

# Aspects of graph homomorphisms and the Hedetniemi Conjecture

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## DECLARATION

I, the undersigned declare that the dissertation, which I hereby submit for the degree **Philosophiae Doctor** at the University of Pretoria, is my own independent work and has not previously been submitted by me or any other person for any degree at this or any other university.

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**Abstract**

The Compactness Theorem for graph colourings, by De Bruijn and Erdős, can be restated as follows. If all the finite subgraphs of a graph  $G$  are homomorphic to a finite complete graph  $K_n$ , then  $G$  is also homomorphic to  $K_n$ . In short, finite complete graphs have the following interesting quality: a graph  $G$  is not homomorphic to a complete graph if and only if some finite subgraph of  $G$  is not homomorphic to said complete graph. There have been many investigations into graphs  $H$  that possess this remarkable characteristic of complete graphs. We further this investigation and describe a graph with finite chromatic number that does not possess the aforementioned quality. Our approach is from a lattice theoretic stand point. That is to say we will study those sets of graphs that are homomorphic to a specific graph. Such sets we call hom-properties, and when a graph possesses

the aforementioned characteristic we will say that it induces or generates a hom-property of finite character. We also study those properties (sets of graphs) that are composed from the union of hom-properties. We do this to gain more insight into the existence of a homomorphism from one graph to another.

Continuing with this study of homomorphisms we describe a technique of constructing, for selected graphs  $G$  and  $H$  bearing the same chromatic number, a graph  $F$  that has the same chromatic number and is homomorphic to both  $G$  and  $H$ . We then apply this technique to solving some special cases of Hedetniemi's Conjecture. The results obtained from this approach extend the results obtained by Burr, Erdős and Lovász, and broaden a result that was obtained by Duffus, Sands and Woodrow, and also by Welzl.

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# 1 Preliminaries

## 1.1 Graph theoretic definitions and notation

The following definitions and notation are predominantly those of [5]. In those instances when the definition is from elsewhere we shall reference the source then.

A *graph*  $G$  is a pair  $(V(G), E(G))$ , where the first,  $V(G)$ , referred to as the *vertex set* of  $G$ , is a non-empty set, and the second,  $E(G)$ , referred to as the *edge set* of  $G$ , is a possibly empty set of 2-element subsets of  $V(G)$ . The *order* of  $G$  refers to  $|V(G)|$ , the cardinal number of  $V(G)$ .

The elements of  $V(G)$  are the *vertices* of  $G$ , while the elements of  $E(G)$  are the *edges* of  $G$ . For all  $\{x, y\} \in E(G)$ , we shall write  $xy$  instead of  $\{x, y\}$ . Furthermore, we make no distinction between  $xy$  and  $yx$ .

For a graph  $G$  we call  $x$  and  $y$  *adjacent* vertices or say  $x$  is *adjacent to*  $y$  if  $xy \in E(G)$ . If  $xy \notin E(G)$ , then we say  $x$  and  $y$  are *non-adjacent* vertices. Given  $x \in V(G)$  the *neighbours* of  $x$  in  $G$  are those vertices of  $G$  that are adjacent to  $x$ , and the *neighbourhood* of  $x$  in  $G$ , denoted by  $N_G(x)$  or  $N(x)$ , is the set of all neighbours of  $x$  in  $G$ .

A graph  $G'$  is a *subgraph* of a graph  $G$ , denoted  $G' \subseteq G$ , if  $V(G') \subseteq V(G)$  and  $E(G') \subseteq E(G)$ . We shall also refer to  $G'$  as an *internal subgraph* of  $G$ . When  $V(G')$  is a proper subset of  $V(G)$  or  $E(G')$  is a proper subset of  $E(G)$ , then we call  $G'$  a *proper subgraph* of  $G$ . A subgraph  $G'$  of  $G$ , is an *internal induced subgraph* of  $G$ , written  $G' \leq G$ , if, for all  $x, y \in V(G')$ ,  $xy \in E(G)$  implies  $xy \in E(G')$ . Given any non-empty subset  $A$  of  $V(G)$ , the induced subgraph of  $G$  whose vertex set is  $A$  will be called the *subgraph of  $G$  induced by  $A$* , written  $G[A]$ . Sometimes, given a subgraph  $G'$  of a graph  $G$ , we may write  $G[G']$  instead of  $G[V(G')]$ . For a graph  $G$ , a vertex  $u$  of  $G$  and an edge  $e$  of  $G$ , we write  $G - u$  to denote the subgraph of  $G$  induced by the set  $V(G) \setminus \{u\}$  and write  $G - e$  to denote the graph obtained from  $G$  after the

removal of  $e$  from the edge set of  $G$ .

Two graphs  $G$  and  $H$  are said to be *isomorphic* if there exists a bijective mapping  $\phi : V(G) \rightarrow V(H)$  such that  $uv \in E(G)$  if and only if  $\phi(u)\phi(v) \in E(H)$ . Such a map we call an *isomorphism*. We say  $G$  and  $H$  are *non-isomorphic* if they are not isomorphic. If  $G''$  is isomorphic to  $G'$ , an internal subgraph of  $G$ , we shall also refer to  $G''$  as a subgraph of  $G$ , similarly for induced subgraphs.

A *path*  $P$  in a graph  $G$ , is a sequence of vertices of  $G$  such that consecutive vertices are adjacent in  $G$  and no vertex in this sequence is repeated. The first and last members of this sequence, when they exist, are called the *end vertices* of  $P$ . Any vertex of  $P$  that is not an end vertex is called an *internal vertex* of  $P$ . If  $u$  and  $v$  are end vertices of a path  $P$  then we on occasion refer to  $P$  as a  $u - v$  path. The *length* of  $P$  is  $|V(P)| - 1$ . A *cycle* in  $G$  is a path in  $G$  with adjacent end vertices. The *length* of a cycle is the order of the cycle. An *even cycle* is a cycle of even order, while an *odd cycle* is one of odd order.

A graph  $G$  is *connected* if, for all  $u, v \in V(G)$ , there exists a  $u - v$  path. It is *disconnected* if it is not connected. A *component* of a graph  $G$  is a connected subgraph of  $G$  that is not a proper subgraph of any connected subgraph of  $G$ . Given a component  $G'$  of a graph  $G$  we use  $G - G'$  to represent the graph  $G[V(G) \setminus V(G')]$ , provided  $V(G) \setminus V(G') \neq \emptyset$ .

A complete graph is one in which any two distinct vertices are adjacent. For a positive integer  $n$  we shall use  $K_n$  and  $C_n$  to represent a complete graph of order  $n$  and a cycle of order  $n$ , respectively. Here  $C_n$  is seen as an autonomous graph, a “cycle in itself”. We use the notation  $K_{\mathbb{N}_0}$  to represent the complete graph whose vertex set is the set  $\{1, 2, \dots\}$ .

The *disjoint union*  $G \sqcup H$  of graphs  $G$  and  $H$  with disjoint vertex sets is that graph with vertex set  $V(G \sqcup H) = V(G) \cup V(H)$  and edge set  $E(G \sqcup H) = E(G) \cup E(H)$ . The disjoint union of more than two graphs is defined similarly.

The *join*  $G \vee H$  of graphs  $G$  and  $H$  with disjoint vertex sets is that graph with vertex set  $V(G \vee H) = V(G) \cup V(H)$  and edge set

$$E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}.$$

The *direct product*  $G \times H$  of graphs  $G$  and  $H$  is a graph whose vertex set  $V(G \times H) = V(G) \times V(H)$  and whose edge set is such that two vertices  $(x_1, x_2)$  and  $(y_1, y_2)$  are adjacent if and only if  $x_1y_1 \in E(G)$  and  $x_2y_2 \in E(H)$ . Since this is the only graph product we deal with we shall refer to it as the *product*. The *Mycielski construct* of a graph  $G$ , denoted by  $M(G)$ , is the graph obtained by first introducing a new vertex  $w$  and then introducing, for each  $x \in V(G)$ , a new vertex  $x'$  and making it adjacent to  $w$  and to all the neighbours of  $x$  in  $G$ . The *Hajós construction* [13] on graphs  $G$  and  $H$  is implemented by deleting an edge  $xy$  in  $G$  and an edge  $uv$  in  $H$ , identifying the vertices  $x$  and  $u$ , and adding the edge  $yv$ . A graph obtained from this construction we call a *Hajós construct*.

For a positive integer  $n$  and a graph  $G$ , an *n-colouring* of  $G$  is an assignment of colours to the vertices of  $G$ , where one colour from  $n$  colours is given to each vertex. A *proper n-colouring* of  $G$  is an  $n$ -colouring of  $G$  in a manner such that no two adjacent vertices are coloured the same. Thus a *proper n-colouring* of  $G$  is a mapping  $f : V(G) \rightarrow \{1, 2, \dots, n\}$  such that  $f(u) \neq f(v)$  for all  $uv \in E(G)$ . Since such a mapping naturally induces a partition of the vertex set of  $G$  into  $n$  independent subsets, we shall also refer to any partition of  $V(G)$  into  $n$  such subsets as a proper  $n$ -colouring. Given any proper  $n$ -colouring  $V_1, \dots, V_n$  of  $G$ , the set  $V_i$  ( $1 \leq i \leq n$ ) of vertices of  $G$  is called a *colour class*. A graph  $G$  is *n-colourable* if there exists a proper  $n$ -colouring of  $G$ . The *chromatic number*  $\chi(G)$  of a graph  $G$  is the least integer  $m$  such that  $G$  admits a proper  $m$ -colouring. A graph  $G$  with  $\chi(G) = n$  is an *n-chromatic graph*. A graph  $G$  is *n-critical*,  $n \geq 2$ , if, for all edges  $e \in E(G)$ ,  $\chi(G - e) < \chi(G)$ . When the chromatic number of  $G$  is



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known we will simply say  $G$  is *critical*.

The *clique number*  $\omega(G)$  of a graph  $G$  is the largest order of all complete subgraphs of  $G$ .

## 1.2 Lattice theoretic definitions and notation

The following definitions and notation are consistent with [6].

By *partial order* we mean a binary relation,  $\leq$ , on some set  $P$ , that is *reflexive*, *antisymmetric* and *transitive*. Where, for  $x, y, z \in P$ , reflexive means  $x \leq x$ , antisymmetric means if  $x \leq y$  and  $y \leq x$  then  $x = y$ , and transitive means that if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ . By *partially ordered set* (or *poset*) we mean a pair  $\langle P, \leq \rangle$ , where  $\leq$  is a partial order on  $P$ . On occasion we write  $P$  for the partially ordered set  $\langle P, \leq \rangle$ . This we do when no ambiguity can arise regarding the partial order on  $P$ . We say that  $x \in P$  is a *minimal element* in  $\langle P, \leq \rangle$  if whenever  $y \leq x$ , for some  $y \in P$ , then we have that  $y = x$ . We say that  $x \in P$  is a *maximal element* in  $\langle P, \leq \rangle$  if whenever  $y \geq x$ , for some  $y \in P$ , then we have that  $y = x$ . An element  $x \in P$  is called the *least element* in  $\langle P, \leq \rangle$  if  $x \leq y$  for all  $y \in P$ . An element  $x \in P$  is called the *greatest element* in  $\langle P, \leq \rangle$  if  $x \geq y$  for all  $y \in P$ . Two elements  $x$  and  $y$  in  $P$  are *incomparable* if  $x \not\leq y$  and  $y \not\leq x$ .

Given any subset  $S$  of a poset  $\langle P, \leq \rangle$ , we say  $x \in P$  is an *upper bound* of  $S$  if  $x \geq y$  for all  $y \in S$ . We say  $x \in P$  is a *lower bound* of  $S$  if  $x \leq y$  for all  $y \in S$ . The *join* of  $S$ ,  $\bigvee S$ , also known as the *least upper bound* of  $S$ , if it exists, is the least element in the set of upper bounds of  $S$ . The *meet* of  $S$ ,  $\bigwedge S$ , also known as the *greatest lower bound* of  $S$ , if it exists, is the greatest element in the set of lower bounds of  $S$ . For any  $x, y \in P$ , we write  $x \vee y$  and  $x \wedge y$  for the join and meet, respectively, of the set  $\{x, y\}$ , when these exist.

A poset  $\langle P, \leq \rangle$  is called a *lattice* if the join and meet of any two elements in  $P$  exist. At times we shall use the notation  $\langle P, \vee, \wedge \rangle$  to mean that a poset  $P$  is a lattice whose joins and meets are defined by  $\vee$  and  $\wedge$ , respectively. A poset  $P$  is said to be a *complete lattice* if the join and meet of all subsets of  $P$  exist. A lattice  $L$  is *modular* if for all  $x, y, z \in L$  such that  $x \leq y$  then  $x \vee (z \wedge y) = (x \vee z) \wedge y$ . A lattice  $L$  is *distributive* if  $x \vee (z \wedge y) = (x \vee z) \wedge (x \vee y)$  for all  $x, y, z \in L$ .

A subset  $M$  of a lattice  $L$ , together with the partial order  $\leq$  inherited from  $L$ , is a *sublattice* of  $L$  if the join and meet of any two elements in  $\langle M, \leq \rangle$  exist and are the join and meet of said elements in  $L$ . That is, for all  $x, y \in M$  we have

$$x \vee_M y = x \vee_L y \quad \text{and} \quad x \wedge_M y = x \wedge_L y,$$

where  $\vee_M, \wedge_M$  and  $\vee_L, \wedge_L$  represent the joins and meets in  $M$  and  $L$ , respectively. We call  $M$  a *join-semi-sublattice* of  $L$  if,

$$x \vee_M y = x \vee_L y \quad \text{for all } x, y \in M$$

and a *meet-semi-sublattice* of  $L$  if,

$$x \wedge_M y = x \wedge_L y \quad \text{for all } x, y \in M.$$

An element  $x$  in a lattice  $L$  is *join-reducible* if there exist distinct elements  $y, z \in L$  such that  $x = y \vee z$  and  $y \neq x \neq z$ . We say  $x$  is *join-irreducible* if  $x$  is not join-reducible and  $x$  is not the least element in  $L$ . Similarly,  $x$  is a *meet-reducible* element in  $L$  if there exist distinct elements  $y, z \in L$  such that  $x = y \wedge z$  and  $y \neq x \neq z$ . If  $x$  is not meet-reducible and it is not the greatest element in  $L$  then we call  $x$  a *meet-irreducible* element of  $L$ .

A closure operator is a mapping  $c: \langle P, \leq \rangle \longrightarrow \langle P, \leq \rangle$  such that, for all  $x, y \in P$ ,

1.  $x \leq c(x)$  (extensive),
2.  $x \leq y$  implies  $c(x) \leq c(y)$  (monotone),
3.  $c(c(x)) = c(x)$  (idempotent).

Let  $L_1$  and  $L_2$  be lattices, then a mapping  $f: L_1 \longrightarrow L_2$  is a *lattice homomorphism* if, for all  $a, b \in L_1$ ,

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

### 1.3 Summary

The existence of a homomorphism from one graph to another has been the subject of much discussion in graph theory. At times when presented with two graphs one can quickly deduce that there does not exist a homomorphism from the one graph to the other. This can be done by identifying certain qualities possessed by the one graph that are absent in the other. One such instance is when the chromatic number of a graph  $G$  is greater than that of a graph  $H$ . That is to say, in such a case one can be confident that there does not exist a homomorphism from  $G$  to  $H$ . But even in such a case we may still not be sure if a homomorphism from  $H$  to  $G$  exists. Thus determining whether a homomorphism exists from one graph to another can be a daunting task, especially when dealing with infinite graphs. The Compactness Theorem for graph colourings, by De Bruijn and Erdős, is but one attempt at alleviating this burden. They proved that the existence of a homomorphism from a graph  $G$  to a finite complete graph is guaranteed by the existence of a homomorphism from every finite subgraph of  $G$  to this complete graph. Granted, this would not be a practical exercise were one presented with say two infinite graphs. But it does shed some light on this subject. It was later proved by A. Salomaa that the existence of a homomorphism from a graph  $G$  to a finite graph  $H$  is guaranteed by the existence of a homomorphism from every finite subgraph of  $G$  to the graph  $H$ . This naturally inspires the following question: If every finite subgraph of a graph  $G$  is homomorphic to an infinite graph  $H$ , is  $G$  homomorphic to  $H$ . If  $G$  is finite, then the answer is obvious. In fact, the answer to this question is ‘not necessarily’. There are examples of graphs  $G$  and  $H$  that are such that every finite subgraph of  $G$  is homomorphic to  $H$  but  $G$  is not homomorphic to  $H$ . Thus when dealing with infinite graphs the existence of a homomorphism seems somewhat challenging. We limit our investigation to countable graphs, and our approach involves the analysis of sets of graphs that are

homomorphic to a specified graph. For a countable graph  $H$ , we define a hom-property  $\rightarrow H$ , as the set of all countable graphs that are homomorphic to  $H$ . We then say a hom-property  $\rightarrow H$  is of finite character if and only if, whenever  $G$  is a countable graph whose finite subgraphs are homomorphic to  $H$ , then  $G$  is homomorphic to  $H$ . One reason for this approach is to try and gain a better understanding of the existence of a homomorphism from a graph  $G$  to a graph  $H$  by also considering all those graphs, finite and infinite graphs that are homomorphic to  $H$ . We are not just concerned with those hom-properties  $\rightarrow H$  that are of finite character but also those which are not. Our intention is to try and identify as many traits as possible in either the graph  $H$  or the set of graphs  $\rightarrow H$  that enable or prevent  $\rightarrow H$  from being of finite character. Although we have not successfully been able to characterise those hom-properties  $\rightarrow H$ , where  $H$  is countable, which are of finite character, we hope the results presented here may aid in the fulfilment of such an endeavour.

In the last chapter we shift our attention to the long-standing conjecture by Hedetniemi, which states that the chromatic number of the direct product of two finite graphs is equal to the minimum of the chromatic numbers of the two graphs. Many special cases of this conjecture have been solved, but not many enough to bring us close to settling this conjecture. In keeping with the theme of homomorphisms, we settle some special cases of this conjecture by constructing, from two graphs  $G$  and  $H$ , a graph  $F$  that is homomorphic to both  $G$  and  $H$ , and whose chromatic number is equal to the minimum of the chromatic numbers of  $G$  and  $H$ . The reason such an approach works will be dealt with later. This is not a very common technique due to its many limitations. One limitation is that it imposes too many restrictions on the graphs  $G$  and  $H$ . The results we obtain are extensions of a result by Burr, Erdős and Lovász, which states that the Hedetniemi Conjecture holds whenever the graph with the least chromatic number, say  $G$ , is such that all its

vertices are contained in a complete subgraph of order  $\chi(G) - 1$ . In addition these results broaden the findings of Duffus, Sands and Woodrow, and also by Welzl. The result by these authors states that the Hedetniemi Conjecture holds whenever  $G$  and  $H$  are connected graphs with equal chromatic number, and both contain a complete subgraph whose order is one below the chromatic number of both graphs. We then end this chapter by highlighting the link between hom-properties and the Hedetniemi Conjecture.

## 2 Hom-properties of finite character

### 2.1 Introduction

The graphs dealt with in this chapter are countable unlabelled simple graphs. By ‘simple’ we mean loop-less, undirected graphs without parallel edges. In addition we make no distinction between isomorphic graphs. Although we are dealing with unlabelled graphs we will on occasion assign labels to their vertices. This we will do only better to describe the vertex set and edge set of the graph in question. We denote the set of all such graphs by  $\mathcal{I}$ . By a property we will mean any subset  $\mathcal{P}$  of  $\mathcal{I}$ .

We mainly study properties (subsets of  $\mathcal{I}$ ) that contain graphs that are all homomorphic to a specified graph  $H$ . Such properties we will call hom-properties induced or generated by  $H$ . This chapter, in some way or another, revolves around these properties. We are interested in whether the existence of a homomorphism from a graph  $G$  to a graph  $H$  is guaranteed by the existence of a homomorphism from each finite subgraph of  $G$  to the graph  $H$ . To be more precise we are interested in those graphs  $H$  which satisfy the following. Whenever a graph  $G$  is such that there exist homomorphisms from each of its finite subgraphs to  $H$  then there exists a homomorphism from  $G$  to  $H$ . When given such a graph  $H$  we say that the set of all graphs for which there exists a homomorphism into  $H$ , that is the hom-property induced by  $H$ , is of finite character. We will give some necessary and sufficient conditions for a hom-property induced by a graph  $H$  to be of finite character. We shall also give an example of a countable graph with finite chromatic number that does not possess this previously mentioned quality. This concept of finite character relates very closely to the Compactness Theorem for graph colourings by De Bruijn and Erdős. We shall in addition mention some related results from other authors.

As an introduction the first two sections are concerned with graph ho-

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momorphisms and hom-properties. This is followed by a discussion on hom-properties of finite character, and then lastly, we take a look at those properties that are unions of hom-properties. For these properties we will describe the lattice they induce and discuss some of the meet and join-irreducible elements in this lattice.



## 2.2 Homomorphisms

In this section we define what we mean by a graph homomorphism, and mention some well known properties of these mappings. The end of this section will be on cores of graphs.

**Definition 1.** [2] *A property is any subset  $\mathcal{P}$  of  $\mathcal{I}$ .*

**Definition 2.** [2] *A property  $\mathcal{P}$  is an induced-hereditary property or i-h property, for short, if  $G \in \mathcal{P}$  implies  $H \in \mathcal{P}$  for all  $H \leq G$ .*

Let  $\mathbb{L}$  be the set of all induced-hereditary properties.

**Proposition 1.** *The pair  $\langle \mathbb{L}, \subseteq \rangle$ , where  $\subseteq$  means ‘to be a subset of’, is a poset.*

*Proof.* Let  $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathbb{L}$ . Clearly  $\mathcal{P} \subseteq \mathcal{P}$ , and if  $\mathcal{P} \subseteq \mathcal{Q}$  and  $\mathcal{Q} \subseteq \mathcal{P}$  then  $\mathcal{P} = \mathcal{Q}$ . Also, if  $\mathcal{P} \subseteq \mathcal{Q}$  and  $\mathcal{Q} \subseteq \mathcal{R}$  then  $\mathcal{P} \subseteq \mathcal{R}$ . All these are a result of set theoretic properties.  $\square$

**Proposition 2.** *The poset  $\langle \mathbb{L}, \subseteq \rangle$  is a complete lattice, where joins and meets are defined by set union and intersection, respectively. That is, the poset  $\langle \mathbb{L}, \subseteq \rangle$  is the lattice  $\langle \mathbb{L}, \cup, \cap \rangle$ .*

*Proof.* Let  $\mathbb{S} \subseteq \mathbb{L}$ , then  $\bigcup \mathbb{S}, \bigcap \mathbb{S} \in \mathbb{L}$ . For all properties  $\mathcal{Q} \in \mathbb{S}$ ,  $\mathcal{Q} \subseteq \bigcup \mathbb{S}$  and  $\bigcap \mathbb{S} \subseteq \mathcal{Q}$ . Therefore  $\bigcup \mathbb{S}$  is an upper bound of  $\mathbb{S}$ , and  $\bigcap \mathbb{S}$  is a lower bound of  $\mathbb{S}$ .

Suppose  $\mathcal{P} \in \mathbb{L}$  is an upper bound of  $\mathbb{S}$ . Since  $\mathcal{Q} \subseteq \mathcal{P}$  for all  $\mathcal{Q} \in \mathbb{S}$  it follows that  $\bigcup \mathbb{S} \subseteq \mathcal{P}$ , therefore  $\bigcup \mathbb{S} = \mathcal{P}$ .

Now suppose  $\mathcal{P} \in \mathbb{L}$  is a lower bound of  $\mathbb{S}$ . Then  $\mathcal{P} \subseteq \mathcal{Q}$  for all  $\mathcal{Q} \in \mathbb{S}$ . From this it follows that  $\mathcal{P} \subseteq \bigcap \mathbb{S}$ , therefore  $\mathcal{P} = \bigcap \mathbb{S}$ .  $\square$

In keeping with [2] we obtain the following.

**Proposition 3.** *The lattice  $\mathbb{L}$  is distributive.*

*Proof.* Let  $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathbb{L}$ . Then

$$\mathcal{P} \vee (\mathcal{Q} \wedge \mathcal{R}) = \mathcal{P} \cup (\mathcal{Q} \cap \mathcal{R}) = (\mathcal{P} \cup \mathcal{Q}) \cap (\mathcal{P} \cup \mathcal{R}) = (\mathcal{P} \vee \mathcal{Q}) \wedge (\mathcal{P} \vee \mathcal{R})$$

and

$$\mathcal{P} \wedge (\mathcal{Q} \vee \mathcal{R}) = \mathcal{P} \cap (\mathcal{Q} \cup \mathcal{R}) = (\mathcal{P} \cap \mathcal{Q}) \cup (\mathcal{P} \cap \mathcal{R}) = (\mathcal{P} \wedge \mathcal{Q}) \vee (\mathcal{P} \wedge \mathcal{R}).$$

Thus  $\mathbb{L}$  is a distributive lattice.  $\square$

**Definition 3.** For all  $G, H \in \mathcal{I}$ , a mapping  $f: V(G) \rightarrow V(H)$  preserves edges if  $f$  is such that  $uv \in E(G)$  implies  $f(u)f(v) \in E(H)$ . The mapping  $f$  preserves non-edges if  $uv \notin E(G)$  implies  $f(u)f(v) \notin E(H)$ .

**Definition 4.** [2] For all  $G, H \in \mathcal{I}$ , a homomorphism from  $G$  to  $H$  is a mapping  $f: V(G) \rightarrow V(H)$  that preserves edges. We write  $G \rightarrow H$  to represent the existence of a homomorphism from  $G$  to  $H$ . On occasion, instead of  $G \rightarrow H$ , we will say  $G$  is homomorphic to  $H$ .

**Definition 5.** Two graphs  $G$  and  $H$  are homomorphically equivalent if  $G \rightarrow H$  and  $H \rightarrow G$  (written  $G \sim H$ ).

**Lemma 1.** Let  $\varphi$  be a homomorphism from  $G$  to  $H$ . The restriction of  $\varphi$  to any subset  $A$  of  $V(G)$  is a homomorphism from  $G[A]$  to  $H$ .

*Proof.* Let  $\varphi$  be a homomorphism from  $G$  to  $H$ , and let  $A$  be any subset  $V(G)$ . Then the mapping  $\gamma: A \rightarrow V(H)$  defined, for all  $x \in A$ , by  $\gamma(x) = \varphi(x)$  is the restriction of  $\varphi$  to  $A$ . Given any adjacent vertices  $u$  and  $v$  of  $G[A]$  it follows that  $\gamma(u)\gamma(v) = \varphi(u)\varphi(v) \in E(H)$ . Thus  $\gamma$  is a homomorphism.  $\square$

**Lemma 2.** If  $G \rightarrow H$  then  $G' \rightarrow H$  for all subgraphs  $G'$  of  $G$ .

*Proof.* Let  $G \rightarrow H$  for some  $G, H \in \mathcal{I}$ . Then there exists a homomorphism  $\varphi$  from  $G$  to  $H$ . Let  $G' \subseteq G$  then  $V(G') \subseteq V(G)$ . It follows by Lemma 1 that the restriction of  $\varphi$  to  $V(G')$ , call it  $f$ , is a homomorphism from  $G[V(G')]$  to  $H$ . Therefore  $f$  preserves the edges of  $G'$  as well, thus it is also a homomorphism from  $G'$  to  $H$ .  $\square$

**Lemma 3.** *If  $G \rightarrow H$  and  $H \rightarrow F$  then  $G \rightarrow F$ .*

*Proof.* Let  $G, H, F \in \mathcal{I}$  be such that  $G \rightarrow H$  and  $H \rightarrow F$ . Then there exist homomorphisms  $\varphi$  and  $\gamma$  from  $G$  to  $H$  and from  $H$  to  $F$ , respectively. We claim that the composite function  $\gamma \circ \varphi$  is a homomorphism from  $G$  to  $F$ . Let  $uv \in E(G)$ , then  $\varphi(u)\varphi(v) \in E(H)$  and consequently  $\gamma(\varphi(u))\gamma(\varphi(v)) \in E(F)$ . Thus  $\gamma \circ \varphi$  is a homomorphism from  $G$  to  $F$ .  $\square$

**Lemma 4.** *If  $G \in \mathcal{I}$  is such that each of its components is homomorphic to a graph  $H \in \mathcal{I}$  then  $G \rightarrow H$ .*

*Proof.* Let  $G$  and  $H$  satisfy the above conditions. Then the mapping from  $V(G)$  to  $V(H)$  whose restriction to each component of  $G$  is a homomorphism from said component to  $H$  is a homomorphism from  $G$  to  $H$ .  $\square$

**Lemma 5.** *For all  $G \in \mathcal{I}$  we have  $\chi(G) \leq n$ , where  $n \in \mathbb{N}$ , if and only if  $G \rightarrow K_n$ .*

*Proof.* Select any graph  $G \in \mathcal{I}$  such that  $\chi(G) \leq n$  for some  $n \in \mathbb{N}$ , and let  $v_1, \dots, v_n$  be the  $n$  vertices of  $K_n$ . Since  $\chi(G) \leq n$  it follows that there exist an integer  $j \leq n$  such that  $V(G)$  can be partitioned into  $j$  colour classes  $V_1, \dots, V_j$ . Then the mapping  $\varphi: V(G) \rightarrow V(K_n)$  that satisfies  $\varphi(V_i) = v_i$  for all  $1 \leq i \leq j$ , is a homomorphism from  $G$  to  $K_n$ .

Let  $G \in \mathcal{I}$  be such that  $G \rightarrow K_n$  for some  $n \in \mathbb{N}$ . Then there exists a homomorphism  $\varphi$  from  $G$  to  $K_n$ . Allow  $v_1, \dots, v_n$  to be the  $n$  vertices of  $K_n$ . In addition, for each  $1 \leq i \leq n$ , let  $\varphi^{-1}(v_i) = \{u \in V(G) \mid \varphi(u) = v_i\}$ . Then the set  $\{\varphi^{-1}(v_i) \mid 1 \leq i \leq n \text{ and } \varphi^{-1}(v_i) \neq \emptyset\}$  is a partition of  $V(G)$  into at most  $n$  colour classes. Therefore  $\chi(G) \leq n$ .  $\square$

**Lemma 6.** *If  $G \rightarrow H$  for some  $G, H \in \mathcal{I}$  then  $\chi(G) \leq \chi(H)$ .*

*Proof.* Let  $G \rightarrow H$ . If  $\chi(H)$  is not finite we are done. So, assume  $\chi(H)$  is finite. Then  $H \rightarrow K_{\chi(H)}$  by Lemma 5. From this and Lemma 3 we obtain  $G \rightarrow K_{\chi(H)}$ . Therefore, by Lemma 5, we obtain  $\chi(G) \leq \chi(H)$ .  $\square$

Notice that  $\chi(G) \leq \chi(H)$  does not imply  $G \rightarrow H$ . As an example the Mycielski construct of  $C_5$ ,  $M(C_5)$ , also known as the Grötsch graph [5], has chromatic number 4, yet  $K_3$ , which is 3-chromatic, is not homomorphic to it.

**Lemma 7.** *If  $\omega(G) > \omega(H)$  then  $G \not\rightarrow H$ .*

*Proof.* Let  $G$  and  $H$  be graphs in  $\mathcal{I}$  such that  $\omega(G) > \omega(H)$ . Then  $G$  contains a complete subgraph of order  $\omega(G)$ . Call this subgraph  $G'$ . Suppose, to the contrary, that  $G \rightarrow H$ . Let  $\gamma$  be a homomorphism from  $G$  to  $H$ . Then, for all distinct vertices  $u, v \in V(G')$ ,  $\gamma(u)\gamma(v) \in E(H)$  since  $uv \in E(G')$ . Therefore  $H[\{\gamma(u) \mid u \in V(G')\}]$  is a complete graph of order  $\omega(G)$ . Which implies  $\omega(H) \geq \omega(G)$ , clearly a contradiction.  $\square$

**Lemma 8.** *If  $\varphi$  is a homomorphism from a countable connected graph  $G$  to a countable graph  $H$ , then  $\varphi$  maps  $G$  to one component of  $H$ .*

*Proof.* Let  $G, H \in \mathcal{I}$  be such that  $G$  is connected and  $G \rightarrow H$ . If  $H$  is connected then we are done, so assume  $H$  is not connected. Suppose there exists a homomorphism  $\varphi$  that maps  $G$  to at least 2 components of  $H$ . Then there exist vertices  $u$  and  $v$  in  $G$  such that  $\varphi(u)$  and  $\varphi(v)$  belong to different components of  $H$ .  $G$  being connected, there exists a  $u - v$  path  $P$  in  $G$ . As a result there exist adjacent vertices  $x$  and  $y$  in  $P$  which are such that  $\varphi(x)$  and  $\varphi(y)$  belong to different components of  $H$ . Therefore  $\varphi(x)\varphi(y) \notin E(H)$  and thus  $\varphi$  does not preserve the edge  $xy$ . Which implies that  $\varphi$  is not a homomorphism. Clearly this is a contradiction.  $\square$

**Definition 6.** [18, 2] *For a finite graph  $G \in \mathcal{I}$ , the core of  $G$ , denoted  $C(G)$ , is a subgraph  $H$  of  $G$  such that  $G \rightarrow H$  and  $G \not\rightarrow H'$  for all proper subgraphs  $H'$  of  $H$ . When  $C(G) = G$  we simply say  $G$  is a core.*

From this definition a finite graph is not a core if and only if it is homomorphic to a proper subgraph of itself. Next we define the core of an infinite graph in  $\mathcal{I}$ .

**Definition 7.** For an infinite graph  $G \in \mathcal{I}$ , the core of  $G$ , denoted  $C(G)$ , is a subgraph  $H$  of  $G$  that is non-isomorphic to  $G$  and is such that  $G \longrightarrow H$  and  $G \not\rightarrow H'$  for all proper subgraphs  $H'$  of  $H$ .

This definition does not allow an infinite graph  $G \in \mathcal{I}$  to be a core of itself. By Lemma 3 any graph  $G$  that has a core satisfies  $G' \longrightarrow C(G)$  for all subgraphs  $G'$  of  $G$ .

### Examples of Cores

1. All finite complete graphs.
2. All  $(k + 1)$ -critical graphs,  $k \in \mathbb{N}$ .
3. The Mycielski construct of an odd cycle.
4. The graph  $K_1 \vee G$ , where  $G$  is a  $(k + 1)$ -critical graph,  $k \in \mathbb{N}$ .
5. The Petersen graph.

The following results were obtained from [18].

**Lemma 9.** Every graph  $G \in \mathcal{I}$  that is homomorphic to a finite subgraph of itself has a core.

*Proof.* Let  $G \in \mathcal{I}$  have a finite subgraph  $H$  such that  $G \longrightarrow H$ . Define  $\mathcal{S}$  to be the set of all subgraphs of  $H$  that  $G$  is homomorphic to. That is  $\mathcal{S} = \{H' \subseteq H \mid G \longrightarrow H'\}$ . Since  $H \longrightarrow H$  it follows that  $\mathcal{S}$  is non-empty. The relation  $\subseteq$ , to be a subgraph of, is a partial order on  $\mathcal{S}$ , hence  $\mathcal{S}$  is a poset. Since  $\mathcal{S}$  is finite it follows that it has a minimal element, thus  $G$  has a core.  $\square$

Not all graphs in  $\mathcal{I}$  have a core. The graph  $G = (K_1 \sqcup K_2 \sqcup K_3 \sqcup \dots)$  depicted in Figure 1 does not have a core [18].

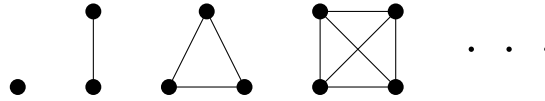


Figure 1:  $G = (K_1 \sqcup K_2 \sqcup K_3 \sqcup \dots)$

This is because the set of subgraphs of  $G$  that  $G$  is homomorphic to, call this set  $\mathcal{S}$ , has elements of the form  $K_i \sqcup K_j \sqcup \dots$ , where  $j > i \geq 1$ . Therefore given any member  $K_i \sqcup K_j \sqcup \dots$  of  $\mathcal{S}$ , there exists a homomorphism from  $G$  to  $K_j \sqcup \dots$ , a proper subgraph of  $K_i \sqcup K_j \sqcup \dots$ . Thus  $\mathcal{S}$  has no minimal element. Consequently  $G$  has no core.

There are however infinite graphs in  $\mathcal{I}$  that do have cores. The graph,  $H = K_3 \sqcup C_5 \sqcup C_7 \sqcup \dots$  depicted in Figure 2, is constructed by taking the disjoint union of a single copy of each odd cycle. The core of this graph is the complete graph  $K_3$ .

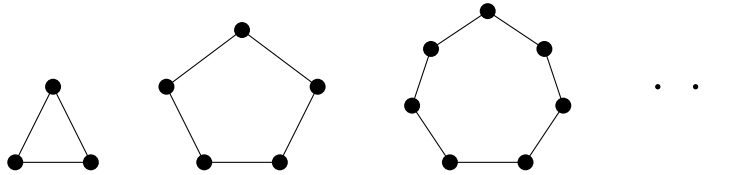


Figure 2: The graph  $H$

### 2.3 Hom-properties

In this section we will define hom-properties and describe the lattice they induce. In addition we will describe the join-irreducible elements in this lattice.

**Definition 8.** [2] *A property  $\mathcal{P} \in \mathbb{L}$  is a hom-property if there exists a graph  $G \in \mathcal{I}$  such that, for all graphs  $H \in \mathcal{I}$ ,*

$$H \in \mathcal{P} \text{ if and only if } H \longrightarrow G.$$

Whence we shall write  $\rightarrow G$  for  $\mathcal{P}$ .

It follows that  $\rightarrow G$  is a hom-property for all  $G \in \mathcal{I}$ . Let  $Hom = \{\rightarrow G \mid G \in \mathcal{I}\}$ .

**Lemma 10.** *For all graphs  $G, H \in \mathcal{I}$*

$$\rightarrow(G \sqcup H) = (\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D},$$

where  $\mathcal{D} = \{F \in \mathcal{I} \mid \text{each component of } F \text{ belongs to } \rightarrow G \text{ or } \rightarrow H\}$ .

*Proof.* Let  $F \in \rightarrow(G \sqcup H)$ , then there exists a homomorphism  $\varphi$  from  $F$  to  $G \sqcup H$ . If  $F$  is connected then  $\varphi$  maps  $F$  into some component of  $G$  or  $H$ . This follows by Lemma 8. Therefore  $F \in \rightarrow G$  or  $F \in \rightarrow H$ , thus  $F \in (\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D}$ . If  $F$  is not connected then the restriction of  $\varphi$  to each component of  $F$  is a homomorphism to a component of  $G$  or  $H$ . This follows by the application of Lemma 1 and Lemma 8. Then  $F \in \mathcal{D}$ , hence  $F \in (\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D}$ . Thus we obtain  $\rightarrow(G \sqcup H) \subseteq (\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D}$ .

Now suppose  $F \in (\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D}$ . If  $F \in \rightarrow G$  then  $F \longrightarrow G$ , therefore  $F \longrightarrow (G \sqcup H)$  by Lemma 4. We obtain the same result for  $F \in \rightarrow H$ . So assume  $F \in \mathcal{D}$ , then each component of  $F$  is homomorphic to  $G$  or  $H$ . Therefore each component of  $F$  is homomorphic to  $G \sqcup H$ . By Lemma 4 we obtain  $F \longrightarrow (G \sqcup H)$ , thus  $F \in \rightarrow(G \sqcup H)$ . As a result  $(\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D} \subseteq \rightarrow(G \sqcup H)$ . Therefore  $(\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D} = \rightarrow(G \sqcup H)$ .  $\square$

**Proposition 4.** *The pair  $\langle Hom, \subseteq \rangle$  is a lattice, where joins and meets are described as follow,*

$$\rightarrow G \vee \rightarrow H = \rightarrow(G \sqcup H)$$

and

$$\rightarrow G \wedge \rightarrow H = \rightarrow G \cap \rightarrow H = \rightarrow(G \times H).$$

*Proof.* Clearly  $(\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D}$ , where  $\mathcal{D}$  is as described above, is an upper bound of  $\rightarrow G$  and  $\rightarrow H$  in  $Hom$ . Suppose  $\rightarrow F$  is an upper bound of  $\rightarrow G$  and  $\rightarrow H$ . Since  $\rightarrow F$  is additive it follows that  $\mathcal{D} \subseteq \rightarrow F$ , hence  $(\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D} \subseteq \rightarrow F$ , therefore

$$\rightarrow(G \sqcup H) = (\rightarrow G) \cup (\rightarrow H) \cup \mathcal{D} = \rightarrow G \vee \rightarrow H.$$

What remains now is to show that

$$\rightarrow G \wedge \rightarrow H = \rightarrow G \cap \rightarrow H = \rightarrow(G \times H).$$

Since the meet of two properties in  $Hom$  as in  $\mathbb{L}$  is defined by their intersection we are only required to show that  $\rightarrow G \cap \rightarrow H = \rightarrow(G \times H)$ . Let  $F \in \rightarrow(G \times H)$  then  $F \rightarrow (G \times H)$ . Now let  $\varphi_1 : V(G \times H) \rightarrow V(G)$  and  $\varphi_2 : V(G \times H) \rightarrow V(H)$  be mappings defined as follows. For all  $(u, v) \in V(G \times H)$ ,

$$\varphi_1((u, v)) = u \text{ and } \varphi_2((u, v)) = v.$$

Then  $\varphi_1$  and  $\varphi_2$  are homomorphisms, so  $(G \times H) \rightarrow G$  and  $(G \times H) \rightarrow H$ . By Lemma 3 we obtain  $F \rightarrow G$  and  $F \rightarrow H$ , therefore  $F \in (\rightarrow G \cap \rightarrow H)$ .

Assume  $F \in (\rightarrow G \cap \rightarrow H)$ , then there exist homomorphisms  $\gamma_1$  and  $\gamma_2$  from  $F$  to  $G$  and from  $F$  to  $H$ , respectively. The mapping  $\gamma : V(F) \rightarrow V(G \times H)$  defined by  $\gamma(u) = (\gamma_1(u), \gamma_2(u))$  is a homomorphism from  $F$  to  $G \times H$ , therefore  $F \in \rightarrow(G \times H)$ , completing the proof.  $\square$



Next we prove that  $Hom$  is in fact a distributive lattice. For this we will employ the theorem below, taken from [6].

**Theorem 1.** *A lattice  $L$  is distributive if and only if, for all  $a, b, c \in L$ ,*

$$a \vee b = c \vee b \text{ and } a \wedge b = c \wedge b \text{ implies } a = c.$$

**Theorem 2.** *The lattice  $Hom$  is distributive.*

*Proof.* Let  $\rightarrow G_1, \rightarrow G_2, \rightarrow G_3$  be properties in  $Hom$  satisfying

$$\rightarrow G_1 \vee \rightarrow G_2 = \rightarrow G_3 \vee \rightarrow G_2$$

and

$$\rightarrow G_1 \wedge \rightarrow G_2 = \rightarrow G_3 \wedge \rightarrow G_2.$$

We show that  $\rightarrow G_1 = \rightarrow G_3$ . Our proof shall be by contradiction. So assume  $\rightarrow G_1 \neq \rightarrow G_3$ , then there exists a graph  $G$  belonging to only one of these properties. Without loss of generality let  $G \in \rightarrow G_1$ . Now assume  $G \in \rightarrow G_2$ , then

$$G \in (\rightarrow G_1 \cap \rightarrow G_2) = (\rightarrow G_3 \cap \rightarrow G_2).$$

This, of course, implies  $G \in \rightarrow G_3$ , which is a contradiction. Therefore  $G$  does not belong to  $\rightarrow G_2$ . Since  $G \in \rightarrow G_1$  it follows, by Proposition 4, that  $G \in (\rightarrow G_1 \vee \rightarrow G_2)$ , and therefore  $G \in (\rightarrow G_3 \vee \rightarrow G_2)$ . By Lemma 10 we have

$$\rightarrow G_3 \vee \rightarrow G_2 = (\rightarrow G_3) \cup (\rightarrow G_2) \cup \mathcal{D},$$

where  $\mathcal{D}$  is described in a similar fashion as earlier. Since  $G \notin \rightarrow G_3$  and  $G \notin \rightarrow G_2$  it follows that  $G \in \mathcal{D}$  and at least one component of  $G$  does not belong to  $\rightarrow G_3$ . Let  $G_\alpha$  be such a component. Then  $G_\alpha \in \rightarrow G_2$  and  $G_\alpha \notin \rightarrow G_3$ . In addition  $G_\alpha \in \rightarrow G_1$  since  $G_\alpha \leq G \in \rightarrow G_1$ . As a result

$$G_\alpha \in (\rightarrow G_1 \cap \rightarrow G_2) = (\rightarrow G_3 \cap \rightarrow G_2),$$

which implies  $G_\alpha \in \rightarrow G_3$ , a contradiction. Thus our initial assumption is false. It follows that  $Hom$  is a distributive lattice.  $\square$

The following two lemmas were obtained from Lemma 5.11 of [6].

**Lemma 11.** *An element  $x$  from a distributive lattice  $L$  is meet-irreducible in  $L$  if and only if whenever  $x \geq b \wedge c$  for some elements  $b$  and  $c$  from  $L$  it follows that  $x \geq b$  or  $x \geq c$ .*

*Proof.* Suppose  $x$  is meet-irreducible and  $x \geq b \wedge c$ . Then  $x = x \vee (b \wedge c) = (x \vee b) \wedge (x \vee c)$  so that  $x = x \vee b$  or  $x = x \vee c$  by the meet-irreducibility of  $x$ . But then it follows that  $x \geq b$  or  $x \geq c$ .

Suppose for the converse that from  $x \geq b \wedge c$  it follows that  $x \geq b$  or  $x \geq c$  and that  $x = b \wedge c$ . Then  $x \leq b$  and  $x \leq c$  follow from this equation while  $x \geq b$  or  $x \geq c$  follows from the given condition, hence  $x = b$  or  $x = c$  as required.  $\square$

**Lemma 12.** *An element  $x$  from a distributive lattice  $L$  is join-irreducible in  $L$  if and only if whenever  $x \leq b \vee c$  for some elements  $b$  and  $c$  from  $L$  it follows that  $x \leq b$  or  $x \leq c$ .*

*Proof.* Suppose  $x$  is join-irreducible and  $x \leq b \vee c$ . Then  $x = x \wedge (b \vee c) = (x \wedge b) \vee (x \wedge c)$ , therefore  $x = x \wedge b$  or  $x = x \wedge c$ , giving us  $x \leq b$  or  $x \leq c$ .

Suppose for the converse that from  $x \leq b \vee c$  it follows that  $x \leq b$  or  $x \leq c$  and that  $x = b \vee c$ . Then  $x \geq b$  and  $x \geq c$  hence  $x = b$  or  $x = c$ , completing our proof.  $\square$

Next we identify join-irreducible elements in  $Hom$  and save the discussion of meet-irreducible elements in  $Hom$  for later.

**Proposition 5.** *The join-irreducible elements in  $Hom$  are precisely those hom-properties  $\rightarrow G$  such that  $G$  is connected.*

*Proof.* We first show that if  $G$  is disconnected then  $\rightarrow G$  is join-reducible. Let  $G$  be a disconnected graph. If  $G_1$  and  $G_2$  are two components of  $G$  such that  $G_1 \rightarrow G_2$ , then  $G \rightarrow (G - G_1)$  by Lemma 4. Since  $(G - G_1) \rightarrow G$  it

follows that  $\rightarrow G = \rightarrow(G - G_1)$ . Therefore it is sufficient to consider only those disconnected graphs  $G$  whose components are mutually non-homomorphic.

Let  $G$  be a disconnected graph whose components are mutually non-homomorphic and let  $G_1$  and  $G_2$  be any two components of  $G$ . Now consider  $\rightarrow(G - G_1)$  and  $\rightarrow(G - G_2)$ . The first does not contain the graph  $G_1$  while the second does and the second does not contain the graph  $G_2$  while the first does. Therefore these two properties are incomparable. Furthermore each is a proper subset of the property  $\rightarrow G$  and

$$\rightarrow(G - G_1) \vee \rightarrow(G - G_2) = \rightarrow((G - G_1) \sqcup (G - G_2)) = \rightarrow G,$$

which, of course, implies that  $\rightarrow G$  is join-reducible.

Now we prove that if  $G$  is connected, then  $\rightarrow G$  is join-irreducible. Assume, for a proof by contradiction, that  $\rightarrow G$  is join-reducible, then there exist graphs  $H$  and  $F$  such that  $\rightarrow H, \rightarrow F \subset \rightarrow G$  and

$$\rightarrow H \vee \rightarrow F = \rightarrow G,$$

which implies  $\rightarrow(H \sqcup F) = \rightarrow G$  by Proposition 4. Which, in turn, implies that  $G \in \rightarrow(H \sqcup F)$ . Therefore  $G$  is homomorphic to  $H \sqcup F$ . Since  $G$  is connected it follows by Lemma 8 that  $G$  is homomorphic to  $H$  or  $F$ . Without loss of generality let  $G$  be homomorphic to  $H$  then  $G \in \rightarrow H$ . Therefore  $\rightarrow G \subseteq \rightarrow H$ , which is a contradiction.  $\square$

## 2.4 Hom-properties of finite character

Before introducing the notion of a hom-property of finite character, we feel it's best to begin with a well known theorem by De Bruijn and Erdős. In 1951, De Bruijn and Erdős obtained the following result.

**Theorem 3.** [7] *If every finite subgraph of a graph  $G$  is  $n$ -colourable then so is  $G$ .*

This theorem, also known as the Compactness Theorem (for graph colourings), was also proven by Hattingh in [15] using semantical entailment in Logic. A formulation of this theorem, one which suits our purposes, is the following.

**Theorem 4.** [7] *If, for every finite subgraph  $F$  of  $G$ ,  $F \rightarrow K_n$  then  $G \rightarrow K_n$ .*

That is, if every finite subgraph of a graph  $G$  is homomorphic to a complete graph  $K_n$  then  $G$  is also homomorphic to  $K_n$ . This, of course, raises the following question: For which graphs  $H$  whenever all the finite subgraphs of a graph  $G$  are homomorphic to  $H$  implies that  $G$  is also homomorphic to  $H$ . This question was partly answered by Salomaa in 1981 when he proved the following.

**Theorem 5.** [20] *For all finite graphs  $H$ , if every finite subgraph  $F$  of a graph  $G$  is such that  $F \rightarrow H$ , then  $G \rightarrow H$ .*

This result was also obtained by Bauslaugh in [1]. The graphs  $H$  with the property described above induce hom-properties  $\rightarrow H$  which we call hom-properties of finite character. Given any graph  $G \in \mathcal{I}$ , let

$$\mathcal{F}(G) = \{H \in \mathcal{I} \mid H \leq G \text{ and } H \text{ is finite}\}.$$

Formally, we say the following.

**Definition 9.** For a graph  $H \in \mathcal{I}$ , the hom-property  $\rightarrow H$  is of finite character if

$$G \longrightarrow H \iff F \longrightarrow H \text{ for all } F \in \mathcal{F}(G)$$

.

In light of this definition, the above results can be formulated as follows.

**Theorem 6.** [7],[15] For all positive integers  $n$ , the hom-property  $\rightarrow K_n$  is of finite character.

**Theorem 7.** [20],[1] For all finite graphs  $H$ , the hom-property  $\rightarrow H$  is of finite character.

**Corollary 1.** If  $H$  is any countable graph which is homomorphically equivalent to a finite graph, then the hom-property  $\rightarrow H$  is of finite character.

*Proof.* Let  $G, H \in \mathcal{I}$  be such that  $G$  is finite and  $G \sim H$ . Then it follows that  $G \longrightarrow H$  and  $H \longrightarrow G$ . Given any graph  $F \in \rightarrow H$ , then we have that  $F \longrightarrow H$ . By Lemma 3, we obtain, from  $F \longrightarrow H$  and  $H \longrightarrow G$ , that  $F \longrightarrow G$ . This implies  $F \in \rightarrow G$ , from which we obtain that  $\rightarrow H \subseteq \rightarrow G$ .

Next consider a graph  $F \in \rightarrow G$ . By definition it follows that  $F \longrightarrow G$ . This together with  $G \longrightarrow H$  implies, by Lemma 3, that  $F \longrightarrow H$ . Which, in turn, implies that  $F \in \rightarrow H$ . Thus we that  $F \in \rightarrow H$ , which yields  $\rightarrow G \subseteq \rightarrow H$ .

From the above argument we that  $\rightarrow H = \rightarrow G$ . By Theorem 7 it follows that  $\rightarrow H$  is of finite character.  $\square$

By Corollary 1, the graph  $H$ , represented in Figure 2, induces a hom-property of finite character. But what of the hom-property induced by the graph  $G$  illustrated in Figure 1? The hom-property induced by this graph  $G$  is not of finite character. To see this consider the complete graph  $K_{\mathbb{N}_0}$  whose vertices are the set of positive integers. Clearly every finite subgraph of  $K_{\mathbb{N}_0}$  is homomorphic to a component of  $G$ , but  $K_{\mathbb{N}_0}$  is not. Thus  $\rightarrow G$  is

not of finite character. Note that  $G$  does not have finite chromatic number. Later we will construct a graph with finite chromatic number that induces a hom-property that is not of finite character. We still have not been able to characterize those infinite graphs  $H \in \mathcal{I}$  that induce hom-properties with finite character.

### 2.4.1 Necessary and sufficient conditions for a hom-property to be of finite-character

We discuss some necessary and sufficient conditions for a hom-property to be of finite character. Before we do so, we introduce some definitions.

**Definition 10.** [19] *A graph  $G \in \mathcal{I}$  has the extension property if for all disjoint finite subsets  $U$  and  $V$  of  $V(G)$ , there exists a vertex  $x \in V(G)$  such that  $x \notin U \cup V$ , and  $x$  is adjacent to every vertex in  $U$  and none in  $V$ .*

**Definition 11.** [3] *A graph  $G \in \mathcal{I}$  has the weak extension property if for all finite subsets  $U$  of  $V(G)$ , there exists a vertex  $x \in V(G)$  such that  $x \notin U$ , and  $x$  is adjacent to every vertex in  $U$ .*

The Rado graph  $R$  [19] is that countable graph whose vertex set  $V(R) = \{1, 2, \dots\}$  and whose edge set  $E(R)$  is described as follows. For any two vertices  $x$  and  $y$  in  $V(R)$  where  $x < y$ ,  $x$  is adjacent to  $y$  if and only if  $y$  has a 1 in the  $x$ th position of its binary expansion.

**Theorem 8.** [19] *A graph  $G$  is isomorphic to the Rado graph if and only if  $G$  has the extension property.*

**Theorem 9.** [19] *If  $H$  has the weak extension property, then, for all countable graphs  $G$ ,  $G \subseteq H$ .*

*Proof.* Let  $H$  be a graph that satisfies the weak extension property, and let  $G \in \mathcal{I}$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots\}$ . We build an injective homomorphism  $\varphi : G \rightarrow H$ . Set  $\varphi(v_1)$  to be any vertex of  $H$ . Suppose, for some integer  $n \geq 1$ , that  $\varphi(v_1), \dots, \varphi(v_n)$  have been determined, then we obtain  $\varphi(v_{n+1})$  as follows. By the weak extension property of  $H$ , there exists a vertex  $x \in V(H)$  such that  $x$  is not in  $\{\varphi(v_1), \dots, \varphi(v_n)\}$  and  $x$  is adjacent to every vertex of  $\{\varphi(v_1), \dots, \varphi(v_n)\}$ . Set  $\varphi(v_{n+1}) = x$ , and complete the construction of  $\varphi$  by recursion. From the construction it is clear that  $\varphi$  is an injective homomorphism, thus  $G \subseteq H$ .  $\square$

The following theorem gives some sufficient condition for a graph  $H$  to induce a hom-property of finite character.

**Theorem 10.** *If a graph  $H \in \mathcal{I}$  satisfies one of the conditions below, then  $\rightarrow H$  is of finite character.*

1.  $H$  is finite.
2.  $H$  has a finite core.
3.  $\omega(H) = \chi(H) = n$  for some positive integer  $n$ .
4.  $H$  has the weak extension property.
5.  $H$  has the extension property, that is,  $H$  is the Rado graph.

*Proof.* By Theorem 7, it follows that 1 is a sufficient condition. Conditions 2 and 3 imply that  $H$  is homomorphically equivalent to a finite graph. From whence it follows, by Corollary 1, that the hom-property induced by  $H$  is of finite character.

Suppose a graph  $H \in \mathcal{I}$  satisfies the weak extension property. Next consider a graph  $G \in \mathcal{I}$  such that  $\mathcal{F}(G) \subseteq \rightarrow H$ . Then, by Theorem 9, we have that  $G \subseteq H$ , from which we obtain that  $G \rightarrow H$ . Which means that  $G \in \rightarrow H$ , and thus the hom-property  $\rightarrow H$  is of finite character.

Now suppose a graph  $H \in \mathcal{I}$  satisfies the extension property. Then, by Theorem 8, it follows that  $H$  is isomorphic to the Rado graph, which is universal in the set of countable graphs. From whence it follows that  $\rightarrow H$  is of finite character.  $\square$

**Lemma 13.** *If  $H \in \mathcal{I}$  does not have a clique number, then  $\rightarrow H$  is not of finite character.*



*Proof.* Let  $H \in \mathcal{I}$  be such that  $H$  does not have a clique number, and assume  $\rightarrow H$  is of finite character. Then for every clique of size  $\alpha$  in  $H$  there exists a clique of size  $\beta$  with  $\beta > \alpha$ . Now let  $\gamma = |V(H)|$ , and consider  $K_\gamma$ . Clearly every finite subgraph of  $K_\gamma$  belongs to  $\rightarrow H$ . Since  $\rightarrow H$  is of finite character we have  $K_\gamma \rightarrow H$ , which implies  $K_\gamma$  is a subgraph of  $H$ , and in turn implies  $\omega(H) = \gamma$ . This is a contradiction.  $\square$

From this lemma it follows that a necessary condition for a hom-property  $\rightarrow H$  to be of finite character is that  $H$  must have a clique number. Below we present a definition that gives a description of a property in terms of those finite graphs that are forbidden from the property.

**Definition 12.** For a given set of graphs  $\mathcal{S}$  we define the property  $-\mathcal{S}$  of  $\mathcal{S}$ -free graphs by

$$G \in -\mathcal{S} \text{ if and only if } S \not\leq G \text{ for every } S \in \mathcal{S}.$$

The theorem below gives a necessary and sufficient condition for a hom-property to be of finite character.

**Theorem 11.** Let  $H \in \mathcal{I}$ , then the hom-property  $\rightarrow H$  is of finite character if and only if  $\rightarrow H = -\mathcal{F}$  for some set  $\mathcal{F}$  of finite graphs.

*Proof.* Suppose that, for some  $H \in \mathcal{I}$ , the hom-property  $\rightarrow H$  is of finite character. Next, let  $\mathcal{F} = \{F \in \mathcal{I} \mid F \not\leq \rightarrow H \text{ and } F \text{ is finite}\}$ . We will show that  $\rightarrow H = -\mathcal{F}$ .

First we show that  $\rightarrow H \subseteq -\mathcal{F}$ . Let  $G \in \rightarrow H$  and suppose that  $G \notin -\mathcal{F}$ . Then there exists a graph  $F \in \mathcal{F}$  such that  $F \leq G$ . This, of course, means that  $F \rightarrow G$ , which implies that  $F \rightarrow H$  since  $G \rightarrow H$ . Therefore we have that  $F \in \rightarrow H$ . This is a contradiction, thus  $\rightarrow H \subseteq -\mathcal{F}$ .

Next we show that  $-\mathcal{F} \subseteq \rightarrow H$ . Let  $G \in -\mathcal{F}$ , then  $F \not\leq G$  for all  $F \in \mathcal{F}$ . This implies that  $G' \notin \mathcal{F}$  for all  $G' \in \mathcal{F}(G)$ . So, for every  $G' \in \mathcal{F}(G)$ , we

have (by the definition of  $\mathcal{F}$ ) that  $G' \in \rightarrow H$  or  $G'$  is infinite. But  $G'$  is finite since it belongs to the set  $\mathcal{F}(G)$ . Therefore  $G' \in \rightarrow H$ , which implies that  $\mathcal{F}(G) \subseteq \rightarrow H$ . Since  $\rightarrow H$  is of finite character it follows that  $G \in \rightarrow H$ , thus  $-\mathcal{F} \subseteq \rightarrow H$ .

Now suppose that  $\rightarrow H = -\mathcal{F}$  for some set  $\mathcal{F}$  of finite graphs. We will show that a graph  $G \in \rightarrow H$  if and only if  $\mathcal{F}(G) \subseteq \rightarrow H$ .

That is, we will show that this assumption implies that  $\rightarrow H$  is of finite character. Let  $G \in \rightarrow H$ , then  $G \rightarrow H$ . This together with  $G' \rightarrow G$  for all  $G' \in \mathcal{F}(G)$  implies that  $G' \rightarrow H$  for all  $G' \in \mathcal{F}(G)$ . Therefore  $G' \in \rightarrow H$  for all  $G' \in \mathcal{F}(G)$ . From this it follows that  $\mathcal{F}(G) \subseteq \rightarrow H$ .

Now assume, for some  $G \in \mathcal{I}$ , that  $\mathcal{F}(G) \subseteq \rightarrow H$ . If  $G \notin \rightarrow H$  then there exists a finite graph  $F \in \mathcal{F}$  such that  $F \leq G$ . This implies that  $F$  belongs to the set  $\mathcal{F}(G)$ . Thus an  $F$ , an element of  $\mathcal{F}(G)$ , does not belong to  $\rightarrow H$ . This is a contradiction, thus  $G \in \rightarrow H$ .  $\square$

### 2.4.2 A hom-property that is not of finite character

Earlier, we mentioned a graph that induces a hom-property that is not of finite character. The graph referred to is the graph whose components are finite complete graphs. This graph does not have a finite chromatic number. What was of interest to us was whether there exists a graph with finite chromatic number that induces a hom-property of finite character. It turns out that such a graph exists, and its existence is guaranteed by the existence of a connected graph that has finite chromatic number and does not have a finite core. This statement is embodied in the following theorem. We use the notation  $\mathcal{I}_f$  to denote the set of all finite graphs in  $\mathcal{I}$ .

**Theorem 12.** *There exists a connected graph with finite chromatic number but without a finite core if and only if there exists a graph  $G$  with finite chromatic number but with  $\rightarrow G$  not of finite character.*

*Proof.* We prove the equivalence of the negations of the two statements.

Suppose each connected graph  $G \in \mathcal{I}$  with finite chromatic number has a finite core. Let  $G$  be a graph in  $\mathcal{I}$  such that  $\chi(G) = n$  for some  $n \in \mathbb{N}$ . Now let  $\mathcal{F} = \{F \in \mathcal{I}_f \mid F \notin \rightarrow G\}$ . Then  $\mathcal{F} \neq \emptyset$  since  $K_{n+1} \in \mathcal{F}$ . We claim that  $\rightarrow G = -\mathcal{F}$ .

First we show that  $\rightarrow G \subseteq -\mathcal{F}$ . Let  $H$  be a graph in  $\rightarrow G$  and assume  $H \notin -\mathcal{F}$ . Then there exists a (finite) graph  $F$  in  $\mathcal{F}$  such that  $F \leq H$ . Therefore  $F \rightarrow H$ , which implies  $F \in \rightarrow G$ , and in turn implies  $F \notin \mathcal{F}$ . This, of course, is a contradiction, thus the assumption that  $H \notin -\mathcal{F}$  is false. This gives us  $\rightarrow G \subseteq -\mathcal{F}$ .

Now we show that  $-\mathcal{F} \subseteq \rightarrow G$ . But before we do this we prove that  $\chi(H) \leq n$  for all  $H \in -\mathcal{F}$ . Let  $H$  be a graph in  $-\mathcal{F}$ , and suppose  $\chi(H) > n$ . Then there exists a graph  $H^* \in \mathcal{I}_f$  such that  $H^* \leq H$  and  $\chi(H^*) > n$ . From this it follows that  $H^* \not\rightarrow G$ , and thus that  $H^* \notin \rightarrow G$ . Therefore we have that  $H^* \in \mathcal{F}$ . But this and  $H^* \leq H$  imply that  $H \notin -\mathcal{F}$ , which is, clearly,

a contradiction. Thus we have that  $\chi(H) \leq n$  for all  $H \in -\mathcal{F}$ .

We are now ready to prove that  $-\mathcal{F} \subseteq \rightarrow G$ . Suppose, to the contrary, that there exists a graph  $H \in -\mathcal{F}$  such that  $H \notin \rightarrow G$ . Then  $H \not\rightarrow G$ , which implies that some component of  $H$  is not homomorphic to  $G$ . Let  $H^*$  be such a component, then  $H^* \not\rightarrow G$ . By the previous paragraph  $\chi(H)$  is finite therefore  $\chi(H^*)$  is finite. This together with our initial assumption imply that  $H^*$  has a finite core  $C(H^*)$ . Now  $H[V(C(H^*))] \not\rightarrow G$ , otherwise  $H[V(C(H^*))] \rightarrow G$  together with  $H^* \rightarrow H[V(C(H^*))]$  would imply that  $H^* \rightarrow G$ , yielding a contradiction. By the definition of  $\mathcal{F}$  and the finiteness of  $H[V(C(H^*))]$ , it follows that  $H[V(C(H^*))] \in \mathcal{F}$ . This implies that  $H \notin -\mathcal{F}$ , yielding a contradiction. Therefore  $\rightarrow G$  is of finite character.

Next we prove the converse. Let  $\rightarrow G$  be of finite character for all graphs  $G \in \mathcal{I}$  with finite chromatic number. Then assume  $G \in \mathcal{I}$ , a connected graph with finite chromatic number, does not have a finite core. Now let  $H$  be the disjoint union of all finite induced subgraphs of  $G$ . Clearly  $H$  is a graph in  $\mathcal{I}$  with finite chromatic number. Now consider the property  $\rightarrow H$ . By our initial assumption  $\rightarrow H$  is of finite character, thus  $G \in \rightarrow H$  since all finite induced subgraphs of  $G$  belong to  $\rightarrow H$ . Which means  $G$  is homomorphic to  $H$ . But since  $G$  is connected it follows that  $G$  is homomorphic to some component of  $H$ . Since all components of  $H$  are finite induced subgraphs of  $G$ , we have  $G$  is homomorphic to some finite induced subgraph of itself, thus implying that  $G$  does in fact have a finite core. This is a contradiction.  $\square$

We will now construct a connected graph  $G$  which has a finite chromatic number but does not have a finite core. Having done so we use this graph to obtain a graph that induces a hom-property that is not of finite character. In [27], Welzl takes planar graphs  $F_2^n$ ,  $n \in \mathbb{Z}^+$ , known as “flowers”, three of which are shown in Figure 3, and constructs graphs  $S_2^n$ ,  $n \in \mathbb{Z}^+$ , which are called “super-flowers”.

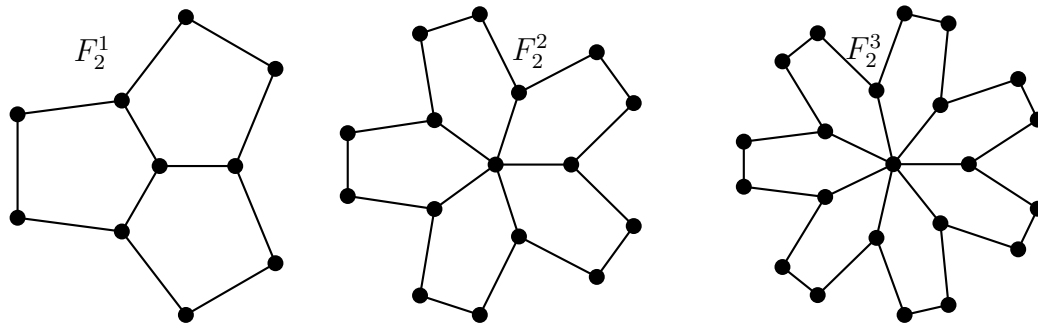


Figure 3: The flowers  $F_2^1, F_2^2$ , and  $F_2^3$

Figure 4 shows an alternative drawing of the flowers  $F_2^1, F_2^2$ , and  $F_2^3$ .

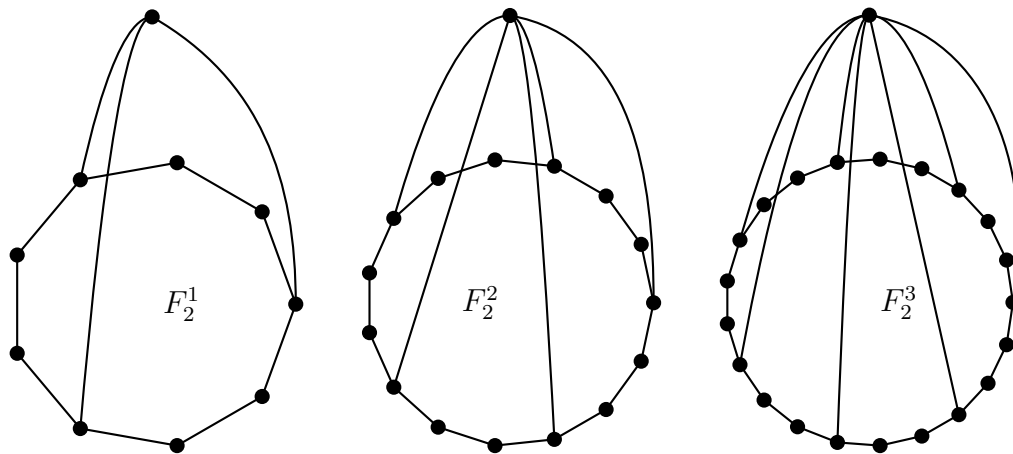


Figure 4: The flowers  $F_2^1, F_2^2$ , and  $F_2^3$

These super-flowers, some of which are depicted in Figure 5 have the following properties. For each positive integer  $n$ ,

1.  $S_2^n$  is a connected finite graph,
2.  $|V(S_2^n)| < |V(S_2^{n+1})|$ ,
3.  $S_2^n$  is  $K_3$ -free,

4.  $\chi(S_2^n) = 3$ ,
5.  $S_2^n \longrightarrow K_3$ ,
6. Every pair of vertices of  $S_2^n$  lie on a cycle of length 5.

Below is a picture of the super-flowers  $S_2^1, S_2^2$ , and  $S_2^3$ .

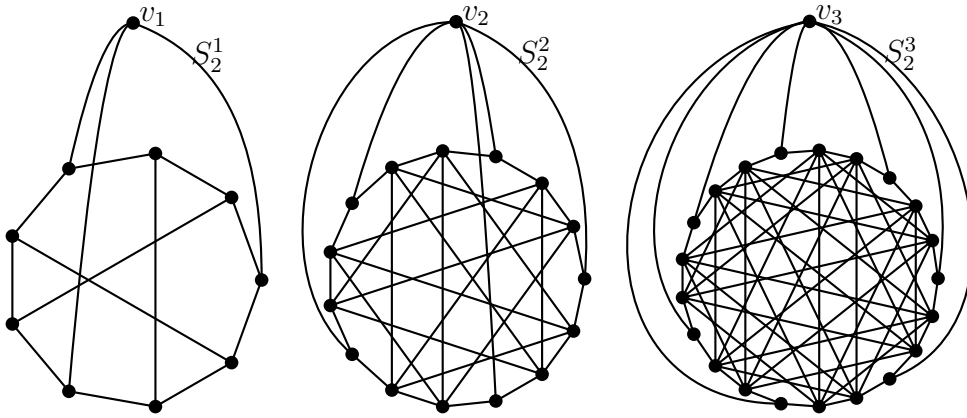


Figure 5: The super-flowers  $S_2^1, S_2^2$ , and  $S_2^3$

We only need the above mentioned properties of these graphs. The proof that the graphs we shall construct have no finite cores will use the properties of these super-flowers given in the following two lemmas.

**Lemma 14.** *For all graphs  $G \in \mathcal{I}$ ,  $S_2^n \longrightarrow G$  if and only if  $\omega(G) \geq 3$  or  $S_2^n$  is a subgraph of  $G$ .*

*Proof.* Let  $G$  be a graph in  $\mathcal{I}$ . If  $\omega(G) \geq 3$  or  $S_2^n$  is a subgraph of  $G$  then  $S_2^n \longrightarrow G$ . Suppose  $S_2^n \longrightarrow G$ , and let  $\varphi$  be a homomorphism from  $S_2^n$  to  $G$ . If  $\varphi(x) \neq \varphi(y)$  for all vertices  $x \neq y$  in  $S_2^n$  then  $S_2^n$  is a subgraph of  $G$ . So assume  $\varphi(x) = \varphi(y)$  for some two vertices  $x, y \in S_2^n$ . Since  $\varphi$  is a homomorphism it follows that  $x$  and  $y$  are non-adjacent vertices in  $S_2^n$ . By property 6 it follows that there exist two adjacent vertices  $w$  and  $z$  in  $S_2^n$

such that  $wx, zy \in E(S_2^n)$ . Therefore  $G[\{\varphi(x), \varphi(w), \varphi(z)\}]$ , the subgraph of  $G$  induced by  $\{\varphi(x), \varphi(w), \varphi(z)\}$ , is isomorphic to  $K_3$ , thus  $\omega(G) \geq 3$ .  $\square$

**Lemma 15.** *For all positive integers  $m > n$ ,  $S_2^m \not\rightarrow S_2^n$ .*

*Proof.* Suppose, to the contrary, that for some positive integers  $m > n$  we have  $S_2^m \rightarrow S_2^n$ . Then by Lemma 14 we have that  $S_2^m$  is a subgraph of  $S_2^n$  or  $\omega(S_2^n) \geq 3$ . By property 3 we have that  $S_2^m$  is a subgraph of  $S_2^n$ . This implies that  $|V(S_2^m)| \leq |V(S_2^n)|$ , which contradicts property 2.  $\square$

Note that the graph  $\sqcup_{n \geq 1} S_2^n$  has chromatic number 3.

**Theorem 13.** *The graph  $\sqcup_{n \geq 1} S_2^n$  does not have a finite core.*

*Proof.* Suppose  $\sqcup_{n \geq 1} S_2^n$  does in fact have a finite core, then there exists an integer  $m \geq 1$  such that  $C(\sqcup_{n \geq 1} S_2^n)$  is a subgraph of  $S_2^1 \sqcup S_2^2 \sqcup \dots \sqcup S_2^m$ , from which we arrive at  $\sqcup_{n \geq 1} S_2^n \rightarrow (S_2^1 \sqcup S_2^2 \sqcup \dots \sqcup S_2^m)$ . This, of course, implies that  $S_2^{m+1} \rightarrow (S_2^1 \sqcup S_2^2 \sqcup \dots \sqcup S_2^m)$ . Since  $S_2^{m+1}$  is connected it follows that  $S_2^{m+1} \rightarrow S_2^j$  for some integer  $j$  with  $1 \leq j \leq m$ . But by Lemma 15 we have that  $S_2^{m+1} \not\rightarrow S_2^j$ . This is a contradiction, thus  $\sqcup_{n \geq 1} S_2^n$  does not have a finite core.  $\square$

For all  $n \in \mathbb{Z}^+$  there is a vertex (drawn at the top of our superflower diagrams) in  $S_2^n$  referred to as the *nucleus*. For all  $n \in \mathbb{Z}^+$  let  $v_n$  be the nucleus of  $S_2^n$ . Then let  $G$  be the graph with vertex set

$$V(G) = \{x\} \cup V(S_2^1) \cup V(S_2^2) \cup V(S_2^3) \cup \dots,$$

and edge set

$$E(G) = \{xv_i \mid i \in \mathbb{Z}^+\} \cup E(S_2^1) \cup E(S_2^2) \cup E(S_2^3) \cup \dots,$$

where  $x$  is an entirely new vertex. Then  $G$  is connected  $K_3$ -free graph with chromatic number 3. For all  $i \in \mathbb{Z}^+$  we shall refer to the subgraph of  $G$  induced by the set  $V(S_2^i)$  as the  $i$ th *bulb* of  $G$ .

**Theorem 14.** *The graph  $G$  does not have a finite core.*

*Proof.* Suppose  $G$  has a finite core  $C(G)$ , then the vertices of  $C(G)$  come from a finite number of bulbs of  $G$ . Thus for all integers  $m > |V(C(G))|$ , the  $m$ th bulb is homomorphic to  $C(G)$ , that is  $S_2^m \rightarrow C(G)$ . Which by Lemma 14 implies that  $\omega(C(G)) \geq 3$  or  $S_2^m$  is a subgraph of  $C(G)$ . Since  $G$  is  $K_3$ -free it follows that its subgraph  $C(G)$  is also  $K_3$ -free, thus  $\omega(C(G)) < 3$ , from which follows that  $S_2^m$  is a subgraph of  $C(G)$ . But  $|V(S_2^m)| > |V(C(G))|$  for all integers  $m > |V(C(G))|$ . This is clearly a contradiction, thus  $G$  does not possess a finite core.  $\square$

By Theorem 12 and 14 it is clear that there exists a graph  $H \in \mathcal{I}$ , with finite chromatic number, such that  $\rightarrow H$  is not of finite character. Next we construct two such graphs  $H$ , one connected and the other disconnected. Consider the following graphs:

The graph  $F$ : For each  $n \in \mathbb{Z}^+$ , let  $F_n$  be the graph obtained by adding a new vertex  $x_n$  to the disjoint union  $\sqcup_{i=1}^n S_2^i$ , and making  $x_n$  adjacent to the nucleus  $v_i$  of  $S_2^i$  for all  $1 \leq i \leq n$ . Then let  $F$  be the disjoint union of all the  $F_n$ 's, that is

$$F = \sqcup_{n \geq 1} F_n.$$

Then  $F$  is a disconnected graph with clique number 2 and chromatic number 3.

The graph  $F^*$ : Let  $F^*$  be the graph obtained from the disjoint union  $\sqcup_{n \geq 1} S_2^n$  by adding, for each integer  $j \in \mathbb{Z}^+$ , a new vertex  $x_j$  to this union, and joining this vertex to the nucleus  $v_i$  of  $S_2^i$  for each  $i \leq j$ . We shall refer to the set



$\{x_j \mid j \in \mathbb{Z}^+\}$  as the *vine* of  $F^*$ . Just as in  $G$ , for all  $i \in \mathbb{Z}^+$  we shall refer to the subgraph of  $F^*$  induced by the set  $V(S_2^i)$  as the  $i$ th *bulb* of  $F^*$ .

Then  $F^*$  is a connected graph with clique number 2 and chromatic number 3. In addition all finite subgraphs of  $G$  belong to  $\rightarrow F$  and  $\rightarrow F^*$ . We will show that  $G \notin \rightarrow F$ , and  $G \notin \rightarrow F^*$ , proving that  $\rightarrow F$  and  $\rightarrow F^*$  are not of finite character.

**Theorem 15.** *The hom-property  $\rightarrow F$  is not of finite character.*

*Proof.* We show that  $G \not\rightarrow F$ . Suppose, to the contrary, that  $G \rightarrow F$ . Then  $G$  is homomorphic to some component of  $F$  since  $G$  is connected. But all components of  $F$  are finite subgraphs of  $G$ , which implies that  $G$  has a finite core. This is a contradiction.  $\square$

**Theorem 16.** *The hom-property  $\rightarrow F^*$  is not of finite character.*

*Proof.* We show that  $G \not\rightarrow F^*$ . Suppose that  $G \rightarrow F^*$ , and let  $\phi$  be a homomorphism of  $G$  to  $F^*$ . We claim that no vertex belonging to a bulb of  $G$  is mapped to the vine of  $F^*$ . Assume, to the contrary, that, for some  $i \geq 1$ , there exists a vertex  $u_1$  in the  $i$ th bulb of  $G$  such that  $\phi(u_1) = x_j$  for some  $j \geq 1$ . Then there exists a vertex  $u_2$  in the  $i$ th bulb of  $G$  that is adjacent to  $u_1$  in  $G$ . By property 3 there exist vertices  $u_3, u_4, u_5$  in the same bulb of  $G$  such that  $u_2u_3, u_3u_4, u_4u_5, u_5u_1 \in E(G)$ . Therefore, for some integers  $k, l \geq 1$ , we have  $\phi(u_2) = v_k$  and  $\phi(u_5) = v_l$ . Clearly  $k \neq l$  since  $k = l$  implies that  $\omega(F^*) \geq 3$ , which is a contradiction. At this point one can see that, for all integers  $m \neq k$  and  $n \neq l$ , the vertex  $\phi(u_3)$  does not belong to the  $m$ th bulb of  $F^*$ , and the vertex  $\phi(u_4)$  does not belong to the  $n$ th bulb of  $F^*$ . From this it follows that  $\phi(u_3)$  is not in the  $k$ th bulb of  $F^*$ , since this would imply that  $\phi(u_4)$  is also in the  $k$ th bulb, which would contradict the result  $k \neq l$ . Similarly  $\phi(u_4)$  is not in the  $l$ th bulb of  $F^*$ . Thus  $\phi(u_3)$  and  $\phi(u_4)$

lie on the vine of  $F^*$ . This implies that the 5-cycle  $G[\{u_1, u_2, u_3, u_4, u_5\}]$  is homomorphic to the 2-chromatic subgraph  $F^*[\{x_i \mid i \in \mathbb{Z}^+\} \cup \{v_i \mid i \in \mathbb{Z}^+\}]$  of  $F^*$ . This contradiction proves our claim.

Now suppose the vertex  $x$  of  $G$  is mapped to some  $j$ th bulb of  $F^*$ . Then it follows by the above claim that, for every integer  $k > j$ , the  $k$ th bulb of  $G$  is mapped to the  $j$ th bulb of  $F^*$ . That is  $S_2^k \rightarrow S_2^j$ . This clearly contradicts Lemma 15, therefore  $\phi(x)$  is not in a bulb of  $F^*$ . Thus  $\phi(x)$  is in the vine of  $F^*$ . Therefore  $\phi(x) = x_j$  for some integer  $j > 1$ . Which means that  $\phi(v_{j+1}) = v_k$  for some  $k \leq j$ . By the above claim we have  $S_2^{j+1} \rightarrow S_2^k$ . This again is a contradiction, thus our initial assumption is false, proving that  $G \not\rightarrow F^*$ .  $\square$

**Theorem 17.** *If  $G \in \mathcal{I}$  is a connected graph with finite chromatic number and no finite core, then  $\rightarrow F$ , where  $F$  is the disjoint union of all graphs in  $\mathcal{F}(G)$ , is not of finite character.*

*Proof.* Let  $G \in \mathcal{I}$  be a connected graph with finite chromatic number, and no finite core. Assume that  $\rightarrow F$ , where  $F$  is the disjoint union of all graphs in  $\mathcal{F}(G)$ , is of finite character. Clearly  $\mathcal{F}(G) \subseteq \rightarrow F$ . Since  $\rightarrow F$  is of finite character, we have  $G \in \rightarrow F$ , therefore  $G \rightarrow F$ . Because  $G$  is connected it follows that  $G$  is homomorphic to some component of  $F$ . But all components of  $F$  belong to  $\mathcal{F}(G)$ , therefore  $G$  is homomorphic to a finite subgraph of itself, and thus has a finite core. This is a contradiction, thus the assumption that  $\rightarrow F$  is of finite character is false.  $\square$

**Theorem 18.** *A connected graph  $G \in \mathcal{I}$  with finite chromatic number has a finite core if and only if  $\rightarrow F$ , where  $F$  is the disjoint union of all finite subgraphs of  $G$ , has finite character.*

*Proof.* Let  $G \in \mathcal{I}$  be a connected graph such that  $\chi(G)$  is finite, and let  $F$  be the disjoint union of all graphs in  $\mathcal{F}(G)$ .

Suppose that  $G$  has a finite core  $C(G)$ , then by Theorem 7 we have that  $\rightarrow G = \rightarrow C(G)$  is of finite character. We shall prove that  $\rightarrow F$  is of finite character by showing that  $\rightarrow F = \rightarrow C(G)$ . Since  $C(G) \in \mathcal{F}(G)$  we have that  $C(G) \rightarrow F$ . In addition to this we have  $F' \rightarrow C(G)$  for all  $F' \in \mathcal{F}(G)$ , which gives  $F \rightarrow C(G)$ . Therefore  $\rightarrow F = \rightarrow C(G)$ .

Now let  $\rightarrow F$  be of finite character, then  $G$  possesses a finite core, otherwise by Theorem 17 we would obtain that  $\rightarrow F$  is not of finite character. This completes the proof.  $\square$

Finally we note that all hom-properties generated by graphs with chromatic number at most 2 are of finite character by Theorem 10(3), since they have clique number and chromatic number equal to 1 or 2.

## 2.5 Hom-hereditary properties

In this section we investigate properties that are unions of hom-properties. We study the lattice they induce and characterize some of the join and meet-irreducible elements in this lattice. Some of the results in this section are taken from [2].

**Definition 13.** *A property  $\mathcal{P} \subseteq \mathcal{I}$  is hom-hereditary if, for all  $H \in \mathcal{P}$ ,  $\rightarrow H \subseteq \mathcal{P}$ .*

Note that each hom-hereditary property  $\mathcal{P}$  is an induced-hereditary property. That is, for a hom-hereditary property  $\mathcal{P}$  whenever  $G \in \mathcal{P}$  it follows that  $H \in \mathcal{P}$  for all  $H \leq G$ . This means that all hom-hereditary properties are members of  $\mathbb{L}$ . We quickly prove this. Let  $G$  belong to a hom-hereditary property  $\mathcal{P}$ , and let  $H$  be an induced subgraph of  $G$ . Then  $H$  belongs to  $\rightarrow G$  because  $H \rightarrow G$ . Therefore, by definition of hom-hereditary property, we have that  $\rightarrow G \subseteq \mathcal{P}$ . From this it follows that  $H \in \mathcal{P}$ .

This above definition implies that all hom-properties are hom-hereditary. We let  $\mathbb{H}$  be the set of all hom-hereditary properties in  $\mathbb{L}$ .

**Lemma 16.**  *$\mathbb{H}$  is generated by Hom through taking the union of elements in Hom, that is, for all  $\mathcal{S} \subseteq \text{Hom}$ ,*

$$\bigcup \mathcal{S} \in \mathbb{H},$$

and for all properties  $\mathcal{P} \in \mathbb{H}$  there exists  $\mathcal{S} \subseteq \text{Hom}$  such that

$$\mathcal{P} = \bigcup \mathcal{S}.$$

*Proof.* Let  $\mathcal{S} \subseteq \text{Hom}$ , and consider any graph  $G \in \bigcup \mathcal{S}$ . Then  $G \in \rightarrow H$  for some  $\rightarrow H \in \mathcal{S}$ , therefore  $\rightarrow G \subseteq \rightarrow H$ , and as a consequence  $\rightarrow G \subseteq \bigcup \mathcal{S}$ , hence  $\bigcup \mathcal{S} \in \mathbb{H}$ .

Now let  $\mathcal{P} \in \mathbb{H}$ , and let  $\mathcal{S} = \{\rightarrow H \mid H \in \mathcal{P}\}$ . Then  $\bigcup \mathcal{S} = \bigcup_{H \in \mathcal{P}} \rightarrow H$ , and by definition we have  $\bigcup_{H \in \mathcal{P}} \rightarrow H \subseteq \mathcal{P}$ . For all  $H \in \mathcal{P}$  we have  $H \in \rightarrow H \subseteq \bigcup_{H \in \mathcal{P}} \rightarrow H$ , thus  $\mathcal{P} \subseteq \bigcup_{H \in \mathcal{P}} \rightarrow H$ , yielding  $\mathcal{P} = \bigcup \mathcal{S}$ .  $\square$

Given any property  $\mathcal{P} \subseteq \mathcal{I}$  it follows that  $\bigcup_{H \in \mathcal{P}} \rightarrow H = \bigcup \mathcal{S}$  where  $\mathcal{S} = \{\rightarrow H \mid H \in \mathcal{P}\}$ , thus, by Lemma 16, we have that  $\bigcup_{H \in \mathcal{P}} \rightarrow H \in \mathbb{H}$  since  $\mathcal{S} \subseteq \text{Hom}$ .

**Lemma 17.**  $\mathbb{H}$  is a sublattice of  $\mathbb{L}$ .

*Proof.* To prove that  $\mathbb{H}$  is a sublattice of  $\mathbb{L}$  we shall show that, for all  $\mathcal{P}, \mathcal{Q} \in \mathbb{H}$ , we have  $\mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q} \in \mathbb{H}$ . Let  $\mathcal{P}, \mathcal{Q} \in \mathbb{H}$ , and let  $G \in \mathcal{P} \cup \mathcal{Q}$ , and  $H \in \mathcal{P} \cap \mathcal{Q}$ . Then, without loss of generality  $G \in \mathcal{P}$ , and hence  $\rightarrow G \subseteq \mathcal{P} \subseteq \mathcal{P} \cup \mathcal{Q}$ . In addition  $H \in \mathcal{P}$  and  $H \in \mathcal{Q}$ , therefore  $\rightarrow H \subseteq \mathcal{P}$  and  $\rightarrow H \subseteq \mathcal{Q}$ . Thus we have  $\rightarrow H \subseteq \mathcal{P} \cap \mathcal{Q}$ . This proves that  $\mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q} \in \mathbb{H}$ .  $\square$

**Lemma 18.**  $\mathbb{H}$  is complete and distributive.

*Proof.* Let  $\mathbb{S} \subseteq \mathbb{H}$ , and let  $G \in \bigcup \mathbb{S}$  and  $H \in \bigcap \mathbb{S}$ . Then  $G \in \mathcal{P}$  for some  $\mathcal{P} \in \mathbb{S}$ , thus  $\rightarrow G \subseteq \mathcal{P}$ , and as a consequence  $\rightarrow G \subseteq \bigcup \mathbb{S}$ , yielding  $\bigcup \mathbb{S} \in \mathbb{H}$ . In addition we have  $H \in \mathcal{Q}$  for all  $\mathcal{Q} \in \mathbb{S}$ , which implies  $\rightarrow G \subseteq \mathcal{Q}$  for all  $\mathcal{Q} \in \mathbb{S}$ , and in turn implies  $\rightarrow G \subseteq \bigcap \mathbb{S}$ , offering  $\bigcap \mathbb{S} \in \mathbb{H}$ . Thus  $\mathbb{H}$  is a complete lattice.

$\mathbb{H}$  is distributive because it is a sublattice of the distributive lattice  $\mathbb{L}$ .  $\square$

Let  $f : \mathfrak{P}(\mathcal{I}) \rightarrow \mathbb{H}$ , where  $\mathfrak{P}(\mathcal{I})$  is the power set of  $\mathcal{I}$ , be the mapping defined as follows, for all  $\mathcal{P} \in \mathfrak{P}(\mathcal{I})$ ,

$$f(\mathcal{P}) = \bigcup_{H \in \mathcal{P}} \rightarrow H.$$

Then one can with relative ease show that  $f$  does not preserve meets, and thus fails to be a lattice homomorphism.

**Lemma 19.** For all  $G \in \mathcal{I}$ ,  $\rightarrow G$  is join irreducible in  $\mathbb{H}$ .

*Proof.* Suppose that, for some  $G \in \mathcal{I}$ ,  $\rightarrow G$  is join-reducible in  $\mathbb{H}$ . Then there exist  $\mathcal{P}, \mathcal{Q} \in \mathbb{H}$  such that  $\mathcal{P} \neq \mathcal{Q}$ ,  $\mathcal{P} \subset \rightarrow G$ ,  $\mathcal{Q} \subset \rightarrow G$ , and  $\rightarrow G = \mathcal{P} \cup \mathcal{Q}$ .

Which implies  $G \in \mathcal{P}$  or  $G \in \mathcal{Q}$ . Without loss of generality let  $G \in \mathcal{P}$ . Then  $\rightarrow G \subseteq \mathcal{P}$  since  $\mathcal{P} \in \mathbb{H}$ . This contradicts  $\mathcal{P} \subset \rightarrow G$ , thus our initial assumption is false.  $\square$

A property  $\mathcal{P} \in \mathbb{H}$  is called *compact* if, for every  $\mathbb{S} \subseteq \mathbb{H}$ ,

$$\mathcal{P} \subseteq \bigvee \mathbb{S} \implies \mathcal{P} \subseteq \bigvee \mathbb{T} \text{ for some finite } \mathbb{T} \subseteq \mathbb{S}.$$

**Theorem 19.**  $\mathcal{P}$  is a compact element in  $\mathbb{H}$  if and only if

$$\mathcal{P} = \bigcup_{G \in \mathcal{F}} \rightarrow G$$

for some finite  $\mathcal{F} \subseteq \mathcal{I}$ .

*Proof.* Suppose  $\mathcal{P}$  is a compact property in  $\mathbb{H}$ . Let  $\mathbb{Q} = \{\rightarrow G \mid G \in \mathcal{P}\}$ ; then  $\mathbb{Q} \subseteq \mathbb{H}$  and  $\mathcal{P} \subseteq \bigvee \mathbb{Q}$ , in fact, by Lemma 16,  $\mathcal{P} = \bigvee \mathbb{Q}$ . By our initial hypothesis it follows that there exists a finite subset  $\mathbb{T}$  of  $\mathbb{Q}$  such that  $\mathcal{P} \subseteq \bigvee \mathbb{T}$ . Therefore, for some positive integer  $n$ , we have  $\mathbb{T} = \{\rightarrow G_1, \dots, \rightarrow G_n\}$ , with  $\rightarrow G_i \in \mathbb{Q}$  for all  $1 \leq i \leq n$ . Thus we have

$$\mathcal{P} \subseteq \bigvee \mathbb{T} = \bigcup_{i=1}^n \rightarrow G_i \subseteq \bigcup_{G \in \mathcal{P}} \rightarrow G = \bigvee \mathbb{Q} = \mathcal{P},$$

which implies

$$\mathcal{P} = \bigcup_{i=1}^n \rightarrow G_i = \bigcup_{G \in \mathcal{F}} \rightarrow G,$$

with  $\mathcal{F} = \{G_1, \dots, G_n\}$

Now suppose  $\mathcal{P} = \bigcup_{G \in \mathcal{F}} \rightarrow G$  for some finite  $\mathcal{F} \subseteq \mathcal{I}$ , and allow  $\mathbb{S} \subseteq \mathbb{H}$  to be such that  $\mathcal{P} \subseteq \bigvee \mathbb{S}$ . Then, for all  $G \in \mathcal{F}$ , we have  $G \in \mathcal{P}$ , and hence  $G \in \mathcal{Q}_G$  for some property  $\mathcal{Q}_G \in \mathbb{S}$ , and, since  $\mathbb{S} \subseteq \mathbb{H}$ , we have  $\rightarrow G \subseteq \mathcal{Q}_G$ . Therefore

$$\mathcal{P} = \bigcup_{G \in \mathcal{F}} \rightarrow G \subseteq \bigcup_{G \in \mathcal{F}} \mathcal{Q}_G = \bigvee_{G \in \mathcal{F}} \mathcal{Q}_G,$$

proving that  $\mathcal{P}$  is compact.  $\square$

By Theorem 19 it follows that all hom-properties are compact elements of  $\mathbb{H}$ . We shall denote the set of compact elements in  $\mathbb{H}$  by  $K(\mathbb{H})$ .

**Theorem 20.** *The lattice  $\mathbb{H}$  is algebraic, that is, for all  $\mathcal{P} \in \mathbb{H}$ ,*

$$\mathcal{P} = \bigvee \{\mathcal{K} \in K(\mathbb{H}) \mid \mathcal{K} \subseteq \mathcal{P}\}.$$

*Proof.* Let  $\mathcal{P} \in \mathbb{H}$ , then with the aid of Lemma 16 we have

$$\mathcal{P} = \bigcup_{G \in \mathcal{P}} \rightarrow G \subseteq \bigvee \{\mathcal{K} \in K(\mathbb{H}) \mid \mathcal{K} \subseteq \mathcal{P}\}.$$

We will now prove the reverse inclusion. Let  $H \in \bigvee \{\mathcal{K} \in K(\mathbb{H}) \mid \mathcal{K} \subseteq \mathcal{P}\}$ , then  $H \in \mathcal{K}$  for some  $\mathcal{K} \in \{\mathcal{K} \in K(\mathbb{H}) \mid \mathcal{K} \subseteq \mathcal{P}\}$  since this supremum is a union. Therefore  $\rightarrow H \subseteq \mathcal{K} \subseteq \mathcal{P}$ , which implies  $H \in \mathcal{P}$ , and in turn implies  $\bigvee \{\mathcal{K} \in K(\mathbb{H}) \mid \mathcal{K} \subseteq \mathcal{P}\} \subseteq \mathcal{P}$ .  $\square$

For a nontrivial property  $\mathcal{P}$  in  $\mathbb{L}$  we define the set of *minimal forbidden subgraphs* of  $\mathcal{P}$ , denoted by  $F(\mathcal{P})$ , as

$$F(\mathcal{P}) = \{G \in \mathcal{I} \mid G \notin \mathcal{P} \text{ and all finite proper induced subgraphs of } G \text{ are in } \mathcal{P}\}.$$

We note that  $F(\mathcal{P}) = \emptyset$  if and only if  $\mathcal{P}$  is of finite character.

**Lemma 20.** *For a nontrivial property  $\mathcal{P} \in \mathbb{L}$  we have  $G \in \mathcal{P}$  if and only if no induced subgraph of  $G$  is in  $F(\mathcal{P})$ .*

*Proof.* Let  $\mathcal{P}$  be a nontrivial property in  $\mathbb{L}$ .

Now let  $G \in \mathcal{P}$  and suppose some induced subgraph  $G'$  of  $G$  belongs to  $F(\mathcal{P})$ . Since  $\mathcal{P} \in \mathbb{L}$  we have  $G' \in \mathcal{P}$ , which contradicts the assumption that  $G' \in F(\mathcal{P})$ .

Now let  $G \in \mathcal{I}$  be such that no induced subgraph of  $G$  belongs to  $F(\mathcal{P})$ , and suppose  $G \notin \mathcal{P}$ . We will show that these two assumptions imply that every finite proper induced subgraph of  $G$  belongs to  $\mathcal{P}$ . This, of course, will imply  $G \in F(\mathcal{P})$ , and thus yield a contradiction.

So, suppose  $G'$  is a finite proper induced subgraph of  $G$  that does not belong to  $\mathcal{P}$ . Let  $\mathcal{S} = \{F \leq G' \mid F \notin \mathcal{P}\}$ , then this set is not empty since  $G' \in \mathcal{S}$ , and  $K_1 \notin \mathcal{S}$  since  $\mathcal{P}$  is nontrivial. Then  $\langle \mathcal{S}, \leq \rangle$  is a finite partially ordered set, and thus has a minimal element, say  $H$ , satisfying  $H \notin \mathcal{P}$  and every finite proper induced subgraph of  $H$  belongs to  $\mathcal{P}$ . This implies  $H \in F(\mathcal{P})$  with  $H \leq G' \leq G$ , that is  $H \leq G$ . This contradicts the assumption that no induced subgraph of  $G$  belongs to  $F(\mathcal{P})$ . Thus we have that every finite proper induced subgraph of  $G$  belongs to  $\mathcal{P}$ . This completes our proof.  $\square$

**Lemma 21.** *Let  $\mathcal{P} \in \mathbb{L}$  be such that  $F(\mathcal{P}) = \{K_n\}$ , where  $n > 1$  is a positive integer. Then  $\mathcal{P} \in \mathbb{H}$ .*

*Proof.* Let  $\mathcal{P}$  be as stated above, and let  $G \in \mathcal{P}$ . Then, by Lemma 20,  $\omega(G) < n$ , and thus  $\rightarrow G$  contains only graphs whose clique numbers are less than  $n$ . By Lemma 20, again, we have  $\rightarrow G \subseteq \mathcal{P}$ , thus  $\mathcal{P} \in \mathbb{H}$ .  $\square$

**Lemma 22.** *Let  $\mathcal{P} \in \mathbb{L}$  be such that  $F(\mathcal{P})$  is a set of finite graphs, each of which is not a complete graph. Then  $\mathcal{P} \notin \mathbb{H}$ .*

*Proof.* Let  $\mathcal{P}$  be as stated above, and assume, for a proof by contradiction, that  $\mathcal{P} \in \mathbb{H}$ . Let  $n = \min\{\chi(H) \mid H \in F(\mathcal{P})\}$ , then there is a graph  $H_0 \in F(\mathcal{P})$  with  $\chi(H_0) = n$  and  $|V(H)| > |V(K_n)|$  for all  $H \in F(\mathcal{P})$ . Therefore no induced subgraph of  $K_n$  belongs to  $F(\mathcal{P})$ , and thus by Lemma 20, we have  $K_n \in \mathcal{P}$ . Since  $\mathcal{P} \in \mathbb{H}$  it follows that  $\rightarrow K_n \subseteq \mathcal{P}$ . But  $H_0 \in \rightarrow K_n$  since  $\chi(H_0) = n$ . This implies that  $H_0$  belongs to  $\mathcal{P}$ , and not to  $F(\mathcal{P})$ , which is a clear contradiction.  $\square$

For an easy example of Lemma 22 consider the following. Let  $\mathcal{P} \in \mathbb{L}$  be such that  $F(\mathcal{P}) = \{C_5\}$ , then  $K_3$  has no induced subgraphs that belong to  $\{C_5\}$ , therefore  $K_3 \in \mathcal{P}$ . Thus  $\mathcal{P} \in \mathbb{H}$  would imply  $\rightarrow K_3 \in \mathcal{P}$ , and in turn imply  $C_5 \in \mathcal{P}$ , which is a contradiction, therefore  $\mathcal{P} \notin \mathbb{H}$ .



**Lemma 23.** *Let  $\mathcal{P} \in \mathbb{L}$  be such that  $F(\mathcal{P})$  is a set of finite graphs. Then  $\mathcal{P} \in \mathbb{H}$  implies  $H$  is a core for all  $H \in F(\mathcal{P})$ , and at least one  $H \in F(\mathcal{P})$  is a complete graph.*

*Proof.* Let  $\mathcal{P}$  be a property with the attributes stated above. Assume  $\mathcal{P}$  belongs to  $\mathbb{H}$ , then by Lemma 22 it follows that some graph  $H \in F(\mathcal{P})$  is a complete graph. Then assume that there exists a graph  $G \in F(\mathcal{P})$  that is not a core. Consider  $C(G)$ , which is a proper induced subgraph of  $G$ . From  $G \in F(\mathcal{P})$  it follows that  $C(G) \in \mathcal{P}$ . But then  $\mathcal{P} \in \mathbb{H}$  implies  $\rightarrow C(G) \subseteq \mathcal{P}$ , which implies  $G \in \mathcal{P}$  since  $G \rightarrow C(G)$ . This is a contradiction, thus the assumption that  $G$  is not a core is false.  $\square$

Lemma 23 implies that if the above described property  $\mathcal{P}$  belongs to  $\mathbb{H}$  then every graph  $H \in F(\mathcal{P})$  is connected.

**Lemma 24.** *Let  $\mathcal{P}, \mathcal{Q} \in \mathbb{L}$  be such that  $F(\mathcal{Q}) \subseteq F(\mathcal{P})$ , then  $\mathcal{P} \subseteq \mathcal{Q}$ .*

*Proof.* Let  $\mathcal{P}, \mathcal{Q}$  be properties as described above, and let  $G \in \mathcal{P}$ , then, by Lemma 20, no induced subgraph of  $G$  belongs to  $F(\mathcal{P})$ , and hence no induced subgraph of  $G$  belongs to  $F(\mathcal{Q})$  since  $F(\mathcal{Q}) \subseteq F(\mathcal{P})$ . Thus by Lemma 20 we have  $G \in \mathcal{Q}$ , which yields  $\mathcal{P} \subseteq \mathcal{Q}$ .  $\square$

**Lemma 25.** *Let  $\mathcal{P} \in \mathbb{L}$  be such that  $F(\mathcal{P}) = \{K_n\}$ , where  $n > 1$  is a positive integer. Then  $\mathcal{P} \in \mathbb{H}$  and  $\mathcal{P}$  is a meet-irreducible with respect to  $\mathbb{L}$ .*

*Proof.* Let  $\mathcal{P}$  be a property as described above. We know that  $\mathcal{P} \in \mathbb{H}$  by Lemma 21. Let  $\mathcal{Q}, \mathcal{R} \in \mathbb{L}$  be such that  $\mathcal{P}$  is a proper subset of both properties. Then each of these two properties contains at least one graph that is not in  $\mathcal{P}$ , and, by Lemma 20, each of these graphs contains an induced subgraph isomorphic to  $K_n$ . Thus  $K_n$  belongs to both  $\mathcal{Q}$  and  $\mathcal{R}$ , and as a result  $K_n \in \mathcal{Q} \cap \mathcal{R} = \mathcal{Q} \wedge \mathcal{R}$ . Therefore  $\mathcal{P} \subset \mathcal{Q} \wedge \mathcal{R}$ , proving that  $\mathcal{P}$  is indeed meet-irreducible with respect to  $\mathbb{L}$ .  $\square$

**Lemma 26.** *Let  $\mathcal{P} \in \mathbb{H}$  be such that  $F(\mathcal{P})$  is a set of finite graphs. Then  $\mathcal{P}$  is a join-irreducible element of  $\mathbb{H}$ .*

*Proof.* Let  $\mathcal{P}$  be a property as described above, and assume, to the contrary, that  $\mathcal{P}$  is join-reducible. Then there exist proper subsets,  $\mathcal{Q}, \mathcal{R} \in \mathbb{H}$ , of  $\mathcal{P}$ , satisfying  $\mathcal{Q} \not\subseteq \mathcal{R}$ ,  $\mathcal{R} \not\subseteq \mathcal{Q}$ , and  $\mathcal{Q} \cup \mathcal{R} = \mathcal{P}$ . Thus there exist  $G \in \mathcal{Q}$ , and  $F \in \mathcal{R}$  such that  $G \notin \mathcal{R}$  and  $F \notin \mathcal{Q}$ . Since  $\mathcal{P}$  is the union of  $\mathcal{Q}$  and  $\mathcal{R}$  it follows that  $G$  and  $F$  belong to  $\mathcal{P}$ . From this we can deduce that no induced subgraph of  $G \sqcup F$  belongs to  $F(\mathcal{P})$ , since if such a subgraph existed it would imply that either an induced subgraph of  $G$  or  $F$  belongs to  $F(\mathcal{P})$ , or there exists a disconnected graph in  $F(\mathcal{P})$ . The first would contradict  $G \in \mathcal{P}$  or  $F \in \mathcal{P}$  respectively, while the second would contradict Lemma 23. Thus by Lemma 20 we have  $G \sqcup F$  belongs to  $\mathcal{P}$ , which implies  $G \sqcup F$  belongs to  $\mathcal{Q}$  or  $\mathcal{R}$ . Without loss of generality let  $G \sqcup F \in \mathcal{Q}$ , then this implies  $F \in \mathcal{Q}$ , which is a contradiction.  $\square$

**Lemma 27.** *Let  $\mathcal{P} \in \mathbb{L}$  be such that  $F(\mathcal{P})$  is a set of finite graphs. Then  $\mathcal{P} \in \mathbb{H}$  if and only if, for all  $H \in F(\mathcal{P})$ ,  $H \rightarrow F$  for any  $F \in \mathcal{I}$  implies that  $H' \leq F$  for some  $H' \in F(\mathcal{P})$ .*

*Proof.* Let  $\mathcal{P}$  be a property with the attributes stated above. If  $\mathcal{P} \in \mathbb{H}$  we assume that for some graph  $H \in F(\mathcal{P})$  there exists a graph  $F \in \mathcal{I}$  such that  $H \rightarrow F$  and  $H' \not\leq F$  for all  $H' \in F(\mathcal{P})$ . Then, by Lemma 20,  $F \in \mathcal{P}$ , and as a consequence  $\rightarrow F \subseteq \mathcal{P}$ . This implies  $H \in \mathcal{P}$ , which is a contradiction.

Now let  $\mathcal{P}$  be such that, for all  $H \in F(\mathcal{P})$ ,  $H \rightarrow F$  for any  $F \in \mathcal{I}$  implies that  $H' \leq F$  for some  $H' \in F(\mathcal{P})$ . Then assume that  $\mathcal{P} \notin \mathbb{H}$ . Then there exists a graph  $F \in \mathcal{P}$  such that  $\rightarrow F \not\subseteq \mathcal{P}$ . Which means that there exists a graph  $G \in \rightarrow F$  such that  $G \notin \mathcal{P}$ . Therefore, by Lemma 20, there exists a graph  $H \in F(\mathcal{P})$  such that  $H \leq G$ . Since  $G \rightarrow F$  it follows that  $H \rightarrow F$ , and by an earlier assumption we have that  $H' \leq F$  for some graph

$H' \in F(\mathcal{P})$ . Which implies, by Lemma 20, that  $F \notin \mathcal{P}$ . This is, clearly, a contradiction.  $\square$

Let  $\mathcal{P} \subseteq \mathcal{I}$  be such that  $F(\mathcal{P})$  is a set of finite cores, then  $\longrightarrow$  is a partial order on  $F(\mathcal{P})$ . Antisymmetry follows from the fact that the elements in  $F(\mathcal{P})$  are cores, and transitivity follows from the transitive property of homomorphisms. Thus  $F(\mathcal{P})$  may contain minimal elements.

**Lemma 28.** *Let  $\mathcal{P} \in \mathbb{H}$  be such that  $F(\mathcal{P})$  is a set of finite graphs with at least two minimal elements. Then  $\mathcal{P}$  is a meet-reducible element of  $\mathbb{H}$ .*

*Proof.* Let  $\mathcal{P}$  be a property as described above, and let  $H', H'' \in F(\mathcal{P})$  be any two minimal elements. Then  $H'$  and  $H''$  are incomparable in  $F(\mathcal{P})$ , that is  $H' \not\rightarrow H''$  and  $H'' \not\rightarrow H'$ . Which means  $H'' \notin \rightarrow H'$  and  $H' \notin \rightarrow H''$ . Let  $\mathcal{Q} = \mathcal{P} \cup \rightarrow H'$  and  $\mathcal{R} = \mathcal{P} \cup \rightarrow H''$ , then  $\mathcal{Q}, \mathcal{R} \in \mathbb{H}$ , and  $\mathcal{P}$  is a proper subset of  $\mathcal{Q}$  and  $\mathcal{R}$  since  $H' \in \mathcal{Q}$ ,  $H'' \in \mathcal{R}$ , and  $H', H'' \notin \mathcal{P}$ . In addition we have  $\mathcal{Q} \not\subseteq \mathcal{R}$  and  $\mathcal{R} \not\subseteq \mathcal{Q}$  since  $H'$  and  $H''$  belong to different properties.

We claim that  $\mathcal{P} = \mathcal{Q} \cap \mathcal{R}$ , which we intend to prove by contradiction. Assume that  $\mathcal{P} \neq \mathcal{Q} \cap \mathcal{R}$ , then  $\mathcal{P} \subset \mathcal{Q} \cap \mathcal{R}$ , and hence there exists a graph  $F \in \mathcal{Q} \cap \mathcal{R}$  such that  $F \notin \mathcal{P}$ . This means  $F \in \rightarrow H'$  and  $F \in \rightarrow H''$ . Since  $F \notin \mathcal{P}$  it follows, by Lemma 20, that  $H^* \leq F$  for some  $H^* \in F(\mathcal{P})$ . This implies  $H^* \rightarrow H'$  and  $H^* \rightarrow H''$ , which in turn implies  $H' \rightarrow H^*$  and  $H'' \rightarrow H^*$  since  $H'$  and  $H''$  are minimal elements of  $F(\mathcal{P})$ . From this we obtain  $H' \rightarrow H''$  and  $H'' \rightarrow H'$ , which is a contradiction. This completes the proof.  $\square$

**Lemma 29.** *Let  $\mathcal{P} \in \mathbb{H}$  be such that  $F(\mathcal{P})$  is a set of finite graphs containing a least element. Then  $\mathcal{P}$  is a meet-irreducible element of  $\mathbb{H}$ .*

*Proof.* Let  $\mathcal{P}$  be a property with the attributes stated above, and let  $H$  be the least element of  $F(\mathcal{P})$ . Allow  $\mathcal{Q}$  and  $\mathcal{R}$  to be any two properties in  $\mathbb{H}$  such that  $\mathcal{P} \subset \mathcal{Q}$  and  $\mathcal{P} \subset \mathcal{R}$ . Then there exist graphs  $G \in \mathcal{Q}$  and  $F \in \mathcal{R}$

such that  $G, F \notin \mathcal{P}$ . By Lemma 20 there exists  $H', H'' \in F(\mathcal{P})$  such that  $H' \leq G$  and  $H'' \leq F$ . Since  $H$  is the least element in  $F(\mathcal{P})$  it follows that  $H \rightarrow H'$  and  $H \rightarrow H''$ . Which means  $H \rightarrow G$  and  $H \rightarrow F$ , and thus  $H \in \rightarrow G \subseteq \mathcal{Q}$  and  $H \in \rightarrow F \subseteq \mathcal{R}$ . Therefore we obtain  $H \in \mathcal{Q} \cap \mathcal{R}$ . Since  $H \notin \mathcal{P}$  it follows that  $\mathcal{P}$  is a proper subset of  $\mathcal{Q} \cap \mathcal{R}$ , and hence  $\mathcal{P}$  is meet-irreducible in  $\mathbb{H}$ .  $\square$

**Lemma 30.** *Let  $\mathcal{P} \in \mathbb{H}$  be such that  $F(\mathcal{P})$  is a set of finite graphs. Then  $\mathcal{P}$  is meet-irreducible in  $\mathbb{H}$  if and only if, for all  $G, H \in F(\mathcal{P})$ , there exists  $F \in F(\mathcal{P})$  such that  $F \rightarrow G$  and  $F \rightarrow H$ .*

*Proof.* Let  $\mathcal{P}$  be a property with the attributes stated above, and assume, first, that for some  $G, H \in F(\mathcal{P})$ , there does not exist  $F \in F(\mathcal{P})$  such that  $F \rightarrow G$  and  $F \rightarrow H$ . Then, proceeding as we did in the proof of Lemma 28, let  $\mathcal{Q} = \mathcal{P} \cup \rightarrow H$  and  $\mathcal{R} = \mathcal{P} \cup \rightarrow G$ . Then  $\mathcal{Q}, \mathcal{R} \in \mathbb{H}$ , and  $\mathcal{P} \subset \mathcal{Q}$ ,  $\mathcal{P} \subset \mathcal{R}$ ,  $\mathcal{Q} \not\subseteq \mathcal{R}$ , and  $\mathcal{R} \not\subseteq \mathcal{Q}$ .

We will show that  $\mathcal{P} = \mathcal{Q} \cap \mathcal{R}$ . Assume, to the contrary, that  $\mathcal{P} \neq \mathcal{Q} \cap \mathcal{R}$ , then  $\mathcal{P} \subset \mathcal{Q} \cap \mathcal{R}$ , and hence there exists a graph  $F \in \mathcal{Q} \cap \mathcal{R}$  such that  $F \notin \mathcal{P}$ . This means  $F \in \rightarrow H$  and  $F \in \rightarrow G$ . Since  $F \notin \mathcal{P}$  it follows, by Lemma 20, that  $H^* \leq F$  for some  $H^* \in F(\mathcal{P})$ . This implies  $H^* \rightarrow H$  and  $H^* \rightarrow G$ , and contradicts our the assumption that there does not exist a graph in  $F(\mathcal{P})$  that is homomorphic to both  $G$  and  $H$ . Therefore  $\mathcal{P} = \mathcal{Q} \cap \mathcal{R}$ , and  $\mathcal{P}$  is meet-reducible.

Now let  $\mathcal{P}$  be such that, for all  $G, H \in F(\mathcal{P})$ , there exists  $F \in F(\mathcal{P})$  such that  $F \rightarrow G$  and  $F \rightarrow H$ . Then allow  $\mathcal{Q}$  and  $\mathcal{R}$  to be any two properties in  $\mathbb{H}$  such that  $\mathcal{P} \subset \mathcal{Q}$  and  $\mathcal{P} \subset \mathcal{R}$ . Then there exist graphs  $G \in \mathcal{Q}$  and  $H \in \mathcal{R}$  such that  $G, H \notin \mathcal{P}$ . By Lemma 20 there exists  $H', H'' \in F(\mathcal{P})$  such that  $H' \leq G$  and  $H'' \leq H$ . By our initial assumption there exists a graph  $F \in F(\mathcal{P})$  it follows that  $F \rightarrow H'$  and  $F \rightarrow H''$ . Which means  $F \rightarrow G$  and  $F \rightarrow H$ , and thus  $F \in \rightarrow G \subseteq \mathcal{Q}$  and  $F \in \rightarrow H \subseteq \mathcal{R}$ . Therefore we

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obtain  $F \in \mathcal{Q} \cap \mathcal{R}$ . Since  $F \notin \mathcal{P}$  it follows that  $\mathcal{P}$  is a proper subset of  $\mathcal{Q} \cap \mathcal{R}$ , and hence  $\mathcal{P}$  is meet-irreducible in  $\mathbb{H}$ . This completes the proof.  $\square$

## 3 Results on the Hedetniemi Conjecture

### 3.1 Introduction

In 1966 Hedetniemi conjectured that the chromatic number of the direct product of two finite graphs is equal to the minimum of the chromatic number of the factors. That is, for all finite graphs  $G$  and  $H$ ,

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

This conjecture has enjoyed much attention over the years, and our interest in it is to consider some special cases for which it holds. By simply considering the mappings  $f : V(G \times H) \rightarrow V(G)$  and  $g : V(G \times H) \rightarrow V(H)$  defined, for all  $(u, v) \in V(G \times H)$ , by

$$f((u, v)) = u \quad \text{and} \quad g((u, v)) = v,$$

one can easily show that the following inequality holds:

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}.$$

These mappings  $f$  and  $g$ , which are projections to the first and second coordinate respectively, are homomorphisms from  $G \times H$ . Therefore  $G \times H \rightarrow G$  and  $G \times H \rightarrow H$ , which implies that  $\chi(G \times H) \leq \chi(G)$  and  $\chi(G \times H) \leq \chi(H)$ , and thus establishes the inequality mentioned earlier. The difficulty with this conjecture lies in proving that

$$\chi(G \times H) \geq \min\{\chi(G), \chi(H)\}.$$

Below is an equivalent formulation of Hedetniemi's conjecture.

**Hedetniemi's Conjecture.** [16] *For all positive integers  $n$  and all finite graphs  $G$  and  $H$ , if  $\chi(G) = \chi(H) = n$ , then  $\chi(G \times H) = n$ .*

Many special cases for which this conjecture holds have been studied, but we pay special attention to those cases where at least one of the graphs  $G$  and  $H$  has a clique number that is one below its chromatic number. In short, our concern is with the following conjecture.

**Conjecture 1.** *For all positive integers  $n$ , and all graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n+1$ , if  $G$  is connected and  $\omega(G) = n$ , then  $\chi(G \times H) = n+1$ .*

The reason we ignore the case where  $\chi(G) = \omega(G)$  is simply because in such a scenario it can quite easily be shown that the Hedetniemi Conjecture holds. Now, the fascination with this aforementioned conjecture arises from a theorem by Duffus and Sauer. The hypothesis of which, if true, would enable us to place a lower bound on the the chromatic number of the direct product of two finite graphs. Here is the result by Duffus and Sauer.

**Theorem 21.** [9] *If Conjecture 1 holds, then  $\chi(G \times H) \geq \frac{n}{2}$ , for all finite graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n$ .*

We do not settle Conjecture 1, but the results mentioned here, we hope, bring us closer to doing so.

All graphs considered in this chapter, except in Section 3.7, are simple finite graphs. In Section 3.7 we shall include infinite simple graphs. The focus of this chapter is on two of many notable well-known results pertaining to Hedetniemi's conjecture. Both these results are linked to Conjecture 1. The first of these two results is due to Burr, Erdős and Lovász, who showed that when every vertex of a graph  $G$  with  $\chi(G) = n+1$  is contained in an  $n$ -clique, then  $\chi(G \times H) = n+1$  whenever  $\chi(H) = n+1$ . The second, by Duffus, Sands and Woodrow, and, obtained independently by Welzl, states that the same is true when  $G$  and  $H$  are connected graphs each with clique number  $n$ . We offer new results which improve, or at least extend, the aforementioned two results. Of the new results, the main one reads as follows: If  $G$  is a graph

with  $\chi(G) = n + 1$  and has the property that the subgraph of  $G$  induced by those vertices of  $G$  that are not contained in an  $n$ -clique is homomorphic to an  $(n + 1)$ -critical graph  $H$ , then  $\chi(G \times H) = n + 1$ . We will show that this result is an improvement of the result by the first authors. In addition it implies a special case of the result obtained by the second set of authors, and covers some cases neglected by their result. Our approach will employ a construction of a graph  $F$ , with chromatic number  $n + 1$ , that is homomorphic to  $G$  and  $H$ .

The following two theorems, informally mentioned earlier, are the inspiration behind the results we will later introduce.

**Theorem 22.** (Burr, Erdős and Lovász [4]) *For all positive integers  $n$  and all graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n + 1$ , if every vertex of  $G$  lies in an  $n$ -clique then  $\chi(G \times H) = n + 1$ .*

**Theorem 23.** (Duffus, Sands and Woodrow [8], Welzl [28]) *For all positive integers  $n$  and all connected graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n + 1$ , if  $\omega(G) = \omega(H) = n$ , then  $\chi(G \times H) = n + 1$ .*

From these results two questions immediately arise. The first, inspired by Theorem 22, is how many of the vertices of  $G$  are required to be in an  $n$ -clique. The second, arising from Theorem 23, is whether we can obtain the same result if  $\omega(H) \leq n - 1$ . Both of these questions we will partially answer, but we require that some restrictions be placed upon  $G$  and  $H$ . Our answer to both is as follows. If those vertices of  $G$  not belonging to an  $n$ -clique induce a subgraph of  $G$  that is homomorphic to an  $(n + 1)$ -critical subgraph of  $H$ , then the clique number of  $H$  may be relaxed to below  $n$ , and the number of vertices of  $G$  not in an  $n$ -clique need not be prescribed.

What Theorem 22 and Theorem 23 have in common is that they require at least one of the graphs  $G$  and  $H$  to have a clique number precisely one less



than the chromatic number. This is what establishes their link to Conjecture 1.

Many of the proofs in this chapter involve, for graphs  $G$  and  $H$ , the construction of a graph  $F$  that is homomorphic to  $G$  and  $H$ , and has the same chromatic number as  $G$  and  $H$ . The following lemma sheds light on why such an approach would be successful.

**Lemma 31.** *Let  $G$  and  $H$  be graphs with  $\chi(G) = \chi(H) = n$ , then  $\chi(G \times H) = n$  if and only if there exists a graph  $F$  with  $\chi(F) = n$  satisfying  $F \rightarrow G$  and  $F \rightarrow H$ .*

*Proof.* Let  $G$  and  $H$  be graphs such that  $\chi(G) = \chi(H) = n$ .

Assume that  $\chi(G \times H) = n$ , then let  $F = G \times H$ . Therefore  $F \rightarrow G$  and  $F \rightarrow H$ , and we are done.

Now suppose there exists a graph  $F$  with  $\chi(F) = n$  satisfying  $F \rightarrow G$  and  $F \rightarrow H$ . We know from the previous chapter that  $\rightarrow(G \times H) = \rightarrow G \cap \rightarrow H$ . Since  $F \rightarrow G$  and  $F \rightarrow H$ , it follows that  $F \in (\rightarrow G \cap \rightarrow H)$ . Therefore  $F \in \rightarrow(G \times H)$ , which implies that  $F \rightarrow G \times H$ . From this it follows that  $n = \chi(F) \leq \chi(G \times H) \leq n$ , which yields  $\chi(G \times H) = n$ .  $\square$

The first difficulty with such an approach is that the graph  $F$  may not exist. The second is in identifying or constructing such a graph  $F$  for any two graphs  $G$  and  $H$ . This is probably why this technique has mostly been neglected. As one would expect with any construction, there are many limitations. The biggest drawback with the constructions we employ is that in order for them to be effective in proving any special case of the Hedetniemi Conjecture, they require that at least one of the two graphs  $G$  and  $H$  has a clique number precisely one below the chromatic number. In addition, though maybe not necessarily, they enforce some restrictions on the other factor of the direct product. There maybe a way around this second draw-

back, one which we haven't been able to find, but as one will see there is no way around the first, unless the constructions are heavily modified.

For the interested reader, an alternative proof of Theorem 22, involving uniquely  $n$ -colourable graphs, can be found in [8]. M. El-Zahar and N. Sauer have offered alternative proofs of Theorem 22 and Theorem 23 in [10], using an approach that is neither utilized here nor in [4], [8], and [28]. For more results concerning the Hedetniemi Conjecture we recommend [23] and [21].

### 3.2 The Hajós-type construct

We begin this section by introducing a construction we call the Hajós-type construction. This construction is reminiscent of the Hajós Construction [13], hence its name. Much like the Hajós Construction, the idea is to obtain, from two graphs  $G$  and  $H$ , a new graph  $F$  that resembles both  $G$  and  $H$ , and satisfies  $\chi(F) \geq \min\{\chi(G), \chi(H)\}$ . In the next section we will use this construction repeatedly in order to obtain a graph  $F$  that is homomorphic to  $G$  and to  $H$ . This will be accomplished with the aid of a special blend of restrictions placed upon the graphs  $G$  and  $H$ .

Let  $G$  and  $H$  be graphs,  $e = uv$  an edge of  $H$ , and  $X \subseteq V(G)$  an independent set of vertices. In addition, for each vertex  $x \in X$ , let  $H_x$  be a copy of  $H - e$ , and let  $N'_{G-X}(x) = \{y \in V(G - X) \mid y \in N_G(x)\}$ . That is,  $N'_{G-X}(x)$  is the set of vertices of  $G - X$  that are neighbours of  $x$  in  $G$ . Furthermore, for each vertex  $w$  in  $H$ , the vertex in  $H_x$  which corresponds to  $w$  is renamed  $w_x$ . Thus the vertices in  $H_x$  that correspond to the vertices  $u$  and  $v$  in  $H$  are  $u_x$  and  $v_x$ , respectively. For each  $x \in X$ , replace  $x$  with  $H_x$ , and make each vertex  $y \in N'_{G-X}(x)$  adjacent to precisely one of the vertices  $u_x$  and  $v_x$  of  $H_x$ . Any resulting graph we refer to as a *Hajós-type construct (Ht-construct)* of  $G$  and  $H$  obtained through  $X$  and  $e$ .

This definition allows for all the vertices in  $N'_{G-X}(x)$ , the former neighbourhood of  $x$ , to be made adjacent to only one of the vertices  $u_x$  and  $v_x$  of  $H_x$ . As obvious as this may be, it is worth mentioning since if this scenario occurs at every vertex  $x \in X$  then the Ht-construct obtained in this manner has an induced subgraph that is isomorphic to  $G$ . The graph in Figure 8, in the example that follows, illustrates this.

For an example, consider the two graphs  $G$  and  $H$  depicted in Figure 6 below, where  $H$  is the complete graph on four vertices. Let  $X = \{x_1, x_2\}$  and  $e = uv \in E(H)$ .

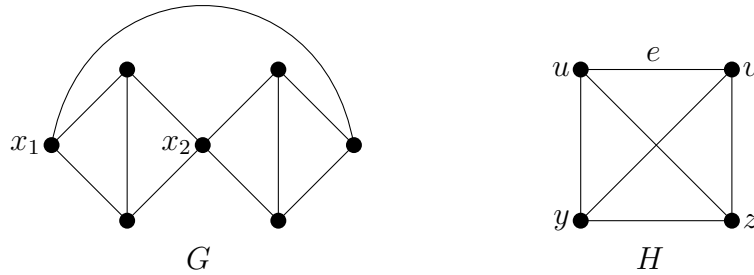


Figure 6: The graphs  $G$  and  $H$  with  $X = \{x_1, x_2\}$  and  $e = uv$

The graphs in Figure 7 and Figure 8 are both Ht-constructs of the graphs  $G$  and  $H$ , shown in Figure 6. Both are obtained through the independent set  $X$  of  $G$  and the edge  $e$  of  $H$ . The graph in Figure 8 was obtained by making all the former neighbours of  $x_1$  adjacent to only  $v_{x_1}$ , and by making all the former neighbours of  $x_2$  adjacent to only  $u_{x_2}$ . This graph has an induced subgraph that is isomorphic to  $G$ .

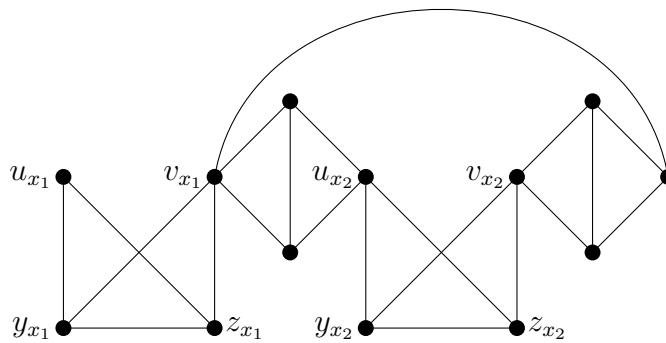


Figure 7: An Ht-construct of  $G$  and  $H$  obtained through  $X$  and  $e$

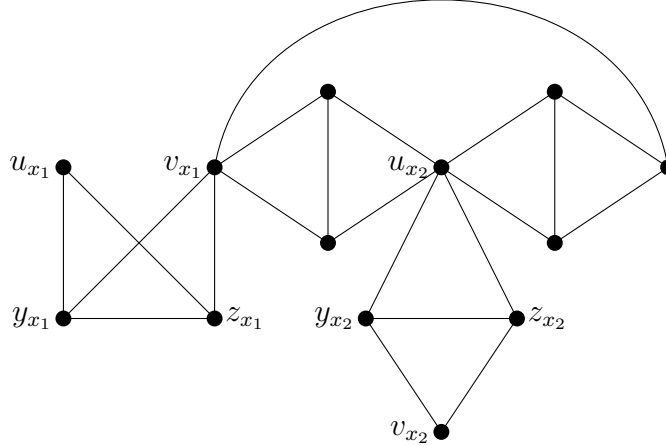


Figure 8: Another Ht-construct of  $G$  and  $H$  obtained through  $X$  and  $e$

**Lemma 32.** *Let  $G$  be a graph with  $\chi(G) = n \geq 2$ , and  $H$  be an  $n$ -critical graph. If  $X$  is an independent set of vertices of  $G$  then every Ht-construct of  $G$  and  $H$ , obtained through  $X$  and any edge  $e$  of  $H$ , has chromatic number  $n$ .*

*Proof.* Let  $G$ ,  $H$ , and  $X$  be as described above, then let  $e = uv$  be an edge of  $H$ , and  $F_X$  be any Ht-construct of  $G$  and  $H$  obtained through  $X$  and  $e$ .

Suppose that  $\chi(F_X) < n$ , then there exists a proper  $(n - 1)$ -colouring  $C$  of  $F_X$ . For each  $x \in X$ , we have that  $\chi(H_x) = n - 1$ , which implies that the restriction of  $C$  to the vertices of  $V(H_x)$  in  $F_X$  is a proper  $(n - 1)$ -colouring of  $H_x$ . From this it follows that  $C(u_x) = C(v_x)$  since  $H_x + u_x v_x$  is  $n$ -critical. Thus, for each  $x \in X$ , we have that  $C(u_x) = C(v_x)$ . This implies that the colouring  $C^* : V(G) \rightarrow \{1, \dots, n\}$  defined by

$$C^*(y) = \begin{cases} C(y) & \text{if } y \in V(G) \setminus X, \text{ and} \\ C(u_y) & \text{if } y \in X, \end{cases}$$

is a proper  $(n - 1)$ -colouring of  $G$ . Clearly this is a contradiction, therefore

the assumption that  $\chi(F_X) < n$  is false.

Next we show that  $\chi(F_X) = n$ . This we achieve by providing a proper  $n$ -colouring  $D$  of  $F_X$ . Since  $\chi(G) = n$  it follows that there exists a proper  $n$ -colouring  $C$  of  $G$ . For each  $x \in X$ , there exists a proper  $(n - 1)$ -colouring  $C_x$  of  $H_x$ . In addition,  $C_x$  satisfies  $C_x(u_x) = C_x(v_x)$ . For each  $x \in X$ , permit  $C_x$  to be such that  $C_x(u_x) = C_x(v_x) = C(x)$ . Then let  $D : V(F_X) \rightarrow \{1, \dots, n\}$  be the mapping defined, for all  $y \in V(F_X)$ , by

$$D(y) = \begin{cases} C(y) & \text{if } y \in V(G) \setminus X, \text{ and} \\ C_x(y) & \text{if } y \in V(H_x). \end{cases}$$

Then  $D$  is a proper  $n$ -colouring of  $F_X$ . □

The graphs in Figure 6 have chromatic number 4, and one can easily show that the chromatic number of the graphs in Figure 7 and Figure 8 is also 4.

### 3.3 An improvement of the Burr, Erdős and Lovász Theorem

The special case of the Hedetniemi Conjecture considered by Burr, Erdős and Lovász is for those  $(n + 1)$ -chromatic graphs  $G$  of which each vertex belongs to an  $n$ -clique of  $G$ . Their result was that, for such graphs  $G$  and for all  $(n + 1)$ -chromatic graphs  $H$ ,

$$\chi(G \times H) = n + 1.$$

In this section we prove that the same result can be obtained without requiring all vertices of  $G$  to belong to an  $n$ -clique. More specifically we require that those vertices that do not belong to an  $n$ -clique of  $G$  be such that they induce a subgraph of  $G$  whose chromatic number is at most the clique number of  $H$ . This is the main result of this section. This result also proves a special case of Theorem 23 which is embodied in Corollary 4 towards the end of this section. We build up to this result by first introducing a weaker result from which the result by the aforementioned authors follows immediately. The proofs in this section employ the repeated application of the Ht-construction.

**Theorem 24.** *Let  $G$  and  $H$  be graphs with  $\chi(G) = \chi(H) = n + 1 \geq 3$  such that  $H$  is critical, and  $\omega(H) \leq n = \omega(G)$ . Let  $A$  be the set of vertices of  $G$  that are not contained in an  $n$ -clique. If there exists an  $(n + 1)$ -colouring  $V_1, \dots, V_{n+1}$  of  $V(G)$  satisfying  $A \subseteq V_n \cup V_{n+1}$ , then  $\chi(G \times H) = n + 1$ .*

*Proof.* Let  $G$ ,  $H$ ,  $A$ , and  $V_1, \dots, V_{n+1}$  be as described above. We will construct a graph  $F$  with  $\chi(F) = n + 1$ , satisfying  $F \rightarrow G$  and  $F \rightarrow H$ . This will imply, by Lemma 31, that  $\chi(G \times H) = n + 1$ . First, we will recursively construct graphs  $F_1, F_2, \dots, F_{n-1}$  in such a way that, for each  $2 \leq i \leq n - 1$ , the graph  $F_i$  is an Ht-construct of  $F_{i-1}$  and  $H$ , while  $F_1$  is an Ht-construct of  $G$  and  $H$ .

Begin by fixing an edge  $e = uv$  of  $H$ . Then let  $F_1$  be the graph obtained as follows: replace each  $x \in V_1$  with  $H_x$ , where  $H_x$  is a copy of  $H - e$  as described earlier, then, for each  $y \in N'_{G-V_1}(x)$ , make  $y$  adjacent to  $v_x$  if  $y \in V_j$  for some odd  $j > 1$ , otherwise make  $y$  adjacent to  $u_x$ . Then, clearly,  $F_1$  is an Ht-construct of  $G$  and  $H$ , and so, by Lemma 32,  $\chi(F_1) = (n + 1)$ .

Next suppose, for some  $i$  with  $2 \leq i \leq n - 1$ , that  $F_{i-1}$  has been constructed. We construct  $F_i$  from  $F_{i-1}$  and  $H$  using  $V_i$  and  $e$  as follows: For each  $x \in V_i$ , replace  $x$  in  $F_{i-1}$  with  $H_x$ , then make  $u_x$  adjacent to those  $y \in N'_{F_{i-1}-V_i}(x)$  that belong to a  $V_j$  with an even  $j > i$ , then make  $v_x$  adjacent to those  $y \in N'_{F_{i-1}-V_i}(x)$  that belong to a  $V_j$  with an odd  $j > i$ . For the remaining  $y \in N'_{F_{i-1}-V_i}(x)$  make  $y$  adjacent to  $v_x$  if  $i$  is even, and make  $y$  adjacent to  $u_x$  if  $i$  is odd. This construction ensures, among other things, that all vertices  $y \in N'_{F_{i-1}-V_i}(x)$  are made adjacent to either  $u_x$  or  $v_x$ .

It is useful to note that, for all  $1 \leq i \leq n - 1$ , each vertex  $u_x$  of  $F_i$  is not adjacent to a vertex in  $V_j$  for all odd  $j > i$ . In addition, for all  $1 \leq i \leq n - 1$ , each vertex  $v_x$  of  $F_i$  is not adjacent to a vertex in  $V_j$  for even  $j > i$ . Clearly,  $F_i$  is an Ht-construct, and thus, by Lemma 32, it follows that  $\chi(F_i) = n + 1$ .

Let  $F = F_{n-1}$ ,  $U = \{u_x \in V(F) \mid x \in V_1 \cup \dots \cup V_{n-1}\}$ , and  $V = \{v_x \in V(F) \mid x \in V_1 \cup \dots \cup V_{n-1}\}$ . If  $n + 1$  is odd then, for every vertex  $x \in V_n$ , we have that  $N_F(x) \subseteq V_{n+1} \cup U$ , and, for every vertex  $x \in V_{n+1}$ , we have that  $N_F(x) \subseteq V_n \cup V$ . If  $n + 1$  is even then, for every vertex  $x \in V_n$ , we have  $N_F(x) \subseteq V_{n+1} \cup V$ , and, for every vertex  $x \in V_{n+1}$ , we have  $N_F(x) \subseteq V_n \cup U$ .

In addition, we have that  $U$  and  $V$  are independent subsets of  $V(F)$ . The independence of  $U$ : Suppose, to the contrary, that there exist two vertices  $u_{x_1}, u_{x_2} \in U$  such that  $u_{x_1}u_{x_2} \in E(F)$ . Then there exist integers  $1 \leq i < k \leq n - 1$  such that  $x_1 \in V_i$  and  $x_2 \in V_k$ . Suppose that  $k$  is even, then when  $F_k$  was constructed  $u_{x_2}$  was made adjacent only to those  $y \in N'_{F_{k-1}-V_k}(x_2)$  that belong to a  $V_j$  such that  $j$  is even and  $j > k$ . This, of course, implies that  $u_{x_1}$  is not adjacent to  $u_{x_2}$ , which is a contradiction. Thus  $k$  must be



odd, in which case,  $u_{x_1}$  is adjacent to  $x_2$  in  $F_{k-1}$ , and thus adjacent to  $x_2$  in  $F_i$ . Since  $k$  is odd, it follows by an earlier remark that the vertex  $u_{x_1}$  in the graph  $F_i$  is not adjacent to any vertex in  $V_k$ , and thus is not adjacent to  $x_2$ . This, clearly, is a contradiction.

The independence of  $V$ : Suppose, to the contrary, that  $v_{x_1}v_{x_2} \in E(F)$  for some two vertices  $v_{x_1}, v_{x_2} \in V$ . Then there exist integers  $1 \leq i < k \leq n - 1$  such that  $x_1 \in V_i$  and  $x_2 \in V_k$ . Assume  $k$  is odd, then when  $F_k$  was constructed  $v_{x_2}$  was made adjacent to only those  $y \in N'_{F_{k-1}-V_k}(x_2)$  that belong to a  $V_j$  such that  $j > k$  and  $j$  is odd. This implies that  $v_{x_1}$  is not adjacent to  $v_{x_2}$ , which is a contradiction. Thus  $k$  must be even, in which case,  $v_{x_1}$  is adjacent to  $x_2$  in  $F_{k-1}$ , and thus adjacent to  $x_2$  in  $F_i$ . Since  $k$  is even, it follows by an earlier remark that in the graph  $F_i$  we have that  $v_{x_1}$  is not adjacent to all vertices in  $V_k$ , and so  $v_{x_1}$  is not adjacent to  $x_2$ . This, of course, is a contradiction.

Finally, we prove that  $F \longrightarrow G$  and  $F \longrightarrow H$ . Let  $\phi : V(F) \longrightarrow V(H)$  be the mapping defined as follows. For all  $x \in V_1 \cup \dots \cup V_{n-1}$ , the mapping  $\phi$  satisfies  $\phi(w_x) = w$ . That is, every vertex of  $F$  that belongs to a subgraph of  $F$  induced by the vertices of  $H_x$ , where  $x$  belongs to  $V_1 \cup \dots \cup V_{n-1}$ , is mapped by  $\phi$  to the vertex in  $H$  to which it corresponds. Therefore  $\phi(U) = u$  and  $\phi(V) = v$ . Thus an edge of  $F$  that belongs to a subgraph induced by the vertices of some  $H_x$  is preserved through  $\phi$  by its corresponding edge in  $H$ . In addition, an edge of  $F$  that is incident with a vertex in  $U$  and a vertex in  $V$  is preserved by the edge  $uv$  in  $H$ . If  $n + 1$  is even, then  $\phi(V_n) = u$  and  $\phi(V_{n+1}) = v$ . If  $n + 1$  is odd, then  $\phi(V_n) = v$  and  $\phi(V_{n+1}) = u$ . From this it follows that an edge of  $F$  incident with a vertex in  $V_n$  and a vertex in  $V_{n+1}$  is preserved by the edge  $uv$  in  $H$ . Those edges in  $F$  that are incident with a vertex in  $V_n \cup V_{n+1}$  and a vertex in  $V(F) \setminus (V_n \cup V_{n+1})$  are also preserved by the edge  $uv$  in  $H$ . Thus  $\phi$  is a homomorphism from  $F$  to  $H$ .

Next we show that  $F$  is homomorphic to  $G$ . Since  $\chi(H_x) = n$ , there

exists a homomorphism  $\gamma$  from  $H_x$  to  $K_n$ , and  $\gamma$  can be chosen to be such that  $\gamma(u_x) = \gamma(v_x)$ . For each  $x \in V_1 \cup \dots \cup V_{n-1}$ , let  $\gamma_x$  be the homomorphism  $\gamma$  that maps  $H_x$  into a complete subgraph  $K_n$  of  $G$  that  $x$  belongs to with  $\gamma_x(u_x) = \gamma_x(v_x) = x$ . Finally, let  $\phi : V(F) \rightarrow V(G)$  be the mapping defined as follows: for each  $x \in V_1 \cup \dots \cup V_{n-1}$ , the restriction of  $\phi$  to  $H_x$  is  $\gamma_x$ , and, for each  $x \in V_n \cup V_{n+1}$ ,  $\phi(x) = x$ . Then  $\phi$  is a homomorphism from  $F$  to  $G$ .

This completes our proof.  $\square$

The following corollary shows that the result obtained by Burr, Erdős and Lovász can be arrived at quite easily from Theorem 24.

**Corollary 2.** [4] *Let  $G$  and  $H$  be graphs with  $\chi(G) = \chi(H) = n + 1$ ,  $\omega(G) = n$ . Let  $A$ , the set of vertices of  $G$  that are not contained in any  $n$ -clique of  $G$ , be empty. Then  $\chi(G \times H) = n + 1$ .*

*Proof.* Let  $G$ ,  $H$  and  $A$  be as described above, and assume that  $A = \emptyset$ . Then, for each  $(n + 1)$ -colouring  $V_1, \dots, V_{n+1}$  of  $V(G)$ , we have that  $A \subseteq V_n \cup V_{n+1}$ . In addition, we have that  $H$  has an  $(n + 1)$ -critical subgraph  $H'$ . Thus, by Theorem 24, we obtain that  $\chi(G \times H') = n + 1$ . This implies that  $\chi(G \times H) = n + 1$  since  $\chi(G \times H') \leq \chi(G \times H)$ .  $\square$

**Theorem 25.** *Let  $G$  and  $H$  be graphs with  $\chi(G) = \chi(H) = n + 1 \geq 3$ ,  $\omega(H) \leq n = \omega(G)$ , and  $H$  is critical. Let  $A$  be the set of vertices of  $G$  that are not contained in any  $n$ -clique of  $G$ . If  $\chi(G[A]) \leq \omega(H)$  then  $\chi(G \times H) = n + 1$ .*

*Proof.* Let  $G$ ,  $H$ , and  $A$  be as described above, and assume that  $\chi(G[A]) \leq \omega(H)$ . We will prove that  $\chi(G \times H) = n + 1$  by constructing a graph  $F$ , with chromatic number  $n + 1$ , that is homomorphic to  $G$  and  $H$ .

Let  $u, v, w_3, \dots, w_k$ , where  $k = \omega(H)$ , be the vertices of a complete subgraph of  $H$ . Also, let  $A_1, \dots, A_r$ , where  $r = \chi(G[A])$ , be a proper  $r$ -colouring of  $G[A]$ . From our initial assumption it follows that  $r \leq k$ . If  $\chi(G[V(G) \setminus A]) = n + 1$  then, by Corollary 2, we obtain that  $\chi(G[V(G) \setminus A] \times H) = n + 1$

which implies  $\chi(G \times H) = n + 1$ . Thus we may assume that  $\chi(G[V(G) \setminus A]) = n$ . Let  $V_1, \dots, V_n$  be a proper  $n$ -colouring of  $G[V(G) \setminus A]$ .

Next, we recursively construct graphs  $F_1, F_2, \dots, F_n$  in such a way that, for all integers  $i$  with  $2 \leq i \leq n$ , the graph  $F_i$  is an Ht-construct of  $F_{i-1}$  and  $H$ , where  $F_1$  is an Ht-construct of  $G$  and  $H$ . For each vertex  $x \in V(G) \setminus A$  let  $H_x$  be a copy of the graph  $H - uv$  as described before. The graph  $F_1$  is obtained as follows. For each  $x \in V_1$  replace  $x$  with  $H_x$ , then make  $v_x$  adjacent to all those  $y \in N'_{G-V_1}(x)$  that belong to  $A_1$  or to a  $V_j$  with an odd  $j > 1$ . Lastly make  $u_x$  adjacent to those  $y \in N'_{G-V_1}(x)$  that belong to  $A \setminus A_1$  or to a  $V_j$  with an even  $j > 1$ . We remark that all the former neighbours of  $x$  are made adjacent to either  $u_x$  or  $v_x$ . It follows that  $F_1$  is an Ht-construct of  $G$  and  $H$ , and so, by Lemma 32,  $\chi(F_1) = (n + 1)$ .

Suppose that, for some integer  $i - 1$  with  $1 \leq i - 1 \leq n - 1$ , the graphs  $F_1, F_2, \dots, F_{i-1}$  have been constructed. We construct  $F_i$  from  $F_{i-1}$  and  $H$  through  $V_i$  and  $uv$  as follows. For each  $x \in V_i$ , replace  $x$  in  $F_{i-1}$  with  $H_x$ , and make  $v_x$  adjacent to all those  $y \in N'_{F_{i-1}-V_i}(x)$  that belong to  $A_1$  or to a  $V_j$  with an odd  $j > i$ . Next make  $u_x$  adjacent to those  $y \in N'_{F_{i-1}-V_i}(x)$  that belong to  $A \setminus A_1$  or to a  $V_j$  with an even  $j > i$ . Finally, for the remaining  $y \in N'_{F_{i-1}-V_i}(x)$ , we proceed as follows. Make  $y$  adjacent to  $v_x$  if  $i$  is even, and make  $y$  adjacent to  $u_x$  if  $i$  is odd. This construction ensures that all vertices  $y \in N'_{F_{i-1}-V_i}(x)$  are made adjacent to either  $u_x$  or  $v_x$ . Note that, for all integers  $i$  with  $1 \leq i \leq n - 2$ , each vertex  $x \in V_i$  has a neighbour in  $V_{n-1}$  and another in  $V_n$ , in fact in every  $V_j$  with  $j \neq i$ . This is because  $G[V(G) \setminus A]$  contains exactly all those vertices of  $G$  which are in a clique of order  $n$ . We point out the following. For all integers  $i$  with  $1 \leq i \leq n - 1$ , each vertex  $u_x$  of  $F_i$  is not adjacent to a vertex in  $V_j$  for all odd  $j > i$ . In addition, for all integers  $i$  with  $1 \leq i \leq n - 1$ , each vertex  $v_x$  of  $F_i$  is not adjacent to a vertex in  $V_j$  for all even  $j > i$ .

By Lemma 32, we have that  $\chi(F_2) = \dots = \chi(F_{n-1}) = \chi(F_n) = n + 1$ . Let

$F = F_n$ , then let  $U = \{u_x \in V(F) \mid x \in V(G) \setminus A\}$ , and  $V = \{v_x \in V(F) \mid x \in V(G) \setminus A\}$ . The sets  $U$  and  $V$  are independent sets of vertices of  $F$ .

To see that  $U$  is independent, consider the following: Suppose, to the contrary, that there exist two vertices  $u_{x_1}, u_{x_2} \in U$  such that  $u_{x_1}u_{x_2} \in E(F)$ . Then there exist integers  $i$  and  $k$  with  $1 \leq i < k \leq n$  such that  $x_1 \in V_i$  and  $x_2 \in V_k$ . Suppose that  $k$  is even, then when  $F_k$  was constructed  $u_{x_2}$  was made adjacent only to those  $y \in N'_{F_{k-1}-V_k}(x_2)$  that belong to  $A \setminus A_1$  or to a  $V_j$  with an even  $j > k$ . This, of course, implies that  $u_{x_1}$  is not adjacent to  $u_{x_2}$ , which is a contradiction. Thus  $k$  must be odd, in which case,  $u_{x_1}$  is adjacent to  $x_2$  in  $F_{k-1}$ , and thus adjacent to  $x_2$  in  $F_i$ . Since  $k$  is odd, it follows by an earlier remark that the vertex  $u_{x_1}$  in the graph  $F_i$  is not adjacent to any vertex in  $V_k$ , and thus is not adjacent to  $x_2$  in the graph  $F_i$ . This, clearly, is a contradiction. Using a similar argument, it can be shown that  $V$  is also independent.

Now we are ready to prove that  $F \rightarrow G$  and  $F \rightarrow H$ . Let  $\phi : V(F) \rightarrow V(H)$  be the mapping defined, for all  $z \in V(F)$ , by

$$\phi(z) = \begin{cases} w & \text{if } z = w_x \text{ for some } x \in V(G) \setminus A; \\ u & \text{if } z \in A_1; \\ v & \text{if } z \in A_2; \\ w_j & \text{if } z \in A_j \text{ where } j > 2. \end{cases}$$

Therefore it follows that  $\phi(U) = u$ ,  $\phi(V) = v$ , and  $\phi(A) = \{u, v, w_3, \dots, w_k\}$ . Thus any edge of  $F$  that is incident with two vertices in  $A$  or is incident with a vertex in  $A$  and a vertex in  $V(F) \setminus A$  is preserved by one of the edges of the complete subgraph of  $H$  induced by the set of vertices  $\{u, v, w_3, \dots, w_k\}$ . Those edges in  $F$  that are incident with a vertex in  $U$  and a vertex in  $V$  are preserved in  $H$  by the edge  $uv$ . An edge of  $F$  belonging to a subgraph of  $F$  induced by the vertices of some  $H_x$  is preserved by the edge in  $H$  to which it corresponds. Thus  $\phi$  is a homomorphism from  $F$  to  $H$ .

For each  $H_x$ , let  $\gamma_x$  be a homomorphism that maps  $H_x$  into an  $n$ -clique of  $G$  that contains  $x$ . Choose  $\gamma_x$  to be such that  $\gamma_x(u_x) = \gamma_x(v_x) = x$ . Then let  $\gamma : V(F) \rightarrow V(G)$  be the mapping defined as follows: for each  $x \in V_1 \cup \dots \cup V_n$ , the restriction of  $\gamma$  to  $V(H_x)$  is  $\gamma_x$ , and the restriction of  $\gamma$  to  $A$  is the identity mapping. Then  $\gamma$  is a homomorphism from  $F$  to  $G$ . This completes our proof.  $\square$

**Corollary 3.** *Let  $G$  and  $H$  be graphs with  $\chi(G) = \chi(H) = n + 1$ ,  $\omega(G) = n$ . Let  $A$ , the set of vertices of  $G$  that are not contained in an  $n$ -clique of  $G$ , be such that  $\chi(G[A]) \leq 2$ . Then  $\chi(G \times H) = n + 1$ .*

*Proof.* Let  $G$ ,  $H$  and  $A$  be as described above, and assume that  $\chi(G[A]) \leq 2$ . Then it follows that  $\chi(G[A]) \leq 2 \leq \omega(H)$ . With the aid of Theorem 25, we obtain that  $\chi(G \times H) = n + 1$ .  $\square$

As promised, we offer a corollary which proves a special case of Theorem 23. It bears a striking similarity to Theorem 23. The difference is that this corollary has an additional requirement. It asks that  $G$  and  $H$  be critical. The criticality of these graphs is required to ensure that  $\chi(G[A]) \leq \omega(H)$ , which makes room for the application of Theorem 25.

**Corollary 4.** *Let  $G$  and  $H$  be connected  $(n + 1)$ -critical graphs with  $\omega(G) = \omega(H) = n$ . Then  $\chi(G \times H) = n + 1$ .*

*Proof.* Let  $G$  and  $H$  be as described above, then let  $A$  be the set of all vertices not contained in an  $n$ -clique of  $G$ . We may assume  $A$  is not empty, since otherwise we would have  $\chi(G \times H) = n + 1$  by Corollary 2. Since  $G$  is critical it follows that  $\chi(G[A]) \leq n = \omega(H)$ , and thus, by Theorem 25, we have that  $\chi(G \times H) = n + 1$ .  $\square$

In [25], Toft gives a complete list of all 4-critical graphs with at most 9 vertices. The complete graph  $K_4$  is in this list. Each of the graphs in Toft's

list has clique number 3, and is such that the set of vertices not contained in a 3-clique induces a graph with chromatic number at most 2. There are 30 graphs on this list, and the graph  $T_{30}$  exhibited in Figure 9 is one of them.

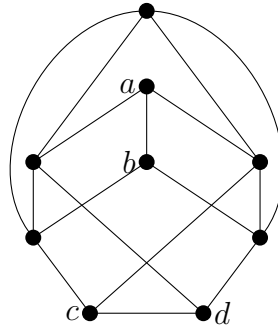


Figure 9: The graph  $T_{30}$

Let  $G$ , a graph with chromatic number 4, be such that one of the 30 aforementioned graphs is a subgraph of  $G$ . Then, by Corollary 3, we have that  $\chi(G \times H) = 4$  for all graphs  $H$  with  $\chi(H) = 4$ . One could have easily arrived at this conclusion by using the result by M. El-Zaher and N. Sauer obtained in [10], which is as follows.

**Theorem 26.** [10] *If  $G$  and  $H$  are graphs such that  $\chi(G) = \chi(H) = 4$  then  $\chi(G \times H) = 4$ .*

An obvious limitation of this theorem is that it applies only to graphs with chromatic number 4. In the next section we construct families of graphs that have chromatic number at least 4 and satisfy the hypothesis of Corollary 3. We do this in order to illustrate the strength of this corollary.

### 3.4 The generalized Mycielski construct

In this section we utilise the generalised Mycielski Construction and the Hajós Construction to obtain, for each integer  $n \geq 4$ , a family of graphs  $\mathcal{F}_n$ , such that, for each graph  $G \in \mathcal{F}_n$ , we have that  $\chi(G) = n$ ,  $\omega(G) = n - 1$ , and most importantly, the set  $A$  of vertices of  $G$  that are not contained in an  $(n - 1)$ -clique induces a subgraph  $G[A]$  satisfying  $\chi(G[A]) = 2$ . Clearly these graphs satisfy the hypothesis of Corollary 3. The intention here is to show that Corollary 3 can be applied to an infinite number of graphs. For our purposes, we will only apply the generalised Mycielski Construction to complete graphs. The construction given below is consistent with that given in [11].

Given positive integers  $m$  and  $r$ , let  $K_m$  be a complete graph with vertex set  $V(K_m) = \{x_1, x_2, \dots, x_m\}$ . Then the *generalised Mycielski construct*  $M_r(K_m)$  of  $K_m$  is obtained as follows. For each integer  $i$  with  $1 \leq i \leq r$ , let

$$X_i = \{x_1^i, x_2^i, \dots, x_m^i\}$$

be a set of new vertices. In addition, let  $X_1, X_2, \dots, X_r$  be pairwise disjoint. Next, let

$$E_1 = \bigcup_{i=1}^m \{x_i^1 x_j^1 \mid i \neq j\}$$

and, for each integer  $k$  with  $2 \leq k \leq r$ , let

$$E_k = \bigcup_{i=1}^m \{x_i^k x_j^{k-1} \mid i \neq j\}.$$

Finally, let

$$E_{r+1} = \{zy \mid y \in X_r\},$$

where  $z$  is an additional vertex. Then let  $M_r(K_m)$  be the graph with vertex set

$$V(M_r(K_m)) = V(K_m) \cup \{z\} \cup \left( \bigcup_{i=1}^r X_i \right)$$

and edge set

$$E(M_r(K_m)) = E(K_m) \cup \left( \bigcup_{i=1}^{r+1} E_i \right).$$

Note that, for positive integers  $m$  and  $r$  with  $m \geq 2$ , the graph  $M_r(K_m)$  is such that the sets  $X_1, \dots, X_r$  are independent sets of vertices, and the neighbourhoods in  $M_r(K_m)$  of vertices in  $X_1 \cup \dots \cup X_r \cup \{z\}$  are as follows: For a vertex  $x \in X_1$ , we have that  $N(x) \subset V(K_m) \cup X_2$ , while for a vertex  $x \in X_i$  where  $1 < i < r$ , we have that  $N(x) \subset X_{i-1} \cup X_{i+1}$ , and, for a vertex  $x \in X_r$ , we have that  $N(x) \subset X_{r-1} \cup \{z\}$ . Lastly, we have that  $N(z) = X_r$ .

Figure 10 shows the graph  $M_3(K_3)$

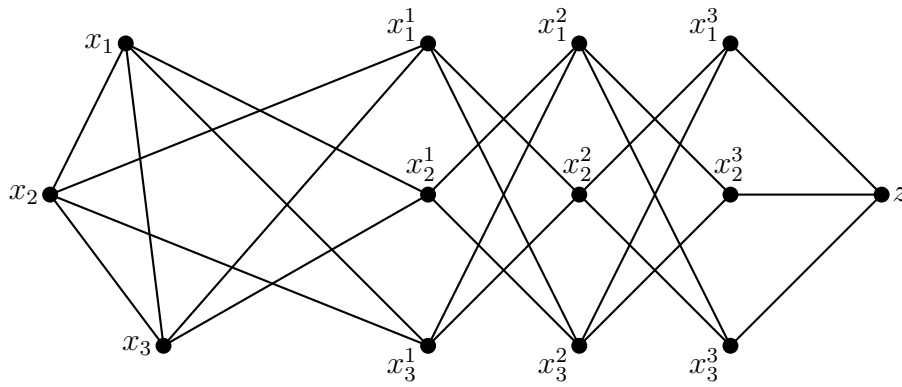


Figure 10:  $M_3(K_3)$

**Lemma 33.** *For all positive integers  $m$  and  $r$  with  $m \geq 3$ , the graph  $M_r(K_m)$ , with vertices as described earlier, is such that*

- (i)  $\omega(M_r(K_m)) = m$ ,
- (ii) each vertex of  $V(K_m) \cup X_1$  belongs to an  $m$ -clique, and
- (iii) the subgraph of  $M_r(K_m)$  induced by the set  $X_2 \cup \dots \cup X_r \cup \{z\}$  has chromatic number 2.



*Proof.* Let  $m$  and  $r$  be as defined above, and consider the graph  $M_r(K_m)$ . The subgraph of  $M_r(K_m)$  induced by the set of vertices  $\{x_1, x_2, \dots, x_m\}$  is a complete graph of order  $m$ , therefore the vertices of  $V(K_m)$  belong to an  $m$ -clique. For any integer  $i$  with  $1 \leq i \leq m$ , the subgraph of  $M_r(K_m)$  induced by the set of vertices  $\{x_1, \dots, x_{i-1}, x_i^1, x_{i+1}, \dots, x_m\}$  is a complete graph of order  $m$ , therefore each vertex of  $X_1$  belongs to an  $m$ -clique. From this it follows that (ii) holds.

Next we prove (i). By (ii) it follows that  $\omega(M_r(K_m)) \geq m$ . For a contradiction, suppose that  $M_r(K_m)$  contains a clique  $K$  of order  $m + 1$ . Then  $z$  is not contained in  $K$  otherwise this would imply that the other  $m$  vertices of  $K$  belong to  $X_r$  since  $N(z) = X_r$ . This, in turn, would imply that  $K$  is not a  $(m + 1)$ -clique since  $X_r$  is an independent set of vertices, which, of course would be a contradiction. We also have that no vertex of  $K$  is contained in the set  $X_2 \cup \dots \cup X_r$ . The reason is as follow. If some vertex of  $K$  belonged to  $X_j$  for some integer  $j$  such that  $2 \leq j \leq r$ , then at least two vertices of  $K$  belong to one of the independent sets  $X_{j-1}$  and  $X_{j+1}$  since  $m \geq 3$ . But then this implies that  $K$  is not complete, which is a contradiction. Since  $K$  is complete it follows that no two vertices of  $K$  belong to the independent set  $X_1$ . Therefore we have that one vertex of  $K$  belongs to  $X_1$  and the remaining  $m$  vertices belong to  $V(K_m)$ . Thus, for some integer  $i$  with  $1 \leq i \leq m$ , the vertex  $x_i^1$  belongs to  $K$ . This vertex, however, is not adjacent to  $x_i$  which is also contained in  $K$ , which implies that  $K$  is not complete. This is a contradiction, hence the assumption that  $M_r(K_m)$  has a clique of order  $m + 1$  is false. Thus  $\omega(M_r(K_m)) = m$ , which proves (i).

To prove (iii) we show that the subgraph of  $M_r(K_m)$  induced by the set  $X_2 \cup \dots \cup X_r \cup \{z\}$  is homomorphic to a path of length  $r$ . Let  $P = (w_2, w_3, \dots, w_{r+1})$  be a path. Clearly  $P$  has length  $r$ . Next define the mapping  $f : X_2 \cup \dots \cup X_r \cup \{z\} \longrightarrow V(P)$  as follows. Let  $f(z) = w_{r+1}$  and, for each integer  $j$  with  $2 \leq j \leq r$ , let the image of the restriction of  $f$  to  $X_j$  be

the vertex  $w_j$ . Then  $f$  is a homomorphism from the subgraph of  $M_r(K_m)$  induced by the set  $X_2 \cup \dots \cup X_r \cup \{z\}$  to  $P$ , which proves (iii).  $\square$

The following theorem, obtained in 1985 by Tuza and Rödl, is given without proof.

**Theorem 27.** [26] *For all  $r \geq 1$  and  $m \geq 2$ , the graph  $M_r(K_m)$  is  $(m + 1)$ -critical.*

Below we prove the existence of the families of graphs  $\mathcal{F}_n, n \geq 4$  mentioned earlier.

**Theorem 28.** *Given an integer  $n \geq 4$  and an integer  $r \geq 2$ , there exists a graph  $G_r^n$  such that,*

$$(i) \chi(G_r^n) = n$$

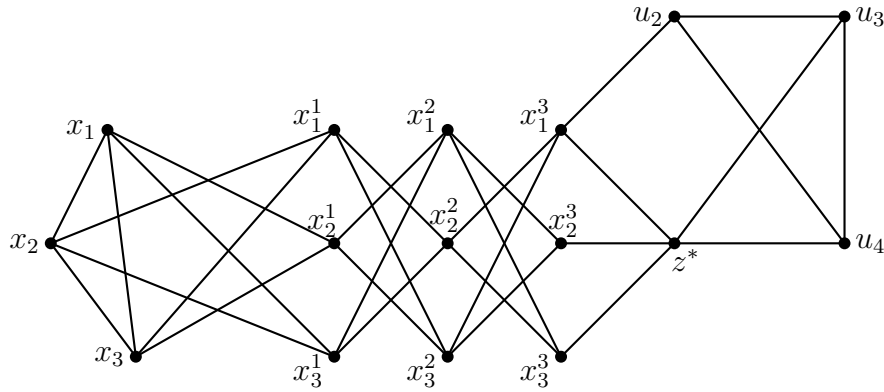
$$(ii) \omega(G_r^n) = n - 1$$

(iii) *The set  $A$  of vertices of  $G_r^n$  that do not belong to an  $(n - 1)$ -clique induces a graph with chromatic number 2. That is  $\chi(G_r^n[A]) = 2$ .*

*Proof.* Let  $n$  and  $r$  be positive integers such that  $n \geq 4$  and  $r \geq 2$ . Let  $H$  be a complete graph of order  $n$  with vertex set  $V(H) = \{u_1, u_2, \dots, u_n\}$ , and consider the graph  $M_r(K_{n-1})$  with vertices described as done earlier in this section. By Theorem 27, we have that  $M_r(K_{n-1})$  is  $n$ -critical. Let  $G_r^n$  be the Hajós construct of  $M_r(K_{n-1})$  and  $H$  obtained as follows:

1. Delete the edge  $u_1u_2$  of  $H$  and the edge  $zx_1^r$  of  $M_r(K_{n-1})$ ,
2. identifying  $z$  and  $u_1$  to form a new vertex  $z^*$ , and
3. make  $x_1^r$  adjacent to  $u_2$ .

Then we have that  $\chi(G_r^n) = n$ ,  $\omega(G_r^n) = n - 1$ , and  $X_2 \cup \dots \cup X_r$  is the set of those vertices of  $G_r^n$  that are not contained in an  $(n - 1)$ -clique. Furthermore, the subgraph of  $G_r^n$  induced by this set has chromatic number 2. This completes our proof.  $\square$


 Figure 11:  $G_3^4$ 

An illustration of the graph  $G_3^4$  is given in Figure 11. For each integer  $n \geq 4$ , let  $\mathcal{F}_n = \{G_r^n \mid r \geq 2\}$ .

**Theorem 29.** *Given an integer  $n \geq 4$  and a graph  $H$  with  $\chi(H) = n$ , then, for all  $G \in \mathcal{F}_n$ ,*

$$\chi(G \times H) = n.$$

*Proof.* Let  $H$  be a graph with  $\chi(H) = n \geq 4$ , and select any graph  $G \in \mathcal{F}_n$ . Then, by Corollary 3, it follows that  $\chi(G \times H) = n$ .  $\square$

### 3.5 The Hajós-*Pe*-string construct

In this section we introduce yet another construction which we call the Hajós-*Pe*-string construct. When paired with the Ht-construct it can be used to construct, for graphs  $G$  and  $H$  with chromatic number  $n$ , a graph  $F$ , with chromatic number  $n$ , that is homomorphic to both  $G$  and  $H$ , provided that  $G$  and  $H$  satisfy some special set of conditions. Essentially, this construction is a two step process, the second of which resembles Hajós's construction.

For an  $(n + 1)$ -critical graph  $H$ ,  $n \geq 2$ , and a path  $P = (x_1, x_2, \dots, x_{k+1})$  in  $H$  with  $k \geq 2$ , permit  $H_i$ , for each  $1 \leq i \leq k$ , to be a copy of the graph  $H - x_i x_{i+1}$ . In addition, for each vertex  $w$  in  $H$ , the vertex in  $H_i$  that corresponds to  $w$  is renamed  $w^i$ . Thus the vertices in  $H_i$  that correspond to the vertices  $x_1, \dots, x_i, x_{i+1}, \dots, x_{k+1}$  in  $H$  are the vertices  $x_1^i, \dots, x_i^i, x_{i+1}^i, \dots, x_{k+1}^i$ , respectively. Next, starting with  $\bigsqcup_{j=1}^k H_j$ , for each integer  $1 \leq j \leq k - 1$ , identify the vertex  $x_{j+1}^j$  of  $H_j$  with the vertex  $x_{j+1}^{j+1}$  of  $H_{j+1}$  to form a new vertex  $x'_{j+1}$ . This new graph we shall refer to as the *P-string of H with respect to P*.

For an example consider the odd cycle  $C_5$  in Figure 12 with vertex set  $V(C_5) = \{x_1, x_2, x_3, x_4, x_5\}$ .

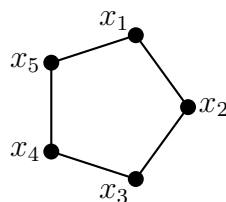


Figure 12: The graph  $C_5$

The *P*-string of  $C_5$ , where  $P = (x_1, x_2, x_3)$ , is depicted in Figure 13. Note that *P*-string of  $C_5$ , where  $P = (x_2, x_3, x_4)$ , is isomorphic to the graph in Figure 13.

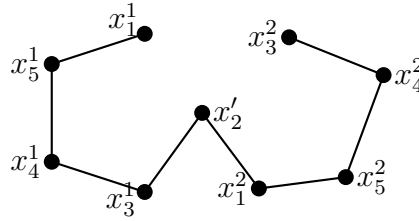


Figure 13: The  $P$ -string of  $C_5$  where  $P = (x_1, x_2, x_3)$

Let  $G$  be an  $(n + 1)$ -chromatic graph, and let  $e = ab$  be an edge of  $G$ , then the *Hajós-Pe-string construct*, also written as *HPes-construct*, of  $G$  and  $H$ , is obtained by “replacing” the edge  $e$  of  $G$  with the  $P$ -string of  $H$  as follows: First, identify the vertex  $a$  of  $G - ab$  with the vertex  $x_1^1$  of the  $P$ -string of  $H$ . We rename this newly formed vertex  $a$ . Next, we make the vertex  $b$  of  $G - ab$  adjacent to the vertex  $x_{k+1}^k$  of the  $P$ -string of  $H$ . This replacement of the edge  $e = ab$  with a  $P$ -string in such a manner that  $b$  is made adjacent to  $x_{k+1}^k$  shall also be referred to as *hanging the  $P$ -string on  $b$*  or *attaching the  $P$ -string on  $a$* .

The *HPes-construct* of the complete graph  $K_3$  and  $C_5$ , where  $e$  is any edge of  $K_3$ ,  $C_5$  is the graph in Figure 12, and  $P = (x_1, x_2, x_3)$ , is the odd cycle  $C_{11}$ .

For a set  $\mathcal{P}$  of paths in  $H$  of length at least 2, and a set  $E$  of edges of  $G$ , consider the graph obtained by replacing, as described above, each and every edge in  $E$  with a  $P$ -string of  $H$  for some  $P \in \mathcal{P}$ . Let these replacements be done in such a way that for all  $P \in \mathcal{P}$  there exists an edge  $e \in E$  such that the edge  $e$  was replaced with the  $P$ -string of  $H$ . We call this graph the *Hajós- $\mathcal{P}E$ -string-construct* or *HPEs-construct* of  $G$  and  $H$ .

For an example, consider the complete graph  $K_3$  and the the 5-cycle  $C_5$  mentioned recently. Let  $\mathcal{P} = \{(x_1, x_2, x_3), (x_2, x_3, x_4)\}$  and let  $E$  be any two edges of  $K_3$ . Then the *HPEs-construct* of  $K_3$  and  $C_5$  is the odd cycle  $C_{19}$ .

**Lemma 34.** *Let  $G$  and  $H$  be connected graphs such that  $\chi(G) = \chi(H) = n + 1 \geq 3$  and  $H$  is critical. If  $\mathcal{P}$  is a set paths of  $H$ , each of length at least 2, and  $E \subseteq E(G)$  is a set of edges of  $G$ , then every  $H\mathcal{P}E$ s-construct of  $G$  and  $H$  has chromatic number  $n + 1$ .*

*Proof.* Let  $G$ ,  $H$ ,  $\mathcal{P}$ , and  $E$  be as described above, and let  $F$  be an  $H\mathcal{P}E$ s-construct of  $G$  and  $H$ . Assume that  $\chi(F) < n + 1$ , and let  $P = (x_1, x_2, \dots, x_{k+1})$  be a path in  $\mathcal{P}$ . Then there exists an edge  $e = ab \in E$  that was replaced by hanging the  $P$ -string of  $H$ , without loss of generality, on  $b$ . Let  $C : V(F) \rightarrow \{1, \dots, n\}$  be a proper  $n$ -colouring of  $F$ . The subgraph of  $F$  induced by the vertices of the  $P$ -string of  $H$  hung on  $b$  is composed of graphs that are each isomorphic to  $H - x_i x_{i+1}$  for some integer  $1 \leq i \leq k$ . Since  $H$  is  $(n + 1)$ -critical it follows, for each  $1 \leq i \leq k$ , that every proper  $n$ -colouring of  $H - x_i x_{i+1}$  assigns the same colour to the the vertices  $x_i$  and  $x_{i+1}$ . Therefore  $C$  satisfies

$$C(a) = C(x'_2) = C(x'_3) = \dots = C(x'_k) = C(x^k_{k+1}).$$

Since  $x^k_{k+1}$  is adjacent to  $b$  in  $F$  it follows that  $C(x^k_{k+1}) \neq C(b)$ , and hence  $C(a) \neq C(b)$ . We can therefore conclude, for all edges  $ab \in E$ , that  $C(a) \neq C(b)$ . This implies that the restriction of  $C$  to  $V(G)$  is a proper  $n$ -colouring of  $G$ , which, of course, is a contradiction since  $\chi(G) = n + 1$ . Therefore it follows that  $\chi(F) \geq n + 1$ .

Now we show that  $\chi(F) \leq n + 1$ . Let  $C$  be an  $(n + 1)$ -colouring of  $G$ . Associate each edge  $e = ab \in E$  with the path  $P = (x_1, x_2, \dots, x_{k+1})$  in  $\mathcal{P}$  that is related to the  $P$ -string that replaced  $e$ . For each edge  $ab \in E$  and its related path  $P$ , let  $b$  be the vertex which the  $P$ -string of  $H$  was hung on, then assign a proper  $n$ -colouring  $D_P^e$  to the  $P$ -string of  $H$  that satisfies

$$C(a) = D_P^e(x^1_1) = D_P^e(x^1_2) = D_P^e(x^1_3) = \dots = D_P^e(x^1_k) = D_P^e(x^k_{k+1}).$$

Such an  $n$ -colouring exists since all  $P$ -strings discussed here are  $n$ -colourable. These  $n$ -colourings  $D_P^e$  are not restricted to using only the colours in  $\{1, 2, \dots, n\}$

but can use any  $n$  colours from the set  $\{1, 2, \dots, n, n+1\}$  as long as the above equality has been met. Now, for all vertices  $u \in V(F)$ , let  $D : V(F) \rightarrow \{1, \dots, n+1\}$  be defined as follows

$$D(u) = \begin{cases} C(u) & \text{if } u \in V(G), \text{ and} \\ D_p^e(u) & \text{if } u \text{ belongs to the } P\text{-string of } H \text{ that replaces } e. \end{cases}$$

Then  $D$  is a proper  $(n+1)$ -colouring of  $F$ .

From the two above arguments we can conclude that  $\chi(F) = n+1$ .  $\square$

This construction will be used in the next section to improve the main result of Section 3.3.

### 3.6 An extension on the Duffus, Sands and Woodrow Theorem

In this section we improve upon the main result of Section 3.3. This improvement will serve to extend Theorem 22 and enhance Theorem 23.

Before we present the third main result, we urge you to first consider the following two graphs. The first graph,  $G_5$ , is exhibited in Figure 14. One can be easily verify from the figure that  $\omega(G_5) = 4$ .

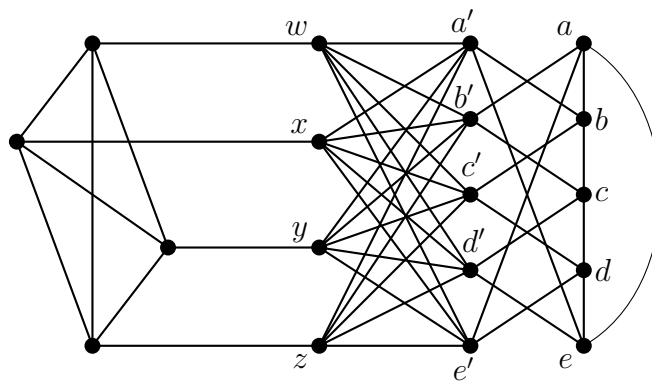


Figure 14:  $G_5$

To convince you that  $\chi(G_5) = 5$ , we submit to you the following. Let  $A = \{w, x, y, z, a', b', c', d', e', a, b, c, d, e\}$ , then, by close observation, it is clear that the induced subgraphs  $G_5[A \setminus \{x, y, z\}]$ ,  $G_5[A \setminus \{w, y, z\}]$ ,  $G_5[A \setminus \{w, x, z\}]$  and  $G_5[A \setminus \{w, x, y\}]$  are isomorphic to the Mycielski construct  $M_1(C_5)$  of  $C_5$ , here denoted by  $M(C_5)$ , which has chromatic number 4 and is triangle-free. Thus every 4-colouring of  $G_5[A]$  requires the vertices  $w, x, y, z$  to be assigned the same colour. Therefore any attempt to achieve a proper 4-colouring  $G_5$  would force a proper 3-colouring of  $G[V(G) \setminus A]$ , which, of course cannot be done.



The second graph,  $G^*$ , which is shown in Figure 15, was obtained by first taking the disjoint union of the Mycielski construct  $M(C_5)$  of  $C_5$ , and a new vertex  $v$ . Then  $v$  was made adjacent to all vertices of  $M(C_5)$ . From this, it can be easily deduced that  $\chi(G^*) = 5$  and  $\omega(G^*) = 3$ .

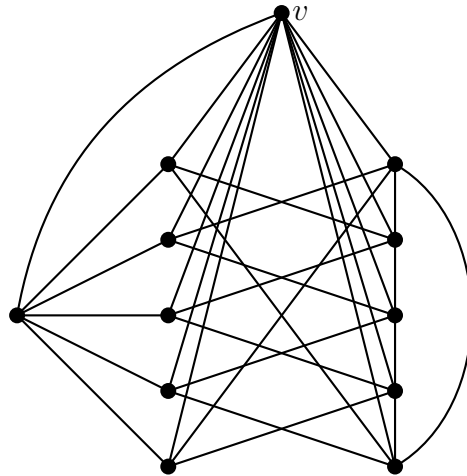


Figure 15:  $G^*$

From the results discussed so far, one is unable to determine whether  $\chi(G_5 \times G^*) = 5$  or not. What is interesting about these two graphs is that  $A$ , the set of vertices of  $G_5$  not contained in a 4-clique of  $G_5$ , induces a subgraph of  $G_5$  that is homomorphic to  $G^*$ . The result that follows asserts that  $\chi(G_5 \times G^*) = 5$ . The proof of which utilises the Ht-construction and the HPEs-construction.

**Theorem 30.** *Let  $G$  and  $H$  be connected graphs such that  $\chi(G) = \chi(H) = n + 1 \geq 3$ ,  $\omega(H) \leq n = \omega(G)$ , and  $H$  is critical. Let  $A$  be the set of vertices of  $G$  that are not contained in any  $n$ -clique of  $G$ . If  $G[A] \rightarrow H$  then  $\chi(G \times H) = n + 1$ .*

*Proof.* Let  $G$ ,  $H$ , and  $A$  be as described above, and assume that  $G[A] \rightarrow H$ .

We will prove that  $\chi(G \times H) = n + 1$  by constructing a graph  $F$ , with chromatic number  $n + 1$ , that is homomorphic to  $G$  and  $H$ .

Note that if  $A = \emptyset$  then we can arrive at  $\chi(G \times H) = n + 1$  by the use of Corollary 2, therefore we can assume that  $|A| \geq 1$ . We remark that the graph  $G[A]$  may or may not be connected. Let  $f : A \rightarrow V(H)$  be a homomorphism from  $G[A]$  to  $H$ , and let  $B$  be the subset of  $A$  that contains those vertices of  $G$  that are adjacent to at least one vertex in  $V(G) \setminus A$ . Note that if  $G[A]$  is disconnected then each component of  $G[A]$  contains at least one vertex that belongs to  $B$ . For some integer  $k \geq 1$ , we can partition  $B$  into  $k$  sets  $B_1, \dots, B_k$  satisfying the following: For all  $b, b' \in B$ , we have that

$$f(b) = f(b') \text{ if and only if } b \text{ and } b' \text{ belong to the same class.}$$

From this it follows that there exist  $k$  vertices  $u, v_2, \dots, v_k \in V(H)$  such that  $u \in f(B_1)$  and  $v_i \in f(B_i)$  for all  $2 \leq i \leq k$ . This means that  $f(B) = \{u, v_2, \dots, v_k\}$ . For all  $2 \leq i \leq k$ , let  $P_i$  be a shortest (in length)  $uv_i$ -path in  $H$ . For each  $2 \leq i \leq k$ , such a path  $P_i$  exists since  $H$  is connected.

Next, let  $\mathcal{P}$  be the set of all these paths, then partition it into  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , where  $\mathcal{P}_1$  is the set of those paths in  $\mathcal{P}$  of length 1. Finally let  $v \in V(H)$  be any vertex adjacent to  $u$ . We point out that  $v$  may belong to the set  $\{v_2, \dots, v_k\}$ , and  $\mathcal{P}_1$  or  $\mathcal{P}_2$  may be empty, possibly both. If  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are empty, then  $f(B) = u$ . Later, each path in  $\mathcal{P}_2$  will be used to create  $P$ -strings of  $H$  if  $\mathcal{P}_2$  is not empty.

Now, consider the subgraph  $G[V(G) \setminus A]$  of  $G$ . If  $\chi(G[V(G) \setminus A]) = n + 1$  then, by Corollary 2, we have that  $\chi(G[V(G) \setminus A] \times H) = n + 1$  which implies that  $\chi(G \times H) = n + 1$ . Therefore we may assume that  $\chi(G[V(G) \setminus A]) = n$ . Let  $V_1, \dots, V_n$  be a  $n$ -colouring of  $G[V(G) \setminus A]$ .

First, we recursively construct graphs  $F_1, F_2, \dots, F_n$  in such a way that, for each  $2 \leq i \leq n$ , the graph  $F_i$  is an Ht-construct of  $F_{i-1}$  and  $H$ , where  $F_1$  is an Ht-construct of  $G$  and  $H$ . For each vertex  $x \in V(G) \setminus A$ , let  $H_x$  be a

copy of the graph  $H - uv$  as described earlier on in this chapter. The graph  $F_1$  is obtained as follows. For each  $x \in V_1$ , replace  $x$  with  $H_x$ , then make  $v_x$  adjacent to all those  $y \in N'_{G-V_1}(x)$  that belong to  $B_1$  or to a  $V_j$  with an odd  $j > 1$ . Lastly make  $u_x$  adjacent to those  $y \in N'_{G-V_1}(x)$  that belong to  $B \setminus B_1$  or to a  $V_j$  with an even  $j > 1$ . We point out that the set  $B_1$  is not empty but the set  $B \setminus B_1$  could possibly be empty. Then  $F_1$  is an Ht-construct of  $G$  and  $H$  obtained through  $V_1$  and  $uv$ , and so, by Lemma 32,  $\chi(F_1) = n + 1$ .

Suppose, for some  $i$  with  $2 \leq i \leq n$ , that the graph  $F_{i-1}$  has been constructed. We construct  $F_i$  from  $F_{i-1}$  and  $H$  using  $V_i$  and  $uv$  as follows. For each  $x \in V_i$ , replace  $x$  in  $F_{i-1}$  with  $H_x$ , and make  $v_x$  adjacent to all those  $y \in N'_{F_{i-1}-V_i}(x)$  that belong to  $B_1$  or to a  $V_j$  with an odd  $j > i$ . Next make  $u_x$  adjacent to those  $y \in N'_{F_{i-1}-V_i}(x)$  that belong to  $B \setminus B_1$  or to a  $V_j$  with an even  $j > i$ . Finally, for the remaining  $y \in N'_{F_{i-1}-V_i}(x)$ , make  $y$  adjacent to  $v_x$  if  $i$  is even, and make  $y$  adjacent to  $u_x$  if  $i$  is odd. Clearly  $F_i$  is an Ht-construct of  $F_{i-1}$  and  $H$  obtained through  $V_i$  and  $uv$ . Thus we have that  $\chi(F_i) = n + 1$ .

By repeated application of Lemma 32 we obtain that  $\chi(F_n) = n + 1$ . We are now at the final step of our construction. Let  $E$  be the set of all edges of  $F_n$  that connect a vertex of  $B$  to a vertex not in  $A$ , that is

$$E = \{wy \in E(F_n) \mid w \in V(F_n) \setminus A \text{ and } y \in B\}.$$

Thus the set  $E$  is not empty since  $|A| \geq 1$  and  $G$  is connected. Notice that if  $wy \in E$  and  $y \in B \setminus B_1$  then  $w = u_x$  for some  $x \in V(G) \setminus A$ . For each integer  $i$  with  $1 \leq i \leq k$  let  $E_i = \{wy \in E \mid y \in B_i\}$ . Then the sets  $E_1, \dots, E_k$  are mutually disjoint, and  $E = E_1 \cup \dots \cup E_k$ . If  $\mathcal{P}_2 = \emptyset$  then we let  $F$  be the graph  $F_n$ , and we are done. If  $\mathcal{P}_2 \neq \emptyset$  then we proceed as follows. For each integer  $2 \leq i \leq k$ , replace each edge  $xy \in E_i$ , where  $y \in B_i$ , with a  $P_i$ -string of  $H$  hung on  $y$  if and only if  $P_i \in \mathcal{P}_2$ . It should be clear that if, for some  $2 \leq j \leq k$ , the path  $P_j$  belongs to  $\mathcal{P}_1$  then none of the edges in  $E_j$  are

replaced. We call this new graph  $F$ . For some  $E^* \subset E$ , the graph  $F$  is an  $HP_2(E^*)_s$ -construct of  $F_n$  and  $H$ , and thus, by Lemma 34,  $\chi(F) = n + 1$ .

Let  $U = \{u_x \in V(F) \mid x \in V(G) \setminus A\}$ , and  $V = \{v_x \in F \mid x \in V(G) \setminus A\}$ . The sets  $U$  and  $V$  are independent sets of vertices of  $F$ . Suppose there exist vertices  $u_x, u_{x'} \in U$  such that  $u_x u_{x'} \in E(F)$ . Then there exist integers  $1 \leq i < j \leq n$  such that  $x \in V_i$  and  $x' \in V_j$ . Suppose  $j$  is even, then when  $F_j$  was constructed the vertex  $u_{x'}$  was made adjacent to only those  $y \in N'_{F_{j-1}-V_j}(x')$  that belong to  $B \setminus B_1$  or to a  $V_t$  with an even  $t > j$ , of which  $u_x$  is not a member. This contradicts the assumption that  $u_x$  and  $u_{x'}$  are adjacent, thus  $j$  is odd. This implies that  $u_x$  is adjacent to  $x'$  in the graph  $F_i$ . But during the construction of  $F_i$ ,  $u_x$  was not made adjacent to any vertex belonging to a  $V_t$  with an odd  $t > i$ , which implies that  $u_x$  is not adjacent to  $x'$ . This is clearly a contradiction, thus  $U$  is in fact an independent set of vertices of  $F$ . Using a similar argument it can also be shown that  $V$  is an independent set of vertices of  $F$ .

Next we describe a homomorphism  $\phi$  from  $F$  to  $H$ . For each path  $P_i = (u = x_1, x_2, \dots, x_{r+1} = v_i)$  in  $\mathcal{P}_2$  let  $T_i$  be the  $P_i$ -string of  $H$ . The mapping  $\phi_{P_i} : V(T_i) \rightarrow V(H)$  defined, for all  $z \in V(T_i)$ , by

$$\phi_{P_i}(z) = \begin{cases} x_j & \text{if } z = x'_j \text{ for some } 2 \leq j \leq r, \text{ and} \\ y & \text{if } z = y^k \text{ for some } y \in V(H) \text{ and some } k \leq r \end{cases}$$

is a homomorphism of  $T_i$  to  $H$ . Let  $\phi : V(F) \rightarrow V(H)$  be the mapping defined as follows: the restriction of  $\phi$  to  $A$  is the homomorphism  $f$  described in the beginning, the restriction of  $\phi$  to a  $P_i$ -string of  $H$  is the homomorphism  $\phi_{P_i}$ , and, for each vertex  $z$  in  $F$  that belongs to an  $H_x$  for some  $x \in V(G) \setminus A$ , let  $\phi(z)$  be the vertex in  $H$  that corresponds to  $z$ . Thus  $\phi(U) = u$  and  $\phi(V) = v$ . It follows that  $\phi$  is a homomorphism from  $F$  to  $H$ .

Lastly we give a description of a homomorphism  $\gamma$  from  $F$  to  $G$ . For each  $H_x$ , let  $\gamma_x$  be a homomorphism that maps  $H_x$  into an  $n$ -clique of  $G$  that contains  $x$ . Select  $\gamma_x$  to be such that  $\gamma_x(u_x) = \gamma_x(v_x) = x$ .

For each edge  $wy \in E$ , where  $y \in B \setminus B_1$ , that was replaced by some  $P_i$ -string, we have that  $w = u_x$  for some  $x \in V(G) \setminus A$ . Let  $P_i = (u = x_1, x_2, \dots, x_{r+1} = v_i)$ , and let  $T_{wy}$  be that subgraph of  $F$  which is the  $P_i$ -string that replaced the edge  $wy$ . Define the mapping  $\gamma_{wy} : V(T_{wy}) \rightarrow V(G)$  as follows. For all  $z \in V(T_{wy})$ ,

$$\gamma_{wy}(z) = \begin{cases} \gamma(x_j) & \text{if } z = x'_j \text{ for some } 2 \leq j \leq r, \text{ and} \\ \gamma_x(y) & \text{if } z \notin \{x'_2, \dots, x'_r\}. \end{cases}$$

Then  $\gamma_{wy}$  is a homomorphism. Finally, define  $\gamma : V(F) \rightarrow V(G)$  as follows: the restriction of  $\gamma$  to  $A$  is the identity mapping, the restriction of  $\gamma$  to a  $P_i$ -string of  $H$  that replaced an edge  $wy \in E$  is the homomorphism  $\gamma_{wy}$ , and, the restriction of  $\gamma$  to any  $V(H_x)$  is  $\gamma_x$ . Then  $\gamma$  is a homomorphism from  $F$  to  $G$ . This completes our proof.  $\square$

After having considered this result it seems natural to ask the following. What if for two graphs  $G$  and  $H$ ,  $G[A]$  is not homomorphic to  $H$  but instead a subgraph of  $H$  is homomorphic to  $G[A]$ . The next result is the best answer we can give.

We offer one last result which is obtained by further increasing the restrictions on the graphs  $G$  and  $H$ . The proof of which exploits the existence of a homomorphism from  $G[A]$  to  $H$ . For this final result we will require two definitions; the first of these two concepts is introduced in [14]. A graph  $G$  with  $\chi(G) = n$  is *uniquely  $n$ -colourable* if there is exactly one partition of its vertex set  $V(G)$  into  $n$  independent sets. A graph  $G$  is said to be a *homomorphic image* of a graph  $H$  if there exists a homomorphism  $\phi$  from  $H$  onto  $G$  satisfying the following: for all edges  $xy \in E(G)$  there exists an edge  $wv \in E(H)$  such that  $\phi(u) = x$  and  $\phi(v) = y$ . We note that this concept is essentially the same as the one defined in [17].

**Theorem 31.** *Let  $G$  and  $H$  be connected graphs such that  $\chi(G) = \chi(H) = n + 1$ ,  $\omega(G) = n$ , and  $H$  is  $(n + 1)$ -critical. In addition, let  $A$ , the set*

of vertices of  $G$  that are not contained in an  $n$ -clique of  $G$ , be such that  $\chi(G[A]) = n$ . If there exists a uniquely  $n$ -colourable subgraph  $H'$  of  $H$  with  $\chi(H') = n$  that is such that  $G[A]$  is a homomorphic image of  $H'$ , then  $\chi(G \times H) = n + 1$ .

*Proof.* Let  $G, H, H'$  and  $A$  be as described above. Permit  $\phi$  to be a homomorphism from  $H'$  to  $G[A]$  and allow  $B \subseteq A$  to be the set of those vertices of  $G$  that are adjacent to a vertex in  $V(G) \setminus A$ . Finally let  $G'$  be that graph with vertex set  $V(G') = (V(G) \setminus A) \cup V(H')$  and edge set

$$E(G') = E(G[V(G) \setminus A]) \cup E(H') \cup N,$$

where  $N = \{xy \mid x \in V(G) \setminus A, y \in V(H'), \phi(y) \in B \text{ and } x\phi(y) \in E(G)\}$ . It follows immediately that  $G' \longrightarrow G$ : This can be seen from the mapping  $\gamma : V(G') \longrightarrow V(G)$  whose restriction to  $V(G) \setminus A$  is the identity mapping and whose restriction to  $V(H')$  is  $\phi$ . From this it follows that  $\chi(G') \leq \chi(G) = n + 1$ . We claim that  $\chi(G') = n + 1$ . Suppose this is not so, then  $\chi(G') = n$  since  $\omega(G') = n$ . From this it follows that there exists a proper  $n$ -colouring  $C$  of  $G'$ . In addition,  $|C(\phi^{-1}(x))| = 1$  for all  $x \in A$ , since the assumption that  $|C(\phi^{-1}(x))| > 1$  for some  $x \in A$  implies that there exists an alternative proper  $n$ -colouring of  $H'$ , one in which the vertices of  $\phi^{-1}(x)$  have been assigned at least two different colours, which would imply that  $H'$  is not uniquely  $n$ -colourable. But then  $|C(\phi^{-1}(x))| = 1$  for all  $x \in A$ , which implies that the mapping  $C' : V(G) \longrightarrow \{1, 2, \dots, n\}$  defined below is a proper  $n$ -colouring of  $G$ . For all  $x \in V(G)$ , let

$$C'(x) = \begin{cases} C(x) & \text{if } x \in V(G) \setminus A \text{ and} \\ C(\phi^{-1}(x)) & \text{if } x \in A. \end{cases}$$

This is clearly a contradiction, thus it follows that  $\chi(G') = n + 1$ . By Theorem 30 it follows that  $\chi(G' \times H) = n + 1$  since the vertices of  $G'$  which are not contained in an  $n$ -clique induce a subgraph of  $G'$  that is homomorphic to  $H$ . From  $\chi(G \times H) \geq \chi(G' \times H)$  we obtain that  $\chi(G \times H) = n + 1$ .  $\square$

If the hypothesis of Theorem 31 permitted  $\chi(G[A])$  to be less than  $n$  and  $H'$  to be uniquely  $\chi(G[A])$ -colourable, then we would be faced with the difficulty of proving that the colouring  $C$  in the proof of Theorem 31 is such that  $|C(\phi^{-1}(x))| = 1$  for all  $x \in A$ . The reason for this is that the restriction of  $C$  to  $V(H')$  may use more colours than  $\chi(G[A])$  which would not allow us to use the unique  $\chi(G[A])$ -colourability of  $H'$ .

One of the biggest shortcomings of Theorem 31 seems to be the restriction that  $G[A]$ , the subgraph of  $G$  induced by the set  $A$ , must have chromatic number  $n$ . But this is not so devastating once one considers the theorem below.

**Conjecture 2.** *For all positive integers  $n$ , and all graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n + 1$ , if  $G$  is a connected graph with  $\omega(G) = n$  and  $\chi(G[A]) = n$ , where  $A$  is the set of those vertices of  $G$  not contained within an  $n$ -clique, then  $\chi(G \times H) = n + 1$ .*

**Theorem 32.** *Conjecture 1 is true if and only if Conjecture 2 is true.*

*Proof.* Clearly Conjecture 1 implies Conjecture 2. Suppose that Conjecture 2 holds and let  $G$  be any connected graph with  $\chi(G) = n + 1$ ,  $\omega(G) = n$ , and  $\chi(G[A]) < n$ . We will prove that  $\chi(G \times H) = n + 1$  for all graphs  $H$  with  $\chi(H) = n + 1$ . If  $\chi(G[A]) \leq 2$  then, by Corollary 3, we have that  $\chi(G \times H) = n + 1$  for all graphs  $H$  with  $\chi(H) = n + 1$ . Therefore we may assume that  $2 < \chi(G[A]) < n$ . Let  $H^*$  be an  $(n + 1)$ -critical graph with  $\omega(H^*) < n$ . If one does not exist then it follows, by Theorem 23, that  $\chi(G \times H) = n + 1$  for all graphs  $H$  with  $\chi(H) = n + 1$ . So assume that such a graph  $H^*$  exists. Next, let  $uv$  be an edge of  $H^*$ , and let  $x \in A$  and  $y \in V(G) \setminus A$  be adjacent vertices in  $G$ . Now let  $G'$  be the Hajós construct of  $G$  and  $H^*$  obtained as follows:

- First, delete the edges  $uv$  and  $yx$ ,

- then, identify the vertices  $u$  and  $y$ , and name this vertex  $z$ ,
- and lastly, make  $v$  and  $x$  adjacent.

Let  $A'$  be the union of  $A$  and the set  $V(H^*) \setminus \{u\}$ , then  $A'$  is the set of those vertices of  $G'$  that do not belong to an  $n$ -clique of  $G'$ . Furthermore, we have that  $\chi(G') = n + 1$ , and  $\chi(G'[A']) = n$ .

We claim that  $G' \rightarrow G$ . Since  $H^*$  is  $(n + 1)$ -critical and  $y$  lies in an  $n$ -clique of  $G$ , it follows that the subgraph of  $G'$  induced by  $V(H^*) \setminus \{u\}$  is homomorphic to an  $n$ -clique of  $G$  that contains  $y$ . Thus there exists such a homomorphism  $\gamma$  that maps  $v$  to  $y$ . Let  $\phi : V(G') \rightarrow G$  be the mapping defined as follows: The restriction of  $\phi$  to those vertices of  $G'$  that do not belong to  $(V(H^*) \setminus \{u\}) \cup \{z\}$  is the identity mapping, while the restriction of  $\phi$  to those vertices of  $G'$  that belong to  $V(H^*) \setminus \{u\}$  is  $\gamma$ , and  $\phi(z) = y$ . Then  $\phi$  is a homomorphism.

By the assumption that Corollary 2 holds, we have that  $\chi(G' \times H) = n + 1$  for all graphs  $H$  with  $\chi(H) = n + 1$ . From the above claim we have  $\chi(G \times H) \geq \chi(G' \times H) = n + 1$  for all graphs  $H$  with  $\chi(H) = n + 1$ . This, of course, implies Conjecture 1.  $\square$

Thus when dealing with graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n + 1$ , and  $\omega(G) = n$ , we can limit our investigation to only those graphs  $G$  that satisfy  $\chi(G[A]) = n$ .



### 3.7 Hom-properties and the Hedetniemi Conjecture

We end this chapter with a brief discussion on the link between hom-properties and the Hedetniemi Conjecture. Let  $Hom^* = \{\rightarrow G \mid \chi(G) \text{ is finite}\}$ , then  $Hom^* \subseteq Hom$ . Let  $\rightarrow G$  and  $\rightarrow H$  be properties in  $Hom^*$ , then  $G \sqcup H$  and  $G \times H$  are of finite chromatic number, therefore  $\rightarrow(G \sqcup H)$  and  $\rightarrow(G \times H)$  belong to  $Hom^*$ . From this it follows that  $Hom^*$  is a sublattice of  $Hom$ . We shift our focus to meet-irreducible elements in  $Hom$ . The following are known meet-irreducible elements in  $Hom$ :  $\rightarrow K_1, \rightarrow K_2, \rightarrow K_3$  and  $\rightarrow K_4$ . Having not managed to characterize the meet-irreducible elements in  $Hom$ , we offer a theorem which we hope brings us a step closer to identifying these elements. This theorem is equivalent to the result mentioned by Tardif in [22].

**Theorem 33.** *The Hedetniemi Conjecture is true if and only if  $\rightarrow K_n$  is meet-irreducible in  $Hom^*$  for all  $n \in \mathbb{N}$ .*

*Proof.* Let the Hedetniemi Conjecture be true and assume that, for some  $n$ ,  $\rightarrow K_n$  is meet-reducible. Then there exist countable graphs  $G$  and  $H$ , of finite chromatic number, such that  $\rightarrow K_n \subset \rightarrow G, \rightarrow H$  and  $\rightarrow(G \times H) = \rightarrow G \wedge \rightarrow H = \rightarrow K_n$ . Which implies  $(G \times H) \rightarrow K_n$  and  $K_n \rightarrow (G \times H)$ . Thus  $n = \chi(G \times H) = \min\{\chi(G), \chi(H)\}$ , which suggests  $\chi(G) = n$  or  $\chi(H) = n$ . Without loss of generality let  $\chi(G) = n$ , then  $G$  is homomorphic to  $K_n$  and consequently  $\rightarrow G \subseteq \rightarrow K_n$ , a contradiction.

Conversely, suppose that  $\rightarrow K_n$  is meet-irreducible, for all  $n \in \mathbb{N}$ , in  $Hom^*$ , and assume that, for some  $n$ , there exist graphs  $G$  and  $H$  such that  $n = \chi(G \times H) < \min\{\chi(G), \chi(H)\}$ . Then

$$\rightarrow K_n \subset \rightarrow(G \sqcup K_n) \text{ and } \rightarrow K_n \subset \rightarrow(H \sqcup K_n).$$

Now

$$(G \sqcup K_n) \times (H \sqcup K_n) = (G \times H) \sqcup (G \times K_n) \sqcup (K_n \times H) \sqcup (K_n \times K_n)$$

where

$$\chi(G \times H) = \chi(G \times K_n) = \chi(K_n \times H) = \chi(K_n \times K_n) = n,$$

since  $\chi(K_n \times F) = n$  for all graphs  $F$  with  $\chi(F) \geq n$ . From this it follows that  $\chi((G \sqcup K_n) \times (H \sqcup K_n)) = n$ , therefore

$$\rightarrow((G \sqcup K_n) \times (H \sqcup K_n)) \subseteq \rightarrow K_n. \text{ Since } K_n \leq K_n \times K_n \leq (G \sqcup K_n) \times (H \sqcup K_n),$$

it follows that  $K_n$  is homomorphic to  $(G \sqcup K_n) \times (H \sqcup K_n)$ .

Thus  $\rightarrow K_n \subseteq \rightarrow((G \sqcup K_n) \times (H \sqcup K_n))$  and as a consequence

$$\rightarrow(G \sqcup K_n) \wedge \rightarrow(H \sqcup K_n) = \rightarrow((G \sqcup K_n) \times (H \sqcup K_n)) = \rightarrow K_n,$$

which implies that  $\rightarrow K_n$  is meet-reducible, a contradiction.  $\square$

As stated by Tardif and Zhu in [24], Hedetniemi's conjecture asserts that if  $G$  and  $H$  are such that  $G \not\rightarrow K_n$  and  $H \not\rightarrow K_n$ , then  $G \times H \not\rightarrow K_n$ . The following definition taken from [12] is been modified to apply only to countable graphs.

**Definition 14.** [12] *A graph  $M \in \mathcal{I}$  is multiplicative if whenever two graphs  $G$  and  $H$  in  $\mathcal{I}$  are such that  $G \not\rightarrow M$  and  $H \not\rightarrow M$ , then  $G \times H \not\rightarrow M$ .*

From this definition it follows that Hedetniemi's conjecture asserts that all finite complete graphs are multiplicative. The following theorem is due to Häggkvist, Hell, Miller and Neumann Lara.

**Theorem 34.** [12] *All cycles  $C_n, n \geq 3$  are multiplicative.*

The following theorem is implied by the results obtained by Sauer in [21].

**Theorem 35.** [12] *A graph  $M \in \mathcal{I}$  with finite chromatic number is multiplicative if and only if  $\rightarrow M$  is meet-irreducible in  $\text{Hom}^*$ .*

*Proof.* Let  $M \in \mathcal{I}$  be such that  $\chi(M)$  is finite.

Now, let that  $M$  be multiplicative and assume that  $\rightarrow M$  is meet-reducible in  $Hom^*$ . Then there exist graphs  $\rightarrow G, \rightarrow M \in Hom^*$  such that  $\rightarrow M \subset \rightarrow G, \rightarrow H$  and  $\rightarrow M = \rightarrow G \wedge \rightarrow H$ . From which we have  $G \not\rightarrow M, H \not\rightarrow M$  but  $G \times H \rightarrow M$ . This contradicts the assumption that  $M$  is multiplicative.

Now, assume that  $\rightarrow M$  is meet-irreducible in  $Hom^*$ . Next let  $G$  and  $H$  be any two graphs in  $\mathcal{I}$  such that  $G \not\rightarrow M$  and  $H \not\rightarrow M$ . Now suppose that  $G \times H \rightarrow M$ . We claim that this assumption implies that

$$\rightarrow(G \sqcup M) \cap \rightarrow(H \sqcup M) = \rightarrow M. \quad (1)$$

We know that

$$\rightarrow(G \sqcup M) \cap \rightarrow(H \sqcup M) = \rightarrow((G \sqcup M) \times (H \sqcup M)), \quad (2)$$

and since the direct product distributes over the disjoint union, we have that

$$\rightarrow((G \sqcup M) \times (H \sqcup M)) = \rightarrow(A \sqcup B \sqcup C \sqcup D), \quad (3)$$

where  $A = G \times H, B = G \times M, C = M \times H$  and  $D = M \times M$ . By (2) and (3), it follows that to prove (1) it is sufficient the following:

$$\rightarrow(A \sqcup B \sqcup C \sqcup D) = \rightarrow M. \quad (4)$$

Let  $F$  be a graph in  $\rightarrow M$ , then  $F \rightarrow M$ , which implies that  $F \rightarrow D = M \times M$ . From this it follows that  $F \in \rightarrow(A \sqcup B \sqcup C \sqcup D)$ . Now suppose that  $K$  is a graph in  $\rightarrow(A \sqcup B \sqcup C \sqcup D)$ , then  $F$  is homomorphic to at least one of the four graphs  $A, B, C$  and  $D$ . Note that these four graphs are all homomorphic to  $M$ . The graph  $A$  is homomorphic to  $M$  by an assumption we made earlier. From all this it follows that  $F \rightarrow M$ , which implies that  $F \in \rightarrow M$ .

By the above argument (4) is established, from which (1) follows. But clearly (1) implies that  $\rightarrow(G \sqcup M) \wedge \rightarrow(H \sqcup M) = \rightarrow M$ . Since  $\rightarrow M$  is meet-irreducible it follows, without loss of generality, that  $\rightarrow M = \rightarrow(G \sqcup M)$ ,

which yields  $G \sqcup M \longrightarrow M$ , and in turn produces  $G \longrightarrow M$ . This, of course, is a contradiction.  $\square$

It follows that the hom-property  $\rightarrow C_n$  is meet-irreducible, for all integers  $n \geq 3$ , in  $\text{Hom}^*$ .

Lastly we present equivalent forms of the Hedetniemi conjecture, some of which are expressed in terms of hom-properties.

**Theorem 36.** *The following statements are equivalent:*

- (a) *For all finite graphs  $G, H \in \mathcal{I}$  we have  $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ .*
- (b) *For all  $G, H \in \mathcal{I}$  with finite chromatic number we have*  

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$
- (c) *For all  $n \in \mathbb{N}$  the hom-property  $\rightarrow K_n$  is meet-irreducible in  $\langle \text{Hom}^*, \wedge, \vee \rangle$ .*
- (d) *For every two hom-properties  $\rightarrow G$  and  $\rightarrow H$  in  $\text{Hom}^*$  if*  

$$\rightarrow K_n = \rightarrow G \cap \rightarrow H \text{ then } \rightarrow K_n = \rightarrow G \text{ or } \rightarrow K_n = \rightarrow H.$$
- (e) *For all  $n \in \mathbb{N}$  if  $\rightarrow G \cap \rightarrow H \subseteq \rightarrow K_n$  then  $\rightarrow G \subseteq \rightarrow K_n$  or  $\rightarrow H \subseteq \rightarrow K_n$ .*
- (f) *For all  $n \in \mathbb{N}$  and every two  $n$ -critical graphs  $G$  and  $H$  there exists an  $n$ -critical graph  $F$  such that  $F \longrightarrow G$  and  $F \longrightarrow H$ .*
- (g) *For all  $n \in \mathbb{N}$  and any graphs  $G$  and  $H$  with  $\chi(G) = \chi(H) = n + 1$  and  $\omega(G) = \omega(H) = n$ , there exists a graph  $F$  of chromatic number  $n + 1$  that is homomorphic to  $G$  and  $H$ .*

*Proof.* It is not too difficult to see that (a) and (b) are equivalent. By Theorem 33 we have (b) and (c) are equivalent, therefore (a) and (c) are equivalent. The equivalence of (c) and (d) follows by the definition of meet-irreducibility in a lattice.

We show that (b) and (e) are equivalent. Assume (b), then  $\rightarrow G \cap \rightarrow H \subseteq \rightarrow K_n$  implies  $\chi(G \times H) \leq n$ . By our assumption  $\chi(G) \leq n$  or  $\chi(H) \leq n$ . Therefore  $G \rightarrow K_n$  or  $H \rightarrow K_n$ . Thus  $\rightarrow G \subseteq \rightarrow K_n$  or  $\rightarrow H \subseteq \rightarrow K_n$ . Now assume (e). Given  $G, H \in \mathcal{I}$  we have that  $\rightarrow(G \times H) \subseteq \rightarrow K_{\chi(G \times H)}$ . By (e) we obtain that  $G \rightarrow K_{\chi(G \times H)}$  or  $H \rightarrow K_{\chi(G \times H)}$ . From which we obtain that  $\chi(G) \leq \chi(G \times H)$  or  $\chi(H) \leq \chi(G \times H)$ . This, of course, implies that  $\chi(G \times H) \geq \min\{\chi(G), \chi(H)\}$ . Therefore  $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ .

Now we show that (b) and (f) are equivalent. Let  $G, H \in \mathcal{I}$  be  $n$ -critical and assume (b). Then  $\chi(G \times H) = n$ , therefore  $G \times H$  has an  $n$ -critical subgraph  $F$ . Since  $F \rightarrow (G \times H) \rightarrow G$  and  $(G \times H) \rightarrow H$ , it follows by Lemma 3 that  $F \rightarrow G$  and  $F \rightarrow H$ . Now assume (f) and let  $G, H \in \mathcal{I}$ . Without loss of generality let  $\chi(G) \leq \chi(H)$ . Then, for all positive integers  $n \leq \chi(G)$ , there exist  $n$ -critical subgraphs of  $G$  and  $H$ . By (f) there exists, for each  $n$ , an  $n$ -critical graph  $F$  that is homomorphic to  $G$  and  $H$ , and as a result is homomorphic to  $G \times H$ . Thus  $\chi(G \times H) \geq n$  for all positive integers  $n \leq \chi(G)$ . As a consequence  $\chi(G \times H) = \chi(G) = \min\{\chi(G), \chi(H)\}$ .

To show that (c) and (g) are equivalent we begin by showing that if (c) does not hold then (g) does not hold. So assume, for some  $n \in \mathbb{N}$ , that  $\rightarrow K_n$  is meet-reducible. Then there exist graphs  $G$  and  $H$  such that  $\rightarrow K_n = \rightarrow G \wedge \rightarrow H$  with  $\rightarrow K_n \subset \rightarrow G$  and  $\rightarrow K_n \subset \rightarrow H$ . Let  $G'$  and  $H'$  be graphs in  $\rightarrow G$  and  $\rightarrow H$ , respectively, with chromatic number  $n + 1$ . We claim  $\omega(G'), \omega(H') < n + 1$ . Without loss of generality assume  $\omega(G') = n + 1$ , then we have  $H' \rightarrow K_{n+1} \rightarrow G' \rightarrow G$  and  $H' \rightarrow H$ . Which implies  $H' \in \rightarrow G \wedge \rightarrow H = \rightarrow K_n$ , and in turn implies  $H' \rightarrow K_n$ . Surely, this is a contradiction since  $\chi(H') = n + 1$ . Let  $G^* = G' \sqcup K_n$  and  $H^* = H' \sqcup K_n$ . Then  $G^*$  and  $H^*$  satisfy the conditions in (g). We argue that by our initial hypothesis there can not exist a graph  $F$  of chromatic number  $n + 1$  that is homomorphic to  $G^*$  and  $H^*$ . Suppose such a graph  $F$  existed, then  $F$  would

be homomorphic to  $G$  and  $H$  since  $G^* \rightarrow G$  and  $H^* \rightarrow H$ . Which would imply  $F \in (\rightarrow G \wedge \rightarrow H) = \rightarrow K_n$ , and thus imply  $F \rightarrow K_n$ . Which again is a contradiction since  $\chi(F) = n + 1$ .

Now we show that (c) implies (g). Assume  $\rightarrow K_n$  is meet-irreducible for all  $n \in \mathbb{N}$  and let  $G$  and  $H$  be any graphs in  $\mathcal{I}$  with  $\chi(G) = \chi(H) = n + 1$  and  $\omega(G) = \omega(H) = n$ . Then  $K_n \rightarrow G$  and  $K_n \rightarrow H$  therefore  $K_n \in \rightarrow(G \times H)$ . Which produces  $\rightarrow K_n \subseteq \rightarrow(G \times H) = \rightarrow G \wedge \rightarrow H$ . Therefore  $\rightarrow K_n \subset \rightarrow(G \times H)$  since  $\rightarrow K_n$  is meet-irreducible. Thus there exists a graph  $F \in \rightarrow(G \times H)$  which does not belong to  $\rightarrow K_n$ . Therefore  $\chi(F) = n + 1$  and  $F$  is homomorphic to  $G$  and  $H$ . This completes our proof.  $\square$

We point out the similarity between (g) and Theorem 23. The statement of Theorem 23 reads exactly like (g) but with the restriction that  $G$  and  $H$  are connected. Although far from obvious, this added condition placed upon  $G$  and  $H$  drastically limits the strength of Theorem 23. Evidenced by Theorem 36, (g) implies the Hedetniemi Conjecture, one reason for this is that the direct product distributes over the disjoint union. Suppose that  $G$  and  $H$  are any two graphs both with chromatic number  $n + 1$ . We can not look to Theorem 23 to deduce whether  $\chi(G \times H) = n + 1$  unless these graphs are both connected and have clique number  $n$ . But if (g) holds it would guarantee us that  $\chi(G \times H) = n + 1$ . The reason is as follows. By letting  $G^* = G \sqcup K_n$  and  $H^* = H \sqcup K_n$ , we have that  $\chi(G^* \times H^*) = n + 1$ . Since the direct product distributes over the disjoint union, we have that

$$(G \sqcup K_n) \times (H \sqcup K_n) = (G \times H) \sqcup (K_n \times H) \sqcup (G \times K_n) \sqcup (K_n \times K_n),$$

from which we obtain that  $\chi(G \times H) = n + 1$  since  $\chi(K_n \times H) = \chi(G \times K_n) = \chi(K_n \times K_n) = n$ .

Our hope is that further investigations into hom-properties might bring us closer to settling the Hedetniemi Conjecture.

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