A letter to the Editor about:
"Machado, M.A.G. and Costa A.F.B. (2014). Some comments regarding the synthetic chart. *Communications in Statistics - Theory and Methods*, 43 (14), 2897-2906"

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Dear Editor,

We recently read the abovementioned paper by Machado and Costa (2014). However, before going into the details, two of the referees requested us to clarify the difference between the four types of the synthetic $\bar{X}$ charts. Hence, in Table 1, we provide the operation of the non-side sensitive (NSS), standard side-sensitive (SSS), revised side-sensitive (RSS) and modified side-sensitive (MSS) synthetic $\bar{X}$ charts that were first proposed by Wu and Spedding (2000), Davis and Woodall (2002), Machado and Costa (2014) and Shongwe and Graham (2016a), respectively. The control charting regions for each of these types of schemes are given as Step (1) in Table 1. According to Shongwe and Graham (2016a)'s zero-state and steady-state empirical analysis, the MSS synthetic $\bar{X}$ chart has a better performance than the other types of Shewhart synthetic $\bar{X}$ charts.

Machado and Costa (2014) is an interesting paper, since, apart from the fact that it point out a very important mistake done by Davis and Woodall (2002) regarding the computation of the stationary probabilities vectors, the authors also proposed a new type of a synthetic $\bar{X}$ chart (i.e. the RSS scheme) and compared its steady-state performance to the NSS scheme. Although Machado and Costa (2014) has some valuable contributions, we point out a correction regarding the NSS scheme's illustrative example done in their p. 2899, which reads as follows:

"Even though Davis and Woodall (2002) declared that the stationary probabilities should be obtained with the process in control, they obtained $\boldsymbol{S}$ with A = Pr[|Z| < k | Z ~ N(d;1)], that is, the stationary probabilities were computed with $\mu = \mu_0 + d\sigma_{\bar{X}}$, the out-of-control value of the process mean. For instance, when (L, k, d) = (5, 2.263, 2), the $\boldsymbol{ARL'}$ = (5.105, 2.806, 3.196, 3.885), $\boldsymbol{S}_0$ = (0.9162, 0.0279, 0.0279, 0.0279), and $\boldsymbol{S}_1$ = (0.4339, 0.1886, 0.1886, 0.1886); $\boldsymbol{S}_0$ is the $\boldsymbol{S}$ vector computed with A = Pr[|Z| < k | Z ~ N(0;1)] and $\boldsymbol{S}_1$ is the $\boldsymbol{S}$ vector computed with A =

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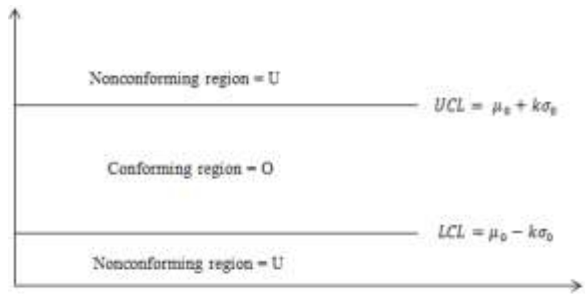
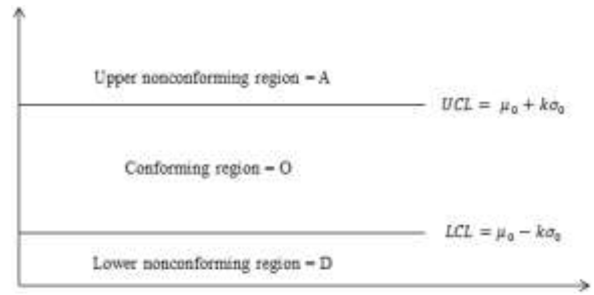
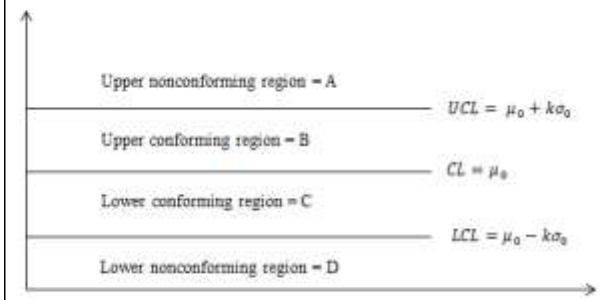
Table 1: Operation of the non-side sensitive (NSS), standard side-sensitive (SSS), revised side-sensitive (RSS) and modified side-sensitive (MSS) synthetic $\bar{X}$ charts

<table>
<tr><th>Step</th><th>NSS</th><th>SSS</th><th>RSS</th><th>MSS</th></tr>
<tr><td>(1)</td><td>
</td><td colspan="2">
</td><td>
</td></tr>
<tr><td>(2)</td><td colspan="4">Set the control limit of the CRL sub-chart (i.e. H), or equivalently, start at H = 1 and increase it accordingly.</td></tr>
<tr><td>(3)</td><td colspan="4">Compute the corresponding k so that the target in-control ARL_0 is attained. Hence the control limits of the $\bar{X}$ sub-chart are $UCL/LCL = {}_0 \pm k\sigma_0$.</td></tr>
<tr><td>(4)</td><td colspan="4">Wait until the next inspection time, take a random sample of size n and calculate the sample mean $\bar{X}_i$.</td></tr>
<tr><td>(5)</td><td colspan="4">If $LCL < \bar{X}_i < UCL$, the i^{th} sample is conforming, hence return to Step (4); otherwise go to Step (6).</td></tr>
<tr><td>(6)</td><td>If $\bar{X}_i \leq LCL$ or $\bar{X}_i \geq UCL$ go to Step (7).</td><td colspan="3">If $\bar{X}_i \leq LCL$ go to Step (7a), or if $\bar{X}_i \geq UCL$ go to Step (7b).</td></tr>
<tr><td>(7)</td><td>Calculate CRL^{S1} and if $CRL^{S1} \leq H$ go to Step (8); otherwise return to Step (4).</td><td>(7a) Calculate CRL_L^{S2} and if $CRL_L^{S2} \leq H$ go to Step (8); otherwise return to Step (4).

(7b) Calculate CRL_U^{S2} and if $CRL_U^{S2} \leq H$ go to Step (8); otherwise return to Step (4).</td><td>(7a) Calculate CRL_L^{S3} and if $CRL_L^{S3} \leq H$ go to Step (8); otherwise return to Step (4).

(7b) Calculate CRL_U^{S3} and if $CRL_U^{S3} \leq H$ go to Step (8); otherwise return to Step (4).</td><td>(7a) Calculate CRL_L^{S4} and if $CRL_L^{S4} \leq H$ go to Step (8); otherwise return to Step (4).

(7b) Compute CRL_U^{S4} and if $CRL_U^{S4} \leq H$ go to Step (8); otherwise return to Step (4).</td></tr>
<tr><td>(8)</td><td colspan="4">Issue an OOC signal and then take necessary corrective action to find and remove the assignable causes. Then return to Step (4).</td></tr>
</table>

CRL^{S1}: Number of *conforming* samples that fall in region ‘O’; which are in between any two consecutive nonconforming samples that fall on region ‘U’.
CRL_L^{S2}: Number of (either *conforming* or *nonconforming*) samples that fall in regions ‘O’ and ‘A’; which are in between the two consecutive nonconforming samples that fall on region ‘D’.
CRL_U^{S2}: Number of (either *conforming* or *nonconforming*) samples that fall in regions ‘O’ and ‘D’; which are in between the two consecutive nonconforming samples that fall on region ‘A’.
CRL_L^{S3}: Number of *conforming* samples that fall in region ‘O’; which are in between the two consecutive nonconforming samples that fall on region ‘D’.
CRL_U^{S3}: Number of *conforming* samples that fall in region ‘O’; which are in between the two consecutive nonconforming samples that fall on region ‘A’.
CRL_L^{S4}: Number of *conforming* samples that fall in regions ‘C’; which are in between the two consecutive nonconforming samples that fall on region ‘D’.
CRL_U^{S4}: Number of *conforming* samples that fall in regions ‘B’; which are in between the two consecutive nonconforming samples that fall on region ‘A’.
Note that each computation of the *CRL* value above, includes the nonconforming sample at the end, so that the absence of any nonconforming sample means *CRL* = 1.

[**Source:** Shongwe and Graham (2016b)]

Pr[|Z| < k | Z ~ N(d;1)]. The steady-state ARL is given by $\underline{\boldsymbol{S}_0'\boldsymbol{ARL} = 5.0}$. In their Table 2, Davis and Woodall (2002) present the value of $\underline{\boldsymbol{S}_1'\boldsymbol{ARL} = 4.1}$. Table 1 shows the $\boldsymbol{S}_0'\boldsymbol{ARL}$s and $\boldsymbol{S}_1'\boldsymbol{ARL}$s for L = 1, 5, and 10 and d varying from 0 to 3.0 in steps of 0.1. Depending on the magnitude of the shift, the percentage difference between $\boldsymbol{S}_1'\boldsymbol{ARL}$ and $\boldsymbol{S}_0\boldsymbol{ARL}$ ranges from 0% to 37%.”

The elements of the paragraph that are underlined are incorrect. These are: (i) dimension of the vectors, (ii) stationary probabilities vector, (iii) $\boldsymbol{ARL}$ vector and (iv) actual *ARL* values; we discuss each of these next.

(i) Dimension of the vectors

Since as stated by Machado and Costa (2014, p.2899), the dimension of the essential transition probabilities matrix (eTPM) of the NSS synthetic $\bar{X}$ chart is equal to (L+1)×(L+1), hence the corresponding $\boldsymbol{S}$ and $\boldsymbol{ARL}$ vectors must have a dimension of (L+1). However, we see that while they consider L = 5, yet the dimension of $\boldsymbol{S}$ and $\boldsymbol{ARL}$ vectors is not equal to 6.

(ii) Stationary probabilities vectors

It is clear that $\boldsymbol{S}_0'$ and $\boldsymbol{S}_1'$ are incorrect due to their dimension. Here, we give the expressions and the corresponding authors that computed the stationary probabilities vectors of the NSS synthetic chart (with those by Knoth (2016) pointed out by one of the referees and are called the cyclical and conditional quasi-stationary distribution vectors, with ϱ the largest (in magnitude) eigenvalue of the eTPM), where $\delta \equiv d = \frac{\mu_1 - \mu_0}{\sigma_0}$:

Machado and Costa (2014): $$\frac{1}{1+LB(\delta)}\left(1, B(\delta), B(\delta), \ldots, B(\delta)\right) \quad (1)$$

Knoth (2016): $$\left(1-\frac{A(\delta)}{\varrho}\right)\left(\frac{\varrho}{B(\delta)}, 1, \frac{A(\delta)}{\varrho}, \left(\frac{A(\delta)}{\varrho}\right)^2, \ldots, \left(\frac{A(\delta)}{\varrho}\right)^{L-1}\right) \quad (2)$$

Knoth (2016): $$\left(A(\delta)^L, B(\delta), B(\delta)A(\delta), B(\delta)A(\delta)^2, \ldots, B(\delta)A(\delta)^{L-1}\right) \quad (3)$$

As stated by Machado and Costa (2014), Davis and Woodall (2002) mistakenly used $\delta \neq 0$ to calculate the stationary probabilities vector. Thus, Machado and Costa (2014, p.2899) was supposed to calculate the incorrect stationary probabilities vector of Davis and Woodall (2002) as one of those given in Table 2, Panel (b). The correct ones were supposed to be given by any of the corresponding vectors in Table 2, Panel (c). Note that Table 2, Panel (a) corresponds to each of the methods in Equations (1) to (3), respectively.

(iii) *ARL* vectors

There are two ways to write the closed-form expressions of the *ARL* vector of the NSS synthetic $\bar{X}$ chart (with that in Equation (4) derived by one of the referees and that in Equation (5) reported

Table 2: The difference between the empirical stationary probabilities and their corresponding *ARL* values

(a) Technique	(b) $'_1(\delta = 2)$	(c) $'_0(\delta = 0)$	(d) $'_1(2) \cdot \boldsymbol{ARL}(2)$	(e) $'_0(0) \cdot \boldsymbol{ARL}(2)$
M&C (2014)	(0.3354,0.1329,0.1329,0.1329,0.1329,0.1329)	(0.8943,0.0211,0.0211,0.0211,0.0211,0.0211)	4.0	5.1
Knoth (2016)	(0.3529,0.1881,0.1527,0.1240,0.1007,0.0817)	(0.8980,0.0213,0.0208,0.0204,0.0199,0.0195)	3.9	5.1
Knoth (2016)	(0.0802,0.3963,0.2392,0.1444,0.0872,0.0526)	(0.8873,0.0236,0.0231,0.0225,0.0220,0.0215)	3.2	5.0

M&C (2014) – Machado and Costa (2014)

in Shongwe and Graham (2016b)). These look notionally different, however, they yield the same empirical values, where $r(\delta) = B(\delta)(1 - A(\delta)^L)$ and $q(\delta) = 1 - A(\delta) - A(\delta)^L + A(\delta)^{L+1}$:

$$\left(\frac{1}{r(\delta)} + \frac{1}{B(\delta)}, \frac{1}{r(\delta)}, \frac{1 + A(\delta)^L(A(\delta)^{-1} - 1)}{r(\delta)}, \dots, \frac{1 + A(\delta)^L\left(A(\delta)^{-(L-1)} - 1\right)}{r(\delta)}\right) \quad (4)$$

$$\frac{1}{q(\delta)}(2 - A(\delta)^L, 1, 1 + A(\delta)^{L-1} - A(\delta)^L, 1 + A(\delta)^{L-2} - A(\delta)^L, \dots, 1 + A(\delta)^2 - A(\delta)^L, 1 + A(\delta) - A(\delta)^L). \quad (5)$$

For (*L*, *k*, *d*) = (5, 2.263, 2), the correct empirical *ARL* vector using either Equation (4) or (5) is given by $\boldsymbol{ARL}'$(2) = (5.2669, 2.7435, 2.8879, 3.1271, 3.5233, 4.1797).

(iv) Actual *ARL* values

Using the *ARL* vector in Equation (4) or (5) and the stationary probabilities vectors in Equation (1) to (3) yield the incorrect actual *ARL* values (‘$\boldsymbol{S}_1'(2) \cdot \boldsymbol{ARL}(2)$’, that were supposed to be reported by Davis and Woodall (2002)) in Table 2, Panel (d) and the corresponding correct actual *ARL* values (‘$\boldsymbol{S}_1'(0) \cdot \boldsymbol{ARL}(2)$’, that were supposed to be calculated by Machado and Costa (2014)) are given in Table 2, Panel (e).

Our findings sound a cautionary note to the use of the values presented in Machado and Costa (2014, p.2899). Note though, one of the referees pointed out that Machado and Costa (2014) might have confused (*L*, *k*, *d*) = (5, 2.263, 2) with (*L*, *k*, *d*) = (3, 2.164, 2), which yields the stationary probabilities and *ARL* vectors in the quoted paragraph above.

Finally, for an empirical and theoretical discussion of other Shewhart synthetic-type monitoring schemes, we refer the reader to Shongwe and Graham (2016c, d); however, for a contrasting point of view on Shewhart synthetic charts, we refer the reader to Knoth (2016).

Thank you for your attention.

Sincerely,

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Acknowledgements

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