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# THE QUEST FOR A CHARACTERIZATION OF HOM-PROPERTIES OF FINITE CHARACTER

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**Abstract**

A *graph property* is a set of (countable) graphs. A *homomorphism* from a graph $G$ to a graph $H$ is an edge-preserving map from the vertex set of $G$ into the vertex set of $H$; if such a map exists, we write $G \to H$. Given any graph $H$, the *hom-property* $\to H$ is the set of $H$*-colourable graphs*, i.e., the set of all graphs $G$ satisfying $G \to H$. A graph property $\mathcal{P}$ is *of finite character* if, whenever we have that $F \in \mathcal{P}$ for every finite induced subgraph $F$ of a graph $G$, then we have that $G \in \mathcal{P}$ too. We explore some of the relationships of the property attribute of being of finite character to other property attributes such as being *finitely-induced-hereditary*, being *finitely determined*, and being *axiomatizable*. We study the hom-properties of finite character, and prove some necessary and some sufficient conditions on $H$ for $\to H$ to be of finite character. A notable (but known) sufficient condition is that $H$ is a finite graph, and our new model-theoretic proof of this compactness result extends from hom-properties to all axiomatizable properties. In our quest to find an intrinsic characterization of those $H$ for which $\to H$ is of finite character, we find an example of an infinite connected graph with no finite core and chromatic number 3 but with hom-property not of finite character.

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## 1. Preliminaries

All graphs considered are simple, undirected and unlabelled, and have countable vertex sets. (Sometimes graphs of arbitrary cardinality will also be mentioned.) The symbol $\mathcal{I}$ denotes the class of all such graphs.

A *property* (of graphs) is a class (here always non-empty) of graphs, closed under isomorphisms. If two graphs are isomorphic, we refer to any one of them as a *clone* of the other. To avoid potential conceptual problems with proper classes or large numbers of clones, we may select a particular subset Skel($\mathcal{I}$), a *skeleton* of the class $\mathcal{I}$, with elements one specific graph chosen from each isomorphism class in $\mathcal{I}$. (Since clones share all their graph properties, and since we have for many purposes, in a property, no reason to distinguish between clones, this move is unproblematic.) Properties of graphs can therefore be considered as *sets* of graphs. In general we follow [10] for graph-theoretic notions and [4] for notation and terminology on properties. We use the symbol $\sqcup$ for the disjoint union of graphs or sets, and write $H \leq G$ to indicate that $H$ is an induced subgraph of $G$; $H \subseteq G$ to indicate that $H$ is a subgraph of $G$; $H < G$ to indicate that $H \leq G$ and $H \not\cong G$; $H \subset G$ to indicate that $H \subseteq G$ and $H \not\cong G$.

A property $\mathcal{P}$ is an *induced-hereditary property of graphs*, or an *induced-hereditary property* for short, if, whenever $G \in \mathcal{P}$ and $H \leq G$, then $H \in \mathcal{P}$ too. Note that, for any graph $G$, the set of graphs $\leq G := \{H \mid H \leq G\}$ is an induced-hereditary property.

Following [5], we say that the graph property $\mathcal{P}$ is *of finite character* if whenever for a graph $G$ we have that, for every finite $F \leq G$, $F \in \mathcal{P}$, then we have $G \in \mathcal{P}$ too. Note that, for an induced-hereditary property $\mathcal{P}$, we have that $\mathcal{P}$ is of finite character if and only if the equivalence "$G \in \mathcal{P}$ if and only if every finite induced subgraph of $G$ is in $\mathcal{P}$" holds. Examples of such properties can be obtained by forbidding sets of finite graphs: we shall prove in the next section that all such properties are of finite character (and even "finitely determined", Theorem 1), and that, in fact, a large class of properties of finite character can be characterized in this way.

A *homomorphism* from a graph $G$ to a graph $H$ is an edge-preserving map from the vertex set of $G$ into the vertex set of $H$; if such a map exists, we write

$G \to H$. Given any graph $H$, the induced-hereditary *hom-property* $\to H$, also called the set of *$H$-colourable graphs*, is the set of all graphs $G$ satisfying $G \to H$. Two graphs $G$ and $H$ are *homomorphically equivalent* if $G \to H$ and $H \to G$ (written $G \sim H$); and the set of all graphs homomorphically equivalent to $H$ is denoted by $[H]$.

If $G$ is a (not necessarily induced) subgraph of $H$ and $H \to G$, then $G$ and $H$ are homomorphically equivalent. Since $\to G \subseteq \to H$ if and only if $G \to H$, it is clear that homomorphically equivalent graphs determine the same hom-properties. A *core* is a graph that is not homomorphic to a proper subgraph of itself. It is well known (see for example Proposition 1.32 of [13]) that every finite graph is homomorphically equivalent to a unique core; uniqueness having the meaning here of "up to isomorphism".

In [1], Bauslaugh compares different definitions of a core of an infinite structure. For our purposes, we say that a countable graph $H$ *has a core*, say $C(H)$, when $C(H) \subseteq H$ and $H \to C(H)$, while there is no $J \subset C(H)$ with $H \to J$. Note that $H$ and any such $C(H)$ are homomorphically equivalent and that, if $H$ is denumerable, $C(H)$ need not be unique.

In this paper we study properties, and in particular hom-properties, of finite character. In Section 2, those induced-hereditary (and even somewhat weaker) properties which are of finite character are characterized. These include the hom-properties $\to H$, where $H$ is homomorphically equivalent to a finite graph, which will be studied in particular in Section 3. We then find the focus on hom-properties segueing into scrutiny of the more general *axiomatizable properties.* It is natural to ask for a characterization of the hom-properties $\to H$ which are of finite character in terms of intrinsic attributes of the graph $H$, like its chromatic, clique and independence numbers, connectedness, and possession of a finite core. This question is elaborated upon in Section 4.

## 2. Finitely Determined Properties

The first characterization we offer is valid for a wider class of properties of finite character than the induced-hereditary ones; we first define this wider class. Given any graph $G \in \mathcal{I}$, let

$$\mathcal{F}(G) = \{H \in \mathcal{I} \mid H \leq G \text{ and } H \text{ is finite}\}.$$

We say that $\mathcal{P} \subseteq \mathcal{I}$ is a *finitely-induced-hereditary property* if, for all $G \in \mathcal{P}$, $\mathcal{F}(G) \subseteq \mathcal{P}$. Here is a very simple example of a property that is finitely-induced-hereditary without being induced-hereditary: let $\mathcal{P} = \{F \mid F \text{ is a finite graph }\} \cup \{G\}$, where $G$ is (for example) the two-way infinite path.

Note that a finitely-induced-hereditary property $\mathcal{P}$ is therefore of finite character if, for all $G \in \mathcal{I}$, we have $G \in \mathcal{P}$ if and only if $\mathcal{F}(G) \subseteq \mathcal{P}$. We call a property

$\mathcal{P}$ *finitely determined* when $\mathcal{P}$ is finitely-induced-hereditary and of finite character, i.e., when $G \in \mathcal{P}$ if and only if $\mathcal{F}(G) \subseteq \mathcal{P}$. For a given set of graphs $\mathcal{S}$ we define the *property* $-\mathcal{S}$ *of* $\mathcal{S}$*-free graphs* by

$$G \in -\mathcal{S} \text{ if and only if } S \not\leq G \text{ for every } S \in \mathcal{S}.$$

In our first theorem the "finiteness" in "finitely determined" relates directly to the finiteness of the graphs in the set $\mathcal{B}$ of "banned" graphs.

**Theorem 1.** *A property* $\mathcal{P}$ *is finitely determined if and only if* $\mathcal{P} = -\mathcal{B}$ *for some set* $\mathcal{B}$ *of finite graphs.*

***Proof.*** Suppose that $\mathcal{P}$ is finitely determined and let $\mathcal{B} = \{F \in \mathcal{I} \mid F \notin \mathcal{P}$ and $F$ is finite$\}$. We will prove that $\mathcal{P} = -\mathcal{B}$.

First we show that $\mathcal{P} \subseteq -\mathcal{B}$. Let $G \in \mathcal{P}$ and suppose $G \notin -\mathcal{B}$. Then there exists a graph $F \in \mathcal{B}$ such that $F \leq G$, therefore $F \in \mathcal{F}(G)$. Since $\mathcal{P}$ is finitely-induced-hereditary, it follows that $\mathcal{F}(G) \subseteq \mathcal{P}$, which then implies that $F \in \mathcal{P}$. This is a contradiction, thus $\mathcal{P} \subseteq -\mathcal{B}$.

Next we show that $-\mathcal{B} \subseteq \mathcal{P}$. Let $G \in -\mathcal{B}$. Then $F \not\leq G$ for all $F \in \mathcal{B}$. This implies that $G' \notin \mathcal{B}$ for all $G' \in \mathcal{F}(G)$. So, for every $G' \in \mathcal{F}(G)$, we have (by the definition of $\mathcal{B}$) that $G' \in \mathcal{P}$ or $G'$ is infinite — and hence $G' \in \mathcal{P}$, implying that $\mathcal{F}(G) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is of finite character it follows that $G \in \mathcal{P}$, thus $-\mathcal{B} \subseteq \mathcal{P}$.

Now suppose that $\mathcal{P} = -\mathcal{B}$ for some set $\mathcal{B}$ of finite graphs. We will show that a graph $G \in \mathcal{P}$ if and only if $\mathcal{F}(G) \subseteq \mathcal{P}$.

Let $G \in \mathcal{P}$, and suppose $G' \notin \mathcal{P}$ for some $G' \in \mathcal{F}(G)$. Then there exists $F \in \mathcal{B}$ such that $F \leq G'$. But then this means $F \leq G$, and implies $G \notin -\mathcal{B}$, which is a contradiction. Thus we have $\mathcal{F}(G) \subseteq \mathcal{P}$.

Now assume, for some $G \in \mathcal{I}$, that $\mathcal{F}(G) \subseteq \mathcal{P}$. If $G \notin \mathcal{P}$ then there exists $F \in \mathcal{B}$ such that $F \leq G$. Which means $F$, an element of $\mathcal{F}(G)$, does not belong to $\mathcal{P}$. This is a contradiction, thus $G \in \mathcal{P}$. ■

## 3. The Compactness Theorems

We would like to see a characterization, if possible, by intrinsic attributes of $H$, of those hom-properties $\to H$ which are of finite character. In the next subsection some first steps are taken in this direction. The De Bruijn-Erdős Theorem of [9], which is known as the Compactness Theorem (for graph colourings), states that, if every $F \in \mathcal{F}(G)$ is $n$-colourable then so is $G$, i.e., for every graph $G$ we have that $G \to K_n$ if $F \to K_n$ for every $F \in \mathcal{F}(G)$. This result has also been proven model-theoretically by Hattingh in [11], while a more general result in which $K_n$ is replaced by any finite graph, is proven by Salomaa in [17] and Bauslaugh in [3]. All the proofs known to us of this more general result use some or other axiom or theorem in ZF set theory, and it was pointed out in [7] that many results of this kind are indeed equivalent to the Boolean Prime Ideal Theorem (known as the BPI, see for example [8]). In Section 3.1 we shall prove a result of this kind for any graph that is homomorphically equivalent to a finite graph using the Compactness Theorem (for first-order predicate logic). Section 3.2 generalizes the compactness result for hom-properties to arbitrary axiomatizable properties.

By the way (in contrast to the profound result that, for a finite $H$, $\to H$ is of finite character), the fact that (again for a finite $H$) $\leq H$ is of finite character is tritely trivial and utterly useless: for an infinite $G$ both $\mathcal{F}(G) \not\subseteq \leq H$ and $G \not\leq H$.

### 3.1. A compactness theorem for *H*-colourings

We continue our efforts of the preceding section by establishing a simple intrinsic attribute of $H$ which suffices for $\to H$ to be a hom-property of finite character: $H$ is finite, or at least is homomorphically equivalent to a finite graph. As remarked above, each hom-property $\to K_n$ is of finite character. This is usually proved by a set-theoretical argument, as in the source [9]. To the best of our knowledge the first proof using any compactness result in formal logic occurs in [11], p. 28. Employing the Compactness Theorem for first-order predicate logic with equality (saying that a set of sentences has a model if and only if every finite subset of that set has a model — see [14] for an 18-page guide to model theory) we generalize this from $K_n$ to any finite graph $H$. This generalization was achieved by Salomaa [17] and Bauslaugh [3] too, employing essentially different proof strategies. Our method of proof, however, will lead in the next subsection to a result with many other applications.

**Theorem 2.** *If $H$ is any finite graph, then the hom-property $\to H$ is of finite character, i.e., if $G$ is any countable graph for which every finite induced subgraph is $H$-colourable, then $G$ itself is $H$-colourable.*

***Proof.*** Consider a countable graph $G$ and a finite graph $H$ such that for every $F \in \mathcal{F}(G)$ there exists a homomorphism $f_F : F \to H$. If $G$ is finite, then there is nothing to prove. So we assume that $V(G) = \{g_1, g_2, \ldots\}$ is denumerable, while $V(H) = \{h_1, h_2, \ldots, h_n\}$ is finite.

We define the first-order predicate language $L$ (suitable for expressing the mathematics of this context) by the following specification of its signature vocabulary, from which sentences are constructed in the standard way:

(1) a denumerable set of individual variables $x, y, z, \ldots$ or $x_1, x_2, \ldots$ for arbitrary elements in the domain of any interpretation of $L$;

(2) a denumerable set of individual constant symbols, $V(G) \sqcup V(H)$; (please allow some innocuous ambiguity: $g_i$ and $h_i$ are simultaneously symbols of $L$ and vertices of $G$ and $H$, respectively);

(3) two unary predicate (property) symbols, $V_G$ (for being an element of some subset of the domain which is $\subseteq$-comparable to $V(G)$), and $V_H$ (for being a vertex of $H$);

(4) three binary relation symbols, $=$ (for standard equality), $E_G$ (for adjacency in the interpretation of $V_G$), and $E_H$ (for adjacency in $H$); and

(5) one unary function symbol, $f$ (to name a function from the domain), which induces a homomorphism $G \to H$ or $f_F : F \to H$ for some $F \in \mathcal{F}(G)$.

Next, we specify a finite set $T$ of sentences of $L$ (all without any occurrence of an individual constant symbol $g_i$ or $h_i$) which express that for any interpretation $\texttt{int}$ of (the set of non-logical symbols of) the language $L$,

(i) $\texttt{int}(V_G)$ carries a graph structure;
(ii) similarly, $\texttt{int}(V_H)$ carries a graph structure; and
(iii) $\texttt{int}(f)|\texttt{int}(V_G)$ is a graph homomorphism from the graph on $\texttt{int}(V_G)$ into the graph on $\texttt{int}(V_H)$:

(i) $(\forall x)[\neg E_G(x,x)]$
"$\texttt{int}(E_G)$ is irreflexive"
$(\forall x)(\forall y)[E_G(x,y) \to \{E_G(y,x) \wedge V_G(x) \wedge V_G(y)\}]$
"$\texttt{int}(E_G)$ is a symmetric binary relation on $\texttt{int}(V_G)$"

(ii) Exactly like (i), with every $G$ replaced by $H$.

(iii) $(\forall x)[\neg\{V_G(x) \wedge V_H(x)\}]$
"$\texttt{int}(V_G)$ and $\texttt{int}(V_H)$ are disjoint"
$(\forall x)[V_G(x) \to V_H(f(x))]$
"$\texttt{int}(f)|\texttt{int}(V_G)$ maps $\texttt{int}(V_G)$ into $\texttt{int}(V_H)$"
$(\forall x)[V_H(x) \to f(x) = x]$
"since $\texttt{int}(f)$ needs to be defined on the whole domain ([16], p. 227; [14], p. 2), it is innocuously the identity on $\texttt{int}(V_H)$)"
$(\forall x)(\forall y)[E_G(x,y) \to E_H(f(x), f(y))]$
"$\texttt{int}(f)|\texttt{int}(V_G)$ is a graph homomorphism, i.e., preserves edges, from $\texttt{int}(V_G)$ into $\texttt{int}(V_H)$".

Let us name the set of the eight sentences of $L$ above — of which we may think as "the general theory" of our $G$-$H$-configuration — as $T :=$ (i) $\cup$ (ii) $\cup$ (iii).

Now we specify a denumerable set, $D(G)$ (called the "diagram" of graph $G$), consisting of all those "diagrammatic" or "literal" sentences of $L$ — i.e., atomic sentences or their negations — which are true in $G$ under the obvious interpretations of the $G$-related non-logical symbols of $L$. (The notion of the "diagram" of a mathematical structure was introduced and used very fruitfully by Robinson [16].) $D(G)$ has as elements all and only the sentences of the following forms:

$V_G(g_i)$ for all $g_i \in V(G)$;
$E_G(g_i, g_j)$ and $E_G(g_j, g_i)$ for all $g_i, g_j \in V(G)$ for which $g_i g_j \in E(G)$;
$\neg E_G(g_i, g_j)$ and $\neg E_G(g_j, g_i)$ for the cases $g_i g_j \notin E(G)$;
$\neg(g_i = g_j)$ for the cases $i \neq j$.

Similarly, we have $D(H)$, a finite set of sentences:
$V_H(h_i)$ for $i = 1, 2, \ldots, n$;
$E_H(h_i, h_j)$ and $E_H(h_j, h_i)$ whenever $h_i h_j \in E(H)$;
$\neg E_H(h_i, h_j)$ and $\neg E_H(h_j, h_i)$ whenever $h_i h_j \notin E(H)$;
$\neg(h_i = h_j)$ whenever $i \neq j$.

Finally, we have a single sentence, $S$, ensuring that we should have $\mathtt{int}(V_H) = \{\mathtt{int}(h_1), \mathtt{int}(h_2), \ldots, \mathtt{int}(h_n)\}$, a set with precisely $n$ elements:

$$S = (\forall x)[V_H(x) \to \{(x = h_1) \vee (x = h_2) \vee \cdots \vee (x = h_n)\}].$$

The denumerable set of sentences, hopefully consistent, $K := T \cup D(G) \cup D(H) \cup \{S\}$ can now be seen as our "complete knowledge base", expressing in $L$ all that we assume and something that we hope about our $G$-$H$-configuration if we just add — outside $L$ now — the meta-level information that for every $F \in \mathcal{F}(G)$, $F \to H$. This latter information will below ensure that every finite subset of $K$ has a model. By compactness, $K$ then has a model, yielding $G \to H$.

Let $J$ indicate any finite subset of $K$. We must specify an interpretation of every non-logical symbol occurring in $J$ which is a model of $J$, i.e., under which every sentence in $J$ is true. No harm is done by giving an interpretation of all the non-logical symbols of the language, since its restriction to those occurring in $J$ will do what is necessary, while the exposition is simplified. The interpretations of $L$ may differ for different $J$, but here only when the finite sets $W$ of symbols $g_i \in V(G)$ which actually occur in two different instances of $J$ are different: $W(J_1) \neq W(J_2)$; then we also write $J_{W_1} \neq J_{W_2}$. So we index the different interpretations of $L$ here needed — denumerably many — by the set of all finite subsets $W$ of $V(G)$, and write $\mathtt{int}_W$ for the interpretation corresponding to $W$.

Now let $W$ be any finite subset of the denumerable set $V(G)$. First, we need to specify the underlying domain of the interpretation $\mathtt{int}_W$ of $L$. If $W = \emptyset$, then this domain is $V(G) \sqcup V(H)$. For this $W$ the interpretation function $\mathtt{int}_\emptyset$ on the set of non-logical symbols of $L$ is the following:
$\mathtt{int}_\emptyset(g_i) = g_i$ for every $i \in \{1, 2, \ldots\}$;
$\mathtt{int}_\emptyset(h_i) = h_i$ for every $i \in \{1, 2, \ldots, n\}$;
$\mathtt{int}_\emptyset(V_G) = \{g_1\}$;
$\mathtt{int}_\emptyset(V_H) = \{h_1, h_2, \ldots, h_n\}$;
$\mathtt{int}_\emptyset(=)$ is standard set-theoretical equality;
$\mathtt{int}_\emptyset(E_G) = \emptyset$;
$\mathtt{int}_\emptyset(E_H) = E(H)$; and
$\mathtt{int}_\emptyset(f) = \{(g_1, h_1)\} \cup \{(g_i, g_i) \mid i = 2, 3, \ldots\} \cup id_{V(H)}$.

We claim that every sentence in the finite set $J_\emptyset$ is true under the interpretation $\mathtt{int}_\emptyset$ (of all the symbols in $L$ and therefore of all the symbols occurring anywhere in $J_\emptyset$). To see this, note first that no symbol $g_i$ occurs anywhere in $J_\emptyset$; hence $J_\emptyset$ is disjoint from the diagram $D(G)$ of $G$. Going carefully through the list of the rest of the sentences of $K$, namely $T \cup D(H) \cup \{S\}$ (containing $J_\emptyset$), one can verify that each of them is satisfied by the interpretation $\mathtt{int}_\emptyset$. Hence $\mathtt{int}_\emptyset$ restricted to the symbols occurring in $J_\emptyset$ yields a model of $J_\emptyset$; this holds for any finite subset $J_\emptyset$ of $K$ in which there is no occurrence of any symbol $g_i$.

Next, we consider any finite subset $J_W$ of $K$ where $W$ is now the non-empty (finite) set of all and only those symbols $g_i \in V(G)$ which have an occurrence in some sentence in $J_W$. Then $G[W] \in \mathcal{F}(G)$, and by assumption there exists a homomorphism $f_W : G[W] \to H$. We specify the interpretation $\mathtt{int}_W$ of all the non-logical symbols of language $L$ (with domain still $V(G) \sqcup V(H)$): For all symbols $s \notin \{V_G, E_G, f\}$ (i.e., for all symbols $s \in \{g_1, g_2, \ldots, h_1, h_2, \ldots, h_n, V_H, =, E_H\}$)

$\mathtt{int}_W(s) = \mathtt{int}_\emptyset(s)$;

$\mathtt{int}_W(V_G) = W$;

$\mathtt{int}_W(E_G) = E(G[W])$; and

$\mathtt{int}_W(f) = f_W \cup \{(g_i, g_i) \mid g_i \in V(G) \text{ and } g_i \notin W\} \cup id_{V(H)}$.

Careful checking through the list of all those sentences of $K$ which could possibly occur in $J_W$ (i.e., those in $T \cup D(H) \cup \{S\}$ together with $D(G)|W$, i.e., $V_G(g_i)$ for all $g_i \in W$; $E_G(g_i, g_j)$ and $E_G(g_j, g_i)$ for all cases $g_i g_j \in E(G[W])$; $\neg E_G(g_i, g_j)$ and $\neg E_G(g_j, g_i)$ for all cases where $g_i, g_j \in W$ and $g_i g_j \notin E(G[W])$; and $\neg(g_i = g_j)$ for all cases where $g_i, g_j \in W$ and $i \neq j$), verifies that $\mathtt{int}_W$ satisfies all these sentences and hence all the sentences in $J_W$. Hence $\mathtt{int}_W$ restricted to the symbols occurring in $J_W$ yields a model of $J_W$.

So, every finite subset of $K$ has a model and (by the compactness of predicate logic) so has $K$. Now, finally, we have an interpretation, say $\mathtt{int}$, of all the symbols of $L$, satisfying $K$, with underlying domain (countable, if you like, by the Downward Löwenheim-Skolem Theorem ([16], p. 20; [14], p. 5)) containing $V(G) \sqcup V(H)$ and with $\mathtt{int}(f)\,|\,V(G) : G \to H$. ■

Remember that a countable graph $H$ that is homomorphically equivalent to a finite graph $F$, satisfies $\to H = \to F$. Hence Theorem 2 applies to such graphs and we have

**Corollary 3.** *If $H$ is any countable graph which is homomorphically equivalent to a finite graph, then the hom-property $\to H$ is of finite character.*

We note that the sufficient condition of this corollary, namely "$H$ is homomorphically equivalent to a finite graph", amounts in some sense to be an intrinsic structural condition on $H$, since it is equivalent to "$H$ is homomorphic to a finite subgraph of itself".

**Corollary 4.** *If $H$ is any graph (with any cardinality whatsoever) which is homomorphically equivalent to a finite graph, then the hom-property $\rightarrow H$ is of finite character in the class of all graphs of arbitrary cardinality.*

***Proof.*** The proof of Theorem 2 succeeds unchanged for $V(G)$ of any cardinality whatsoever. In language $L$ the set of individual constant symbols may have arbitrary cardinality ([14], p. 2 and [16], p. 3), and the relevant compactness theorem ([14], p. 7, by Robinson somewhat idiosyncratically called the "Principle of Localization" in [16] on p. 21) holds for a set of sentences of any cardinality. ■

**Corollary 5** (Bauslaugh [3])**.** *If $H$ is any directed graph which is homomorphically equivalentto a finite graph, then the hom-property $\rightarrow H$ is of finite character.*

***Proof.*** With slight changes to the formulation and the proof of Theorem 2 it holds for directed graphs too. ■

The last corollary above alerts one to the fact that the proof technique of Theorem 2 can be used, perhaps unlike the proof techniques employed in [3] and [17], to prove compactness results for more situations. The flexibility created by the rich expressive power of the formal language $L$ allows interesting variants of extra attributes to be imposed upon the crucial homomorphisms $f_F : F \rightarrow H$ and $f : G \rightarrow H$ – while maintaining the global structure of the compactness proof above intact. These extra attributes need just to be expressible as finitely many sentences of $L$.

Here is a simple example. A homomorphism $f : G \rightarrow H$ is called *full* when it preserves (not only adjacency, but also) non-adjacency:

$(\forall x)(\forall y)[\neg E_G(x, y) \rightarrow \neg E_H(f(x), f(y))]$,

or, contra-positively,

$(\forall x)(\forall y)[E_H(f(x), f(y)) \rightarrow E_G(x, y)]$,

i.e., adjacency between $f$-image vertices in $H$ comes from adjacency between every pair of corresponding pre-image vertices in $G$. Note that a full homomorphism need not be an isomorphism, since it can map the vertices of an independent set in $G$ all to a single vertex in $H$. Adding the single sentence for fullness of $f$ to the set of "axioms" $T$(iii) in the proof of Theorem 2 yields a new theorem: If every finite induced subgraph of $G$ is fully $H$-colourable, then $G$ itself is fully $H$-colourable.

In Corollary 5 "directed graph" may mean that we are dealing with three types of adjacency:

$E^0$: no direction; $E^1$: one direction; $E^2$: both directions.

It is easy to see how the axioms $T$ and the diagrams $D(G)$ and $D(H)$ (in the proof of Theorem 2) have to be adapted to coerce $f$ into preserving each of the three types of adjacency separately. Similarly, one can force $f$ to preserve labels, (say "colours") on vertices or on edges, multiple edges, etc. It should be clear that a multitude of potentially interesting variants of Theorem 2 hold with the same proof blueprint. This spoor is followed in the next subsection.

### 3.2. Axiomatizable properties

Careful metamathematical scrutiny of the previous subsection reveals that all the graph attributes considered there are expressible by sentences of (some variant of) the formal language $L$. A graph property $\mathcal{P}$ is called *axiomatizable* in a suitable language $L$ of first-order predicate logic when there exists a sentence $Ax(\mathcal{P})$ of $L$ such that, for any $G \in \mathcal{I}$, $G \in \mathcal{P}$ if and only if $G$ is a model of (i.e., satisfies) $Ax(\mathcal{P})$.

In this subsection we also want to consider a weakening of the attribute of being of finite character that a graph property may (not quite) have. We say that graph property $\mathcal{P}$ is *of weakly finite character* if whenever for a graph $G$ we have that for every finite $F \leq G$, $F \in \mathcal{P}$, then there exists a $G' \in \mathcal{P}$ such that $G \leq G'$. Another way to express that $\mathcal{P}$ is of weakly finite character is as follows: If all the finite induced subgraphs of some graph $G$ are in $\mathcal{P}$, then each of them as well as $G$ are induced subgraphs of some graph in $\mathcal{P}$. It is immediately clear that if $\mathcal{P}$ is of weakly finite character and induced-hereditary, then $\mathcal{P}$ is of finite character.

Looking back now at the proof of the compactness Theorem 2, we see that the signature (i.e., the set of non-logical symbols) of the language $L$ employed syntactically, has been chosen to allow axiomatization of the hom-property $\to H$, namely "has a homomorphism into the finite graph $H$". The relevant $Ax(\to H)$ can there be taken to be the conjunction of all the sentences in the finite set $T \cup D(H) \cup \{S\}$. We are guided to the following generalization of Theorem 2.

**Theorem 6.** *Axiomatizable properties are of weakly finite character.*

***Proof.*** Assume that property $\mathcal{P}$ is axiomatized by $Ax(\mathcal{P})$ in a suitable language $L$, and that $G$ is a graph (of any cardinality — but only the infinite case is of interest) such that every element of $\mathcal{F}(G)$ satisfies $Ax(\mathcal{P})$, i.e., $\mathcal{F}(G) \subseteq \mathcal{P}$. Define $K := \{Ax(\mathcal{P})\} \cup D(G)$, giving complete information about $G$, and also saying of any model of $K$ (which we hope exists) that its "$G$-part" (a clone of $G$) is an induced subgraph of that element of $\mathcal{P}$. Among the conjuncts making up $Ax(\mathcal{P})$ we surely have the two sentences in $T$(i), ensuring that this $G$-part is a graph.

To ensure that $K$ has a model, it suffices to show that every finite subset $J$ of $K$, called $J_W$ (with $W$ the finite subset of $V(G)$ occurring as symbols in $J$), has a model — and $G[W]$ obliges. (Should $W$ be empty, pick any $v \in V(G)$ and add the sentence $v = v$ to $J$, then taking $W = \{v\}$.) Let the model of $K$ be the graph $G' = (\mathtt{int}(V_G), \mathtt{int}(E_G))$, then $G = (V(G), E(G)) \leq G' \in \mathcal{P}$. ■

How many axiomatizable properties could there be? The answer is: countably many, since there are at most denumerably many sentences $Ax(\mathcal{P})$ in $L$.

To illustrate Theorem 6, we need some concepts from domination theory as used, for example, in [12]. A *dominating vertex* of a graph is a vertex that is adjacent to every other vertex of the graph. Generalizing, consider a graph $G$ and $D \subseteq V(G)$. We say that $D$ is a *dominating set of vertices* of $G$ when every vertex in $V(G) \setminus D$ is adjacent to at least one element of $D$. Using Theorem 6, we shall now show that a certain property about the existence of finite dominating sets of vertices, called ${}_mD_n$, can be axiomatized by a sentence $Ax({}_mD_n)$, ensuring that it is of weakly finite character. (This property is obviously not induced-hereditary.)

Choose two natural numbers $m$ and $n$, with $m \geq 3$ and $1 \leq n \leq m$. A graph $G$ has the graph property ${}_mD_n$ when the following holds: if $|V(G)| \geq m$, then there exists a non-empty subset $D \subseteq V(G)$ with $|D| \leq n$ such that every vertex in $V(G) \setminus D$ is adjacent to at least one element of $D$, i.e., $D$ is a dominating set of vertices of $G$ of cardinality at most $n$, and, for every $u_i \in D$, $u_i$ together with its neighbourhood induce a star in $G$.

In the language $L$ with signature $x, y, z, \ldots;\ g_1, g_2, \ldots;\ V_G, =, E_G$, we construct the sentence $Ax({}_mD_n)$ as the conjunction of the two sentences in $T$(i) (saying that any interpretation of $(V_G, E_G)$ that models these sentences is a graph) and the following sentence, $Ax({}_mD_n)$, (saying that such an interpretation has the property ${}_mD_n$):

$$\left\{(\exists x_1)(\exists x_2)\cdots(\exists x_m)\left[\bigwedge_{i=1}^{m}\{V_G(x_i)\} \wedge \bigwedge_{i,j=1,i\neq j}^{m}\{\neg(x_i = x_j)\}\right]\right\} \rightarrow$$

$$\left\{(\exists u_1)(\exists u_2)\cdots(\exists u_n)\left[\bigwedge_{i=1}^{n}\{V_G(u_i)\} \wedge (\forall y)\left(\left[\bigwedge_{i=1}^{n}\{\neg(y = u_i)\} \wedge V_G(y)\right] \rightarrow \bigvee_{i=1}^{n}\{E_G(y, u_i)\}\right)\right.\right.$$

$$\left.\left.\wedge(\forall z_1)(\forall z_2)\left(\left[V_G(z_1) \wedge V_G(z_2) \wedge \bigvee_{i=1}^{n}\{E_G(z_1, u_i) \wedge E_G(z_2, u_i)\}\right] \rightarrow \neg\{E_G(z_1, z_2)\}\right)\right]\right\}$$

We note that any graph with fewer than $m$ vertices satisfies $Ax({}_mD_n)$. By the grace of Theorem 6 we now have

**Corollary 7.** *The property ${}_mD_n$ is axiomatizable, and hence of weakly finite character.*

We mention, without proof, another property which is axiomatizable and hence of weakly finite character: A graph is called *flexible* when it has only one vertex or has an automorphism with some non-fixed vertices.

## 4. Intrinsic Attributes of $H$ and Finite Character of $\to H$

In Section 3 we demonstrated that the intrinsic attribute of $H$ of being finite (or at least homomorphically equivalent to a finite graph) is sufficient for the hom-property $\to H$ to be of finite character. This section deals further with the logical relationships between other intrinsic attributes of $H$ and combinations of such attributes on the one hand, and the attribute of being of finite character of $\to H$ on the other hand. Intrinsic attributes of $H$ (beyond finiteness, now) that join the dance are the following: having a (possibly finite) core; having a clique number (perhaps equal to the chromatic number); having an independence number; being perfect; and being connected.

In Section 4.1 we give an attribute of $H$ — a chain condition, not strictly intrinsic! — that is both sufficient and necessary for $\to H$ to be of finite character. Section 4.2 posits three equivalent intrinsic attributes of $H$ sufficient for the finite character of $\to H$, while Section 4.3 proves two necessary intrinsic attributes of $H$ when $\to H$ is of finite character. The final Section 4.4 unravels some of the complex logical entanglements between different combinations of intrinsic attributes of $H$ and $\to H$ being of finite character, or not.

### 4.1. A sufficient and necessary attribute

To characterize the graphs $H$ with $\to H$ of finite character, one could reasonably expect some "transcending finiteness" condition, other than the definition. The one that we found, and offer in the next result, uses finite graphs which form chains with respect to the induced subgraph relation. The condition apparently does not offer a short proof for Theorem 2 since it seems not to be easy to determine whether every finite graph $H$ satisfies it. Also, the condition is not expressed in terms of only intrinsic structural attributes of $H$.

In the sequel the notation $A \leq^+ B$ will be used to denote the fact that $A$ is an *internal* induced subgraph of $B$, i.e., $A$ is an induced subgraph of $B$ and $V(A) \subseteq V(B)$.

**Theorem 8.** *Let $H$ be any countable graph. The following two conditions are equivalent.*

1. *The hom-property $\to H$ is of finite character.*
2. *The following chain condition holds:*
   *For all ascending sequences of finite graphs $H_1 \leq^+ H_2 \leq^+ \cdots$ of which each admits a homomorphism $H_i \to H$, the limit graph $\bigcup_{i\geq 1} H_i$ also admits a homomorphism $(\bigcup_{i\geq 1} H_i) \to H$.*

***Proof.*** 1. implies 2.: Suppose the hom-property $\to H$ is of finite character. Let, for each $i \geq 1$, $H_i$ be a finite graph satisfying $H_1 \leq^+ H_2 \leq^+ \cdots$ and suppose that each $H_i$ admits a homomorphism $H_i \to H$. Then $G := \bigcup_{i\geq 1} H_i$ is a graph of which each finite induced subgraph $F$ is an induced subgraph of some $H_k$. But then $F \to H$ since $H_k \to H$. Therefore, since $\to H$ is a property of finite character, the required homomorphism $G = (\bigcup_{i\geq 1} H_i) \to H$ exists.

2. implies 1.: Let $H$ be any graph which satisfies the given chain condition and let $G$ be any countable graph which satisfies the condition that $F \to H$ for every finite subgraph $F$ of $G$. Then we show that $G \to H$: Label the vertices of $G$ so that $V(G) = \{v_1, v_2, \ldots\}$ and consider the ascending sequence of finite induced subgraphs of $G$ induced by the initial segments of $V(G)$, i.e., for each $i \geq 1$ we take $H_i = G[\{v_1, v_2, \ldots, v_i\}]$. Then, by our assumptions on $G$ we have, for the ascending chain $H_1 \leq^+ H_2 \leq^+ \cdots$ of graphs that $H_i \to H$ for each $i$ and hence, by our assumptions on $H$ there is a homomorphism $(\bigcup_{i\geq 1} H_i) \to H$. But $\bigcup_{i\geq 1} H_i = G$ so that the required homomorphism $G \to H$ exists, completing the proof. ∎

### 4.2. Sufficient attributes

Our next two theorems offer, respectively, some sufficient and some necessary conditions for a hom-property to be of finite character. In them, we say that a countable graph $H$ *has a clique number* if $K_{\aleph_0} \leq H$ or the largest order of a complete (induced) subgraph of $H$ is finite. The clique number $\omega(H)$ of $H$ is then, respectively, $\aleph_0$ or the relevant natural number. A subset of the vertex set $V(H)$ of a graph $H$ of which no two vertices are adjacent in $H$ is called an *independent* set of vertices. Also, $H$ *has an independence number* if the (edgeless) graph $\overline{K_{\aleph_0}} \leq H$ or the largest order of an induced edgeless subgraph of $H$, i.e., one of the form $\overline{K}$ for a complete $K$, is finite.

**Theorem 9.** *Let $n \in \{1, 2, \ldots, \aleph_0\}$. For any countable graph $H$ the following statements are equivalent and suffice for $\to H$ to be a hom-property of finite character.*

1. *$H \in [K_n]$, i.e., $H$ is hom-equivalent to $K_n$.*
2. *$K_n$ is a core of $H$.*
3. *$H$ has a clique number $\omega(H)$ and $\chi(H) = \omega(H) = n$.*

***Proof.*** 1. implies 2.: Assume that $H \to K_n$ and $K_n \to H$. The latter homomorphism ensures that $K_n \leq H$ and also that there can be no $J \subset K_n$ with $H \to J$. Hence $K_n$ is a core of $H$.

2. implies 3.: Assume that $K_n$ is a core of $H$: $H \to K_n$, $K_n \leq H$ (and there is no $J \subset K_n$ with $H \to J$). Then $K_n$ is the clique of maximum cardinality in $H$ (since otherwise we would have $H \not\to K_n$) and $\omega(H) = n$. Since $H \to K_n$, $\chi(H) \leq n$; but since $K_n \leq H$, also $\chi(H) \geq n$. Hence $\chi(H) = n$.

3. implies 1.: Assume that $\chi(H) = \omega(H) = n$. Then $\chi(H) = n$ implies that $H \to K_n$, while $\omega(H) = n$ implies that $K_n \to H$. So, $H \sim K_n$ and $H \in [K_n]$.

Now assume statement 1, saying that $H$ is homomorphically equivalent to $K_n$. If $n$ is finite, then Corollary 3 (which is independent from this theorem) ensures that $\to H$ is a hom-property of finite character. If $n = \aleph_0$, then the remark that $[K_{\aleph_0}] = \{H \in \mathcal{I} \mid \to H = \mathcal{I}\}$ ensures that $\to H$ is a hom-property of finite character. ■

By analogy to the definition of a finite perfect graph, as for example in [10], we say that a countable graph $H$ is *perfect* when it has a clique number and for every $G \leq H$ we have that $\chi(G) = \omega(G)$. Obviously, any perfect graph $H$ satisfies condition 3 in Theorem 9. Hence perfect graphs induce hom-properties of finite character.

**Corollary 10.** *If a graph $H$ is perfect, then $\to H$ is a hom-property of finite character.*

### 4.3. Necessary attributes

**Theorem 11.** *If $\to H$ is a hom-property of finite character, then the graph $H$ has a clique number and an independence number.*

***Proof.*** Suppose the hom-property $\to H$ is of finite character. Then, if $K_{\aleph_0} \not\leq H$ and $H$ has complete (induced) subgraphs of arbitrary large finite order, we use the graph $G = K_{\aleph_0}$ to show that $\to H$ is not of finite character: Every finite (induced) subgraph of $G$ is in $\to H$ (it is in fact an induced subgraph of $H$), yet $G$ is not in $\to H$. Hence $K_{\aleph_0} \leq H$ or there is a largest finite value of $n$ such that $K_n \leq H$, i.e., $H$ has a clique number.

Similarly, if $\overline{K_{\aleph_0}} \not\leq H$ and $H$ has edgeless induced subgraphs of arbitrary large finite order, we can use the graph $G = \overline{K_{\aleph_0}}$ to show that $\to H$ is not of finite character. ■

Three remarks are in order.

1. If $\to H$ is of finite character, then by definition, any $G$ whose finite induced subgraphs are all in $\to H$ needs to be in $\to H$ too. This has to be satisfied in particular by the graphs $G = K_{\aleph_0}$ and $G = \overline{K_{\aleph_0}}$ and the structure of the induced subgraphs of these two graphs make it possible to express the necessary conditions of Theorem 11 in terms of the existence of graph parameters, clique and independence numbers.

2. One can now easily construct a hom-property $\to H$ which is *not* of finite character by taking $H$ as a graph which has (induced) complete subgraphs of arbitrary large order without containing $K_{\aleph_0}$ too, for example $H = \bigsqcup_n K_n$, the disjoint union of all finite complete graphs. Such a graph does not satisfy the necessary condition given in Theorem 11 of having a clique number and hence

$\to H$ is not of finite character. ($H$ has independence number $\aleph_0$ so that $\overline{H}$ has clique number $\aleph_0$, but no independence number.)

3. Ruaan Kellerman ([15]) pointed out to us that (in contrast to the strong condition on $H$ in Theorem 9(3) — which suffices for $\to H$ to be a hom-property of finite character) the converse of Theorem 11 is not true. There are indeed graphs with a clique number and an independence number for which the hom-properties fail to be of finite character: Take, for example, for each integer $i \geq 1$, a connected finite graph $M_i$ with $\chi(M_i) = i$, $\omega(M_i) = 2$ and $M_i \leq M_{i+1}$; the Mycielski construction (see [6]) delivers exactly such graphs. Next let $A = \bigsqcup_{i\geq 1} M_i$ and let $B$ be the limit of the ascending sequence $M_1 \leq^+ M_2 \leq^+ \cdots$ of graphs. Then it is easy to check that every finite (induced) subgraph $F$ of $B$ satisfies $F \to A$, indeed, it satisfies $F \leq M_i \leq A$ for some $i$. Yet $B \not\to A$ since homomorphisms preserve connectivity and $B$ is connected; so in order for $B \to A$ to exist, $B \to M_i$ must exist for some component $M_i$ of $A$ and this is impossible since this would imply that $\chi(B) \leq \chi(M_i)$ while $\chi(B) = \aleph_0$ and $\chi(M_i) = i$. Hence $\to A$ is not of finite character. Note that the graph $A$ has chromatic number $\aleph_0$ although it has a small clique number (in fact, it is only 2). An example with small chromatic number seems harder to find; we shall exhibit one in Section 4.4.

### 4.4. Entangled attributes

As indicated previously, in this final subsection we disentangle some of the intricate logical relationships between intrinsic attributes of $H$ and the attribute of $\to H$ of being of finite character. The possibility of characterizing hom-properties of finite character is now linked to the question of the existence of cores for (infinite) graphs. Note that in [2] it is shown that every (di)graph $H$ of which $\to H$ is of finite character has a core. In our first result of the subsection, we use the notation $\mathcal{I}_f$ to denote the set of all finite graphs.

**Theorem 12.** *There exists a connected graph with finite chromatic number but without a finite core if and only if there exists a graph $G$ with finite chromatic number but with $\to G$ not of finite character.*

***Proof.*** We prove the equivalence of the negations of the two statements. Suppose each connected graph $G \in \mathcal{I}$ with finite chromatic number has a finite core. Let $G$ be a graph in $\mathcal{I}$ such that $\chi(G) = n$ for some $n \in \mathbb{N}$. Now let $\mathcal{B} = \{F \in \mathcal{I}_f \mid F \notin \to G\}$. Then $\mathcal{B} \neq \emptyset$ since $K_{n+1} \in \mathcal{B}$. We claim that $\to G = -\mathcal{B}$.

First we show that $\to G \subseteq -\mathcal{B}$. Let $H$ be a graph in $\to G$ and assume $H \notin -\mathcal{B}$. Then there exists a (finite) graph $F$ in $\mathcal{B}$ such that $F \leq H$. Therefore $F \to H$, which implies $F \in \to G$, and in turn implies $F \notin \mathcal{B}$. This, of course, is a contradiction, thus the assumption that $H \notin -\mathcal{B}$ is false. This gives us $\to G \subseteq -\mathcal{B}$.

Now we show that $-\mathcal{B} \subseteq \to G$. But before we do this we prove that $\chi(H) \leq n$ for all $H \in -\mathcal{B}$. Let $H$ be a graph in $-\mathcal{B}$, and suppose $\chi(H) > n$. Then there exists a graph $H^* \in \mathcal{I}_f$ such that $H^* \leq H$ and $\chi(H^*) > n$. ¿From this it follows that $H^* \not\to G$, and thus that $H^* \notin \to G$. Therefore we have that $H^* \in \mathcal{B}$. But this and $H^* \leq H$ imply that $H \notin -\mathcal{B}$, which is, clearly, a contradiction. Thus we have that $\chi(H) \leq n$ for all $H \in -\mathcal{B}$.

We are now ready to prove that $-\mathcal{B} \subseteq \to G$. For a proof by contradiction, suppose there exists a graph $H \in -\mathcal{B}$ such that $H \notin \to G$. Therefore $H \not\to G$, which implies that some component of $H$ is not homomorphic to $G$. Let $H^*$ be such a component, then $H^* \not\to G$. By the previous paragraph $\chi(H)$ is finite therefore $\chi(H^*)$ is finite. This together with our initial assumption imply that $H^*$ has a finite core $C(H^*)$. Now $H[C(H^*)] \not\to G$, otherwise $H[C(H^*)] \to G$ together with $H^* \to H[C(H^*)]$ would imply that $H^* \to G$, yielding a contradiction. By the definition of $\mathcal{B}$ and the finiteness of $H[C(H^*)]$, it follows that $H[C(H^*)] \in \mathcal{B}$. This implies that $H \notin -\mathcal{B}$, yielding a contradiction. Therefore $\to G$ is of finite character.

Next we prove the converse. Let $\to G$ be of finite character for all graphs $G \in \mathcal{I}$ with finite chromatic number. Then assume $G \in \mathcal{I}$, a connected graph with finite chromatic number, does not have a finite core. Now let $H$ be the disjoint union of all finite induced subgraphs of $G$. Clearly $H$ is a graph in $\mathcal{I}$ with finite chromatic number. Now consider the property $\to H$. By our initial assumption $\to H$ is of finite character, thus $G \in \to H$ since all finite induced subgraphs of $G$ belong to $\to H$. Which means $G$ is homomorphic to $H$. But since $G$ is connected it follows that $G$ is homomorphic to some component of $H$. Since all components of $H$ are finite induced subgraphs of $G$, we have $G$ is homomorphic to some finite induced subgraph of itself, thus implying that $G$ does in fact have a finite core. This is a contradiction. ■

We shall construct a connected graph $G$ which has a finite chromatic number but does not have a finite core, but first we need some preliminaries. In [18], Welzl takes planar graphs $F_2^n$, $n \in \mathbb{Z}^+$, known as "flowers", and constructs graphs $S_2^n$, $n \in \mathbb{Z}^+$, which are called "super-flowers". These super-flowers have the following properties. For each positive integer $n$,

1. $S_2^n$ is a connected finite graph;
2. $|V(S_2^n)| < \left|V(S_2^{n+1})\right|$;
3. $S_2^n$ is $K_3$-free;
4. $\chi(S_2^n) = 3$;
5. $S_2^n \to K_3$;
6. Every pair of vertices of $S_2^n$ lie on a cycle of length 5.

In Figure 1 the super-flowers $S_2^i$ for $i = 1, 2, 3$ are depicted.

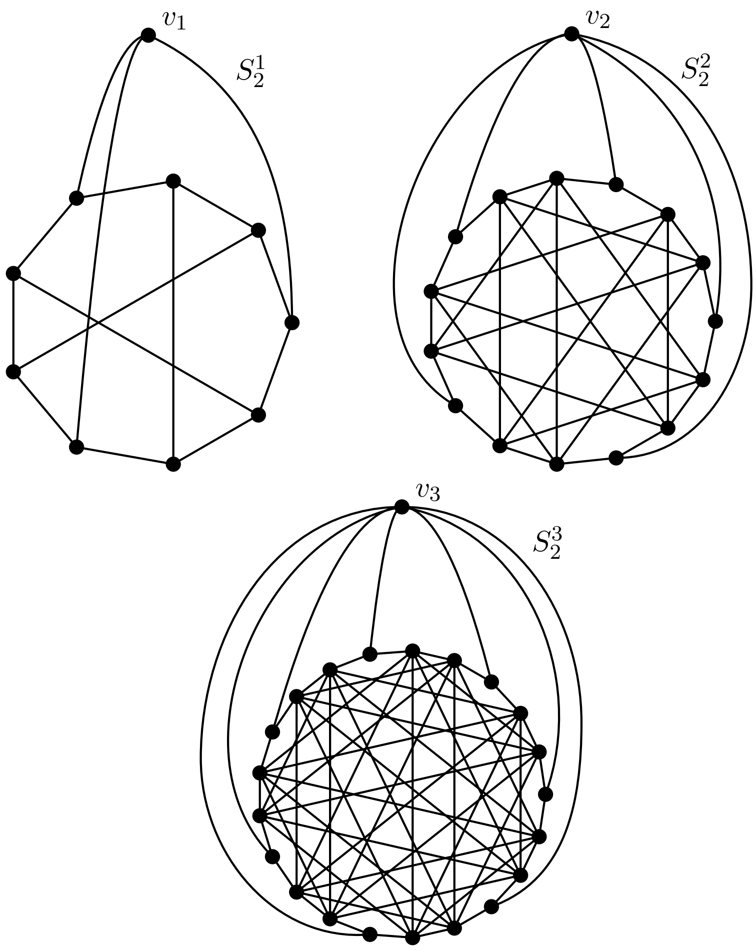


Figure 1. The super-flowers $S_2^1$, $S_2^2$ and $S_2^3$.

We only need the above mentioned properties of these graphs. The proof that the graphs we shall construct have no finite cores will use the properties of these super-flowers given in the following two lemmas.

**Lemma 13.** *For all graphs* $G \in \mathcal{I}$, $S_2^n \to G$ *if and only if* $\omega(G) \geq 3$ *or* $S_2^n$ *is a subgraph of* $G$.

***Proof.*** Let $G$ be a graph in $\mathcal{I}$. If $\omega(G) \geq 3$ or $S_2^n$ is a subgraph of $G$ then $S_2^n \to G$. Suppose $S_2^n \to G$, and let $\varphi$ be a homomorphism from $S_2^n$ to $G$. If $\varphi(x) \neq \varphi(y)$ for all vertices $x \neq y$ in $S_2^n$ then $S_2^n$ is a subgraph of $G$. So assume $\varphi(x) = \varphi(y)$ for some two vertices $x, y \in S_2^n$. Since $\varphi$ is a homomorphism it follows that $x$ and $y$ are non-adjacent vertices in $S_2^n$. By property 6 it follows that there exist two adjacent vertices $w$ and $z$ in $S_2^n$ such that $wx, zy \in E(S_2^n)$. Therefore $G[\{\varphi(x), \varphi(w), \varphi(z)\}]$, the subgraph of $G$ induced by $\{\varphi(x), \varphi(w), \varphi(z)\}$, is isomorphic to $K_3$, thus $\omega(G) \geq 3$. ■

**Lemma 14.** *For all positive integers $m > n$, $S_2^m \not\to S_2^n$.*

***Proof.*** Suppose, to the contrary, that for some positive integers $m > n$ we have $S_2^m \to S_2^n$. Then by Lemma 13 we have that $S_2^m$ is a subgraph of $S_2^n$ or $\omega(S_2^n) \geq 3$. By property 3 we have that $S_2^m$ is a subgraph of $S_2^n$. This implies that $|V(S_2^m)| \leq |V(S_2^n)|$, which contradicts property 2. ■

Note that the graph $\bigsqcup_{n\geq 1} S_2^n$ has chromatic number 3. We are now ready to prove

**Theorem 15.** *The graph $\bigsqcup_{n\geq 1} S_2^n$ does not have a finite core.*

***Proof.*** Suppose $\bigsqcup_{n\geq 1} S_2^n$ does in fact have a finite core, then there exists integers $m > n$ such that $C(\bigsqcup_{n\geq 1} S_2^n)$ is a subgraph of $S_2^n \sqcup S_2^{n+1} \sqcup \cdots \sqcup S_2^m$, from which we arrive at $\bigsqcup_{n\geq 1} S_2^n \to (S_2^n \sqcup S_2^{n+1} \sqcup \cdots \sqcup S_2^m)$. This, of course, implies that $S_2^{m+1} \to (S_2^n \sqcup S_2^{n+1} \sqcup \cdots \sqcup S_2^m)$. Since $S_2^{m+1}$ is connected it follows that $S_2^{m+1} \to S_2^j$ for some integer $n \leq j \leq m$. But by Lemma 14 we have that $S_2^{m+1} \not\to S_2^j$. This is a contradiction, thus $\bigsqcup_{n\geq 1} S_2^n$ does not have a finite core. ■

For all $n \in \mathbb{Z}^+$ there is a vertex (drawn at the top of our superflower diagrams in Figure 1) in $S_2^n$ referred to as the *nucleus*. For all $n \in \mathbb{Z}^+$ let $v_n$ be the nucleus of $S_2^n$. Then let $G$ be the graph with vertex set

$$V(G) = \{x\} \cup V(S_2^1) \cup V(S_2^2) \cup V(S_2^3) \cup \cdots ,$$

and edge set

$$E(G) = \{xv_i \mid i \in \mathbb{Z}^+\} \cup E(S_2^1) \cup E(S_2^2) \cup E(S_2^3) \cup \cdots ,$$

where $x$ is an entirely new vertex. Then $G$ is connected $K_3$-free graph with chromatic number 3. For all $i \in \mathbb{Z}^+$ we shall refer to the subgraph of $G$ induced by the set $V(S_2^i)$ as the $i$'th *bulb* of $G$.

**Theorem 16.** *The graph $G$ does not have a finite core.*

***Proof.*** Suppose $G$ has a finite core $C(G)$, then the vertices of $C(G)$ come from a finite number of bulbs of $G$. Thus for all integers $m > |V(C(G))|$, the $m$'th bulb is homomorphic to $C(G)$, that is $S_2^m \to C(G)$. Which by Lemma 13 implies that $\omega(C(G)) \geq 3$ or $S_2^m$ is a subgraph of $C(G)$. Since $G$ is $K_3$-free it follows that its subgraph $C(G)$ is also $K_3$-free, thus $\omega(C(G)) < 3$, from which follows that $S_2^m$ is a subgraph of $C(G)$. But $|V(S_2^m)| > |V(C(G))|$ for all integers $m > |V(C(G))|$. This is clearly a contradiction, thus $G$ does not possess a finite core. ■

By Theorems 12 and 16 it is clear that there exists a graph $H \in \mathcal{I}$, with finite chromatic number, such that $\to H$ is not of finite character. Next we construct two such graphs $H$, one connected and the other disconnected. Consider the following graphs.

The graph $F$: For each $n \in \mathbb{Z}^+$, let $F_n$ be the graph obtained by adding a new vertex $x_n$ to the disjoint union $\bigsqcup_{i=1}^{n} S_2^i$, and making $x_n$ adjacent to the nucleus $v_i$ of $S_2^i$ for all $1 \le i \le n$. Then let $F$ be the disjoint union of all the $F_n$'s, that is

$$F = \bigsqcup_{n \ge 1} F_n.$$

Then $F$ is a disconnected graph with clique number 2 and chromatic number 3.

The graph $F^*$: Let $F^*$ be the graph obtained from the disjoint union $\bigsqcup_{n \ge 1} S_2^n$ by adding, for each integer $j \in \mathbb{Z}^+$, a new vertex $x_j$ to this union, and joining this vertex to the nucleus $v_i$ of $S_2^i$ for each $i \le j$. We shall refer to the set $\{x_j \mid j \in \mathbb{Z}^+\}$ as the *vine* of $F^*$. Just as in $G$, for all $i \in \mathbb{Z}^+$ we shall refer to the subgraph of $F^*$ induced by the set $V(S_2^i)$ as the $i$'th *bulb* of $F^*$.

Then $F^*$ is a connected graph with clique number 2 and chromatic number 3. In addition all finite subgraphs of $G$ belong to $\to F$ and $\to F^*$. We will show that $G \notin \to F$, and $G \notin \to F^*$, proving that $\to F$ and $\to F^*$ are not of finite character.

**Theorem 17.** *The hom-property $\to F$ is not of finite character.*

***Proof.*** We show that $G \not\to F$. Suppose, to the contrary, that $G \to F$. Then $G$ is homomorphic to some component of $F$ since $G$ is connected. But all components of $F$ are finite subgraphs of $G$, which implies that $G$ has a finite core. This is a contradiction. ∎

**Theorem 18.** *The hom-property $\to F^*$ is not of finite character.*

***Proof.*** We show that $G \not\to F^*$. Suppose that $G \to F^*$, and let $\phi$ be a homomorphism of $G$ to $F^*$. We claim that no vertex belonging to a bulb of $G$ is mapped to the vine of $F^*$. Assume, to the contrary, that, for some $i \ge 1$, there exists a vertex $u_1$ in the $i$'th bulb of $G$ such that $\phi(u_1) = x_j$ for some $j \ge 1$. Then there exists a vertex $u_2$ in the $i$'th bulb of $G$ that is adjacent to $u_1$ in $G$. By property 6 there exist vertices $u_3, u_4, u_5$ in the same bulb of $G$ such that $u_2u_3, u_3u_4, u_4u_5, u_5u_1 \in E(G)$. Therefore, for some integers $k, l \ge 1$, we have $\phi(u_2) = v_k$ and $\phi(u_5) = v_l$. Clearly $k \ne l$ since $k = l$ implies that $\omega(F^*) \ge 3$, which is a contradiction. At this point one can see that, for all integers $m \ne k$ and $n \ne l$, the vertex $\phi(u_3)$ does not belong to the $m$'th bulb of $F^*$, and the vertex $\phi(u_4)$ does not belong to the $n$'th bulb of $F^*$. From this it follows that $\phi(u_3)$ is not in the $k$'th bulb of $F^*$, since this would imply that $\phi(u_4)$ is also in the $k$'th bulb, which would contradict the result $k \ne l$. Similarly $\phi(u_4)$ is not in the $l$'th bulb of $F^*$. Thus $\phi(u_3)$ and $\phi(u_4)$ lie on the vine of $F^*$. Which implies that the 5-cycle $G[\{u_1, u_2, u_3, u_4, u_5\}]$ is homomorphic to the 2-chromatic subgraph $F^*[\{x_i \mid i \in \mathbb{Z}^+\} \cup \{v_i \mid i \in \mathbb{Z}^+\}]$ of $F^*$. This contradiction proves our claim.

Now suppose the vertex $x$ of $G$ is mapped to some $j$'th bulb of $F^*$. Then it follows by the above claim that, for every integer $k > j$, the $k$'th bulb of $G$ is mapped to the $j$'th bulb of $F^*$. That is $S_2^k \to S_2^j$. This clearly contradicts Lemma 14, therefore $\phi(x)$ is not in a bulb of $F^*$. Thus $\phi(x)$ is in the vine of $F^*$. Therefore $\phi(x) = x_j$ for some integer $j > 1$. Which means that $\phi(v_{j+1}) = v_k$ for some $k \leq j$. By the above claim we have $S_2^{j+1} \to S_2^k$. This again is a contradiction, thus our initial assumption is false, proving that $G \not\to F^*$. ■

**Theorem 19.** *If $G \in \mathcal{I}$ is a connected graph with finite chromatic number and no finite core, then $\to F$, where $F$ is the disjoint union of all graphs in $\mathcal{F}(G)$, is not of finite character.*

***Proof.*** Let $G \in \mathcal{I}$ be a connected graph with finite chromatic number, and no finite core. Assume that $\to F$, where $F$ is the disjoint union of all graphs in $\mathcal{F}(G)$, is of finite character. Clearly $\mathcal{F}(G) \subseteq \to F$. Since $\to F$ is of finite character, we have $G \in \to F$, therefore $G \to F$. Because $G$ is connected it follows that $G$ is homomorphic to some component of $F$. But all components of $F$ belong to $\mathcal{F}(G)$, therefore $G$ is homomorphic to a finite subgraph of itself, and thus has a finite core. This is a contradiction, thus the assumption that $\to F$ is of finite character is false. ■

**Theorem 20.** *A connected graph $G \in \mathcal{I}$ with finite chromatic number has a finite core if and only if $\to F$, where $F$ is the disjoint union of all finite subgraphs of $G$, has finite character.*

***Proof.*** Let $G \in \mathcal{I}$ be a connected graph such that $\chi(G)$ is finite, and let $F$ be the disjoint union of all graphs in $\mathcal{F}(G)$.

Suppose that $G$ has a finite core $C(G)$, then by Theorem 2 we have that $\to G = \to C(G)$ is of finite character. We shall prove that $\to F$ is of finite character by showing that $\to F = \to C(G)$. Since $C(G) \in \mathcal{F}(G)$ we have that $C(G) \to F$. In addition to this we have $F' \to C(G)$ for all $F' \in \mathcal{F}(G)$, which gives $F \to C(G)$. Therefore $\to F = \to C(G)$.

Now let $\to F$ be of finite character, then $G$ possesses a finite core, otherwise by Theorem 19 we would obtain that $\to F$ is not of finite character. This completes the proof. ■

Finally we note that all hom-properties generated by graphs with chromatic number at most 2 are of finite character by Theorem 9(3), since they have clique number and chromatic number equal to 1 or 2.

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