^2 - \bar{z}^2z^3 \\ z^1z^2 - (z^3)^2 & 0 & (z^1)^2 - z^2z^3 & \bar{z}^2z^3 - z^1\bar{z}^3 & z^3\bar{z}^3 - z^1\bar{z}^1 & \bar{z}^1z^3 - z^1\bar{z}^2 \\ (z^2)^2 - z^1z^3 & z^2z^3 - (z^1)^2 & 0 & z^1\bar{z}^2 - z^2\bar{z}^3 & z^1\bar{z}^3 - \bar{z}^1z^2 & z^1\bar{z}^1 - z^2\bar{z}^2 \\ z^3\bar{z}^3 - z^2\bar{z}^2 & z^1\bar{z}^3 - \bar{z}^2z^3 & z^2\bar{z}^3 - z^1\bar{z}^2 & 0 & \bar{z}^1\bar{z}^2 - (\bar{z}^3)^2 & (\bar{z}^2)^2 - \bar{z}^1\bar{z}^3 \\ \bar{z}^1z^3 - z^2\bar{z}^3 & z^1\bar{z}^1 - z^3\bar{z}^3 & \bar{z}^1z^2 - z^1\bar{z}^3 & (\bar{z}^3)^2 - \bar{z}^1\,\bar{z}^2 & 0 & \bar{z}^2\bar{z}^3 - (\bar{z}^1)^2 \\ \bar{z}^2z^3 - \bar{z}^1z^2 & z^1\bar{z}^2 - \bar{z}^1z^3 & z^2\bar{z}^2 - z^1\bar{z}^1 & \bar{z}^1\bar{z}^3 - (\bar{z}^2)^2 & (\bar{z}^1)^2 - \bar{z}^2\bar{z}^3 & 0 \end{pmatrix} \tag{3–44}$$

The radial independence of this non-commutativity matrix Θ_w is manifest in the spherical coordinate system[8] and this ascertains conformal invariance of the theory [32]. This matrix will play a central role in encoding the deformation of the spacetime coordinates on the gravity side.

3.5 THE STAR PRODUCT

Next we discuss the twist in module/representation of the HA. In accordance with section (3.2.0.2) a compatibly twisted module product is $\mu_F(x^i \otimes x^j)$ and this clearly means x^i and x^j are no longer commutative. We transfer this non-commutativity of module the elements to a star product so that x^i and x^j once again commute. The star product is defined as

$$x^i \star x^j = \mu_{F_w}(x^i \otimes x^j) = (F_w)^{j\ i}_{\ k\ l} x^k x^l \tag{3–45}$$

with the last equality reminiscent of the quantum plane relations discussed in (2.3). In fact this star product obeys the RTT relations (2–21) because

$$x^i \star x^j = F^{j\ i}_{\ k\ l} x^k\ x^l = [\ (F)^{j\ i}_{\ k\ l}\ ][\ (F^{-1})^{l\ k}_{\ m\ n}\ ]\ x^n \star x^m = R^{j\ i}_{\ n\ m}\ x^n \star x^m \tag{3–46}$$

The use of F_w rather than its inverse F_w^{-1} may seem unorthodox but an attractive feature sought after in a star product —especially with solution-generating techniques in mind—is one that administers the deformation simply by promoting the products between fields of the undeformed theory in order to obtain the deformed fields (up to a factor). The availability of this luxury rests in that F_w satisfies the YBE, thus the star product it defines is associative. It turns out that promoting the product in the superpotential of $\mathcal{N} = 4$ SYM to a star product defined via F_w gives the correct w-deformed superpotential, hence the choice.

The module of interest is parameterized by the coordinates z^i thus it suffices to consider the star product at the level of coordinates rather than that of [general] functions. For the β and w deformations associativity of star product is ingrained since their respective twists satisfy the YBE hence

$$(z^i \star z^j) \star z^k = F_{12}F_{13}F_{23}z^iz^jz^k = F_{23}F_{13}F_{12}z^iz^jz^k = z^i \star (z^j \star z^k) \tag{3–47}$$

And in the matrix representation (where the map μ is matrix multiplication) the star product corresponding to the w-deformation manifests as

$$z^i \star z^j = \mu(F_w \triangleright z^i \otimes z^j) = \tilde{C}\Big[(w+1)\,\Big(\mathbb{1}\ z^i \otimes \mathbb{1}\ z^j\Big) + w\,\Big(V\ z^i \otimes U\ z^j\Big)\Big] \tag{3–48}$$

8 The form of this said matrix is bulky hence it does not appear here.

where $\tilde{C} = [(2w+1)(w^2+w+1)]^{\frac{1}{3}}$ is the normalization constant. In the basis where

$$z^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;, \; z^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } z^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3–49}$$

the star product on the coordinates is

$$z^1 \star z^2 = \tilde{C}\Big[(1+2w)\; z^1 z^2\Big] \tag{3–50}$$

$$z^2 \star z^1 = \tilde{C}\Big[(1+w)\; z^2 z^1 + w\; z^3 z^3\Big] \tag{3–51}$$

$$z^3 \star z^3 = \tilde{C}\Big[(1+w)\; z^3 z^3 + w\; z^2 z^1\Big]. \tag{3–52}$$

We conclude then that

$$[z^1, z^2]_\star = z^1 \star z^2 - z^2 \star z^1 = 2w\; z^2 \star z^1 - \left(\frac{1+2w}{1+w}\right) z^3 \star z^3 \tag{3–53}$$

Having an associative star product defined via the twist F_w and the 2-tensor Θ_w^{IJ} to capturing commutation relations of coordinates we are ready to consider the w-deformation on the gravity side.

Part II

THE GRAVITY SIDE

4

GENERALIZED GEOMETRY

Thematic of the AdS-CFT correspondence conjecture, the first part of this paper was devoted to field theory [CFT], this point marks the second part, the gravity [AdS]. In treating the symmetries of the respective field theories Hopf Algebras were the language of choice; in understanding the corresponding gravity side: *Generalized Complex Geometry* [GCG]. This framework is appealing in that it places spacetime metric and B-field on equal footing. This matches string theory view that the metric and B-field are different excitation modes of the same string.

4.1 AN OVERVIEW

In order to properly place GCG it is meaningful to briefly consider the two [manifold] definitions: symplectic and complex manifolds.

A *symplectic* manifold is a smooth manifold M of even dimension containing a non-degenerate 2-form ω which is both smooth and closed. ω is a linear map $\omega : T(M) \to T^*(M)$, called a *symplectic form* [1] and it satisfies $\omega^* = -\omega$. Here T and T^* are the tangent and co-tangent bundles respectively.

Similarly if an even-dimensional smooth manifold N is equipped with $J : T(N) \to T(N)$ such that $J^2 = -1$ then the pair (N, J) is an *almost complex* manifold and J is called an *almost complex structure*. If J is compatible across coordinate systems then we drop the "almost" and the pair is said to be a *complex* manifold [42]. This is where GCG enters the picture as it first replaces tangent and cotangent bundles with a sum of the two

$$T(M)\ ,\ T^*(M) \to T(M) \oplus T^*(M) \tag{4–1}$$

whose sections are

$$X + \xi \ \in T \oplus T^* \text{ with } X \in T \text{ and } \xi \ \in \ T^*. \tag{4–2}$$

Secondly it generalizes the two above-mentioned structures by replacing them with a Generalized Complex Structure [GCS] in which symplectic and complex structures are contained as special cases [17] [22] [23]. Thus a GCS on an n-dimensional manifold M is an endomorphism $\mathcal{J} : T(M) \oplus T^*(M) \to T(M) \oplus T^*(M)$ such that $\mathcal{J}^2 = -\mathbb{1}_{2n}$ and $\mathcal{J}^* = -\mathcal{J}$. A GCS can be constructed from a complex structure J in the following way

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \tag{4–3}$$

1 or structure.

and also by means of a symplectic structure ω

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{4–4}$$

The GCS $\mathcal{J}$ is also required to satisfy the hermiticity condition $\mathcal{J}^t \mathcal{G} \mathcal{J} = \mathcal{G}$ where

$$\mathcal{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4–5}$$

is a natural metric on $T \oplus T^*$. In flat $\mathbb{C}_3$ with coordinates $z^i, \bar{z}^j$ and $i, j = 1, 2, 3$, the holomorphic volume form Ω and the Kähler form J will be needed to define a metric $g_{i\bar{j}} = -iJ_{i\bar{j}}$. These forms are given by

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 \text{ and } J = \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^{\bar{j}} \tag{4–6}$$

and since $\mathbb{C}_3$ is a six-dimensional manifold we know that it is spanned by a basis of polyforms with maximum degree of 6. So then there is a unique form of dimension six, the volume form. Thus the six dimensional forms $J \wedge J \wedge J$ and $\Omega \wedge \Omega$ have to be individually proportional to the volume form, hence to one another explicitly in the fashion:

$$J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}. \tag{4–7}$$

This manifold can be described from the generalized geometric view where the tangent bundle T is replaced with a 12 dimensional generalized tangent bundle $T \oplus T^*$, the space on which we will define the corresponding GCS. We first note that the natural metric $\mathcal{G}$ which defines the hermiticity condition for the GCS reduces natural $SO(6,6)$ structure on $T \oplus T^*$ to $O(6,6)$ thus spinors will transform under Cliff$(6,6)$. The Clifford map

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \longleftrightarrow \quad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k}_{\alpha\beta} \tag{4–8}$$

allows a one to one relation between spinors to forms so that the chirality of spinors is manifest as the degree [even or odd] of the forms. Thus to an element of $T \oplus T^*$ a gamma matrix Γ_I will be assigned

$$\Gamma_I = \iota_I \text{ and } \Gamma_{I+6} = dz^I \text{ with } I = 1, 2, \ldots, 6 \tag{4–9}$$

under the set ordering $\{\iota_I, dz^I\} = \{\iota_1, \bar{\iota}_1, \iota_2, \ldots, dz^3, d\bar{z}^3\}$.

4.2 PURE SPINORS

The missing ingredients to make a full transition to GCG are *pure spinors*. These are the special forms used to define the conditions for a supersymmetric in GCG. More formally

Definition 4.2.1. A pure spinor Φ is a (poly-)differential form whose annihilator, $\mathbb{L}_\Phi$,in $T \oplus T^* \otimes \mathbb{C}$ has complex dimension 6 and whose inner product is nowhere vanishing on the manifold, that is $\langle \bar{\Phi}, \Phi \rangle \neq 0$

The annihilator of spinor A in $T \oplus T^* \otimes \mathbb{C}$ is the set defined as [16]

$$\mathbb{L}_A = \{(X + \xi) \in T \oplus T^* | (X + \xi) \cdot A = 0\}, \tag{4–10}$$

here the action of a polyform $(X + \xi)$ on a spinor Φ is understood as

$$(X + \xi) \cdot \Phi = X^m \iota_{\partial_m} \Phi + \xi_m dx^m \wedge \Phi. \tag{4–11}$$

On the other hand the inner product on polyforms, $\langle\ ,\ \rangle$, in component form is given by

$$\langle A, B\rangle = \sum(-1)^{[\frac{n}{2}]} A_n \wedge B_{6-n} \tag{4–12}$$

where A_n and B_{6-n} are components of the forms A and B corresponding to the degree[2]. The existence of two closed pure spinors that also are compatible guarantees an $\mathcal{N} = 2$ supersymmetric background. A pair of pure spinors $\Phi_\pm$is compatible if it satisfies the following [14]:

- $\langle\Phi_-, X\Phi_+\rangle = \langle\bar{\Phi}_-, X\Phi_+\rangle = 0\ \forall\ X\ \in T \oplus T^*$ — Mukai pairing
- $\langle\bar{\Phi}_+, X\Phi_+\rangle = \langle\bar{\Phi}_-, X\Phi_-\rangle$ — Equal norms
- If they satisfy the first two requirements then they define a metric; this metric is required to be positive definite.

For flat 6d space the spinors are

$$\Phi^0_- = \Omega = dz^1 \wedge dz^2 \wedge dz^3, \tag{4–13}$$

$$\Phi^0_+ = e^{-iJ} = 1 + \frac{1}{2}\sum_i dz^i \wedge d\bar{z}^i + \frac{1}{4}\sum_i dz^i \wedge d\bar{z}^i \wedge dz^{i+1} \wedge d\bar{z}^{i+1} + \frac{1}{8} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge dz^3 \wedge d\bar{z}^3 \tag{4–14}$$

and the elements of their annihilator are respectively given by

$$L^-_i = dz^i \wedge\ , \qquad L^-_{\bar{i}} = \iota_{\bar{i}} \tag{4–15}$$

$$L^+_i = dz^i \wedge + 2\iota_{\bar{i}}\ , \qquad L^+_{\bar{i}} = d\bar{z}^{\bar{i}} \wedge + 2\iota_i \tag{4–16}$$

The pure spinors Φ^0_- and Φ^0_+ are compatible and closed and thus define an $\mathcal{N} = 2$ background, hence flat space is a solution to SUGRA.

We now can consider the bosonic fields. By combining sections of $T \oplus T^*$ we can construct two-indexed anti-symmetric gamma matrices

$$\Gamma_{IJ} = \frac{1}{2}(\iota_I\iota_J - \iota_J\iota_I) = \iota_I \wedge \iota_J\ , \qquad \Gamma_{I,J+6} = \frac{1}{2}(\iota_I dz^J - dz^J\iota_I) \tag{4–17}$$

$$\Gamma_{I+6\ J+6} = \frac{1}{2}(dz^I dz^J - dz^J dz^I) = dz^I \wedge dz^J\ , \qquad \Gamma_{I+6,J} = \frac{1}{2}(dz^I\iota_J - \iota_J dz^I) \tag{4–18}$$

which are useful in obtaining the GCS through the Mukai pairing

$$\mathcal{J}_{\pm MN} = \langle\bar{\Phi}_\pm, \Gamma_{MN}\Phi_\pm\rangle. \tag{4–19}$$

These GCS define a 12-dimensional generalized metric via

$$\mathcal{G}_{MN} = -\mathcal{J}_{+ML}\mathcal{J}_{-}{}^{L}{}_{N} \tag{4–20}$$

We know that contained in the generalized metric are the 6-dimensional [complex] spacetime metric g and B-field B [16] since

$$\mathcal{G}^M{}_N = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B^{-1} & Bg^{-1} \end{pmatrix} \tag{4–21}$$

2 [.] selects the integer part.

From g we obtain the dilaton $e^{2\phi} = \sqrt{|det(g)|}$; this is the complete NS-NS sector. The generalized metric defined from the flat pure spinors is

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \quad \text{with } g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4–22}$$

This is flat space in complex coordinates, with no B-field and no dilaton.

4.3 THE TWIST

On the field theory side we used a 2-tensor non-commutativity parameter Θ as a container for the deformation via a twist and defined a star product which transmits the deformation to fields of the undeformed theory by simply replacing regular product. Here we propose that Θ will play a similar role as on field theory side and also the star product will manifest on the wedge product as a deformed wedge product $\wedge_\star$[3]. Its action on polyforms is

$$dz^I \wedge_\star dz^J = \left(1 - \frac{i}{2}\Theta^{KL}\iota_K \wedge \iota_L\right) dz^I \wedge dz^J = dz^I \wedge dz^J - i\Theta^{IJ} \tag{4–23}$$

That this star wedge product does not anticommute is evident but the non-anticommutativity is governed by Θ^{IJ}. This approach allows a transparent transition of the twist from the field theory side on to the geometry on the gravity side. The deformations at hand are of the bivector type hence they act to deform in the fashion $\Phi_\pm = e^{\beta^{IJ}\iota_I\wedge\iota_J}\Phi^0_\pm$. The star wedge product on mixed forms is understood to mean

$$dz^I \wedge_\star d\bar{z}^{\bar{J}} = dz^I \wedge d\bar{z}^{\bar{J}} + \Theta^{I\bar{J}} \tag{4–24}$$

4.3.1 *β-deformed pure spinors*

We now promote the anticommuting wedge product in the flat space pure spinors to the star wedge product in order to deform them. The simplified result is

$$\Phi^\beta_- = \Phi^0_- + \beta\, \mathrm{d}(z^1z^2z^3) \tag{4–25}$$

$$\Phi^\beta_+ = \Phi^0_+ - \frac{\beta}{4}\left[\bar{z}^1\bar{z}^2\, dz^1 \wedge dz^2 + \bar{z}^1z^2\, dz^1 \wedge d\bar{z}^2 + z^1\bar{z}^2\, d\bar{z}^1 \wedge dz^2 + z^1z^2\, d\bar{z}^1 \wedge \bar{z}^2 \; + cyclic\right] \tag{4–26}$$

and by following the GCG prescription of extracting the metric, B-field and dilaton from the generalized metric one obtains an $\mathcal{N} = 2$ background. It is from this background that we recover the $\mathcal{N} = 1$ real-β Lunin-Maldacena background [36] by introducing a stack of D3-branes. Hence such an $\mathcal{N} = 2$ background is said to be a precursor of the Lunin-Maldacena background. A very simplistic viewpoint, the fact that the deformed spinors define an $\mathcal{N} = 2$ background means that the deformation (star wedge product) commutes with the exterior derivative.

3 or "star wedge product".

4.3.2 *w-deformed pure spinors*

We again begin with the flat space pure spinors and promote the wedge product to the non-anticommuting star wedge product to obtain

$$\Phi_-^w = \Phi_-^0 - iw[(z^2z^3 - (z^1)^2)dz^1 \wedge + (z^3z^1 - (z^2)^2)dz^2 + (z^1z^2 - (z^3)^2)dz^3] \tag{4–27}$$

$$\begin{aligned}\Phi_+^w = \Phi_+^0 &+ \frac{iw}{4}\big[(z^3\bar{z}^3 - z^2\bar{z}^2)dz^1 \wedge d\bar{z}^1 + ((\bar{z}^3)^2 - \bar{z}^1\bar{z}^2)dz^1 \wedge dz^2 \\ &+ (z^1\bar{z}^3 - \bar{z}^2z^3)dz^1 \wedge d\bar{z}^2 + (z^2\bar{z}^3 - z^1\bar{z}^2)dz^1 \wedge d\bar{z}^3 + (z^1z^2 - (z^3)^2)d\bar{z}^1 \wedge d\bar{z}^2 + \text{cyclic}\big] \\ &+ \frac{iw}{8}\big[(z^1\bar{z}^1 - z^2\bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 - (\bar{z}^2\bar{z}^3 - (\bar{z}^1)^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge dz^3 \\ &+ (\bar{z}^1z^3 - z^1\bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 - (z^1\bar{z}^3 - \bar{z}^1z^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 \\ &+ (z^2z^3 - (z^1)^2)dz^1 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + \text{cyclic}\big]\end{aligned} \tag{4–28}$$

These forms are still pure spinors because their inner product is no-zero and their annihilator sets $\mathbb{L}_\pm$ are of dimension 6. The elements of these sets are

$$L^-_{(1)} = dz^1 - iw\big[(z^1z^2 - (z^3)^2)\iota_2 - (z^1z^3 - (z^2)^2)\iota_3\big] \tag{4–29}$$

$$L^-_{(\bar{1})} = \iota_{\bar{1}} \tag{4–30}$$

$$L^+_{(\bar{1})} = d\bar{z}^1 + \Big[\big(-2 + iw(z^3\bar{z}^3 - z^2\bar{z}^2)\big)\iota_1 + iw(z^2\bar{z}^3 - z^1\bar{z}^2)\iota_3 \tag{4–31}$$

$$- iw(z^3\bar{z}^2 - z^1\bar{z}^3)\iota_2 - iw(\bar{z}^1\bar{z}^3 - (\bar{z}^2)^2)\iota_{\bar{3}} + iw(\bar{z}^1\bar{z}^2 - (\bar{z}^3)^2)\iota_{\bar{2}}\Big] \tag{4–32}$$

$$L^+_{(1)} = dz^1 + \Big[\big(2 - iw(z^3\bar{z}^3 - z^2\bar{z}^2)\big)\,\iota_{\bar{1}} - iw(z^3\bar{z}^2 - z^2\bar{z}^1)\iota_{\bar{3}} \tag{4–33}$$

$$+ iw(z^2\bar{z}^3 - z^3\bar{z}^1)\iota_{\bar{2}} + iw(z^1z^3 - (z^2)^2)\iota_3 - iw(z^1z^2 - (z^3)^2)\iota_2\Big] \tag{4–34}$$

together with their cyclic permutations. They have a vanishing exterior derivative and both their Mukai norms are equal to 1. So they guarantee an $\mathcal{N} = 2$ background. Since the star wedge product $\wedge_*$ was defined from the w-deformed field theory, we propose that this $\mathcal{N} = 2$ background is the NS-NS precursor of the dual w-deform theory.

5

W-DEFORMED BACKGROUND

5.1 W-DEFORMED GCS

From the two w-deformed pure spinors we define the two GCS which produce the 12-dimensional generalized metric. The GCS corresponding to Φ_-^w is defined as

$$\mathcal{J}_{-\ N}^{M} = \begin{pmatrix} J_-^{(ul)} & J_-^{(ur)} \\ 0 & -J_-^{(ul)} \end{pmatrix} \tag{5–1}$$

and the one corresponding to Φ_+ as

$$\mathcal{J}_{+\ L}^{K} = \begin{pmatrix} J_+^{(ul)} & J_+^{(ur)} \\ J_+^{(ur)} & (J_+^{(ul)})^T \end{pmatrix} \tag{5–2}$$

The J blocks are presented explicitly in Appendix [B.1] where we have used (ul) = upper-left and (ur) = upper-right to denote the respective blocks. The generalized metric is defined in 4–21 and from it we can extract the spacetime metric g_w and B-field B_w of the NS-NS sector precursor solution. The 10-dimensional metric is obtained by concatenating the 4-dimensional Minkowski metric to g_w, a metric of w-deformed $\mathbb{C}_3$. The 6-dimensional part of the 10d-spacetime metric, in string frame and complex coordinates, takes the form

$$ds^2 = G\Big[g_{ii}\, dz^i dz^i + g_{i\bar{i}}\, dz^i d\bar{z}^{\bar{i}} + g_{i\ i+1}\, dz^i dz^{i+1} + g_{i\ \overline{i+1}}\, dz^i d\overline{z}^{\overline{i+1}} + c.c.\Big] \tag{5–3}$$

where the G factor is given by

$$\begin{aligned} G^{-1} = 1+w^2\Big[&z_1^2\bar{z}_1^2 + z_2^2\bar{z}_2^2 + z_2^3\bar{z}_3^2 + z_1\bar{z}_1z_2\bar{z}_2 + z_2\bar{z}_2z_3\bar{z}_3 + z_1\bar{z}_1z_3\bar{z}_3 - z_1z_2\bar{z}_3^2 \\ &- z_2z_3\bar{z}_1^2 - z_3z_1\bar{z}_2^2 - z_1^2\bar{z}_2\bar{z}_3 - z_2^2\bar{z}_3\bar{z}_1 - z_3^2\bar{z}_1\bar{z}_2\Big] \end{aligned} \tag{5–4}$$

The components of the metric are stowed away in Appendix [B.2] as are those of the B-field, which B-field has the form

$$B = \frac{1}{2}\Big[B_{ii}\, dz^i \wedge d\bar{z}^{\bar{i}} + B_{i\ i+1}\, dz^i \wedge d\bar{z}^{i+1} + B_{i\ i-1}\, dz^i \wedge d\bar{z}^{i-1} + \mathrm{c.c}\Big] \tag{5–5}$$

and the dilaton is obtained $e^{\phi} = \sqrt{det(g)} = G$[1]. The full 10d-metric in real coordinates in the Einstein frame is

$$ds^2 = \frac{1}{\sqrt[4]{G}}\Big[-dt^2 + \sum_{i=1}^{3} dx_i^2 + G\big[g_{ij}^{rr} dr_i dr_j + g_{ij}^{r\phi} dr_i d\phi_j + g_{ij}^{\phi\phi} d\phi_i d\phi_j\big]\Big] \tag{5–6}$$

1 where G is given in (5–4).

The metric and B-field components as expressed in terms of real coordinates are contained in Appendix [B.2].

In probing the R-R sector we follow [39] [1] to take the near-horizon limit so that the background $AdS_5 \times S^5_w$ arises where only the AdS_5 part of the geometry has radial dependence. That is to say the deformed S^5_w is a sphere of constant radius. We thus denote the radius of AdS_5 by R. This means we truly are dealing with a 5-sphere. We use the coordinates $(\alpha, \theta, \phi_1, \phi_2, \phi_3)$ to parametrize the 5-sphere and thus obtain that the factor G is given by

$$G^{-1} = 1 + w^2 R^4 \Big[1 - s_\alpha^2 c_\alpha^2 - s_\alpha^4 s_\theta^2 c_\theta^2 - 2 c_\alpha s_\alpha^3 c_\theta s_\theta^2 C_2 - 2 s_\alpha^2 c_\alpha^2 s_\theta c_\theta C_1 - 2 c_\alpha s_\alpha^3 c_\theta^2 s_\theta C_3 \Big] \tag{5–7}$$

and the metric can be written as

$$g_{\mu\nu} = G \tilde{g}_{\mu\nu} \tag{5–8}$$

where the components $\tilde{g}_{\mu\nu}$ are recorded in Appendix [B.3] together with all the R-R fields which were computationally affordable to the resources available. Of course from the stand point of GCG this is sufficient but as a confirmation the metric was tested and found to satisfy the IIB SUGRA e.o.m in Appendix[A.3].

5.2 CONCLUSION AND FUTURE WORK

We have studied the role of the Hopf twist on the field theory side and its manifestation on the gravity side in the Generalized Geometry framework and have found that this approach it fruitful at least when the twist satisfies the YBE. No energy was expended in studying the geometry which was produced from the w-deformation. The particular twist used herein possesses features that by-pass the would-be pitfalls. The fact that F_w satisfies the quasitriangularity axioms implied the YBE (at the level of the twist), which in turn guaranteed associativity of the star product defined. One can consider twists that give rise to R-matrix which are quasitriangular structures with the twists not being quasitriangular structures themselves. These cases require that we either know the action of the co-product, Δ, at the group level or that we express the twist in terms of the generators of the algebra. The latter approach has issues with uniqueness. Is there a unique way to express the twist in terms of the generators of the algebra? If not, how are they many ways related to one another? These questions are at the forefront of future work.

We also narrowed our scope to real deformation parameter w. These cases were used as a ”controlled environment” on which to test the Hopf Twist-Generalized Geometry approach since they are well-understood. Ultimately we would like to study the full (q, h)-deformed theory and thus the full twist [45]. This would mean considering mathematical structures that generalize Hopf algebras and quasi-Hopf algebras are suspected to make an appearance in course of this endeavour. A useful starting point to test this approach is to study twists whose deformation parameter is allowed to take on complex values.

A

APPENDIX A

A.1 GRASSMANN COORDINATES AND INTEGRATION

Grassmann coordinates are fermionic in nature and thus anticommute. The highest power of a coordinate and its conjugate to not vanish is

$$\theta^2 = \theta^\alpha \theta_\alpha \text{ and } \bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \tag{A–1}$$

The combination ($\theta^2\ \bar{\theta}^2$) is legal and useful for notational brevity. It is to be understood by means of (A–1). The rule for Grassmann integration in the case where α takes only one value, a single coordinate θ^1, is

$$\int d\theta^1\ \theta^1 = 1 \text{ and } \int d\theta^1\ 1 = 0 \tag{A–2}$$

Grassmann integration on the fermionic superspace coordinate has a similar effect that differentiation has on regular spacetime coordinates. If $\alpha = 1,2$ then there are two coordinates, θ^1 and θ^2, and we define the measure of superspace to be

$$d\theta^2 = -\frac{1}{4}\epsilon_{\alpha\beta} d\theta^\alpha d\theta^\beta \tag{A–3}$$

here $\epsilon_{\alpha\beta}$ is

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A–4}$$

Then $d\theta^1 d\theta^2 = -d\theta^2 d\theta^1$ from which we deduce that

$$\int d\theta^2 \theta^2 = 1 \tag{A–5}$$

The conclusions arrived at above are true also for the conjugate coordinate(s) and this allows us to be concise in our notation since we have the luxury of defining

$$\int d^4\theta \equiv \int d^2\theta \int d^2\bar{\theta}^2 \tag{A–6}$$

A.2 PROOFS

Noether's theorem:
Theorem. *Every continuous symmetry of the Lagrangian, $\mathcal{L}$, gives rise to a conserved current $j^\mu(x)$, that is*

$$\partial_\mu j^\mu(x) = 0 \tag{A–7}$$

Proof. Let $\mathcal{L}(\phi^i)$ be a Lagrangian which depends on a set of fields $\{\phi^i\}$ and also let $\delta\phi^i$ be a combined transformation of the fields such that $\mathcal{L}$ is invariant. By varying the Lagrangian, we obtain:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^i}\delta\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta(\partial_\mu\phi^i) \tag{A–8}$$

Using the Leibniz product rule of derivatives:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta\phi^i\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\right)\delta\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\partial_\mu(\delta\phi^i) \tag{A–9}$$

and invoking the *action* principle we reduce (A–8) to

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\right)\right]\delta\phi^i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta\phi^i\right] = 0 \tag{A–10}$$

From Lagrangian mechanics, the first term of (A–10) is the Euler-Lagrange equation and is itself zero. Thus we have that second term is also zero. Thus

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta\phi^i\right] = 0 \tag{A–11}$$

This is exactly (A–7)

$$\partial_\mu j^\mu = 0 \text{ with } j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta\phi^i \tag{A–12}$$

Hence the invariance of $\mathcal{L}$ with respect to $\delta\phi^i$, the combined transformation of the fields ϕ^i has led to a conserved current j^μ. To this conserved current we associate a conserved charge by noting that

$$\partial_\mu j^\mu = \frac{\partial j^0}{\partial t} + \nabla\cdot\vec{j} = 0 \tag{A–13}$$

and then we can define the conserved charge Q by

$$Q = \int_{\mathbb{R}^3} d^3x\frac{\partial j^0}{\partial t} = -\int_{\mathbb{R}^3} d^3x\nabla\cdot\vec{j} = 0$$

□

Preservation of Quasitriangularity in Twisting:
Theorem. *If the pair (H, R) is a quasitriangular Hopf Algebra and χ is a twist - 2-cocyclic and counital - then (H_χ, R_χ) , the twisted HA, is also quasitriangular having a co-product $\Delta_\chi(a) = \chi\Delta(a)\chi^{-1}$. And $R_\chi = \chi R\chi^{-1}$*

Proof.

$$\begin{aligned}
(\Delta_\chi\otimes id)R_\chi &= R_{\chi_{13}}R_{\chi_{23}}\\
&= \chi(\Delta\otimes id)\chi^{-1}R_\chi\\
&= \chi(\Delta\otimes id)\chi^{-1}\chi R\chi^{-1}\\
&= \chi(\Delta\otimes id)R\chi^{-1}\\
&= \chi(R_{13}R_{23})\chi^{-1}\\
&= (\chi R_{13}\chi^{-1})(\chi R_{23}\chi^{-1})\\
&= R_{\chi_{13}}R_{\chi_{23}}
\end{aligned}$$

□

A.3 IIB SUGRA EQUATIONS

The NS-NS sector field equations:

$$R_{MN} = \frac{1}{2}\partial_M \Phi \partial_N \Phi + \frac{1}{4}e^{-\Phi} H_{MRS} H_N^{RS} - \frac{1}{48} e^{-\Phi} H_{RST} H^{RST} \tag{A–14}$$
$$H = dB$$

$$\frac{1}{2}\nabla^M \partial_M \Phi = -\frac{1}{24\sqrt{G}} H_{MNR} H^{MNR} \tag{A–15}$$

and

$$D^P(e^{-\frac{\Phi}{2}} H_{MNP}) = \frac{1}{2}(D^P\Phi) e^{-\frac{\Phi}{2}} H_{MNP} \tag{A–16}$$

The R-R sector field equations [In string frame]:

- The 5-form field strength:

$$F_{(5)} = \omega_{AdS_5} + \omega_{S^5_w} \text{ with } \omega_{S^5_w} = G s_\alpha^3 c_\alpha s_\theta c_\theta \tag{A–17}$$

- Three-form field e.o.m:

$$F_{MNP} = -\frac{R}{24} D_M \sqrt{g} e^{-2\Phi} \epsilon_{NPQRS} H^{QRS} \tag{A–18}$$

and

$$H_{MNP} = \frac{R}{24} D_M \sqrt{g} \epsilon_{NPQRS} H^{QRS} \tag{A–19}$$

- The Einstein equations:

$$\begin{aligned} R_{MN} = {} & -2D_M \partial_N \Phi - \frac{1}{4} g_{MN} D^P \partial_P \Phi + \frac{1}{2} g_{MN} \partial_P \Phi \partial^P \Phi + \frac{1}{96} e^{2\Phi} F_{MPQRS} F_N^{PQRS} \\ & + \frac{1}{4}(H_{MPQ} H_N^{PQ} + e^{2\Phi} F_{MPQ} F_N^{PQ}) - \frac{1}{48} g_{MN} (H_{MNP} H^{MNP} + e^{2\Phi} F_{MNP} F^{MNP}) \end{aligned} \tag{A–20}$$

- The dilaton equation:

$$D^M \partial_M e^{-2\Phi} = -\frac{1}{6}(F_{MNP} F^{MNP} - e^{-2\Phi} H_{MNP} H^{MNP}) \tag{A–21}$$

B

APPENDIX B

B.1 GENERALIZED COMPLEX STRUCTURES

Here are the block matrices that constitute the GCS for the w-deformed pure spinors $\Phi^w_\pm$

$$J_-^{(ul)} = i\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B–1}$$

$$J_-^{(ur)} = w\begin{pmatrix} 0 & 0 & ((z_3)^2 - z_1z_2) & 0 & (z_1z_3 - (z_2)^2) & 0 \\ 0 & 0 & 0 & ((\bar{z}_3)^2 - \bar{z}_1\bar{z}_2) & 0 & (\bar{z}_1\bar{z}_3 - (\bar{z}_2)^2) \\ -((z_3)^2 - z_1z_2) & 0 & 0 & 0 & -(z_2z_3 - (z_1)^2) & 0 \\ 0 & -((\bar{z}_3)^2 - \bar{z}_1\bar{z}_2) & 0 & 0 & 0 & -(\bar{z}_2\bar{z}_3 - (\bar{z}_1)^2) \\ -(z_1z_3 - (z_2)^2) & 0 & (z_2z_3 - (z)^2) & 0 & 0 & 0 \\ 0 & -(\bar{z}_1\bar{z}_3 - (\bar{z}_2)^2) & 0 & (\bar{z}_2\bar{z}_3 - (\bar{z}_1)^2) & 0 & 0 \end{pmatrix} \tag{B–2}$$

$$J_+^{(ul)} = \frac{w}{2}\begin{pmatrix} (z_3\bar{z}_3 - z_2\bar{z}_2) & 0 & -(z_2\bar{z}_3 - \bar{z}_1z_3) & ((z_3)^2 - z_1z_2) & (\bar{z}_2z_3 - \bar{z}_1z_2) & (z_1z_3 - (z_2)^2) \\ 0 & (z_3\bar{z}_3 - z_2\bar{z}_2) & ((\bar{z}_3)^2 - \bar{z}_1\bar{z}_2) & (z_1\bar{z}_3 - \bar{z}_2z_3) & (\bar{z}_1\bar{z}_3 - (\bar{z}_2)^2) & (z_2\bar{z}_3 - z_1\bar{z}_2) \\ (z_1\bar{z}_3 - \bar{z}_2z_3) & -((z_3)^2 - z_1z_2) & -(z_3\bar{z}_3 - z_1\bar{z}_1) & 0 & -(\bar{z}_1z_3 - z_1\bar{z}_2) & -(z_2z_3 - (z_1)^2) \\ ((\bar{z}_3)^2 - \bar{z}_1\bar{z}_2) & (z_2\bar{z}_3 - \bar{z}_1z_3) & 0 & (z_3\bar{z}_3 - z_1\bar{z}_1) & (\bar{z}_1\bar{z}_3 - (\bar{z}_1)^2) & \\ z_2\bar{z}_3 - z_1\bar{z}_2 & -(z_1z_3 - (z_2)^2) & -(z_1\bar{z}3 - \bar{z}_1z_2) & (z_2z_3 - (z_1)^2) & (z_2\bar{z}_2 - z_1\bar{z}_1) & 0 \\ -(\bar{z}_1\bar{z}_3 - (\bar{z}_2)^2) & (\bar{z}_2z_3 - \bar{z}_1z_2) & (\bar{z}_2\bar{z}_3 - (\bar{z}_1)^2) & -(\bar{z}_1z_3 - z_1\bar{z}_2) & 0 & (z_2\bar{z}_2 - z_1\bar{z}_1) \end{pmatrix} \tag{B–3}$$

$$J_+^{(ur)} = i\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{B–4}$$

B.2 NS-NS FIELDS OF w-DEFORMED PRECURSOR SOLUTION

The 6d metric components in complex coordinates have the form

$$
\begin{aligned}
g_{\overline{ii+1}} &= w^2\Big[(z_i^2 - z_{i+1}z_{i-1})(\bar{z}_{i+1}^2 - \bar{z}_{i+2}\bar{z}_i)\Big] \\
g_{ii} &= w^2\Big[z_i(\bar{z}_{i+1}^3 + \bar{z}_{i-1}^3) + \bar{z}_i(z_{i+1}\bar{z}_{i-1}^2 + z_{i-1}\bar{z}_{i+1}^2) \\
&\quad - \bar{z}_{i+1}\bar{z}_{i-1}(z_{i-1}\bar{z}_{i-1} + z_{i+1}\bar{z}_{i+1}) - 2z_i\bar{z}_i\bar{z}_{i+1}\bar{z}_{i-1}\Big] \\
g_{i\ i} &= w^2\Big[2z_i^2\bar{z}_i^2 + z_{i-1}^2\bar{z}_{i-1}^2 + z_{i+1}^2\bar{z}_{i-1}^2 + 2z_{i+1}\bar{z}_{i+1}z_{i-1}\bar{z}_{i-1} \\
&\quad + z_i\bar{z}_i(z_{i-1}\bar{z}_{i-1} + z_{i+2}\bar{z}_{i+2}) \\
&\quad - \left[2\, z_i^2\bar{z}_2\bar{z}_3 + \bar{z}_i(z_{i+1}^2\bar{z}_{i-1} + z_{i-1}^2\bar{z}_{i+1}) + \text{c.c.}\right]\Big] \\
g_{i\bar{i}} &= 2 + w^2\Big[2z_i^2\bar{z}_i^2 + z_{i+1}^2\bar{z}_{i+1}^2 + z_{i-1}^2\bar{z}_{i-1}^2 + 2z_{i+1}\bar{z}_{i+1}z_{i-1}\bar{z}_{i-1} \\
&\quad + z_i\bar{z}_i(z_{i-1}\bar{z}_{i-1} + z_{i+1}\bar{z}_{i+1}) \\
&\quad - \left[2z_i^2\bar{z}_{i+1}\bar{z}_{i-1} + z_{i+1}^2\bar{z}_i\bar{z}_{i-1} + z_{i-1}^2\bar{z}_i\bar{z}_{i+1} + \text{c.c.}\right]\Big] \\
g_{\bar{i}\bar{j}} &= \overline{g_{ij}} \quad \text{and} \quad g_{\bar{j}i} = \overline{g_{i\bar{j}}}
\end{aligned}
$$

while the B-field components assume the form

$$
\begin{aligned}
B_{ii} &= -B_{\bar{i}i} = \frac{iw}{4}G(z_{i-1}\bar{z}_{i-1} - z_{i+1}\bar{z}_{i+1}) \\
B_{ii+1} &= -B_{i+1i} = \frac{iw}{4}G(\bar{z}_i\bar{z}_{i+1} - (\bar{z}_{i-1})^2) \\
B_{\overline{ii+1}} &= -B_{\overline{i+1}i} = \frac{iw}{4}G(z_i\bar{z}_{i-1} - z_{i-1}\bar{z}_{i+1})
\end{aligned}
$$

Upon converting to real coordinates using

$$z^j = r_j e^{i\phi_j},\ ,\bar{z}^j = r_j e^{-i\phi_j} \tag{B–5}$$

the 6d-metric components then become:

- rr-components

$$
\begin{aligned}
g_{11}^{rr} &= 1 + \frac{w^2}{2}\Big[(2r_1^4 + r_2^4 + r_3^4 + r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 - C_{31}r_1r_3^3 - C_{12}r_1r_2^3 \\
&\quad + C_1(r_2r_3^3 + r_2^3r_3 - 2r_1^2r_2r_3) - 3C_2r_1r_2^2r_3 - 3C_3r_1r_2r_3^2\Big] \qquad \text{(B–6)} \\
g_{12}^{rr} &= \frac{w^2}{2}\Big[r_1r_2^3 + r_2r_1^3 + C_{1,2}r_1^2r_2^2 + C_3r_3^4 - C_2(r_2^3r_3 + r_1^2r_2r_3) \\
&\quad - C_1(r_1^3r_3 + r_1r_2^2r_3)\Big] \qquad \text{(B–7)}
\end{aligned}
$$

- $r\phi$ components [note that $g^{\phi r} = g^{r\phi}$]

$$g^{r\phi}_{11} = -\frac{w^2}{2} r_1 \Big[S_1(r_2 r_3^3 + r_2^3 r_3 + 2r_1^2 r_2 r_3) + S_2 r_1 r_2^2 r_3 + S_3 r_1 r_2 r_3^2 + S_{3,1} r_1 r_3^3 - S_{1,2} r_1 r_2^3 \Big] \tag{B–8}$$

$$g^{r\phi}_{12} = -\frac{w^2}{2} r_2 \Big[S_1(r_1^3 r_3 - r_1 r_2^2 r_3) - S_3 r_3^4 - S_2(r_2^3 r_3 + r_1^2 r_2 r_3) - S_{1,2} r_1^2 r_2^2 \Big] \tag{B–9}$$

$$g^{r\phi}_{13} = -\frac{w^2}{2} r_3 \Big[S_1(r_1^3 r_2 - r_1 r_2 r_3^2) - S_2 r_2^4 - S_3(r_2 r_3^2 + r_1^2 r_2 r_3) + S_{3,1} r_1^2 r_3^2 \Big] \tag{B–10}$$

- $\phi\phi$-components:

$$g^{\phi\phi}_{11} = r_1^2 + \frac{w^2}{2} r_1^2 \Big[2r_1^4 + r_2^4 + r_3^4 + r_1^2 r_2^2 + r_1^2 r_3^2 + 2r_2^2 r_3^2 - C_1(r_2 r_3^3 + r_2^3 r_3 - 6r_1^2 r_2 r_3) - C_2 r_1 r_2^2 r_3 - C_3 r_1 r_2 r_3^2 + C_{1,2} r_1 r_2^3 + C_{3,1} r_1 r_3^3 \Big] \tag{B–11}$$

$$g^{\phi\phi}_{12} = \frac{w^2}{2} r_1 r_2 \Big[2r_1 r_2 r_3^2 - r_1 r_2^3 - r_1^3 r_2 - C_3 r_3^4 + C_2(r_1^2 r_2 r_3 - r_2^2 r_3) + C_1(r_1 r_2^2 r_3 - r_1^3 r_3) + C_{1,2} r_1^2 r_2^2 \Big] \tag{B–12}$$

The symmetry of the metric in its indices and $\mathbb{Z}_3$-cyclicity allows us to obtain the remaining unrecorded components. For brevity the following short hand notation has been used

$$C_i = cos(2\phi_i - \phi_{i+1} - \phi_{i-1}) \qquad C_{i,j} = cos(3\phi_i - 3\phi_j) \tag{B–13}$$

$$S_i = sin(2\phi_i - \phi_{i+1} - \phi_{i-1}) \qquad S_{i,j} = sin(3\phi_i - 3\phi_j) \tag{B–14}$$

The B-field components:

$$B_{r_1,r_2} = -\frac{Gw}{2} r_3 [r_1 S_1 + r_2 S_2 + r_3 S_3] \tag{B–15}$$

$$B_{r_1,\phi_1} = \frac{Gw}{2} r_1 [r_3^2 - r_2^2] \tag{B–16}$$

$$B_{r_1,\phi_2} = \frac{Gw}{2} r_2 [-r_1 r_2 + C_3 r_3^3 - C_2 r_2 r_3 + C_1 r_1 r_3] \tag{B–17}$$

$$B_{r_1,\phi_3} = \frac{Gw}{2} r_3 [r_1 r_3 + C_3 r_2 r_3 - C_2 r_2^2 - C_1 r_1 r_2] \tag{B–18}$$

$$B_{\phi_1,\phi_2} = \frac{Gw}{2} r_1 r_2 r_3 [r_3 S_3 - r_2 S_2 - r_1 S_1] \tag{B–19}$$

B.3 R-R SECTOR OF W-DEFORMED 5-SPHERE

Below is a record of the independent metric components, the rest can be obtain using the symmetry in indices of the metric and B-field.

The S^5_w real metric components

$$g_{\alpha\alpha} = \frac{w^2R^6}{2}\Big[(4c_\alpha^3 - 3c_\alpha)s_\alpha c_\theta^2 s_\theta C_3 - s_\theta c_\alpha s_\alpha s_\theta^2 C_{21} + (4c_\alpha^4 - 3c_\alpha^3 + 1)c_\theta s_\theta C_1$$
$$+ (4c_\alpha^3 - 3c_\alpha)s_\alpha s_\theta^2 c_\theta C_2 - (c_\alpha s_\alpha c_\theta^3)C_{31} + 2c_\alpha^2 s_\alpha^2 c_\theta^2 s_\theta^2 + 1\Big] + R^6 \tag{B–20}$$

$$g_{\alpha\theta} = \frac{w^2R^6}{2}\Big[s_\alpha^2 c_\alpha^2 c_\theta^2 s_\theta C_{31} - [(2s_\alpha^4 - s_\alpha^2)c_\theta^2 - s_\alpha^4]s_\theta C_2 - s_\alpha^2 c_\alpha^2 s_\theta^2 c_\theta C_{21} - s_\alpha^2 c_\alpha^2 s_\theta^2 c_\theta C_3$$
$$+ (2c_\alpha s_\alpha^3 c_\theta^2 + c_\alpha^3 s_\alpha)C_1 + s_\theta c_\theta c_\alpha s_\alpha^3(2c_\theta^2 - 1)\Big] \tag{B–21}$$

$$g_{\alpha\phi_1} = \frac{w^2R^6}{2}\Big[s_\alpha^2 c_\alpha^2 c_\theta^3 S_{31} + 2c_\alpha^2 s_\alpha^2 s_\theta^3 S_{21} + c_\alpha^2 s_\alpha^2 c_\theta^2 s_\theta S_3 + (c_\alpha^2 + 1)c_\alpha s_\alpha c_\theta s_\theta s_1 + c_\alpha^2 s_\alpha^2 s_\theta^2 c_\theta\Big] \tag{B–22}$$

$$g_{\alpha\phi_2} = \frac{w^2R^6}{2}\Big[s_\alpha^2 c_\alpha^2 s_\theta^3 - (s_\alpha^4 c_\theta s_\theta^4 + s_\alpha^2 c_\alpha^2 c_\theta s_\theta^2)S_2 + (c_\alpha^2 - s_\alpha^2 s_\theta^2)c_\alpha s_\alpha s_\theta c_\theta S_1 - s_\alpha^4 s_\theta c_\theta^4\Big] \tag{B–23}$$

$$g_{\alpha\phi_3} = \frac{w^2R^6}{2}\Big[s_\alpha^2 c_\alpha^2 c_\theta^3 S_{31} - (s_\alpha^4 c_\theta^4 + s_\alpha^2 c_\alpha^2 c_\theta^2)S_3 + c_\alpha c_\theta s_\alpha^3 s_\theta^3 S_1 + s_\alpha^4 s_\theta^4 c_\theta s_2\Big] \tag{B–24}$$

$$g_{\theta\theta} = \frac{w^2R^6}{2}\Big[(s_\alpha^2 s_\theta^2 + s_\theta^2 - s_\alpha^2)c_\alpha s_\alpha^3 s_\theta C_3 - c_\alpha^3 s_\alpha^3 s_\theta c_\theta^2 C_{21} - c_\alpha^3 s_\alpha^3 c_\theta s_\theta^2 C_{31} + (1 - 2s_\alpha^2 s_\theta^2 c_\theta^2)s_\alpha^2$$
$$+ (c_\alpha s_\alpha^3 c_\theta^3 - c_\alpha s_\alpha^5 c_\theta s_\theta^2)C_2 + (s_\alpha^4 + c_\alpha^2 + 1)s_\alpha^2 c_\theta s_\theta C_1 - s_\alpha^6 c_\theta s_\theta C_{32}\Big] + s_\alpha^2 R^2 \tag{B–25}$$

$$g_{\theta\phi_1} = \frac{w^2R^6}{2}\Big[c_\alpha^3 s_\alpha^3 c_\theta S_{21} - c_\alpha^3 s_\alpha^3 s_\theta c_\theta^2 S_{31} + (c_\alpha^3 s_\alpha^3 s_\theta c_\theta^2 - c_\alpha s_\alpha^5 s_\theta^3)S_2$$
$$+ (c_\theta^2 - s_\theta^2)c_\alpha s_\alpha^5 c_\theta S_3 + (1 - 2s_\theta^2)c_\alpha^4 s_\alpha^2 S_1\Big] \tag{B–26}$$

$$g_{\theta\phi_2} = \frac{w^2R^6}{2}\Big[c_\theta s_\theta^2 c_\alpha^3 s_\alpha^3 S_{21} - s_\alpha^6 s_\theta^2 c_\theta^2 S_{32} + \left[c_\alpha s_\alpha^5 s_\theta^3(s_\theta^2 - 2) - c_\alpha s_\alpha^3 s_\theta c_\theta^2\right] S_2$$
$$+ (1 - s_\alpha^2 s_\theta^2)c_\alpha^2 s_\alpha^2 s_\theta^2 S_1 - c_\alpha s_\alpha^5 c_\theta s_\theta^4\Big] \tag{B–27}$$

$$g_{\theta\phi_3} = \frac{w^2R^6}{2}\Big[c_\alpha s_\alpha^5 c_\theta^4 s_\theta S_2 - s_\alpha^6 c_\theta^2 s_\theta^2 S_{32} - c_\theta^2 s_\theta c_\alpha^3 s_\alpha^3 S_{31} + \left[c_\alpha s_\alpha^3 c_\theta s_\theta^2 - (c_\theta^2 + 2)c_\alpha s_\alpha^5 c_\theta^3\right] S_3$$
$$+ (1 - c_\theta^2 c_\alpha^2 s_\alpha^2)s_\alpha^2 c_\alpha^2 c_\theta^2 S_1\Big] \tag{B–28}$$

$$g_{\phi_1\phi_1} = \frac{w^2R^6}{2}\Big[s_\alpha^3c_\alpha^3s_\theta^3C_{21} - (1+5c_\alpha^2)c_\theta s_\theta s_\alpha^2c_\alpha^2C_1 - c_\alpha^3s_\alpha^3c_\theta s_\theta^2C_2 + c_\alpha^3s_\alpha^3c_\theta^3C_{31} - c_\alpha^3s_\alpha^3c_\theta^2s_\theta C_3 + (s_\alpha^2+2c_\alpha^6)c_\alpha^2\Big] + c_\alpha^2R^2 \tag{B–29}$$

$$g_{\phi_1\phi_2} = \frac{w^2R^6}{2}\Big[(c_\alpha^2 - s_\alpha^2s_\theta^2)c_\alpha s_\alpha^3c_\theta s_\theta^2C_2 + (s_\alpha^2s_\theta^2 - c_\alpha^2)c_\alpha^2s_\alpha^2c_\theta s_\theta C_1 + c_\alpha^3s_\alpha^3s_\theta^3C_{21} - c_\alpha s_\theta s_\alpha^5c_\theta^4C_3 - (1+s_\alpha^2s_\theta^2-3s_\alpha^2)\Big] \tag{B–30}$$

$$g_{\phi_1\phi_3} = \frac{w^2R^6}{2}\Big[(c_\alpha^2 - s_\alpha^2c_\theta^2)c_\alpha s_\theta s_\alpha^3c_\theta^2C_3 + (s_\alpha^2c_\theta^2 - c_\alpha^2)c_\alpha^2s_\alpha^2c_\theta s_\theta C_1 + c_\alpha^3s_\alpha^3c_\theta^3C_{31} - c_\alpha s_\alpha^5c_\theta s_\theta^4C_2 + (2s_\alpha^2s_\theta^2 - s_\alpha^2c_\theta^2 - c_\alpha^2)\Big] \tag{B–31}$$

$$g_{\phi_2\phi_2} = \frac{w^2R^6}{2}\Big[s_\alpha^6c_\theta^3s_\theta^3C_{32} + c_\alpha^3s_\alpha^3s_\theta^3C_{21} - (1-5s_\alpha^2s_\theta^2)c_\alpha s_\alpha^3c_\theta s_\theta^2C_2 - c_\alpha s_\alpha^5c_\theta^2s_\theta^3C_3 - c_\alpha^2s_\alpha^4c_\theta s_\theta^3C_1 + (2s_\alpha^4s_\theta^4 + s_\alpha^4c_\theta^2s_\theta^2 + c_\alpha^2s_\alpha^2s_\theta^2 + s_\alpha^4c_\theta^4 + 2c_\alpha^2s_\alpha^2c_\theta^2 + c_\alpha^4)\Big] + s_\alpha^2s_\theta^2R^2 \tag{B–32}$$

$$g_{\phi_2\phi_3} = \frac{w^2R^6}{2}\Big[(c_\theta^2 - s_\theta^2)c_\alpha s_\alpha^5s_\theta^2c_\theta C_2 + (s_\theta^2 - c_\theta^2)c_\alpha s_\alpha^5s_\theta c_\theta^2C_3 - s_\alpha^6c_\theta^3s_\theta^3C_{32} - c_\alpha^4s_\alpha^2c_\theta s_\theta C_1 - (s_\alpha^2s_\theta^2 + s_\alpha^2c_\theta^2 - 2c_\alpha^2)s_\alpha^4c_\theta^2s_\theta^2\Big] \tag{B–33}$$

$$g_{\phi_3\phi_3} = \frac{w^2R^6}{2}\Big[s_\alpha^6c_\theta^3s_\theta^3C_{32} - (1+5c_\theta^2s_\theta^2)s_\alpha^3c_\theta^2c_\alpha s_\theta - c_\theta^3s_\theta s_\alpha^2c_\alpha^2C_1 - c_\alpha s_\alpha^5c_\theta^3s_\theta^2C_2c_\alpha^3s_\alpha^3c_\theta^3C_{31} + (2s_\alpha^6c_\theta^6 - s_\alpha^4s_\theta^4 + s_\alpha^2s_\theta^2)\Big] + s_\alpha^2c_\theta^2R^2 \tag{B–34}$$

The real B-field components:

$$
\begin{aligned}
B_{ii} &= 0 \\
B_{\alpha\theta} &= wR^4G[s_\alpha^2s_2s_\theta + s_\alpha^2s_3c_\theta + c_\alpha s_\alpha s_1] \\
B_{\alpha\phi_1} &= wR^4G[2c_\alpha s_\alpha s_\theta^2 - c_\alpha s_\alpha] \\
B_{\alpha\phi_2} &= wR^4G[(s_\alpha^2c_2c_\theta + c_\alpha s_\alpha)s_\theta^2 + (-s_\alpha^2c_3c_\theta^2 - c_\alpha s_\alpha c_1c_\theta)s_\theta] \\
B_{\alpha\phi_3} &= wR^4G[s_\alpha^2c_2c_\theta s_\theta^2 + (c_\alpha s_\alpha c_1c_\theta - s_\alpha^2c_3c_\theta^2)s_\theta - c_\alpha s_\alpha c_\theta^2] \\
B_{\theta\alpha} &= wR^4G[-s_\alpha^2s_2s_\theta - s_\alpha^2s_3c_\theta - c_\alpha s_\alpha s_1] \\
B_{\theta\phi_1} &= wR^4G[(2c_\alpha^2s_\alpha^2c_\theta - c_\alpha s_\alpha^3c_2)s_\theta - c_\alpha s_\alpha^3c_3c_\theta + c_\alpha^2s_\alpha^2c_1] \\
B_{\theta\phi_2} &= wR^4G[-c_\alpha s_\alpha^3c_2s_\theta^3 + (c_\alpha s_\alpha^3c_3c_\theta + (s_\alpha^2 - s_\alpha^4)c_1)s_\theta^2 + (s_\alpha^2 - 2s_\alpha^4)c_\theta s_\theta] \\
B_{\theta\phi_3} &= wR^4G[(c_\alpha s_\alpha^3c_2c_\theta^2 + (s_\alpha^2 - 2s_\alpha^4)c_\theta)s_\theta - c_\alpha s_\alpha^3c_3c_\theta^3 + (s_\alpha^2 - s_\alpha^4)c_1c_\theta^2] \\
B_{\phi_1\alpha} &= wR^4G[c_\alpha s_\alpha - 2c_\alpha s_\alpha s_\theta^2] \\
B_{\phi_1\theta} &= wR^4G[(c_\alpha s_\alpha^3c_2 - 2c_\alpha^2s_\alpha^2c_\theta)s_\theta + c_\alpha s_\alpha^3c_3c_\theta - c_\alpha^2s_\alpha^2c_1]
\end{aligned}
$$

$$
\begin{aligned}
B_{\phi_1\phi_2} &= wR^4G[(c_\alpha s_\alpha^3 s_3 c_\theta^2 - c_\alpha^2 s_\alpha^2 s_1 c_\theta)s_\theta - c_\alpha s_\alpha^3 s_2 c_\theta s_\theta^2] \\
B_{\phi_1\phi_3} &= wR^4G[(c_\alpha s_\alpha^3 s_3 c_\theta^2 + c_\alpha^2 s_\alpha^2 s_1 c_\theta)s_\theta - c_\alpha s_\alpha^3 s_2 c_\theta s_\theta^2] \\
B_{\phi_2\alpha} &= wR^4G[(-s_\alpha^2 c_2 c_\theta - c_\alpha s_\alpha)s_\theta^2 + (s_\alpha^2 c_3 c_\theta^2 + c_\alpha s_\alpha c_1 c_\theta)s_\theta] \\
B_{\phi_2\theta} &= wR^4G[c_\alpha s_\alpha^3 c_2 s_\theta^3 + (-c_\alpha s_\alpha^3 c_3 c_\theta - c_\alpha^2 s_\alpha^2 c_1)s_\theta^2 + (s_\alpha^4 - c_\alpha^2 s_\alpha^2)c_\theta s_\theta] \\
B_{\phi_2\phi_3} &= wR^4G[c_\alpha s_\alpha^3 s_2 c_\theta s_\theta^2 + (c_\alpha^2 s_\alpha^2 s_1 c_\theta - c_\alpha s_\alpha^3 s_3 c_\theta^2)s_\theta] \\
B_{\phi_2\phi_3} &= wR^4G[(c_\alpha^2 s_\alpha^2 s_1 c_\theta - c_\alpha s_\alpha^3 s_3 c_\theta^2)s_\theta - c_\alpha s_\alpha^3 s_2 c_\theta s_\theta^2] \\
B_{\phi_3\alpha} &= wR^4G[-s_\alpha^2 c_2 c_\theta s_\theta^2 + (s_\alpha^2 c_3 c_\theta^2 - c_\alpha s_\alpha c_1 c_\theta)s_\theta + c_\alpha s_\alpha c_\theta^2] \\
B_{\phi_1\theta} &= wR^4G[((s_\alpha^4 - c_\alpha^2 s_\alpha^2)c_\theta - c_\alpha s_\alpha^3 c_2 c_\theta^2)s_\theta + c_\alpha s_\alpha^3 c_3 c_\theta^3 - c_\alpha^2 s_\alpha^2 c_1 c_\theta^2] \\
B_{\phi_3\phi_1} &= wR^4G[c_\alpha s_\alpha^3 s_2 c_\theta s_\theta^2 + (-c_\alpha s_\alpha^3 s_3 c_\theta^2 - c_\alpha^2 s_\alpha^2 s_1 c_\theta)s_\theta] \\
B_{\phi_3\phi_2} &= wR^4G[c_\alpha s_\alpha^3 s_2 c_\theta s_\theta^2 + (c_\alpha s_\alpha^3 s_3 c_\theta^2 - c_\alpha^2 s_\alpha^2 s_1 c_\theta)s_\theta]
\end{aligned}
$$

The End

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