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Common fixed point results of a pair of generalized (ψ,φ)-contraction mappings in modular spaces

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Abstract

In this paper, we establish the existence of a common fixed point of almost generalized contractions on modular spaces. As an application, we present some fixed and common fixed point results for such mappings on modular spaces endowed with a graph. The existence of fixed and common points of mappings satisfying generalized contractive conditions of integral type is also obtained in such spaces. Some examples are presented to support the results obtained herein. Our results generalize and extend various comparable results in the existing literature.

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1 Introduction

Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory. The classical Banach contraction principle is one of the most useful results in nonlinear analysis. It ensures the existence and uniqueness of the fixed point of nonlinear operators satisfying the strict contraction condition. It also shows that the fixed point can be approximated by means of a Picard iteration. Due to its applications in mathematics and other related disciplines, the Banach contraction principle has been generalized in many directions. Extensions of the Banach contraction principle have been obtained either by generalizing the domain of the mapping (see, *e.g.*, [1, 2]) or by extending the contractive condition on the mappings [3, 4]. The existence of fixed points in ordered metric spaces has been studied by Ran and Reurings [5], Theorem 2.1. Subsequently, Nieto and Rodríguez-López [6] extended the results in [5], Theorem 2.1 for nondecreasing mappings and applied them to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Since then, a number of results have been proved in the framework of ordered metric spaces. In 2008, Jachymski [7] investigated a new approach in metric fixed point theory by replacing the order structure with a graph structure on a metric space. In this way, the results proved in ordered metric spaces are generalized (see for details [7] and the references therein). Abbas and Nazir [8] obtained some fixed point results for a power graphic contraction pair endowed with a graph. Beg and Butt [9] proved fixed point theorems for set-valued mappings on a metric space with a graph. In this direction, we refer to [10–12] and the references mentioned therein.

The concept of a modular space was initiated by Nakano [13] and was redefined and generalized by Musielak and Orlicz [14]. In addition to it, the most important development of this theory is due to Mazur and Musielak, Luxemburg and Turpin (see [15–17]). The fixed point theory in modular function spaces has recently got a great deal of attention of researchers, for example, Khamsi [18] (see also [10, 11, 15, 17, 19–26]). Kuaket and Kumam [27] and Mongkolkeha and Kumam [28–30] proved some fixed and common fixed point results for generalized contraction mappings in modular spaces. Also, Kumam [22] obtained some fixed point theorems for nonexpansive mappings in arbitrary modular spaces.

The aim of this paper is to prove common fixed point results of a pair of mappings satisfying an almost generalized (ψ,φ)-contraction condition in the setting of modular spaces. We provide an example to show that our results are a substantial generalization of comparable results in the existing literature. As an application of the results obtained herein, we obtain fixed and common fixed point results in the framework of modular space endowed with a directed as well as undirected graph. Some examples are presented to support the results proved herein. The existence of common fixed points of mappings satisfying a contractive condition of integral type is also obtained in such spaces.

2 Preliminaries

In the sequel the letters $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integer numbers, respectively. Let (X,d) be a metric space and $T: X \to X$. A point $x \in X$ is called a fixed point of T iff $Tx = x$. A mapping $T: X \to X$ is called a Picard operator (PO). If

(1) $F(T) = \{x \in X : Tx = x\} = \{z\}$,

(2) for any $x_0 \in X$, the Picard iteration $x_n = T^n x_0$ converges to z.

A sequence as in the above definition is called a sequence of successive approximations of T starting from x_0.

The Banach contractive condition forces the mappings to be continuous. It is natural to ask if there do or do not exist weaker contractive conditions that ensure the existence and uniqueness of a fixed point but do not imply the continuity of mappings. Kannan [23], by considering a weaker contractive conditions, proved the existence of a fixed point for a mapping that can have a discontinuity. Following Kannan's result, a lot of papers were devoted to obtaining fixed point or common fixed point theorems for various classes of contractive type conditions that do not require the continuity of the mappings; see, for example, [24] and [31].

The following definition is more suitable in this context.

Definition 2.1 Let (X,d) be a metric space. A map $T: X \to X$ is called an almost contraction or a δ-weak contraction if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(x,Ty)$$

for any $x, y \in X$.

This concept was introduced by Berinde as a 'weak contraction' in [3]. But in [4], Berinde renamed the 'weak contraction' as an 'almost contraction', which is more appropriate.

Berinde [3] proved some fixed point results for an almost contraction in the setting of a complete metric space and generalized the results in [23, 24], and [31].

Recently Babu *et al.* [32] considered the class of mappings that satisfy 'condition (B)' as follows.

Let (X,d) be a metric space. A map $T:X\to X$ is said to satisfy '*condition* (B)' if there exist a constant $\delta\in]0,1[$ and some $L\geq 0$ such that

$$d(Tx,Ty)\leq \delta d(x,y)+L\min\bigl\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\bigr\}$$

for all $x,y\in X$.

They proved the following fixed point theorem.

Theorem 2.2 ([32], Theorem 2.3) *Let (X,d) be a complete metric space and $T:X\to X$ be a map satisfying condition* (B). *Then T has a unique fixed point.*

Afterwards Berinde [33] introduced the concept of a generalized almost contraction as follows.

Let (X,d) be a metric space. A map $T:X\to X$ is called a generalized almost contraction if there exist a constant $\delta\in]0,1[$ and some $L\geq 0$ such that

$$d(Tx,Ty)\leq \delta M_1(x,y)+L\min\bigl\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\bigr\},$$

where

$$M_1(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}\bigl[d(x,Ty)+d(y,Tx)\bigr]\right\}.$$

Theorem 2.3 *Let (X,d) be a complete metric space and $T:X\to X$ a generalized almost contraction. Then T has a unique fixed point.*

A point $y\in X$ is called a point of coincidence *of two self-mappings f and T on X if there exists a point $x\in X$ such that $y=Tx=fx$. The point x is called coincidence point of a pair (f,T).*

Abbas *et al.* [34] introduced a generalization of 'condition (B)' for a pair of self-maps and obtained a unique *point of coincidence*. Ciric *et al.* [35] extended the concept of the generalized almost contraction to two mappings and obtained some common fixed point results in a complete metric space.

Consistent with [14], some basic facts and notations needed in this paper are recalled as follows.

Definition 2.4 Let X be an arbitrary vector space. A functional $\rho:X\to[0,\infty)$ is called a modular if, for any x, y in X, the following conditions hold:

(m_1) $\rho(x)=0$ if and only if $x=0$;
(m_2) $\rho(\alpha x)=\rho(x)$ for every scalar α with $|\alpha|=1$;
(m_3) $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$, whenever $\alpha+\beta=1$, and $\alpha,\beta\geq 0$.

If (m_3) is replaced with $\rho(\alpha x + \beta y) \le \alpha^s \rho(x) + \beta^s \rho(y)$ where $\alpha^s + \beta^s = 1$, $\alpha, \beta \ge 0$, and $s \in (0,1]$, then ρ is called s-convex modular. If $s = 1$, then we say that ρ is convex modular.

The following are some consequences of condition (m_3).

Remark 2.5 [20]

(r_1) For $a, b \in \mathbb{R}$ with $|a| < |b|$ we have $\rho(ax) < \rho(bx)$ for all $x \in X$.
(r_2) For $a_1, \ldots, a_n \in \mathbb{R}_+$ with $\sum_{i=1}^n a_i = 1$, we have

$$\rho\left(\sum_{i=1}^n a_i x_i\right) \le \sum_{i=1}^n \rho(x_i) \quad \text{for any } x_1, \ldots, x_n \in X.$$

Proposition 2.6 [30] *Let X_ρ be a modular space. If $a, b \in \mathbb{R}_+$ with $b \ge a$, then $\rho(ax) \le \rho(bx)$.*

A mapping $\rho : \mathbb{R} \to [0, \infty]$ defined by $\rho(x) = \sqrt{|x|}$ is a trivial example of a modular functional.

The vector space X_ρ given by

$$X_\rho = \{x \in X; \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

is called a modular space. Generally, the modular ρ is not sub-additive and therefore does not behave as a norm or a distance. One can associate to a modular an F-norm. One can associate to a modular an F-norm.

The modular space X_ρ can be equipped with an F-norm defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \le \alpha\right\}.$$

When ρ is convex modular, then

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \le 1\right\}$$

defines a norm on the modular space X_ρ, and this is called the Luxemburg norm.

Define the ρ-ball, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x, r) = \{h \in X_\rho; \rho(x - h) \le r\}.$$

A function modular is said to satisfy:

(a) the Δ_2-type condition if there exists $K > 0$ such that for any $x \in X_\rho$, we have $\rho(2x) \le K\rho(x)$;
(b) the Δ_2-condition if $\rho(2x_n) \to 0$ as $n \to \infty$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 2.7 A sequence $\{x_n\}$ in modular space X_ρ is said to be:

(t_1) ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$;
(t_2) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent. Note that ρ-convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality. In fact, one can show that this will happen if and only if ρ satisfies the Δ_2-condition. We know that [25] the norm and modular convergence are also the same when we deal with the Δ_2-type condition.

In the sequel, suppose the modular function ρ is convex and satisfies the Δ_2-type condition.

Mongkolkeha and Kumam [30] proved the existence of a fixed point generalized weak contractive mapping in modular space as follows.

Theorem 2.8 *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$. Suppose that there exist continuous and monotone nondecreasing functions $\psi, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. If for any $x, y \in X_\rho$, the following condition holds*:

$$\psi\big(\rho(Tx - Ty)\big) \leq \psi\big(m(x,y)\big) - \varphi\big(m(x,y)\big), \tag{1}$$

where

$$m(x,y) = \max\left\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(x-Ty)) + \rho(\frac{1}{2}(y-Tx))}{2}\right\},$$

then T has a unique fixed point.

Definition 2.9 Let X_ρ be a modular space and $T : X_\rho \to X_\rho$ be a self-map. We say that T is ρ-continuous when if $\rho(x_n - x) \to 0$, then $\rho(Tx_n - Tx) \to 0$ as $n \to \infty$.

3 Common fixed point of almost generalized (ψ, φ)-contraction

We set $\Psi = \{\psi : [0,\infty) \to [0,\infty) : \psi$ a continuous nondecreasing function and $\psi(t) = 0$ if and only if $t = 0\}$ and $\Phi = \{\varphi : [0,\infty) \to [0,\infty) : \varphi$ a lower-semi continuous function and $\varphi(t) = 0$ if and only if $t = 0\}$.

In this section, we obtain common fixed point results for a pair of mappings satisfying the generalized (ψ, φ)-contractive condition in the framework of a modular space.

Theorem 3.1 *Let X_ρ be a ρ-complete modular space and $S, T : X_\rho \to X_\rho$. Suppose that there exists $L \geq 0$ such that for any $x, y \in X_\rho$, the following condition holds*:

$$\psi\big(\rho(Sx - Ty)\big) \leq \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big) + L\psi\big(N(x,y)\big), \tag{2}$$

where $\psi \in \Psi$, $\varphi \in \Phi$, $M(x,y) = \max\{\rho(x-y), \rho(x-Sx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Sx)) + \rho(\frac{1}{2}(x-Ty))}{2}\}$ and $N(x,y) = \min\{\rho(x-Sx), \rho(y-Ty), \rho(y-Sx), \rho(x-Ty)\}$. Then S and T have a unique common fixed point provided that one of the mappings S or T is ρ-continuous.

Proof Let x_0 be a given point in X_ρ. We construct a sequence $\{x_n\}$ for $n \geq 0$ by a two step iterative process thus:

$$\begin{aligned} x_{2n+2} &= Tx_{2n+1}, \\ x_{2n+1} &= Sx_{2n}. \end{aligned} \tag{3}$$

We divide the proof into the following steps.

Step 1. Prove that $\rho(x_n - x_{n+1}) \to 0$ as $n \to \infty$.

From Remark 2.5, the properties of the functions ψ and φ and substituting $x = x_{2n}$ and $y = x_{2n+1}$ in (2), we have

$$\begin{aligned}\psi\big(\rho(x_{2n+1} - x_{2n+2})\big) &= \psi\big(\rho(Sx_{2n} - Tx_{2n+1})\big)\\ &\leq \psi\big(M(x_{2n}, x_{2n+1})\big) - \varphi\big(M(x_{2n}, x_{2n+1})\big) + L\psi\big(N(x_{2n}, x_{2n+1})\big),\end{aligned}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\bigg\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n} - Sx_{2n}), \rho(x_{2n+1} - Tx_{2n+1}),\\ &\quad \frac{\rho(\frac{1}{2}(x_{2n+1} - Sx_{2n})) + \rho(\frac{1}{2}(x_{2n} - Tx_{2n+1}))}{2}\bigg\}\\ &= \max\bigg\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2}),\\ &\quad \frac{\rho(\frac{1}{2}(x_{2n+1} - x_{2n+1})) + \rho(\frac{1}{2}(x_{2n} - x_{2n+2}))}{2}\bigg\}\\ &\leq \max\bigg\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2}),\\ &\quad \frac{\rho(x_{2n} - x_{2n+1}) + \rho(x_{2n+1} - x_{2n+2})}{2}\bigg\}\\ &= \max\big\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2})\big\}\end{aligned}$$

and

$$\begin{aligned}N(x_{2n}, x_{2n+1}) &= \min\big\{\rho(x_{2n} - Sx_{2n}), \rho(x_{2n+1} - Tx_{2n+1}), \rho(x_{2n} - Tx_{2n+1}),\\ &\quad \rho(x_{2n+1} - Sx_{2n})\big\}\\ &= \min\big\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2}), \rho(x_{2n} - x_{2n+2}),\\ &\quad \rho(x_{2n+1} - x_{2n+1})\big\}\\ &= \min\big\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2}),\\ &\quad \rho(x_{2n} - x_{2n+2}), 0\big\}\\ &= 0.\end{aligned}$$

Hence we have

$$\begin{aligned}\psi\big(\rho(x_{2n+1} - x_{2n+2})\big) &\leq \psi\big(\max\big\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2})\big\}\big)\\ &\quad - \varphi\big(\max\big\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2})\big\}\big).\end{aligned} \tag{4}$$

We now consider the following case.

If $\max\{\rho(x_{2n} - x_{2n+1}), \rho(x_{2n+1} - x_{2n+2})\} = \rho(x_{2n+1} - x_{2n+2})$ for some n, then using the definition of φ, (4) becomes

$$\psi\big(\rho(x_{2n+1} - x_{2n+2})\big) \leq \psi\big(\rho(x_{2n+1} - x_{2n+2})\big) - \varphi\big(\rho(x_{2n+1} - x_{2n+2})\big) < \psi\big(\rho(x_{2n+1} - x_{2n+2})\big),$$

a contradiction. Consequently

$$\max\{\rho(x_{2n}-x_{2n+1}),\rho(x_{2n+1}-x_{2n+2})\}=\rho(x_{2n}-x_{2n+1}).$$

Thus from (4), we have

$$\begin{aligned}\psi(\rho(x_{2n+1}-x_{2n+2}))&\leq\psi(\rho(x_{2n}-x_{2n+1}))-\varphi(\rho(x_{2n}-x_{2n+1}))\\&<\psi(\rho(x_{2n}-x_{2n+1})).\end{aligned}\tag{5}$$

Continuing this way, we obtain

$$\begin{aligned}\psi(\rho(x_{2n+1}-x_{2n}))&\leq\psi(\rho(x_{2n-1}-x_{2n}))-\varphi(\rho(x_{2n-1}-x_{2n}))\\&<\psi(\rho(x_{2n-1}-x_{2n})).\end{aligned}\tag{6}$$

From (5) and (6), it follows that $\{\rho(x_n-x_{n+1})\}$ is monotone decreasing and bounded below. Therefore, there is $r\geq 0$ such that

$$\lim_{n\to\infty}\rho(x_n-x_{n+1})=r.$$

On taking the limit as $n\to\infty$ on both sides of the inequality (5), we have

$$\psi(r)\leq\psi(r)-\varphi(r),$$

which implies that $\varphi(r)=0$, that is, $r=0$. Hence

$$\lim_{n\to\infty}\rho(x_n-x_{n+1})=0.\tag{7}$$

Step 2. Now we show that $\{x_n\}$ is a ρ-Cauchy sequence.

It is sufficient to show that $\{x_{2n}\}$ is a ρ-Cauchy sequence. Assume the contrary. Then there exists $\varepsilon>0$ such that we can find two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers satisfying $n_k>m_k\geq k$ such that the following inequalities hold:

$$\rho(x_{2n_k}-x_{2m_k})\geq\varepsilon,\qquad\rho(2(x_{2n_{k-1}}-x_{2m_k}))<\varepsilon.\tag{8}$$

From (8) and Remark 2.5, it follows that

$$\begin{aligned}\varepsilon&\leq\rho(x_{2n_k}-x_{2m_k})\\&=\rho(x_{2n_k}-x_{2n_k-1}+x_{2n_k-1}-x_{2m_k})\\&\leq\rho(2(x_{2n_k}-x_{2n_k-1}))+\rho(2(x_{2n_k-1}-x_{2m_k}))\\&<\varepsilon+\rho(2(x_{2n_k}-x_{2n_k-1})).\end{aligned}$$

On taking the limit as $k\to\infty$, we obtain

$$\lim_{k\to\infty}\rho(x_{2n_k}-x_{2m_k})=\varepsilon.\tag{9}$$

Using $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in (2), we have

$$\begin{aligned}\psi\big(\rho(x_{2n_k+1} - x_{2m_k})\big) &= \psi\big(\rho(Sx_{2n_k} - Tx_{2m_k-1})\big) \\ &\leq \psi\big(M(x_{2n_k}, x_{2m_k-1})\big) - \varphi\big(M(x_{2n_k}, x_{2m_k-1})\big) \\ &\quad + L\psi\big(N(x_{2n_k}, x_{2m_k-1})\big),\end{aligned} \tag{10}$$

where

$$\begin{aligned}M(x_{2n_k}, x_{2m_k-1}) &= \max\bigg\{\rho(x_{2n_k} - x_{2m_k-1}), \rho(x_{2n_k} - Sx_{2n_k}), \rho(x_{2m_k-1} - Tx_{2m_k-1}), \\ &\quad \frac{\rho(\frac{1}{2}(x_{2m_k-1} - Sx_{2n_k})) + \rho(\frac{1}{2}(x_{2n_k} - Tx_{2m_k-1}))}{2}\bigg\} \\ &= \max\bigg\{\rho(x_{2n_k} - x_{2m_k-1}), \rho(x_{2n_k} - x_{2n_k+1}), \rho(x_{2m_k-1} - x_{2m_k}), \\ &\quad \frac{\rho(\frac{1}{2}(x_{2m_k-1} - x_{2n_k+1})) + \rho(\frac{1}{2}(x_{2n_k} - x_{2m_k}))}{2}\bigg\}\end{aligned} \tag{11}$$

and

$$\begin{aligned}N(x_{2n_k}, x_{2m_k-1}) &= \min\big\{\rho(x_{2n_k} - Sx_{2n_k}), \rho(x_{2m_k-1} - Tx_{2m_k-1}), \\ &\quad \rho(x_{2m_k-1} - Sx_{2n_k}), \rho(x_{2n_k} - Tx_{2m_k-1})\big\} \\ &= \min\big\{\rho(x_{2n_k} - x_{2n_k+1}), \rho(x_{2m_k-1} - x_{2m_k}), \\ &\quad \rho(x_{2m_k-1} - x_{2n_k+1}), \rho(x_{2n_k} - x_{2m_k})\big\}.\end{aligned} \tag{12}$$

Also from (8) and Remark 2.5, it follows that

$$\begin{aligned}\rho(x_{2n_k+1} - x_{2m_k}) &= \rho(x_{2n_k+1} - x_{2n_k} + x_{2n_k} - x_{2n_k-1} + x_{2n_k-1} - x_{2m_k}) \\ &\leq \rho\big(2(x_{2n_k+1} - x_{2n_k} + x_{2n_k} - x_{2n_k-1})\big) + \rho\big(2(x_{2n_k-1} - x_{2m_k})\big) \\ &\leq \rho\big(4(x_{2n_k+1} - x_{2n_k})\big) + \rho\big(4(x_{2n_k} - x_{2n_k-1})\big) + \rho\big(2(x_{2n_k-1} - x_{2m_k})\big) \\ &< \rho\big(4(x_{2n_k+1} - x_{2n_k})\big) + \rho\big(4(x_{2n_k} - x_{2n_k-1})\big) + \varepsilon.\end{aligned} \tag{13}$$

Also, from (11) we have

$$\begin{aligned}\rho(x_{2n_k} - x_{2m_k-1}) &= \rho(x_{2n_k} - x_{2n_k-1} + x_{2n_k-1} - x_{2m_k} + x_{2m_k} - x_{2m_k-1}) \\ &\leq \rho\big(2(x_{2n_k} - x_{2n_k-1} + x_{2m_k} - x_{2m_k-1})\big) + \rho\big(2(x_{2n_k-1} - x_{2m_k})\big) \\ &\leq \rho\big(4(x_{2n_k} - x_{2n_k-1})\big) + \rho\big(4(x_{2m_k} - x_{2m_k-1})\big) + \rho\big(2(x_{2n_k-1} - x_{2m_k})\big) \\ &< \rho\big(4(x_{2n_k} - x_{2n_k-1})\big) + \rho\big(4(x_{2m_k} - x_{2m_k-1})\big) + \varepsilon.\end{aligned} \tag{14}$$

Note that

$$\begin{aligned}\rho\left(\frac{1}{2}(x_{2m_k-1} - x_{2n_k+1})\right) &= \rho\left(\frac{1}{2}(x_{2m_k-1} - x_{2m_k} + x_{2m_k} - x_{2n_k+1})\right) \\ &\leq \rho(x_{2m_k-1} - x_{2m_k}) + \rho(x_{2m_k} - x_{2n_k+1})\end{aligned}$$

$$\begin{aligned}
&= \rho(x_{2m_k-1} - x_{2m_k}) \\
&\quad + \rho(x_{2m_k} - x_{2n_k-1} + x_{2n_k-1} - x_{2n_k} + x_{2n_k} - x_{2n_k+1}) \\
&\le \rho(x_{2m_k-1} - x_{2m_k}) + \rho\bigl(2(x_{2n_k-1} - x_{2m_k})\bigr) \\
&\quad + \rho\bigl(2(x_{2n_k-1} - x_{2n_k} + x_{2n_k} - x_{2n_k+1})\bigr) \\
&\le \rho(x_{2m_k-1} - x_{2m_k}) + \rho\bigl(2(x_{2n_k-1} - x_{2m_k})\bigr) + \rho\bigl(4(x_{2n_k-1} - x_{2n_k})\bigr) \\
&\quad + \rho\bigl(4(x_{2n_k} - x_{2n_k+1})\bigr) \\
&< \varepsilon + \rho(x_{2m_k-1} - x_{2m_k}) + \rho\bigl(4(x_{2n_k-1} - x_{2n_k})\bigr) \\
&\quad + \rho\bigl(4(x_{2n_k} - x_{2n_k+1})\bigr). \qquad (15)
\end{aligned}$$

By Remark 2.5 and Proposition 2.6, we get

$$\begin{aligned}
\rho\left(\frac{1}{2}(x_{2n_k} - x_{2m_k})\right) &= \rho\left(\frac{1}{2}(x_{2n_k} - x_{2n_k-1} + x_{2n_k-1} - x_{2m_k})\right) \\
&\quad \rho(x_{2n_k} - x_{2n_k-1}) + \rho(x_{2n_k-1} - x_{2m_k}) \\
&\le \rho(x_{2n_k} - x_{2n_k-1}) + \rho\bigl(2(x_{2n_k-1} - x_{2m_k})\bigr) \\
&< \rho(x_{2n_k} - x_{2n_k-1}) + \varepsilon. \qquad (16)
\end{aligned}$$

From (14), (15), (16), and arranging (11), we obtain

$$\begin{aligned}
&M(x_{2n_k}, x_{2m_k-1}) \\
&\quad = \max\Biggl\{\rho(x_{2n_k} - x_{2m_k-1}), \rho(x_{2n_k} - x_{2n_k+1}), \rho(x_{2m_k} - x_{2m_k-1}), \\
&\quad \frac{\rho(\frac{1}{2}(x_{2m_k-1} - x_{2n_k+1})) + \rho(\frac{1}{2}(x_{2n_k} - x_{2m_k}))}{2}\Biggr\} \\
&\quad < \max\Biggl\{\rho\bigl(4(x_{2n_k} - x_{2n_k-1})\bigr) + \rho\bigl(4(x_{2m_k} - x_{2m_k-1})\bigr) + \varepsilon, \\
&\quad \rho(x_{2n_k} - x_{2n_k+1}), \rho(x_{2m_k} - x_{2m_k-1}), \\
&\quad \frac{\varepsilon + \rho(x_{2m_k-1} - x_{2m_k}) + \rho(4(x_{2n_k-1} - x_{2n_k})) + \rho(4(x_{2n_k} - x_{2n_k+1})) + \rho(x_{2n_k} - x_{2n_k-1}) + \varepsilon}{2}\Biggr\}. \qquad (17)
\end{aligned}$$

Taking the limit as $k \to \infty$ on both sides of (10) and by using (7), (12), (13), (17), and Proposition 2.6, we get

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon),$$

a contradiction. Hence $\{x_{2n}\}$ is a ρ-Cauchy sequence.

Step 3. We prove the existence of a fixed point of one mapping.

As X_ρ is a ρ-complete, there exists a $z \in X_\rho$ such that $\rho(x_n - z) \to 0$ as $n \to \infty$. Assume that S is ρ-continuous. From $\rho(x_{2n} - z) \to 0$ as $n \to \infty$, we have $\rho(Sx_{2n} - Sz) \to 0$, that is, $\rho(x_{2n+1} - Sz) \to 0$ as $n \to \infty$. By the uniqueness of the limit we obtain $Sz = z$. Thus z is a fixed point of S.

Step 4. We prove that z is a fixed point of a mapping T.

By (2) and Remark 2.5, we get

$$\psi\big(\rho(Sx_{2n} - Tz)\big) \le \psi\big(M(x_{2n}, z)\big) - \varphi\big(M(x_{2n}, z)\big) + L\psi\big(N(x_{2n}, z)\big), \tag{18}$$

where

$$\begin{aligned} M(x_{2n}, z) &= \max\bigg\{\rho(x_{2n} - z), \rho(x_{2n} - Sx_{2n}), \rho(z - Tz), \\ &\quad \frac{\rho(\frac{1}{2}(x_{2n} - Tz)) + \rho(\frac{1}{2}(z - Sx_{2n}))}{2}\bigg\} \\ &= \max\bigg\{\rho(x_{2n} - z), \rho(x_{2n} - x_{2n+1}), \rho(z - Tz), \\ &\quad \frac{\rho(\frac{1}{2}(x_{2n} - Tz)) + \rho(\frac{1}{2}(z - x_{2n+1}))}{2}\bigg\} \\ &\le \max\bigg\{\rho(x_{2n} - z), \rho(x_{2n} - x_{2n+1}), \rho(z - Tz), \\ &\quad \frac{\rho(x_{2n} - z) + \rho(z - Tz) + \rho(\frac{1}{2}(z - x_{2n+1}))}{2}\bigg\} \end{aligned} \tag{19}$$

and

$$\begin{aligned} N(x_{2n}, z) &= \min\big\{\rho(x_{2n} - Sx_{2n}), \rho(z - Tz), \rho(x_{2n} - Tz), \rho(z - Sx_{2n})\big\} \\ &= \min\big\{\rho(x_{2n} - x_{2n+1}), \rho(z - Tz), \rho(x_{2n} - Tz), \rho(z - x_{2n+1})\big\}, \end{aligned} \tag{20}$$

on taking the limit $n \to \infty$ by (18)-(20), we get

$$\psi\big(\rho(z - Tz)\big) \le \psi\big(\rho(z - Tz)\big) - \varphi\big(\rho(z - Tz)\big),$$

which implies that $\varphi(\rho(z - Tz)) = 0$ and so $\rho(z - Tz) = 0$. Then we obtain $Tz = z$. Thus z is a fixed point of T. Hence, z is a common fixed point of S and T.

Similarly, if we suppose that T is ρ-continuous, then we get the same result.

Step 5. To prove the uniqueness of a common fixed point of two mappings.

We assume that w is an another common fixed point, that is, $w = Sw$, $w = Tw$, and $w \neq z$,

$$\begin{aligned} \psi\big(\rho(z - w)\big) &= \psi\big(\rho(Sz - Tw)\big) \\ &\le \psi\big(M(z, w)\big) - \varphi\big(M(z, w)\big) + L\psi\big(N(z, w)\big), \end{aligned} \tag{21}$$

where

$$M(z, w) = \max\bigg\{\rho(z - w), \rho(z - Sz), \rho(w - Tw), \frac{\rho(\frac{1}{2}(w - Sz)) + \rho(\frac{1}{2}(z - Tw))}{2}\bigg\} \tag{22}$$

and

$$N(z, w) = \min\big\{\rho(z - Sz), \rho(w - Tw), \rho(w - Sz), \rho(z - Tw)\big\}. \tag{23}$$

By (21)-(23) we have

$$\begin{aligned}\psi\bigl(\rho(z-w)\bigr) &\le \psi\bigl(\rho(z-w)\bigr)-\varphi\bigl(\rho(z-w)\bigr)\\ &< \psi\bigl(\rho(z-w)\bigr),\end{aligned}$$

a contradiction. Hence $z = w$. □

The following results are obtained directly from Theorem 3.1.

Corollary 3.2 *Let S, T be self-mappings on a ρ-complete modular space X_ρ such that for any $x, y \in X_\rho$, the following condition holds*:

$$\psi\bigl(\rho(Sx-Ty)\bigr) \le \psi\bigl(M(x,y)\bigr)-\varphi\bigl(M(x,y)\bigr), \tag{24}$$

where $\psi \in \Psi$, $\varphi \in \Phi$, $M(x,y) = \max\{\rho(x-y), \rho(x-Sx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}\}$. Then S and T have a unique common fixed point provided that one of the mappings S or T is ρ-continuous.

Corollary 3.3 *Let S, T be self-mappings on a ρ-complete modular space X_ρ such that for any $x, y \in X_\rho$, the following condition holds*:

$$\rho(Sx-Ty) \le M(x,y)-\varphi\bigl(M(x,y)\bigr), \tag{25}$$

where $\varphi \in \Phi$, and $M(x,y) = \max\{\rho(x-y), \rho(x-Sx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}\}$. Then S and T have a unique common fixed point provided that one of the mappings S or T is ρ-continuous.

Corollary 3.4 *Let S, T be self-mappings on a ρ-complete modular space X_ρ. Suppose that there exist $k \in [0,1)$ and $L \ge 0$ such that for any $x, y \in X_\rho$, the following condition holds*:

$$\rho(Sx-Ty) \le kM(x,y)+LN(x,y), \tag{26}$$

where $M(x,y) = \max\{\rho(x-y), \rho(x-Sx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}\}$ and $N(x,y) = \min\{\rho(x-Sx), \rho(y-Ty), \rho(y-Sx), \rho(x-Ty)\}$. Then S and T have a unique common fixed point provided that one of the mappings S or T is ρ-continuous.

Define $\digamma = \{\xi : \mathbb{R}_+ \to \mathbb{R}_+ : \varphi$ is a Lebesgue integral mapping which is summable, nonnegative and satisfies $\int_0^\varepsilon \xi(t)\,dt > 0$, for each $\varepsilon > 0\}$.

Corollary 3.5 *Let S, T be self-mappings on a ρ-complete modular space X_ρ. Suppose that there exist $k \in [0,1)$ and $L \ge 0$ such that for any $x, y \in X_\rho$, the following condition holds*:

$$\int_0^{\rho(Sx-Ty)} \xi(t)\,dt \le k\int_0^{M(x,y)} \xi(t)\,dt + L\int_0^{N(x,y)} \xi(t)\,dt, \tag{27}$$

where $M(x,y) = \max\{\rho(x-y), \rho(x-Sx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Sx))+\rho(\frac{1}{2}(x-Ty))}{2}\}$ and $N(x,y) = \min\{\rho(x-Sx), \rho(y-Ty), \rho(y-Sx), \rho(x-Ty)\}$. Then S and T have a unique common fixed point provided that one of the mappings S or T is ρ-continuous.

Proof Take $\psi(t) = \int_0^t \xi(t)\,dt$ and $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$. The result then follows from Theorem 3.1. □

Corollary 3.6 *Let T be a self-mapping on a ρ-complete modular space X_ρ which satisfies the following inequality*:

$$\psi\bigl(\rho(Tx-Ty)\bigr) \le \psi\bigl(m(x,y)\bigr) - \varphi\bigl(m(x,y)\bigr) + L\psi\bigl(n(x,y)\bigr) \tag{28}$$

for all $x,y \in X_\rho$, where $\psi \in \Psi$, $\varphi \in \Phi$,

$$m(x,y) = \max\left\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(y-Tx)) + \rho(\frac{1}{2}(x-Ty))}{2}\right\},$$

and $n(x,y) = \min\{\rho(x-Tx), \rho(y-Ty), \rho(y-Tx), \rho(x-Ty)\}$. Then T has a unique fixed point.

4 Common fixed points on modular spaces with a directed graph

Let X_ρ be a ρ-modular space and $\Delta = \{(x,x) : x \in X\}$ denote the diagonal of $X_\rho \times X_\rho$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph such that $\Delta \subseteq E(G)$. Also assume that G has no parallel edges and G is a weighted graph in the sense that each edge (x,y) is assigned the weight $\rho(x-y)$. Since ρ is a modular functional on X_ρ, the weight assigned to each vertex x to vertex y does not need to be zero and whenever a zero weight is assigned to some edge (x,y), it reduces to a (x,x) having weight 0. The graph G is identified with the pair $(V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ of vertices such that $x = x_0, \dots, x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1,2,\dots,k\}$.

Recall that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E\bigl(G^{-1}\bigr) = \bigl\{(x,y) \in X \times X : (y,x) \in E(G)\bigr\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, and with this convention we have

$$E(\widetilde{G}) = E(G) \cup E\bigl(G^{-1}\bigr).$$

Let G_x be the component of G consisting of all the edges and vertices which are contained in some path in G beginning at x. In $V(G)$, we define the relation R in the following way.

For $x,y \in V(G)$, we have $x\,R\,y$ if and only if there is a path in G from x to y. If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the equivalence class $[x]_G$ in $V(G)$ defined by the relation R is $V(G_x)$.

Let X_ρ be a modular space endowed with a graph G and $S, T : X_\rho \to X_\rho$. We set

$$X_{ST} = \bigl\{x \in X : (x, Sx) \in E(G) \text{ and } (Sx, TSx) \in E(G)\bigr\}.$$

Theorem 4.1 *Let X_ρ be a ρ-complete modular space endowed with a directed graph G, and S, T self-maps on X_ρ. Suppose that the following conditions hold*:

(i) *If $\{x_n\}$ is a sequence in X_ρ such that $\rho(x_n - x) \to 0$ and $(x_{2n}, x_{2n+1}) \in E(G)$ for all $n \geq 0$, then there exists a subsequence $\{x_{2n_p}\}$ of $\{x_{2n}\}$ such that*

(i$_\text{a}$) *T is ρ-continuous and $(x, x_{2n_p+1}) \in E(G)$ for all $p \geq 0$ or*

(i$_\text{b}$) *S is ρ-continuous and $(x_{2n_p}, x) \in E(G)$ for all $p \geq 0$.*

(ii) *There is a sequence $\{x_n\}$ in X_ρ such that*

$$(x_{2n}, Sx_{2n}) \in E(G) \quad \textit{implies that} \quad (x_{2n+2}, Sx_{2n+2}) \in E(G) \quad \textit{and}$$
$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \quad \textit{implies that} \quad (x_{2n+3}, Tx_{2n+3}) \in E(G).$$

(iii) *For any $(x, y) \in E(G)$, S and T satisfy* (2) *in Theorem* 3.1.

(iv) *X_{ST} is nonempty.*

Then S and T have a common fixed point.

Proof Let x_0 be a given point in X_{ST}, then $(x_0, Sx_0) \in E(G)$ and $(Sx_0, TSx_0) \in E(G)$, that is, $(x_1, Tx_1) \in E(G)$. From (ii), it follows that $(x_2, Sx_2) \in E(G)$ and $(x_3, Tx_3) \in E(G)$. Continuing this way, we can obtain a sequence $\{x_n\}$ in X_ρ such that $(x_{2n}, Sx_{2n}) \in E(G)$ and $(x_{2n+1}, Tx_{2n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Also, $(x_{2n}, x_{2n+1}) \in E(G)$ and $(x_{2n+1}, x_{2n+2}) \in E(G)$ for all $n \in \mathbb{N}$. Using arguments as in the proof of Theorem 3.1, we obtain

$$\rho(x_n - x_{n+1}) \to 0, \quad n \to \infty. \tag{29}$$

Also, $\{x_n\}$ is a ρ-Cauchy sequence. Since X_ρ is ρ-complete, there exists a $\rho(x_n - x^*) \to 0$ as $n \to \infty$.

Now we show that x^* is a common fixed point of S and T. As $\rho(x_n - x^*) \to 0$ and $(x_{2n}, x_{2n+1}) \in E(G)$, there exists a subsequence $\{x_{2n_p}\}$ of $\{x_{2n}\}$. Assume that (i$_\text{a}$) holds. Then T is ρ-continuous and $(x^*, x_{2n_p+1}) \in E(G)$. Thus

$$\rho\bigl(Tx_{2n_p+1} - Tx^*\bigr) = \rho\bigl(x_{2n_p+2} - Tx^*\bigr) \to 0, \quad p \to \infty.$$

This implies that $Tx^* = x^*$. From (iii) and Remark 2.5, we have

$$\begin{aligned} \psi\bigl(\rho\bigl(Sx^* - Tx_{2n_p+1}\bigr)\bigr) \leq{}& \psi\bigl(M\bigl(x^*, x_{2n_p+1}\bigr)\bigr) - \varphi\bigl(M\bigl(x^*, x_{2n_p+1}\bigr)\bigr) \\ &+ L\psi\bigl(N\bigl(x^*, x_{2n_p+1}\bigr)\bigr), \end{aligned} \tag{30}$$

where

$$\begin{aligned} M\bigl(x^*, x_{2n_p+1}\bigr) ={}& \max\biggl\{\rho\bigl(x^* - x_{2n_p+1}\bigr), \rho\bigl(x^* - Sx^*\bigr), \rho(x_{2n_p+1} - Tx_{2n_p+1}), \\ & \frac{\rho(\frac{1}{2}(x^* - Tx_{2n_p+1})) + \rho(\frac{1}{2}(x_{2n_p+1} - Sx^*))}{2}\biggr\} \\ ={}& \max\biggl\{\rho\bigl(x^* - x_{2n_p+1}\bigr), \rho\bigl(x^* - Sx^*\bigr), \rho(x_{2n_p+1} - x_{2n_p+2}), \end{aligned}$$

$$\left.\frac{\rho(\frac{1}{2}(x^* - x_{2n_p+2})) + \rho(\frac{1}{2}(x_{2n_p+1} - Sx^*))}{2}\right\}$$

$$\leq \max\left\{\rho(x^* - x_{2n_p+1}), \rho(x^* - Sx^*), \rho(x_{2n_p+1} - x_{2n_p+2}),\right.$$
$$\left.\frac{\rho(\frac{1}{2}(x^* - x_{2n_p+2})) + \rho(x_{2n_p+1} - x^*) + \rho(x^* - Sx^*)}{2}\right\} \tag{31}$$

and

$$N(x^*, x_{2n_p+1}) = \min\{\rho(x^* - Sx^*), \rho(x_{2n_p+1} - Tx_{2n_p+1}),$$
$$\rho(x_{2n_p+1} - Sx^*), \rho(x^* - Tx_{2n_p+1})\}. \tag{32}$$

On taking the limit as $p \to \infty$ (30)-(32), we have

$$\psi(\rho(Sx^* - x^*)) \leq \psi(\rho(Sx^* - x^*)) - \varphi(\rho(Sx^* - x^*)),$$

a contradiction. Thus $Sx^* = x^*$ and hence x^* is a common fixed point of S and T. Similarly, the result follows if we suppose that S is ρ-continuous and $(x_{2n_p}, x^*) \in E(G)$. □

We note that Theorem 4.1 does not guarantee the uniqueness of a common fixed point.

To obtain the uniqueness, an additional assumption as given in the following theorem is required.

Theorem 4.2 *In addition to the conditions of Theorem* 4.1, *assume that for any two common fixed point* x^*, y^* *of* S *and* T, *there exists* $z \in X_\rho$ *such that* $(x^*, z) \in E(G)$ *and* $(z, y^*) \in E(G)$. *Then* $x^* = y^*$.

Proof Let x^*, y^* be common fixed points of S and T, then $(x^*, z) \in E(G)$ and $(z, y^*) \in E(G)$. As G is a directed graph, $(x^*, y^*) \in E(G)$. From (iii), we obtain

$$\psi(\rho(x^* - y^*)) = \psi(\rho(Sx^* - Ty^*))$$
$$\leq \psi(M(x^*, y^*)) - \varphi(M(x^*, y^*)) + L\psi(N(x^*, y^*)), \tag{33}$$

where

$$M(x^*, y^*) = \max\left\{\rho(x^* - y^*), \rho(x^* - Sx^*), \rho(y^* - Ty^*),\right.$$
$$\left.\frac{\rho(\frac{1}{2}(y^* - Sx^*)) + \rho(\frac{1}{2}(x^* - Ty^*))}{2}\right\} \tag{34}$$

and

$$N(x^*, y^*) = \min\{\rho(x^* - Sx^*), \rho(y^* - Ty^*), \rho(y^* - Sx^*), \rho(x^* - Ty^*)\}. \tag{35}$$

By (33)-(35) we have

$$\psi(\rho(x^* - y^*)) \leq \psi(\rho(x^* - y^*)) - \varphi(\rho(x^* - y^*)) < \psi(\rho(x^* - y^*)),$$

a contradiction. Hence $x^* = y^*$. □

Example 4.3 Let $X_\rho = \mathbb{R}$, $\rho(x) = |x|$ for all $x \in X_\rho$ and $E(G) = \{(x,y) : x,y \in [0,1]\}$. Define the mappings $S, T : X \to X$ as follows:

$$Sx = \frac{x}{3}, \qquad Tx = \frac{x}{4}, \quad x \in X_\rho.$$

Take $\psi(t) = \frac{t}{3}$ and $\varphi(t) = \frac{t}{6}$ for all $t > 0$ and $L \geq \frac{1}{5}$. It is verified that all the conditions of Theorem 3.1 and Theorem 4.2 are satisfied. Moreover, 0 is a common fixed point of S and T.

The following results are obtained directly from Theorem 4.1.

Corollary 4.4 *Let X_ρ be a ρ-complete modular space endowed with a directed graph G, and S, T self-maps on X_ρ. Suppose that the following conditions hold*:

(i) *If $\{x_n\}$ is a sequence in X_ρ such that $\rho(x_n - x) \to 0$ and $(x_{2n}, x_{2n+1}) \in E(G)$ for all $n \geq 0$, then there exists a subsequence $\{x_{2n_p}\}$ of $\{x_{2n}\}$ such that*

   ($\mathrm{i_a}$) *T is ρ-continuous and $(x, x_{2n_p+1}) \in E(G)$ for all $p \geq 0$ or*

   ($\mathrm{i_b}$) *S is ρ-continuous and $(x_{2n_p}, x) \in E(G)$ for all $p \geq 0$.*

(ii) *There is a sequence $\{x_n\}$ in X_ρ such that*

$$(x_{2n}, Sx_{2n}) \in E(G) \quad \textit{implies that} \quad (x_{2n+2}, Sx_{2n+2}) \in E(G) \quad \textit{and}$$

$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \quad \textit{implies that} \quad (x_{2n+3}, Tx_{2n+3}) \in E(G).$$

(iii) $\psi(\rho(Sx - Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y))$, *for any* $(x,y) \in E(G)$.

(iv) *X_{ST} is nonempty.*

Then S and T have a common fixed point.

Corollary 4.5 *Let X_ρ be a ρ-complete modular space endowed with a directed graph G, and S, T self-maps on X_ρ. Suppose that the following conditions hold*:

(i) *If $\{x_n\}$ is a sequence in X_ρ such that $\rho(x_n - x) \to 0$ and $(x_{2n}, x_{2n+1}) \in E(G)$ for all $n \geq 0$, then there exists a subsequence $\{x_{2n_p}\}$ of $\{x_{2n}\}$ such that*

   ($\mathrm{i_a}$) *T is ρ-continuous and $(x, x_{2n_p+1}) \in E(G)$ for all $p \geq 0$ or*

   ($\mathrm{i_b}$) *S is ρ-continuous and $(x_{2n_p}, x) \in E(G)$ for all $p \geq 0$.*

(ii) *There is a sequence $\{x_n\}$ in X_ρ such that*

$$(x_{2n}, Sx_{2n}) \in E(G) \quad \textit{implies that} \quad (x_{2n+2}, Sx_{2n+2}) \in E(G) \quad \textit{and}$$

$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \quad \textit{implies that} \quad (x_{2n+3}, Tx_{2n+3}) \in E(G).$$

(iii) $\rho(Sx - Ty) \leq M(x,y) - \varphi(M(x,y))$ *for any* $(x,y) \in E(G)$.

(iv) *X_{ST} is nonempty.*

Then S and T have a common fixed point.

Corollary 4.6 *Let X_ρ be a ρ-complete modular space endowed with a directed graph G, and S, T self-maps on X_ρ. Suppose that the following conditions hold*:

(i) *If* $\{x_n\}$ *is a sequence in* X_ρ *such that* $\rho(x_n - x) \to 0$ *and* $(x_{2n}, x_{2n+1}) \in E(G)$ *for all* $n \geq 0$, *then there exists a subsequence* $\{x_{2n_p}\}$ *of* $\{x_{2n}\}$ *such that*

(i_a) T *is* ρ*-continuous and* $(x, x_{2n_p+1}) \in E(G)$ *for all* $p \geq 0$ *or*

(i_b) S *is* ρ*-continuous and* $(x_{2n_p}, x) \in E(G)$ *for all* $p \geq 0$.

(ii) *There is a sequence* $\{x_n\}$ *in* X_ρ *such that*

$$(x_{2n}, Sx_{2n}) \in E(G) \quad \textit{implies that} \quad (x_{2n+2}, Sx_{2n+2}) \in E(G) \quad \textit{and}$$

$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \quad \textit{implies that} \quad (x_{2n+3}, Tx_{2n+3}) \in E(G).$$

(iii) *There exist* $k \in [0,1)$ *and* $L \geq 0$ *such that*

$$\rho(Sx - Ty) \leq kM(x,y) + LN(x,y)$$

for any $(x,y) \in E(G)$.

(iv) X_{ST} *is nonempty.*

Then S *and* T *have a common fixed point.*

Corollary 4.7 *Let* X_ρ *be a* ρ*-complete modular space endowed with a directed graph* G, *and* S, T *self-maps on* X_ρ. *Suppose that the following conditions hold*:

(i) *If* $\{x_n\}$ *is a sequence in* X_ρ *such that* $\rho(x_n - x) \to 0$ *and* $(x_{2n}, x_{2n+1}) \in E(G)$ *for all* $n \geq 0$, *then there exists a subsequence* $\{x_{2n_p}\}$ *of* $\{x_{2n}\}$ *such that*

(i_a) T *is* ρ*-continuous and* $(x, x_{2n_p+1}) \in E(G)$ *for all* $p \geq 0$ *or*

(i_b) S *is* ρ*-continuous and* $(x_{2n_p}, x) \in E(G)$ *for all* $p \geq 0$.

(ii) *There is a sequence* $\{x_n\}$ *in* X_ρ *such that*

$$(x_{2n}, Sx_{2n}) \in E(G) \quad \textit{implies that} \quad (x_{2n+2}, Sx_{2n+2}) \in E(G) \quad \textit{and}$$

$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \quad \textit{implies that} \quad (x_{2n+3}, Tx_{2n+3}) \in E(G).$$

(iii) *There exists* $k \in [0,1)$ *and* $L \geq 0$ *such that*

$$\int_0^{\rho(Sx-Ty)} \xi(t)\,dt \leq k \int_0^{M(x,y)} \xi(t)\,dt + L \int_0^{N(x,y)} \xi(t)\,dt$$

for each $(x,y) \in E(G)$.

(iv) X_{ST} *is nonempty.*

Then S *and* T *have a common fixed point.*

Proof The proof follows from Theorem 4.1 by taking $\psi(t) = \int_0^t \xi(t)\,dt$ and $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$. □

5 Fixed point results on modular spaces involving undirected graph

Recently, Öztürk *et al.* [10] obtained some fixed point results for mappings satisfying a contractive condition of integral type in modular spaces endowed with a graph using the C_ρ-graph and being orbitally G_ρ-continuous. To apply this property, we modify for $C = 1$ as follows.

Definition 5.1 Let $\{T^n x\}$ be a sequence in X_ρ for some $x \in X_\rho$ such that $\rho(T^n x - x^*) \to 0$ for $x^* \in X_\rho$ and $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Then a graph G is called a modified C_ρ-graph if there exists a subsequence $\{T^{n_p}x\}$ of $\{T^n x\}$ such that $(T^{n_p}x, x^*) \in E(G)$ for $p \in \mathbb{N}$.

Definition 5.2 A mapping $T : X_\rho \to X_\rho$ is called orbitally G_ρ-continuous if for all $x, y \in X_\rho$ and any sequence $(n_p)_{p\in\mathbb{N}}$ of positive integers, $\rho(T^{n_p}x - y) \to 0$ and $\rho(T(T^{n_p}x) - T(y)) \to 0$ as $p \to \infty$.

Next, we define an almost generalized (ψ, φ)-$\widetilde{G}$-contraction mapping.

Definition 5.3 Let X_ρ be a modular space endowed with undirected graph G. A self-mapping T on X_ρ is called an almost generalized (ψ, φ)-G-contraction mapping if:

(i) T preserves edges of G.

(ii) There exists $L \ge 0$ such that

$$\psi\bigl(\rho(Tx - Ty)\bigr) \le \psi\bigl(m(x,y)\bigr) - \varphi\bigl(m(x,y)\bigr) + L\psi\bigl(n(x,y)\bigr)$$

holds for all $(x, y) \in E(G)$.

Remark 5.4 Let X_ρ be a modular space endowed with a graph G and $T : X_\rho \to X_\rho$ an almost generalized (ψ, φ)-G-contraction mapping. If there exists $x_0 \in X_\rho$ such that $Tx_0 \in [x_0]_{\widetilde{G}}$, then the following statements hold:

(i) T is both an almost generalized (ψ, φ)-G^{-1}-contraction mapping and an almost generalized (ψ, φ)-$\widetilde{G}$-contraction mapping.

(ii) $[x_0]_{\widetilde{G}}$ is T-invariant and $T_{|[x_0]_{\widetilde{G}}}$ is an almost generalized (ψ, φ)-$\widetilde{G}_{x_0}$-contraction mapping.

Theorem 5.5 *Let X_ρ be ρ-complete modular space endowed with a graph G and $T : X_\rho \to X_\rho$. If the following statements hold*:

(i) *G is weakly connected and modified C_ρ-graph*;

(ii) *T is an almost generalized (ψ, φ)-$\widetilde{G}$-contraction*;

(iii) *$X_T = \{x \in X : (x, Tx) \in E(G)\}$ is nonempty,*

then T is a PO.

In Theorem 5.5, if we replace the condition that G is a modified C_ρ-graph with modified orbitally G_ρ-continuity of T, then we have the following theorem.

Theorem 5.6 *Let X_ρ be a ρ-complete modular space endowed with a graph G, and $T : X_\rho \to X_\rho$ an almost generalized (ψ, φ)-$\widetilde{G}$-contraction mapping and modified orbitally G_ρ-continuous. If X_T is nonempty and the graph G is weakly connected, then T is a PO.*

Example 5.7 Let $X_\rho = \mathbb{R}$, $\rho(x) = |x|$ for all $x \in X_\rho$, and

$$E(\widetilde{G}) = \bigl\{(0,0), (0,1), (1,1), (1,3), (2,2), (0,3), (2,3), (3,3)\bigr\}.$$

Define $T : X_\rho \to X_\rho$ by

$$Tx = \begin{cases} 0, & x \in \{0,1\}, \\ 1, & x \in \{2,3\}, \\ \frac{x}{2}, & \text{otherwise}. \end{cases}$$

Clearly, G is weakly connected and a C_ρ-graph, X_T is nonempty, and T is orbitally G_ρ-continuous and an almost generalized (ψ,φ)-$\widetilde{G}$-contraction mapping where $\psi(t) = \frac{t}{3}$, $\varphi(t) = \frac{t}{6}$, and $L \geq 0$. Note that T does not satisfy inequality (25). Indeed, we have

$$\psi\bigl(\rho(T1 - T2)\bigr) \leq \psi\bigl(m(1,2)\bigr) - \varphi\bigl(m(1,2)\bigr) + L\psi\bigl(n(1,2)\bigr),$$

where $m(1,2) = 1$, $n(1,2) = 0$, and so $\frac{1}{3} \leq \frac{1}{6}$ is obtained, an absurd statement. Also, Corollary 3.6 is not applicable in this case, but all the conditions of Theorem 5.5 and Theorem 5.6 are satisfied.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Abbas, M, Ilic, D: Common fixed points of generalized almost nonexpansive mappings. Filomat **24**(13), 11-18 (2010)
2. Aydi, H, Amor, SH, Karapınar, E: Berinde-type generalized contractions on partial metric spaces. Abstr. Appl. Anal. **2013**(10), Article ID 312479 (2013)
3. Berinde, V: Approximation fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum **9**(1), 43-53 (2004)
4. Berinde, V: General constructive fixed point theorems for Ciric-type almost contractions in metric spaces. Carpath. J. Math. **24**(2), 10-19 (2008)
5. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some application to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
6. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
7. Jachymski, J: The contraction principle for mappings on a metric space endowed with a graph. Proc. Am. Math. Soc. **136**, 1359-1373 (2008)
8. Abbas, M, Nazir, T: Common fixed point of a power graphic contraction pair in partial metric spaces endowed with a graph. Fixed Point Theory Appl. **2013**, 20 (2013). doi:10.1186/1687-1812-2013-20
9. Beg, I, Butt, AR: Fixed point of set-valued graph contractive mappings. J. Inequal. Appl. **2013**, 252 (2013). doi:10.1186/1029-242X-2013-252
10. Öztürk, M, Abbas, M, Girgin, E: Fixed points of mappings satisfying contractive condition of integral type in modular spaces endowed with a graph. Fixed Point Theory Appl. **2014**, 220 (2014). doi:10.1186/1687-1812-2014-220
11. Öztürk, M, Girgin, E: On some fixed-point theorems for ψ-contraction on metric space involving a graph. J. Inequal. Appl. **2014**, 39 (2014). doi:10.1186/1029-242X-2014-39
12. Asl, JH, Mohammadi, B, Rezapour, S, Vaezpour, SM: Some fixed point results generalized quasi-contractive multifunctions on graphs. Filomat **27**(2), 311-315 (2013)
13. Nakano, H: Modulared Semi-Ordered Linear Spaces. Tokyo Math. Book Ser., vol. 1. Maruzen, Tokyo (1950)
14. Musielak, T, Orlicz, W: On modular spaces. Stud. Math. **18**, 49-65 (1959)
15. Mazur, S, Orlicz, W: On some classes of linear spaces. Stud. Math. **17**, 97-119 (1958). (Reprinted in Wladyslaw Orlicz Collected Papers, pp. 981-1003, PWN, Warszawa (1988).)
16. Luxemburg, WAJ: Banach function spaces. Thesis, Delft Inst. of Techn. Asser., The Netherlands (1955)

17. Turpin, P: Fubini inequalities and bounded multiplier property in generalized modular spaces. Special issue dedicated to Wladyslaw Orlicz on the occasion of his seventy-fifth birthday. Comment. Math. **1**, 331-353 (1978)
18. Khamsi, MA, Kozolowski, WK, Reich, S: Fixed point theory in modular function spaces. Nonlinear Anal. **14**, 935-953 (1990)
19. Khamsi, MA: A convexity property in modular function spaces. Math. Jpn. **44**, 269-279 (1996)
20. Beygmohammadi, M, Razani, A: Two fixed-point theorems for mappings satisfying a general contractive condition of integral type in the modular space. Int. J. Math. Math. Sci. **2010**, Article ID 317107 (2010)
21. Kumam, P: Some geometric properties and fixed point theorem in modular spaces. In: Garcia Falset, J, Fuster, L, Sims, B (eds.) Fixed Point Theorem and Its Applications, pp. 173-188. Yokohama Publishers, Yokohama (2004)
22. Kumam, P: Fixed point theorems for nonexpansive mappings in modular spaces. Arch. Math. **40**(4), 345-353 (2004)
23. Kannan, R: Some results on fixed points. Bull. Calcutta Math. Soc. **60**, 71-76 (1968)
24. Chartterjea, SK: Fixed point theorem. C. R. Acad. Bulgare Sci. **25**, 727-730 (1972)
25. Benavides, TD: Fixed point theorems for uniformly Lipschitzian mappings and asymptotically regular mappings. Nonlinear Anal. **32**, 15-27 (1998)
26. Benavides, TD, Khamsi, MA, Samadi, S: Uniformly Lipschitzian mappings in modular function spaces. Nonlinear Anal. **46**, 267-278 (2001)
27. Kuakket, K, Kumam, P: Fixed points of asymptotic pointwise contractions in modular spaces. Appl. Math. Lett. **24**(11), 1795-1798 (2004)
28. Mongkolkeha, C, Kumam, P: Fixed point and common fixed point theorems for generalized weak contraction mappings of integral type in modular spaces. Int. J. Math. Math. Sci. **2011**, Article ID 705943 (2011)
29. Mongkolkeha, C, Kumam, P: Common fixed points for generalized weak contraction mappings in modular spaces. Sci. Math. Jpn. (Online) **e-2012**, 117-127 (2012)
30. Mongkolkeha, C, Kumam, P: Some fixed point results for generalized weak contraction mappings in modular spaces. Int. J. Anal. **2013**, Article ID 247378 (2013)
31. Zamfirescu, T: Fixed point theorems in metric spaces. Arch. Math. **23**, 292-298 (1972)
32. Babu, GVR, Sandhya, ML, Kameswari, MVR: A note on a fixed point theorem of Berinde on weak contractions. Carpath. J. Math. **24**(1), 8-12 (2008)
33. Berinde, V: Some remark on a fixed point theorems for Ciric-type almost contractions. Carpath. J. Math. **25**(2), 157-162 (2009)
34. Abbas, M, Babu, GVR, Alemayehu, GN: On common fixed of weakly compatible mappings satisfying generalized condition (B). Filomat **25**(2), 9-19 (2011)
35. Ciric, L, Abbas, M, Saadati, R, Hussain, N: Common fixed points of almost generalized contractive mappings in ordered metric spaces. Appl. Math. Comput. **217**, 5784-5789 (2011)