Analysis and dynamically consistent nonstandard discretization for a rabies model in humans and dogs

M. Chapwanya, J. M. -S. Lubuma, and Y. A. Terefe

Department of Mathematics and Applied Mathematics,

University of Pretoria

Abstract

Rabies is a fatal disease in dogs as well as in humans. A possible model to represent rabies transmission dynamics in human and dog populations is presented. The next generation matrix operator is used to determine the threshold parameter $\mathcal{R}_0$, that is the average number of new infective individuals produced by one infective individual introduced into a completely susceptible population. If $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable, while it is unstable and there exists a locally asymptotically stable endemic equilibrium when $\mathcal{R}_0 > 1$. A nonstandard finite difference scheme that replicates the dynamics of the continuous model is proposed. Numerical tests to support the theoretical analysis are provided.

1 Introduction

Rabies, also known as hydrophobia of human, is an acute disease that progresses to coma or death after the first symptom appearers by attacking the central nervous system of the victim. It is a viral disease that mainly affects carnivores and bats, although it can affect any mammals. Rabies transmits from animals to animals or animals to humans through bites or scratches [10]. The most common wild animals to transmit rabies are raccoons, skunks, bats, foxes and coyotes while domestic animals such as cats, cattle and dogs can also transmit rabies [9].

A major observation during infection is a change in personality of the rabid animal such as approach humans without fear or hesitation. Many rabid animals are incapable of swallowing their saliva. The incubation period, the time between exposure to the rabies virus and the development of sign varies from one week to one year.

The primary route of humans infection with the rabies is through a bite from a rabid dog. Rabies transmission is also possible when a claw scratch or mucosal surface (e.g. mouse, nose, eye) is contaminated with the saliva from a rabid dog. Rabies is responsible for the death of 50,000 to 60,000 people annually although effective vaccines for pre and post-exposed individuals are available. More than 99 percent of these death occur in the developing countries where the disease is endemic in domestic dog population. There is no treatment after clinical symptoms are observed [3, 10].

Domestic dogs in most African countries are the major source of rabies disease for humans. Understanding the dynamics of rabies between humans and domestic dogs would therefore important to design a more effective control strategy that could reduce human deaths. In the controlling strategies of the disease, mathematical models play significant roles.

A model to study the spread of rabies among foxes is given in [17]. Here, we develop a model to study the rabies transmission between human and dog populations. In each population of our study, we have susceptible (S), exposed (E), and infectious (I) compartments. In the human population, we have additional compartment denoted by (V) containing indi-

viduals removed by vaccination.

The proposed model, like most mathematical models in applied sciences, cannot be solved by analytical techniques. Here, we design a nonstandard finite difference (NSFD) scheme to approximate the solution of the continuous model. The scheme is constructed based on two Mickens' rules [15] and as singled out in [2]. To the best of our knowledge, this is the first time to present the rabies model between human and dog populations with its dynamically consistent numerical approximation.

This article is organized as follows. The model is formulated in Section 2. Equilibrium points and their stability analysis are given in Section 3. In Section 4, a dynamically consistent NSFD scheme is designed and its numerical simulations are illustrated in Section 5. Concluding remarks on our findings and possible future research directions are indicated in the last Section.

2   Model formulation

To formulate the model, we make the following assumptions:

1. there is no transmission of the disease from rabid humans to susceptible humans;

2. there is no contacts between rabid humans and susceptible dogs;

3. the contacts among susceptible and exposed humans, and among susceptible and exposed dogs don't give rise to infection;

4. there is no vaccination for susceptible dogs before and after exposed to the disease.

Based on these assumptions, the dynamics of the disease between humans and dogs is given in Fig 1.
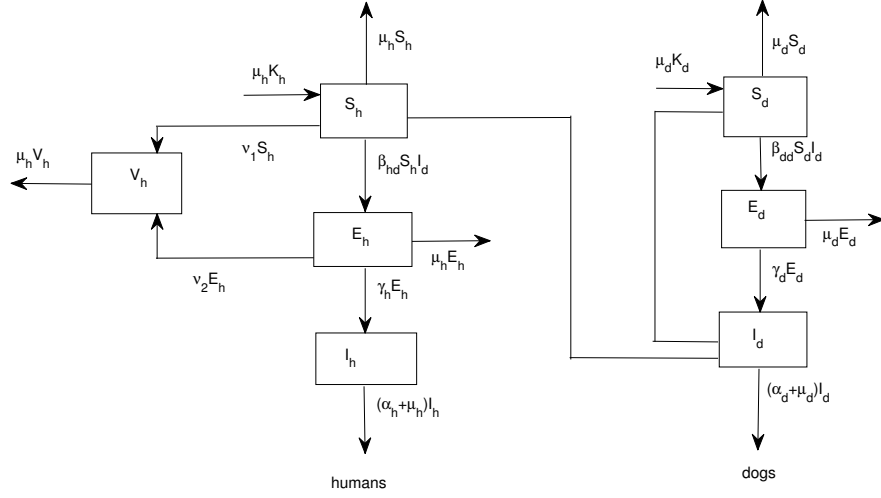
Figure 1: Flow chart

Using system of differential equations, the flow chart given in Fig 1 reads as follows:

$$S_h' = \mu_h K_h - \beta_{hd} S_h I_d - \mu_h S_h - \nu_1 S_h \tag{2.1}$$

$$E_h' = \beta_{hd} S_h I_d - (\mu_h + \gamma_h + \nu_2) E_h \tag{2.2}$$

$$I_h' = \gamma_h E_h - (\alpha_h + \mu_h) I_h \tag{2.3}$$

$$V_h' = \nu_1 S_h + \nu_2 E_h - \mu_h V_h \tag{2.4}$$

$$S_d' = \mu_d K_d - \beta_{dd} S_d I_d - \mu_d S_d \tag{2.5}$$

$$E_d' = \beta_{dd} S_d I_d - (\mu_d + \gamma_d) E_d \tag{2.6}$$

$$I_d' = \gamma_d E_d - (\alpha_d + \mu_d) I_d, \tag{2.7}$$

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where all parameters are nonnegative and the description of these parameters are given in Table 1. Moreover, $N_h = S_h + E_h + I_h + V_h$ and $N_d = S_d + E_d + I_d$. Furthermore N_h and N_d satisfy the equations

$$N_h' = \mu_h K_h - \mu_h N_h - \alpha_h I_h \tag{2.8}$$

and

$$N_d' = \mu_d K_d - \mu_d N_d - \alpha_d I_d, \tag{2.9}$$

respectively. Nonnegative solutions N_h for (2.8) and N_d for (2.9) satisfy the conservation laws

$$\mu_h K_h - (\mu_h + \alpha_h) N_h \leq N_h' \leq \mu_h (K_h - N_h) \tag{2.10}$$

and

$$\mu_d K_d - (\mu_d + \alpha_d) N_d \leq N_d' \leq \mu_d (K_d - N_d), \tag{2.11}$$

respectively [16]. By applying Gronwall inequality for differential equations [18], from above and below of (2.10) and (2.11), we get for $t \geq 0$,

$$\frac{\mu_h K_h}{\mu_h + \alpha_h} - \left(\frac{\mu_h K_h}{\mu_h + \alpha_h} - N_h^0 \right) e^{-(\mu_h + \alpha_h)t} \leq N_h(t) \leq K_h - (K_h - N_h^0) e^{-\mu_h t}$$

and

$$\frac{\mu_d K_d}{\mu_d + \alpha_d} - \left(\frac{\mu_d K_d}{\mu_d + \alpha_d} - N_d^0 \right) e^{-(\mu_d + \alpha_d)t} \leq N_d(t) \leq K_d - (K_d - N_d^0) e^{-\mu_d t},$$

respectively. Thus for $0 \leq N_h^0 \leq K_h$ and $0 \leq N_d^0 \leq K_d$, we get respectively,

$$0 \leq N_h(t) \leq K_h \quad \text{and} \quad 0 \leq N_d(t) \leq K_d.$$

Theorem 2.1 *The epidemiological model* $(2.1) - (2.7)$ *defines a dynamical system on the biologically feasible region*

$$\begin{aligned} \Omega \;=\; & \{(S_h, E_h, I_h, V_h, S_d, E_d, I_d) \in \mathbb{R}_+^7 : 0 \leq S_h + E_h + I_h + V_h = N_h \leq K_h, \\ & 0 \leq S_d + E_d + I_d = N_d \leq K_d\}. \end{aligned}$$

Parameters and variables	Description	Parameters and variables	Description
K_h	human carrying capacity	S_h	susceptible human
E_h	exposed human	I_h	Infectious human
V_h	vaccinated human pre-exposed and post-exposed	μ_h	birth or natural death human
β_{hd}	contact rate between S_h and I_d	ν_1	rate of vaccination for S_h
γ_h	rate of transfer from E_h to I_h	ν_2	rate of vaccination for E_h
α_h	death rate of human due to rabies	K_d	dog carrying capacity
S_d	susceptible dog	E_d	exposed dog
I_d	infectious dog	μ_d	birth or natural death rate of dogs
β_{dd}	contact rate between S_d and I_d	γ_d	rate of transfer from E_d to I_d
α_d	death rate of dogs induced by rabies		

Table 1: Description of variables and parameters.

Proof: The theorem can be proved by showing that Ω is positively invariant. Thus for any initial data in Ω, no trajectory will leave Ω by crossing through one of its faces (see [5]).

3 Equilibrium points and their stability analysis

To determine the equilibrium points of $(2.1) - (2.7)$, we set

$$\mu_h K_h - \beta_{hd} S_h I_d - \mu_h S_h - \nu_1 S_h = 0 \tag{3.1}$$

$$\beta_{hd} S_h I_d - (\mu_h + \gamma_h + \nu_2) E_h = 0 \tag{3.2}$$

$$\gamma_h E_h - (\alpha_h + \mu_h) I_h = 0 \tag{3.3}$$

$$\nu_1 S_h + \nu_2 E_h - \mu_h V_h = 0 \tag{3.4}$$

$$\mu_d K_d - \beta_{dd} S_d I_d - \mu_d S_d = 0 \tag{3.5}$$

$$\beta_{dd} S_d I_d - (\mu_d + \gamma_d) E_d = 0 \tag{3.6}$$

$$\gamma_d E_d - (\alpha_d + \mu_d) I_d = 0. \tag{3.7}$$

From (3.7), we obtain

$$E_d = \frac{(\alpha_d + \mu_d)I_d}{\gamma_d}. \tag{3.8}$$

Inserting Equation (3.8) into Equation (3.6), we get

$$\beta_{dd}S_dI_d - \frac{(\gamma_d + \mu_d)(\alpha_d + \mu_d)I_d}{\gamma_d} = 0. \tag{3.9}$$

This gives $I_d = 0$ or $\beta_{dd}S_d - \frac{(\gamma_d+\mu_d)(\alpha_d+\mu_d)}{\gamma_d} = 0$.

If $I_d = 0$, then

$$E_0 \equiv (S_h, E_h, I_h, V_h, S_d, E_d, I_d) = \left(\frac{\mu_h K_h}{\mu_h + \nu_1}, \; 0, \; 0, \; \frac{\nu_1 K_h}{\mu_h + \nu_1}, \; K_d, \; 0, \; 0 \right)$$

is the disease-free equilibrium obtained by using appropriate substitutions in the subsequent equations.

To find the basic reproduction number which is very important in the qualitative analysis of the model, we use the method of next generation matrix discussed in [6, 20]. For this, we consider the vector function

$$F = \begin{pmatrix} \beta_{hd}S_hI_d \\ 0 \\ \beta_{dd}S_dI_d \\ 0 \end{pmatrix},$$

which is the rate at which disease compartments increase in size, but the vector function

$$V = \begin{pmatrix} (\gamma_h + \mu_h + \nu_2)E_h \\ -\gamma_h E_h + (\alpha_h + \mu_h)I_h \\ (\mu_d + \gamma_d)E_d \\ -\gamma_d E_d + (\alpha_d + \mu_d)I_d \end{pmatrix}$$

is the rate at which the disease compartments decrease due to the disease. By using Diekmann et al [6] and van Driessche et al [20], the next generation matrix is given by

$$K = J_F(J_V)^{-1},$$

where

$$J_F = \begin{pmatrix} 0 & 0 & 0 & \frac{\beta_{hd}\mu_h K_h}{\mu_h + \nu_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_{dd}K_d \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the Jacobian matrix of F at E_0 and

$$J_V = \begin{pmatrix} \gamma_h + \mu_h + \nu_2 & 0 & 0 & 0 \\ -\gamma_h & \alpha_h + \mu_h & 0 & 0 \\ 0 & 0 & \gamma_d + \mu_d & 0 \\ 0 & 0 & -\gamma_d & \alpha_d + \mu_d \end{pmatrix}$$

is the Jacobian of V at E_0. The basic reproduction number denoted by $\mathcal{R}_0$ is defined as the expected number of secondary cases produced in a completely susceptible population by a typical infected individual during its entire period of infectious [4]. Numerically, it is defined as the spectral radius of K. That is $\mathcal{R}_0 = \rho(K)$ and explicitly,

$$\mathcal{R}_0 = \frac{\gamma_d \beta_{dd} K_d}{(\gamma_d + \mu_d)(\alpha_d + \mu_d)}. \tag{3.10}$$

Remark 3.1 *In Equation (3.10), we can see that $\mathcal{R}_0$ is completely defined in terms of the parameters of the dog population. In the eradication strategy of the disease, the action to decrease the value of $\mathcal{R}_0$ is advised to considered with respect to these parameters.*

If $I_d > 0$,   from (3.9)

$$S_d = \frac{(\gamma_d + \mu_d)(\alpha_d + \mu_d)}{\gamma_d \beta_{dd}} \qquad \text{or} \qquad S_d = \frac{K_d}{\mathcal{R}_0}.$$

Then by back substitution,

$$E_\infty = (S_h^*, E_h^*, I_h^*, V_h^*, S_d^*, E_d^*, I_d^*)$$

is the endemic equilibrium, where

$$
\begin{aligned}
S_h^* &= \frac{\mu_h \beta_{dd} K_h}{\mu_d \beta_{hd} (\mathcal{R}_0 - 1) + (\mu_h + \nu_1) \beta_{dd}}, \\
E_h^* &= \frac{\mu_d \mu_h \beta_{hd} K_h (\mathcal{R}_0 - 1)}{(\mu_h + \gamma_h + \nu_2) [\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd}]}, \\
I_h^* &= \frac{\mu_d \mu_h \gamma_h \beta_{hd} K_h (\mathcal{R}_0 - 1)}{(\alpha_h + \mu_h)(\mu_h + \gamma_h + \nu_2)[\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd}]}, \\
V_h^* &= \frac{K_h}{\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd}} \left(\nu_1 \beta_{dd} + \frac{\nu_2 \mu_d \beta_{hd}(\mathcal{R}_0 - 1)}{\mu_h + \gamma_h + \nu_2} \right),
\end{aligned}
$$

$$S_d^* = \frac{K_d}{\mathcal{R}_0}, \qquad E_d^* = \frac{\mu_d(\alpha_d + \mu_d)(\mathcal{R}_0 - 1)}{\gamma_d \beta_{dd}}, \qquad \text{and} \qquad I_d^* = \frac{\mu_d}{\beta_{dd}}(\mathcal{R}_0 - 1).$$

For $\mathcal{R}_0 > 1$, we obtain unique endemic equilibrium.

Theorem 3.2 *The disease-free equilibrium point E_0 is globally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.*

Proof: To show that E_0 is globally asymptotically stable, we use the LaSalle Invariance Principle [12].

Let $E(t) = (S_h(t), E_h(t), I_h(t), V_d(t), S_d(t), E_d(t), I_d(t)) \in \mathbb{R}_+^7$ for $t > 0$, then we consider a function defined by

$$
\begin{aligned}
U(E) &= \gamma_d E_d + (\mu_d + \gamma_d) I_d \\
\dot{U}(E) &= \gamma_d E_d' + (\mu_d + \gamma_d) I_d' \\
&= \gamma_d \beta_{dd} S_d I_d - (\mu_d + \gamma_d)(\alpha_d + \mu_d) I_d \\
&\leq (\gamma_d \beta_{dd} K_d - (\mu_d + \gamma_d)(\alpha_d + \mu_d)) I_d \\
&\leq (\mu_d + \gamma_d)(\alpha_d + \mu_d)(\mathcal{R}_0 - 1) I_d \\
&< 0
\end{aligned}
$$

for $\mathcal{R}_0 < 1$ and $E \neq E_0$. Hence, U is a positive definite Lyapunov function and E_0 is locally asymptotically stable for $\mathcal{R}_0 < 1$ while unstable for $\mathcal{R}_0 > 1$. By using the procedure mention in [12, 19], it is not difficult to show that the set $A = \{E_0\}$ is the maximum invariant set in $\mathcal{M} = \{E : \dot{U} = 0\}$ for $\mathcal{R}_0 < 1$. This completes the proof. $\square$

Remark 3.3 *By using the Hartman-Grobman theorem [18], we can prove that E_0 is hyperbolic, locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.*

For $I_d > 0$ and $\mathcal{R}_0 > 1$, the stability property of E_∞ is given in the next theorem.

Theorem 3.4 *For $\mathcal{R}_0 > 1$, the endemic equilibrium point E_∞ is locally asymptotically stable.*

Proof: To prove this theorem, we use Hartman-Grobman theorem [18]. The Jacobian matrix of the right-side of $(2.1) - (2.7)$ at E_∞ is

$$
\begin{pmatrix}
-(\beta_{hd}I_d^* + \mu_h + \nu_1) & 0 & 0 & 0 & 0 & 0 & -\beta_{hd}S_h^* \\
\beta_{hd}I_d^* & -(\mu_h + \gamma_h + \nu_2) & 0 & 0 & 0 & 0 & \beta_{hd}S_h^* \\
0 & \gamma_h & -(\alpha_h + \mu_h) & 0 & 0 & 0 & 0 \\
\nu_1 & \nu_2 & 0 & -\mu_h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(\beta_{dd}I_d^* + \mu_d) & 0 & -\beta_{dd}S_d^* \\
0 & 0 & 0 & 0 & \beta_{dd}I_d^* & -(\mu_d + \gamma_d) & \beta_{dd}S_d^* \\
0 & 0 & 0 & 0 & 0 & \gamma_d & -(\alpha_d + \mu_d)
\end{pmatrix}.
$$

Clearly, the trace of this matrix is negative. We need to show that all eigenvalues of $J(E_\infty)$ have negative real parts. For this, let r be an eigenvalue of $J(E_\infty)$ and the characteristic equation is

$$det\left(rI - J(E_\infty)\right) = 0,$$

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which leads into the equation

$$[r + (\beta_{hd}I_d^* + \mu_h + \nu_1)] (r + (\mu_h + \gamma_h + \nu_2)) [r + \mu_h]\{[r + (\beta_{dd}I_d^* + \mu_d)](r + (\mu_d + \gamma_d))[r + (\alpha_d + \mu_d)]$$
$$- (r + \mu_d)\gamma_d\beta_{dd}S_d^*\} = 0. \tag{3.11}$$

All real parts of r in Equation (3.11) are negative if and only if all r in

$$[r + (\beta_{dd}I_d^* + \mu_d)](r + (\mu_d + \gamma_d))[r + (\alpha_d + \mu_d)] - (r + \mu_d)\gamma_d\beta_{dd}S_d^* = 0 \tag{3.12}$$

have negative real parts. By using values of I_d^* and S_d^* with $\mathcal{R}_0$ in (3.10), Equation (3.12) can be written as

$$r^3 + a_2r^2 + a_1r + a_0 = 0, \tag{3.13}$$

where,

$$
\begin{aligned}
a_2 &= \alpha_d + \mu_d(\mathcal{R}_0 + 2) + \gamma_d \\
a_1 &= \mu_d\mathcal{R}_0(\alpha_d + 2\mu_d + \gamma_d) \\
a_0 &= \mu_d(\mu_d + \gamma_d)(\alpha_d + \mu_d)(\mathcal{R}_0 - 1).
\end{aligned}
$$

Clearly $a_2 > 0$, because the parameters in the expression are positive. It is also possible to show that $a_2a_1 > a_0 > 0$ for $\mathcal{R}_0 > 1$. Then by Routh-Hurwitz condition [8], all real parts of r in (3.13) are negative. Hence, the endemic equilibrium point, E_∞ is locally asymptotically stable for $\mathcal{R}_0 > 1$. $\quad\square$.

4 Nonstandard finite difference scheme

In this section, we want to design a nonstandard finite difference scheme for $(2.1) - (2.7)$ that preserves all properties we discussed in the continuous model. For this numerical approximation, we replace the continuous time $t \in [0, \infty)$ with $t_n = n\Delta t$, $n = 0, 1, 2, ...$, where Δt is the time step-size. Thus, we are interested in finding approximate solutions

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S_h^n, E_h^n, I_h^n, V_h^n, S_d^n, E_d^n, I_d^n of S_h, E_h, I_h, V_h, S_d, E_d, I_d at time $t = t_n$, respectively. Having these in mind, we propose the following NSFD scheme:

$$\frac{S_h^{n+1} - S_h^n}{\phi(\Delta t)} = \mu_h K_h - \beta_{hd} S_h^{n+1} I_d^n - (\mu_h + \nu_1) S_h^{n+1} \tag{4.1}$$

$$\frac{E_h^{n+1} - E_h^n}{\phi(\Delta t)} = \beta_{hd} S_h^{n+1} I_d^n - (\mu_h + \gamma_h + \nu_2) E_h^{n+1} \tag{4.2}$$

$$\frac{I_h^{n+1} - I_h^n}{\phi(\Delta t)} = \gamma_h E_h^{n+1} - (\alpha_h + \mu_h) I_h^{n+1} \tag{4.3}$$

$$\frac{V_h^{n+1} - V_h^n}{\phi(\Delta t)} = \nu_1 S_h^{n+1} + \nu_2 E_h^{n+1} - \mu_h V_h^{n+1} \tag{4.4}$$

$$\frac{S_d^{n+1} - S_d^n}{\phi(\Delta t)} = \mu_d K_d - \beta_{dd} S_d^{n+1} I_d^n - \mu_d S_d^{n+1} \tag{4.5}$$

$$\frac{E_d^{n+1} - E_d^n}{\phi(\Delta t)} = \beta_{dd} S_d^{n+1} I_d^n - (\mu_d + \gamma_d) E_d^{n+1} \tag{4.6}$$

$$\frac{I_d^{n+1} - I_d^n}{\phi(\Delta t)} = \gamma_d E_d^{n+1} - (\alpha_d + \mu_d) I_d^{n+1}, \tag{4.7}$$

where the function $\phi(\Delta t)$ is obtained from the exact solutions of the conservation laws (2.8) and (2.9) to satisfies the condition

$$\phi \equiv \phi(\Delta t) = \Delta t + \mathcal{O}((\Delta t)^2). \tag{4.8}$$

In the approximation, positive constants are left as positive constants. From Equations (2.10) and (2.11), by using Grownwall inequality for difference equations, we choose

$$\phi(\Delta t) = \frac{e^{\mu \Delta t} - 1}{\mu} \quad or \quad \phi(\Delta t) = \frac{e^{(\mu + \alpha)\Delta t} - 1}{\mu + \alpha},$$

respectively, where $\mu = \max\{\mu_h, \mu_d\}$ and $\alpha = \max\{\alpha_h, \alpha_d\}$ (see [15, 19]). These functions satisfy Equation (4.8).

Remark 4.1 *Note that in the constructed NSFD scheme, two Mickens rules [15] are reinforced as highlighted in [2]: the nonlinear terms are approximated in a nonlocal way and the usual standard denominator Δt of the discrete derivative is replaced by the complex denominator function $\phi(\Delta t)$.*

By adding $(4.1) - (4.4)$ and $(4.5) - (4.7)$, we obtain

$$\frac{N_h^{n+1} - N_h^n}{\phi(\Delta t)} = \mu_h K_h - \mu_h N_h^{n+1} - \alpha_h I_h^{n+1} \tag{4.9}$$

and

$$\frac{N_d^{n+1} - N_d^n}{\phi(\Delta t)} = \mu_d K_d - \mu_d N_d^{n+1} - \alpha_d I_d^{n+1}, \tag{4.10}$$

respectively. From (4.9) and (4.10), the inequalities

$$\mu_h K_h - (\mu_h + \alpha_h) N_h^{n+1} \leq \frac{N_h^{n+1} - N_h^n}{\phi(\Delta t)} \leq \mu_h K_h - \mu_h N_h^{n+1} \tag{4.11}$$

and

$$\mu_d K_d - (\mu_d + \alpha_h) N_d^{n+1} \leq \frac{N_d^{n+1} - N_d^n}{\phi(\Delta t)} \leq \mu_d K_d - \mu_d N_d^{n+1} \tag{4.12}$$

are obtained, respectively. By using the Gronwall inequality for discrete dynamical systems [18], solutions for (4.9) and (4.10) are positive and bounded. More precisely, $0 \leq N_h^0 \leq K_h$ and $0 \leq N_d^0 \leq K_d$, lead into the inequalities $0 \leq N_h^n \leq K_h$ and $0 \leq N_d^n \leq K_d$, respectively for $n \in \mathbb{N}$.

Theorem 4.2 *The NSFD scheme* $(4.1) - (4.7)$ *defines a discrete dynamical system on the biologically feasible domain*

$$\begin{aligned}
\Omega \ = \ & \{(S_h, E_h, I_h, V_h, S_d, E_d, I_d) \in \mathbb{R}^7 : S_h, E_h, I_h, V_h, S_d, E_d, I_d \geq 0, \\
& N_h, N_d > 0, \ 0 \leq S_h + E_h + I_h + V_h = N_h \leq K_h, \ 0 \leq S_d + E_d + I_d = N_d \leq K_d \}.
\end{aligned}$$

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For further computational purpose, we rearrange the system $(4.1) - (4.7)$ into

$$S_h^{n+1} = \frac{\mu_h \phi(\Delta t) K_h + S_h^n}{1 + (\beta_{hd} I_d^n + \mu_h + \nu_1)\phi(\Delta t)} \tag{4.13}$$

$$E_h^{n+1} = \frac{\beta_{hd}\phi(\Delta t) S_h^{n+1} I_d^n + E_h^n}{1 + (\mu_h + \gamma_h + \nu_2)\phi(\Delta t)} \tag{4.14}$$

$$I_h^{n+1} = \frac{\gamma_h \phi(\Delta t) E_h^{n+1} + I_h^n}{1 + (\alpha_h + \mu_h)\phi(\Delta t)} \tag{4.15}$$

$$V_h^{n+1} = \frac{(\nu_1 S_h^{n+1} + \nu_2 E_h^{n+1})\phi(\Delta t) + V_h^n}{1 + \mu_h \phi(\Delta t)} \tag{4.16}$$

$$S_d^{n+1} = \frac{\mu_d \phi(\Delta t) K_d + S_d^n}{1 + (\beta_{dd} I_d^n + \mu_d)\phi(\Delta t)} \tag{4.17}$$

$$E_d^{n+1} = \frac{\beta_{dd}\phi(\Delta t) S_d^{n+1} I_d^n + E_d^n}{1 + (\mu_d + \gamma_d)\phi(\Delta t)} \tag{4.18}$$

$$I_d^{n+1} = \frac{\gamma_d \phi(\Delta t) E_d^{n+1} + I_d^n}{1 + (\alpha_d + \mu_d)\phi(\Delta t)} \tag{4.19}$$

From Equations $(4.13) - (4.19)$, we can see that the NSFD scheme $(4.1) - (4.7)$ gives positive solutions for any positive initial data. Moreover, the algorithm runs by following the Gauss-Seidel cycle on $(4.13) - (4.19)$.

To determine the fixed points, we set $(4.13) - (4.19)$ into the following form:

$$\frac{\mu_h \phi(\Delta t) K_h + S_h}{1 + (\beta_{hd} I_d + \mu_h + \nu_1)\phi(\Delta t)} = S_h \tag{4.20}$$

$$\frac{\beta_{hd}\phi(\Delta t) S_h I_d + E_h}{1 + (\mu_h + \gamma_h + \nu_2)\phi(\Delta t)} = E_h \tag{4.21}$$

$$\frac{\gamma_h \phi(\Delta t) E_h + I_h}{1 + (\alpha_h + \mu_h)\phi(\Delta t)} = I_h \tag{4.22}$$

$$\frac{(\nu_1 S_h + \nu_2 E_h)\phi(\Delta t) + V_h}{1 + \mu_h \phi(\Delta t)} = V_h \tag{4.23}$$

$$\frac{\mu_d \phi(\Delta t) K_d + S_d}{1 + (\beta_{dd} I_d + \mu_d)\phi(\Delta t)} = S_d \tag{4.24}$$

$$\frac{\beta_{dd}\phi(\Delta t) S_d I_d + E_d}{1 + (\mu_d + \gamma_d)\phi(\Delta t)} = E_d \tag{4.25}$$

$$\frac{\gamma_d \phi(\Delta t) E_d + I_d}{1 + (\alpha_d + \mu_d)\phi(\Delta t)} = I_d. \tag{4.26}$$

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From (4.26), we get

$$E_d = \frac{(\alpha_d + \mu_d)I_d}{\gamma_d}.$$ (4.27)

Plugging (4.27) into (4.25), we obtain

$$\beta_{dd}S_d I_d = \frac{(\alpha_d + \mu_d)(\mu_d + \gamma_d)I_d}{\gamma_d}.$$ (4.28)

From (4.28), we get

$$I_d = 0 \ \text{ or } \ S_d = \frac{(\alpha_d + \mu_d)(\mu_d + \gamma_d)}{\gamma_d \beta_{dd}} = \frac{K_d}{\mathcal{R}_0^2}.$$

If $I_d = 0$, then

$$E_0 = \left(\frac{\mu_h K_h}{\mu_h + \nu_1}, 0, 0, \frac{\nu_1 K_h}{\mu_h + \nu_1}, K_d, 0, 0 \right)$$

is the fixed point associated with the disease-free equilibrium point. This confirms that the scheme preserves the disease-free equilibrium point.

If $I_d > 0$, by using (3.10) and (4.20) $-$ (4.28), $E_\infty = (S_h^*, E_h^*, I_h^*, V_h^*, S_d^*, E_d^*, I_d^*)$ is the endemic fixed point for $\mathcal{R}_0 > 1$, where

$$S_d^* = \frac{K_d}{\mathcal{R}_0},$$

$$I_d^* = \frac{\mu_d(\mathcal{R}_0 - 1)}{\beta_{dd}},$$

$$E_d^* = \frac{\mu_d(\alpha_d + \mu_d)(\mathcal{R}_0 - 1)}{\gamma_d \beta_{dd}},$$

$$S_h^* = \frac{\mu_h \beta_{dd} K_h}{\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd}},$$

$$E_h^* = \frac{\mu_d \mu_h \beta_{hd} K_h(\mathcal{R}_0 - 1)}{(\mu_h + \gamma_h + \nu_2)(\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd})},$$

$$I_h^* = \frac{\mu_d \mu_h \gamma_h \beta_{hd} K_h(\mathcal{R}_0 - 1)}{(\alpha_h + \mu_h)(\mu_h + \gamma_h + \nu_2)(\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd})},$$

$$V_h^* = \frac{K_h}{\mu_d \beta_{hd}(\mathcal{R}_0 - 1) + (\mu_h + \nu_1)\beta_{dd}} \left(\nu_1 \beta_{dd} + \frac{\nu_2 \mu_d \beta_{hd}(\mathcal{R}_0 - 1)}{\mu_h + \gamma_h + \nu_2} \right).$$

Hence, the NSFD scheme given in (4.1) $-$ (4.7) also preserves the endemic equilibrium point. In the coming two theorems, we will check the stability properties of the fixed points.

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Theorem 4.3 *The disease-free fixed point E_0 of $(4.1) - (4.7)$ is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof: The Jacobian matrix of $(4.13) - (4.19)$ at E_0 is given by

$$
\begin{pmatrix}
\frac{1}{1+(\mu_h+\nu_1)\phi} & 0 & 0 & 0 & 0 & 0 & x \\[2mm]
0 & \frac{1}{1+(\mu_h+\gamma_h+\nu_2)\phi} & 0 & 0 & 0 & 0 & y \\[2mm]
0 & z & \frac{1}{1+(\alpha_h+\mu_h)\phi} & 0 & 0 & 0 & 0 \\[2mm]
\frac{\nu_1}{1+\mu_h\phi}\frac{\phi}{1+(\mu_h+\nu_1)\phi} & \frac{\nu_2}{1+\mu_h\phi}\frac{\phi}{1+(\mu_h+\gamma_h+\nu_2)\phi} & 0 & \frac{1}{1+\mu_h\phi} & 0 & 0 & 0 \\[2mm]
0 & 0 & 0 & 0 & \frac{1}{1+\mu_d\phi} & 0 & -\frac{\beta_{dd}K_d\phi}{1+\mu_d\phi} \\[2mm]
0 & 0 & 0 & 0 & 0 & \frac{1}{1+(\mu_d+\gamma_d)\phi} & \frac{\beta_{dd}K_d\phi}{1+(\mu_d+\gamma_d)\phi} \\[2mm]
0 & 0 & 0 & 0 & 0 & \frac{\gamma_d\phi}{1+(\alpha_d+\mu_d)\phi}\frac{1}{1+(\mu_d+\gamma_d)\phi} & \frac{1}{1+(\mu_d+\gamma_d)\phi}
\end{pmatrix},
$$

where

$$
\begin{aligned}
x &= \frac{-\mu_h\beta_{hd}K_h\phi}{(1+(\mu_h+\nu_1)\phi)(\mu_h+\nu_1)} \\[2mm]
y &= \frac{\mu_h\beta_{hd}K_h\phi}{(1+(\mu_h+\gamma_h+\nu_2)\phi)(\mu_h+\nu_1)} \\[2mm]
z &= \frac{\gamma_h\phi}{1+(\alpha_h+\mu_h)\phi}\frac{1}{1+(\mu_h+\gamma_h+\nu_2)\phi}
\end{aligned}
$$

The corresponding characteristic equation is

$$
det\left(rI - J(E_0)\right) = 0.
$$

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This leads into

$$\left(r - \frac{1}{1 + (\mu_h + \nu_1)\phi}\right)\left(r - \frac{1}{1 + (\mu_h + \gamma_h + \nu_2)\phi}\right)\left(r - \frac{1}{1 + (\alpha_h + \mu_h)\phi}\right)$$
$$\left(r - \frac{1}{1 + \mu_h\phi}\right)\left(r - \frac{1}{1 + \mu_d\phi}\right)\left(r^2 + a_1 r + a_0\right) = 0, \qquad (4.29)$$

where

$$a_1 = -\left(\frac{1}{1 + (\mu_d + \gamma_d)\phi} + \frac{1}{1 + (\alpha_d + \mu_d)\phi}\right)$$

$$a_0 = \frac{1}{(1 + (\mu_d + \gamma_d)\phi)(1 + (\alpha_d + \mu_d)\phi)}\left(1 - \frac{\mathcal{R}_0^2(\mu_d + \gamma_d)(\alpha_d + \mu_d)\phi^2}{1 + (\mu_d + \gamma_d)\phi}\right).$$

The stability properties of E_0 depends on the roots of

$$r^2 + a_1 r + a_0 = 0.$$

By using [13], the roots lie inside the unit circle if and only if $|a_0| < 1$ and $|a_1| < 1 + a_0$. Clearly $|a_0| < 1$ and $|a_1| < 1 + a_0$ if $\mathcal{R}_0 < 1$. Hence, E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. $\quad\square$.

Theorem 4.4 *The disease-free fixed point E_0 is globally attractor if $\mathcal{R}_0 < 1$ and $\frac{\mu_d + \gamma_d}{\beta_{dd}} \leq 1$.*

Proof: From Equation (4.18), we have

$$\begin{aligned}
I_d^{n+1} &= \frac{\gamma_d\phi E_d^{n+1} + I_d^n}{1 + (\alpha_d + \mu_d)\phi} \\
&\leq \frac{(\gamma_d\phi K_d + 1)I_d^n}{1 + (\alpha_d + \mu_d)\phi} \\
&= \left(\frac{\frac{(\alpha_d + \mu_d)(\mu_d + \gamma_d)\mathcal{R}_0\phi}{\beta_{dd}} + 1}{1 + (\alpha_d + \mu_d)\phi}\right) I_d^n.
\end{aligned}$$

Thus, $I_d^{n+1} \leq D I_d^n$, where

$$D = \frac{\frac{(\alpha_d + \mu_d)(\mu_d + \gamma_d)\mathcal{R}_0^2\phi}{\beta_{dd}} + 1}{1 + (\alpha_d + \mu_d)\phi}.$$

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For $\mathcal{R}_0 < 1$ and $\frac{\mu_d+\gamma_d}{\beta_{dd}} \leq 1,\ \ 0 < D < 1$, the sequence $(I_d^n)_{n\geq 0}$ tends to zero for any initial value $0 \leq I_d^0 \leq K_d$. Using this sequence in Equations $(4.12) - (4.17)$, we can show that if
$$E_n = (S_h^n, E_h^n, I_h^n, V_h^n, S_d^n, E_d^n, I_d^n),$$
$$\lim_{n\to\infty} E_n = E_0.$$
This implies that E_0 is globally attractor. $\square$.

Theorem 4.5 *The disease-free fixed point E_0 is globally asymptotically stable for $\mathcal{R}_0 < 1$ and* $\frac{\mu_d+\gamma_d}{\beta_{dd}} \leq 1$.

Proof: This is a result obtained from Theorem 4.2 and Theorem 4.3.

Theorem 4.6 *The endemic fixed point E_∞ of $(4.1) - (4.7)$ is locally asymptotically stable for $\mathcal{R}_0 > 1$.*

Proof: To prove this theorem, we need to show that all roots of the characteristic polynomial related to the Jacobian matrix of $(4.12) - (4.18)$ at E_∞ lie inside the unit circle. To show this, we have the Jacobian matrix at E_∞ and read as

$$
\begin{pmatrix}
b & 0 & 0 & 0 & 0 & 0 & c \\
d & \frac{1}{1+(\mu_h+\gamma_h+\nu_2)\phi} & 0 & 0 & 0 & & e \\
0 & f & \frac{1}{1+(\alpha_h+\mu_h)\phi} & 0 & 0 & 0 & 0 \\
g & h & 0 & \frac{1}{1+\mu_h\phi} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{1+(\beta_{dd}I_d^*+\mu_d)\phi} & 0 & -\frac{(\mu_d\phi K_d+S_d^*)\beta_{dd}\phi}{(1+(\beta_{dd}I_d^*+\mu_d)\phi)^2} \\
0 & 0 & 0 & \frac{\beta_{dd}\phi I_d^*}{1+(\mu_d+\gamma_d)\phi}\frac{1}{1+(\beta_{dd}I_d^*+\mu_d)\phi} & \frac{1}{1+(\mu_d+\gamma_d)\phi} & \frac{\beta_{dd}\phi S_d^*}{1+(\mu_d+\gamma_d)\phi} \\
0 & 0 & 0 & 0 & \frac{\gamma_d\phi}{1+(\alpha_d+\mu_d)\phi}\frac{1}{1+(\mu_d+\gamma_d)\phi} & \frac{1}{1+(\alpha_d+\mu_d)\phi}
\end{pmatrix}
$$

where,

$$b = \frac{1}{1 + (\beta_{hd}I_d^* + \mu_h + \nu_1)\phi}$$

$$c = -\frac{(\mu_h\phi K_h + S_h^*)\beta_{hd}\phi}{(1 + (\beta_{hd}I_d^* + \mu_h + \nu_1)\phi)^2}$$

$$d = \frac{\beta_{hd}\phi I_d^*}{1 + (\mu_h + \gamma_h + \nu_2)\phi} \frac{1}{1 + (\beta_{hd}I_d^* + \mu_h + \nu_1)\phi}$$

$$e = \frac{\beta_{hd}\phi S_h^*}{1 + (\mu_h + \gamma_h + \nu_2)\phi}$$

$$f = \frac{\mu_h\phi}{1 + (\alpha_h + \mu_h)\phi} \frac{1}{1 + (\mu_h + \gamma_h + \nu_2)\phi}$$

$$g = \frac{\nu_1\phi}{1 + \mu_h\phi} \frac{1}{1 + (\beta_{hd}I_d^* + \mu_h + \nu_1)\phi}$$

$$h = \frac{\nu_2\phi}{1 + \mu_h\phi} \frac{1}{1 + (\mu_h + \gamma_h + \nu_2)\phi}.$$

For an eigenvalue r of E_∞, the characteristic equation is given by

$$det(rI - J(E_\infty)) = 0.$$

By observing the given Jacobian matrix, the stability properties of E_∞ are determined based on the roots of the cubic equation

$$(r - \frac{1}{1 + (\beta_{dd}I_d^* + \mu_d)\phi})\{(r - \frac{1}{1 + (\mu_d + \gamma_d)\phi})(r - \frac{1}{1 + (\alpha_d + \mu_d)\phi}) - \frac{\gamma_d\beta_{dd}\phi^2 S_d^*}{(1 + (\mu_d + \gamma_d)\phi)^2(1 + (\alpha_d + \mu_d)\phi)}\}$$

$$+ \frac{(\mu_d\phi K_d + S_d^*)\gamma_d\beta_{dd}^2\phi^3 I_d^*}{(1 + (\beta_{dd}I_d^* + \mu_d)\phi)^3(1 + (\mu_d + \gamma_d)\phi)^2(1 + (\alpha_d + \mu_d)\phi)} = 0,$$

the other roots of the characteristic equation lie inside the unit circle. This expression can be simplified into

$$r^3 + a_2 r^2 + a_1 r + a_0 = 0, \tag{4.30}$$

where,

$$a_2 = -\left(\frac{1}{1 + (\mu_d + \gamma_d)\phi} + \frac{1}{1 + (\alpha_d + \gamma_d)\phi} + \frac{1}{1 + (\beta_{dd}I_d^* + \mu_d)\phi}\right)$$

$$a_1 = \left(\frac{1}{1 + (\beta_{dd}I_d^* + \mu_d)\phi}\right)\left(\frac{1}{1 + (\mu_d + \gamma_d)\phi} + \frac{1}{1 + (\alpha_d + \gamma_d)\phi}\right)$$

$$+ \frac{1}{(1 + (\mu_d + \gamma_d)\phi)(1 + (\alpha_d + \gamma_d)\phi)}\left(1 - \frac{\frac{\gamma_d \beta_{dd} K_d}{\mathcal{R}_0}\phi^2}{1 + (\mu_d + \gamma_d)\phi}\right)$$

$$a_0 = \frac{1}{(1 + (\beta_{dd}I_d^* + \mu_d)\phi)}\frac{1}{(1 + (\mu_d + \gamma_d)\phi)}\frac{1}{(1 + (\alpha_d + \gamma_d)\phi)}\left(\frac{\frac{\gamma_d \beta_{dd} K_d}{\mathcal{R}_0}\phi^2}{1 + (\mu_d + \gamma_d)\phi} - 1\right) + c,$$

where,

$$c = \frac{(\mu_d \phi K_d + S_d^*)\gamma_d \beta_{dd}^2 I_d^* \phi^2}{(1 + (\beta_{dd}I_d^* + \mu_d)\phi)^3 (1 + (\alpha_d + \mu_d)\phi)(1 + (\mu_d + \gamma_d)\phi)^2}.$$

By using [13], E_∞ is locally asymptotically stable if the the roots of (4.30) lie inside the unit circle. This is true if and only if

$$|a_0 + a_2| < 1 + a_1 \quad and \quad |a_1 - a_0 a_2| < 1 - a_0^2.$$

From the given values, we can check that $|a_0 + a_2| < 1 + a_1$ and $|a_1 - a_0 a_2| < 1 - a_0^2$ for $\mathcal{R}_0 > 1$. Hence, E_∞ is locally asymptotically stable for $\mathcal{R}_0 > 1$. $\square$

5 Numerical Simulation

In this section, we give numerical experiments for our nonstandard finite difference scheme (4.1)-(4.7). It is known that standard finite difference methods do not always preserve the dynamics of the corresponding differential equation (see [2]). The designed numerical scheme, for arbitrary values of the system parameters, provides accurate numerical solutions to the original model. To show this, we arbitrarily choose the data given in Table 2 with in the domain of parameters to show the consistency of the scheme with the theoretical analysis of the continuous model. By using Eqn (3.10), the corresponding value of $\mathcal{R}_0 \simeq 3.3 > 1$. Hence, by using Theorem 4.5, we can infer that all solutions will converge to the endemic fixed point

E_∞. More precisely, the performance of the NSFD scheme $(4.1)-(4.7)$ is illustrated in Fig 2-3, which display the local stability of endemic fixed point for $\mathcal{R}_0 > 1$. For $\mathcal{R}_0 < 1$, analogously the disease-free fixed point is globally asymptotically stable.

Variable	value	parameter	value
K_h	1000	β_{hd}	0.3
S_h0	900	μ_h	0.2
E_h0	0	α_h	0.1
I_h0	0	γ_h	0.2
V_h0	10	ν_1	0.2
K_d	100	ν_2	0.1
S_d0	80	β_{dd}	0.5
E_d0	5	μ_d	0.1
I_d0	5	γ_d	0.2
		α_d	0.2

Table 2: parameter values.

6   Conclusion

The qualitative analysis of the rabies model is given. It is proved that the disease free equilibrium is globally stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$. The endemic equilibrium is locally asymptotically stable for $\mathcal{R}_0 > 1$.

Nonstandard finite difference techniques were developed empirically for solving practical problems in applied sciences before two decades by R. E. Mickens. Having this in our mind, we designed a NSFD scheme which is dynamically consistent with the original rabies model. Numerical simulations of the NSFD scheme are given from Fig 2-3.

The given model is relatively simple and it captures some basic features of epidemiological
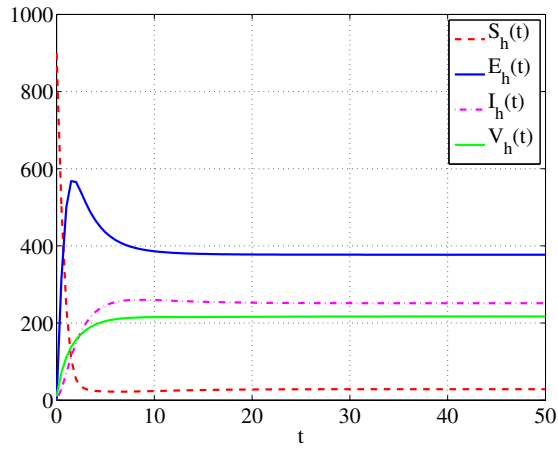
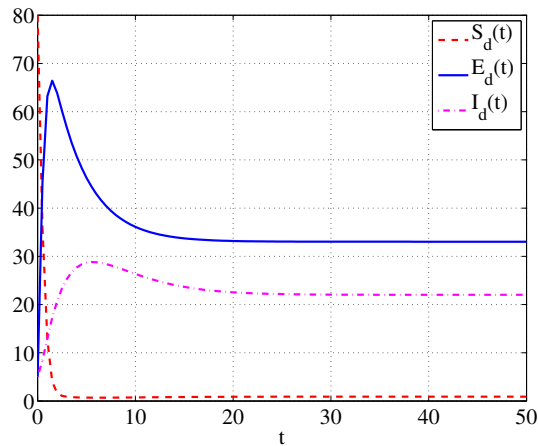Figure 2: Numerical simulation of human population

Figure 3: Numerical simulation of dog population

models. When the rabies virus enters the central nervous system, it induces behavioral changes in its host. If it enters in the spinal cord, it causes paralysis. Moreover, if it enters the limbic system, the dog become aggressive, lose their sense of direction and territorial behavior and wander about in a more or less random way. For such cases, our future interest is to add

a diffusion term in the infected dog compartment and to add vaccination parameters in the susceptible dog population. Fitting real data to estimate the parameters is also our interest for future research. The extension of the model will also include the qualitative analysis and the numerical simulation.

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