Synthetic and runs-rules charts combined with an $\bar{X}$ chart: Theoretical discussion

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Abstract

A synthetic and runs-rules $\bar{X}$ charts that are combined with a basic $\bar{X}$ chart are called a Synthetic-$\bar{X}$ and improved runs-rules $\bar{X}$ charts, respectively. This paper gives the zero-state and steady-state theoretical results of the Synthetic-$\bar{X}$ and improved runs-rules $\bar{X}$ monitoring schemes. The Synthetic-$\bar{X}$ and IRR schemes can each be classified into four different categories i.e. (i) non-side-sensitive, (ii) standard side-sensitive, (iii) revised side-sensitive and, (iv) modified side-sensitive. In this paper, we first give the operation and secondly, the general form of the transition probability matrices for each of the categories. Thirdly, in steady-state, we show that for each of the categories, the three methods that are widely used in the literature to compute the initial probability vectors result in different probability expressions (or values). Fourthly, we derive the closed-form expressions of the average run-length (*ARL*) vectors for each of the categories, so that, by multiplying each of these *ARL* vectors with the zero-state and steady-state initial probability vectors, yield the zero-state and steady-state *ARL* expressions. Finally, we formulate the closed-form expressions of the extra quadratic loss function for each of the categories.

Keywords: Average run-length, Steady-state, Transition probability matrix (TPM), Zero-state.

1. Introduction

Wu and Spedding[1] first proposed a synthetic $\bar{X}$ chart as an attempt to improve the Shewhart $\bar{X}$ chart in detecting small and moderate process shifts. A synthetic $\bar{X}$ chart is a combination of the basic $\bar{X}$ chart and a conforming run-length (*CRL*) chart. Then Wu et al.[2] further increased the sensitivity of the synthetic $\bar{X}$ chart so that it may be sensitive to both small and large shifts by proposing a Synthetic-$\bar{X}$ chart, which is combination of the Wu and Spedding[1]'s synthetic $\bar{X}$ chart and a basic $\bar{X}$ chart. A variety of categories for the Synthetic-$\bar{X}$ schemes now exist, see Table I in Shongwe and Graham[3].

Similar endeavors were undertaken to improve the performance of the Shewhart $\bar{X}$ chart using supplementary signaling rules. Klein[4] showed that his side-sensitive *w*-of-(*w*+*v*) scheme (where the integers $w \geq 2$ and $v \geq 0$) not only improves the performance of the $\bar{X}$ chart in detecting small shifts, but also outperforms the corresponding non-side-sensitive scheme of Derman and Ross[5]. To further sensitize Klein[4]'s rules, in effectively detecting both small and large process shifts, Khoo and Ariffin[6]

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proposed the improved runs-rules (IRR) schemes which are a combination of the $\bar{X}$ chart and the signaling rules of Klein[4]. Since Davis and Woodall[7] showed that a synthetic chart is a special case of runs-rules charts with a head-start feature; Shongwe and Graham[3] conducted an investigation into the performance of the Synthetic-$\bar{X}$ and IRR $\bar{X}$ schemes. For instance, in their Figure 3, it is clear that grouping these schemes according to the categories defined in their Table I, shows that, in zero-state, the Synthetic-$\bar{X}$ schemes outperform their corresponding IRR $\bar{X}$ schemes, however, in steady-state, in each corresponding category, the IRR and Synthetic-$\bar{X}$ schemes perform exactly the same, with the modified side-sensitive ones being the best performing schemes. In Tables I and II, we give the operation of the Synthetic-$\bar{X}$ and IRR $\bar{X}$ schemes, respectively. Moving forward, we only consider runs-rules schemes with w = 2 so that $v = H$-1 and $w+v = H$+1.

In most studies in the area of Statistical Process Control and Monitoring (SPCM), researchers typically use the average run-length (*ARL*) to measure the performance of a control chart at some specific shift, and this is given by

$$ARL(\delta) = \boldsymbol{\xi}_{(1\times M)} \cdot \boldsymbol{ARL}_{(M\times 1)} = \boldsymbol{\xi}_{(1\times M)} \cdot (\mathbf{I} - \mathbf{Q}(\delta))^{-1}\mathbf{1} \quad (1)$$

where $\boldsymbol{\xi}_{(1\times M)}$ is the initial probability vector (depending on whether a zero-state or a steady-state analysis is being considered). Hence, Shongwe and Graham[3] used Equation (1) to compute zero-state and steady-state *ARL*s (denoted by *ZSARL* and *SSARL*, respectively) of each of the schemes using SAS® 9.3 programs. The authors gave examples with H = 1 and 5, respectively, in order to show how to construct the transition probability matrices (TPMs) of each of the schemes.

In this paper, we make a further contribution to the theory and application of Synthetic-$\bar{X}$ and IRR $\bar{X}$ monitoring schemes by:

- Giving the general form of each of the scheme's TPM for any $H > 0$;
- Deriving the closed-form expressions for the $\boldsymbol{ARL}_{(M\times 1)}$ in Equation (1);
- Deriving the closed-form expressions of the *ZSARL*;
- Show that the three methods, to compute the steady-state initial probability vector, yield different initial probabilities, however, the resulting *SSARL* values are approximately equal;
- Deriving the closed-form expressions of the *SSARL*;
- Finally, using the *ZSARL* and *SSARL* expressions, we give the closed-form expressions of the overall performance statistic, i.e. the extra quadratic loss (*EQL*), given in Shongwe and Graham[3]'s Equations (4) and (5).

Table I: Operation of the four different Synthetic-$\bar{X}$ charts

Step	SC1 scheme	SC2 scheme	SC3 scheme	SC4 scheme
(1)	Set the control limit of the *CRL* sub-chart (i.e. *H*). Set to some value and compute the corresponding $_2$ so that the target ARL_0 is attained. Hence the control / warning limits of the $\bar{X}$ sub-chart are given by: $UCL/LCL = \mu_0 \pm \ \sigma_0$ and $UWL/LWL = \mu_0 \pm \ _2\sigma_0$.			
(2)	Wait until the next inspection time, take a random sample of size *n* and calculate the sample mean $\bar{X}_i$.			
(3)	If $\bar{X}_i \leq LCL$ or $\bar{X}_i \geq UCL$ go to Step (7).			
(4)	If $LWL < \bar{X}_i < UWL$, the i^{th} sample is conforming, hence return to Step (2); otherwise go to Step (5).			
(5)	If $LCL < \bar{X}_i \leq LWL$ or $UWL \leq \bar{X}_i < UCL$ go to Step (6).	If $LCL < \bar{X}_i \leq LWL$ go to Step (6a), or if $UWL \leq \bar{X}_i < UCL$ go to Step (6b).	If $LCL < \bar{X}_i \leq LWL$ go to Step (6a), or if $UWL \leq \bar{X}_i < UCL$ go to Step (6b).	If $LCL < \bar{X}_i \leq LWL$ go to Step (6a), or if $UWL \leq \bar{X}_i < UCL$ go to Step (6b).
(6)	Calculate CRL^{SC1} and if $CRL^{SC1} \leq H$ go to Step (7); otherwise return to Step (2).	(6a) Calculate CRL_L^{SC2} and if $CRL_L^{SC2} \leq H$ go to Step (7); otherwise return to Step (2). (6b) Calculate CRL_U^{SC2} and if $CRL_U^{SC2} \leq H$ go to Step (7); otherwise return to Step (2).	(6a) Calculate CRL_L^{SC3} and if $CRL_L^{SC3} \leq H$ go to Step (7); otherwise return to Step (2). (6b) Calculate CRL_U^{SC3} and if $CRL_U^{SC3} \leq H$ go to Step (7); otherwise return to Step (2).	(6a) Calculate CRL_L^{SC4} and if $CRL_L^{SC4} \leq H$ go to Step (7); otherwise return to Step (2). (6b) Compute CRL_U^{SC4} and if $CRL_U^{SC4} \leq H$ go to Step (7); otherwise return to Step (2).
(7)	Issue an out-of-control (OOC) signal and then take necessary corrective action to find and remove the assignable causes. Then return to Step (2).			

CRL^{SC} : Number of *conforming* samples that fall in region 'O'; which are in between any two consecutive nonconforming samples that fall on region 'U', see Shongwe and Graham[3]'s Figure 1(a).
CRL_L^{SC2}: Number of (either *conforming* or *nonconforming*) samples that fall in regions 'O' and 'A'; which are in between the two consecutive nonconforming samples that fall on region 'D', see Shongwe and Graham[3]'s Figure 1(b).
CRL_U^{SC2}: Number of (either *conforming* or *nonconforming*) samples that fall in regions 'O' and 'D'; which are in between the two consecutive nonconforming samples that fall on region 'A', see Shongwe and Graham[3]'s Figure 1(b).
CRL_L^{SC3}: Number of *conforming* samples that fall in region 'O'; which are in between the two consecutive nonconforming samples that fall on region 'D', see Shongwe and Graham[3]'s Figure 1(b).
CRL_U^{SC3}: Number of *conforming* samples that fall in region 'O'; which are in between the two consecutive nonconforming samples that fall on region 'A', see Shongwe and Graham[3]'s Figure 1(b).
CRL_L^{SC4}: Number of *conforming* samples that fall in region 'C'; which are in between the two consecutive nonconforming samples that fall on region 'D', see Shongwe and Graham[3]'s Figure 1 (c).
CRL_U^{SC4}: Number of *conforming* samples that fall in region 'B'; which are in between the two consecutive nonconforming samples that fall on region 'A', see Shongwe and Graham[3]'s Figure 1 (c).

Note that each computation of the *CRL* value above, includes the nonconforming sample at the end, so that the absence of any nonconforming sample means *CRL*=1.

Table II: Operation of the four different improved runs-rules $\bar{X}$ charts

Step	IRR1 scheme	IRR2 scheme	IRR3 scheme	IRR4 scheme
(1)	Specify the values of w and v. Set k to some value and compute the corresponding $_2$ so that the target ARL_0 is attained. Hence the control / warning limits of the $\bar{X}$ sub-chart are given by: $UCL/LCL = \mu_0 \pm \quad \sigma_0$ and $UWL/LWL = \mu_0 \pm k_2\sigma_0$.			
(2)	Wait until the next inspection time, take a random sample of size n and calculate the sample mean $\bar{X}_i$.			
(3)	If $\bar{X}_i \leq LCL$ or $\bar{X}_i \geq UCL$ go to Step (7).			
(4)	If $LCL < \bar{X}_i < UCL$, the i^{th} sample is conforming, hence return to Step (2).			
(5)	If $LCL < \bar{X}_i \leq LWL$ or $UWL \leq \bar{X}_i < UCL$ go to Step (6).	If $LCL < \bar{X}_i \leq LWL$ go to Step (6a), or if $UWL \leq \bar{X}_i < UCL$ go to Step (6b).	If $LCL < \bar{X}_i \leq LWL$ go to Step (6a), or if $UWL \leq \bar{X}_i < UCL$ go to Step (6b).	If $LCL < \bar{X}_i \leq LWL$ go to Step (6a), or if $UWL \leq \bar{X}_i < UCL$ go to Step (6b).
(6)	*Out of the next* (w+v−1) *consecutive samples*: (consider Shongwe and Graham[3]'s Figures 1(a), (b), (c))			
	If $(w-1)$ nonconforming samples fall in any region U which are separated by v conforming samples in region O, go to Step (7); otherwise return to Step (2). - See Figure 1(a)	(6a) If $(w-1)$ nonconforming samples fall in region D and are separated by at most v samples (conforming or nonconforming) that fall in regions O and A, go to Step (7); otherwise return to Step (2). (6b) If $(w-1)$ nonconforming samples fall in region A and are separated by at most v samples (conforming or nonconforming) that fall in regions O and D, go to Step (7); otherwise return to Step (2). - See Figure 1(b)	(6a) If $(w-1)$ nonconforming samples fall in region D and are separated by at most v conforming samples that fall in region O, go to Step (7); otherwise return to Step (2). (6b) If $(w-1)$ nonconforming samples fall in region A and are separated by at most v conforming samples that fall in region O, go to Step (7); otherwise return to Step (2). - See Figure 1(b)	(6a) If $(w-1)$ nonconforming samples fall in region D and are separated by at most v conforming samples that fall in region C, go to Step (7); otherwise return to Step (2). (6b) If $(w-1)$ nonconforming samples fall in region A and are separated by at most v conforming samples that fall in region B, go to Step (7); otherwise return to Step (2). - See Figure 1(c)
(7)	Issue an OOC signal and then take necessary corrective action to find and remove the assignable causes. Then return to Step (2).			

The empirical results have already been discussed in Shongwe and Graham[3], thus motivated by the work in Section 2 and the Appendix of Lim and Cho[8] and Machado and Costa[9], respectively, in this paper, our objective is to give the theoretical results or closed-form expressions that can equivalently be used instead of Equation (1) for each of the schemes. The rest of the paper is organized as follows: In Section 2, we give the general form of the TPMs. In Section 3, we derive the closed-form expressions of the *ARL* vectors. Then in Section 4, we give the zero-state initial probability vectors and derive the corresponding *ZSARL* and *ZSEQL* expressions for each of the schemes. In Section 5, steady-state initial probability vectors are derived and their corresponding *SSARL* and *SSEQL* expressions. In Section 6, we give some concluding remarks.

2. General transition probability matrices

Using the methodology outlined in Shongwe and Graham[3]'s Appendix, then for any $H > 0$, we obtain the general form of the TPMs of each of the schemes as shown in Table III, Panels (a) to (d).

From the TPMs above, when we remove the shaded elements on the TPMs of the SC2, SC3 and SC4 schemes, this results in the following remark.

Remark 1: Relation of the steady-state IRR and Synthetic-$\bar{X}$ schemes

When the effect of a head-start feature is removed, that is, when the process starts and stays in-control (IC) for a very long time, the TPMs of the SC2, SC3 and SC4 schemes reduce to those of the *1-of-1* or *2-of-*(*H*+1) IRR2, IRR3 and IRR4 schemes, respectively. That is, in steady-state, SC2≡IRR2, SC3≡IRR3 and SC4≡IRR4 which means that the dimensions of the essential TPMs given in Shongwe and Graham[3]'s Equation (A.3b) are then given by

$$M = \tau = \begin{cases} (H^2 + H + 1) & \text{for IRR2, SC2} \\ (2H + 1) & \text{for IRR3, IRR4, SC3, SC4.} \end{cases} \qquad (2)$$

For instance, in Machado and Costa[9], the steady-state TPM of the SC3 scheme is actually that given here in Table III, Panel (c), without the shaded elements.

3. *ARL* vectors

Here, we illustrate how to compute the $\boldsymbol{ARL}_{(M\times 1)}$ in Equation (1), which needs to be multiplied by the initial probability vector $\boldsymbol{\xi}_{(1\times M)}$ in order to calculate the *ZSARL* and *SSARL* expressions. This procedure is done recursively and based on the patterns of these examples as *H* increases; we formulate the general form of the *ARL* vectors. Firstly, for the SC1 / IRR1 scheme,

Table III: The general form of the TPMs of the schemes listed in Tables I and II

(a) SC1 / IRR1

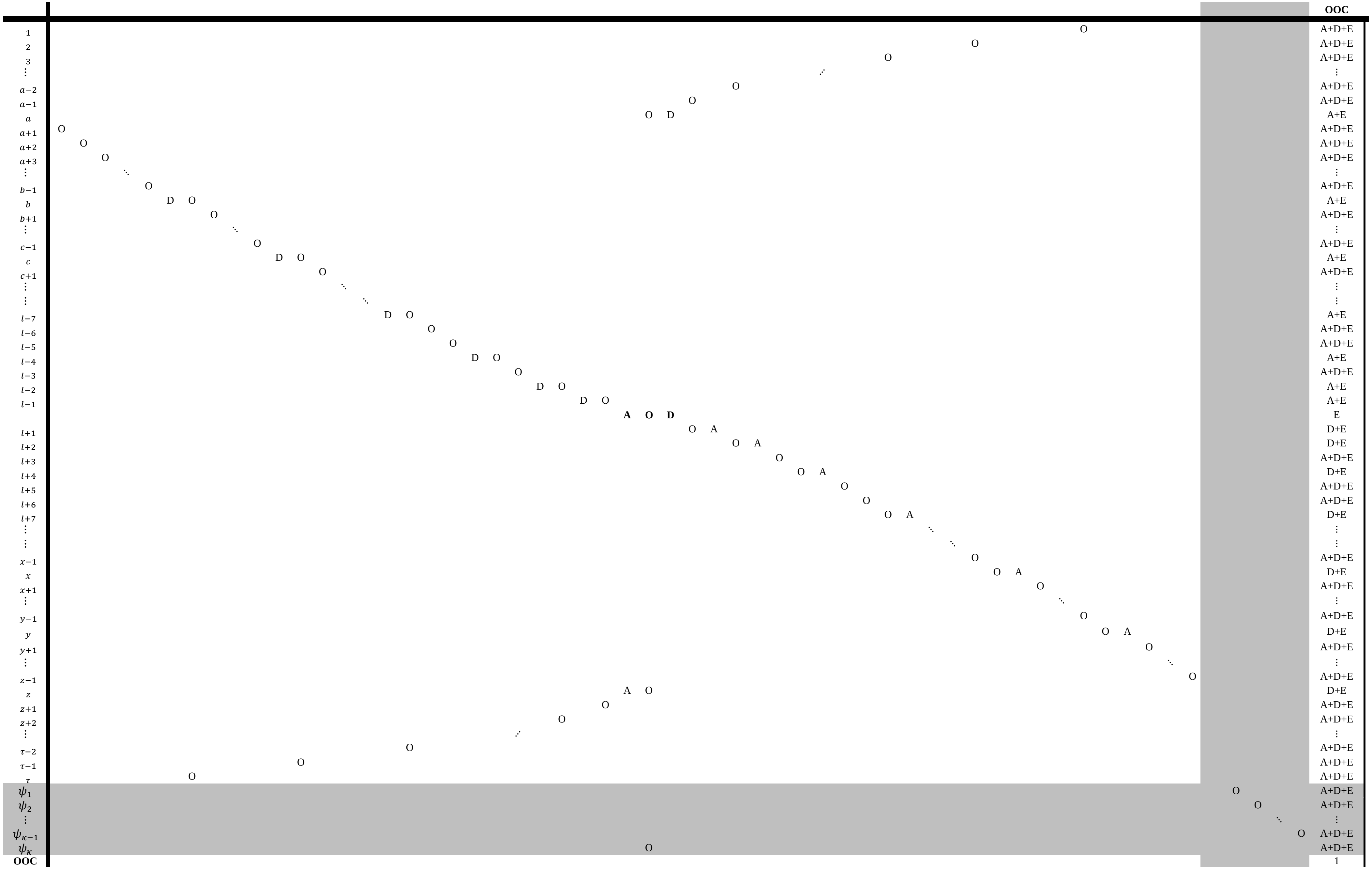
		2	3	4	5	$\cdots$	$\tau-3$	$\tau-2$	$\tau-1$	τ	OOC
	θ_O	θ_U				$\cdots$					θ_E
2			θ_O			$\cdots$					$\theta_U+\theta_E$
3				θ_O		$\cdots$					$\theta_U+\theta_E$
4					θ_O	$\cdots$					$\theta_U+\theta_E$
5						$\ddots$					$\theta_U+\theta_E$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\vdots$
$\tau-3$								θ_O			$\theta_U+\theta_E$
$\tau-2$									θ_O		$\theta_U+\theta_E$
$\tau-1$										θ_O	$\theta_U+\theta_E$
τ	θ_O										$\theta_U+\theta_E$
OOC											1

(b) SC2 / IRR2

For the SC2 / IRR2 schemes, we define the following dummy variables to facilitate in easily writing the general form of the TPM:

$$
\begin{gathered}
a = H \\
b = H + (H-1) \\
c = H + (H-1) + (H-2) \\
d = H + (H-1) + (H-2) + (H-3) \\
\vdots \\
l = \frac{\tau+1}{2} \\
\vdots \\
x = (\tau+1) - c \\
y = (\tau+1) - b \\
z = (\tau+1) - a.
\end{gathered}
$$

Finally, within the TPM, ‘A’ denotes ‘θ_A’, that is, ‘A’ ≡ ‘θ_A’. Similarly, ‘D’ ≡ ‘θ_D’, ‘O’ ≡ ‘θ_O’ and ‘E’ ≡ ‘θ_E’.


OOC
1 A+D+E
2 A+D+E
3 A+D+E
a−2 A+D+E
a−1 A+D+E
a A+E
a+1 A+D+E
a+2 A+D+E
a+3 A+D+E
b−1 A+D+E
b A+E
b+1 A+D+E
c−1 A+D+E
c A+E
c+1 A+D+E
l−7 A+E
l−6 A+D+E
l−5 A+D+E
l−4 A+E
l−3 A+D+E
l−2 A+E
l−1 A+E
E
l+1 D+E
l+2 D+E
l+3 A+D+E
l+4 D+E
l+5 A+D+E
l+6 A+D+E
l+7 D+E
x−1 A+D+E
x D+E
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z+2 A+D+E
τ−2 A+D+E
τ−1 A+D+E
τ A+D+E
ψ1 A+D+E
ψ2 A+D+E
ψκ−1 A+D+E
ψκ A+D+E
OOC 1

(c) SC3 / IRR3

Let $l = \frac{\tau+1}{2}$ then it follows that the TPM may be written as follows

	1	2	$\cdots$	$l-4$	$l-3$	$l-2$	$l-1$		$l+1$	$l+2$	$l+3$	$l+4$	$\cdots$	$\tau-2$	$\tau-1$	τ	ψ_1	ψ_2	ψ_3	ψ_4	$\cdots$	$\psi_{\kappa-2}$	$\psi_{\kappa-1}$	ψ_κ	OOC
1								θ_O	θ_D																$\theta_A + \theta_E$
2	θ_O								θ_D																$\theta_A + \theta_E$
3		θ_O							θ_D																$\theta_A + \theta_E$
$\vdots$			$\ddots$						$\vdots$																$\vdots$
$l-3$				θ_O					θ_D																$\theta_A + \theta_E$
$l-2$					θ_O				θ_D																$\theta_A + \theta_E$
$l-1$						θ_O			θ_D																$\theta_A + \theta_E$
							θ_A	θ_O	θ_D																θ_E
$l+1$							θ_A			θ_O															$\theta_D + \theta_E$
$l+2$							θ_A				θ_O														$\theta_D + \theta_E$
$l+3$							θ_A					θ_O													$\theta_D + \theta_E$
$\vdots$							$\vdots$						$\ddots$												$\vdots$
$\tau-3$							θ_A							θ_O											$\theta_D + \theta_E$
$\tau-2$							θ_A								θ_O										$\theta_D + \theta_E$
$\tau-1$							θ_A									θ_O									$\theta_D + \theta_E$
τ							θ_A	θ_O																	$\theta_D + \theta_E$
ψ_1																		θ_O							$\theta_A + \theta_D + \theta_E$
ψ_2																			θ_O						$\theta_A + \theta_D + \theta_E$
ψ_3																				θ_O					$\theta_A + \theta_D + \theta_E$
$\vdots$																					$\ddots$				$\vdots$
$\psi_{\kappa-3}$																						θ_O			$\theta_A + \theta_D + \theta_E$
$\psi_{\kappa-2}$																							θ_O		$\theta_A + \theta_D + \theta_E$
$\psi_{\kappa-1}$																								θ_O	$\theta_A + \theta_D + \theta_E$
ψ_κ								θ_O																	$\theta_A + \theta_D + \theta_E$
OOC																									1

(d) SC4 / IRR4

Let $l = \frac{\tau+1}{2}$ then it follows that the TPM may be written as follows

	1	2	⋯	$l-3$	$l-2$	$l-1$		$l+1$	$l+2$	$l+3$	⋯	$\tau-2$	$\tau-1$	τ	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	⋯	$\psi_{\kappa-3}$	$\psi_{\kappa-2}$	$\psi_{\kappa-1}$	ψ_κ	OOC
1							$\theta_B+\theta_C$	θ_D																	$\theta_A+\theta_E$
2	θ_B						θ_C	θ_D																	$\theta_A+\theta_E$
3		θ_B					θ_C	θ_D																	$\theta_A+\theta_E$
⋮			⋱				⋮	⋮																	⋮
$l-2$				θ_B			θ_C	θ_D																	$\theta_A+\theta_E$
$l-1$					θ_B		θ_C	θ_D																	$\theta_A+\theta_E$
						θ_A	$\theta_B+\theta_C$	θ_D																	θ_E
$l+1$						θ_A	θ_B		θ_C																$\theta_D+\theta_E$
$l+2$						θ_A	θ_B			θ_C															$\theta_D+\theta_E$
⋮						⋮	⋮				⋱														⋮
$\tau-3$						θ_A	θ_B					θ_C													$\theta_D+\theta_E$
$\tau-2$						θ_A	θ_B						θ_C												$\theta_D+\theta_E$
$\tau-1$						θ_A	θ_B							θ_C											$\theta_D+\theta_E$
τ						θ_A	$\theta_B+\theta_C$																		$\theta_D+\theta_E$
ψ_1																θ_B	θ_C								$\theta_A+\theta_D+\theta_E$
ψ_2							θ_C	θ_D										θ_B							$\theta_A+\theta_E$
ψ_3						θ_A	θ_B												θ_C						$\theta_D+\theta_E$
⋮						⋮	⋮													⋱					⋮
$\psi_{\kappa-5}$							θ_C	θ_D													θ_B				$\theta_A+\theta_E$
$\psi_{\kappa-4}$						θ_A	θ_B															θ_C			$\theta_D+\theta_E$
$\psi_{\kappa-3}$							θ_C	θ_D															θ_B		$\theta_A+\theta_E$
$\psi_{\kappa-2}$						θ_A	θ_B																	θ_C	$\theta_D+\theta_E$
$\psi_{\kappa-1}$							$\theta_B+\theta_C$	θ_D																	$\theta_A+\theta_E$
ψ_κ						θ_A	$\theta_B+\theta_C$																		$\theta_D+\theta_E$
OOC																									1

$$\boldsymbol{ARL}(\delta)=(\mathbf{I}-\mathbf{Q}(\delta))^{-1}\mathbf{1}=\begin{pmatrix}\phi=\varpi_1(\delta)\\ \varpi_2(\delta)\\ \varpi_3(\delta)\\ \varpi_4(\delta)\\ \vdots\\ \varpi_{H-2}(\delta)\\ \varpi_{H-1}(\delta)\\ \varpi_H(\delta)\\ \varpi_{H+1}(\delta)\end{pmatrix}=\frac{1}{1-\theta_O(1+\theta_U\theta_O^{H-1})}\begin{pmatrix}1+\theta_U\sum_{i=0}^{H-1}\theta_O^i\\ 1\\ 1+\theta_U\sum_{i=H-1}^{H-1}\theta_O^i\\ 1+\theta_U\sum_{i=H-2}^{H-1}\theta_O^i\\ \vdots\\ 1+\theta_U\sum_{i=3}^{H-1}\theta_O^i\\ 1+\theta_U\sum_{i=2}^{H-1}\theta_O^i\\ 1+\theta_U\sum_{i=1}^{H-1}\theta_O^i\end{pmatrix}, \quad (3)$$

where $\mathbf{Q}_{(H+1)\times(H+1)}$ is given in Table III, Panel (a).

Secondly, for the SC2 / IRR2 scheme, the procedure is explained in Remark 2.

Remark 2: An *ARL* vector of the SC2 / IRR2 scheme

The closed-form expression of the $\boldsymbol{ARL}_{(M\times1)}$ of the SC2 / IRR2 scheme (if it exists) is not as easily apparent as that of the other schemes. This is mainly caused by the fact that the dimension of the other schemes increase in a linear manner, whereas that of the SC2 / IRR2 scheme increases in a quadratic manner; see Equation (2) as well as Shongwe and Graham[3]'s Table AI. Thus, for the purpose of this study, we assume that the $\boldsymbol{ARL}_{(M\times1)}$ is given by

$$\boldsymbol{ARL}(\delta)=\left(\mathbf{I}-\mathbf{Q}_{(H^2+2H+1)\times(H^2+2H+1)}\right)^{-1}\mathbf{1}$$

where $\mathbf{Q}_{(H^2+2H+1)\times(H^2+2H+1)}$ is given in Table III, Panel (b) and keeping in mind Remark 1.

Thirdly, for the SC3 / IRR3 scheme, $\mathfrak{D}(\delta)=1-\theta_O-\theta_A\theta_O^H-\theta_D\theta_O^H-\theta_A\theta_D\sum_{i=0}^{2H-1}\theta_O^i$ and $\mathbf{Q}_{(3H+1)\times(3H+1)}$ is given in Table III, Panel (c) and taking into account Remark 1,

$$\boldsymbol{ARL}(\delta) = \left(\mathbf{I} - \mathbf{Q}_{(3H+1)\times(3H+1)}\right)^{-1}\mathbf{1} = \begin{pmatrix} \varpi_1(\delta) \\ \varpi_2(\delta) \\ \vdots \\ \varpi_{H-3}(\delta) \\ \varpi_{H-2}(\delta) \\ \varpi_{H-1}(\delta) \\ \varpi_H(\delta) \\ \phi = \varpi_{H+1}(\delta) \\ \varpi_{H+2}(\delta) \\ \varpi_{H+3}(\delta) \\ \varpi_{H+4}(\delta) \\ \varpi_{H+5}(\delta) \\ \vdots \\ \varpi_{2H}(\delta) \\ \varpi_{2H+1}(\delta) \\ \psi_1 = \varpi_{2H+2}(\delta) \\ \varpi_{2H+3}(\delta) \\ \varpi_{2H+4}(\delta) \\ \varpi_{2H+5}(\delta) \\ \vdots \\ \varpi_{3H}(\delta) \\ \varpi_{3H+1}(\delta) \end{pmatrix} \tag{4}$$

$$
= \frac{1}{\mathfrak{D}(\delta)} \begin{pmatrix}
\left(1+\theta_A\theta_O\sum_{i=0}^{H-2}\theta_O^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_O^i\right) \\
\left(1+\theta_A\theta_O^2\sum_{i=0}^{H-3}\theta_O^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_O^i\right) \\
\vdots \\
\left(1+\theta_A\theta_O^{H-3}\sum_{i=0}^{2}\theta_O^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_O^i\right) \\
\left(1+\theta_A\theta_O^{H-2}\sum_{i=0}^{1}\theta_O^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_O^i\right) \\
(1+\theta_A\theta_O^{H-1})\left(1+\theta_D\sum_{i=0}^{H-1}\theta_O^i\right) \\
1+\theta_D\sum_{i=0}^{H-1}\theta_O^i \\
\left(1+\theta_A\sum_{i=0}^{H-1}\theta_O^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_O^i\right) \\
1+\theta_A\sum_{i=0}^{H-1}\theta_O^i \\
(1+\theta_D\theta_O^{H-1})\left(1+\theta_A\sum_{i=0}^{H-1}\theta_O^i\right) \\
\left(1+\theta_D\theta_O^{H-2}\sum_{i=0}^{1}\theta_O^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_O^i\right) \\
\left(1+\theta_D\theta_O^{H-3}\sum_{i=0}^{2}\theta_O^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_O^i\right) \\
\vdots \\
\left(1+\theta_D\theta_O^2\sum_{i=0}^{H-3}\theta_O^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_O^i\right) \\
\left(1+\theta_D\theta_O\sum_{i=0}^{H-2}\theta_O^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_O^i\right) \\
1-\theta_A\theta_D\left(\sum_{i=0}^{H-1}\theta_O^i\right)^2 \\
1+\theta_A\theta_O^{H-1}+\theta_D\theta_O^{H-1}-\theta_A\theta_D\left[1+\theta_O\left(2+\theta_O\left(3+\theta_O\left(\ldots\left((H-2)+\theta_O\left((H-1)+\theta_O\left((H-2)+\theta_O\left(\ldots\left(3+\theta_O(2+\theta_O-\theta_O^3)\right)\ldots\right)\right)\right)\right)\ldots\right)\right)\right)\right] \\
1+\theta_A\theta_O^{H-2}\sum_{i=0}^{1}\theta_O^i+\theta_D\theta_O^{H-2}\sum_{i=0}^{1}\theta_O^i+\theta_A\theta_D\left(\sum_{i=H-2}^{H-1}\theta_O^i-\sum_{i=0}^{H-3}\theta_O^i\right)\left(\sum_{i=0}^{H-1}\theta_O^i\right) \\
1+\theta_A\theta_O^{H-3}\sum_{i=0}^{2}\theta_O^i+\theta_D\theta_O^{H-3}\sum_{i=0}^{2}\theta_O^i+\theta_A\theta_D\left(\sum_{i=H-3}^{H-1}\theta_O^i-\sum_{i=0}^{H-4}\theta_O^i\right)\left(\sum_{i=0}^{H-1}\theta_O^i\right) \\
\vdots \\
1+\theta_A\theta_O^2\sum_{i=0}^{H-3}\theta_O^i+\theta_D\theta_O^2\sum_{i=0}^{H-3}\theta_O^i+\theta_A\theta_D\left(\sum_{i=2}^{H-1}\theta_O^i-\sum_{i=0}^{1}\theta_O^i\right)\left(\sum_{i=0}^{H-1}\theta_O^i\right) \\
1+\theta_A\theta_O\sum_{i=0}^{H-2}\theta_O^i+\theta_D\theta_O\sum_{i=0}^{H-2}\theta_O^i+\theta_A\theta_D\left(\sum_{i=1}^{H-1}\theta_O^i-\sum_{i=0}^{0}\theta_O^i\right)\left(\sum_{i=0}^{H-1}\theta_O^i\right)
\end{pmatrix}.
$$

Finally, for the SC4 / IRR4 scheme, $\mathbf{Q}_{(4H)\times(4H)}$ is given in Table III, Panel (d) and taking into account Remark 1, with

$$
\mathfrak{D}(\delta) = 1-\theta_A\left(\theta_C+\theta_D+\theta_D\theta_C\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_B\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_C-\theta_A\theta_B^H\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_D\theta_C^H-\theta_A\sum_{j=1}^{H-1}\theta_B^j\left(\theta_C+\theta_D+\theta_D\theta_C\sum_{i=0}^{H-1}\theta_C^i\right),
$$

$$\boldsymbol{RL}(\delta)=\left(\mathbf{I}-\mathbf{Q}_{(4H)\times(4H)}\right)^{-1}\mathbf{1}=\begin{pmatrix}\varpi_1(\delta)\\ \varpi_2(\delta)\\ \vdots\\ \varpi_{H-3}(\delta)\\ \varpi_{H-2}(\delta)\\ \varpi_{H-1}(\delta)\\ \varpi_{H}(\delta)\\ \phi=\varpi_{H+1}(\delta)\\ \varpi_{H+2}(\delta)\\ \varpi_{H+3}(\delta)\\ \varpi_{H+4}(\delta)\\ \varpi_{H+5}(\delta)\\ \vdots\\ \varpi_{2H}(\delta)\\ \varpi_{2H+1}(\delta)\\ \psi_1=\varpi_{2H+2}(\delta)\\ \varpi_{2H+3}(\delta)\\ \varpi_{2H+4}(\delta)\\ \varpi_{2H+5}(\delta)\\ \varpi_{2H+6}(\delta)\\ \varpi_{2H+7}(\delta)\\ \varpi_{2H+8}(\delta)\\ \vdots\\ \varpi_{4H-3}(\delta)\\ \varpi_{4H-2}(\delta)\\ \varpi_{4H-1}(\delta)\\ \varpi_{4H}(\delta)\end{pmatrix}=\frac{1}{\mathfrak{D}(\delta)}\begin{pmatrix}\left(1+\theta_A\theta_B\sum_{i=0}^{H-2}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \left(1+\theta_A\theta_B^2\sum_{i=0}^{H-3}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \vdots\\ \left(1+\theta_A\theta_B^{H-3}\sum_{i=0}^{2}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \left(1+\theta_A\theta_B^{H-2}\sum_{i=0}^{1}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ (1+\theta_A\theta_B^{H-1})\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ 1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\\ \left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ 1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\\ (1+\theta_D\theta_C^{H-1})\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \left(1+\theta_D\theta_C^{H-2}\sum_{i=0}^{1}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \left(1+\theta_D\theta_C^{H-3}\sum_{i=0}^{2}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \vdots\\ \left(1+\theta_D\theta_C^{2}\sum_{i=0}^{H-3}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \left(1+\theta_D\theta_C\sum_{i=0}^{H-2}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ 1-\theta_A\theta_D\left(\sum_{i=0}^{H-1}\theta_B^i\right)\left(\sum_{i=0}^{H-1}\theta_C^i\right)\\ (1+\theta_A\theta_B^{H-1})\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ (1+\theta_D\theta_C^{H-1})\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \left(1+\theta_A\theta_B^{H-2}\sum_{i=0}^{1}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \left(1+\theta_D\theta_C^{H-2}\sum_{i=0}^{1}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \left(1+\theta_A\theta_B^{H-3}\sum_{i=0}^{2}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \left(1+\theta_D\theta_C^{H-3}\sum_{i=0}^{2}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \vdots\\ \left(1+\theta_A\theta_B^{2}\sum_{i=0}^{H-3}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \left(1+\theta_D\theta_C^{2}\sum_{i=0}^{H-3}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\\ \left(1+\theta_A\theta_B\sum_{i=0}^{H-2}\theta_B^i\right)\left(1+\theta_D\sum_{i=0}^{H-1}\theta_C^i\right)\\ \left(1+\theta_D\theta_C\sum_{i=0}^{H-2}\theta_C^i\right)\left(1+\theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\end{pmatrix}. \tag{5}$$

4. Zero-state mode

The *ZSARL* is the product of the initial probability vector(s) in Shongwe and Graham[3]'s Equation (A.7) and the $\boldsymbol{ARL}$s in Equations (3), (4) and (5), that is, $ZSARL = \mathbf{q} \cdot \boldsymbol{ARL}(\delta)$ yields the closed-form expressions given in Table IV.

Remark 3: *ZSARL* for the SC2 / IRR2 schemes

From Remark 2, it follows that since the closed-form expression of the *ARL* vector for the SC2 / IRR2 scheme is not easily obtainable (if it exists), the closed-form expression of the *ZSARL* is also not easily obtainable. Therefore, to obtain the *ZSARL* values of the SC2 / IRR2 schemes, for any *H*, it may be computed directly using some software by using the standard Markov chain equation,

$$ZSARL = \mathbf{q} \cdot \left(\mathbf{I} - \mathbf{Q}_{((H^2+2H+1)\times(H^2+2H+1))}\right)^{-1} \mathbf{1}.$$

Next, using the *ZSARL* expressions, the *ZSEQL* expressions are given in Table V – where δ_{min} and δ_{max} denote the lower and upper bounds on the range of shift values, respectively.

5. Steady-state mode

There are three exact methods that are mostly used in the SPCM literature to compute the steady-state probability vector (SSPV) and these are as follows:

(i) Crosier[10]'s cyclic steady-state method – denoted by SSPV1

Crosier[10]'s cyclic steady-state method is a two-stage procedure used when a TPM is not ergodic. Firstly, we need to compute $\mathbf{P}^*$ from the $\mathbf{P}$ in Shongwe and Graham[3]'s Equation (A1). The matrix $\mathbf{P}^*$ is obtained by altering $\mathbf{P}$ so that the control statistic is reset to the '*initial state*' whenever it goes into an '*OOC state*'. That is, the component with the value one on the last row of the TPM is altered so that the value of one is moved to the respective initial state, i.e.

$$\begin{cases} \eta_1 & \text{for IRR1, SC1} \\ \eta_{\frac{(\tau+1)}{2}} & \text{for IRR2, IRR3, IRR4, SC2, SC3, SC4.} \end{cases}$$

That is, for SC1 / IRR1

$$\mathbf{P} = \begin{array}{c|ccccc} & \phi & \eta_2 & \cdots & \eta_\tau & \text{OOC} \\ \hline \phi & & & & & \\ \eta_2 & & & \mathbf{Q}_{\tau\times\tau} & & \mathbf{r}_{\tau\times 1} \\ \vdots & & & & & \\ \eta_\tau & & & & & \\ \text{OOC} & 0 & 0 & \cdots & 0 & 1 \end{array} \quad \text{and} \quad \mathbf{P}^* = \begin{array}{c|ccccc} & \phi & \eta_2 & \cdots & \eta_\tau & \text{OOC} \\ \hline \phi & & & & & \\ \eta_2 & & & \mathbf{Q}_{\tau\times\tau} & & \mathbf{r}_{\tau\times 1} \\ \vdots & & & & & \\ \eta_\tau & & & & & \\ \text{OOC} & 1 & 0 & \cdots & 0 & 0 \end{array}$$

Table IV: The *ZSARL* expressions of the *1-of-1* or *2-of-(H+1)* improved runs-rules and Synthetic-$\bar{X}$ charts

IRR1:	$\frac{{}_U \sum_{i=0}^{H-1} {}_O^{i}}{- {}_O(1 \quad {}_U {}_O^{H-1})}$
IRR2:	See Remark 3
IRR3:	$\frac{(1 \quad {}_A \sum_{i=0}^{H-1} {}_O^{i})(1 \quad {}_D \sum_{i=0}^{H-1} {}_O^{i})}{- {}_O - {}_A {}_O^{H} - {}_D {}_O^{H} - {}_A {}_D \sum_{i=0}^{2H-1} {}_O^{i}}$
IRR4:	$\frac{(1 \quad {}_A \sum_{i=0}^{H-1} {}_B^{i})(1 \quad {}_D \sum_{i=0}^{H-1} {}_C^{i})}{- {}_A(\theta_C \quad {}_D \quad {}_D \; {}_C \sum_{i=0}^{H-1} {}_C^{i}) - {}_B(1 \quad {}_D \sum_{i=0}^{H-1} {}_C^{i}) - {}_C - {}_A {}_B^{H}(1 \quad {}_D \sum_{i=0}^{H-1} {}_C^{i}) - {}_D {}_C^{H} - {}_A \sum_{j=1}^{H-1} {}_B^{j}(\theta_C \quad {}_D \quad {}_D \; {}_C \sum_{i=0}^{H-1} {}_C^{i})}$
SC1:	$\frac{}{- {}_O(1 \quad {}_U {}_O^{H-1})}$
SC2:	See Remark 3
SC3:	$\frac{- {}_A {}_D(\sum_{i=0}^{H-1} {}_O^{i})^2}{- {}_O - {}_A {}_O^{H} - {}_D {}_O^{H} - {}_A {}_D \sum_{i=0}^{2H-1} {}_O^{i}}$
SC4:	$\frac{- {}_A {}_D(\sum_{i=0}^{H-1} {}_B^{i})(\sum_{i=0}^{H-1} {}_C^{i})}{- {}_A(\theta_C \quad {}_D \quad {}_D \; {}_C \sum_{i=0}^{H-1} {}_C^{i}) - {}_B(1 \quad {}_D \sum_{i=0}^{H-1} {}_C^{i}) - {}_C - {}_A {}_B^{H}(1 \quad {}_D \sum_{i=0}^{H-1} {}_C^{i}) - {}_D {}_C^{H} - {}_A \sum_{j=1}^{H-1} {}_B^{j}(\theta_C \quad {}_D \quad {}_D \; {}_C \sum_{i=0}^{H-1} {}_C^{i})}$

Table V: The *ZSEQL* expressions of the *1-of-1* or *2-of-*(*H+1*) improved runs-rules and Synthetic-$\bar{X}$ charts

IRR1:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{1 \quad \theta_U\sum_{i=0}^{H-1}\theta_O^i}{1-\theta_O(1 \quad \theta_U\theta_O^{H-1})}\right)d\delta$
IRR2:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\mathbf{q}\cdot\left(\mathbf{I}-\mathbf{Q}_{((H^2+H+1)\times(H^2+H+1))}\right)^{-1}\mathbf{1}\right)d\delta$
IRR3:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{\left(1 \quad \theta_A\sum_{i=0}^{H-1}\theta_O^i\right)\left(1 \quad \theta_D\sum_{i=0}^{H-1}\theta_O^i\right)}{1-\theta_O-\theta_A\theta_O^H-\theta_D\theta_O^H-\theta_A\theta_D\sum_{i=0}^{2H-1}\theta_O^i}\right)d\delta$
IRR4:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{\left(1 \quad \theta_A\sum_{i=0}^{H-1}\theta_B^i\right)\left(1 \quad \theta_D\sum_{i=0}^{H-1}\theta_C^i\right)}{1-\theta_A\left(\theta_C \quad \theta_D \quad \theta_D\theta_C\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_B\left(1 \quad \theta_D\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_C-\theta_A\theta_B^H\left(1 \quad \theta_D\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_D\theta_C^H-\theta_A\sum_{j=1}^{H-1}\theta_B^j\left(\theta_C \quad \theta_D \quad \theta_D\theta_C\sum_{i=0}^{H-1}\theta_C^i\right)}\right)d\delta$
SC1:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{1}{1-\theta_O(1 \quad \theta_U\theta_O^{H-1})}\right)d\delta$
SC2:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\mathbf{q}\cdot\left(\mathbf{I}-\mathbf{Q}_{((H^2+2H+1)\times(H^2+2H+1))}\right)^{-1}\mathbf{1}\right)d\delta$
SC3:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{1-\theta_A\theta_D\left(\sum_{i=0}^{H-1}\theta_O^i\right)^2}{1-\theta_O-\theta_A\theta_O^H-\theta_D\theta_O^H-\theta_A\theta_D\sum_{i=0}^{2H-1}\theta_O^i}\right)d\delta$
SC4:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{1-\theta_A\theta_D\left(\sum_{i=0}^{H-1}\theta_B^i\right)\left(\sum_{i=0}^{H-1}\theta_C^i\right)}{1-\theta_A\left(\theta_C \quad \theta_D \quad \theta_D\theta_C\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_B\left(1 \quad \theta_D\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_C-\theta_A\theta_B^H\left(1 \quad \theta_D\sum_{i=0}^{H-1}\theta_C^i\right)-\theta_D\theta_C^H-\theta_A\sum_{j=1}^{H-1}\theta_B^j\left(\theta_C \quad \theta_D \quad \theta_D\theta_C\sum_{i=0}^{H-1}\theta_C^i\right)}\right)d\delta$

whereas for SC2, SC3, SC4, IRR2, IRR3, IRR4

	η_1	...	$\eta_{\left(\frac{\tau+1}{2}\right)-1}$	ϕ	$\eta_{\left(\frac{\tau+1}{2}\right)+1}$	...	η_τ	OOC
η_1								
$\vdots$								
$\eta_{\left(\frac{\tau+1}{2}\right)-1}$								
ϕ				$\mathbf{Q}_{\tau\times\tau}$				$\mathbf{r}_{\tau\times 1}$
$\eta_{\left(\frac{\tau+1}{2}\right)+1}$								
$\vdots$								
η_τ								
OOC	0	...	0	0	0	...	0	1

and

*

	η_1	...	$\eta_{\left(\frac{\tau+1}{2}\right)-1}$	ϕ	$\eta_{\left(\frac{\tau+1}{2}\right)+1}$	...	η_τ	OOC
η_1								
$\vdots$								
$\eta_{\left(\frac{\tau+1}{2}\right)-1}$								
ϕ				$\mathbf{Q}_{\tau\times\tau}$				$\mathbf{r}_{\tau\times 1}$
$\eta_{\left(\frac{\tau+1}{2}\right)+1}$								
$\vdots$								
η_τ								
OOC	0	...	0	1	0	...	0	0

Once * has been determined, we use it to find the (τ+1)×1 probability vector **p** such that the following equation is satisfied

$$\mathbf{p} \quad {}^{*\prime}\mathbf{p} \quad \text{subject to} \quad \mathbf{1}'\mathbf{p} \quad 1.$$

Secondly, the SSPV1 is given by

$$\mathbf{s} \quad (\mathbf{1}'\mathbf{z})^{-1}\cdot\mathbf{z} \tag{6}$$

where **z** is the τ×1 vector obtained from **p** by deleting the (τ+1)$^{\text{th}}$ component associated with the absorbing state.

(ii) Champ[11]'s simplified steady-state method – denoted by SSPV2

Champ[11] stated that the Crosier[10]'s cyclic steady-state method may also be calculated using Equation (6), however with

$$\mathbf{z}_{(\tau\times 1)} \quad (\mathbf{G}-\mathbf{Q}')^{-1}\cdot\mathbf{u},$$
$$\mathbf{u}_{(\tau\times 1)} \quad (1\ \ 0\ \ 0\ldots 0)^{T},$$

$$\mathbf{G}_{(\tau\times\tau)} \quad \begin{pmatrix} 2 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(iii) Adjusted version of Crosier[10]'s cyclic steady-state method – denoted by SSPV3

A number of authors in the literature prefer to use the adjusted version of Crosier[10]'s method. The SSPV3 is obtained by dividing each element of $\mathbf{Q}(\delta = 0)$ (i.e. while the process is IC) by its corresponding row sum, so that the '*new*' $\mathbf{Q}$ is called the conditional essential TPM, denoted by $\mathbf{Q}_C$. That is, $\mathbf{Q}_C$ is the altered version of $\mathbf{Q}$ so that the '*new*' essential TPM is ergodic. Thus, the SSPV3 is a vector such that

$$\mathbf{s}\cdot\mathbf{Q}_C \quad \mathbf{s} \text{ subject to } \textstyle\sum_{i=1}^{\tau} s_i = 1.$$

In Tables VI, VII, VIII and IX, we give steady-state initial probability vectors for each scheme when H = 1, 2 and 3. Since the process must be IC when calculating $\mathbf{s}$, we let $\theta_1 = \theta_A = \theta_D$, $\gamma \quad \theta_B \quad \theta_C \quad \frac{1}{2}\theta_O$, $c_0 \quad \frac{\theta_U}{\theta_O+\theta_U}$, $c_1 \quad \frac{\theta_O}{\theta_O+\theta_1}$, $c_2 \quad \frac{\theta_1}{\theta_O+\theta_1}$, $c_3 \quad \frac{\theta_O}{\theta_O+2\theta_1}$, $c_5 \quad \frac{\gamma}{2\gamma+\theta_1}$ and c_7 $\frac{2\gamma}{2\gamma+2\theta_1}$.

These three SSPV methods were checked for a large number of H values that cannot be displayed here due to space limitation, and noticed that, although they empirically yield different initial probability vectors, the resulting empirical *SSARL*s are approximately equal. Hence, for the sake of uniformity, we opted to use SSPV3 to formulate the general form of the steady-state initial probability vectors. Note though, any of these methods would have been equally fine. Machado and Costa[9] also computed $\mathbf{s}$ using the SSPV3 method for the SC3 scheme.

Thus, for the SC1 / IRR1 schemes

$$\mathbf{s}_{(1\times\tau)} \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{H-1} \\ s_H \\ s_{H+1} \end{pmatrix}' \quad \frac{1}{1+Hc_0}\begin{pmatrix} 1 \\ c_0 \\ c_0 \\ \vdots \\ c_0 \\ c_0 \\ c_0 \end{pmatrix}'.$$

Table VI: The steady-state initial probability vectors of the SC1 / IRR1 scheme when *H* =1, 2, 3

H		SSPV1	SSPV2	SSPV3
1	$\begin{pmatrix} 1 \\ s_2 \end{pmatrix}'$	$\frac{1}{1+\theta_U}\begin{pmatrix} 1 \\ \theta_U \end{pmatrix}'$	$\frac{1}{2-\theta_O+\theta_U(1-\theta_O)}\begin{pmatrix} 1 \\ \theta_U \end{pmatrix}'$	$\frac{1}{1+c_0}\begin{pmatrix} 1 \\ c_0 \end{pmatrix}'$
2	$\begin{pmatrix} 1 \\ s_2 \\ 3 \end{pmatrix}'$	$\frac{1}{1+\theta_U\sum_{i=0}^{1}\theta_O^i}\begin{pmatrix} 1 \\ \theta_U \\ \theta_U\theta_O \end{pmatrix}'$	$\frac{1}{2-\theta_O+\theta_U(\sum_{i=0}^{1}\theta_O^i-\theta_O^2)}\begin{pmatrix} 1 \\ \theta_U \\ \theta_U\theta_O \end{pmatrix}'$	$\frac{1}{1+2c_0}\begin{pmatrix} 1 \\ c_0 \\ c_0 \end{pmatrix}'$
3	$\begin{pmatrix} 1 \\ 2 \\ s_3 \\ 4 \end{pmatrix}'$	$\frac{1}{1+\theta_U\sum_{i=0}^{2}\theta_O^i}\begin{pmatrix} 1 \\ \theta_U \\ \theta_U\theta_O \\ \theta_U\theta_O^2 \end{pmatrix}'$	$\frac{1}{2-\theta_O+\theta_U(\sum_{i=0}^{2}\theta_O^i-\theta_O^3)}\begin{pmatrix} 1 \\ \theta_U \\ \theta_U\theta_O \\ \theta_U\theta_O^2 \end{pmatrix}'$	$\frac{1}{1+3c_0}\begin{pmatrix} 1 \\ c_0 \\ c_0 \\ c_0 \end{pmatrix}'$

Table VII: The steady-state initial probability vectors of the SC2 / IRR2 scheme when H =1, 2, 3

H		SSPV1	SSPV2	SSPV3
1	$\begin{pmatrix}1\\ s_2\\ 3\end{pmatrix}'$	$\frac{1}{1+\theta_1}\begin{pmatrix}\theta_1\\ 1-\theta_1\\ \theta_1\end{pmatrix}'$	$\frac{1}{(1+\theta_1)(2-\theta_O-\theta_1-\theta_O\theta_1)}\begin{pmatrix}1-\theta_O-\theta_O\theta_1\\ \theta_O(1+\theta_1)\\ \theta_1\end{pmatrix}'$	$\frac{1}{2+2c_1(1-c_3)^{-1}}\begin{pmatrix}1\\ 2c_1(1-c_3)^{-1}\\ 1\end{pmatrix}'$
2	$\begin{pmatrix}1\\ 2\\ 3\\ 4\\ 5\\ 6\\ s_7\end{pmatrix}'$	$\frac{1}{1+(2+\theta_O)(\theta_1+\theta_1^2)}\begin{pmatrix}\theta_1^2\\ \theta_O\theta_1(1+\theta_1)\\ \theta_1\\ 1-\theta_O\theta_1-\theta_O\theta_1^2\\ \theta_1\\ \theta_O\theta_1(1+\theta_1)\\ \theta_1^2\end{pmatrix}'$	$\frac{1}{2-\theta_O(1-\theta_1+3\theta_1^2)-\theta_O^3\theta_1^2(1-\theta_1^2)-\theta_O^2\theta_1(2+\theta_1-\theta_1^3)}\begin{pmatrix}1-\theta_O^3\theta_1^2-\theta_O^2\theta_1(2+\theta_1)-\theta_O(1+\theta_1^2)\\ \theta_O^2\theta_1(1+\theta_O\theta_1)\\ \theta_O\theta_1(1-\theta_O\theta_1^2-\theta_O^2\theta_1^2)\\ \theta_O^2(1-\theta_O\theta_1(\theta_1-1))\\ \theta_O^2\theta_1(1+\theta_1+\theta_O\theta_1)\\ \theta_O(1-\theta_O-\theta_O^2\theta_1-\theta_O\theta_1^2)\\ \theta_O^2\theta_1^2(1+\theta_1+\theta_O\theta_1)\end{pmatrix}'$	$\frac{1}{4+2c_2+2c_1(1-c_3)^{-1}}\begin{pmatrix}c_2\\ 1\\ 1\\ 2c_1(1-c_3)^{-1}\\ 1\\ 1\\ c_2\end{pmatrix}'$
3	$\begin{pmatrix}1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ 11\\ 12\\ s_{13}\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}\theta_O\theta_1^2-\theta_O^2\theta_1^3\\ \theta_O\theta_1^2\\ \theta_O^2\theta_1(1+\theta_1-\theta_O\theta_1^2)\\ \theta_1^2(1-\theta_O\theta_1)\\ \theta_O\theta_1\\ \theta_1(1-\theta_O\theta_1)\\ 1-\theta_O\theta_1-\theta_O^2\theta_1-\theta_O^2\theta_1^2+\theta_O^3\theta_1^3\\ \theta_1(1-\theta_O\theta_1)\\ \theta_O\theta_1\\ \theta_1^2(1-\theta_O\theta_1)\\ \theta_O^2\theta_1(1+\theta_1-\theta_O\theta_1^2)\\ \theta_O\theta_1^2\\ \theta_O\theta_1^2-\theta_O^2\theta_1^3\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}1-\theta_O-2\theta_O^2\theta_1^2+\theta_O^6\theta_1^4-\theta_O^3\theta_1(2+\theta_1)-\theta_O^4(\theta_1^2-\theta_1^4)-\theta_O^5(\theta_1^2-\theta_1^4)\\ \theta_O^2\theta_1^2(1-\theta_O^2\theta_1(\theta_1-1)-\theta_O^3\theta_1^2)\\ \theta_O^3\theta_1(1+\theta_O\theta_1+\theta_O^2\theta_1-\theta_O^3\theta_1^3)\\ \theta_O\theta_1^2-2\theta_O^3\theta_1^4-2\theta_O^4\theta_1^4+\theta_O^6\theta_1^6-\theta_O^5(\theta_1^4-\theta_1^6)\\ \theta_O^2\theta_1(1-\theta_O^2\theta_1(\theta_1-1)-\theta_O^3\theta_1^2)\\ \theta_O\theta_1-2\theta_O^3\theta_1^6-2\theta_O^4\theta_1^3+\theta_O^6\theta_1^5-\theta_O^5(\theta_1^3-\theta_1^5)\\ (\theta_O^2-\theta_O^3\theta_1)(1+\theta_O\theta_1-\theta_O^2\theta_1(\theta_1-1)-\theta_O^3\theta_1^3)\\ \theta_O^2\theta_1(1+\theta_O\theta_1-\theta_O^2\theta_1(\theta_1-1))\\ \theta_O^3\theta_1(1+\theta_1(1+\theta_O+\theta_O^2)-\theta_O^2\theta_1^3(1+\theta_O))\\ \theta_O^2\theta_1^2(1+\theta_O\theta_1-\theta_O^2\theta_1(\theta_1-1))\\ \theta_O(1-\theta_O-\theta_O^3\theta_1-2\theta_O^2\theta_1^2+\theta_O^4\theta_1^4)\\ \theta_O^3\theta_1^2(1+\theta_1(1+\theta_O+\theta_O^2)-\theta_O^2\theta_1^3(1+\theta_O))\\ \theta_O^3\theta_1^2(1+\theta_O\theta_1-\theta_O^2\theta_1(\theta_1-1))\end{pmatrix}'$	$\frac{1}{6+6c_2+2c_1(1-c_3)^{-1}}\begin{pmatrix}c_2\\ c_2\\ 1\\ c_2\\ 1\\ 1\\ 2c_1(1-c_3)^{-1}\\ 1\\ 1\\ c_2\\ 1\\ c_2\\ c_2\end{pmatrix}'$

where for the SSPV1 at H = 3,

$$\mathcal{F} = 1+\theta_1(2+\theta_O+\theta_O^2)+\theta_1^2(2+2\theta_O+\theta_O^2)+\theta_O\theta_1^3(2+2\theta_O+\theta_O^2)$$

and for the SSPV2 at H = 3,

$$\mathcal{F} = 2+2\theta_O^6\theta_1^4-\theta_O^7\theta_1^6-\theta_O^2\theta_1(3\theta_1-2)-\theta_O(1-\theta_1-\theta_1^2)-2\theta_O^3(\theta_1+\theta_1^4)-\theta_6^5\theta_1^2(1-\theta_1^2-\theta_1^4).$$

Table VIII: The steady-state initial probability vectors of the SC3 / IRR3 scheme when H=1, 2, 3

H		SSPV1	SSPV2	SSPV3
1	$\begin{pmatrix}1\\ s_2\\ 3\end{pmatrix}'$	$\frac{1}{1+\theta_1}\begin{pmatrix}\theta_1\\ 1-\theta_1\\ \theta_1\end{pmatrix}'$	$\frac{1}{(2-\theta_O-\theta_1-\theta_O\theta_1)(1+\theta_1)}\begin{pmatrix}1-\theta_O-\theta_O\theta_1\\ \theta_O(1+\theta_1)\\ \theta_1\end{pmatrix}'$	$\frac{1}{2+2c_1(1-c_3)^{-1}}\begin{pmatrix}1\\ 2c_1(1-c_3)^{-1}\\ 1\end{pmatrix}'$
2	$\begin{pmatrix}1\\ 2\\ 3\\ 4\\ s_5\end{pmatrix}'$	$\frac{1}{1+\theta_1\sum_{i=0}^{1}\theta_O^i}\begin{pmatrix}\theta_O\theta_1\\ \theta_1\\ 1-\theta_1\sum_{i=0}^{1}\theta_O^i\\ \theta_1\\ \theta_O\theta_1\end{pmatrix}'$	$\frac{1}{\left(2-\theta_O-\theta_1\left(1-\sum_{i=1}^{1}\theta_O^i+\theta_O^2\right)\right)(1+\theta_1\sum_{i=0}^{1}\theta_O^i)}\begin{pmatrix}1-\theta_O-\theta_O^2\theta_1-\theta_1^2\\ \theta_1(\theta_O+\theta_1+\theta_O\theta_1+\theta_O^2\theta_1)\\ \theta_O(1-\theta_1)(1+\theta_1\sum_{i=0}^{1}\theta_O^i)\\ \theta_1(1+\theta_O\theta_1)\\ \theta_O\theta_1(1+\theta_O\theta_1)\end{pmatrix}'$	$\frac{1}{2+2c_1+2c_1^2(1-c_3)^{-1}}\begin{pmatrix}c_1\\ 1\\ 2c_1^2(1-c_3)^{-1}\\ 1\\ c_1\end{pmatrix}'$
3	$\begin{pmatrix}1\\ 2\\ 3\\ 4\\ 5\\ 6\\ s_7\end{pmatrix}'$	$\frac{1}{1+\theta_1\sum_{i=0}^{2}\theta_O^i}\begin{pmatrix}\theta_O^2\theta_1\\ \theta_O\theta_1\\ \theta_1\\ 1-\theta_1\sum_{i=0}^{2}\theta_O^i\\ \theta_1\\ \theta_O\theta_1\\ \theta_O^2\theta_1\end{pmatrix}'$	$\frac{1}{\left(2-\theta_O-\theta_1\left(1-\sum_{i=1}^{2}\theta_O^i+\theta_O^3\right)\right)(1+\theta_1\sum_{i=0}^{2}\theta_O^i)}\begin{pmatrix}1-\theta_O-\theta_O^3\theta_1-\theta_1^2-\theta_O\theta_1^2\\ \theta_O\theta_1(\theta_O+\theta_1+\theta_O\theta_1+\theta_O^2\theta_1+\theta_O^3\theta_1)\\ \theta_1(\theta_O+\theta_1+\theta_O\theta_1+\theta_O^2\theta_1+\theta_O^3\theta_1)\\ \theta_O(1-\theta_1\sum_{i=0}^{1}\theta_O^i)(1+\theta_1\sum_{i=0}^{2}\theta_O^i)\\ \theta_1(1+\theta_O\theta_1+\theta_O^2\theta_1)\\ \theta_O\theta_1(1+\theta_O\theta_1+\theta_O^2\theta_1)\\ \theta_O^2\theta_1(1+\theta_O\theta_1+\theta_O^2\theta_1)\end{pmatrix}'$	$\frac{1}{2\sum_{i=0}^{2}c_1^2+2c_1^3(1-c_3)^{-1}}\begin{pmatrix}c_1^2\\ c_1\\ 1\\ 2c_1^3(1-c_3)^{-1}\\ 1\\ c_1\\ c_1^2\end{pmatrix}'$

Table IX: The steady-state initial probability vectors of the SC4 / IRR4 scheme when H=1, 2, 3

H		SSPV1	SSPV2	SSPV3
1	$\begin{pmatrix}1\\ s_2\\ 3\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}\theta_1\\ 1-\theta_1\\ \theta_1\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}1-2\gamma(1+\theta_1)\\ 2\gamma(1+\theta_1)\\ \theta_1\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}1\\ 4c_5(1-c_7)^{-1}\\ 1\end{pmatrix}'$
2	$\begin{pmatrix}1\\ 2\\ 3\\ 4\\ s_5\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}\gamma\theta_1\\ \theta_1\\ 1-\theta_1\sum_{i=0}^{1}\gamma^i\\ \theta_1\\ \gamma\theta_1\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}1-\gamma(2+2\theta_1+\theta_1^2)-\gamma^2\theta_1(2+\theta_1)-\theta_1^2\\ \theta_1(2\gamma(1+\theta_1)+2\gamma^2\theta_1+\theta_1)\\ \gamma(2-\theta_1)(1+\theta_1\sum_{i=0}^{1}\gamma^i)\\ \theta_1(1+\gamma\theta_1)\\ \gamma\theta_1(1+\gamma\theta_1)\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}c_5\\ 1\\ (4c_5^2+2c_5)(1-c_7)^{-1}\\ 1\\ c_5\end{pmatrix}'$
3	$\begin{pmatrix}1\\ 2\\ 3\\ 4\\ 5\\ 6\\ s_7\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}\gamma^2\theta_1\\ \gamma\theta_1\\ \theta_1\\ 1-\theta_1\sum_{i=0}^{2}\gamma^i\\ \theta_1\\ \gamma\theta_1\\ \gamma^2\theta_1\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}1-2\gamma(1+\theta_1+\theta_1^2)-2\gamma^2\theta_1(1+\theta_1)-2\gamma^3\theta_1(1+\theta_1)-\gamma^4\theta_1^2-\theta_1^2\\ \gamma\theta_1(2\gamma(1+\theta_1)+2\gamma^2\theta_1+2\gamma^3\theta_1+\theta_1)\\ \theta_1(2\gamma(1+\theta_1)+2\gamma^2\theta_1+2\gamma^3\theta_1+\theta_1)\\ \gamma\left(2-\theta_1\sum_{i=0}^{1}\gamma^i\right)(1+\theta_1\sum_{i=0}^{2}\gamma^i)\\ \theta_1(1+\gamma\theta_1+\gamma^2\theta_1)\\ \gamma\theta_1(1+\gamma\theta_1+\gamma^2\theta_1)\\ \gamma^2\theta_1(1+\gamma\theta_1+\gamma^2\theta_1)\end{pmatrix}'$	$\frac{1}{\mathcal{F}}\begin{pmatrix}c_5^2\\ c_5\\ 1\\ \left(4c_5^3+2\sum_{i=1}^{2}c_5^i\right)(1-c_7)^{-1}\\ 1\\ c_5\\ c_5^2\end{pmatrix}'$

Table IX: (continued)

H	SSPV1	SSPV2	SSPV3
1	$\mathcal{F}=1+\theta_1$	$\mathcal{F}=\big(2-2\gamma-\theta_1(1+2\gamma)\big)(1+\theta_1)$	$\mathcal{F}=2+4c_5(1-c_7)^{-1}$
2	$\mathcal{F}=1+\theta_1\sum_{i=0}^{1}\gamma^i$	$\mathcal{F}=\big(2-2\gamma-\theta_1(1+2\gamma^2)\big)(1+\theta_1\sum_{i=0}^{1}\gamma^i)$	$\mathcal{F}=2\sum_{i=0}^{1}c_5^i+\big(4c_5^2+2c_5\big)(1-c_7)^{-1}$
3	$\mathcal{F}=1+\theta_1\sum_{i=0}^{2}\gamma^i$	$\mathcal{F}=\big(2-2\gamma-\theta_1(1+2\gamma^3)\big)(1+\theta_1\sum_{i=0}^{2}\gamma^i)$	$\mathcal{F}=2\sum_{i=0}^{2}c_5^i+\left(4c_5^3+2\sum_{i=1}^{2}c_5^i\right)(1-c_7)^{-1}$

Table X: The *SSEQL* expressions of the *1-of-1* or *2-of-*(*H+1*) improved runs-rules and Synthetic-$\bar{X}$ charts

IRR1 or SC1:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\frac{\varpi_1(\delta)}{1+Hc_0}+\frac{c_0}{1+Hc_0}\sum_{i=2}^{H+1}\varpi_i(\delta)\right)d\delta$
IRR2 or SC2:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(\mathbf{s}\cdot\left(\mathbf{I}-\mathbf{Q}_{\left((H^2+H+1)\times(H^2+H+1)\right)}(\delta)\right)^{-1}\mathbf{1}\right)d\delta$
IRR3 or SC3:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(s_{H+1}^{\#}\varpi_{H+1}^{\#}(\delta)+\sum_{i=1}^{H}s_i^{\#}\left(\varpi_i^{\#}(\delta)+\varpi_{(2H+2)-i}^{\#}(\delta)\right)\right)d\delta$
IRR4 or SC4:	$\frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}}\delta^2\times\left(s_{H+1}\varpi_{H+1}(\delta)+\sum_{i=1}^{H}s_i\left(\varpi_i(\delta)+\varpi_{(2H+2)-i}(\delta)\right)\right)d\delta$

For the SC2 / IRR2 schemes, with $\tau = H^2 + H + 1$ and using the dummy variables defined in Table III, Panel (b), it follows that

$$\mathbf{s}_{(1\times\tau)} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{a-2} \\ s_{a-1} \\ s_a \\ s_{a+1} \\ s_{a+2} \\ \vdots \\ s_{b-1} \\ s_b \\ s_{b+1} \\ \vdots \\ s_{c-1} \\ s_c \\ \vdots \\ \vdots \\ \vdots \\ s_{l-11} \\ s_{l-10} \\ s_{l-9} \\ s_{l-8} \\ s_{l-7} \\ s_{l-6} \\ s_{l-5} \\ s_{l-4} \\ s_{l-3} \\ s_{l-2} \\ s_{l-1} \\ s_l \\ s_{l+1} \\ s_{l+2} \\ s_{l+3} \\ s_{l+4} \\ s_{l+5} \\ s_{l+6} \\ s_{l+7} \\ s_{l+8} \\ s_{l+9} \\ s_{l+10} \\ s_{l+11} \\ \vdots \\ \vdots \\ \vdots \\ s_x \\ s_{x+1} \\ \vdots \\ s_{y-1} \\ s_y \\ s_{y+1} \\ \vdots \\ s_{z-2} \\ s_{z-1} \\ s_z \\ s_{z+1} \\ s_{z+2} \\ \vdots \\ s_{\tau-1} \\ s_\tau \end{pmatrix}' = \frac{1}{2\left(H + c_2 \sum_{i=1}^{H-1}(H-i) + c_1(1-c_3)^{-1}\right)} \begin{pmatrix} c_2 \\ c_2 \\ \vdots \\ c_2 \\ c_2 \\ 1 \\ c_2 \\ c_2 \\ \vdots \\ c_2 \\ 1 \\ c_2 \\ \vdots \\ c_2 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ c_2 \\ c_2 \\ c_2 \\ 1 \\ c_2 \\ c_2 \\ 1 \\ c_2 \\ 1 \\ 1 \\ 2c_1(1-c_3)^{-1} \\ 1 \\ 1 \\ c_2 \\ 1 \\ c_2 \\ c_2 \\ 1 \\ c_2 \\ c_2 \\ c_2 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ c_2 \\ \vdots \\ c_2 \\ 1 \\ c_2 \\ \vdots \\ c_2 \\ c_2 \\ 1 \\ c_2 \\ c_2 \\ \vdots \\ c_2 \\ c_2 \end{pmatrix}'.$$

For the SC3 / IRR3 schemes,

$$\mathbf{s}_{(1\times\tau)} \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{H-2} \\ s_{H-1} \\ s_H \\ s_{H+1} \\ s_{H+2} \\ s_{H+3} \\ s_{H+4} \\ \vdots \\ s_{2H-1} \\ s_{2H} \\ s_{2H+1} \end{pmatrix}' \quad \frac{1}{2\sum_{i=0}^{H-1} c_1^i + 2c_1^H(1-c_3)^{-1}} \begin{pmatrix} c_1^{H-1} \\ c_1^{H-2} \\ c_1^{H-3} \\ \vdots \\ c_1^2 \\ c_1 \\ 1 \\ 2c_1^H(1-c_3)^{-1} \\ 1 \\ c_1 \\ c_1^2 \\ \vdots \\ c_1^{H-3} \\ c_1^{H-2} \\ c_1^{H-1} \end{pmatrix}'.$$

For the SC4 / IRR4 schemes,

$$\mathbf{s}_{(1\times\tau)} \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{H-2} \\ s_{H-1} \\ s_H \\ s_{H+1} \\ s_{H+2} \\ s_{H+3} \\ s_{H+4} \\ \vdots \\ s_{2H-1} \\ s_{2H} \\ s_{2H+1} \end{pmatrix}' \quad \frac{1}{2\sum_{i=0}^{H-1} c_5^i + \left(2\sum_{i=1}^{H-1} c_5^i + 4c_5^H\right)(1-c_7)^{-1}} \begin{pmatrix} c_5^{H-1} \\ c_5^{H-2} \\ c_5^{H-3} \\ \vdots \\ c_5^2 \\ c_5 \\ 1 \\ \left(2\sum_{i=1}^{H-1} c_5^i + 4c_5^H\right)(1-c_7)^{-1} \\ 1 \\ c_5 \\ c_5^2 \\ \vdots \\ c_5^{H-3} \\ c_5^{H-2} \\ c_5^{H-1} \end{pmatrix}'.$$

Using the general form of the steady-state initial probability vectors given above and the ***ARL*** in Section 3, the following *SSARL* expressions are obtained:

(i) For the SC1 / IRR1 scheme,

$$SSARL(\delta) \quad \mathbf{s}\cdot\boldsymbol{ARL}(\delta) \quad s_1\varpi_1(\delta) + s_2\varpi_2(\delta) + \cdots + s_{H+1}\varpi_{H+1}(\delta)$$

$$\frac{\varpi_1(\delta)}{1+Hc_0} + \frac{c_0}{1+Hc_0}\sum_{i=2}^{H+1}\varpi_i(\delta)\,;$$

(ii) For the SC2 / IRR2 scheme,

$$SSARL(\delta) \quad \mathbf{s}\cdot\boldsymbol{ARL}(\delta) \quad \mathbf{s}\cdot\left(\mathbf{I}-\mathbf{Q}_{\left((H^2+H+1)\times(H^2+H+1)\right)}(\delta)\right)^{-1}\mathbf{1}$$

and when δ 0,

$$SSARL(0) = \mathbf{s}\cdot\boldsymbol{ARL}(0) = \mathbf{s}\cdot\left(\mathbf{I}-\mathbf{Q}_{\left((H^2+H+1)\times(H^2+H+1)\right)}(0)\right)^{-1}\mathbf{1},$$

$\mathbf{Q}_{\left((H^2+H+1)\times(H^2+H+1)\right)}(0)$ greatly simplifies the computations as the resulting closed-form equations are slightly more neat than those from the latter;

(iii) For the SC3 / IRR3 scheme,

$$SSARL(\delta) \quad \mathbf{s}\cdot\boldsymbol{ARL}(\delta) \quad s_{H+1}^{\#}\varpi_{H+1}^{\#}(\delta)+\sum_{i=1}^{H}s_i^{\#}\left(\varpi_i^{\#}(\delta)+\varpi_{(2H+2)-i}^{\#}(\delta)\right)$$

and when δ 0,

$$SSARL(0) \quad \mathbf{s}\cdot\boldsymbol{ARL}(0) \quad s_{H+1}^{\#}\varpi_{H+1}^{\#}(0)+2\sum_{i=1}^{H}s_i^{\#}\varpi_i^{\#}(0)$$

where $\boldsymbol{ARL}(0)$ is the $\boldsymbol{ARL}(\delta)$ with θ_A θ_D, and we substitute δ 0 in Shongwe and Graham[3]'s Equation (A2). To avoid confusion between the SC3 / IRR3 and SC4 / IRR4 expressions, as they have similar structure but different values, from here onwards, we put the hash tag (i.e. '#') for the SC3 / IRR3's *SSARL* expression; and

(iv) For the SC4 / IRR4 scheme,

$$SSARL(\delta) \quad \mathbf{s}\cdot\boldsymbol{ARL}(\delta) \quad s_{H+1}\varpi_{H+1}(\delta)+\sum_{i=1}^{H}s_i\left(\varpi_i(\delta)+\varpi_{(2H+2)-i}(\delta)\right)$$

and when δ 0,

$$SSARL(0) \quad \mathbf{s}\cdot\boldsymbol{ARL}(0) \quad s_{H+1}\varpi_{H+1}(0)+2\sum_{i=1}^{H}s_i\varpi_i(0)$$

where $\boldsymbol{ARL}(0)$ is the $\boldsymbol{ARL}(\delta)$ with θ_A θ_D, θ_B θ_C, and δ 0 in Shongwe and Graham[3]'s Equation (A2). This equality greatly simplifies the computations.

Next, using the *SSARL* expressions, the *SSEQL* expressions are as given in Table X.

[Insert Table X]

Note that although we were unable to compute the closed-form expressions for the SC2 / IRR2 scheme – see Remarks 2 and 3, it is not really worth the effort because the specific shift and overall OOC performance of the SC2 / IRR2 scheme and that of the SC3 / IRR3 scheme are approximately equal throughout, however, with that of the SC3 / IRR3 scheme, being just marginally better than that of the SC2 / IRR2 scheme.

6. Concluding remarks

The main objective of this discussion was to give the general form of the TPMs of the schemes in Tables I and II, then derive their corresponding easy-to-use closed-form expressions to calculate the zero-state and steady-state *ARL*s rather than to use the slightly more complicated Equation (1) which requires a relatively complex statistical or mathematical software to evaluate. The expressions derived here are very important as some authors are not really familiar with Markov chain, hence these will be of great help.

Note though, the equations derived here only hold for a normal distribution, however, these may be modified for other types of monitoring schemes. We intend to extend these general expressions for other synthetic-type and runs-type monitoring schemes in the future.

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