Stochastic quasilinear parabolic equations with non standard growth: Weak and strong solutions

by

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Declaration

I, Ali Zakaria Idriss declare that the thesis, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE:

DATE: 2015-06-22

Dedicace

I dedicate this thesis to my parents, my father Zakaria Idriss Bremé, my mother Halimta Adam (known as Al-halla), my family that I will not mention their names as the list would be exhaustive and the communities of my entire region a ”savoir”, Am-timan (Bahr Azoum or SALAMT) and Haraze (Moscouvette) Mongueigne.

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Acronyms

$\forall$	for all
$\exists$	there exists
$\equiv$	equivalent
$\sum$	summation
$\prod$	product
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\overline{\mathbb{R}}^+$	set of positive extended real numbers : $[0, \infty]$
$[t_0, T]$	closed interval $t_0 \leqslant t \leqslant T$ in $\mathbb{R}$
n	positive integer greater than or equals to 1
d	positive integer greater than or equals to 1
m	positive integer greater than or equals to 1
$\mathbb{R}^d$	Euclidean space or d-dimensional vectors
$\mathbb{D}$	open bounded subset of $\mathbb{R}^n$
$\partial\mathbb{D}$	boundary of $\mathbb{D}$
$\overline{\mathbb{D}}$	closure of $\mathbb{D}$ i.e., the set $\mathbb{D}$ plus its boundary $\partial\mathbb{D}$
$C(\mathbb{D})$	continuous functions from $\mathbb{D}$ to $\mathbb{R}$
$C(\mathbb{D})$	space of continuous functions on $\mathbb{D}$ with compact support in $\mathbb{D}$
$C^k(\mathbb{D})$	space of k-times continuously differentiable function on $\mathbb{D}$
$C^\infty(\mathbb{D})$	infinitely differentiable functions on $\mathbb{D}$
$C_0^\infty(\mathbb{D})$	infinitely differentiable functions with compact support on $\mathbb{D}$, this space is also denoted by $\mathcal{D}(\mathbb{D})$
$C([0, T]; \mathbb{D})$	continuous function $u : [0, T] \to \mathbb{D}$

I_B	characteristic (indicator) function of the set B
C	generic constant that may change from line to line
Ω	non empty sample space(a collection of possible outcomes)
$\mathcal{F}$	σ-algebra of subsets of Ω
$\mathbb{P}$	probability measure defined on Ω
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
W	Wiener process (d-dimensional or infinite dimensional)
$\bar{W}$	prescribed Wiener process (standard d-dimensional or cylindrical)
$(\mathcal{F})_{t \geqslant 0}$	filtration
$\mathbb{E}$	mathematical expectation w.r.t. $\mathbb{P}$
$\mathcal{B}(E)$	Borel σ-algebra of a topological set E
$(E, \mathcal{B}(E))$	measurable space
$X = (X(t), t \geqslant 0)$	stochastic process X
dX	stochastic differential of a process X
$\sigma(X, t \geqslant 0)$	natural filtration of the process X
a.s.	almost surely
a.e.	almost everywhere
a.a.	almost all
$\\| \cdot \\|_E$	norm in a normed space E
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
p	real number such that $1 \leqslant p \leqslant \infty$
p'	the Hölder conjugate of p
$L^p(\mathbb{D})$	L^p-space of equivalent classes of mappings from $\mathbb{D}$ to $\mathbb{R}$
	the usual Lebesgue space
$L^{p'}(\mathbb{D})$	the dual of $L^p(\mathbb{D})$
	$\int_{\mathbb{D}} \|v(x)\|^p dx < \infty$ and $1 \leqslant p < \infty$
$L^\infty(\mathbb{D})$	essentially bounded real-valued measurable functions v on $\mathbb{D}$
$W^{m,p}(\mathbb{D})$	usual Sobolev space
$W_0^{m,p}(\mathbb{D})$	closure of $C_0^\infty(\mathbb{D})$ w.r.t. the norm $\\| \cdot \\|_{W^{m,p}(\mathbb{D})}$
q	real number such that $q \in [1, \infty]$
$W^{-m,p'}(\mathbb{D})$	space dual of the space $W_0^{m,p}(\mathbb{D})$
$L^q(0, T; E)$	L^q-space of equivalent classes $\phi : [0, T] \ni t \to \phi(t) \in E$
	such that $\\|\phi(t)\\|_E \in L^q([0, T])$

$p(\cdot)$	the variable exponent i.e., a measurable function on $\mathbb{D}$ with $p(x) \in [1, \infty]$ for $x \in \mathbb{D}$
$q(\cdot)$	the conjugate exponent function of $p(\cdot)$
ϱ_p	the modular(convex)
ODE	ordinary differential equations
PDE	partial differential equations
SDE	stochastic differential equations
SPDE	stochastic partial differential equations
$L^{p(\cdot)}(\mathbb{D})$	the generalized Lebesgue spaces called also Lebesgue space with variable exponents
$L^{q(\cdot)}(\mathbb{D})$	dual of $L^{p(\cdot)}(\mathbb{D})$
$W^{k,p(\cdot)}(\mathbb{D})$	the generalized Sobolev spaces called sometimes Sobolev spaces with variable exponents
$\mathring{W}^{k,p(\cdot)}(\mathbb{D})$	closure of $C_0^\infty(\mathbb{D})$ in $W^{k,p(x)}(\mathbb{D})$
$W^{-k,q(\cdot)}(\mathbb{D})$	dual space of $\mathring{W}^{k,p(x)}(\mathbb{D})$
$a \wedge b$	minimum of a and b for $a, b \in \mathbb{R}$
$(\cdot, \cdot)$	inner product in $L^2(\mathbb{D})$
$\langle \cdot, \cdot \rangle_{X \times X'}$	duality paring between X and its dual X'
∇	gradient
Δ_{p_k}	p_k-Laplacian, with $p_k \in (1, \infty)$ for $k = 1, 2$
$\Delta_{p(x)}$	$p(x)$-Laplacian

Contents

Title	Existence results for a class of nonlinear stochastic partial differential equations
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Abstract

This thesis consists of two main parts.

The first part concerns the existence of weak probabilistic solutions (called elsewhere martingale solutions) for a stochastic quasilinear parabolic equation of generalized polytropic filtration, characterized by the presence of a nonlinear elliptic part admitting nonstandard growth.
The deterministic version of the equation was first introduced and studied by Samokhin in [178] as a generalized model for polytropic filtration. Our objective is to investigate the corresponding stochastic counterpart in the functional setting of generalized Lebesgue and Sobolev spaces. We establish an existence result of weak probabilistic solutions when the forcing terms do not satisfy Lipschitz conditions and the noise involves cylindrical Wiener processes.

The second part is devoted to the existence and uniqueness results for a class of strongly nonlinear stochastic parabolic partial differential equations.
This part aims to treat an important class of higher-order stochastic quasilinear parabolic equations involving unbounded perturbation of zeroth order. The deterministic case was studied by Brezis and Browder (*Proc. Natl. Acad. Sci. USA*, **76**(1): 38-40, 1979). Our main goal is to provide a detailed study of the corresponding stochastic problem. We establish the existence of a probabilistic weak solution and a unique strong probabilistic solution. The main tools used in this part of the thesis are a regularization through a truncation procedure which enables us to adapt the work of Krylov and Rozosvkii (*Journal of Soviet Mathematics*, **14**: 1233-1277,

1981), combined with analytic and probabilistic compactness results (Prokhorov and Skorokhod Theorems), the theory of pseudomonotone operators, and a Banach space version of Yamada-Watanabe's theorem due to Röckner, Schmuland and Zhang.

The study undertaken in this thesis is in some sense pioneering since both classes of stochastic partial differential equations have not been the object of previous investigation, to the best of our knowledge.

The results obtained are therefore original and constitute in our view significant contribution to the nonlinear theory of stochastic parabolic equations.

Chapter 1

Introduction

Mathematical models of natural and physical phenomena governed by ordinary or partial differential equations (ODEs or PDEs for shorts) have always played a central role in science since the times of Newton and Euler. Among them, parabolic equations have been the object of extensive research in the mathematical community. These equations are of evolution type, time-dependent and often nonlinear; and within their range of important application we find aircraft simulation, non-Newtonian fluids (e.g., electro-rheological fluids), flow through porous media, crud oil extraction, weather prediction, image restoration, etc.

The nonlinearities generally introduce many features such as singularity, blow up, extinction, instability, just to name a few; as opposed to linearity which tends to account for more regularity in the related processes. In view of the prevalence of randomness in nearly every natural phenomena, it became imperative to incorporate that feature in governing partial differential equations. This led to the emerging of stochastic PDEs (SPDEs) with the inception of modern probability at the beginning of the 20^{th} century; a noise involving Brownian or Wiener process is added to the deterministic PDEs. For example, the stochastic Navier-Stokes equations govern the complicated phenomena of turbulence in fluids.

Rigorous mathematical study of SPDEs may be traced back to the pioneering works of Bensoussan and Temam [23], [25] in the early 1970's. These works were followed by the important theses of Viot [206] and Pardoux [155]. Viot deals with important classes of nonlinear SPDEs in the infinite dimensional framework of weak probabilistic solutions using compactness method. Pardoux developed quite general theory of strong solutions for nonlinear equations using monotonicity arguments. This theory is further extended by the fundamental work of Krylov and Rozovskii [123] which concentrates on general studies on concepts of strong probabilistic solutions using compactness and classical monotonicity methods. Further developments are due to Metivier [141], Mikulivicius and Rozovskii in [54], the

works of Krylov and Gyongy [97, 98], Röckner, Schmuland and Zhang [175], Ondreját [152], Da Prato and Zabczyk [64], Peszat and Zabczyk [159], Benssoussan [21], Brzezniak et al. [50, 46, 48, 49, 50, 47], Rozovskiĭ [176], where more references are given therein.
The above mentioned works extended the results of Lions [138], Browder [40], Vishik [207] on various classes of nonlinear parabolic PDEs to their stochastic counterparts. As we mentioned above, the monotonicity and compactness methods were key tools in the progress made.

In this thesis, we initiate the study of some interesting classes of nonlinear parabolic SPDEs driven by Wiener processes and we shall investigate the issues of existence and uniqueness of solutions to these equations. The thesis is mainly divided into two parts. The first part addresses the existence of weak probabilistic solutions to stochastic nonlinear parabolic PDEs with non-Lipschitz coefficients and having a nonstandard growth characterized by the presence of variable power of the gradient of the unknown. And the second part of this thesis deals with the existence and uniqueness of weak and strong probabilistic solutions to strongly nonlinear stochastic parabolic problems, of Brézis-Browder type; the main feature is the presence of a nonlinear unbounded perturbation of zero-th order.

There has been a growing interest in the mathematical study of nonlinear parabolic problems involving the $p(\cdot)$-Laplace operator. These problems arise naturally in the mathematical modeling of several phenomena such as flows of incompressible turbulent fluids or gases in pipes, generalized non-Newtonian fluids, e.g., electrorheological fluids, filtration in porous media, etc... We refer for instance to [114] for the physical processes of filtration in porous media and turbulent gas in pipelines, [205] for glaciology, [70] for related model of turbulent flows and so on. Electrorheological fluids have interesting mechanical properties which can lead to several technical applications such as crud oil, breaks, actuators, clutches, shock absorbers just to name a few. More details regarding their characterization and corresponding problems in their mathematical modeling can be found in [177, 164, 72]. Further development of the results of [177] can be found in Lars Diening's thesis [72].

The influential work of Kovačik and Rákosník [120] contains key properties of the generalized Lebesgue space $L^{p(x)}(\mathbb{D})$ and the corresponding Sobolev space $W^{k,p(x)}(\mathbb{D})$; for latest developments in this direction we refer to the recent monograph [71] by Lars Diening and his coworkers; also earlier works by Edmunds and Rákosník [81, 82] and Fan et al. [87, 88].

The weak solvability in the deterministic case was first studied by Samokhin, in [178]. We refer to [2]-[4], [70, 13, 83, 87, 30] for more details about other important results. The regularity and higher integrability theory have been considered in [177, 137, 56, 9, 11, 30, 158], [101]-[104], [196, 210, 214]. For a localization property of weak solutions for parabolic initial-boundary value problem with nonstandard growth conditions, we refer to [96].

The stochastic case has been considered only recently [7, 19]. We note the work in [19] dealing with one dimensional Wiener process and the dissertation [7] which was devoted to the case of d-dimensional multiplicative noises. It seems to us that the work [8] predated [19].
Despite these contributions, however, the investigation of existence for nonlinear parabolic SPDEs perturbed by cylindrical Wiener processes in the functional setting of generalized Lebesgue and Sobolev spaces has not yet been undertaken in the literature. This is the object of part of the present thesis.

We establish the existence of probabilistic weak solution for a certain class of models that generalize the problems of polytropic elastic filtration perturbed by cylindrical Wiener process in a framework of probabilistic evolution spaces involving the generalized Lebesgue and Sobolev spaces. This study is motivated by the stochastic version of practical engineering problems such as the electrorheological fluids; as indicated earlier. Besides the non standard growth difficulty present in the equation, the nonlinear external forces (depending on the solution), do not satisfy any Lipschitz conditions with respect to the solution. However we impose on the intensity of the noise a linear growth condition. This potentially avoids the occurrence of blow-up (explosion).

As aforementioned, in the second part of the thesis, we investigate the existence and uniqueness of an important class of SPDEs which has so far not been studied by experts in the field; it is the stochastic counterpart of strongly nonlinear parabolic equations which originated from the works of Brezis-Browder [34, 33]. The main feature of these equations is characterized by the presence of nonlinear terms which are unbounded perturbations of zero-th order, making it impossible to treat the resulting problem by directly using methods considered in works cited in the previous paragraphs (for instance [21, 123, 140, 155, 206]). Brezis and Browder introduced a suitable regularization through appropriate truncations and thanks to compactness arguments, they derived the needed existence result. Further advances in the study of these equations are due to Landes and Mustonen; see [129], [130], [131] and [133].

The main results of this part are the construction of a probabilistic weak solution under rather general conditions on the nonlinear intensity of the noise followed by the existence of a probabilistic strong solution for the present problem under Lipschitz conditions on the forcing term. Since a direct approach through Galerkin approximation is hopeless and Itô's formula is prohibited in that case, we therefore introduce a regularization through truncations which reduces the problem to a sequence of problems in the sequel which fits into the framework of quite general SPDEs studied by Krylov and Rozovskii [123]. We establish uniform a priori estimates which enable us to implement analytic and probabilistic compactness results as in the previous part. This leads to the weak probabilistic solution. We close this part by establishing a strong probabilistic solution under Lipschitz condition on the nonlinear intensity of the force. A key tool turns out to be a Banach space version of Yamada-Watanabe's celebrated theorem.

To the best of our knowledge, the results obtained in the present thesis are novel. Beside the novelty of the results, several difficulties which are due to the stochastic nature of the problem and therefore absent in the deterministic case [178, 33, 34], had to be overcome.

1.1 Our Main Results

We now state the key results obtained in the thesis.

1.1.1 Main Results of Part 1

The objective of the first part of the thesis is to study a class of stochastic quasilinear parabolic initial boundary value problem with nonstandard growth subjected to cylindrical Wiener perturbation in the functional setting of generalized Sobolev spaces. Namely, we consider in $(0,T)\times\mathbb{D}$ (where $T<\infty$ and $\mathbb{D}$ is an open bounded subset of $\mathbb{R}^n$ with C^2 boundary $\partial\mathbb{D}$) the stochastic problem

$$du + A(t,u)dt = f(t,u)dt + G(t,u)dW(t) \text{ in } (0,T)\times\mathbb{D}, \tag{1.1}$$

$$u(t,x) = 0, \text{ on } (0,T)\times\partial\mathbb{D}, \tag{1.2}$$

$$u(0,x) = u_0(x), \text{ in } \mathbb{D}, \tag{1.3}$$

where the function $u=u(t,x)$ is unknown alongside with W, the cylindrical Wiener process. The nonlinear terms $f(t,u)$ and $G(t,x,u)$ are known functions, $u_0\in L^2(\mathbb{D})$ and the leading operator A is given by

$$A(t,u) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(\cdot)-2}\frac{\partial u}{\partial x_i}\right).$$

We emphasis that the problem we are considering in this thesis is the first stochastic version of a generalized model for polytropic filtration with a multiplicative noise of cylindrical type. The main result is the existence of weak probabilistic solution under continuity and linear growth of the nonlinear external forcing terms.

The formulation of our results in this subsection involves generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}(\mathbb{D})$, $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$, $\left(W^{1,p(\cdot)}(\mathbb{D})\right)'$, $\mathring{V}(Q)$ which will be defined later. We also postpone other relevant semantics.

The nonlinear external forces $f(t,u)$ and $G(t,x,u)$ are jointly continuous from $(0,T)\times L^2(\mathbb{D}) \longrightarrow \left(\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right)'$ and $(0,T)\times L^2(\mathbb{D}) \longrightarrow \mathcal{L}_2\left(\mathbb{K}, L^2(\mathbb{D})\right)$ respectively and satisfy

$$\|f(t,u)\|_{\left(\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right)'} \leqslant C\left(1+\|u(t)\|_{L^2(\mathbb{D})}\right); \tag{1.4}$$

$$\|G(t,x,u)\|_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))} \leqslant C\left(1+\|u(t)\|_{L^2(\mathbb{D})}\right). \tag{1.5}$$

Here, $\mathcal{L}_2(\mathbb{K},\mathbb{U})$ stands for the space of Hilbert Schmidt operators defined from $\mathbb{K}$ to $\mathbb{U}$, $\left(\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right)'$, the dual of the space $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ and C is a generic constant.

Moreover, the variable exponent function $p(\cdot)$ satisfies

$$2 \leqslant r \leqslant p(\cdot) \leqslant s < \infty. \tag{1.6}$$

Our main result in the first part of the thesis is

Theorem 1. *Under the conditions (1.4), (1.5), (1.6) and $u_0 \in L^2(\mathbb{D})$, the problem (1.1)-(1.3) has a probabilistic weak solution which is a tuple $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leqslant t\leqslant T},\mathbb{P},W,u)$, where*

(1) $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leqslant t\leqslant T},\mathbb{P})$ is a stochastic basis satisfying the usual conditions,

(2) W is an $\mathcal{F}_t$-adapted cylindrical Wiener process evolving on $L^2(\mathbb{D})$,

(3) the process $u(t,\omega)$ is progressively measurable and

$$u \in L^q\left(\Omega,\mathcal{F},\mathbb{P},L^r(0,T;\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right) \cap L^r\left(\Omega,\mathcal{F},\mathbb{P};\mathring{V}(Q_T)\right),\ \forall q\in[2,\infty),$$

(4) $u \in L^q\left(\Omega,\mathcal{F},\mathbb{P},C_w([0,T];L^2(\mathbb{D}))\right)$ and for all $t\in[0,T]$, $\mathbb{P}$-a.s.

$$(u(t),v)+\int_0^t\langle A(s,u),v\rangle ds = (u_0,v)+\int_0^t\langle f(s,u),v\rangle ds + \left(\int_0^t G(s,u(s))\,dW(s),v\right), \tag{1.7}$$

for any $v\in\mathring{W}^{1,p(\cdot)}(\mathbb{D})$; $C_w([0,T];L^2(\mathbb{D}))$ stands for functions which are weakly continuous on $[0,T]$ with values in $L^2(\mathbb{D})$.

Without loss of generality, we suppose given a probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$ with $\bar{W}$, a cylindrical Wiener process prescribed on it and $(\bar{\mathcal{F}}_t)_{t\in[0,T]}$, the $\bar{\mathbb{P}}$-augmentation of the natural filtration generated by $\bar{W}$. Using Galerkin procedure, we construct approximating solutions $(u_m)_{m\in\mathbb{N}}$ and obtain among others, uniform estimates of the forms

$$\bar{\mathbb{E}} \sup_{0\leqslant t\leqslant T} \|u_m(t)\|^q_{L^2(\mathbb{D})} \leqslant C, \ \forall q \in [2,\infty), \tag{1.8}$$

$$\bar{\mathbb{E}}\|u_m\|^r_{\mathring{V}(Q)} \leqslant C, \tag{1.9}$$

$$\bar{\mathbb{E}}\left(\int_0^T \|u_m(t)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}\right)^{\frac{r}{2}} \leqslant C, \tag{1.10}$$

$$\bar{\mathbb{E}} \sup_{0\leqslant|\theta|\leqslant\delta} \int_0^T \|u_m(t+\theta) - u_m(t)\|^2_{W^{-1,q(\cdot)}(\mathbb{D})} dt \leqslant C\delta, \ \forall \delta \in [0,1), \tag{1.11}$$

where $\bar{\mathbb{E}}$ is the mathematical expectation with respect to the probability measure $\bar{\mathbb{P}}$ and $W^{-1,q(\cdot)}(\mathbb{D})$, the dual of $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$. We construct a family of probability measures $\{\Pi_m : m \in \mathbb{N}\}$ on $\mathcal{S} = C([0,T];\mathbb{K})\times L^2(0,T;L^2(\mathbb{D}))\cap C([0,T];W^{-1,q(\cdot)}(\mathbb{D}))$ by $\Pi_m(A) = \bar{\mathbb{P}}\left(\bar{\phi}_m^{-1}(A)\right)$, $\forall A \in \mathcal{B}(\mathcal{S})$, where $\bar{\phi}_m : \bar{\omega} \mapsto (\bar{W}(\cdot,\bar{\omega}), u_m(\cdot,\bar{\omega}))$. Here, $\mathbb{K} = \mathcal{Q}^{\frac{1}{2}}(\mathbb{U})$, where $\mathcal{Q} : \mathbb{U}^* \longrightarrow \mathbb{U}$ is a symmetric nonnegative operator, $\mathbb{U}$ is a separable Banach space and $\mathbb{U}^*$ its dual. Following ideas from [166] and invoking results from [192, Theorem 1, p.80] we show that $(\Pi_{1,m} = \mathcal{L}(u_m))$, the family of laws of the sequence $(u_m)_{m\geqslant 1}$ is tight on $\mathcal{S}_1 = L^2(0,T;L^2(\mathbb{D}))\cap C([0,T];W^{-1,q(\cdot)}(\mathbb{D}))$. Next, thanks to [157, Chap. II, Theorem 3.2] we also show that $(\Pi_{2,m} = \mathcal{L}(\bar{W}))_{m\in\mathbb{N}}$, the law of the cylindrical Wiener process $\bar{W}$ is tight on $\mathcal{S}_2 = C([0,T];\mathbb{K})$. Once more, arguing as in [166], we prove that $\{\Pi_m : m \in \mathbb{N}\}$ is tight in $(\mathcal{S},\mathcal{B}(\mathcal{S}))$. Therefore, applying Prokhorov's compactness procedure, we assert that there exists a subsequence Π_{m_ν}, that weakly converges to a probability measure Π on $\mathcal{S}$. By Skorohod's compactness result [193], we can find a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $\mathcal{S}$-valued random variables (W_{m_ν}, u_{m_ν}) and (W,u) defined on $(\Omega,\mathcal{F},\mathbb{P})$ such that $\mathcal{L}(W_{m_\nu}, u_{m_\nu})$ (resp. $\mathcal{L}(W,u)$), the law of (W_{m_ν}, u_{m_ν}) is $\Pi_{m_\nu} = \mathcal{L}(\bar{W}, u_m)$ (resp. the law of (W,u) is Π) and

$$\begin{aligned}
&W_{m_\nu} \longrightarrow W \text{ in } C([0,T];\mathbb{K}) \ \mathbb{P}-\text{a.s.},\\
&u_{m_\nu} \longrightarrow u \text{ in } L^2(0,T;L^2(\mathbb{D})) \ \mathbb{P}-\text{a.s.},\\
&u_{m_\nu} \longrightarrow u \text{ in } C([0,T];W^{-1,q(\cdot)}(\mathbb{D})) \ \mathbb{P}-\text{a.s.}.
\end{aligned}$$

Next, we prove that the stochastic process W is $\mathbb{K}$-valued $\mathcal{Q}$-Wiener process by constructing the filtration $\mathcal{F}_t = \sigma(N \cup \sigma((W(\tau),u(\tau));\tau \in [0,t]))$, where N is a null set of $\mathcal{F}$ and checking that the finite dimensional distributions of W are Gaussian. We proceed by showing that the processes $(u_{m_\nu})_{m\in\mathbb{N}}$ are solutions of the finite dimensional SDE's

$$u_{m_\nu}(t) + \int_0^t P_{m_\nu}A(s,u_{m_\nu})ds = P_{m_\nu}(u_0) + \int_0^t P_{m_\nu}f(s,u_{m_\nu})ds + \int_0^t P_{m_\nu}G(s,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu},$$

where the injection $\mathcal{J} : \mathbb{U} \longrightarrow \mathbb{K}$ is a Hilbert-Schmidt operator and $\mathcal{Q} = \mathcal{J}\mathcal{J}^*$.

The compactness procedures of Prokhorov and Skorokhod enable us to prove that the sequence u_m (modulo an extraction of a subsequence denoted in the same way) converges in appropriate topologies to the process u satisfying the same uniform estimates. Briefly, it follows from the cited estimates that we can extract a subsequence $(u_{m_\nu})_{\nu\in\mathbb{N}} \subseteq (u_m)_{m\in\mathbb{N}}$ such that

$$\begin{aligned} u_{m_\nu} &\longrightarrow u \ \text{ weakly in } L^q(\Omega,\mathcal{F},\mathbb{P};L^a(0,T;L^2(\mathbb{D}))) \\ &\qquad\cap L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathring{W}^{1,p(\cdot)}(\mathbb{D}))), \forall a\in[2,\infty); \end{aligned} \tag{1.12}$$

$$u_{m_\nu} \longrightarrow u \text{ weakly in } L^r\left(\Omega,\mathcal{F},\mathbb{P};\mathring{V}(Q)\right) \text{ for any } 2\leqslant r\leqslant p(\cdot)\leqslant s<\infty; \tag{1.13}$$

$$A(t,u_{m_\nu}(\omega)) \longrightarrow \chi(\omega) \ \text{ weakly in } \left(\mathring{V}(Q)\right)' \text{ for a.e. } \omega\in\Omega \tag{1.14}$$

$$u_{m_\nu}(T) \longrightarrow \daleth \ \text{ weakly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})). \tag{1.15}$$

Furthermore,

$$u_{m_\nu}(\omega) \longrightarrow u(\omega) \ \text{ weakly } -* \text{ in } L^\infty(0,T;L^2(\mathbb{D}))\,\text{a.e. } \omega. \tag{1.16}$$

Moreover, the process u satisfies the estimates

$$\begin{aligned} &\mathbb{E}\|u\|^q_{L^a(0,T;L^2(\mathbb{D}))} \leqslant C,\ q\in[2,\infty),\ a\in[2,\infty), \\ &\left(\mathbb{E}\|u\|^r_{\mathring{V}(Q)}\right)^{1/r} \leqslant C,\ \text{ for any } 2\leqslant r\leqslant p(\cdot)\leqslant s<\infty, \\ &\mathbb{E}\left(\int_0^T \|u(t)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}dt\right)^{\frac{r}{2}} \leqslant C, \\ &\mathbb{E}\sup_{0\leqslant|\theta|\leqslant\delta<1}\int_0^T \|u(t+\theta)-u(t)\|^2_{W^{-1,q(\cdot)}(\mathbb{D})}dt \leqslant C\delta. \end{aligned}$$

We observe by Vitali's theorem that

$$u_{m_\nu} \longrightarrow u \ \text{ strongly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;L^2(\mathbb{D}))). \tag{1.17}$$

Hence, there exists a subsequence still denoted (u_{m_ν}) such that

$$u_{m_\nu} \longrightarrow u \ \text{ for almost all } (t,\omega) \text{ w.r.t. } d\mathbb{P}\times dt. \tag{1.18}$$

We combine the assumption on f, (1.8), (1.18) with Vitali's theorem to obtain

$$f(\cdot,u_{m_\nu}(\cdot)) \longrightarrow f(\ \cdot,u(\cdot)) \text{ in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2\left(0,T;W^{-1,q(\cdot)}(\mathbb{D})\right)\right).$$

Finally, following essential ideas of [22, 67, 168, 183, 184, 185, 186], we prove that

$$\int_0^T P_{m_\nu}G(t,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu} \longrightarrow \int_0^T G(t,x,u)dW \ \text{ weakly in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right), \tag{1.19}$$

where $\mathcal{J} : L^2(\mathbb{D}) \longrightarrow \mathbb{K}$ is the injection (Hilbert-Schmidt) with $\mathbb{K} = \mathcal{Q}^{\frac{1}{2}}(L^2(\mathbb{D}))$ and $\mathcal{Q} = \mathcal{J}\mathcal{J}^*$.
And lastly, we end by adapting monotonicity and hemicontinuity arguments used in [138, 145, 207] to prove that $\chi(t) = A(t,u)$. Similar ideas have already been used in [7, 8, 21, 155]. We prove that u is a solution of the considered problem in the sense of Theorem 1.

Our results extend some related results developed previously by many authors in [2], [4], [13], [23], [25], [71], [81, 82], [83], [87], [88], [96], [120], [155], [164], [177], [178], [183], [214] and many more in the references therein. The results of this part in the finite dimensional setting of the noise have been the object of the publication [8]. We also addressed there the issue of existence of strong solution.

1.1.2 Main Results of Part 2

The main purpose of the second part of the thesis is to investigate weak and strong solvability of the stochastic counterpart of strongly nonlinear parabolic equations with unbounded perturbation of zero-th order in the sense of Brézis-Browder [34, 33].

Let $\mathbb{D} \subset \mathbb{R}^m$, $m \geqslant 1$ be an open bounded subset with sufficiently regular boundary $\partial\mathbb{D}$. We consider the stochastic initial boundary value problem

$$(P)\begin{cases} du + [A_t(u) + g(t,x,u)]\,dt &= f(t)dt + G(t,x,u)dW(t) \quad \text{in } Q_T = (0,T)\times\mathbb{D} \\ u(x,0) &= 0 \quad \text{in } \mathbb{D} \\ \dfrac{\partial^j u}{\partial N^j} &= 0 \quad \text{on } (0,T)\times\partial\mathbb{D}, \end{cases}$$

where the processes u, W (a m-dimensional Wiener process which could have been infinite-dimensional of cylindrical type) and the probability space on which they are defined are unknown, the linear term f, the nonlinear term G and the nonlinear perturbation g are given; $\dfrac{\partial^j u}{\partial N^j} = 0$ is the Dirichlet boundary condition where $\dfrac{\partial^j u}{\partial N^j}$ is the j^{th} normal derivative of u with $0 \leqslant j \leqslant m-1$; A_t is a nonlinear elliptic operator of order $2m$ in the generalized divergence form, that is,

$$A_t(u) = \sum_{|\beta|\leqslant m} (-1)^{|\beta|} D^\beta A_\beta(x,t,u,Du,\ldots,D^m u),$$

where the coefficient functions A_β satisfies the Carathéodory conditions.
Note that $W = (W_1,\ldots,W_m)$, where each W_i, $i = 1,\ldots,m$ is a standard one dimensional Wiener process and $dW = (dW_1,\ldots,dW_m)$ stands for the m-dimensional white noise.

1.1.2.1 Existence of weak probabilistic solution

We formulate the structure conditions on A_β. Let $2 \leq p < \infty$ with $p' = \frac{p}{p-1}$.

(i) There exists $c_0 > 0$, $h_0 \in L^{p'}(Q)$ such that

$$|A_\beta(t,x,\xi)| \leqslant c_0\{|\xi|^{p-1} + h_0(t,x)\}, \ \forall (t,x,\xi) \in [0,T] \times \mathbb{D} \times \mathbb{R}^n.$$

(ii) For all $(t,x) \in Q = [0,T] \times \mathbb{D}$, all lower-order jets $\eta \in R^{N_1}$, and $\gamma \neq \zeta$ in $\mathbb{R}^{N_2}$, we have

$$\sum_{|\beta|=m} \left[A_\beta(t,x,\eta,\gamma) - A_\beta(t,x,\eta,\zeta)\right](\gamma_\beta - \zeta_\beta) > 0.$$

(iii) There exists $c_1 > 0$, $h_1 \in L^1(Q)$ such that for all $(t,x) \in Q$ and all $\xi \in \mathbb{R}^n$, we have

$$\sum_{|\beta| \leqslant m} A_\beta(t,x,\xi)\xi_\beta \geq c_1|\xi|^p - h_1(t,x).$$

(iv) $g(t,x,u)$ is measurable in (t,x), and continuous in u. There exists a continuous nondecreasing function $h : \mathbb{R} \longrightarrow \mathbb{R}$ with $h(0) = 0$ such that for all $(t,x) \in Q$, $r \in \mathbb{R}$, and a fixed constant C, we have
$rg(t,x,r) \geq 0$; $|g(t,x,r)| \leqslant |h(r)|$.

(v) For the nonlinear intensity of the noise G, we assume that there exists a positive constant C such that
$G(\cdot,u) : (0,T) \longrightarrow (L^2(\mathbb{D}))^m$, measurable, a.e. t, $G(t,\cdot) : L^2(\mathbb{D}) \longrightarrow (L^2(\mathbb{D}))^m$, continuous, and G satisfies

$$\|G(t,x,u)\|_{(L^2(\mathbb{D}))^m} \leqslant C(1 + \|u(t)\|_{L^2(\mathbb{D})}).$$

(vi) We assume that $f \in L^{p'}(0,T; L^{p'}(\mathbb{D}))$.

Our main result in this part is

Theorem 2. *Under the conditions* (i)-(vi)*,* (P) *has a probabilistic weak solution which is understood as a system* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P}, W, u)$*, where*

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $\mathcal{F}_t$ *is a filtration on it,*

(2) W *is a d-dimensional* $\mathcal{F}_t$*- standard Wiener process,*

(3) $(t,\omega) \to u(t,\omega)$ *is progressively measurable,*

(4) $u \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0,T; L^2(\mathbb{D}))) \cap L^p(\Omega, \mathcal{F}, \mathbb{P}, L^p(0,T; W_0^{m,p}(\mathbb{D})))$,

(5) for all $t \in [0,T]$, $u(t)$ satisfies the integral identity

$$\begin{aligned}(u(t), v) &+ \int_0^t \langle A_s(u), v\rangle ds + \int_0^t \int_D g(s,x,u)v dx\, ds \\ &= \int_0^t \langle f(s), v\rangle ds + \left(\int_0^t G(s, u(s))\, dW(s), v\right), \forall v \in W_0^{m,p}(\mathbb{D}),\ \mathbb{P} - a.s.\end{aligned} \tag{1.20}$$

The initial step of the proof of this theorem is a suitable regularization of problem (P) on intermediary probability space $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$ with a prescribed Wiener process $\bar{W}(t)$. Letting g_k be truncations of the function g at heights $k \in \mathbb{N}$, our regularized problem reads as follows:

$$(P_k) \begin{cases} du + [A_t(u) + g_k(t,x,u)]\, dt = f(t)dt + G(t,x,u)d\bar{W}(t) & \text{in } Q_T \\ u(x,0) = 0, & \text{in } \mathbb{D} \\ \dfrac{\partial^j u}{\partial N^j} = 0, & \text{on } \partial(0,T) \times \mathbb{D}. \end{cases}$$

By the properties of truncations, we have that $g_k \in L^\infty(Q_T)$, $\mathbb{P}$-a.s. Moreover, from the definition of g_k and the assumption $g(t,x,r)r \geq 0$, one can obviously check that $g_k(t,x,r)r \geq 0$. Now, problem (P_k) is more regular than problem (P) in the sense that the L^∞-norm of the nonlinear terms g_k is under control; this is in sharp contrast with the unboundedness of g.

(H_1) $(\mathcal{A}, G)$-Coercitivity condition: there exist $c_3 > 0$, $c_4 \in \mathbb{R}$ and there exists an $(\bar{\mathcal{F}}_t)$-adapted process $h_1 \in L^1(Q \times \bar{\Omega})$ such that for all $(t,x) \in Q$, $v \in W_0^{m,p}(\mathbb{D})$

$$2\int_{\mathbb{D}} \mathcal{A}(t,v)v dx + ||G(t,v)||^2_{(L^2(\mathbb{D}))^m} - h_1(t) + c_3||v(t)||^p_{W_0^{m,p}(\mathbb{D})} \leqslant c_4||v(t)||^2_{L^2(\mathbb{D})} \text{ on } \bar{\Omega}.$$

Here $(L^2(\mathbb{D}))^m$ denotes the m copies of the space $L^2(\mathbb{D})$.

(H_2) Hemicontinuity: for all $t \in [0,T]$, $\bar{\omega} \in \bar{\Omega}$ and $u, v, w \in W_0^{m,p}(\mathbb{D})$ the map $\lambda \mapsto \langle \mathcal{A}(t, u + \lambda v), w\rangle$ is continuous on $\mathbb{R}$. Here $W^{-m,p'}(\mathbb{D})$ is the dual of $W_0^{m,p}(\mathbb{D})$ and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W_0^{m,p}(\mathbb{D})$ and $W^{-m,p'}(\mathbb{D})$.

Here $L^{p'}(\mathbb{D})$ denotes the dual of $L^p(\mathbb{D})$ and we need to make notation clear, instead of using the map $\omega \mapsto G(t,x,v,\bar{\omega})$ we write $G(t,v)$. This also applies to other functions used throughout.

We know that any strictly monotone operator is monotone hence pseudomonotone.
Under assumptions (i)-(vi), (H_1), (H_2), problem (P_k) satisfies the conditions of existence from [123]. Thus, for each fixed $k \in \mathbb{N}$, the truncated problem (P_k) has a strong solution u_k

in the sense of Krylov and Rozovskii [123, pp: 1252-1253].

Our next task is to derive some key a priori estimates for the sequence (u_k) satisfying problem (P_k). These estimates are summarized in

Lemma 1. *Under the conditions of Theorem 2, for each $k \in \mathbb{N}$, u_k satisfies*

$$\bar{\mathbb{E}} \sup_{0\leqslant t\leqslant T} \|u_k(t)\|^2_{L^2(\mathbb{D})} + \bar{\mathbb{E}} \int_0^T \|u_k(t)\|^p_{W_0^{m,p}(\mathbb{D})} dt \leqslant C. \tag{1.21}$$

Furthermore, for any $q \geq 2$

$$\bar{\mathbb{E}} \sup_{t\in[0,T]} \|u_k(t)\|^q_{L^2(\mathbb{D})} \leqslant C, \tag{1.22}$$

and for sufficiently small $\delta > 0$;

$$\bar{\mathbb{E}} \sup_{|\theta|\leqslant\delta} \int_0^T \|u_k(t+\theta) - u_k(t)\|^{p'}_{W^{-m,p'}(\mathbb{D})} dt \leqslant C\delta^{1/(p-1)}; \tag{1.23}$$

the constant C is independent of k.

This lemma is a crucial stepping stone leading to probabilistic compactness results. For that we introduce the product space $S = C\left([0,T];\mathbb{R}^m\right) \times L^2(0,T;L^2(\mathbb{D}))$ and $\mathcal{B}(S)$ the σ-algebra of its Borel sets. For each k, we construct the probability measure Λ_k on S, the push-forward of $\bar{\mathbb{P}}$ by the mapping $\varphi_k : \bar{\omega} \mapsto \left(\bar{W}(.,\bar{\omega}), u_k(.,\bar{\omega})\right)$ defined on $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$ and taking values in $(S, \mathcal{B}(S))$; that is $\Lambda_k(A) = \bar{\mathbb{P}}(\varphi_k^{-1}(A))$ for all $A \in \mathcal{B}(S)$. We prove that $\{\Lambda_k\}_{k=1}^\infty$ is tight in $(S, \mathcal{B}(S))$. Therefore Prokhorov's compactness theorem (see [161]) comes in force and enables us claim that $\{\Lambda_k\}_{k=1}^\infty$ weakly converges (in the sense of measures) to a probability measure Λ on S; we denote by $\{\Lambda_{k_i}\}_{i=1}^\infty$ a corresponding subsequence converging to Λ. Next another equally powerful compactness result due to Skorokhod (see [194]) implies that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and pairs of random variables (W_{k_i}, u_{k_i}) and (W, u) on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in S such that the probability law of (W_{k_i}, u_{k_i})(resp. (W,u)) is Λ_{k_i}(resp. Λ) and

$$(W_{k_i}(.,\omega), u_{k_i}(.,\omega)) \longrightarrow (W(.,\omega), u(.,\omega)) \text{ in } S, \text{ as } i \longrightarrow \infty, \ \mathbb{P}-\text{a.s.},$$

Next, we introduce the filtration $\mathcal{F}_t$ by setting $\mathcal{F}_t = \sigma\{(W(s), u(s)) : 0 \leq s \leq t\}$. It turns out according to similar reasoning used in [22], [168] and [185] that W is a d-dimensional $\mathcal{F}_t$-standard Wiener process. Following these references, one can also prove that

$$u_{k_i}(t) + \int_0^t \left[A_s(u_{k_i}(s)) + g_{k_i}(s, u_{k_i})\right] ds = \int_0^t f(s)ds + \int_0^t G(s, u_{k_i}) dW_{k_i}(s), \ \mathbb{P}-\text{a.s.}, \tag{1.24}$$

as an equality between random variables with values in $W^{-m,p'}(\mathbb{D})$.

Owing to (1.24), (1.21), (1.22) can be applied to u_{k_i}. Thus, there exists a new subsequence of (u_{k_i}) which we still denote by the same symbol (u_{k_i}) such that

$$u_{k_i} \rightharpoonup u \text{ in } L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(0,T; W_0^{m,p}(\mathbb{D}))) \cap L^q\left(\Omega, \mathcal{F}, \mathbb{P}; L^r\left(0,T; L^2(\mathbb{D})\right)\right), \tag{1.25}$$

with $p,\ q$ as above and any $r \in (1, \infty)$.
Thus, there exists a random function $\varpi \in L^{p'}\left(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}\left(0,T; W^{-m,p'}(\mathbb{D})\right)\right)$ such that up to extraction of a subsequence

$$A_t(u_{k_i}(t)) \rightharpoonup \varpi(t) \text{ in } L^{p'}\left(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}\left(0,T; W^{-m,p'}(\mathbb{D})\right)\right). \tag{1.26}$$

Furthermore u satisfies the estimates (1.21) and (1.22). Thanks to the higher integrability (1.22) and Vitali's theorem, we obtain

$$u_{k_i} \longrightarrow u \text{ in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T; L^2(\mathbb{D}))\right) \text{ and almost everywhere in } Q_T \times \Omega. \tag{1.27}$$

Thus, there exists a new subsequence still denoted (u_{k_i}) for simplicity of notation such that for almost every (t, ω) we have

$$u_{k_i} \longrightarrow u \text{ in } L^2(\mathbb{D}) \text{ (with respect to the measure } d\mathbb{P} \times dt). \tag{1.28}$$

Thanks, to De La Vallee Poussin principles, we show that the sequence $\{g_{k_i}(t, x, u_{k_i})\}_{i \in \mathbb{N}}$ is uniformly integrable in $L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_t))$. Then, using the Dunford-Pettis theorem (see next chapter for statement), the properties of truncation T_{k_i}, Vitali's theorem, the convergence (1.28), we deduce that

$$g_{k_i}(u_{k_i}) \longrightarrow g(u) \text{ in } L^1\left(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_T)\right); \tag{1.29}$$

and $g(u) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_T))$.

Following [22, 67, 168, 183, 184, 185, 186], we prove that

$$\int_0^t G(s, u_{k_i}) dW_{k_i}(s) \rightharpoonup \int_0^t G(s,u) dW(s) \text{ in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})\right). \tag{1.30}$$

The remaining assertion $\varpi = A_t(u)$ is established by carefully adapting the approach used in [39, 34, 33] and based on pseudo monotonicity arguments. Then u satisfies the integral identity (1.20) and Theorem 2 follows.

1.1.2.2 Strong solution

We establish the pathwise uniqueness of solutions of the problem (P) and use Yamada-Watanabe's classical result to derive the existence of a strong probabilistic solution.

To do so, we imposed additional condition on $g(t, x, r)$ that we borrowed from [34]. We require that (iv)' The function $g(t,x,r)$ is non-decreasing in r and in addition, the sign condition on g is preserved, i.e. $rg(t,x,r) \geq 0$.

Next, we introduce a function Γ by setting $\Gamma(t,x,r) = \int_0^r g(t,x,s)ds$. It is obvious that Γ is continuous and convex in r and non-negative for all (t,r). By construction we have that $\Gamma(t,x,0) = 0$ and $\Gamma'(t,x,r) = g(t,x,r)$.

We also need an extra condition on G; the Lipschitz condition. Namely, there exists a constant $L > 0$ such that

$$\|G(t,u_1) - G(t,u_2)\|_{(L^2(\mathbb{D}))^m} \leq L\|u_1(t) - u_2(t)\|_{L^2(\mathbb{D})}. \tag{1.31}$$

Our pathwise uniqueness result is

Theorem 3. *Under conditions of Theorem 2 and (H_1), (H_2), (g-nonlinINTRO) and (iv)', problem (P) is pathwise unique.*

We refer to [110, 152, 175, 211] for definition of pathwise uniqueness. The function Γ defined above plays a central role in proving Theorem 3.

Sketch of the proof We denote the weak solution by the couple (W,u) in order to simplify the notation. Let (W,u_1) and (W,u_2) be two weak solutions with u_1 and u_2 in the space $L^p(\Omega,\mathcal{F},\mathbb{P};L^p(0,T;W_0^{m,p}(\mathbb{D}))) \cap L^2(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;L^2(\mathbb{D})))$, $p \geq 2$. For any $v \in L^p(\Omega,\mathcal{F},\mathbb{P};L^p(0,T;W_0^{m,p}(\mathbb{D})) \cap L^\infty(Q_T))$, we substitute u by u_1 and u_2 in

$$d(u(t) - v(t)) = -\left[A_t(u(t)) + g(t,x,u) - f(t) + \frac{\partial v}{\partial t}\right]dt + G(t,x,u)dW. \tag{1.32}$$

and multiply the resulting relations by $u_1 - v$ and $u_2 - v$, respectively. Arguing similarly as in [33], we express v as the mean of u_1 and u_2 i.e., $v = \frac{1}{2}(u_1 + u_2)$. Then thanks to Itô's formula applied to $\|u_1(t) - u_2(t)\|^2_{L^2(\mathbb{D})}$, we get

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\|u_1(t) - u_2(t)\|^2_{L^2(\mathbb{D})} + 2\mathbb{E}\int_0^t \langle u_1 - u_2, A_s(u_1) - A_s(u_2)\rangle ds \\
&\leq 4\mathbb{E}\int_{Q_t}(v - u_1)g(s,x,u_1)dxds + 4\mathbb{E}\int_{Q_t}(v - u_2)g(s,x,u_2)dxds+ \\
&+ 2\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{D}}(u_1 - u_2)[G(s,u_1) - G(s,u_2)]dW_s dx\right| + C\mathbb{E}\int_0^t \|u_1(s) - u_2(s)\|^2_{L^2(\mathbb{D})}ds.
\end{aligned} \tag{1.33}$$

Since by definition, $\Gamma'(t,x,u) = g(t,x,u)$, we use (iv) to obtain the following properties which are straightforward consequences of the convexity of the function Γ:

$$g(t,x,u_j)(v - u_j) \leqslant \Gamma(t,x,v) - \Gamma(t,x,u_j),\ j = 1,2, \tag{1.34}$$

$$\Gamma\left(t,x,\frac{1}{2}(u_1 + u_2)\right) \leq \frac{1}{2}\left\{\Gamma(t,x,u_1) + \Gamma(t,x,u_2)\right\}. \tag{1.35}$$

It follows from (1.31), (1.33), Burkholder-Davis-Gundy, Young's inequalities, (H_1), (1.34), (1.35), the Lipschitz condition on G and condition (ii) that

$$\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}\le C\mathbb{E}\int_0^T\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}dt.$$

Thanks to Gronwall's Lemma, we conclude that for any $t\in[0,T]$, we have $u_1(t)=u_2(t)$, $\mathbb{P}$-a.s.. □

Our last main result is

Theorem 4. *Let the assumptions in Theorem 3,* (iv)' *and (1.31) be satisfied. Then problem* (P) *admits a unique strong solution* u*, in the sense that the system* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]}\mathbb{P},W,u)$ *is a weak solution of* (P) *and* u *is adapted to the filtration generated by the Wiener process* W*.*

The proof of this result follows from the celebrated theorem of Yamada-Watanabe originally proved in [211] (see also [110]), in the finite dimensional case. According to Yamada-Watanabe's theorem, weak probabilistic solution and pathwise uniqueness give rise to the existence of unique probabilistic strong solution. The result has since been established in the infinite-dimensional setting by many authors; we refer to Rockner, Schmuland and Zhang [175] and Ondreját [152] for further details in this direction. It is the version obtained in [175] that enables us to conclude the proof of the theorem. This theorem has been addressed in the infinite dimensional framework of mild solution, by Ondreját [152].

Plan of the thesis

This thesis consists of four chapters.

In Chapter 2, we introduce preliminary materials that will be needed throughout the thesis.
Chapter 3, contains the statement and proof of our first main result on the existence of weak probabilistic solution of a quasilinear stochastic parabolic equation with nonstandard growth and driven by cylindrical Wiener processes; see Theorem 31.

In Chapter 4, we state and prove our second main results on existence and uniqueness of weak and strong probabilistic solutions to the stochastic counterpart of strongly nonlinear parabolic PDE's.

Chapter 2

Preliminaries

2.1 Function spaces

In this chapter we collect well known facts on functions spaces pertaining to standard Lebesgue and Sobolev spaces, some notions from functional analysis, probability theory and Itô's stochastic calculus.

2.2 Standard Lebesgue and Sobolev spaces

First, by a domain in $\mathbb{R}^n$ we mean an open set in n-dimensional real Euclidean space $\mathbb{R}^n$. A typical point in $\mathbb{R}^n$ is denoted by $x = (x_1, \ldots, x_n)$.

Definition 1. Let E be a vector space. A subset $\mathbb{D}$ of E is called convex if

$$\lambda x + (1-\lambda)y \in \mathbb{D}, \text{ whenever } xy \in \mathbb{D}.$$

Definition 2. Let $\mathbb{D}$ be a convex set in $\mathbb{R}^n$. A function $u : \mathbb{D} \to \mathbb{R}$ is convex provided

$$u(\lambda x + (1-\lambda)y) \leqslant \lambda u(x) + (1-\lambda)u(y), \text{ for any } x, y \in \mathbb{D} \text{ and each } \lambda \in [0,1].$$

A special case of this Definition is when one takes $\lambda = \frac{1}{2}$. This yields what we call the midpoint convexity:

$$u\left(\frac{1}{2}x + \frac{1}{2}y\right) \leqslant \frac{1}{2}u(x) + \frac{1}{2}u(y), \text{ for any } x, y \in \mathbb{D}.$$

Let $\mathbb{D}$ be an open bounded subset of $\mathbb{R}^n$. We denote by $L^1_{loc}(\mathbb{D})$ the set of all Lebesgue measurable function $u : \mathbb{D} \longrightarrow \mathbb{R}$ such that $|u|$ is integrable on each compact subset of $\mathbb{D}$. A multi-index α is an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers components α_i. We defined the order of a multi-index α by $|\alpha| = \sum_{i=1}^{n} \alpha_i$. If we set $D_i = \frac{\partial}{\partial x_i}$ for $1 \leqslant i \leqslant n$, then we define the partial derivatives

$$D^{\alpha}u = D_1^{\alpha_1} \dots D_n^{\alpha_n} u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

where $D_i^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x_i}$ for $1 \leqslant i \leqslant n$. D^{α} is the differential operator of order α; so that $D^{0,\dots,0}u = u$. This notation leads us to the following concept of weak derivatives which extends the concept of classical derivatives.

Definition 3. Let $u \in L^1_{loc}(\mathbb{D})$. A function $v_\alpha \in L^1_{loc}(\mathbb{D})$ is called the α-weak or distributional partial derivative of u if

$$\int_{\mathbb{D}} u(x) D^{\alpha}\phi(x)dx = (-1)^{\alpha} \int_{\mathbb{D}} v_{\alpha}(x)\phi(x)dx, \text{ for all } \ \phi \in C_0^{\infty}(\mathbb{D}).$$

Remark 1. The α-weak derivative of a function u if it exists is unique, i.e. if $D^{\alpha}u = v$ and $D^{\alpha}u = \tilde{v}$ then $v = \tilde{v}$.

Definition 4. Let $p \in [1, \infty)$, $\mathbb{D}$ be an open bounded subset of $\mathbb{R}^n$. The space $L^p(\mathbb{D})$ consists of all measurable functions $u : \mathbb{D} \longrightarrow \mathbb{R}$, identified up to pointwise almost everywhere equality such that

$$\int_{\mathbb{D}} |u(x)|^p dx < \infty.$$

We endow $L^p(\mathbb{D})$ with the norm

$$||u||_{L^p(\mathbb{D})} = \left(\int_{\mathbb{D}} |u(x)|^p dx \right)^{1/p}, \text{ if } p \in [1, \infty).$$

A function $u : \mathbb{D} \longrightarrow \mathbb{R}$ is p-power locally integrable i.e., $u \in L^p_{loc}(\mathbb{D})$ if $u \in L^p(V)$ for each $V \subset\subset \mathbb{D}$.
When $p = \infty$, the norm in $L^{\infty}(\mathbb{D})$ is defined by the essential supremum. That is,

$$||u||_{L^{\infty}(\mathbb{D})} = \operatorname*{ess\,sup}_{x \in \mathbb{D}} |u(x)|.$$

For $p \in [1, \infty]$, the space $L^p(\mathbb{D})$ is a Banach space under the norm $|| \cdot ||_{L^p(\mathbb{D})}$.

Next, we introduce some important classical and elementary inequalities which are really fundamental and needed in the sequel. To do so, we first define the Hölder conjugacy.

Definition 5. Let $p \in [1, \infty]$, Hölder the conjugate exponent denoted $p^{'}$ of an exponent $p \in [1, \infty]$ is defined by

$$p^{'} = \begin{cases} \dfrac{p}{p-1} & \text{if } p \in (1, \infty), \\ 1 & \textit{if } p = \infty, \\ \infty & \text{if } p = 1. \end{cases}$$

We note that one can check that $(p^{'})^{'} = p$ and $1/p + 1/p^{'} = 1$. Here we used the notation $\frac{1}{\infty} = 0$. Since we have defined the Hölder conjugate exponent $p^{'}$, before stating Hölder's inequality, first we need to introduce a famous inequality known as Young's inequality. For all $x, y \geqslant 0$ and conjugate exponents $p, p^{'} \in [1, \infty)$, we have

$$xy \leqslant \frac{x^p}{p} + \frac{y^{p^{'}}}{p^{'}}. \tag{2.1}$$

We can now state Hölder's inequality.

Lemma 2. *Let $p, p^{'} \in (1, \infty)$ be conjugate exponents of each other i.e. $\frac{1}{p} + \frac{1}{p^{'}} = 1$. If $u \in L^p(\mathbb{D})$ and $v \in L^{p^{'}}(\mathbb{D})$ then $uv \in L^1(\mathbb{D})$ and*

$$||uv||_{L^1(\mathbb{D})} \leqslant ||u||_{L^p(\mathbb{D})} ||v||_{L^{p^{'}}(\mathbb{D})}.$$

For the duality in Lebesgue and Sobolev spaces we need the following abstract notions from functional analysis.

Let U and V be two Banach spaces. We denote by $\mathcal{L}(U, V)$ the space of bounded linear operator $J : U \longrightarrow V$. We equip this space with the supremum norm

$$||J||_{\mathcal{L}(U,V)} = \sup_{u \in U - \{0\}} \left\{ \frac{||Ju||_V}{||u||_U} \right\}.$$

A linear functional on U is a bounded linear map form U to $\mathbb{R}$. We denote by U^* (sometime we use the notation U') the dual space of U, the space of all linear functional on U. U^* endowed with the norm

$$||J||_{\mathcal{L}(U,\mathbb{R})} = \sup_{u \in U - \{0\}} \left\{ \frac{|Ju|}{||u||_U} \right\}.$$

is a Banach space.

Remark 2. *(i)* If $u \in U$ and $J \in U^*$, we write $J(u) = \langle J, u \rangle$ to denote the duality pairing between U and U^*. By setting $u(J) = J(u)$ i.e. we think of u as acting on the operator J instead of J acting on u.

(ii) We define

$$||J||_{U^*} := \sup_{u \neq 0} \{\langle J, u \rangle ; ||u||_U \leqslant 1\}$$

(iii) A Banach space U is reflexive if $U = U^{**}$ (this equality is understood in the sense of isomorphism). This means that U coincides with its bidual. More precisely, for each $u^{**} \in U^{**}$, there exists $u \in U$ such that

$$\langle u^{**}, u^* \rangle = \langle u, u^* \rangle \textit{ for all } u^* \in U^*.$$

(iv) By the Hahn-Banach theorem, one can identify $u \in U$ with $u^{**} \in U^{**}$ by setting $u(u^{**}) = u^{**}(u)$. The reflexivity of U is done under the identification $i : U \longrightarrow U^{**}$ where $i(u)(J) = J(u)$ for every $J \in U^*$.

Theorem 5. *Let $\mathbb{D} \subset \mathbb{R}^n$ and $p \in [1, \infty)$. Then the map $J : L^{p'}(\mathbb{D}) \longrightarrow (L^p(\mathbb{D}))^*$ which associates to a function $v \in L^{p'}(\mathbb{D})$ the integral*

$$J(u) : v \mapsto \int_{\mathbb{D}} u(x)v(x)dx$$

if it exists, is an isometric isomorphism of $L^{p'}(\mathbb{D})$ onto $(L^p(\mathbb{D}))^*$, the dual space of the space $L^p(\mathbb{D})$ meaning that we have the isomorphism

$$(L^p(\mathbb{D}))^* \approxeq L^{p'}(\mathbb{D}),$$

where p' is the Hölder conjugate of p.

Remark 3. The result of this theorem holds for $p = 1$ we have

$$\left(L^1(\mathbb{D})\right)^* = L^\infty(\mathbb{D}).$$

However, in general, $(L^\infty(\mathbb{D}))^*$, the dual space of $L^\infty(\mathbb{D})$ is much larger than $L^1(\mathbb{D})$.

Example 1. All Hilbert spaces are reflexive. In particular, for $p = 2$, the space $L^2(\mathbb{D})$ coincides with its dual and it is endowed with the scalar product

$$(u, v) = \int_{\mathbb{D}} u(x)v(x)dx.$$

The following result is a corollary of the duality result given in Theorem 5.

Corollary 1. *The space $L^p(\mathbb{D})$ is a Banach space for $p \in [1, \infty)$. $L^p(\mathbb{D})$ is reflexive provided that $p \in (1, \infty)$.*

Remark 4. We notice from Remark 3 that the spaces $L^1(\mathbb{D})$ and $L^\infty(\mathbb{D})$ are non-reflexive Banach spaces.

Next we define what we mean by continuous and compact embedding.

Definition 6. Let U and V be two normed Banach spaces such that $U \subseteq V$. We say that U is continuously embedded into V and we write $U \hookrightarrow V$, if the inclusion map $i : U \to V : u \mapsto u$ is continuous i.e., there exists a constant C such that

$$||u||_V \leqslant C||u||_U \ \forall u \in U.$$

Following similar ideas as in [169, p. 198], we have

Example 2. *(a)* The real line is naturally embedded into the plan, take $U = \mathbb{R}$ and $V = \mathbb{R}^2$ with the respective norms $\|\cdot\|_U = |\cdot|$, the absolute value and $||\cdot||_V = \sqrt{v_1^2 + v_2^2}$ for any $v_1,\ v_2 \in \mathbb{R}$. Consider the projection of the plane into the real line, that is $P : \mathbb{R}^2 \to \mathbb{R} \times \{0\} : (v_1, v_2) \mapsto (v_1, 0)$. We define the identity map (inclusion map) using P by $i : \mathbb{R} \to \mathbb{R}^2 : v_1 \mapsto (v_1, 0)$. We have

$$||v_1||_U = |v_1| = ||(v_1, 0)||_V.$$

Therefore, the best possible constant for the continuous embedding here is $C = 1$.

(b) Set $\mathbb{D} = [0, 1]$ and consider $U = C(\mathbb{D})$ with the supremum norm and $V = L^1(\mathbb{D})$ with the L^1-norm. Let us show that the embedding of U in V is not continuous. We define a function u_n by

$$u_n(x) = \begin{cases} -n^2x + n & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

It is obvious that u_n is continuous on $\mathbb{D}$ since $\lim_{x \to (\frac{1}{n})^-} u_n(x) = 0$ and $\mathbb{D}$ is a closed interval in $\mathbb{R}$. Hence u_n attains its maximum. Thus by computing, we get

$$||u_n||_U = \sup\{|u_n(x)| : x \in \mathbb{D}\} = n.$$

On the other hand

$$||u_n||_V = ||u_n||_{L^1(\mathbb{D})} = \int_{\mathbb{D}} |u_n(x)|dx = \int_0^{\frac{1}{n}} [-n^2x + n]dx = \frac{1}{2}.$$

Hence there is no choice of a constant C for the above definition. In this case we say that the embedding is not continuous.

Definition 7. We say that U is compactly embedded in V, if

1. U is continuously embedded into V, i.e., there exists a constant C such that $||u||_V \leqslant C||u||_U$;
2. the embedding operator(inclusion map) $i : U \hookrightarrow V$ is compact i.e., every bounded sequence in U has a subequence that converges in V.

We have the continuous embedding: $L^{p_2}(\mathbb{D}) \hookrightarrow L^{p_1}(\mathbb{D})$ whenever $p_1 \leqslant p_2$. Moreover we have the following density result as well: $C_0^\infty(\mathbb{D})$ is dense in $L^p(\mathbb{D})$. It worth mentioning that this result fails for $p = \infty$ i.e., $C_0^\infty(\mathbb{D})$ is not dense in $L^\infty(\mathbb{D})$ since L^∞-limit of continuous functions is continuous.

Definition 8. *1. Let U be a Banach space. A sequence $(u_n)_{n\in\mathbb{N}} \subset U$ converges weakly to u if $f(u_n)$ converges to $f(u)$ for every $f \in U^*$.*

2. A sequence $(f_n)_{n\in\mathbb{N}} \subset U^*$ converges weakly-* to f if $f_n(u)$ converges to $f(u)$ for every $u \in U$.

Remark 5. If U is a reflexive Banach space then weak and weak-$*$ convergence are the same.

Definition 9. Let $\mathbb{D} \subset \mathbb{R}^n$ be open, $p \in [1,\infty]$ and $k \in \mathbb{N}$. The Sobolev space $W^{k,p}(\mathbb{D})$ is defined to consist of classes of all real-valued functions $u \in L^p(\mathbb{D})$ such that their α-weak derivatives $D^\alpha u$ exist and belong to $L^p(\mathbb{D})$, for any multi-index α of order $|\alpha| \leqslant k$.

We define the norm in this space by

$$\|u\|_{W^{k,p}(\mathbb{D})} = \left(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|^p_{L^p(\mathbb{D})} \right)^{\frac{1}{p}}, 1 \leqslant p < \infty, \tag{2.2}$$

$$\|u\|_{W^{k,\infty}(\mathbb{D})} = \operatorname{ess} \sup_{0\leqslant|\alpha|\leqslant k} \|D^\alpha u\|_{L^\infty(\mathbb{D})}, p = \infty. \tag{2.3}$$

These spaces were introduce by the Russian mathematician S. L. Sobolev in [195]. For a broader exposition, we refer the reader to [5], [84] and [197].

We need to point out that $W^{0,2}(\mathbb{D}) = L^2(\mathbb{D})$.
For $p = 2$, we denote by $H^k(\mathbb{D})$ the Sobolev space $W^{k,2}(\mathbb{D})$ for any $k \in \mathbb{N}$. We have the following

Theorem 6. (Sobolev, Gagliardo, Nirenberg)
Assume that the domain $\mathbb{D}$ in $\mathbb{R}^n$ is sufficiently smooth. Then for any k and p, we have

$$W^{j+k,p}(\mathbb{D}) \hookrightarrow W^{j,p}(\mathbb{D}), \text{ whenever } p \leqslant q \leqslant \frac{np}{n-kp},$$

$$W^{j+k,p}(\mathbb{D}) \hookrightarrow\hookrightarrow W^{j,p}(\mathbb{D}), \text{ whenever } n > kp \text{ and } q \in [1,\infty],,$$

$$W^{j+k,p}(\mathbb{D}) \hookrightarrow\hookrightarrow C^j(\overline{D}) \text{ whenever } n < kp.$$

The above embedding results hold for $W_0^{k,p}(\mathbb{D})$ since $W_0^{k,p}(\mathbb{D}) \subset W^{k,p}(\mathbb{D})$, and we have the following

Theorem 7. *Let $\mathbb{D}$ be an arbitrary domain in $\mathbb{R}^n$, $k \in \mathbb{N}$ and $p \in [1,\infty)$. Then*

1. *For $kp < n$ we have*

$$W_0^{k,p}(\mathbb{D}) \hookrightarrow L^q(\mathbb{D}) \ \forall q \in [p, p^*], \ p^* = \frac{np}{n-kp}.$$

2. *In the case $kp = n$ we have*

$$W_0^{k,p}(\mathbb{D}) \hookrightarrow L^q(\mathbb{D}) \ \forall q \in [p,\infty).$$

We denote by $W^{-k,p^*}(\mathbb{D})$ the space dual of $W_0^{k,p}(\mathbb{D})$.

2.3 Some analytic and probabilistic theoretical facts

We now introduce some evolution spaces which play an important role in the work. Given a Banach space B, for $1 \leqslant q \leqslant \infty$, we denote by $L^q(0,T;B)$ the set of functions defined on $[0,T]$ and taking values in B. We endow $L^q(0,T;B)$ with the norm

$$\|u\|_{L^q(0,T;B)} = \left(\int_0^T \|u(t)\|_B^q \, dt\right)^{1/q} \quad \text{if } 1 \leqslant q < \infty.$$

When $q = \infty$, the space $L^\infty(0,T;B)$ is the space of all essentially bounded functions on the closed interval $[0,T]$ with values in B with the norm

$$\|u\|_{L^\infty(0,T;B)} = \operatorname*{ess\,sup}_{[0,T]} \|u\|_B \; < \; \infty.$$

We have

Lemma 3. *(cf [121, Theorem 3.3.2, page 132])*
Consider $\mathbb{U}_1$, $\mathbb{U}_2$ two Banach spaces with $\mathbb{U}_1$ separable, reflexive and continuously embedded in $\mathbb{U}_2$. Then, the space $L^p(0,T;\mathbb{U}_1)$ is continuously embedded in $L^q(0,T;\mathbb{U}_2)$ for any $p,q \in [1,\infty]$ with $p \leqslant q$. In addition, $(L^p(0,T;\mathbb{U}_1))^* = L^q(0,T;\mathbb{U}_1^*)$, where, $q = p/(p-1)$ with $p \in (1,\infty)$.

We also have

Proposition 1. *(cf. [121, Proposition 3.3.2, page 134])*
Let $\mathbb{U}$ be a Banach space; if $f \in L^p(0,T;\mathbb{U})$, we have

$$\int_0^T \langle g, f(t)\rangle_{\mathbb{U}\times\mathbb{U}^*} dt = \langle g, \int_0^T f(t)dt\rangle_{\mathbb{U}\times\mathbb{U}^*} \;\; \forall g \in \mathbb{U}^*. \tag{2.4}$$

If $g \in L^p(0,T;\mathbb{U}^*)$, we get

$$\int_0^T \langle g(t), f\rangle_{\mathbb{U}\times\mathbb{U}^*} dt = \langle \int_0^T g(t)dt, f\rangle_{\mathbb{U}\times\mathbb{U}^*} \;\; \forall f \in \mathbb{U}. \tag{2.5}$$

Having defined needed evolutionary spaces, we now introduce a compactness result due to Lions [138, Chap. 1, Lemma 1.3]. The following result is of great importance for the rest of the paper.

Lemma 4. *Let $(g_k)_{k=1,2,\ldots}$ and g be some functions in $L^q(0,T;L^q(\mathbb{D}))$ with $1 \leqslant q \leqslant \infty$ such that*

$$\|g_k\|_{L^q(0,T;L^q(\mathbb{D}))} \leqslant C, \forall k$$

for some positive constant C independent of k and

$$g_k \longrightarrow g \text{ for almost all } (t,x) \in (0,T) \times D \text{ as } k \longrightarrow \infty.$$

Then,

$$g_k \longrightarrow g \text{ weakly in } \; L^q(0,T;L^q(\mathbb{D})) \text{ as } k \to \infty.$$

The next lemma, is a crucial analytic compactness result from [192, sect. 8, Theorem 5].

Lemma 5. *Given some Banach spaces B, F and H with F a subset of H such that B is compactly embedded into F. For any p, $q \in [1,\infty]$, let V be a bounded set in $L^q(0,T;B)$ such that*

$$\lim_{\theta \longrightarrow 0^+} \int_0^{T-\theta} \|v(t+\theta) - v(t)\|_H^p \, dt = 0, \text{ uniformly for all } v \in V.$$

Then V is relatively compact in $L^p(0,T,F)$.

We now introduce some facts from probability theory. Key references are [31], [65], [79] and [190].

Definition 10. Let $(\Omega,\mathcal{F})$ and $(U,\mathcal{U})$ be two measurable spaces. A mapping $X : \Omega \to U$ is measurable or random variable if

$$X^{-1}(A) \in \mathcal{F} \text{ for every } A \in \mathcal{U}.$$

Next, we intend to define filtration, stopping times and martingales. For that purpose, we take our basic structure to be a measurable space $(\Omega,\mathcal{F})$.

Definition 11. A filtration is an increasing family of sub-σ-fields or sub-σ-algebras on a measurable space. That is, given a measurable space $(\Omega,\mathcal{F})$, a filtration is a sequence of σ-algebras $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for each $t \in [0,T]$ and satisfying

$$\mathcal{F}_s \subseteq \mathcal{F}_t \text{ provided that } s \leqslant t.$$

When $T = \infty$ we define $\mathcal{F}_\infty$ as the σ-algebra generated by the infinite union of the $\mathcal{F}_t$'s, which is also contained in $\mathcal{F}$:

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{t\in[0,T)} \mathcal{F}_t\right) \subseteq \mathcal{F}.$$

As a convention we write $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$.
We define $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ and, for $t > 0$, we define $\mathcal{F}_{t^-} = \bigvee_{s<t} \mathcal{F}_s$.
The filtration is said to be right continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space (Ω is an arbitrary set, $\mathcal{F}$ a σ-algebra of subsets of Ω and $\mathbb{P} : \mathcal{F} \to [0,1]$ a probability measure) and let $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ be a filtration of nondecreasing and right continuous family of sub σ-algebra of $\mathcal{F}$ with $\mathcal{F}_0$ containing all the $\mathbb{P}$-null sets.
A filtered probability space (known as a stochastic basis) is a probability space equipped with a filtration.
For an $\mathcal{F}$-measurable function $u : \Omega \to \mathbb{R}$, we define

$$\mathbb{E}(u) := \int_\Omega u(\omega)\mathbb{P}(d\omega)$$

whenever

$$\mathbb{E}(|u|) := \int_\Omega |u(\omega)|\mathbb{P}(d\omega) < \infty.$$

The integral

$$\mathbb{E}(u) := \int_\Omega u(\omega)\mathbb{P}(d\omega)$$

is called the expectation of the random variable X. Throughout we denote by $\mathbb{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$.

We define the conditional expectations with respect to a σ-algebra $\mathcal{G}$ (sub-σ-algebra of $\mathcal{F}$). The concept of conditional expectation is of major importance in the definition of martingale. Suppose again $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$.

Definition 12. We assume that X is an E-valued integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The conditional expectation of X with respect to the σ-field $\mathcal{G}$ (conditional expectation of X given $\mathcal{G}$) is the (a.s. unique) integrable random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying

(i) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable;

(ii) for every $B \in \mathcal{G}$

$$\int_H X\, d\mathbb{P} = \int_H \mathbb{E}[X|\mathcal{G}]\, d\mathbb{P}, \;\; or\; \mathbb{E}[XI_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]I_B], \;\; for\ all\ H \in \mathcal{G}. \tag{2.6}$$

The existence and uniqueness of $\mathbb{E}[X|\mathcal{G}]$ follows from the Radon-Nikodym theorem

Theorem 8. *Let μ be the measure on $\mathcal{G}$ defined by*

$$\mu(H) = \int_H X\, d\mathbb{P}; \;\; H \in \mathcal{G}.$$

Then μ is absolutely continuous with respect to $\mathbb{P}|\mathcal{G}$, so there exists a $\mathbb{P}|\mathcal{G}$-unique $\mathcal{G}$-measurable random variable Y on Ω such that

$$\mu(H) = \int_H Y\, d\mathbb{P}\ for\ all\ H \in \mathcal{G}.$$

Thus the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is a modification (see definition below) of Y. The random variable $Y =: \mathbb{E}[X|\mathcal{G}]$ is indeed unique a.s. with respect to the measure $\mathbb{P}|\mathcal{G}$.

The following result is known as Fatou's lemma and it can be used for positive random variables of finite or infinite type and not integrable as well.

Lemma 6. (**Fatou's lemma**)
Let $X, X_1, X_2, \cdot$ be random variables. We have
(a) if $X_n \geqslant X$ for all $n \in \mathbb{N}$ and $\mathbb{E}(X) > -\infty$, then

$$\mathbb{E}\left(\liminf_n X_n\right) \leqslant \liminf_n \mathbb{E}(X_n);$$

(b) if $X_n \leqslant X$ for all $n \in \mathbb{N}$ and $\mathbb{E}(X) < \infty$, then

$$\limsup_n \mathbb{E}(X_n) \leqslant \mathbb{E}\left(\limsup_n X_n\right);$$

(c) if $|X_n| \leqslant X$ for all $n \in \mathbb{N}$ and $\mathbb{E}(X) < \infty$, then

$$\mathbb{E}\left(\liminf_n X_n\right) \leqslant \liminf_n \mathbb{E}(X_n) \leqslant \limsup_n \mathbb{E}(X_n) \leqslant \mathbb{E}\left(\limsup_n X_n\right).$$

Theorem 9. (**Lebesgue's Dominated Convergence**)
Let $X, Y, X_1, X_2, \ldots$ be random variables such that $|X_n| \leqslant Y$, $\mathbb{E}(Y) < \infty$ and $X_n \to X$ (a.s.). Then $\mathbb{E}(|X|) < \infty$,

$$\mathbb{E}(X_n) \to \mathbb{E}(X) \text{ as } n \to \infty,$$

and

$$\mathbb{E}|X_n - X| \to 0, \text{ as } n \to \infty.$$

We now introduce some needed probabilistic evolution spaces

Let B be a Banach space and let $1 \leqslant p \leqslant \infty$. The space

$$L^p\left(\Omega, \mathcal{F}, \mathbb{P}, L^q\left(0, T; B\right)\right)$$

consists of all random functions $u : [0, T] \times \mathbb{D} \times \Omega \longrightarrow L^q(0, T; B)$ such that u is measurable w.r.t. (t, ω) and for all t, u is measurable w.r.t. $\mathcal{F}_t$. We furthermore endow this space with the norm

$$\|u\|_{L^p(\Omega,\mathcal{F},\mathbb{P},L^q(0,T;B))} = \left(\mathbb{E}\|u\|^p_{L^q(0,T;B)}\right)^{1/p}; \tag{2.7}$$

when $q = \infty$, then the norm in the space $L^p(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0, T; B))$ is given by

$$\|u\|_{L^p(\Omega,\mathcal{F},\mathbb{P},L^\infty(0,T;B))} = \left(\mathbb{E}\ \|u\|^p_{L^\infty(0,T;B)}\right)^{1/p}.$$

Theorem 10. *$L^p(\Omega, \mathcal{F}, \mathbb{P}, L^q(0,T;B))$ with the norm defined in (2.7) is a Banach space.*

We introduce the probabilistic version of Lemma 4 in

Remark 6. The powerful compactness result in lemma 4 remains valid whenever the space $L^q(\Omega, \mathcal{F}, \mathbb{P}; L^q(0,T;L^q(\mathbb{D})))$ is used instead of $L^q(0,T;L^q(\mathbb{D}))$. That is, if the sequence (g_k) satisfies

$$\mathbb{E}\int_0^T \|g_k(t)\|^q_{L^q(\mathbb{D})}dt \leqslant C, \ \forall k \in \mathbb{N}.$$

and in addition if it holds that

$$g_k \longrightarrow g \text{ for almost all } (t,x,\omega) \in (0,T)\times\mathbb{D}\times\Omega \text{ as } k \longrightarrow \infty.$$

Then, we have

$$g_k \longrightarrow g \text{ weakly in } \ L^q(\Omega, \mathcal{F}, \mathbb{P}; L^q(0,T;L^q(\mathbb{D}))).$$

Next, we define stopping times.

Definition 13. Suppose that we are given a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$. A map $\tau : \Omega \longrightarrow [0,\infty]$ is called a **stopping time** with respect to the filtration $\mathcal{F}$ (or an $\mathcal{F}$-stopping time or simply a stopping time if there is no confusion) provided that the event $\{\omega : \tau(\omega) \leqslant t\} \in \mathcal{F}_t$, for all $t \in [0,T]$.

We are going to define stochastic processes and introduce some properties. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 14. A **stochastic process** X defined on a measurable space $(\Omega, \mathcal{F})$ with values in a measurable space $(E, \mathcal{G})$ is a family of random variables $(X(t))_{0\leqslant t\leqslant T}$ with values in E, indexed by $t \in [0,T]$.

(1) For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X(t,\omega)$; $0 \leqslant t \leqslant T$ is the sample path of the process X associated with ω.

(2) X is continuous if its sample paths $X(t,\omega)$ are continuous functions of t, for almost all (almost everywhere) $\omega \in \Omega$.

Definition 15. Suppose that $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ is a filtration of the measurable space $(\Omega, \mathcal{F})$, and X is a stochastic process defined on $(\Omega, \mathcal{F})$ with values in $(E, \mathcal{G})$. Then X is said to be **adapted** to the filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ (or $\mathcal{F}_t$-adapted) if $X(t) \in \mathcal{F}_t$ that is $\mathcal{F}_t$-measurable random variable, for each $t \in [0,T]$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and $(E, \mathcal{E})$ a measurable space, where $\mathcal{E} = \mathcal{B}(E)$ is the borel σ-algebra of subsets of E.

Definition 16. A stochastic process X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be progressively measurable if the map $[0, t] \times \Omega \longrightarrow E$ defined by $(s, \omega) \mapsto X(s, \omega)$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable for every $t \in [0, T]$.

Definition 17. X is $\{\mathcal{F}_t : 0 \leqslant t \leqslant T\}$-martingale if

(i) X is $\{\mathcal{F}_t : t \geqslant 0\}$-adapted;

(ii) X is integrable w.r.t. $\mathbb{P}$ i.e., $X \in L^1(\mathbb{P})$ for all $t \in [0, T]$; that is $\mathbb{E}(|X|) < +\infty$;

(iii) for all $s, t \in [0, T]$ we have $\mathbb{E}\left(X_t|\mathcal{F}_s\right) = X_s$

We introduce one of the main ingredient in the theory of integration which is the concept of square integrable martingales.

Definition 18. A random variable X is said to be **square integrable** if it has a finite second moment (or mean square), that is $\mathbb{E}[X^2] < \infty$. A process $X = \{X(t)\}_{t\in[0,T]}$ is square integrable if $\sup_{t\in[0,T]} \mathbb{E}[X(t)^2] < \infty$.
If the process X satisfies the following:

a) X is a martingale,

b) X is square integrable,

then X is called square integrable martingale.

Definition 19. An adapted process $X = \{X(t)\}_{0\leqslant t\leqslant T}$ with values in E is said to be a local martingale if there exists an increasing sequence of stopping times τ_n, such that

(i) $\tau_n \longrightarrow \infty$ almost surely as $n \longrightarrow \infty$,

(ii) for each n the stopped processes $X(t \wedge \tau_n)$ is uniformly integrable (the definition will follow) martingale in t.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real-valued random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$, the measure μ on $\mathcal{F}$ given by

$$\mathbb{A} \mapsto \mathbb{P}\left(X^{-1}(\mathbb{A})\right); \forall \mathbb{A} \in \mathcal{F},$$

is called the distribution of X, or the law of X.

We now introduce one of the most important theorems used in the construction of Brownian motions. It is due to Kolmogorov's extension theorem. For further proofs and more information about these theorems and definitions we refer to [153].

Theorem 11. *For all $t_1, t_2, \dots, t_k \in [0,T]$, $k \in \mathbb{N}$ let $\nu_{t_1}, \dots, \nu_{t_k}$ be probability measures on $\mathbb{R}^{nk}$ such that*

$$\nu_{t_{\sigma(1)},\dots,\dots t_{\sigma(k)}}(F_1 \times \dots \times F_k) = \nu_{t_1,\dots t_k}\left(F_{\sigma^{-1}(1)} \times \dots \times F_{\sigma^{-1}(k)}\right) \tag{2.8}$$

for all permutations σ on $\{1, 2, \dots, k\}$ and

$$\nu_{t_1,\dots,\dots t_k}(F_1 \times \dots \times F_k) = \nu_{t_1,\dots t_k, t_{k+1},\dots,t_{k+m}}(F_1 \times \dots \times F_k \times \mathbb{R}^n \times \dots \times \mathbb{R}^n) \tag{2.9}$$

for all $m \in \mathbb{N}$, where the set on the right hand side has a total of $m+n$ factors.
Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $X = \{X(t)\}$ on Ω, $X(t) : \Omega \longrightarrow \mathbb{R}^n$, such that

$$\nu_{t_1,\dots,\dots t_k}(F_1 \times \dots \times F_k) = \mathbb{P}\left[X(t_1) \in F_1, \cdots, X(t_k) \in F_k\right],$$

for all $t_i \in [0,T]$, $k \in \mathbb{N}$ and all Borel sets F_i.

Definition 20. Two stochastic processes $X = \{X(t)\}$ and $Y = \{Y(t)\}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E, \mathcal{G})$ are said to be a **modification** of (or a **version** of) each other if

$$\mathbb{P}(\omega : X(t,\omega) = Y(t,\omega)) = 1, \ \ \forall t \in [0,T].$$

Note that if $X(t)$ is a modification of $Y(t)$, then $X(t)$ and $Y(t)$ have the same finite-dimensional distributions.

Next, we introduce another famous theorem of Kolmogorov which can help to justify the existence of a continuous version of Brownian motion:

Theorem 12 (Kolmogorov's continuity theorem)**.** *Suppose that the process $X = X(t)_{t\in[0,T]}$ satisfies the following condition: for all $T > 0$ there exist positives constant α, β, C such that*

$$\mathbb{E}\left[|X(t+h) - X(t)|^\alpha\right] \leqslant C|h|^{1+\beta}; \ 0 \leqslant t, \ h \leqslant T.$$

Then there exists a continuous version of X.

In order to construct Brownian motion we need the following.
Fix $x \in \mathbb{R}^n$ and define

$$p(t,x,y) = (2\pi t)^{-n/2} \exp\left(-\frac{|x-y|_{\mathbb{R}^n}}{2t}\right) \text{ for } y \in \mathbb{R}^n, \, t > 0.$$

If $0 \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_k$ define a measure $\nu_{t_1,\dots,t_k}$ on $\mathbb{R}^n$ by

$$\begin{aligned}&\nu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) = \\ &\int_{F_1 \times \dots \times F_k} p(t_1, x, x_1) p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \cdots dx_k,\end{aligned} \tag{2.10}$$

where $dx = dx_1 \cdots dx_k$ stands for the Lebesgue measure and $p(0,x,y)dy = \delta_x(y)$, is the unit point mass at x. The extension of this definition rests on (2.8). It is clear that $p(t,x,y)$ satisfies $\int_{\mathbb{R}^n} p(t,x,y)\,dy = 1$ for all $t \in [0,T]$, hence property (2.9) is valid. Then we apply Kolmogorov extension Theorem 11 to find a probability space $(\Omega, \mathcal{F}, \mathbb{P}^x)$ and a stochastic process $B = \{B(t) : t \in [0,T]\}$ on Ω (the underlying sample space) such that the finite dimensional distributions on $B(t)$ are given by

$$\mathbb{P}^x\left(B(t_1) \in F_1, \cdots, B(t_k) \in F_k\right) = \int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)\,dx_1\cdots dx_k. \tag{2.11}$$

Definition 21. A process $B = \{B(t) : t \in [0,T]\}$ satisfying properties (2.10) and (2.11) above is called Brownian motion on a measurable space $(\Omega, \mathcal{F})$ with a family of probability measures $\mathbb{P}^x$, i.e, $\mathbb{P}^x(B(0) = x) = 1$, and B is a Brownian motion starting at x under $\mathbb{P}^x$.

The process B defined above, can be characterized by the following properties:

(i) $B(t)$ is a Gaussian process, i.e. for all $0 \leqslant t_1 \leqslant \cdots \leqslant t_k$ the random variable $B = \{B(t) : t \in [t_1, t_k]\}$ has a normal distribution.

(ii) $B(t)$ has independent increments, i.e.

$$B(t_1), B(t_2) - B(t_1), \cdots B(t_k) - B(t_{k-1})$$

are independent for all $0 \leqslant t_1 < t_2 < \cdots < t_k$.

(iii) $B(t)$ has a continuous path almost everywhere.

A stochastic process W is said to be a Wiener process if the following properties are satisfied:

(i) $W(0) = 0$,

(ii) $W(t)$ is almost surely continuous,

(iii) $W(t)$ has independent increments with distribution $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ (for $0 \leqslant s \leqslant t$). $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

It follows from this definition that the distribution of $W(t) - W(s)$ coincides with the distribution of $W(t-s)$ and is normal with mean zero and variance $t-s$.
Condition (iii) means that if $0 \leqslant s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2$ then $W(t_1) - W(s_1)$ and $W(t_2) - W(s_2)$ are independent random variables.

To check whether or not a given proess $W(t)$ is a Wiener process, we need the following necessary and sufficient condition: for and arbitrary n, $0 = t_0 < t_1 < \cdots < t_n = T$, and $z_0, z_1, \ldots, z_n$

$$\mathbb{E} \exp \left\{ i \sum_{k=1}^{n} z_k \left[W(t_k) - W(t_{k-1}) \right] + i z_0 W(t_0) \right\} = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{n} z_k^2 (t_k - t_{k-1}) \right\}.$$

Next we introduce the definition of stochastic integrals of the form

$$\int_0^T X(t)\, dW(t) \tag{2.12}$$

with respect to a standard one dimensional Wiener process W for the process $X = (X(t))_{t \in [0,T]}$ defined on $[0, T]$.
Let X be an $\mathcal{F}_t$-measurable process for each $t \in [0, T]$, for which

$$\int_0^T X^2(t)\, dt < \infty, \text{ almost surely.}$$

Then we can define the Itô integral (2.12) for the process X as follows:

$$\int_0^T X(t)\, dW(t) = \lim_{n \longrightarrow \infty} \sum_{i=0}^{n-1} X(t_i) \left(W(t_{i+1}^n) - W(t_i^n) \right), \tag{2.13}$$

as $|\delta_n| \longrightarrow 0$ and $n \longrightarrow \infty$, where for each n, $\{t_i^n\}$, is a partition of the interval $[0, T]$, and the limit is taken over all partitions with $\delta_n = \max_{1 \leqslant i \leqslant n-1} (t_{i+1}^n - t_i^n)$ is the mesh of the partition $t_i^n = \{t_0^n < t_1^n < \cdots < t_n^n = T\}$ of the interval $[0, T]$; provided that the limit exists. The limit is understood as a.s. in probability in $L^2(\mathbb{D})$-sense.

Theorem 13. *For a process X possessing the above properties, i.e., X is $\mathcal{F}_t$-measurable for each $t \in [0, T]$, for which*

$$\mathbb{E} \int_0^T X^2(t) dt < \infty;$$

the stochastic integrals

$$\int_0^t X(s)\, dW(s)$$

are continuous martingales in t with zero mean, that is

$$\mathbb{E} \int_0^{\cdot} X(t)\, dW(t) = 0.$$

and Itô's isometry holds

$$\mathbb{E} \left(\int_0^t X(s) dW(s) \int_0^t X(r) dW(r) \right) = \int_0^t \mathbb{E}[X^2(s)] ds.$$

In the following result we introduce the Itô's formula for $\varphi(X(t))$. The Itô's formula is the stochastic equivalent of the classical chain rule of differentiation of functions in classical calculus and for more details about the exposition of this topic we refer the reader to the monographs [193], [194], [94] and [85]. In the following, we use the notation

$$|h(t, X(t))|^2 = \mathrm{Tr}(hh^*),$$

where h^* denotes the adjoint of h.

Theorem 14. *Let X have a stochastic differential for $0 \leqslant t \leqslant T$,*

$$dX(t) = b(t, X(t))dt + h(t, X(t))dW(t),$$

where $b(t, X(t))$ is an $\mathbb{R}$-valued progressively measurable process such that

$$\mathbb{E}\left(\int_0^T |b(t, X(t))|dt\right) < \infty$$

and $h(t, X(t))$ is progressively measurable process such that

$$\int_0^T |h(t, X(t))|^2 dt < \infty \ \mathbb{P} - a.s.$$

Suppose that $\varphi(x)$ is once continuously differentiable in t and twice continuously differentiable in x. Then the process $Y(t) = \varphi(X(t))$ also possesses a stochastic differential and is given by

$$\begin{aligned} d\varphi(X(t)) &= \varphi'(X(t))\, dX(t) + \frac{1}{2}\varphi''(X(t))(dX)^2\, dt \\ &= \varphi'(X(t))\, dX(t) + \frac{1}{2}\varphi''(X(t))h^2(t, X(t))\, dt \\ &= \left[\varphi'(X(t))b(t, X(t)) + \frac{1}{2}\varphi''(X(t))h^2(t, X(t))\right] dt + \varphi'(X(t))h(t, X(t))\, dW(t). \end{aligned} \tag{2.14}$$

Equivalently

$$\varphi(X(t)) = \varphi(X(0)) + \int_0^t \varphi'(X(s))\, dX(s) + \frac{1}{2}\int_0^t \varphi''(X(s))h^2(s, X(s))\, ds. \tag{2.15}$$

Formula (2.15) is called Itô's formula for $\varphi(X(t))$.
We shall introduce a useful result which is known as the Burkholder-Davis-Gundy inequality. This gives bounds for the maximum of a martingale in terms of the quadratic variations. The proof of the Burkholder-Davis-Gundy inequality can be found in [171].

Theorem 15. *Let $\tau > 0$ be a stopping time and $X = (X_t)_{t\in[0,T]}$ be a local martingale with $X_0 = 0$. Suppose that $Y(t) = \int_0^t X(s)\,dW(s)$ is the Itô's integral process such that*

$$\mathbb{E}\left(\int_0^\tau X^2(s)\,ds\right)^{p/2} < \infty.$$

Then, For any $1 \leqslant p < \infty$, there exists a positive constant C_p, independent of τ such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\left|\int_0^t X(s)\,dW(s)\right|^p \leqslant C_p\mathbb{E}\left[\int_0^\tau X^2(s)\,ds\right]^{p/2}. \tag{2.16}$$

Furthermore, for continuous local martingales, this statement holds for all $p \in (0,\infty)$.

In what follows, we define and give a characterization of uniform integrability. For that purpose, assume given a triplet $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 22. A family of random variables $(X_n)_{n\geqslant 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called uniformly integrable if

$$\int_{|X_n|>c} |X_n| d\mathbb{P} \to 0 \text{ as } c \to \infty$$

uniformly in $n \geqslant 1$.

We also have

Theorem 16. *A family $(X_n)_{n\geqslant 1}$ is uniformly integrable if for any $\varepsilon > 0$ there exists $\delta > 0$ and a measurable set A with $\mathbb{P}(A) < \delta$ such that*

$$\int_A |X_n| d\mathbb{P} \leqslant \varepsilon$$

uniformly in $n \geqslant 1$.

We are going to introduce a general definition that we will need in the sequel.

Definition 23. A family $(X_n)_{n\in\mathbb{N}} \subset L^p(0,T;B)$ is p-uniformly integrable if $\forall\varepsilon > 0$, there exists $\delta > 0$: $\forall I \subset (0,T)$, $|I| < \delta$ implies that

$$\sup_n \int_I ||X_n(t)||_B^p dt < \varepsilon,$$

or equivalently

$$\lim_{|I|\downarrow 0}\sup_n \int_I ||X_n(t)||_B^p dt = 0;$$

where $|\cdot|$ stands for the Lebesgue measure.

The following result is known as de la vallée Poussin theorem; see [73], where the theory on uniform integrability is well written.

Theorem 17 (de la Vallée Poussin). *A family $(X_n)_{n\geqslant 1} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ is uniformly integrable if and only if there exist a convex even function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(0) = 0$, $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ and*

$$\sup_n \mathbb{E}\left(\Phi \circ |X_n(\omega))|\right) = \sup_n \int_\Omega \Phi(X_n(\omega))d\mathbb{P}(\omega) < \infty.$$

Next, we are going to state another stunning fundamental criterion result on uniform integrability known as the Dunford-Pettis theorem [73, Theorem 3, page 46] and [65, II Theorem T23 p 20].

Theorem 18 (Dunford-Pettis). *Let $(X_n)_{n\geqslant 1} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$. The following statements are equivalent:*
1. $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable;
2. $(X_n)_{n\in\mathbb{N}}$ is relatively compact in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ in the weak topology $\sigma(L^1, L^\infty)$;
3. there exists a subsequence $(X_{n_\nu})_{\nu\in\mathbb{N}} \subseteq (X_n)_{n\in\mathbb{N}}$ that converges in the sense of the topology $\sigma(L^1, L^\infty)$.

We remark from the Dunford-Pettis and de la Vallée Poussin theorems that $(X_n)_{n\in\mathbb{N}} \subset L^1(0, T)$ is uniformly integrable if and only if there exists a convex and increasing function $\Phi : [0, \infty) \to [0, \infty)$ such that $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ and

$$\sup_n \int_0^T (\Phi \circ |X_n|)(t)dt < \infty.$$

Let $\mathcal{E}$ to be a separable complete metric space and $\mathcal{B}(\mathcal{E})$ its Borel σ-field. We have the following definitions of relative compactness and of tightness of probability measures.

Definition 24. A family of probability measures Π_n on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ is said to be relatively compact if from every sequence of elements of Π_n we can extract a subsequence Π_{n_j} such that Π_{n_j} converges weakly to the measure Π. This can also be formulated as follows:
For any continuous and bounded function ϕ on $\mathcal{E}$

$$\lim_{j\longrightarrow\infty} \int_{\mathcal{E}} \phi(s)\, d\Pi_{n_j} \longrightarrow \int_{\mathcal{E}} \phi(s)\, d\Pi.$$

Definition 25. A family of probability measures Π_n on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ is tight if for any $\varepsilon > 0$, we can find a compact subset K_ε of $\mathcal{E}$ such that

$$\mathbb{P}(K_\varepsilon) \;\geqslant\; 1 - \varepsilon \text{ for every } \mathbb{P} \in \Pi_n.$$

Next we state some key compactness results due to Prokhorov [161] and Skorohod [193]. A detailed proof of these results can be found in [64]. It is of paramount importance in the weak probabilistic solvability of stochastic equations.

Theorem 19 (**Prokhorov**). *The family of probability measures* Π_n *is relatively compact if and only if it is tight in* $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$.

The following theorem relates the weak convergence of probability measures and the almost everywhere convergence of random variables.

Theorem 20 (**Skorokhod**). *For any sequence of probability measures* Π_n *on* $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ *which converges weakly to a probability measure* Π, *there exists a probability space* $(\Omega', \mathcal{F}', \mathbb{P}')$ *and random variables* $u, u_1, \ldots, u_n, \ldots$ *with values in* $\mathcal{E}$ *such that the probability law of* u_n *is* Π_n *and that of* u *is* Π *and*

$$\lim_{n\to\infty} \Pi_n = \Pi, \; \mathbb{P}' - a.s.$$

Chapter 3

A quasilinear stochastic parabolic equation with non-standard growth: Probabilistic weak solvability

3.1 Introduction

In the present chapter, we shall be concerned with the study of probabilistic weak solution of the initial boundary value problem for equations of evolution subjected to random perturbations of cylindrical type.

In this chapter, we investigate the problem of existence of weak probabilistic solutions for quasilinear parabolic SPDEs which generalize the equations of polytropic elastic filtration, characterized by the presence of a nonlinear elliptic part admitting nonstandard growth. Let $\mathbb{D}$ be an open bounded domain of the Euclidean space $\mathbb{R}^n$, $n \geq 1$ with C^2 boundary $\partial\mathbb{D}$. We consider the cylindrical domain $Q_T = (0, T) \times \mathbb{D}$ with some given final time $T > 0$. The SPDEs are driven by infinite dimensional Wiener processes of cylindrical type. Precisely speaking, we consider the following initial boundary-value problem

$$du + A(t, u)dt = f(t, u)dt + G(t, u)dW(t) \text{ in } (0, T) \times \mathbb{D}, \tag{3.1}$$

$$u(t, x) = 0, \text{ on } (0, T) \times \partial\mathbb{D}, \tag{3.2}$$

$$u(0, x) = u_0(x), \text{ in } \mathbb{D}, \tag{3.3}$$

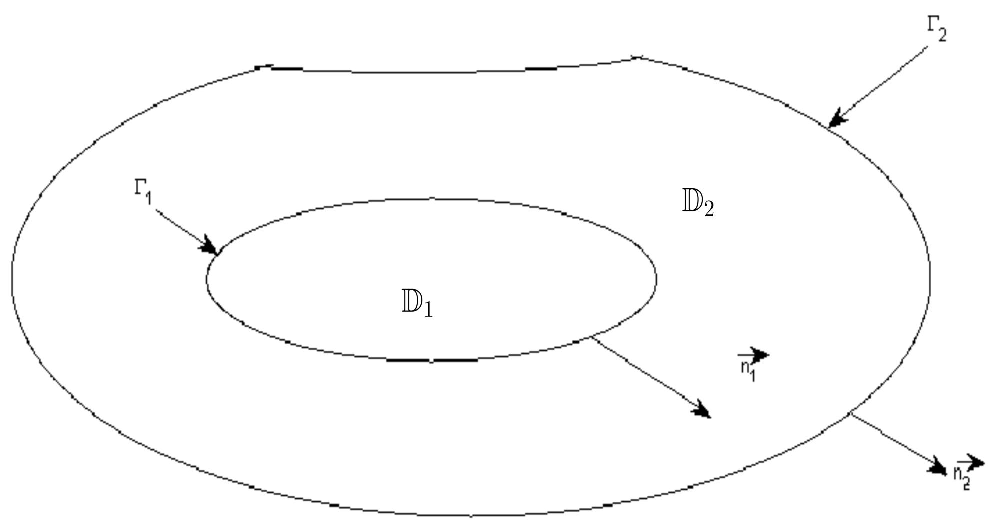
where $u = u(t, x)$ is unknown, the nonlinear terms $f(t, u)$ and $G(t, u)$ are known functions, u_0 is a given function in $L^2(\mathbb{D})$, W is a cylindrical Wiener process evolving on $L^2(\mathbb{D})$ which enters the equation as an unknown and A is a nonlinear operator given by

$$A(t, u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(\cdot)-2} \frac{\partial u}{\partial x_i}\right).$$

A is degenerate nonlinear differential operator which has an additional complicated structure, non homogeneity. This distinguishes it from the well-known $p-$Laplacian Δ_p corresponding to the case when $p(\cdot)$ is a constant function, i.e., $p(\cdot) \equiv p > 1$.

In view of the nature of the operator A, the functional space in which we expect the solution of problem (3.1)-(3.3) to belong involves the space $\mathring{W}^{1,p(x)}(\mathbb{D})$. To see how these spaces arise, we give a motivation through the following example of a transmission problem for a nonlinear elliptic equation.

Let $\mathbb{D} \subset \mathbb{R}^n$ be an open bounded domain with boundary $\partial\mathbb{D}$, $\mathbb{D}_1$ be a proper subdomain of $\mathbb{D}$ with the boundary Γ_1, and $\mathbb{D}_2 = \mathbb{D} \setminus \mathbb{D}_1$. Then $\mathbb{D}_2$ is bounded by Γ_1 and Γ_2. We assume that Γ_1 and Γ_2 are sufficiently smooth. Let $\overrightarrow{n}_1$ (resp. $\overrightarrow{n_2}$) be the field of unit normal vectors to Γ_1(resp. Γ_2) oriented toward the interior of $\mathbb{D}_2$ as shown in the figure below, and $p_1, p_2 \in (1, \infty)$ are constants,



We consider the operator

$$\Delta_{p_k}\varphi = \mathrm{div}\left(\left|\frac{\partial\varphi}{\partial x}\right|^{p_k-2}\nabla\varphi\right), k = 1, 2,$$

where ∇ denotes the gradient. We consider the transmission problem

$$-\Delta_{p_k} u_k = f_k \quad on \quad \mathbb{D}_k, \quad k = 1, 2, \tag{3.4}$$

$$u_1(x) = u_2(x) \quad on \quad \Gamma_1 \tag{3.5}$$

$$\frac{\partial_{\Delta_{p_2}} u_2}{\partial \overrightarrow{n}_1}(x) = \frac{\partial_{\Delta_{p_1}} u_1}{\partial \overrightarrow{n}_1}(x) \quad on \ \Gamma_1 \tag{3.6}$$

$$u_2 = 0 \ on \quad \Gamma_2, \tag{3.7}$$

where,

$$\frac{\partial_{\Delta_{p_k}}\varphi}{\partial\overrightarrow{n}} = \left|\frac{\partial\varphi}{\partial x}\right|^{p_k-2}\nabla\varphi.n.$$

Integrating (3.4) by parts yields

$$\sum_{k=1}^{2}\int_{\mathbb{D}_k} -\Delta_{p_k}u_k u_k\,dx = \sum_{k=1}^{2}\int_{\mathbb{D}_k}\left|\frac{\partial u_k}{\partial x}\right|^{p_k-2}\nabla u_k\nabla u_k\,dx - \sum_{k=1}^{2}\int_{\Gamma_k}\left|\frac{\partial u_k}{\partial x}\right|^{p_k-2}u_k\nabla u_k n_k\,dx.$$

By using the transmission conditions (3.5) and (3.6), we end up with

$$\sum_{k=1}^{2}\int_{\mathbb{D}_k} -\Delta_{p_k}u_k u_k dx = \sum_{k=1}^{2}\int_{\mathbb{D}_k}|\nabla u_k|^{p_k}\,dx. \tag{3.8}$$

We note that the right hand side of (3.8) can be expressed as follows:

$$\|u\|_{\mathring{W}^{1,p(\cdot)}}(\mathbb{D}) = \int_{\mathbb{D}}|\nabla u|^{p(\cdot)}\,dx,$$

where, $p(\cdot) = p_1$ on $\overline{\mathbb{D}}_1$, $p(\cdot) = p_2$ on $\mathbb{D}_2 \cup \Gamma_2$ and we define the function $u = u_1$ in $\mathbb{D}_1$ and u_2 in $\mathbb{D}_2$. The relation (3.8) suggests that it may be reasonable to look for an appropriate weak solution u of problem (3.4)-(3.7) in the space $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$.

The work of Kovačik and Rákosník [120] is the first fundamental paper where key properties of the generalized Lebesgue space $L^{p(x)}(\mathbb{D})$ and the corresponding Sobolev space $W^{k,p(x)}(\mathbb{D})$ were studied with many examples and counter examples on the Sobolev embedding theorems. The influence of their work on the study of nonlinear PDEs is huge. The recent monograph [71] is an up to date account of the latest developments in this direction; we refer also to earlier works by Edmunds and Rákosník [81, 82], Fan et al. [88], Donaldson [74] and Krasnosel'skii [122].

Our aim is to establish the existence of a probabilistic weak solution (also known as martingale solution) for the problem (3.1)-(3.3) in a framework of probabilistic evolution spaces involving the spaces $\mathring{W}^{1,p(x)}(\mathbb{D})$. Besides the non standard growth difficulty present in the operator A, the intensity $G(t,u)$ of the noise does not satisfy the Lipschitz condition with respect to u. For the proof we use a Galerkin approximation scheme combined with some deep analytic (Aubin-Simon's type) and probabilistic compactness results due to Prokhorov and Skorokhod. It should be noted that the non-standard growth of A introduces considerable difficulties in the derivation of uniform a priori estimates for the solutions of the Galerkin approximating equations which were absent in the standard growth case. The framework used in this part of the thesis, in order to prove our first main result is inspired from the work Brzezniak, Goldys and Jegaraj [48] which has proved successful also in [166]; several ideas of the latter reference will be borrowed. It should be noted that the presence of the Hilbert-space valued cylindrical Wiener process W, requires skillful handling due to additional difficulties absent in the finite dimensional case; in particular when proving the tightness of the probability measures generated by the sequence of solution of the Galerkin scheme. Closely related to our work are the previous papers [7], [8, 21, 67, 68, 167, 168] [182]-[187] which considered various SPDEs driven by finite-dimensional Wiener processes.

In the deterministic case, i.e., when $G(t, u) \equiv 0$ in (3.1), Samokhin was the first to study in detail the problem (3.1)-(3.3) for weak solvability in the sense of distributions, in [178]. In the last decade, several authors have studied and obtained many important results on such problems also referred to as problems with non-standard growth conditions. For more details, we refer to [2]-[4], [70, 13, 83, 87], [83], [87] and [214]. The regularity and higher integrability theory have been considered in [177, 137, 56, 9, 11, 30, 158], [101]-[104], [196, 210, 214]. The authors of [87] studied nonlinear parabolic initial boundary value problem with $p(x)$-growth conditions with respect to u and ∇u by introducing a compactness method combined with Galerkin's approximation. For a localization property of weak solutions for parabolic initial-boundary value problem with nonstandard growth conditions, we refer to [96]. Closely related to this problem are nonlinear parabolic with anisotropy; the case of doubly degenerate parabolic equations exhibiting such behavior was initially studied in [183]. We refer to [72, 164, 177] for corresponding problems in the mathematical modeling of electrorheological fluids.

In the case when $p(x) = p$ is a constant function in (3.1), a huge amount of results has been obtained in the deterministic case on the existence and regularity properties of the solutions; the celebrated monograph by Lions [138] has been the key reference in that direction since its publication. The corresponding research in the stochastic case started with the pioneering works of Bensoussan and Temam [25], [23], followed by those of Pardoux [155], Krylov and Rozovskii [123], Krylov and Gyongy [97], [98], [141], [206], and many other authors. All these works use decisively the deterministic monotonicity and compactness methods elaborated in [138].

The results proved in this chapter extend those of [178] and [87] to the stochastic case and can also be seen as generalizations of some of Krylov and Rozovskii's results [123] to stochastic nonlinear parabolic equations with nonstandard growth. The foundation laid in the present chapter should enable the study of the very challenging system of equations of electrorheology; the deterministic theory was rigorously investigated in [72, 164, 177].

Stochastic quasilinear parabolic equations with non-standard growth, driven by infinite dimensional Wiener processes, and with non Lipschitz forces have been investigated for the first time in the paper [8] which originated from the research work of the first author [7]. The current results extend those in [8] to the case of Hilbert-valued cylindrical Wiener processes.

Additional expositions of weak probabilistic (or martingale) solutions can be found in [53, 64], [89]-[93], and [115, 116].

This chapter is organized in the following manner.
In section 2.1, we introduce preliminary materials of function spaces providing necessary tools that are relevant for further development in the rest of the paper. This section includes the definitions of the variable exponents spaces $L^{p(\cdot)}(\mathbb{D})$ and $W^{k,p(\cdot)}(\mathbb{D})$ and their corresponding probabilistic evolution spaces in which our probabilistic weak solutions live; we state several important properties of these spaces; we also define the cylindrical Wiener processes driving our equation. In section 3, we provide the definition of probabilistic weak solution relevant to problem (3.1)-(3.3) and state the main result on the existence of such a solution as a probabilistic system $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$, where $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a stochastic basis i.e., a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $u(t)$ is a process satisfying problem (3.1)-(3.3) in the sense of distributions on $(0,T) \times \mathbb{D}$, $\mathbb{P}$-a.s., and W is a cylindrical Wiener process evolving in $L^2(\mathbb{D})$. Section 4 is devoted to the proof of our main result. For that purpose we introduce an appropriate Galerkin scheme on a fixed probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with a prescribed cylindrical Wiener process $\bar{W}$. We establish crucial a priori estimates for the Galerkin solutions. Combining these estimates with analytic compactness results, we obtain the tightness of probability measures generated by the sequence of Galerkin solutions and the cylindrical Wiener process $\bar{W}$, following [48, 166]. Then Prokhorov and Skorokhod compactness procedures become applicable. And finally, we conclude the proof by a delicate passage to the limit involving arguments of monotonicity.

3.2 Preliminaries and notations

In this section, we shall introduce some useful tools that are needed in the sequel. We specify some function spaces following in particular, ideas from [120, 71]. We collect several technical tools and provide appropriate concepts of modular and norm convergences that are suitable for these spaces and in particular the study of evolution equations.

3.2.1 Generalized Lebesgue and Sobolev spaces

Lebesgue and Sobolev spaces with variable exponent are a particular class of Orlicz spaces which appeared for the first time in the work of Nakano [150]. Further development in this direction is done through the contribution of Musielak [149], and Tsenov [200] and the reference therein. These spaces are popular nowadays due to their generalization of the classical Lebesgue and Sobolev spaces.

We begin this section by fixing some notations that will be used in the sequel. Throughout the paper all sets and functions are supposed to be Lebesgue measurable. By $|A|$ we denote the Lebesgue measure of any subset A of $\mathbb{D}$ and I_A its characteristic function. We denote by $\mathcal{P}(\mathbb{D})$ the set of all measurable functions $p : \mathbb{D} \longrightarrow [1, \infty]$ and by $\|f\|_X$ the norm of f on a

normed space X.
Let $p \in \mathcal{P}(\mathbb{D})$. Then we write $\mathbb{D}_1^p = \{x \in \mathbb{D} : p(x) = 1\}$, $\mathbb{D}_\infty^p = \{x \in \mathbb{D} : p(x) = \infty\}$, $\mathbb{D}_0^p = \mathbb{D} \setminus (\mathbb{D}_1^p \cup \mathbb{D}_\infty^p)$, and we set

$$p_* = \operatorname*{ess\,inf}_{\mathbb{D}_0^p} p(\cdot), \text{ and } p^* = \operatorname*{ess\,sup}_{\mathbb{D}_0^p} p(\cdot) \text{ for } |\mathbb{D}_0^p| > 0, p_* = p^* = 1 \text{ for } |\mathbb{D}_0^p| = 0.$$

We define on E, the set of all extended real valued functions on $\mathbb{D}$, the functional ϱ_p:

$$\begin{aligned}\varrho_p(u) &= \int_{\mathbb{D}} |u|^{p(\cdot)}\,dx \\ &= \int_{\mathbb{D}\setminus\mathbb{D}_\infty^p} |u|^{p(\cdot)}\,dx + \operatorname*{ess\,sup}_{x\in\mathbb{D}_\infty^p} |u|.\end{aligned} \tag{3.9}$$

From this definition, it is clear that ϱ_p take values in $\bar{\mathbb{R}}^+ = [0, \infty]$.

Arguing similarly as in [178] and [120], one can prove that the non-negative functional ϱ_p $(\varrho_p(u) \geqslant 0)$ satisfies:

(i) the null condition:

$$\varrho_p(u) = 0 \text{ if and only if } u = 0;$$

(ii) the symmetric condition:

$$\varrho_p(-u) = \varrho_p(u) \text{ for every function } u \text{ in } E;$$

(iii) the triangular condition:

$$\varrho_p(\alpha_1 u + \alpha_2 v) \leqslant \alpha_1 \varrho_p(u) + \alpha_2 \varrho_p(v) \text{ for every } u, v \in E;$$

and for each $\alpha_m \geqslant 0, m = 1, 2$ with $\alpha_1 + \alpha_2 = 1$.

(iii) the monotonicity condition:
ϱ_p is monotone if

$$\varrho_p(u) \leqslant \varrho_p(v) \text{ for every } u, v \in E \text{ with } |u| \leqslant |v|;$$

and the strict monotonicity is defined similarly.

Therefore ϱ_p generates a modular space:

$$X_{\varrho_p} = \left\{ u \in E : \lim_{\lambda \to 0^+} \varrho_p(\lambda u) = 0 \right\}.$$

Conditions (i) - (iii) characterizes the functional ϱ_p as a convex modular on E in the sense of [178, p. 1] and [148].

We say that ϱ_p is finite if and only if the characteristic or indicator function I_A of a set $A \subset \mathbb{D}$ belongs to the modular space X_{ϱ_p} provided that $A \in \mathcal{B}(\mathbb{D})$ with the Lebesgue measure of A being finite ($|A| < \infty$).

Definition 26. The generalized Lebesgue space denoted by $L^{p(\cdot)}$, consists of classes of functions u defined on $\mathbb{D}$ such that the modular $\varrho_p(\lambda^{-1}u)$ is finite with $\lambda(u(x)) > 0$ (i.e., λ is a function of u). That is,

$$L^{p(x)}(\mathbb{D}) = \left\{ u \in X_\varrho : \varrho_p(\lambda^{-1}u) < \infty, \ \textit{with} \ \lambda(u(x)) > 0 \right\}. \tag{3.10}$$

$L^{p(\cdot)}(\mathbb{D})$ is a Banach space when endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\mathbb{D})} = \inf \left\{ \lambda > 0 : \varrho_p \left(u\lambda^{-1} \right) \leqslant 1 \right\}. \tag{3.11}$$

Example 3. *(a)* As an example of modular function we take

$$\varrho_p(u) = |u|^p, p \in [1, \infty).$$

In this case the modular space associated with ϱ_p is the standard Lebesgue space $L^p(\mathbb{D})$ and moreover,

$$\|u\|_{L^{p(\cdot)}(\mathbb{D})} = \left[\int_{\mathbb{D}} \varrho_p(u) \right]^{1/p} = \|u\|_{L^p(\mathbb{D})}$$

(b) Assume that we are given a linear space $\mathcal{X}$ endowed with a norm $\|\cdot\|_{\mathcal{X}}$. Then, one can define a convex modular in $\mathcal{X}$ by setting

$$\varrho_p(\cdot) = \|\cdot\|_{\mathcal{X}}.$$

It follows from the definition of ϱ_p that the corresponding modular space $\mathcal{X}_{\varrho_p}$ is the normed linear space $\mathcal{X}$ itself. That is,

$$\mathcal{X}_{\varrho_p} = \mathcal{X}.$$

In addition to that, we also have

$$\|u\|_{L^{p(\cdot)}(\mathbb{D})} = \inf \left\{ \lambda > 0 : \varrho_p(\lambda^{-1}u) = \|\lambda^{-1}u\|_{\mathcal{X}} \leqslant 1 \right\} = \|u\|_{\mathcal{X}} = \varrho_p(u).$$

By virtue of this example, we deduce that the notion of a convex modular generalizes the notion of a norm. In a similar fashion, the concept of a modular space extends the notion of normed linear space. Next, we state some important results involving the modular ϱ_p and the induced norm defined in (3.11).

Lemma 7. *(see [87, Theorem 1.3])*
Consider $u \in L^{p(\cdot)}(\mathbb{D})$, then

(a) $\|u\|_{L^{p(\cdot)}(\mathbb{D})} \leqslant 1 (> 1)$ *if and only if* $\varrho_p(u) \leqslant 1 (> 1)$,

(b) If $\|u\|_{L^{p(\cdot)}(\mathbb{D})} < 1$, *then* $\|u\|^{p^*}_{L^{p(\cdot)}(\mathbb{D})} \leqslant \varrho_p(u) \leqslant \|u\|^{p_*}_{L^{p(\cdot)}(\mathbb{D})}$.

(c) If $\|u\|_{L^{p(\cdot)}(\mathbb{D})} > 1$, *then* $\|u\|^{p_*}_{L^{p(\cdot)}(\mathbb{D})} \leqslant \varrho_p(u) \leqslant \|u\|^{p^*}_{L^{p(\cdot)}(\mathbb{D})}$.

We deduce from definitions of the modular, the Luxumberg norm and the results $(b)-(c)$ in Lemma 7 that for any $u \in L^{p(\cdot)}(\mathbb{D})$

$$\min\left\{\|u\|^{p^*}_{L^{p(\cdot)}(\mathbb{D})}, \|u\|^{p_*}_{L^{p(\cdot)}(\mathbb{D})}\right\} \leqslant \varrho_p(u) \leqslant \max\ \left\{\|u\|^{p^*}_{L^{p(\cdot)}(\mathbb{D})}, \|u\|^{p_*}_{L^{p(\cdot)}(\mathbb{D})}\right\}. \tag{3.12}$$

The following result is the generalized Hölder's inequality and it is due to [120] and [178].

Theorem 21. *(see [120, Theorem 2.1] and [178, Theorem 1.1])*
Let $p \in \mathcal{P}(\mathbb{D})$. For any $u \in L^{p(\cdot)}(\mathbb{D})$ and $v \in L^{q(\cdot)}(\mathbb{D})$,

$$\left|\int_{\mathbb{D}} u(x)v(x)dx\right| \leqslant (C_p + 1/p_* - 1/p^*)\, \|u\|_{L^{p(\cdot)}(\mathbb{D})}\ \|v\|_{L^{q(\cdot)}(\mathbb{D})},$$

where, $C_p = \|I_{\mathbb{D}^p_0}\|_{L^\infty(\mathbb{D})} + \|I_{\mathbb{D}^p_1}\|_{L^\infty(\mathbb{D})} + \|I_{\mathbb{D}^p_\infty}\|_{L^\infty(\mathbb{D})}$ *and $q(\cdot)$ is the pointwise conjugate exponent function defined by*

$$q(\cdot) = \begin{cases} \infty \ \textit{if } p(\cdot) = 1, \\ 1 \ \textit{if } p(\cdot) = \infty, \\ \frac{p(\cdot)}{p(\cdot)-1} \ \textit{otherwise.} \end{cases}$$

If $p(x) \neq 1$ and $p(x) \neq \infty$ the above inequality becomes

$$\left|\int_{\mathbb{D}} u(x)v(x)dx\right| \leqslant \left(\frac{1}{p_*} + \frac{1}{q^*}\right) \|u\|_{L^{p(\cdot)}(\mathbb{D})}\ \|v\|_{L^{q(\cdot)}(\mathbb{D})},$$

Here we use the convention that $\dfrac{1}{\infty} = 0$.

We characterize the dual $\left(L^{p(\cdot)}(\mathbb{D})\right)'$ of $L^{p(\cdot)}(\mathbb{D})$, the space of all continuous linear functionals over $L^{p(\cdot)}(\mathbb{D})$ as follows:

Theorem 22. *(see [120, Theorem 2.3] and [178, Theorem 1.3])*
$p \in L^{\infty}(\mathbb{D})$ if and only if for any functional $J \in \left(L^{p(\cdot)}(\mathbb{D})\right)'$ there exists a unique function $v \in L^{q(\cdot)}(\mathbb{D})$ such that for any $u \in L^{p(\cdot)}(\mathbb{D})$

$$J(v)(x) = \int_{\mathbb{D}} u(x)v(x)dx, \text{ with } C_p^{-1}\|v\|_{L^{q(\cdot)}(\mathbb{D})} \leqslant \|J\|_{\left(L^{p(\cdot)}(\mathbb{D})\right)'} \leqslant (C_p + 1/p_* - 1/p^*)\|v\|_{L^{q(\cdot)}(\mathbb{D})}.$$

Theorem 23. *(see [120, Theorem 2.5, 2.6, Corollary 2.7 and Theorem 2.8] and [178, Theorem 1.1]) Let $p \in \mathcal{P}(\mathbb{D})$.*

(i) The space $L^{p(\cdot)}(\mathbb{D})$ endowed with the Luxemburg norm (3.11) is a Banach space.

(ii) $\left(L^{p(\cdot)}(\mathbb{D})\right)' \cong L^{q(\cdot)}(\mathbb{D})$ iff $p \in L^{\infty}(\mathbb{D})$; with $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$.

(iii) The space $L^{p(\cdot)}(\mathbb{D})$ is reflexive iff $1 < r = \operatorname{ess\,inf}_{\mathbb{D}} p(\cdot) \leqslant p(\cdot) \leqslant s = \operatorname{ess\,sup}_{\mathbb{D}} p(\cdot) < \infty$.

(iv) $L^{p_1(\cdot)}(\mathbb{D}) \hookrightarrow L^{p_2(\cdot)}(\mathbb{D})$ for $0 < |\mathbb{D}| < \infty$ and $p_1(\cdot) \leqslant p_2(\cdot)$ a.e in D.

(v) $L^{p(\cdot)}(\mathbb{D})$ is separable for $p \in \mathcal{P}(\mathbb{D}) \cap L^{\infty}(\mathbb{D})$.

Let $\mathbb{D} \subset \mathbb{R}^n, n \geqslant 1$ be open and $p \in \mathcal{P}(\mathbb{D})$ with $p(x) \in [1, \infty]$ for any $x \in \mathbb{D}$ and $k \in \mathbb{N}_0$. We set

$$W^{k,p(\cdot)}(\mathbb{D}) = \left\{u \in L^{p(\cdot)}(\mathbb{D}) : \exists D^{\alpha}u \in L^{p(\cdot)}(\mathbb{D}), \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leqslant k\right\},$$

and we define the norm in this space by

$$\|u\|_{W^{k,p(\cdot)}(\mathbb{D})} = \sum_{|\alpha| \leqslant k} \|D^{\alpha}u(x)\|_{L^{p(\cdot)}(\mathbb{D})}, \tag{3.13}$$

where D^{α} is the α^{th}-weak derivative of u. $W^{k,p(\cdot)}(\mathbb{D})$ is the generalized Sobolev space or sometimes called Sobolev spaces with variable exponents. We set

$$\mathring{W}^{k,p(\cdot)}(\mathbb{D}) = \overline{C_0^{\infty}(\mathbb{D})}^{W^{k,p(\cdot)}(\mathbb{D})}.$$

That is, the closure of $C_0^{\infty}(\mathbb{D})$ (the space of infinitely differentiable functions compactly supported in $\mathbb{D}$) w.r.t. the norm (3.13). Obviously, $\mathring{W}^{k,p(\cdot)}(\mathbb{D})$ is a subspace of $W^{k,p(\cdot)}(\mathbb{D})$.

For $k = 1$, the norm (3.13) becomes

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{D})} = \|u\|_{L^{p(\cdot)}(\mathbb{D})} + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(\mathbb{D})}. \tag{3.14}$$

We endow the space $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ with the norm

$$\|u\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} = \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(\mathbb{D})}. \tag{3.15}$$

Obviously, the norm (3.15) is equivalent to the norm (3.14).

The space $\mathring{W}^{k,p(\cdot)}(\mathbb{D})$ is a proper subspace of $W^{k,p(\cdot)}(\mathbb{D})$, provided that D is a proper subset of $\mathbb{R}^n$. If $k = 0$, since D is open bounded domain of $\mathbb{R}^n$, then the space $W^{0,p(\cdot)}(\mathbb{D})$ coincides with $L^{p(\cdot)}(\mathbb{D})$.

Notice that the generalized Sobolev spaces, $W^{k,p(\cdot)}(\mathbb{D})$ inherit many properties from the corresponding Lebesgue spaces, $L^{p(\cdot)}(\mathbb{D})$. This can be seen in the following results which are of great importance, we refer to [120] for their proofs.

Theorem 24. *([120, Theorem 2.11, 3.1] and [178, Theorem 1.4])*
Let p be in $\mathcal{P}(\mathbb{D})$ and finite. Then

(i) $C_0^\infty(\mathbb{D})$ *is dense in* $L^{p(\cdot)}(\mathbb{D}) = W^{0,p(\cdot)}(\mathbb{D})$*;*

(ii) $W^{k,p(\cdot)}(\mathbb{D})$ *and* $\mathring{W}^{k,p(\cdot)}(\mathbb{D})$ *are separable Banach spaces and for any* $u \in \mathring{W}^{1,p(\cdot)}(\mathbb{D})$*, there exists a positive constant depending only on* $\mathbb{D}$ *such that*

$$\|u\|_{L^{p(\cdot)}(\mathbb{D})} \leqslant C \|\nabla u\|_{L^{p(\cdot)}(\mathbb{D})}, i = 1, 2, \ldots, n;$$

(iii) The spaces $W^{k,p(\cdot)}(\mathbb{D})$ *and* $\mathring{W}^{k,p(\cdot)}(\mathbb{D})$ *are reflexive if* $1 < p_* < p^*$*;*

(iv) If $p_1(\cdot) \leqslant p_2(\cdot)$ *a.e. in* $\mathbb{D}$ *then* $W^{k,p_2(\cdot)}(\mathbb{D})$ *continuously embedded into* $W^{k,p_1(\cdot)}(\mathbb{D})$*.*

Theorem 25. *[178, Theorem 1.5]*
Let $\mathbb{D}$ and p satisfy one of the following conditions:

(i) p *is continuous on* $\bar{\mathbb{D}}$*;*

(ii) there exist numbers p_i *and* r_i *satisfying*

$$p_1 < p_2 < r_1 < p_3 < r_2 < \cdots < p_{m-1} < r_{m-2} < n < p_m < r_{m-1} < r_m,$$

with $p_1 = 1$, $r_m = \infty$, $r_i < np_i/(n - p_i), i = 1, 2, \ldots, m - 1$.
There are also subsets $\mathbb{G}_i \subset \mathbb{D}$, $i = 1, 2, \ldots, m$, *that contain finitely many components with Lipschitzian boundaries such that* $|\mathbb{D} \setminus \cup_{i=1}^m \mathbb{G}_i| = 0$*, the interiors of* $\mathbb{G}_i$ *are mutually disjoint, and* $p_i \leqslant p(x) \leqslant r_i$ *for* $i = 1, 2, \ldots, m$ *and for all* $x \in \mathbb{G}_i$*. Then there is a compact embedding:*

$$\mathring{W}^{1,p(\cdot)}(\mathbb{D}) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\mathbb{D}).$$

Using the generalized Hölder inequality, we can characterize the dual $\left(W^{k,p(\cdot)}(\mathbb{D})\right)'$ by the following.

Theorem 26. *(see [178, Theorem 1.6]) Let $p \in L^{\infty}(\mathbb{D}) \cap \mathcal{P}(\mathbb{D})$. Then for any functional $J \in \left(\mathring{W}^{k,p(\cdot)}(\mathbb{D})\right)'$ there exists a unique system of functions $\{v_\alpha \in L^{q(\cdot)}(\mathbb{D}) : |\alpha| \leqslant k\}$ such that*

$$J(u) = \sum_{|\alpha|\leqslant k} \int_{\mathbb{D}} D^{\alpha} u(x) v_{\alpha}(x)\ dx,\ \ u \in \mathring{W}^{k,p(\cdot)}(\mathbb{D}).$$

Next, we introduce an intermediary space which is important for the construction of the solution to our problem. Let $p \in \mathcal{P}(\mathbb{D})$. We introduce the functional

$$\begin{aligned} \varrho_{p,Q}(u) &= \int_0^T \int_{\mathbb{D}} |u(t,x)|^{p(\cdot)} dxdt \\ &= \int_0^T \int_{\mathbb{D}-\mathbb{D}_\infty} |u(t,x)|^{p(\cdot)} dxdt + \int_0^T \operatorname*{ess\ sup}_{x\in\mathbb{D}_\infty} |u(t,x)| dt \end{aligned} \tag{3.16}$$

Definition 27. The space $\mathring{V}(Q)$ consists of all measurable vector-valued functions $u : [0,T] \longrightarrow \mathring{W}^{1,p(\cdot)}(\mathbb{D})$ with finite norm

$$\|u\|_{\mathring{V}(Q)} = \sum_{i=1}^{n} \left\| \frac{\partial u(t,x)}{\partial x_i} \right\|_{L^{p(\cdot)}(Q)} = \sum_{i=1}^{n} \inf \left\{ \lambda_i > 0 : \int_0^T \varrho_p \left(\lambda_i^{-1} \frac{\partial u}{\partial x_i} \right) dt \leqslant 1 \right\} \tag{3.17}$$

where $Q = [0,T] \times \mathbb{D}$.

We note that the spaces $\mathring{V}(Q_t)$ are introduced similarly.

Also, the following counterpart of the conclusion of Lemma 7 i.e., (3.12) holds for $\varrho_{p,Q}$:

$$\min\left\{ ||u||^{p_*}_{\mathring{V}(Q)}, ||u||^{p^*}_{\mathring{V}(Q)} \right\} \leqslant \sum_{i=1}^{n} \varrho_{p,Q} \left(\frac{\partial u}{\partial x_i} \right) \leqslant \max\left\{ ||u||^{p^*}_{\mathring{V}(Q)}, ||u||^{p^*}_{\mathring{V}(Q)} \right\}. \tag{3.18}$$

Following [178] and [210], and using Theorem 24, we can derive the result below.

Theorem 27. *Let $p \in \mathcal{P}(\mathbb{D})$ with $p^* < \infty$. Then, the space $\mathring{V}(Q)$ is*

(i) a separable Banach space under the norm $\|\cdot\|_{\mathring{V}(Q)}$;

(ii) reflexive if $1 < p_* \leqslant p^* < \infty$.

Alongside with the characterization of $L^{q(\cdot)}(\mathbb{D})$ and $\left(\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right)'$ we characterize as well $\left(\mathring{V}(Q)\right)'$ the dual space of $\mathring{V}(Q)$. We equip $\left(\mathring{V}(Q)\right)'$ with the norm

$$\|u\|_{\left(\mathring{V}(Q)\right)'} := \sup_{\|v\|_{\mathring{V}(Q)}\leqslant 1} |\langle u, v\rangle| := \inf \sum_{|\alpha|\leqslant 1} \int_0^T \|u_\alpha(t)\|_{L^{q(\cdot)}(\mathbb{D})} dt,$$

where the infimum is taken over all possible decompositions

$$u(t) = \sum_{\alpha} D_x^{\alpha} u_{\alpha}(t),\ \ u_{\alpha}(t) \in L^{q(\cdot)}(Q).$$

Following [210, Lemma 2.10 p.316], we have

Lemma 8. *Let $p \in \mathcal{P}(\mathbb{D})$ such that $p \in C(\bar{\mathbb{D}}) \cup L^{\infty}(\mathbb{D})$, and moreover, let $p(\cdot) \geq 2$. Then the space $\mathring{V}(Q)$ is continuously embedded into $L^2(0,T;\mathring{W}^{1,p(\cdot)}(\mathbb{D}))$.*

From the embedding of the generalized Lebesgue and Sobolev spaces in Theorem 23-Theorem 20 and Lemma 8 we have

$$L^s(0,T;\mathring{W}^{1,s}(\mathbb{D})) \hookrightarrow \mathring{V}(Q) \hookrightarrow L^r(0,T;\mathring{W}^{1,r}(\mathbb{D})),$$

where $2 \leqslant r \leqslant p(x) \leqslant s < \infty$. Obviously the embedding of their dual follows:

$$L^{r/(r-1)}(0,T;W^{-1,r/(r-1)}(\mathbb{D})) \hookrightarrow \left(\mathring{V}(Q)\right)' \hookrightarrow L^{s/(s-1)}(0,T;W^{-1,s/(s-1)}(\mathbb{D})).$$

We now define probabilistic evolution spaces. Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We endow the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an increasing filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual condition, that is :

1. $\mathbb{P}$ is complete in $(\Omega, \mathcal{F})$,
2. $\mathcal{F}_0$ contains all null sets of $(\Omega, \mathcal{F}, \mathbb{P})$,
3. the filtration $\mathbb{F}$ is right-continuous.

We define the norm in the space $L^{p(\cdot)}(\Omega, \mathcal{F}, \mathbb{P}; (0,T)\times\mathbb{D})$ by

$$\|u\|_{L^{p(\cdot)}(\Omega,\mathcal{F},\mathbb{P};(0,T)\times\mathbb{D})} = \inf\left\{\lambda > 0 : \mathbb{E}\int_0^T \varrho_p(\lambda^{-1}u)dt \leqslant 1\right\}.$$

We recall the following result known as Schauder's theorem and its main source is [86]. For a Banach space $\mathbb{X}$, we denote by $\mathbb{X}'$, its dual.

Theorem 28. *([86, Theorem 15.3, Chapter 15 page 658])*
Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces with $\mathbb{X}'$, $\mathbb{Y}'$ their respective duals and $\mathcal{T}$ be a bounded linear map. Then $\mathcal{T}$ is compact if and only if its adjoint $\mathcal{T}^* : \mathbb{X}' \to \mathbb{Y}'$ is compact.

From this Theorem, we prove directly the following result.

Lemma 9. *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces such that the embedding $\mathbb{X} \subset \mathbb{Y}$ is compact and dense. Then the embedding $\mathbb{Y}' \subset \mathbb{X}'$ is compact and dense, where $\mathbb{X}'$ and $\mathbb{Y}'$ stand for the dual of $\mathbb{X}$ and $\mathbb{Y}$ respectively.*

Proof. We only prove the compact embedding. For this aim, let $\mathcal{J} : \mathbb{X} \to \mathbb{Y}$ be the canonical injection defined by $\mathcal{J}(x) = x$, $x \in \mathbb{X}$. It is easy to observe that $\mathcal{J}$ is linear and bounded. Obviously, we have $\mathcal{J}^* = \mathcal{J}$. Thus $\mathcal{J}^*$ is also linear and bounded. For $g \in \mathbb{Y}'$, by definition of the adjoint operator, we have that

$$\mathcal{J}^*(g)(x) = \mathcal{J}(g(x)) = g(x), \text{ for all } x \in \mathbb{X}.$$

If $||x||_{\mathbb{X}} \leqslant 1$ then

$$|\mathcal{J}^*(g)(x)| = |g(x)| = |g(\mathcal{J}(x))| \leqslant ||g||_{\mathbb{Y}'} ||\mathcal{J}||_{\mathcal{L}(\mathbb{X},\mathbb{Y})} \text{ for } g \in \mathbb{Y}',$$

where $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of bounded linear mappings from $\mathbb{X}$ to $\mathbb{Y}$. Clearly $\mathcal{J}^*$ is determined by the fixed element x from $\mathbb{X}$. Since by assumption, $\mathcal{J}$ is compact, it follows from Theorem 28 that its adjoint is $\mathcal{J}^* : \mathbb{Y}' \to \mathbb{X}'$ is also compact. By the density assumption $\mathcal{J}^*$ is also an injection. We can see by definition that $\mathcal{J}^*$ is the canonical injection of $\mathbb{Y}'$ into $\mathbb{X}'$. Hence the embedding $\mathbb{Y}' \subset \mathbb{X}'$ is compact. □

We state the following result.

Recall that $\mathcal{L}(\mathbb{X})$ stands for the space of bounded linear operator $\mathcal{T} : \mathbb{X} \longrightarrow \mathbb{X}$. We recall the following result known as the Riesz's theorem and its main source is [86].

Lemma 10. *([86, Lemma 15.5, page 659]) Let $\mathbb{X}$ be a Banach space. Let $T \in \mathcal{L}(\mathbb{X})$, denote $S = I_{\mathbb{X}} - \mathcal{T}$ and $\mathbb{Y} = S(\mathbb{X})$. If $\mathbb{Y}$ is a proper closed subspace of $\mathbb{X}$, then for every $\varepsilon > 0$ there is $x_0 \in B_{\mathbb{X}}$ such that dist$(\mathcal{T}(x_0), \mathcal{T}(\mathbb{Y})) > 1 - \varepsilon$.*

3.2.2 Cylindrical Wiener Processes

In this subsection we state the definition of cylindrical Wiener processes and formulate some of their properties, by reproducing almost verbatim the subsection 2.2 of [166]; for additional information, we refer to [64], and [172].

Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two separable Hilbert spaces with complete orthonormal basis (ONB) $\{e_k\} \subset \mathbb{U}_1$ and $\{f_j\} \subset \mathbb{U}_2$. An Hilbert-Schmidt operator $\mathcal{T} : \mathbb{U}_1 \to \mathbb{U}_2$ is a bounded linear operator satisfying

$$\sum_{k=1}^{\infty} \|\mathcal{T} e_k\|_{\mathbb{U}_2}^2 < \infty.$$

We denote by $\mathcal{L}_2(\mathbb{U}_1, \mathbb{U}_2)$ the space of Hilbert-Schmidt operators from $\mathbb{U}_1$ to $\mathbb{U}_2$. We endow $\mathcal{L}_2(\mathbb{U}_1, \mathbb{U}_2)$ with the scalar product

$$\langle \mathcal{T}_1, \mathcal{T}_2 \rangle_2 := \sum_{k=1}^{\infty} \langle \mathcal{T}_1 e_k, \mathcal{T}_2 e_k \rangle_{\mathbb{U}_2};$$

and the induced norm

$$\|\mathcal{T}\|_{\mathcal{L}_2(\mathbb{U}_1,\mathbb{U}_2)} = \left(\sum_{k=1}^{\infty} ||\mathcal{T} e_k||^2_{\mathbb{U}_2} \right)^{1/2}.$$

The definition of these operators and the norm $|| \cdot ||_{\mathcal{L}_2(\mathbb{U}_1,\mathbb{U}_2)}$ are independent of the choices of the bases, $\{e_k\}$ and $\{f_j\}$; it is enough to notice that

$$\begin{aligned} \sum_{k=1}^{\infty} ||\mathcal{T} e_k||^2_{\mathcal{U}_2} &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle \mathcal{T} e_k, f_j \rangle^2_{\mathbb{U}_2} \\ &= \sum_{j=1}^{\infty} |\mathcal{T}^* f_j|^2_{\mathbb{U}_2}, \end{aligned}$$

and the sequence of operators $\{f_j \otimes e_k\}_{j,k \in \mathbb{N}}$ form a complete ONB in $\mathcal{L}_2(\mathbb{U}_1, \mathbb{U}_2)$.

The following definition is borrowed from [50, Definition 4.1]

Definition 28. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space and $\mathbb{K}$ be a real separable Hilbert space. An $\mathbb{F}$-adapted cylindrical Wiener process on $\mathbb{K}$ is a family $W = (W(t))_{t \in [0,T]}$ of bounded linear operators from $\mathbb{K}$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ [1] such that

1. *for all $t \geq 0$, and $\mathbf{k}_1$, $\mathbf{k}_2 \in \mathbb{K}$, $\mathbb{E}[W(t)\mathbf{k}_1 W(t)\mathbf{k}_2] = \langle \mathbf{k}_1, \mathbf{k}_2 \rangle_{\mathbb{K}}$,*
2. *for each $\mathbf{k} \in \mathbb{K}$, $t \geq 0$, $W(t)\mathbf{k}$ is a real valued $\mathbb{F}$-adapted Wiener process.*

Let $\mathbb{U}$ be a separable Banach space and $\mathcal{Q} : \mathbb{U}' \to \mathbb{U}$ a symmetric and nonnegative operator, that is, $\mathcal{Q}$ satisfies

$$\begin{aligned} &\langle \mathcal{Q}u^*, u^* \rangle \geq 0, \\ &\langle \mathcal{Q}u^*, v^* \rangle = \langle u^*, \mathcal{Q}v^* \rangle, \end{aligned}$$

for any $u^*, v^* \in \mathbb{U}'$. Let $H_{\mathcal{Q}}$ be the completion of $\mathrm{Ran}(\mathcal{Q})$ (the range of $\mathcal{Q}$) with respect to the scalar product $[\cdot, \cdot]_{\mathcal{Q}}$ defined by

$$[\mathcal{Q}u^*, \mathcal{Q}v^*]_{\mathcal{Q}} = \langle \mathcal{Q}u^*, v^* \rangle, \forall u^*, v^* \in \mathrm{Ran}(\mathcal{Q}).$$

Now we introduce the definition of a $\mathcal{Q}$-Wiener process with values in $\mathbb{U}$.

[1] The space of real-valued square integrable random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by $L^2(\Omega, \mathcal{F}, \mathbb{P})$

Definition 29. Let $\mathbb{U}$ be a separable Banach space and $\mathcal{Q} : \mathbb{U}' \to \mathbb{U}$ be a symmetric and nonnegative operator as above. An $\mathbb{F}$-adapted stochastic process W is a $\mathcal{Q}$-Wiener process taking values in $\mathbb{U}$ if

1. $i_{\mathcal{Q}} : H_{\mathcal{Q}} \to \mathbb{U}$ is γ-Radonifying operator,
2. $W(0) = 0$,
3. W has continuous trajectories,
4. W has independent increments,
5. for any $0 \leq s \leq t$ the random variable $\mathbb{Y} = W(t) - W(s)$ is Gaussian with zero mean and covariance $(t-s)\mathcal{Q}$, that is, its characteristic function is of the form

$$\begin{aligned}\varphi_{\mathbb{Y}}(u^*) =& \mathbb{E}\left(\exp[i\mathbb{Y}u^*]\right),\\ =& \exp\left(-\frac{1}{2}[t-s]\langle \mathcal{Q}u^*, u^*\rangle\right), \ u^* \in \mathbb{U}'.\end{aligned}$$

For the setting of the above definition we followed closely [172] and [203]. We refer, for instance, to [46], [151] and [202] and references therein for more information about γ-Radonifying operators and their use in the context of stochastic calculus.

Remark 7. 1. Following [64, Proposition 4.11] (see also [50, Remark 4.2]) we can represent a cylindrical Wiener process on $\mathbb{K}$ as a formal series

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t)e_i(x), t \geq 0 \tag{3.19}$$

where $(\beta_i)_{i\in\mathbb{N}}$ is a family of independent standard 1-dimensional Wiener processes, and $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{K}$. The above series does not converge in the Hilbert space $\mathbb{K}$ but it does in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T]; \mathbb{U}))$ for any Hilbert space $\mathbb{U}$ such that the embedding $\mathbb{K} \subset \mathbb{U}$ is Hilbert-Schmidt. The series admits an $\mathbb{U}$-valued continuous modification $\mathbb{P}$-almost surely.

2. Let $\mathbb{U}$ and $\mathbb{K}$ be such two real separable Hilbert spaces such that the canonical injection $\mathcal{J}$ from $\mathbb{K}$ into $\mathbb{U}$ is Hilbert-Schmidt. Let us denote by $\mathcal{J}^*$ the adjoint of $\mathcal{J}$. It is easy to see that $\mathcal{Q} = \mathcal{J}\mathcal{J}^*$ is a symmetric and nonnegative operator with tr $\mathcal{Q} < \infty$. Thanks to [64, Proposition 4.11] we can view the cylindrical Wiener process W on $\mathbb{K}$ defined by ((3.19)) as a $\mathcal{Q}$-Wiener process with values in $\mathbb{U}$ and

$$\mathcal{Q}^{\frac{1}{2}}(\mathbb{U}) = \mathbb{K}.$$

Conversely, if $\{\zeta_j;\ j\in\mathbb{N}\}$ is an orthonormal basis of $\mathbb{U}$ consisting of eigenfunctions of $\mathcal{J}\mathcal{J}^*$; that is, there exists an increasing family $\{\kappa_j;\ j\in\mathbb{N}\}$ of positive numbers such that $\mathcal{J}\mathcal{J}^*\zeta_j=\kappa_j\zeta_j$. Then, it follows from [64, Theorem 4.3] that W can be written as a formal series

$$W(t)=\sum_{j=1}^{\infty}\beta_j(t)\frac{\mathcal{J}^*\zeta_j(x)}{\sqrt{\lambda_j}},$$

where $\{\beta_j(t)=\sqrt{\lambda_j}\langle W(t),\zeta_j\rangle_{\mathbb{U}};j\in\mathbb{N}\}$ is a sequence of independent real-valued standard Wiener processes. Using the definition of $\{\zeta_j;j\in\mathbb{N}\}$ we can easily check that $\{\xi_j=\frac{\mathcal{J}^*e_j}{\sqrt{\kappa_j}}:j\in\mathbb{N}\}$ forms an orthonormal basis of $\mathbb{K}$. Therefore, W defines a cylindrical Wiener process on $\mathbb{K}$.

3.3 Setting of the problem and formulation of the main result

We consider the operator family $A(t,u)$ which acts from $\overset{\circ}{W}{}^{1,p(\cdot)}(\mathbb{D})$ to $W^{-1,q(\cdot)}(\mathbb{D})$ according to

$$\langle A(t,u),v\rangle \;=\; \sum_{i=1}^{n}\int_{\mathbb{D}}\left|\frac{\partial u(t,x)}{\partial x_i}\right|^{p(\cdot)-2}\frac{\partial u(t,x)}{\partial x_i}\,\frac{\partial v}{\partial x_i}dx,\forall u,v\in\overset{\circ}{W}{}^{1,p(\cdot)}(\mathbb{D}),\forall t\in[0,T],$$

where, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between the generalized Sobolev spaces $\overset{\circ}{W}{}^{1,p(\cdot)}(\mathbb{D})$ and their dual spaces $W^{-1,q(\cdot)}(\mathbb{D})$.

Moreover, from the above definition of A we have

Remark 8. If $u\in\overset{\circ}{V}(Q_T)$ then $A(t,u)\in\left(\overset{\circ}{V}(Q_T)\right)'$.

Here, we introduce some relevant hypotheses on the nonlinear functions f and G in (3.1).

Assumption 29. The function $t\mapsto f(t,u):(0,T)\longrightarrow W^{-1,q(\cdot)}(\mathbb{D})$ is measurable with respect to t for any $u\in L^2(\mathbb{D})$, the mapping $u\mapsto f(t,u)$ is continuous from $L^2(\mathbb{D})$ to $W^{-1,p(\cdot)}(\mathbb{D})$, a.e. $t\in[0,T]$ and there exist a positive constant $C>0$ such that

$$\|f(t,u)\|_{W^{-1,q(\cdot)}(\mathbb{D})}\leqslant C\left(1+\|u(t)\|_{L^2(\mathbb{D})}\right). \tag{3.20}$$

Assumption 30. The function $t\mapsto G(t,u):(0,T)\longrightarrow\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))$ is continuous with respect to t, for any $u\in L^2(\mathbb{D})$ and the mapping $u\mapsto G(t,u)$ is continuous from $L^2(\mathbb{D})$ to $\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))$, a.e. $t\in[0,T]$, and there is a constant C such that

$$\|G(t,u)\|_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}\leqslant C\left(1+\|u(t)\|_{L^2(\mathbb{D})}\right). \tag{3.21}$$

Here, $\mathcal{L}_2(\mathbb{K},\mathbb{U})$ stands for the space of Hilbert Schmidt operators defined from $\mathbb{K}$ to $\mathbb{U}$, $\left(\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right)'$, the dual of the space $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ and C is a generic constant.

Most importantly for the exponent function $p(\cdot)$, we assume that $p(\cdot)$ satisfies

$$1 \leqslant r \leqslant p(\cdot) \leqslant s < \infty. \tag{3.22}$$

Next, we define the concept of weak solution of the problem (3.1)-(3.3). Throughout the chapter, $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{D})$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the generalized Sobolev spaces $W_0^{1,p(\cdot)}(\mathbb{D})$ and $W^{-1,q(\cdot)}(\mathbb{D})$, with $W^{-1,q(\cdot)}(\mathbb{D})$ being the dual of $W_0^{1,p(\cdot)}(\mathbb{D})$ and $q(\cdot)$ the conjugate exponent function of $p(\cdot)$ and C is a generic constant that might change from line to line.

Definition 30. A weak probabilistic solution is a tuple $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,u)$, where

(1) $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},\mathbb{P})$ is a stochastic basis satisfying the usual conditions,

(2) W is an $\mathcal{F}_t$-adapted cylindrical Wiener process evolving on $L^2(\mathbb{D})$,

(3) the process u is progressively measurable and

$$u \in L^q(\Omega,\mathcal{F},\mathbb{P},L^r(0,T;\mathring{W}^{1,p(\cdot)}(\mathbb{D}))) \cap L^r(\Omega,\mathcal{F},\mathbb{P};\mathring{V}(Q_T))$$

for any $q\in[2,\infty)$ and $r=\operatorname{ess\,sup} p(\cdot)$ satisfying *(3.22)*,

(4) $u \in L^q\left(\Omega,\mathcal{F},\mathbb{P},C([0,T];L^2(\mathbb{D}))\right)$,

and for all $t\in[0,T]$, $\mathbb{P}$-almost surely

$$(u(t),v) + \int_0^t \langle A(s,u),v\rangle ds = (u_0,v) + \int_0^t \langle f(s,u),v\rangle ds + \int_0^t (G(s,u),v)\,dW, \tag{3.23}$$

for any $v\in\mathring{W}^{1,p(\cdot)}(\mathbb{D})$.

Remark 9. One should notice that, in our weak probabilistic formulation *(3.23)*, alongside the process u, the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ and the cylindrical Wiener process are part of the solution and therefore unknown as well. For notation purpose, the stochastic integral is interpreted as

$$\int_0^t (G(s,u),v)dW = \sum_{k=1}^{\infty}\int_0^t (G_k(s,u),v)d\beta_k \;\forall v\in\mathring{W}^{1,p(\cdot)}(\mathbb{D}), t\geqslant 0;$$

where $G_k(s,u)=G(s,u)e_k$. We are seeking to solve the equation

$$u(t) = u_0 + \int_0^t A(s,u)ds + \int_0^t f(s,u)ds + \sum_{k=1}^{\infty}\int_0^t G_k(s,u)d\beta_k;$$

where $W=\sum_{k=1}^{\infty} e_k\beta_k$ with $\{e_k\}_{k\geqslant 1}$ an ONB of $\mathbb{K}$ and $(\beta_k)_{k\geqslant1}$ a family of independent one dimensional Brownian motion. .

We state our main result in the following theorem.

Theorem 31. *Assume that (3.20)-(3.22) hold and $u_0 \in L^2(\mathbb{D})$. Then, the problem* (3.1)-(3.3) *admit a probabilistic weak solution in the sense of Definition 30.*

This result is a generalization of those of [21] to stochastic quasilinear parabolic PDE's with nonstandard growth.

3.4 Proof of Theorem 31

Here, we prove the main theorem stated in the previous section. For this purpose, we use Galerkin scheme to construct approximate solutions to (3.1)-(3.3). Next we derive a priori estimates for the approximating solutions of the Galerkin system with assigned cylindrical Wiener process on a prescribed probability space. We use these key estimates to establish the tightness of Galerkin solutions which enable us to apply the Prokhorov-Skorokhod theorem. For the remaining part of the proof, the Prokhorov-Skorokhod compactness method in conjunction with the monotonicity of the operator A yields the desired result.

3.4.1 Galerkin approximating sequence

In this subsection we construct Galerkin approximating sequence related to the problem under consideration. For this purpose, we state the following lemma (see, e.g., [188, Appendix, Lemma A, pages 1852-1854]).

Lemma 11. *For every bounded open set $\mathbb{D}$ of $\mathbb{R}^n$, $n \geqslant 1$, there exists a complete orthonormal system $\{\varphi_j\}_{j\geqslant 1}$ in $L^2(\mathbb{D})$ with $\{\varphi_j\}_{j\geqslant 1} \subset W_0^{1,2}(\mathbb{D}) \cap W_0^{1,p(\cdot)}$ $\forall j$. Fix now a positive integer m and let $H_m := \operatorname{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$. For any process $v(t) \in W_0^{1,p(\cdot)}(\mathbb{D})$, there exists a sequence of processes $(v_m(t))_{m\geqslant 1} \subset H_m$ such that v_m converges to v strongly in $L^r(0, T; W_0^{1,p(\cdot)}(\mathbb{D}))$.*

We proceed with the proof of this result in two steps.

Proof. First step. We consider the case when $2 \leqslant r = \operatorname{ess\,sup}_{\mathbb{D}} p(\cdot) \leqslant p(\cdot)$. In this case if $p(\cdot) \equiv p$, a constant function, then the proof is typically the same as [188]. By virtue of Theorem 24, we have that $W_0^{k,p(\cdot)}(\mathbb{D})$ are separable reflexive Banach spaces. Hence there exists a sequence $\{\phi_j\}_{j\geqslant 1} \subset C_0^\infty(\mathbb{D})$ such that the closure of $\bigcup_{j\geqslant 1} \operatorname{Span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ contains $C_0^\infty(\mathbb{D})$ in $C^1(\overline{\mathbb{D}})$. It follows from Theorem 24 that $W_0^{1,p(\cdot)}(\mathbb{D})$ is continuously embedded in $W_0^{1,r}(\mathbb{D})$ since $2 \leqslant r \leqslant p(\cdot)$ and it follows from Theorem 25 that $W_0^{1,p(\cdot)}(\mathbb{D})$ and $W_0^{1,r}(\mathbb{D})$ are compactly embedded in $L^2(\mathbb{D})$. We have the embedding

$$W_0^{1,2}(\mathbb{D}) \cap W_0^{1,p(\cdot)}(\mathbb{D}) \subset W_0^{1,2}(\mathbb{D}) \cap L^2(\mathbb{D})$$

which motivates us to introduce the space

$$\mathcal{V}(\mathbb{D}) = \{u \text{ measurable}: \ u \in W_0^{1,p(\cdot)}(\mathbb{D}) \cap L^2(\mathbb{D})\}.$$

with the norm

$$||u||_{\mathcal{V}(\mathbb{D})} = ||u||_{L^2(\mathbb{D})} + ||\nabla u||_{L^{p(\cdot)}(\mathbb{D})}.$$

It follows from Theorem 24 that $\mathcal{V}(\mathbb{D})$ is a separable and reflexive Banach space and in addition it is a closed subspace of $W_0^{1,p(\cdot)}(\mathbb{D}) \cap L^2(\mathbb{D})$. Hence there exists a countable set of linearly independent functions $\{\varphi_j\}_{j\geqslant 1} \subseteq C_0^\infty(\mathbb{D})$ such that $\{\varphi_j\}_{j\geqslant 1} \subseteq \mathcal{V}(\mathbb{D}) \subset W_0^{1,p(\cdot)}(\mathbb{D}) \cap L^2(\mathbb{D})$ and it consists of a complete system of $\mathcal{V}(\mathbb{D})$. Without loss of generality (W.l.o.g.), we may take $\{\varphi_j\}_{j\geqslant 1}$ to be the Gram-Schmidt orthogonalization of $\{\phi_j\}_{j\geqslant 1}$. We then have that $\{\varphi_j\}_{j\geqslant 1}$ forms a complete orthonormal system in $L^2(\mathbb{D})$ and satisfying: $\forall v(t) \in W_0^{1,p(\cdot)}(\mathbb{D})$, $t \in [0,T]$ each $\{\phi_j\}_{j\geqslant 1}$ enjoys the representation

$$\phi_j = \sum_{k=1}^{m_j} c_{jk}\varphi_k$$

for some positive constant m_j and c_{jk} are constants; with $\phi_j \in W_0^{1,p(\cdot)}(\mathbb{D})$, $j = 1, 2, \ldots$. For each $m \in \mathbb{N}$, we set

$$H_m := \operatorname{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}.$$

Arguing similarly as in [207], we assert that H_m is a closed, separable subset of $W_0^{1,p(\cdot)}(\mathbb{D})$. We define the norm in H_m by

$$||w||_{H_m} := \sup_{x\in\overline{\mathbb{D}}} |w(x)| + \sup_{x\in\overline{\mathbb{D}}} \sum_{i=1}^{n} \left|\frac{\partial w}{\partial x_i}\right| \text{ for } w \in H_m.$$

W.l.o.g., let us first assume that $v \in \cup_m H_m$. Since $\cup_m H_m$ is dense in $W_0^{1,p(\cdot)}(\mathbb{D})$, it follows that

$$||v||_{W_0^{1,p(\cdot)}(\mathbb{D})} \leqslant K_1||v||_{H_m} \leqslant K_2$$

for some constants $K_1, K_2 > 0$ and for any $\varepsilon = \varepsilon(j) > 0$, a vanishing sequence i.e., $\varepsilon(j) \to 0$ as $j \to \infty$, there exists $M > 0$ such that $\forall j > M$ we have

$$\|v - \phi_j\|_{W_0^{1,p(\cdot)}(\mathbb{D})} < \varepsilon(j). \tag{3.24}$$

Since $\phi_j \in W_0^{1,p(\cdot)}(\mathbb{D})$, there exist an increasing (strictly) sequence $\{j_k\}_{k\geqslant 1}$, i.e., $j_1 < j_2 < \cdot < j_k \to \infty$ and $M_{j_k} > 0$ such that

$$\left\|\phi_j - \sum_{k=1}^{M_{j_k}} c_{j_k m}\varphi_{j_k}\right\|_{W_0^{1,p(\cdot)}(\mathbb{D})} < \varepsilon(j). \tag{3.25}$$

Consequently, combining (3.24) and (3.25) we can deduce that for j sufficiently large

$$\left\| v - \sum_{k=1}^{M_j} c_{j_k m}\phi_k \right\|_{W_0^{1,p(\cdot)}(\mathbb{D})} < \varepsilon(j). \tag{3.26}$$

In the case $v \in W_0^{1,p(\cdot)}(\mathbb{D}) \setminus \cup_{m\geqslant 1} H_m$, similar arguments as in [188, Lemma A, page 1853] can be adapted since $\cup_{j=1}^{\infty}\{\phi_j\}$ is dense in $W_0^{1,p(\cdot)}(\mathbb{D})$ (see e.g., [178, page 655]).

Since $\mathring{V}(Q_T)$ is a reflexive and separable Banach space and by virtue of Theorem 24, we can also have the continuous embedding $L^{p(\cdot)}(\mathbb{D}) \subset L^2(\mathbb{D})$. Then for any $v \in \mathring{V}(Q_T) \cap L^2(Q)$ and fixed $m \geqslant 1$ there exists a sequence $(v_{jm})_{j\geqslant 1} \subset C^1([0,T], H_m)$ such that

$$v_m(t,x) = \sum_{j=1}^{m} v_{jm}(t)\varphi_j(x) \longrightarrow v \text{ in } \mathring{V}(Q_T) \cap L^2(Q); \tag{3.27}$$

$$||v(t) - v_m(t)||^r_{W_0^{1,p(\cdot)}(\mathbb{D})} \longrightarrow 0, \text{ as } m \to \infty \text{ for every } t \in [0,T]. \tag{3.28}$$

The proof of (3.27) follows from adapting similar ideas as in [91] and [188].

Second step. For the case $1 < p(\cdot) < 2$ we have the embedding $W_0^{1,2}(\mathbb{D}) \subset W_0^{1,p(\cdot)}(\mathbb{D})$ and therefore if $p(\cdot) \equiv p$, a constant function, then similar procedures as in [91, 682] and [188, page, 1854] can be used. Since $p(\cdot) \in (1,2)$, then $s = \operatorname{ess\,sup} p(\cdot) < 2$ and we have that $W_0^{1,2}(\mathbb{D}) \subset W_0^{1,r}(\mathbb{D}) \subset W_0^{1,p(\cdot)}(\mathbb{D})$. Hence same procedure as in [91, Lemma 3.1, page 682] can be employed to obtain $\{\phi_j\}$ and $\{\varphi\}$ as above. Applying ideas from [90, page 319] and [217, page 12], we can assert that there exists a sequence $v_m(t) \in C^1([0,T]; H_m)$ such that (3.27) holds true. Thus, this carries out the rest of the proof.

□

Let us assume that we are given a probabilistic system $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{0\leqslant t\leqslant T}, \bar{\mathbb{P}}, \bar{W})$, where $(\bar{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ is the natural filtration generated by the cylindrical Wiener process $\bar{W}$.

We write

$$\bar{W} = \sum_{j=1}^{\infty} w_j \bar{\beta}_j$$

where $\{\bar{\beta}_j\}_{j\in\mathbb{N}}$ is a sequence of independent standard 1-dimensional Wiener processes. We approximate $\bar{W}$ by setting

$$\bar{W}^{(m)} := \sum_{j=1}^{m} w_j \bar{\beta}_j.$$

Following [23, Proposition 1.2, page 103] and [64, p. 99], we infer that for any m, $\bar{W}^{(m)}$ is a stochastic process satisfying the conditions of Definitions 29. Moreover, $\bar{W}^{(m)}$ converges strongly to $\bar{W}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{U})$. We need to point out the following: let $\mathcal{Q} = I$ be the identity operator. It follows from the conclusion of Remark 7 that $\langle \bar{W}(t), w_j\rangle$ $j = 1, 2, \dots$ form real-valued mutually independent Brownian motions. We have that

$$\mathbb{E}|\bar{W}^{(m)}(t)|^2 = \sum_{j=1}^{m} \mathbb{E}|\bar{\beta}_j(t)|^2 = mt.$$

Since the series (3.19) does not satisfy any notion of convergence in $\mathbb{K}$, it is unexpected to assert that $\bar{W}^{(m)}$ converges to a $\mathbb{U}$-valued stochastic process. However, on the other hand if we assume that $\bar{W}$ is an $\mathbb{U}$-valued $\mathcal{Q}^{(m)} = \sum_{j=1}^{m} \kappa_j w_j \otimes w_j$-Wiener process; using $H_{\mathcal{Q}^{(m)}}$, the convenient extension of $Ran(\mathcal{Q}^{(m)})$ and arguing as in [54, Chap 6, page 243] and [64, p. 99]; we notice that, for an arbitrary $\varepsilon > 0$ and $m, n \geqslant 1$,

$$\begin{aligned}
\bar{\mathbb{P}}\left(\sup_{s\leqslant t}\left\|\bar{W}^{(m+n)}(t) - \bar{W}^{(m)}(t)\right\|_{H_{\mathcal{Q}^{(m)}}} > \varepsilon\right) &= \bar{\mathbb{P}}\left(\sup_{s\leqslant t}\left\|\sum_{j=m}^{m+n}\bar{\beta}_j(t)\kappa_j^{\frac{1}{2}}w_j\right\|_{(\mathcal{Q}^{(m)})^{\frac{1}{2}}(\mathbb{U})} > \varepsilon\right)\\
&\leqslant \varepsilon^{-2}\sup_{s\leqslant t}\bar{\mathbb{E}}\left\|\sum_{j=m}^{m+n}\bar{\beta}_j(t)\kappa_j^{\frac{1}{2}}w_j\right\|^2_{(\mathcal{Q}^{(m)})^{\frac{1}{2}}(\mathbb{U})}\\
&\leqslant \frac{t}{\varepsilon^2}\sum_{j=m}^{m+n}\kappa_j;
\end{aligned} \tag{3.29}$$

and recall that $\mathcal{Q}^{(m)} w_j = \kappa_j w_j$ and $\sum_{j=1}^{\infty} \kappa_j < \infty$. Fix n $geqslant1$ and passing to the limit as $m \to \infty$ in (3.29) and then afterward passage to the limit as $n \to \infty$ can be executed; we deduce that the left hand side of (3.29) converges to zero by the arbitrariness of ε. We deduce from this that (see, e.g., [23, Proposition 1.2, page 103]) $\bar{W}^{(m)}$ converges strongly to the cylindrical Wiener process $\bar{W}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T]; \mathbb{U}))$ as $m \to \infty$. We used the fact that for arbitrary $\varphi \in \mathbb{U}$, the sequence of linear operators $w_j \otimes w_j$ is defined by $(w_i \otimes w_j) \cdot \varphi = w_j \langle w_i, \varphi\rangle_{\mathbb{U}}$ to get $\mathcal{Q}^{(m)}$, the corresponding symmetric nonnegative operator defined on $\mathbb{U} = L^2(\mathbb{D})$ by $\mathcal{Q}^{(m)} = \sum_{i,j=1}^{m} (\mathcal{Q} w_i, w_j)(w_i, \varphi) w_j$.

We seek approximating sequence of stochastic processes $\{u_m\}_{j=1}^{\infty}$ solutions of the problem (3.1)-(3.3) in the form

$$u_m(t, x, \bar{\omega}) = \sum_{j=1}^{m} C_{jm}(t, \bar{\omega}) w_j(x). \tag{3.30}$$

The functions $C_{1m}(t, \bar{\omega}), C_{2m}(t, \bar{\omega}), \dots, C_{mm}(t, \bar{\omega})$ are required to solve the system of stochastic ordinary differential equations

$$(du_m(t), w_j) + \langle A(t, u_m), w_j\rangle dt = \langle f(t, u_m), w_j\rangle dt + \int_{\mathbb{D}} \sum_{l=1}^{m} G(t, u_m) w_l w_j dx d\bar{\beta}_l \tag{3.31}$$

for $j = 1, 2, \ldots, m$ and $t \in [0, T]$. This system is supplemented with the initial condition

$$u_{jm}(0) = \int_D u_0(x) w_j(x) dx, \;\; j = 1, 2, \ldots, m. \tag{3.32}$$

We have the initial conditions: $C_{1m}(0, \bar{\omega}) = C_1, \ldots, C_{mm}(0, \bar{\omega}) = C_m$ and $u_0(x) = \sum_{i=1}^{\infty} C_i w_i(x)$, where $\{C_1, C_2, \ldots\}$ are constants defined by $C_i = (u_0, w_i)$ for $i \in \mathbb{N}$. We express explicitly system (3.31) as follows:

$$\begin{aligned} &\sum_{k=1}^{m} dC_{km}(t, \bar{\omega})(w_k, w_j) + \sum_{i=1}^{n} \int_{\mathbb{D}} \left| \sum_{k=1}^{m} C_{km}(t, \bar{\omega}) \frac{\partial w_k}{\partial x_i} \right|^{p(\cdot)-2} \left(\sum_{k=1}^{m} C_{km}(t, \bar{\omega}) \frac{\partial w_k}{\partial x_i} \right) \frac{\partial w_j}{\partial x_i} \, dx \\ &= \langle f(t, u_m), w_j \rangle dt + \sum_{l=1}^{m} \int_0^t \left(G(s, u_m) w_l, w_j \right) d\bar{\beta}_l, \;\; j = 1, \ldots, m. \end{aligned} \tag{3.33}$$

Since the coefficients of the present problem do not need to satisfy any Lipschitz conditions, we rely on results due to [194], to claim existence of the stochastic processes C_{jm}. In order to prove this claim we proceed as follows.

It is well known that the operator family $A : W_0^{1,p(\cdot)}(\mathbb{D}) \to W^{-1,q(\cdot)}(\mathbb{D})$ satisfies the conditions of Leray-Lions [138] i.e.,

(a) The operator family A is continuous from $W_0^{1,p(\cdot)}(\mathbb{D})$ to $W^{-1,q(\cdot)}(\mathbb{D})$,

(b) Claim: A is monotone, i.e., $\langle A(t, u) - A(t, v), u(t) - v(t) \rangle \geqslant 0$.

Remark 10. *It is easy to prove that $A(t, u)$ is continuous with respect to u for almost every $(x, \bar{\omega}) \in \mathbb{D} \times \bar{\Omega}$. The proof of (b) will be dealt with at the end of subsection 3.4.4.1.*

In this part, we shall consider further results along the lines of [217, Lemma 3.5 page 13] and apply the existence results in [194].

Lemma 12. *Under assumptions of Theorem 3.6, for each positive integer $m = 1, 2, \ldots,$ there exists a sequence of random functions $\{u_m\}_{m \geqslant 1}$ with u_m of the form (3.30) satisfying system (3.31) and (3.32).*

Proof. In order to prove the solvability of system (4.21) for C_{km}, $k = 1, 2, \ldots, m$, we denote by

$$\overline{C}(t, \bar{\omega}) = \{C_{1m}(t, \bar{\omega}), C_{2m}(t, \bar{\omega}), \ldots, C_{mm}(t, \bar{\omega})\}$$

and we consider a vector valued mapping $\overline{F}(t, \bar{\omega}, \overline{C}) : [0, T] \times \bar{\Omega} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ defined by

$$\mathbb{R}^m \ni \overline{C}(t, \bar{\omega}) \mapsto \overline{F}(t, \bar{\omega}, \overline{C}(t, \bar{\omega})) = \left\{ \overline{F}_1(t, \bar{\omega}, \overline{C}(t, \bar{\omega})), \ldots, \overline{F}_m(t, \bar{\omega}, \overline{C}(t)) \right\}$$

where the j-th component is given by

$$\overline{F}_j(t,\bar{\omega},\overline{C}(t,\bar{\omega})) = \langle f(t,u_m),w_j\rangle dt + \sum_{l=1}^{m}(G(s,u_m)w_l,w_j)\,d\bar{\beta}_l-$$
$$-\sum_{i=1}^{n}\int_{\mathbb{D}}\left|\sum_{k=1}^{m}C_{km}(t,\bar{\omega})\frac{\partial w_k}{\partial x_i}\right|^{p(\cdot)-2}\left(\sum_{k=1}^{m}C_{km}(t,\bar{\omega})\frac{\partial w_k}{\partial x_i}\right)\frac{\partial w_j}{\partial x_i}\,dxdt,\ \ j=1,\dots,m.$$

For the initial condition we use the notation

$$\overline{u}_m(0) = \{u_{1m}(0), u_{2m}(0),\dots,u_{mm}(0)\}.$$

Then we can study the system of stochastic ordinary differential equations which is a reduced form of (4.21)

$$\sum_{k=1}^{m}d\overline{C}_{km}(t,\bar{\omega})(w_k,w_j) - \overline{F}(t,\bar{\omega},\overline{C}(t,\bar{\omega})) = 0 \tag{3.34}$$

with the initial condition

$$\overline{C}(0,\bar{\omega}) = \overline{u}_m(0).$$

We have

$$\left(\overline{F}(t,\bar{\omega},\overline{C}(t,\bar{\omega})),\overline{C}(t,\bar{\omega})\right)_{\mathbb{R}^m} = \overline{F}(t,\bar{\omega},\overline{C}(t,\bar{\omega}))\overline{C}(t,\bar{\omega}) =$$
$$= \sum_{j=1}^{m}\left[\langle f(t,u_m),w_j\rangle dt + \sum_{l=1}^{m}(G(s,u_m)w_l,w_j)\,d\bar{\beta}_l\right]\overline{C}_{jm}(t,\bar{\omega})-$$
$$-\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{\mathbb{D}}\left|\sum_{k=1}^{m}C_{km}(t,\bar{\omega})\frac{\partial w_k}{\partial x_i}\right|^{p(\cdot)-2}\left(\sum_{k=1}^{m}C_{km}(t,\bar{\omega})\frac{\partial w_k}{\partial x_i}\right)\frac{\partial w_j}{\partial x_i}\,dx\overline{C}_{jm}(t,\bar{\omega}).$$

Now let us fix $m\in\mathbb{N}$ and let $\tau\in(0,T)$ be a sufficiently small parameter and denote the corresponding horizon interval by $\mathcal{G}=[0,\tau]$. In addition to that let us assume w.l.o.g. that there exists a positive constant $\alpha>0$ sufficiently large such that the ball $B_\alpha(0)\subset\mathbb{R}^m$ contains the vector $\overline{C}(t,\bar{\omega})$ and set for the moment $\mathcal{H}=\overline{B_\alpha(0)}$. We can easily observe by assumptions 3.2 and 3.3 and claim b that the function $\overline{F}:\mathcal{G}\times\mathcal{H}\times\bar{\Omega}\longrightarrow\mathbb{R}^m$, $(t,\bar{\omega},\overline{C})\mapsto$

$$\left(\langle f(t,u_m),w_j\rangle dt - \sum_{i=1}^{n}\int_{\mathbb{D}}\left|\sum_{k=1}^{m}C_{km}(t,\bar{\omega})\frac{\partial w_k}{\partial x_i}\right|^{p(\cdot)-2}\left(\sum_{k=1}^{m}C_{km}(t,\bar{\omega})\frac{\partial w_k}{\partial x_i}\right)\frac{\partial w_j}{\partial x_i}\,dx\right)_{j=1,2,\dots,m} +$$
$$+\left(\sum_{l=1}^{m}(G(s,u_m)w_l,w_j)\,d\bar{\beta}_l\right)_{j=1,2,\dots,m}$$

is a continuous function. Further, we can estimate each component $\overline{F}_j$ on $\mathcal{G} \times \mathcal{H} \times \overline{\Omega}$ using among others Hölder and Young inequalities as follows

$$\begin{aligned}
&\left|\overline{F}_j(t, \bar{\omega}, \overline{C})\right| \\
&\leqslant ||f(t, u_m)||_{W^{-1,q(\cdot)}(\mathbb{D})} ||w_j||_{W_0^{1,p(\cdot)}(\mathbb{D})} + \sum_{l=1}^{m} (G(t, u_m) w_l, w_j) d\bar{\beta}_l + \\
&+ \sum_{i=1}^{n} \left[\int_{\mathbb{D}} \left| \left| \sum_{k=1}^{m} C_{km}(t, \bar{\omega}) \frac{\partial w_k}{\partial x_i} \right|^{p(\cdot)-1} \right|^{r'} dx \right]^{1/r'} \left[\int_{D} \left| \frac{\partial w_j}{\partial x_i} \right|^{r} dx \right]^{1/r}
\end{aligned} \tag{3.35}$$

where $r' = \frac{r}{r-1}$ is the conjugate of $r = \operatorname{ess\,sup} p(\cdot)$.
Since $w_j \in W_0^{1,p(\cdot)}(\mathbb{D}) \subset W_0^{1,r}(\mathbb{D})$ and using the fact that the nonlinear terms f and G are jointly continuous and satisfy the nonlinear growth (3.20) and (3.21), we can estimate the right hand side of (3.35) in such a way that

$$\left|\overline{F}_j(t, \bar{\omega}, \overline{C})\right| \leqslant K(\alpha, m) M$$

uniformly on $\mathcal{G} \times \mathcal{H} \times \bar{\Omega}$, where $K(\alpha, m)$ and M are positive constants with M independent on i, j, k, m and α. Thus, under our conditions on f and G, one can apply the Skorohod existence result on stochastic ordinary differential equations (see, for instance [194, page 59]) to system (34) supplemented with its corresponding initial condition to ensure the existence of a distributional random continuous solution $C_{jm}(t, \bar{\omega})$ of system (34) on some closed interval $[0, t_m]$ where t_m is a positive number such that $t_m \leq T$. The uniform estimates for the functions u_m obtained below will imply that the stochastic processes $\{u_m, m \in \mathbb{N}\}$ exist on the entire interval $[0, T]$. Thus, the function $u_m(t, x, \bar{\omega}) = \sum_{j=1}^{m} C_{jm}(t, \bar{\omega}) w_j(x)$ is the desired Galerkin solution of system (3.31) and (3.32).

□

Using the a priori estimate obtained below for the local solution u_m constructed above, we must show that $t_m = T$ for any m and hence this will imply that the stochastic processes $\{u_m, m \in \mathbb{N}\}$ can be extended to the entire interval $[0, T]$.

3.4.2 A priori estimates for the approximate solutions

In this subsection we estimate the Galerkin approximate solutions u_m. To do that, we define a sequence of stopping times τ_k^m by

$$\tau_k^m = \begin{cases} \inf\{t \in [0, T] : \|u_m(t)\|_{L^2(\mathbb{D})} \geqslant k\}, \\ T \text{ if } \{t \in [0, T] : \|u_m(t)\|_{L^2(\mathbb{D})} \geqslant k\} = \emptyset \end{cases}$$

for any positive integer k.
Our first key estimate on the Galerkin approximate solutions u_m is

Lemma 13. *There exists a positive constant C such that for all m*

$$\bar{\mathbb{E}} \sup_{0\leqslant t\leqslant T} \|u_m(t)\|^2_{L^2(\mathbb{D})} \leqslant C, \tag{3.36}$$

$$\bar{\mathbb{E}}\, \|u_m\|^r_{\mathring{V}(Q_T)} \leqslant C, \tag{3.37}$$

where $r = ess\inf_{\mathbb{D}} p(\cdot)$ and $\bar{\mathbb{E}}$ denotes the mathematical expectation w.r.t. the probability measure $\bar{\mathbb{P}}$.

Proof. Let $\sigma \in [0,T]$, we set $\sigma_k = \sigma \wedge \tau_k^m$ where $a \wedge b = \min\{a,b\}$. Thanks to Itô's formula applied to the function $(u_m(\sigma), w_j)^2$, we deduce first from equation (3.31) by integrating it over $[0,\sigma]$ that for all $j = 1,\dots,m$,

$$\begin{aligned}(u_m(\sigma), w_j)^2 = -2\int_0^\sigma (u_m(\tau), w_j)\langle A(\tau,u_m), w_j\rangle d\tau + 2\int_0^\sigma (u_m(\tau), w_j)\langle f(\tau,u_m), w_j\rangle d\tau + \\ + (u_m(0), w_j)^2 + \sum_{l=1}^m \int_0^\sigma (G(\tau,u_m)w_l, w_j)^2\, d\tau + \\ + 2\sum_{l=1}^m \int_0^\sigma (u_m(\tau), w_j)\,(G(\tau,u_m)w_l, w_j)\, d\bar{\beta}_l(\tau).\end{aligned} \tag{3.38}$$

Now it follows using (3.38) (by summing the corresponding equation, (3.38) from $j = 1$ to $j = m$) that

$$\begin{aligned}||u_m(\sigma)||^2_{L^2(\mathbb{D})} = -2\int_0^\sigma \langle A(\tau,u_m), u_m(\tau)\rangle d\tau + 2\int_0^\sigma \langle f(\tau,u_m), u_m(\tau)\rangle d\tau + ||u_m(0)||^2_{L^2(\mathbb{D})} + \\ + \sum_{j=1}^m\sum_{l=1}^m \int_0^\sigma (G(\tau,u_m)w_l, w_j)^2\, d\tau + 2\sum_{l=1}^m \int_0^\sigma (G(\tau,u_m)w_l, u_m)\, d\bar{\beta}_l(\tau).\end{aligned} \tag{3.39}$$

We observe from relation (3.39), Hölder's inequality and the definition of the operator A that

$$\begin{aligned}\|u_m(\sigma)\|^2_{L^2(\mathbb{D})} + 2\int_0^\sigma \sum_{i=1}^n \int_D \left|\frac{\partial u}{\partial x_i}\right|^{p(\cdot)} dx d\tau \leqslant \\ \leqslant 2\|u_m\|_{\mathring{V}(Q_\sigma)} \|f(\tau,u_m)\|_{\left(\mathring{V}(Q_\sigma)\right)'} + \sum_{j=1}^m\sum_{l=1}^m \int_0^\sigma (G(\tau,u_m)w_l, w_j)^2\, d\tau + \\ + 2\sum_{l=1}^m \int_0^\sigma (G(\tau,u_m)w_l, u_m)\, d\bar{\beta}_l + \|u_m(0)\|^2_{L^2(\mathbb{D})}.\end{aligned} \tag{3.40}$$

We begin to estimate each member on the right hand side of (3.40). For that purpose, by virtue of the definition of the Hilbert-Schmidt norm, we have

$$\sum_{j,l=1}^{m}\int_0^{\sigma}\left(G(\tau,u_m)w_l,w_j\right)^2 d\tau \leqslant C\int_0^{\sigma}\sum_{j,l=1}^{m}|G(\tau,u_m)w_l|^2|w_j|^2 d\tau$$
$$\leq C\int_0^{\sigma}\|G(\tau,u_m)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}d\tau \tag{3.41}$$

Owing to Young's inequality, we get for an arbitrary $\varepsilon > 0$

$$\|u_m\|_{\mathring{V}(Q_\sigma)}\|f(\tau,u_m)\|_{\left(\mathring{V}(Q_\sigma)\right)'} \leqslant \varepsilon^r C\|u_m\|^r_{\mathring{V}(Q_\sigma)} + C_\varepsilon\|f(\tau,u_m)\|^{\frac{r}{r-1}}_{(\mathring{V}(Q_\sigma))'} \tag{3.42}$$

where, $r = \operatorname{ess\,inf}_{\mathbb{D}} p(\cdot) \geqslant 2$.
Next, we proceed to obtain an estimate for the integral in the left hand side of (3.40) containing the term $\left|\dfrac{\partial u}{\partial x_i}\right|^{p(\cdot)}$, $i = 1,\dots n$. For this purpose, we use the definitions of the functional ϱ_p and the norm $\|\cdot\|_{\mathring{V}(Q_\sigma)}$

$$\|u_m\|_{\mathring{V}(Q_\sigma)} = \sum_{i=1}^{n}\inf\left\{\lambda_i : \int_0^{\sigma}\varrho_p\left(\lambda_i^{-1}\nabla_i u_m\right)d\tau \leq 1\right\},$$

where, $\nabla_i = \dfrac{\partial}{\partial x_i}$ with $i = 1,\dots m$ and $m \in \mathbb{N}$.
It follows from this and the conditions of Lemma 7 that

$$\varrho_p\left((\nabla_i u_m)/\lambda_i\right) \leq 1 \text{ if and only if } \|(\nabla_i u_m)/\lambda_i\|_{L^{p(\cdot)}(\mathbb{D})} \leq 1.$$

This enables us to draw a consideration of two alternatives. For each $m \geq 1$ and $\sigma \in [0, t\wedge\tau_k^m]$, either $\|u_m\|_{\mathring{V}(Q_\sigma)} \leq 1$, or $\|u_m\|_{\mathring{V}(Q_\sigma)} > 1$.

For the first case i.e., $\|u_m\|_{\mathring{V}(Q_\sigma)} \leq 1$ implies that $\|u_m\|^r_{\mathring{V}(Q_\sigma)} \leqslant 1$ which along with (3.42) yields that

$$\|u_m(\sigma)\|^2_{L^2(\mathbb{D})} \leqslant 2\varepsilon^r C\|u_m\|^r_{\mathring{V}(Q_\sigma)} + C\int_0^{\sigma}\|G(\tau,u_m)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}d\tau$$
$$+ 2C_\varepsilon\|f(\tau,u_m)\|^{r/(r-1)}_{\left(\mathring{V}(Q_\sigma)\right)'} + 2\sum_{l=1}^{m}\int_0^{\sigma}\left(G(\tau,u_m)w_l,u_m\right)d\bar{\beta}_l. \tag{3.43}$$

If $\|u_m\|_{\mathring{V}(Q_\sigma)} > 1$, then we set

$$\delta = \int_0^{\sigma}\int_D\sum_{i=1}^{n}\left|\frac{\partial u_m}{\partial x_i}\right|^{p(x)}dxd\tau.$$

This Definition in combination with the result in Lemma 7 enable us to deduce that $\delta > 1$. The assumption on the growth $p(x)$ provides us with $\frac{1}{p(x)} \leqslant \frac{1}{r} < 1$ which its turn in combination with the fact that $\delta^{-1} < 1$ yield $\delta^{-1/r} \leqslant \delta^{-1/p(x)} < 1$. We have

$$1 = \frac{\sum_{i=1}^{n} \int_0^\sigma \varrho_p(\nabla_i u) d\tau}{\sum_{i=1}^{n} \int_0^\sigma \varrho_p(\nabla_i u) d\tau} = \sum_{i=1}^{n} \int_0^\sigma \varrho_p(\delta^{-1/p(x)} \nabla_i u) d\tau.$$

It follows from the fact that ϱ_p is monotone that

$$\int_0^\sigma \sum_{i=1}^{n} \varrho_p(\delta^{-1/r} \nabla_i u) d\tau \leqslant \int_0^\sigma \sum_{i=1}^{n} \varrho_p(\delta^{-1/p(x)} \nabla_i u) d\tau = 1.$$

This inequality, together with the definition of the norm in the space $\mathring{V}(Q_\sigma)$ enable us to deduce that

$$\|u_m\|_{\mathring{V}(Q_\sigma)} = \sum_{i=1}^{n} \left\| \frac{\partial u_m}{\partial x_i} \right\|_{L^{p(\cdot)}(0,\sigma;\mathbb{D})} \leq \delta^{1/r},$$

hence,

$$\|u_m\|^r_{\mathring{V}(Q_\sigma)} \leqslant \delta.$$

This implies that

$$\|u_m(\sigma)\|^2_{L^2(\mathbb{D})} + 2\|u_m\|^r_{\mathring{V}(Q_\sigma)} \leqslant \|u_m(\sigma)\|^2_{L^2(\mathbb{D})} + 2\int_0^\sigma \int_{\mathbb{D}} \sum_{i=1}^{n} \left| \frac{\partial u_m}{\partial x_i} \right|^{p(\cdot)} dx d\tau.$$

This in conjunction with (3.40) enables us to deduce that

$$\begin{aligned} &\|u_m(\sigma)\|^2_{L^2(\mathbb{D})} + 2\|u_m\|^r_{\mathring{V}(Q_\sigma)} \\ &\leqslant 2\varepsilon^r C \|u_m\|^r_{\mathring{V}((Q_\sigma)} + C\int_0^\sigma \|G(\tau, u_m)\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))} d\tau + \\ &+ 2C_\varepsilon \|f(\tau, u_m)\|^{r/(r-1)}_{\left(\mathring{V}(Q_\sigma)\right)'} + 2\sum_{l=1}^{m} \int_0^\sigma (G(\tau, u_m) w_l, u_m)\, d\bar{\beta}_l. \end{aligned} \tag{3.44}$$

It follows from the embedding result of Lemma 8 that $\mathring{V}(Q_\sigma) \hookrightarrow L^2(0, \sigma; \mathring{W}^{1,p(\cdot)}(\mathbb{D}))$ which implies that $L^2(0, \sigma; W^{-1,q(\cdot)}(\mathbb{D})) \hookrightarrow \left(\mathring{V}(Q_\sigma)\right)'$. Therefore,

$$||f(\tau, u_m)||_{\left(\mathring{V}(Q_\sigma)\right)'} \leqslant C||f(\tau, u_m)||_{L^2(0,\sigma;W^{-1,q(\cdot)}(\mathbb{D}))}$$

Since $r \geqslant 2$, its conjugate $r' = \dfrac{r}{r-1} \leqslant 2$ and we can apply Young's inequality to obtain

$$\begin{aligned} ||f(\tau, u_m)||^{r'}_{\left(\mathring{V}(Q_\sigma)\right)'} \leqslant C\int_0^\sigma \|f(\tau, u_m)\|^{r'}_{W^{-1,q(\cdot)}(\mathbb{D})} d\tau &\leqslant \sigma\varepsilon^r C + C_\varepsilon C \int_0^\sigma \|f(\tau, u_m)\|^2_{W^{-1,q(\cdot)}(\mathbb{D})} d\tau \\ &\leqslant \sigma\varepsilon^r C + C\int_0^\sigma \left[1 + \|u_m(\tau)\|_{L^2(\mathbb{D})}\right]^2 d\tau. \end{aligned} \tag{3.45}$$

Next, to deal with the estimate for the stochastic integral appearing in the right-hand side of (3.44) which requires the application of the Burkholder-Gundy-Davis inequality, we take the supremum over the interval $[0, \sigma_k]$ in both sides of (3.44), apply $\bar{\mathbb{E}}$ and tackle the corresponding inequality term by term. First, it follows from (3.21) that

$$\int_0^{\sigma_k} \|G(t, u_m(t))\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))}\, dt \leqslant C \int_0^{\sigma_k} \left[1 + \|u_m(t)\|_{L^2(\mathbb{D})}\right]^2 dt. \tag{3.46}$$

Using Burkholder-Gundy-Davis inequality, (3.46) and Young's inequality, we get that for any $\eta > 0$,

$$\begin{aligned}
&\bar{\mathbb{E}} \sup_{t\in[0,\sigma_k]} \left| \int_0^t \sum_{l=1}^m (G(\tau, u_m)w_l, u_m)\, d\bar{\beta}_l(\tau) \right| \\
&\leqslant C\bar{\mathbb{E}} \left[\int_0^{\sigma_k} \sum_{l=1}^m (G(t, u_m)w_l, u_m)^2\, dt \right]^{\frac{1}{2}} \\
&\leqslant C\bar{\mathbb{E}} \left[\int_0^{\sigma_k} \|G(t, u_m)\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))} \|u_m(t)\|^2_{L^2(\mathbb{D})} dt \right]^{\frac{1}{2}} \\
&\leqslant C\bar{\mathbb{E}} \sup_{t\in[0,\sigma_k]} \|u_m(t)\|_{L^2(\mathbb{D})} \left(\int_0^{\sigma_k} [1 + \|u_m(t)\|_{L^2(\mathbb{D})}]^2 dt \right)^{1/2} \\
&\leqslant \eta\bar{\mathbb{E}} \sup_{t\in[0,\sigma_k]} \|u_m(t)\|^2_{L^2(\mathbb{D})} + C_\eta \bar{\mathbb{E}} \int_0^{\sigma_k} [1 + \|u_m(t)\|_{L^2(\mathbb{D})}]^2 dt.
\end{aligned} \tag{3.47}$$

By appropriate choices of ε and η; and invoking (3.43) and (3.47) we obtain the following inequality

$$\bar{\mathbb{E}} \sup_{t\in[0,\sigma_k]} \|u_m(t)\|^2_{L^2(\mathbb{D})} + \bar{\mathbb{E}} \|u_m\|^r_{\overset{\circ}{V}(Q_{\sigma_k})} \leqslant TC + C\bar{\mathbb{E}} \int_0^{\sigma_k} \|u_m(t)\|^2_{L^2(\mathbb{D})} dt.$$

Combining with Gronwall's inequality, we deduce that for all $t \in [0, \sigma_k]$ and for all $m, k \geqslant 1$

$$\bar{\mathbb{E}} \sup_{t\in[0,\sigma_k]} \|u_m(t)\|^2_{L^2(\mathbb{D})} + \bar{\mathbb{E}} \|u_m\|^r_{\overset{\circ}{V}(Q_{\sigma_k})} \leqslant C, \tag{3.48}$$

where C is a positive constant independent of m.

We argue as in [6, page 317-318] and [165, page 37-38] to prove that, as $k \longrightarrow \infty$, $\sigma_k \nearrow T$, $\bar{\mathbb{P}}$-a.s. Passing to the limit in (3.48) as $k \longrightarrow \infty$, we deduce from the resulting relation, the conclusion of the desired proof. □

Thus, from inequalities (3.36) and (3.37) follows the existence of the Galerkin solutions u_m over the entire interval $[0, T]$.

Now, we derive the crucial estimate on the high integrability which is important in its own right.

Lemma 14. *Let $q \in (2, \infty)$. Then, there exists $C > 0$ such that for any $m \in \mathbb{N}$*

$$\bar{\mathbb{E}} \sup_{0 \leqslant t \leqslant T} \|u_m(t)\|^q_{L^2(\mathbb{D})} \leqslant C.$$

Proof. We set $X = \|u_m(\sigma)\|^2_{L^2(\mathbb{D})}$ and consider the function $\Psi(\sigma, x) = x^{\frac{q}{2}}$. We have that $\Psi_t(\sigma, x) = \frac{\partial \Psi(t,x)}{\partial t} = 0$, $\Psi'(\sigma, x) = \frac{q}{2} x^{\frac{q}{2}-1}$, and $\Psi''(\sigma, x) = \frac{q}{2}(\frac{q}{2} - 1)x^{\frac{q}{2}-2}$. Here the notation Ψ' stands for the derivative w.r.t. x of the function Ψ, Ψ_x and Ψ'' the second derivative, Ψ_{xx}. By Itô's formula applied to the function $\Psi(\sigma, X) = X^{\frac{q}{2}}$:

$$d\Psi(\sigma, X) = \Psi_\sigma(\sigma, X)d\sigma + \Psi'(\sigma, X)du_m + \frac{1}{2}\Psi''(\sigma, X)(du_m)^2.$$

It follows from this and (3.38) that

$$\begin{aligned}
&\|u_m(\sigma)\|^q_{L^2(\mathbb{D})} + C \int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})} \int_{\mathbb{D}} \sum_{i=1}^n \left|\frac{\partial u_m}{\partial x_i}\right|^{p(x)} dx d\tau \\
&\leqslant \|u_m(0)\|^q_{L^2(\mathbb{D})} + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})} \|f(\tau, u_m)\|_{W^{-1,q(\cdot)}(\mathbb{D})} \|u_m(\tau)\|_{W_0^{1,p(\cdot)}(\mathbb{D})} d\tau \\
&+ C \int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})} \sum_{l=1}^m (G(\tau, u_m)w_l, u_m)\, d\bar{\beta}_l(\tau) + C \int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})} \|G(\tau, u_m)\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))} d\tau \\
&+ C \int_0^\sigma \|u_m(\tau)\|^{q-4}_{L^2(\mathbb{D})} (G(\tau, u_m(\tau)), u_m(\tau))^2\, d\tau. \qquad (3.49)
\end{aligned}$$

By virtue of Lemma 8, we have the embedding $\mathring{V}(Q_\sigma) \subset L^2\left(0, \sigma; W_0^{1,p(\cdot)}(\mathbb{D})\right)$ which implies that there is a constant $C > 0$ such that

$$\|u_m\|_{L^2(0,\sigma;W_0^{1,p(\cdot)}(\mathbb{D}))} \leqslant C||u_m||_{\mathring{V}(Q_\sigma)}. \qquad (3.50)$$

Taking the square and mathematical expectation in both sides of (3.50) we get

$$\bar{\mathbb{E}} \int_0^\sigma ||u_m(t)||^2_{W_0^{1,p(\cdot)}(\mathbb{D})} d\sigma = \bar{\mathbb{E}}\|u_m\|^2_{L^2(0,\sigma;W_0^{1,p(\cdot)}(\mathbb{D}))} \leqslant C\bar{\mathbb{E}}||u_m||^2_{\mathring{V}(Q_\sigma)}. \qquad (3.51)$$

Since (3.51) is a particular case of (3.37), we deduce that

$$\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^2_V d\tau \leq C. \qquad (3.52)$$

We are interested in estimating members of the right hand side of (3.49). We set $V = \mathring{W}^{1,p(\cdot)}(\mathbb{D})$ and V' the dual space of $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$. on account of Assumption 29, the inequalities

(3.36), (3.37), (3.21), (3.52) and Young's inequality, we have

$$\begin{aligned}
&\bar{\mathbb{E}}\int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\|f(\tau,u_m)\|_{V'}\|u_m(\tau)\|_V d\tau\\
&\leqslant \bar{\mathbb{E}}\int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\left[\frac{r-1}{r}\varepsilon^{-\frac{r}{r-1}}\|f(\tau,u_m)\|^{\frac{r}{r-1}}_{V'}+\frac{1}{r}\varepsilon^r\|u_m(\tau)\|^r_V\right]d\tau\\
&\leqslant \frac{r-1}{r\varepsilon^{\frac{r}{r-1}}}C\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\int_0^\sigma \|f(\tau,u_m)\|^{\frac{r}{r-1}}_{V'}d\tau+\frac{1}{r}\varepsilon^r\bar{\mathbb{E}}\int_0^\sigma\|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\|u_m(\tau)\|^r_V d\tau\\
&\leqslant \frac{r-1}{r\varepsilon^{\frac{r}{r-1}}}C\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\int_0^\sigma [1+\|u_m(\tau)\|^2_{L^2(\mathbb{D})}]d\tau+\frac{1}{r}\varepsilon^r\bar{\mathbb{E}}\int_0^\sigma\|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\|u_m(\tau)\|^r_V d\tau\\
&\leqslant \sigma C+C\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q}_{L^2(\mathbb{D})}+\frac{1}{r}\varepsilon^r\bar{\mathbb{E}}\int_0^\sigma\|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\|u_m(\tau)\|^r_V d\tau; \qquad (3.53)
\end{aligned}$$

since

$$\bar{\mathbb{E}}\int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}d\tau \leqslant C(q,\sigma)\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\leqslant C(T)\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q}_{L^2(\mathbb{D})}.$$

We set $H=L^2(\mathbb{D})$ for the simplicity of notation. It follows from (3.21) that

$$\begin{aligned}
\bar{\mathbb{E}}\int_0^\sigma \|u_m(\tau)\|^{q-4}_H\left(G(\tau,u_m),u_m\right)^2d\tau &\leqslant \bar{\mathbb{E}}\int_0^\sigma \|u_m(\tau)\|^{q-2}_H\|G(\tau,u_m)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}d\tau\\
&\leqslant \bar{\mathbb{E}}\int_0^\sigma \|u_m(\tau)\|^{q-2}_H\left[1+\|u_m\|^2_H\right]d\tau\\
&\leqslant \sigma C\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q-2}_H+C\bar{\mathbb{E}}\int_0^\sigma\|u_m(\tau)\|^q_H d\tau\\
&\leqslant C(q,T)\bar{\mathbb{E}}\sup_{\tau\in[0,\sigma]}\|u_m(\tau)\|^{q}_H+C\bar{\mathbb{E}}\int_0^\sigma\|u_m(\tau)\|^q_H d\tau.
\end{aligned} \qquad (3.54)$$

For the stochastic term, we use the Burkholder-Davis-Gundy inequality and (3.54) to get

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{\sigma\in[0,T]}\left|\int_0^\sigma \|u_m(\tau)\|^{q-2}_{L^2(\mathbb{D})}\sum_{l=1}^m\left(G(\tau,u_m)w_l,u_m\right)d\bar{\beta}_l(\tau)\right|\\
&\leqslant C\bar{\mathbb{E}}\left[\int_0^T\|u_m(\tau)\|^{2(q-2)}_{L^2(\mathbb{D})}\left[\sum_{l=1}^m(G(\tau,u_m)w_l,u_m)\right]^2d\tau\right]^{1/2}\\
&\leqslant C\bar{\mathbb{E}}\left[\int_0^T\|u_m(\tau)\|^{2(q-2)}_{L^2(\mathbb{D})}\|G(\tau,u_m)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}\|u_m(\tau)\|^2_{L^2(\mathbb{D})}d\tau\right]^{1/2}\\
&\leqslant C\bar{\mathbb{E}}\left[\int_0^T\left(\|u_m(\tau)\|^{2(q-1)}_{L^2(\mathbb{D})}+\|u_m(\tau)\|^{2q}_{L^2(\mathbb{D})}\right)d\tau\right]^{1/2}\\
&\leqslant CT\bar{\mathbb{E}}\sup_{\sigma\in[0,T]}\|u_m(\sigma)\|^{q-1}_{L^2(\mathbb{D})}+CT\bar{\mathbb{E}}\sup_{\sigma\in[0,T]}\|u_m(\sigma)\|^{q}_{L^2(\mathbb{D})}. \qquad (3.55)
\end{aligned}$$

Therefore, it follows by Young's inequality that

$$\bar{\mathbb{E}} \sup_{\sigma\in[0,T]} \int_0^\sigma \|u_m(\tau)\|_{L^2(\mathbb{D})}^{q-2} \sum_{l=1}^m (G(\tau,u_m)w_l,u_m)\, d\bar{\beta}_l \leqslant C(q,T)\bar{\mathbb{E}} \sup_{\sigma\in[0,T]} \|u_m(\sigma)\|_{L^2(\mathbb{D})}^q. \tag{3.56}$$

Combining the previous estimates of Lemma 18 with (3.52)-(3.56), we deduce from (3.49) that

$$\begin{aligned}
&\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_H^q + \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^{q-2} \sum_{i=1}^n \varrho_{p(\cdot)}(\nabla_i u_m) d\tau \\
&\leqslant C\left(\|u_0\|_{L^2(\mathbb{D})}^q, T\right) + C(T)\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_{L^2(\mathbb{D})}^q + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^q d\tau \\
&+ \frac{1}{r}\varepsilon^r \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_{L^2(\mathbb{D})}^{q-2} \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^r d\tau.
\end{aligned} \tag{3.57}$$

Similarly as in the proof of the above Lemma, we have either $\|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} \leqslant 1$ or $\|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} > 1$. If $\|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} \leqslant 1$ then

$$\|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^s \leqslant \sum_{i=1}^n \varrho_{p(\cdot)}(\nabla_i u_m(\tau)) \leqslant \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^r \leqslant 1.$$

Hence, it follows from this, (3.57) and Young's inequality that

$$\begin{aligned}
\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_H^q \leqslant &\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_H^q + \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^{q-2} \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^s d\tau \\
&\leqslant C\left(\|u_0\|_H^q, T\right) + C(T)\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_H^q + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^q d\tau \\
&+ \frac{1}{r}\varepsilon^r \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^{q-2} \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^r d\tau.
\end{aligned} \tag{3.58}$$

Otherwise $\|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} > 1$ then

$$1 < \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^r \leqslant \sum_{i=1}^n \varrho_{p(\cdot)}(\nabla_i u_m(\tau)) \leqslant \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^s.$$

In this case we have

$$\begin{aligned}
&\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_H^q + \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^{q-2} \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^r d\tau \\
&\leqslant C\left(\|u_0\|_H^q, T\right) + C(T)\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|_H^q + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^q d\tau \\
&+ \frac{1}{r}\varepsilon^r \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|_H^{q-2} \|u_m(\tau)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}^r d\tau.
\end{aligned} \tag{3.59}$$

Using similar reasoning as in the proof of the previous lemma, we deduce that

$$\min\left\{||u_m||^r_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}, ||u_m||^s_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}\right\} \leqslant \max\left\{||u_m||^r_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}, ||u_m||^s_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}\right\} \tag{3.60}$$

where $r = \operatorname{ess\,inf}_{\mathbb{D}} p(\cdot)$ and $s = \operatorname{ess\,sup}_{\mathbb{D}} p(\cdot)$ with r, s satisfying condition (3.22). Combining (3.58) with (3.59) and on account of (3.60) we obtain

$$\begin{aligned} &2\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|^q_H + k_0\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^{q-2}_H \|u_m(\tau)\|^r_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\tau \\ &\leqslant C + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^q_H d\tau + \frac{1}{r}\varepsilon^r C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^{q-2}_H \|u_m(\tau)\|^r_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\tau. \end{aligned} \tag{3.61}$$

We choose a constant $\varepsilon > 0$ in (3.61) so that $C_r\varepsilon^r < k_0$ and hence

$$\begin{aligned} &\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|^q_H + \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^{q-2}_H \|u_m(\tau)\|^r_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\tau \\ &\leqslant C + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^q_H d\tau. \end{aligned} \tag{3.62}$$

Since $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ is embedded in $\mathring{W}^{1,r}(\mathbb{D})$, we also have

$$\begin{aligned} &\bar{\mathbb{E}} \sup_{\tau\in[0,\sigma]} \|u_m(\tau)\|^q_H + \bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^{q-2}_H \|u_m(\tau)\|^r_{\mathring{W}^{1,r}(\mathbb{D})} d\tau \\ &\leqslant C + C\bar{\mathbb{E}} \int_0^\sigma \|u_m(\tau)\|^q_H d\tau. \end{aligned} \tag{3.63}$$

Finally, an application of Gronwall's inequality completes the proof of Lemma 14. □

We state and prove an improvement of (3.52) in the following result.

Lemma 15. *Assume that r satisfies (3.22). Then there exists a constant $C > 0$ such that*

$$\bar{\mathbb{E}}\left(\int_0^T \|u_m(\sigma)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\sigma\right)^{\frac{r}{2}} \leqslant C.$$

Proof. It follows from the embedding in Lemma 8 that

$$\int_0^T \|u_m(\sigma)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\sigma \leqslant C\|u_m\|^2_{V(Q)}.$$

Raising this inequality to the power $\frac{r}{2}$ for any $r \geqslant 2$ and taking mathematical expectation leads to

$$\bar{\mathbb{E}}\left(\int_0^T \|u_m(\sigma)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\sigma\right)^{\frac{r}{2}} \leqslant C\bar{\mathbb{E}}\left(\|u_m\|^2_{V(Q)}\right)^{\frac{r}{2}} = C\bar{\mathbb{E}}\|u_m\|^r_{V(Q)},$$

where $r = \operatorname{ess\,inf} p(\cdot)$ satisfies condition (3.22).
It follows from the previous estimates (3.37) and (3.52) that

$$\bar{\mathbb{E}}\left(\int_0^T \|u_m(\sigma)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} d\sigma\right)^{\frac{r}{2}} \leqslant C.$$

This proves the desired result. □

Lemma 16. *For any $\delta \in [0,1)$, we have*

$$\bar{\mathbb{E}} \sup_{0\leqslant|\theta|\leqslant\delta} \int_0^T \|u_m(t+\theta)-u_m(t)\|^2_{V'}\,dt \leqslant C\delta.$$

Proof. We go back to (3.31), and consider the fact that for any $t \in (0,T)$ we have w_j, $u_m(t) \in H_m \subset W_0^{1,p(\cdot)}(\mathbb{D}) \subset W^{-1,q(\cdot)}(\mathbb{D})$. Let $\theta \geqslant 0$; we extend u_m by zero outside the interval $[0,T]$. The identity (3.31) can be expressed in integral form as an equality between random variables with values in V'. Let $P_m : W^{-1,q(\cdot)}(\mathbb{D}) \longrightarrow H_m$ denotes the projection of $W^{-1,q(\cdot)}(\mathbb{D})$ onto H_m defined by

$$P_m v = \sum_{j=1}^m \langle v, w_j\rangle w_j \;\; \forall v \in W^{-1,q(\cdot)}(\mathbb{D}).$$

Moreover, we deduce that

$$u_m(t+\theta)-u_m(t) = -\int_t^{t+\theta} P_m A(\sigma,u_m)d\sigma + \int_t^{t+\theta} P_m f(\sigma,u_m)d\sigma + \int_t^{t+\theta} \pi_m G(\sigma,u_m)d\bar{W},$$

where the integrands are understood to be zeros for $t+\theta > T$ and $\pi_m G(t,u_m)d\bar{W} = \sum_{l=1}^m \left(G(t,u_m)d\bar{\beta}_l, w_j\right) w_l$.
We proceed by first setting

$$y_t(\theta) = \|u_m(t+\theta)-u_m(t)\|_{V'}.$$

We have

$$y_t(\theta) \leqslant \left\|\int_t^{t+\theta} P_m A(\sigma,u_m)d\sigma\right\|_{V'} + \left\|P_m\int_t^{t+\theta} f(\sigma,u_m)d\sigma\right\|_{V'} + \left\|\int_t^{t+\theta} \pi_m(G(\sigma,u_m)d\bar{W}\right\|_{V'}. \tag{3.64}$$

Since A is bounded from $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ to $W^{-1,q(\cdot)}(\mathbb{D})$, we have by Fubini's theorem and Hö lder's inequality

$$
\begin{aligned}
\left\|\int_t^{t+\theta} P_m A(\sigma,u_m)d\sigma\right\|_{V'} &= \sup_{\varphi\in V:\|\varphi\|_V=1}\int_{\mathbb{D}}\left(\int_t^{t+\theta} P_m A(\sigma,u_m)\varphi(x)d\sigma\right)dx\\
&\leqslant \int_t^{t+\theta}\|P_m A(\sigma,u_m)\|_{W^{-1,q(\cdot)}(\mathbb{D})}d\sigma\\
&\leqslant C\int_t^{t+\theta}\|u_m(\sigma)\|_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}d\sigma\\
&\leqslant C\theta^{1/2}\left(\int_t^{t+\theta}\|u_m(\sigma)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}\right)^{1/2}. \qquad (3.65)
\end{aligned}
$$

We have by Hölder's inequality

$$
\begin{aligned}
\left\|P_m\int_t^{t+\theta} f(\sigma,u_m)d\sigma\right\|_{W^{-1,q(\cdot)}(\mathbb{D})} &\leqslant \int_t^{t+\theta}\|f(\sigma,u_m)\|_{W^{-1,q(\cdot)}(\mathbb{D})}d\sigma\\
&\leqslant C\theta^{1/2}\left(\int_t^{t+\theta}\|f(\sigma,u_m)\|^2_{W^{-1,q(\cdot)}(\mathbb{D})}d\sigma\right)^{1/2}. \qquad (3.66)
\end{aligned}
$$

By taking the square to both sides of relation (3.64) using the elementary inequality $(a+b+c)^2\leqslant 3(a^2+b^2+c^2)$ for any $a>0$, $b>0$, $c>0$ we deduce now using the inequalities (3.65) and (3.66) that

$$
y_t^2(\theta)\leqslant C\theta\int_t^{t+\theta}\|u_m(\sigma)\|_V^2 d\sigma + C\theta\int_t^{t+\theta}\|f(\sigma,u_m)\|^2_{V'}d\sigma + \left\|\int_t^{t+\theta}\pi_m G(\sigma,u_m)d\bar{W}\right\|^2_{V'}.
$$

We fix $\delta<1$ and take the supremum over the interval $\theta\leqslant\delta$ to obtain

$$
\begin{aligned}
\sup_{\theta\leqslant\delta}[y_t(\theta)]^2 \leqslant CT\delta\int_t^{t+\theta}\|u_m(\sigma)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}d\sigma + C\delta\sup_{\theta\leqslant\delta}\int_t^{t+\delta}\|f(\sigma,u_m)\|^2_{V'}d\sigma\\
+\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}\pi_m G(\sigma,u_m)d\bar{W}(\sigma)\right\|^2_{V'}. \qquad (3.67)
\end{aligned}
$$

Integrating this inequality over the interval $[0,T]$ (with $u(t+\theta)=0$ for $t+\theta\notin[0,T]$) and taking mathematical expectation we get

$$
\begin{aligned}
&\bar{\mathbb{E}}\sup_{\theta\leqslant\delta}\int_0^T\|u_m(t+\theta)-u_m(t)\|^2_{V'}dt\\
&\leqslant C\delta\bar{\mathbb{E}}\int_0^T\left(\sup_{\theta\leqslant\delta}\int_t^{t+\delta}\|f(\sigma,u_m)\|^2_{V'}d\sigma\right)dt\\
&+CT\delta\bar{\mathbb{E}}\int_0^T\int_t^{t+\theta}\|u_m(\sigma)\|^2_V d\sigma + \bar{\mathbb{E}}\int_0^T\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}\pi_m G(\sigma,u_m)d\bar{W}\right\|^2_{V'}. \qquad (3.68)
\end{aligned}
$$

We estimate the first term in the right-hand side of (3.68). We have

$$\bar{\mathbb{E}}\int_0^T\left(\int_t^{t+\delta}\|f(\sigma,u_m)\|_{V'}^2d\sigma\right)dt\leqslant CT+C\bar{\mathbb{E}}\int_0^T\|u_m(t)\|_{L^2(\mathbb{D})}^2dt. \tag{3.69}$$

Thanks to (3.52), we have

$$\bar{\mathbb{E}}\int_0^T\left(\int_t^{t+\delta}\|u_m(\sigma)\|_{W_0^{1,p(\cdot)}(\mathbb{D})}^2d\sigma\right)dt\leqslant C\bar{\mathbb{E}}\int_0^T\|u_m(t)\|_{W_0^{1,p(\cdot)}(\mathbb{D})}^2dt\leqslant C.$$

Next, the crucial part is to control the stochastic integral appearing in the last term on the right-hand side of (3.68). To this end, we use martingale's inequality. We have by (3.21) and Fubini's Theorem

$$\bar{\mathbb{E}}\int_0^T\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}\pi_mG(\sigma,u_m)d\bar{W}\right\|_{V'}^2dt\leqslant\int_0^T\bar{\mathbb{E}}\int_t^{t+\delta}\|G(\sigma,u_m)\|_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}^2d\sigma dt.$$

Owing to (3.21), we obtain

$$\begin{aligned}\bar{\mathbb{E}}\int_0^T\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}\pi_mG(\sigma,u_m)d\bar{W}\right\|_{V'}^2dt&\leqslant C\int_0^T\bar{\mathbb{E}}\int_t^{t+\delta}\left[1+\|u_m(\sigma)\|_{L^2(\mathbb{D})}\right]^2d\sigma dt\\&\leqslant T\delta C+C\int_0^T\bar{\mathbb{E}}\int_t^{t+\delta}\|u_m(\sigma)\|_{L^2(\mathbb{D})}^2d\sigma dt\\&\leqslant T\delta C+CT\delta\bar{\mathbb{E}}\sup_{\sigma\in[0,t]}\|u_m(\sigma)\|_{L^2(\mathbb{D})}^2.\end{aligned}$$

Taking into account the estimate (3.36), we deduce that

$$\bar{\mathbb{E}}\int_0^T\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}G(\sigma,u_m(\sigma))d\bar{W}(\sigma)\right\|_{W^{-1,q(\cdot)}(\mathbb{D})}^2dt\leqslant C\delta.$$

We gain by combining all the above estimate and using (3.67) that

$$\bar{\mathbb{E}}\int_0^T[y_t(\theta)]^2dt\leqslant C(T)\delta+C(T)\delta\bar{\mathbb{E}}\sup_{t\in[0,T]}||u_m(t)||_{L^2(\mathbb{D})}^2 \tag{3.70}$$

from which we deduce by taking into account the estimate (3.36), and combining all the above inequalities that

$$\bar{\mathbb{E}}\sup_{\theta\leqslant\delta}\int_0^T\|u_m(t+\theta)-u_m(t)\|_{W^{-1,q(\cdot)}(\mathbb{D})}^2dt\leqslant C\delta.$$

Finally, collecting all the estimates and making a similar reasoning with $\theta<0$, we thus deduce that

$$\bar{\mathbb{E}}\sup_{|\theta|\leqslant\delta}\int_0^T\|u_m(t+\theta)-u_m(t)\|_{W^{-1,q(\cdot)}(\mathbb{D})}^2dt\leqslant C\delta.$$

Thus the proof of the Lemma is complete.

□

As a result of the proof of Lemma 16, we state and prove the

Remark 11. It holds that

$$\mathbb{E}\sup_{|\theta|\leqslant\delta}\|u_m(t+\theta)-u_m(t)\|_{V'}<C\delta\ \forall t\in[0,T].$$

Proof. Since the space $L^2(0,T;W^{-1,q(\cdot)}(\mathbb{D}))$ is continuously embedded in $L^1(0,T;W^{-1,q(\cdot)}(\mathbb{D}))$. On one hand, we have by Cauchy-Schwarz's inequality and Lemma 4.7

$$\int_0^T ||u_m(t+\theta)-u_m(t)||_{W^{-1,q(\cdot)}(\mathbb{D})}dt\leqslant\sqrt{T}||\tau_\theta u_m-u_m||_{L^2(0,T;W^{-1,q(\cdot)}(\mathbb{D}))}\leqslant C\sqrt{T}\sqrt{\delta}.$$

where $\tau_\theta u_m$ stands for the translation function of u_m with $u_m(t+\theta)=0$ for $t+\theta\notin[0,T]$ and $\theta\geqslant 0$. On the other hand Young's inequality implies

$$\sup_{\theta\leqslant\delta}||\tau_\theta u_m(t)-u_m(t)||_{W^{-1,q(\cdot)}(\mathbb{D})}\leqslant\frac{\varepsilon^2}{2}+C_\varepsilon\sup_{\theta\leqslant\delta}||\tau_\theta u_m(t)-u_m(t)||^2_{W^{-1,q(\cdot)}(\mathbb{D})}.$$

Now, making use of (3.67) gives

$$\begin{aligned}\sup_{\theta\leqslant\delta}||\tau_\theta u_m(t)-u_m(t)||_{W^{-1,q(\cdot)}(\mathbb{D})}\leqslant CT\delta\int_t^{t+\theta}\|u_m(\sigma)\|_V^2d\sigma+C\delta\sup_{\theta\leqslant\delta}\int_t^{t+\theta}\|f(\sigma,u_m)\|^2_{V'}d\sigma\\+\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}\pi_mG(\sigma,u_m)d\bar{W}(\sigma)\right\|^2_{V'}\end{aligned}\tag{3.71}$$

since u is extended by zero outside $[0,T]$.
It follows using (3.20) that

$$\begin{aligned}\sup_{\theta\leqslant\delta}\int_t^{t+\theta}||f(\sigma,u_m)||^2_{W^{-1,q(\cdot)}(\mathbb{D})}d\sigma&\leqslant C\sup_{\theta\leqslant\delta}\int_t^{t+\theta}(1+||u_m(\sigma)||_{L^2(\mathbb{D})})^2d\sigma\\&\leqslant TC\delta+C\delta\sup_{0\leqslant t\leqslant T}||u_m(\sigma)||^2_{L^2(\mathbb{D})}.\end{aligned}\tag{3.72}$$

First, we take mathematical expectation in both sides of (3.71) and we estimate the corresponding terms in the right hand side of (3.71). Thus, using the result of Lemma 15, we obtain in particular

$$\bar{\mathbb{E}}\int_t^{t+\theta}\|u_m(\sigma)\|_V^2d\sigma\leqslant\bar{\mathbb{E}}\int_0^T\|u_m(\sigma)\|_V^2d\sigma\leqslant C.\tag{3.73}$$

Using (3.21), we deduce according to Burkholder-Davis-Gundy's inequality that

$$\begin{aligned}\bar{\mathbb{E}}\sup_{\theta\leqslant\delta}\left\|\int_t^{t+\theta}\pi_m G(\sigma,u_m)d\bar{W}\right\|_{V'}^2 &\leqslant \bar{\mathbb{E}}\int_0^T \|G(\sigma,u_m)\|_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}^2 d\sigma \\ &\leqslant C\bar{\mathbb{E}}\int_0^T \left[1+\|u_m(\sigma)\|_{L^2(\mathbb{D})}\right]^2 d\sigma \\ &\leqslant T\delta C + C\bar{\mathbb{E}}\int_0^T \|u_m(\sigma)\|_{L^2(\mathbb{D})}^2 d\sigma \\ &\leqslant T\delta C + CT\delta\bar{\mathbb{E}}\sup_{\sigma\in[0,t]}\|u_m(\sigma)\|_{L^2(\mathbb{D})}^2. \end{aligned} \tag{3.74}$$

Finally, combining estimates (3.72), (3.73) and (3.74), we deduce from (3.71) that

$$\bar{\mathbb{E}}\sup_{\theta\leqslant\delta}||\tau_\theta u_m(t)-u_m(t)||_{W^{-1,q(\cdot)}(\mathbb{D})} \leqslant T\delta C + CT\delta\bar{\mathbb{E}}\sup_{\sigma\in[0,t]}\|u_m(\sigma)\|_{L^2(\mathbb{D})}^2 \tag{3.75}$$

from which we deduce using (3.36) that

$$\bar{\mathbb{E}}\sup_{\theta\leqslant\delta}||\tau_\theta u_m(t)-u_m(t)||_{W^{-1,q(\cdot)}(\mathbb{D})} \leqslant C(T)\delta.$$

We argue similarly for the case $\theta<0$ and thus we obtain

$$\bar{\mathbb{E}}\sup_{|\theta|\leqslant\delta}||\tau_\theta u_m(t)-u_m(t)||_{W^{-1,q(\cdot)}(\mathbb{D})} \leqslant C(T)\delta.$$

Hence follows the proof of Remark 11.

□

3.4.3 Compactness results and tightness criterion

Following Bensoussan [21], we have

Proposition 2. *Let μ_n and ν_n be both sequences of positive real numbers such that $\mu_n,\nu_n\to 0$ as $n\to\infty$. Then*

$$\mathcal{W}_{\mu_n,\nu_n}=\left\{z\in \left| \begin{array}{c} L^2\left(0,T;\mathring{W}^{1,p(\cdot)}(\mathbb{D})\right)\cap L^\infty\left(0,T;L^2(\mathbb{D})\right) \\ \sup_n \dfrac{1}{\nu_n}\sup_{|\theta|\leqslant\mu_n}\left[\int_0^T\|z(t+\theta)-z(t)\|_{W^{-1,q(\cdot)}(\mathbb{D})}^2\,dt\right]^{1/2}<\infty \end{array}\right.\right\}$$

is compactly embedded in $L^2(0,T;L^2(\mathbb{D}))$.

Proof. Since the embedding $\mathring{W}^{1,p(\cdot)}(\mathbb{D})\subset L^2(\mathbb{D})$ is compact, the above proposition is a corollary of [21, Proposition 3.1]. □

Let $\mathcal{Z}_{\mu_n,\nu_n}$ be the set of adapted stochastic processes z such that

$$\mathbb{E}\left[\int_0^T \|z(t)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}dt\right]^{1/2} < \infty, \left[\mathbb{E}\sup_{0\leqslant t\leqslant T}\|z(t)\|^q_{L^2(\mathbb{D})}\right]^{1/q} < \infty;$$

$$\mathbb{E}\sup_n \frac{1}{\nu_n}\sup_{|\theta|\leqslant\mu_n}\left[\int_0^T \|z(t+\theta)-z(t)\|^2_{W^{-1,q(\cdot)}(\mathbb{D})}\,dt\right]^{1/2} < \infty.$$

We extend the process z by zero outside $[0,T]$.

The space $\mathcal{Z}_{\mu_n,\nu_n}$ is a Banach space under the norm

$$\|z(t)\|_{\mathcal{Z}_{\mu_n,\nu_n}} = \mathbb{E}\left[\int_0^T \|z(t)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}dt\right]^{1/2} + \left[\mathbb{E}\sup_{0\leqslant t\leqslant T}\|z\|^q_{L^2(\mathbb{D})}\right]^{1/q} + \\ + \mathbb{E}\sup_n \frac{1}{\nu_n}\sup_{|\theta|\leqslant\mu_n}\left[\int_0^T \|z(t+\theta)-z(t)\|^2_{W^{-1,q(\cdot)}(\mathbb{D})}\,dt\right]^{1/2}.$$

The $\grave{a}$ priori estimates established in the previous Lemmata allow us to assert that for any $q\in[2,\infty)$, and for μ_n, ν_n such that the series $\sum_{n=1}^{\infty}\frac{\sqrt{\mu_n}}{\nu_n}$ converges, the Galerkin solutions $\{u_m : m\in\mathbb{N}\}$ remain in a bounded subset of $\mathcal{Z}_{\mu_n,\nu_n}$, thanks to the embedding $\mathring{V}(Q)\hookrightarrow L^2(0,T;\mathring{W}^{1,p(\cdot)}(\mathbb{D}))$.

Next, we state the tightness property of Π_m generated by the Galerkin solutions u_m and the Wiener process $\bar{W}$. In order to do so, we set

$$\mathcal{S} = C([0,T];\mathbb{K})\times L^2(0,T;L^2(\mathbb{D}))\cap C([0,T];V'),$$

We define the mapping

$$\bar{\phi_m} : \bar{\Omega} \longrightarrow \mathcal{S} : \bar{\omega}\mapsto \left(\bar{W}(.,\bar{\omega}), u_m(.,\bar{\omega})\right).$$

For each $m\geqslant 1$, we set

$$\Pi_m(\mathbb{A}) = \bar{\mathbb{P}}(\phi_m^{-1}(\mathbb{A})), \forall\mathbb{A}\in\mathcal{B}(\mathcal{S}).$$

Having constructed the needed probability measure Π_m; we now discuss the main result of this subsection concerning the tightness of the family Π_m. This is achieved thanks to similar arguments in [166]. Thus, we state and prove the tightness in the

Theorem 32. *The set of laws $\{\Pi_m, m\geqslant 1\}$ on the space $\mathcal{S}$ is tight in $(\mathcal{S},\mathcal{B}(\mathcal{S}))$.*

In order to prove this theorem, we first need to prove that the family of laws of $(u_m)_{m\geqslant 1}$ is tight in $L^2(0,T;L^2(\mathbb{D}))\cap C([0,T];V')$. For this purpose, we define the mappings $\bar{\Psi}_m$: $\bar{\Omega}\longrightarrow\mathcal{S}_1 : \bar{\omega}\mapsto u_m(\cdot,\bar{\omega})$. By similar reasoning as above, for each $m\in\mathbb{N}$, we set

$$\Pi_{1,m}(\mathbb{A}_1) = \mathcal{L}(u_m)(\mathbb{A}_1) = \bar{\mathbb{P}}\left(\bar{\Psi}_m^{-1}(\mathbb{A}_1)\right), \forall\mathbb{A}_1\in\mathcal{B}(\mathcal{S}_1).$$

As an auxiliary step in the proof of Theorem 32, we prove the following tightness property.

Lemma 17. *The family* $\Pi_{1,m}, m \in \mathbb{N}$ *is tight in* $(\mathcal{S}_1, \mathcal{B}(\mathcal{S}_1))$.

Proof. A crucial role is played by the high integrability in Lemma 14 . The Galerkin solution is a subset of $\mathcal{Z}_{\mu_n,\nu_n}$ for an appropriate choice of μ_n and ν_n such that the series $\sum_{n=1}^{\infty} \dfrac{\sqrt{\mu_n}}{\nu_n}$ converges. For instance, we can conveniently choose the sequences μ_n and ν_n by

$$\mu_n = \frac{1}{n^p}, \nu_n = \frac{1}{n} \; with \; p > 4;$$

clearly we note that if $\frac{p}{2} - 1 > 1$, we have

$$\sum_{n\in\mathbb{N}} \frac{\sqrt{\mu_n}}{\nu_n} = \sum_{n\in\mathbb{N}} \frac{1}{n^{\frac{p}{2}-1}} < \infty.$$

Let $\varepsilon > 0$, we set

$$\tilde{\Omega}_\varepsilon = \left\{ \bar{\omega} : u_m(\bar{\omega}) \in B_{\mathcal{W}_{\mu_n,\nu_n}}(L_\varepsilon) \right\}$$

where L_ε is a constant to be chosen later and $B_{\mathcal{W}_{\mu_n,\nu_n}}(L_\varepsilon)$ is the ball in the space $\mathcal{W}_{\mu_n,\nu_n}$ of radius L_ε and centered at the origin. By Chebychev's inequality, we infer that for any $m \in \mathbb{N}$

$$\bar{\mathbb{P}}(\bar{\omega} \notin \tilde{\Omega}_\varepsilon) \leqslant \frac{1}{L_\varepsilon^2} \bar{\mathbb{E}} \|u_m\|^2_{\mathcal{W}_{\mu_n,\nu_n}} \leqslant \frac{C}{L_\varepsilon^2}.$$

Let us choose $L_\varepsilon = \sqrt{\frac{C}{\varepsilon}}$ so that

$$\bar{\mathbb{P}}(\bar{\omega} \notin \tilde{\Omega}_\varepsilon) \leqslant \varepsilon.$$

Since the ball $B_{\mathcal{W}_{\mu_n,\nu_n}}(L_\varepsilon)$ is a compact set of $L^2(0,T;L^2(\mathbb{D}))$, we thus deduce the tightness in $L^2(0,T;L^2(\mathbb{D}))$. We must now prove the tightness in $C([0,T];V')$, where V' stands for $W^{-1,q(\cdot)}(\mathbb{D})$, the dual of $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$.

Since the embedding $L^2(\mathbb{D}) \subset W^{-1,q(\cdot)}(\mathbb{D})$ is continuous, it follows from

$$\bar{\mathbb{P}}(\bar{\omega} : \sup_{0\leqslant t\leqslant T} \|u_m(t,\bar{\omega})\|_{L^2(\mathbb{D})} > R) \leqslant \mathbb{E} \sup_{0\leqslant t\leqslant T} \|u_m(t,\bar{\omega})\|_{L^2(\mathbb{D})} > R) < \frac{C}{R^2}$$

that $\{u_m(t) : m \in \mathbb{N}\}$ is bounded in $W^{-1,q(\cdot)}(\mathbb{D})$. We thus easily prove that $\{u_m(t) : m \in \mathbb{N}\}$ is relatively compact in $W^{-1,q(\cdot)}(\mathbb{D})$. Moreover, by the conclusion of [192, Theorem 1, page 71] the time criterion results of Lemma 19 and its Remark 11 are the same as the uniform equicontinuity of Arzela-Ascoli. Using the Arzela-Ascoli characterization of compact sets in $C(0,T,W^{-1,q(\cdot)}(\mathbb{D}))$ (see e.g.,[192, Lemma 1, page 71]), it is therefore equivalent to say that $\{ u_m : m \in \mathbb{N}\}$ is relatively compact in $C(0,T,W^{-1,q(\cdot)}(\mathbb{D}))$. Hence we claim that $\Pi_{1,m}$ is tight on $C([0,T];W^{-1,q(\cdot)}(\mathbb{D}))$. Thus, we conclude that the laws of the family $\{u_m, m \in \mathbb{N}\}$ is tight on $\mathcal{S}_1$. This proves Lemma 17. $\square$

We now proceed with the proof of Theorem 32.

Proof. Following similar ideas as in [166], we note that there exists a negligible (null) subset $\bar{\Omega}'$ of $\bar{\Omega}$ such that $\bar{W}(\cdot, \bar{\omega}) \in \mathcal{S}_2$, where $\mathcal{S}_2 = C([0,T]; \mathbb{K})$ for any $\bar{\omega} \in \bar{\Omega}'^c$. Let $\mathcal{L}(\bar{W})$ denotes the law of the process $\bar{W}$ on $\mathcal{S}_2$. We introduce the sequence of probability laws on $\mathcal{S}_2$ as follows:

$$\Pi_{2,m} = \mathcal{L}(\bar{W}), \forall m \geqslant 1.$$

Hence, by definition, we assert that the constructed sequence $(\Pi_{2,m})_{m \in \mathbb{N}}$ belongs to $\mathbb{S}_2$, where $\mathbb{S}_2$ denotes the collection of all probability measures on $(\mathcal{S}_2, \mathcal{B}(\mathcal{S}_2))$ with $\mathcal{B}(\mathcal{S}_2)$ being the Borel σ-field of $\mathcal{S}_2$. We equip the space $\mathcal{S}_2$ with the induced metric

$$\Theta(u, v) = \sup_{t \in [0,T]} \|u(t) - v(t)\|_{\mathbb{K}}, \forall u, v \in \mathbb{K}. \tag{3.76}$$

One can check that a sequence in $\mathcal{S}_2$ is convergent if and only if it converges uniformly. The space $\mathcal{S}_2$ is a complete separable metric space i.e. a Polish space.

The weak convergence of measures generates a metric d_∞ defined by

$$d_\infty(\mu, \nu) = \inf\{\delta > 0 : \mu(B) \leqslant \nu(N_\delta) + \delta, \forall B \in \mathcal{B}(\mathcal{S}_2)\},$$

where $N_\delta := \{u \in \mathcal{S}_2 : \exists v \in B, d(u, v) < \delta\}$ for any subset B of $\mathcal{S}_2$. The definition of the function d_∞ can be found for e.g. in the monographs [29], [78] and [157]. It is easy to prove that d_∞ is a metric(see [29, p. 72] for the proof). The function d_∞ is called the Prokhorov metric on $\mathbb{S}_2$ induced by d. Since $\mathcal{S}_2$ is separable, then convergence measured in the Prokhorov metric d_∞ is the same as weak convergence of measures in $(\mathbb{S}_2, \mathcal{B}(\mathbb{S}_2))$. In addition, we deduce that $(\mathbb{S}_2, d_\infty)$ is separable. See for instance [161], [29, p. 72-73], [78] and [157]. $(\mathbb{S}_2, d_\infty)$ is complete (see for example [161, Lemma 1.4, p. 169]).

Arguing similarly as in [166], we use the result in [157, Chap. II, Theorem 3.2] to deduce that the family of laws $\{\Pi_{2,m} : m = 1, 2, \ldots\}$ is tight on $\mathcal{S}_2$. Thus, combining this with the tightness result of Lemma 17, we deduce that the family of laws $\{\Pi_m : m = 1, 2, \ldots\}$ form a tight sequence of probability measures in $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. This enables us to conclude the proof of Theorem 32. $\square$

The tightness result proved in Theorem 32 in combination with the powerful compactness result of Prokhorov enable us to extract a subsequence (Π_{m_ν}) that weakly converges to Π on $\mathcal{S}$. Next, Skorokhod compactness result enables us to assert that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and pairs of $\mathcal{S}$-valued random variables (W_{m_ν}, u_{m_ν}) and (W, u) defined on

$(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\Pi_{m_\nu} = \mathcal{L}(W_{m_\nu}, u_{m_\nu}) \tag{3.77}$$

$$W_{m_\nu} \longrightarrow W \text{ in } C([0,T];\mathbb{K}),\ as\ \nu \to \infty\ \mathbb{P}-a.s., \tag{3.78}$$

$$u_{m_\nu} \longrightarrow u \text{ in } L^2(0,T;L^2(\mathbb{D})),\ as\ \nu \to \infty\ \mathbb{P}-a.s., \tag{3.79}$$

$$u_{m_\nu} \longrightarrow u \text{ in } C([0,T];W^{-1,q(\cdot)}(\mathbb{D})),\ as\ \nu \to \infty\ \mathbb{P}-a.s., \tag{3.80}$$

$$\Pi = \mathcal{L}(W, u) \tag{3.81}$$

and that $\Pi_{m_\nu} = \mathcal{L}(u_{m_\nu}, \bar{W})$.

In order to construct our driving cylindrical Wiener process, we need to show that the process W is a $\mathcal{Q}$-Wiener process with values in $\mathbb{K}$ and adapted to the natural filtration $\left(\tilde{\mathcal{F}}_t\right)_{0\leqslant t\leqslant T}$ generated by the pair (W, u), i.e.,

$$\tilde{\mathcal{F}}_t = \sigma\left((W(\tau), u(\tau)); 0 \leqslant \tau \leqslant t\right).$$

We set

$$\mathcal{F}_t = \sigma\left(N \cup \tilde{\mathcal{F}}_\tau; 0 \leqslant \tau \leqslant t\right),$$

where N is the set of null sets of $\mathcal{F}$. Indeed, this filtration satisfies the usual conditions. Now, let $\mathcal{Q} = \mathcal{J}\mathcal{J}^*$ with the injection $\mathcal{J} : L^2(\mathbb{D}) \longrightarrow \mathbb{K}$ being a Hilbert-Schmidt operator. Following for instance [166](where [191, Lemma 139, p. 105] was crucially used), we can easily check that for each ν the sequences of stochastic processes $(W_{m_\nu}(t); t \in [0,T])_{m\in\mathbb{N}}$ forms $\mathbb{K}$-valued $\mathcal{Q}$-Wiener processes defined on the probability space $(\Omega, \mathcal{F}; \mathbb{P})$ and that for each $\tau \in [0,t)$ with $t \leqslant T$, the increment $W_{m_\nu}(t) - W_{m_\nu}(\tau)$ are independent of the σ-algebra $\mathcal{F}_\tau^{m_\nu} = \sigma\left((W_{m_\nu}(l), u_{m_\nu}(l)) : l \in [0,\tau]\right)$; and the pair (W_{m_ν}, u_{m_ν}) satisfies the integral form of equation (3.31). Namely,

$$\begin{aligned} u_{m_\nu}(t) + \int_0^t P_{m_\nu}\left(A(\tau, u_{m_\nu})\right) d\tau = u_{m_\nu}(0) + \int_0^t P_{m_\nu}\left(f(\tau, u_{m_\nu})\right) d\tau + \\ + \int_0^t P_{m_\nu}\left(G(\tau, u_{m_\nu}) \circ \mathcal{J}^{-1}\right) dW_{m_\nu}(\tau), \end{aligned} \tag{3.82}$$

where W_{m_ν} is written in an informal manner as the following series:

$$W_{m_\nu}(t) = \sum_{k=1}^{\infty} \beta_{m_\nu}^k(t)\mathcal{J}e_k, t \geqslant 0; \tag{3.83}$$

$$P_{m_\nu}(dW_{m_\nu}(t)) = \sum_{k=1}^{m_\nu} \mathcal{J}e_k d\beta_{m_\nu}^k(t), t \geqslant 0; \tag{3.84}$$

with $(\beta_{m_\nu}^k(t))_{k\in\mathbb{N}}$ being a family of mutually independent standard 1-dimensional Wiener processes given by

$$\beta_{m_\nu}^k(t) = \frac{1}{\sqrt{\kappa_k}}\langle W_{m_\nu}(t), w_k\rangle$$

and $(e_k)_{k\in\mathbb{N}}$ is the ONB of $\mathbb{K}$ given by

$$e_k = \frac{\mathcal{J}^* w_k}{\sqrt{\kappa_k}}.$$

Note from (3.83) and (3.84) that

$$\begin{aligned}\int_0^t P_m G(\tau, u_{m_\nu}) d\bar{W} = \int_0^t P_{m_\nu} G(\tau, u_{m_\nu}) \circ \mathcal{J}^{-1} dW_{m_\nu} &= \sum_{k=1}^{m_\nu} \int_0^t G(\tau, u_{m_\nu}) e_k d\beta^k_{m_\nu}(\tau)\\ &= \sum_{k=1}^{m_\nu} \int_0^t G(\tau, u_{m_\nu}) w_k d\bar{\beta}_k(\tau),\end{aligned}$$

where $G(t, u_{m_\nu}) \circ \mathcal{J}^{-1} \in \mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))$ with

$$||G(t, u_{m_\nu})||_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))} = ||G(t, u_{m_\nu}) \circ \mathcal{J}^{-1}||_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}.$$

To check that the limiting process W is a $\mathcal{Q}$-Wiener process with values in $\mathbb{K}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we proceed by considering characterization of Wiener process via their characteristic functions. It is sufficient to show that W has the right finite dimensional Gaussian distributions to be a $\mathcal{Q}$-Wiener process in $\mathbb{K}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\tau_0 = 0 < \tau_1 < \cdots < \tau_l = T$ be a partition of $[0, T]$. We will show that $W(\tau_{i+1}) - W(\tau_i)$ is independent of $\mathcal{F}_{\tau_i}$ where $\mathcal{F}_t = \sigma\left((W(s), u(s)); s \in [0, t]\right), t \in [0, T]$. We will also show that the finite dimensional distribution of W is Gaussian. To this end we will compute the characteristic function of $W(\tau_{i+1}) - W(\tau_i)$. For each $u \in \mathbb{K}$ and $(\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l$ we have

$$\mathbb{E}\exp\left\{ i\sum_{j=1}^{l} \gamma_j \langle W^j_{m_\nu} - W^{j-1}_{m_\nu}, u\rangle_{\mathbb{K}} \right\} = \exp\left\{ -\frac{1}{2}\sum_{j=1}^{l} \gamma_j^2 (\tau_j - \tau_{j-1}) \langle \mathcal{Q}u, u\rangle_{\mathbb{K}} \right\}, \tag{3.85}$$

where $W^j_{m_\nu} = W_{m_\nu}(\tau_j)$ and $i = \sqrt{-1}$ is the imaginary unit.
Using the convergence (3.78), we can pass to the limit in (3.85) and thanks to Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}\exp\left\{ i\sum_{j=1}^{l} \gamma_j \langle W(\tau_j) - W(\tau_{j-1}), u\rangle_{\mathbb{K}} \right\} = \exp\left\{ -\frac{1}{2}\sum_{j=1}^{l} \gamma_j^2 (\tau_j - \tau_{j-1}) \langle \mathcal{Q}u, u\rangle_{\mathbb{K}} \right\}. \tag{3.86}$$

Let $(\varphi_j)_{j=1,\ldots,l}$ and $(\phi_j)_{j=1,\ldots,l}$ and ϕ be continuous and bounded real functions with $\varphi_j : V' \longrightarrow \mathbb{R}$ and $\phi, \phi_j : \mathbb{K} \longrightarrow \mathbb{R}$. We know that $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$ if X and Y are independent. We can see that for each $\nu \in \mathbb{N}$

$$\begin{aligned}&\mathbb{E}\left[\prod_{j=1}^{l} \varphi_j(u_{m_\nu}(h_j)) \prod_{j=1}^{l} \phi_j(W_{m_\nu}(h_j)) \phi(W_{m_\nu}(t) - W_{m_\nu}(\tau))\right] =\\ &\mathbb{E}\left[\prod_{j=1}^{l} \varphi_j(u_{m_\nu}(h_j)) \prod_{j=1}^{l} \phi_j(W_{m_\nu}(h_j))\right] \mathbb{E}\left[\phi(W_{m_\nu}(t) - W_{m_\nu}(\tau))\right].\end{aligned}$$

Using convergence (3.78), the fact that φ_j, ϕ_j and ϕ are continuous and bounded, the Lebesgue dominated convergence theorem can be applied; hence we get

$$\mathbb{E}\left[\prod_{j=1}^{N}\varphi_j(u(h_j))\prod_{j=1}^{N}\phi_j(W(h_j))\phi(W(t)-W(\tau))\right] =$$
$$\mathbb{E}\left[\prod_{j=1}^{N}\varphi_j(u(h_j))\prod_{j=1}^{N}\phi_j(W(h_j))\right]\mathbb{E}\left[\phi(W(t)-W(\tau))\right].$$

Indeed, we can deduce from (3.85) and (3.86) that for each $v \in \mathbb{N}$

$$W(t) = \sum_{k=1}^{\infty}\beta_k(t)w_k,\ t \geqslant 0; \tag{3.87}$$

where $(\beta_k)_{k\in\mathbb{N}}$ is a family of independent real valued standard Wiener processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

3.4.4 Existence of probabilistic weak solution

In this subsection, we establish some convergence properties of the subsequence (u_{m_ν}) obtained in the previous section. Following similar reasoning as in [48, Remark 4.4, page 19] we may assume that the subsequence obtained from the previous subsection (u_{m_ν}) is a family of H_m-valued processes. Therefore,

$$\Pi_{m_\nu} = \mathcal{L}(W_m, u_m) \text{ on } C([0,T];H_m)$$

since the Borel subsets of $C([0,T];H_m)$ are considered as well Borel subsets of $L^2(0,T;L^2(\mathbb{D}))\cap C([0,T];W^{-1,q(\cdot)}(\mathbb{D}))$. Thus, we infer from this reasoning and the estimates obtained in the previous sections, Lemma 12 - Lemma 16, that by equality of laws on $C([0,T];H_m)$ we have the subsequence (u_{m_ν}) satisfies the same estimates as the original sequence (u_m):

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{m_\nu}(t)\|^q_{L^2(\mathbb{D})} \leqslant C,\ q \geqslant 2; \tag{3.88}$$

$$\mathbb{E}\|u_{m_\nu}\|^r_{\mathring{V}(Q)} \leqslant C,\ \text{for any } 2 \leqslant r \leqslant p(\cdot) \leqslant s < \infty; \tag{3.89}$$

$$\mathbb{E}\left(\int_0^T \|u_{m_\nu}(t)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})}dt\right)^{\frac{r}{2}} \leqslant C; \tag{3.90}$$

$$\tag{3.91}$$

It follows from the cited estimates that the full sequence of approximate solution of problem (3.1)-(3.3) contains a subsequence $\{u_{m_\nu} : \nu = 1, 2, \ldots\}$ denoted in the same way as the previous subsequence such that

$$u_{m_\nu} \longrightarrow u \quad \text{weakly in} \quad L^q(\Omega, \mathcal{F}, \mathbb{P}; L^a(0,T;L^2(\mathbb{D}))),\ \forall a \in [2,\infty); \tag{3.92}$$

$$u_{m_\nu} \longrightarrow u \text{ weakly in } \ L^r\left(\Omega, \mathcal{F}, \mathbb{P}; \mathring{V}(Q)\right) \text{ for any } 2 \leqslant r \leqslant p(\cdot) \leqslant s < \infty; \tag{3.93}$$

$$A(t, u_{m_\nu}(\omega)) \longrightarrow \chi(\omega) \quad \text{weakly in} \quad \left(\mathring{V}(Q)\right)' \text{ for a.e. } \omega \in \Omega \tag{3.94}$$

$$u_{m_\nu}(T) \longrightarrow \daleth \quad \text{weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})). \tag{3.95}$$

It follows from the embedding in lemma 8 and (3.93) that

$$u_{m_\nu} \longrightarrow u \quad \text{weakly in} \quad L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T; \mathring{W}^{1,p(\cdot)}(\mathbb{D}))). \tag{3.96}$$

Using the properties of the operator A and the previous a priori estimates in combination with the results in Remark 8, we also have

$$\mathbb{E}\|A(t, u_{m_\nu})\|_{\left(\mathring{V}(Q)\right)'} \leqslant C;$$

consequently, using this inequality and the embedding result in Lemma 8, we deduce that

$$\mathbb{E}\|A(t, u_{m_\nu})\|_{L^2\left(0,T;W^{-1,q(\cdot)}(\mathbb{D})\right)} \leqslant C.$$

Furthermore,

$$u_{m_\nu}(\omega) \longrightarrow u(\omega) \quad \text{weakly} \quad -* \text{ in } \quad L^\infty(0,T;L^2(\mathbb{D})) \text{ a.e. } \omega. \tag{3.97}$$

Moreover, it is straightforward to show that estimates (3.88), (3.89) and (**??**) for the sequence u_{m_ν} lead to corresponding estimates for the process u. That is,

$$\begin{aligned}
&\mathbb{E}\|u\|^q_{L^a(0,T;L^2(\mathbb{D}))} \leqslant C,\ q \in [2,\infty),\ a \in [2,\infty),\\
&\left(\mathbb{E}\|u\|^r_{\mathring{V}(Q)}\right)^{1/r} \leqslant C, \text{ for any } 2 \leqslant r \leqslant p(\cdot) \leqslant s < \infty,\\
&\mathbb{E}\left(\int_0^T \|u(t)\|^2_{\mathring{W}^{1,p(\cdot)}(\mathbb{D})} dt\right)^{\frac{r}{2}} \leqslant C,
\end{aligned}$$

We can use (3.88) to prove that the sequence $\left(\int_0^T ||u_m(t)||^2_{L^2(\mathbb{D})} dt\right)_{\nu \in \mathbb{N}}$ is bounded $\mathbb{P}$-a.s.. For that purpose, we consider the increasing function $\psi : \mathbb{R} \to \mathbb{R}^+$ defined by $\psi(u_{m_\nu}) = ||u_{m_\nu}||^q_{L^2(0,T;L^2(\mathbb{D}))}$ for any $q \geqslant 4$. We have

$$\frac{\psi(u_{m_\nu})}{||u_{m_\nu}||_{L^2(0,T;L^2(\mathbb{D}))}} = \infty,\ as\ ||u_{m_\nu}||_{L^2(0,T;L^2(\mathbb{D}))} \to \infty.$$

It follows from

$$\mathbb{E}||u_{m_\nu}||_{L^1(0,T;L^2(\mathbb{D}))} = \mathbb{E}\int_0^T ||u_{m_\nu}(t)||_{L^2(\mathbb{D})}dt \leqslant C\mathbb{E}\sup_{0\leqslant t\leqslant T}||u_{m_\nu}(t)||_{L^2(\mathbb{D})} \leqslant C,$$

that the sequence $(||u_{m_\nu}||_{L^2(0,T;L^2(\mathbb{D}))})_{\nu\in\mathbb{N}}$ is a bounded subset of $L^1(\Omega,\mathcal{F},\mathbb{P})$. It follows from the high integrability in Lemma 14 that

$$\begin{aligned}\sup_\nu\int_\Omega \psi(u_{m_\nu})d\mathbb{P} = \sup_\nu \mathbb{E}\left(||u_{m_\nu}||_{L^2(0,T;L^2(\mathbb{D}))}\right)^q &= \sup_\nu \mathbb{E}\left(\int_0^T ||u_{m_\nu}(t)||^2_{L^2(\mathbb{D})}dt\right)^{q/2} \\ &\leqslant C\sup_\nu \mathbb{E}\sup_{0\leqslant t\leqslant T}||u_{m_\nu}(t)||^q_{L^2(\mathbb{D})} < \infty.\end{aligned}$$

Therefore, the sufficient condition for uniform integrability [190, Lemma 3, page 188] enables us to deduce that, $\int_0^T\|u_{m_\nu}(t)\|^2_{L^2(\mathbb{D})}dt$ is uniformly integrable w.r.t. $\mathbb{P}$. This enable us to deduce from the Vitali convergence theorem that

$$u_{m_\nu} \longrightarrow u \text{ strongly in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;L^2(\mathbb{D}))). \tag{3.98}$$

Hence, there exists a subsequence still denoted u_{m_ν} in order to simplify notation such that

$$u_{m_\nu} \longrightarrow u \text{ in } L^2(\mathbb{D}) \text{ for almost all } (t,\omega) \text{ w.r.t. } \mathbb{P}\times dt. \tag{3.99}$$

Thanks to (3.23) and (3.88), we derive from (3.99) and Vitali's theorem that

$$f(\cdot,u_{m_\nu}(\cdot)) \longrightarrow f(\,\cdot,u(\cdot)) \text{ in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2\left(0,T;W^{-1,q(\cdot)}(\mathbb{D})\right)\right).$$

In particular, for fixed j, since $w_j\in \mathring{W}^{1,p(\cdot)}(\mathbb{D})$, we get that

$$\langle f(\cdot,u_{m_\nu}(\cdot)),w_j\rangle \longrightarrow \langle f(\cdot,u(\cdot)),w_j\rangle \text{ in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T)), \tag{3.100}$$

Similarly, using the condition on G, (3.21), we prove that

$$P_{m_\nu}G(t,u_{m_\nu}) \longrightarrow G(t,u) \text{ in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D})))\right). \tag{3.101}$$

Finally, we must prove that

$$\int_0^t P_{m_\nu}G(s,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu} \longrightarrow \int_0^t G(s,u)dW \text{ weakly in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right) \tag{3.102}$$

which can be deduced from the following convergence

$$\int_0^t G(s,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu} \longrightarrow \int_0^t G(s,u)dW \text{ weakly in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right) \tag{3.103}$$

for any $t \in [0,T]$.
For that purpose, we intend to use integration by parts in order to prove the convergence of the stochastic integral

$$\int_0^t G(t, u_{m_\nu}) \circ \mathcal{J}^{-1} dW_{m_\nu}(t).$$

In order to do that, we need to use the notion of regularization techniques from harmonic analysis. First, we need to emphasis that the integral on the left does not depend on the injection $\mathcal{J}$. But since the integrand is not smooth with respect to t, we deal with it by introducing a suitable regularization (or mollification), G^ε. Here we simply extend G as zero on $\mathbb{R} \setminus [0,T]$ and denote the extension also by G in order to define G^ε on all of $[0,T]$. That is, we restrict the range of integration to the interval $[0,T]$. Let ϱ be a standard mollifier. For any $\varepsilon > 0$ we define the mollification of G by

$$G^\varepsilon(t,u) = \frac{1}{\varepsilon}\int_0^T \varrho\left(\frac{\tau - t}{\varepsilon}\right) G(\tau, u) d\tau \ \forall \tau \in [0,T] \text{ and } 0 < \varepsilon < \operatorname{dist}(\tau, \partial([0,T])).$$

Note that G^ε exists (as a Bochner integral see [18, Page 9] for the definition) and defines (and it coincides a.e. with) a continuous function $\varrho_\varepsilon * G : [0,\infty) \to L^2(\mathbb{D})$. Moreover, one can merely check that G^ε is smooth in t and continuous in u. We refer to the monograph of Arendt and co-authors [18] for detailed information regarding Fourier multipliers and convolution (see, e.g., [18, Section 1.3, page 21 and page 486]) and mollification (see [18, Page 23]) of Banach spaces-valued functions (see, e.g., [18, page 489]). Among other things, we can easily prove from (3.21) that the intensity of the noise $G(t, u_m(t))$ remains in a bounded subset of the space $L^2(\Omega, \mathcal{F}, P; L^2(0,T; \mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D})))$ and we have the uniform estimate for G^ε

$$\mathbb{E}\int_0^T \|G^\varepsilon(t,u)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))} dt \leqslant \mathbb{E}\int_0^T \|G(t,u)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))} dt. \tag{3.104}$$

Moreover, it follows from the definition of G^ε that

$$G^\varepsilon(.,u) \longrightarrow G(.,u) \text{ in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2\left(0,T; \mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))\right)\right), \ as\ \varepsilon \to 0; \tag{3.105}$$

and that

$$\mathbb{E}\int_0^T ||G^\varepsilon(t,u_{m_\nu}) - G^\varepsilon(t,u)||^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))} dt \leqslant C\mathbb{E}\int_0^T ||G(t,u_{m_\nu}) - G(t,u)||^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))} dt.$$

Arguing similarly as in [7, 8, 22, 23, 68, 168, 182, 183, 184, 185, 186, 199], we must prove that

$$\int_0^T G(s, u_{m_\nu}) \circ \mathcal{J}^{-1} dW_{m_\nu} \to \int_0^T G(s,u) dW \ \text{ weakly in } \ L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})\right) \tag{3.106}$$

from which (3.103) follows.
Next we use integration by parts to get

$$\sum_{k=1}^{m_\nu}\int_0^T G^\varepsilon(t,u_{m_\nu})e_k d\beta^k_{m_\nu} = \sum_{k=1}^{m_\nu} G^\varepsilon(T,u_{m_\nu})e_k\beta^k_{m_\nu}(T) - \sum_{k=1}^{m_\nu}\int_0^T (G^\varepsilon)'(t,u_{m_\nu})e_k\beta^k_{m_\nu}(t)dt. \tag{3.107}$$

Owing to (3.21) and (3.99), we have that

$$G^\varepsilon(\cdot,u_{m_\nu}) \longrightarrow G^\varepsilon(\cdot,u), \quad \text{almost everywhere in } (0,T)\times\Omega \text{ as } \nu\to\infty. \tag{3.108}$$

It follows from (3.108) and (3.78) that

$$\sum_{k=1}^{m_\nu} G^\varepsilon(\cdot,u_{m_\nu})e_k\beta^k_{m_\nu}(\cdot) \longrightarrow \sum_{k=1}^{\infty} G^\varepsilon(\cdot,u)w_k\beta_k(\cdot) = G^\varepsilon(\cdot,u)W(\cdot), \tag{3.109}$$

for almost all $\omega\in\Omega$.
From the definition of G^ε, it can straightforward be seen that the mapping $u\mapsto (G^\varepsilon)'(t,u)$ is continuous from $L^2(\mathbb{D})$ to $\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))$, a.e. $t\in[0,T]$. Hence, using (3.78), (3.107) and (3.109), we are able to deduce from (3.107) after passing to the limit as $\nu\to\infty$ that

$$\int_0^T G^\varepsilon(t,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu}(t) \longrightarrow (G^\varepsilon(T,u)W(T) - \int_0^T (G^\varepsilon)'(t,u)W(t)dt, \tag{3.110}$$

for almost all $\omega\in\Omega$, thanks to the fact that the function $(G^\varepsilon)'(t,\cdot)$ is still continuous w.r.t. to the second variable. The right hand sight of (3.110) is equal to $\int_0^T G^\varepsilon(t,u)dW(t)$. By Fubini's theorem, Burkholder-Davis-Gundy's inequality and (3.21), we have

$$\mathbb{E}\left\|\int_0^T G^\varepsilon(t,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu}(t)\right\|^2_{L^2(\mathbb{D})} \leqslant \mathbb{E}\left[\int_0^T \|G^\varepsilon(t,u_{m_\nu})\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}dt\right] \leqslant C \tag{3.111}$$

since $||G^\varepsilon(t,u_{m_\nu}\circ\mathcal{J}^{-1}||_{\mathcal{L}_2(\mathbb{K},\mathbb{U})} = ||G^\varepsilon(t,u_{m_\nu}||_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}$.
Similarly as in (3.111), by Burkholder-Davis-Gundy inequality, we have the estimates

$$\mathbb{E}\left\|\int_0^T G(t,u_{m_\nu})\circ\mathcal{J}^{-1}dW^k_{m_\nu}(t)\right\|^2_{L^2(\mathbb{D})} \leqslant \mathbb{E}\left[\int_0^T \|G(t,u_{m_\nu})\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}dt\right] \leqslant C. \tag{3.112}$$

We define a function $\psi(x)=x^2$, for any $x\in\mathbb{R}^+$. Then $\lim_{x\to\infty}\frac{\psi(x)}{x}=\infty$. It follows from (3.112) and the results of Lemma 18 that the sequence $(M_\nu)_{\nu\in\mathbb{N}}$ is uniformly integrable, where

$$M_\nu = \int_0^T G^\varepsilon(t,u_{m_\nu})dW_{m_\nu}(t).$$

Since the sequence $(M_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D}))$, it follows from this uniform integrability, (3.110) and relying on Vitali's theorem, we can assert that for any $a\in(1,2]$, we have the convergence

$$M_\nu \longrightarrow \int_0^T G^\varepsilon(t,u)dW(t) \text{ weakly in } \ L^a\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right), \text{ as } \ \nu\to\infty. \tag{3.113}$$

It follows from (3.111), convergence (3.108) and the result of Lemma 12, that for a fixed ε and letting $\nu\to\infty$

$$\int_0^T G^\varepsilon(t,u_{m_\nu})dW_{m_\nu}(t) \longrightarrow \int_0^T G^\varepsilon(t,u)dW(t), \text{ weakly in } \ L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right). \tag{3.114}$$

That is, for all $\Xi\in L^2\left(\Omega,\mathcal{F},\mathbb{P},L^2(\mathbb{D})\right)$

$$\mathbb{E}\sum_{k=1}^{m_\nu}\int_0^T(\Xi,G^\varepsilon(t,u_{m_\nu})e_k)d\beta^k_{m_\nu}(t) \longrightarrow \mathbb{E}\int_0^T(\Xi,G^\varepsilon(t,u))dW(t) \tag{3.115}$$

Thus, we can find $\kappa\in L^\infty\left(\Omega,\mathcal{F},\mathbb{P}\right)$ such that for all $\Xi\in L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right)$

$$\mathbb{E}\int_0^T(\Xi,G^\varepsilon(t,u_{m_\nu}))dW_{m_\nu}(t) \longrightarrow \mathbb{E}(\Xi,\kappa). \tag{3.116}$$

Besides (3.102), (3.103) and (3.106), it remains to show that

$$\kappa=\mathbb{E}\int_0^T G^\varepsilon(t,u)dW(t). \tag{3.117}$$

Therefore, for fixed ε we let ν tends to ∞ to have, for any $\Xi\in L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right)$

$$\mathbb{E}\int_0^T\left(\Xi,G^\varepsilon(t,u_{m_\nu})\circ\mathcal{J}^{-1})\right)dW_{m_\nu}(t) \longrightarrow \mathbb{E}\int_0^T\left(\Xi,G^\varepsilon(t,u)dW(t)\right). \tag{3.118}$$

We have

$$\mathbb{E}\left(\Xi,\int_0^T G(t,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu}(t)\right)-\mathbb{E}\left(\Xi,\int_0^T G(t,u)W(t)\right)=I_1+I_2+I_3, \tag{3.119}$$

where Ξ is an arbitrary element of $L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right)$ and we consider $I_i, i=1,2,3$, the three integrals separately

$$\begin{aligned}
I_1&=\mathbb{E}\left(\Xi,\int_0^T[G(t,u_{m_\nu})-G^\varepsilon(t,u_{m_\nu})]\circ\mathcal{J}^{-1}dW_{m_\nu}(t)\right),\\
I_2&=\mathbb{E}\left(\Xi,\int_0^T[G^\varepsilon(t,u)-G(t,u)]dW(t)\right),\\
I_3&=\mathbb{E}\left(\Xi,\int_0^T G^\varepsilon(\tau,u_{m_\nu})\circ\mathcal{J}^{-1}dW_{m_\nu}(t)\right)-\mathbb{E}\left(\Xi,\int_0^T G^\varepsilon(t,u)dW(t)\right).
\end{aligned}$$

In order to find the limiting candidate of the stochastic integral, we must prove that $I_i, i = 1, 2, 3$ converge to zero as $\varepsilon \to 0$ and $\nu \to \infty$. For I_1, we apply Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities to get

$$\begin{aligned} I_1 &\leqslant \mathbb{E}\left[\|\Xi\|_{L^2(\mathbb{D})} \left\| \int_0^T [G(t, u_{m_\nu}) - G^\varepsilon(t, u_{m_\nu})] \circ \mathcal{J}^{-1} dW_{m_\nu}(t) \right\|_{L^2(\mathbb{D})} \right] \\ &\leqslant C\mathbb{E}\left[\int_0^T \|G(t, u_{m_\nu}) - G^\varepsilon(t, u_{m_\nu})\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))} dt \right]^{\frac{1}{2}}. \end{aligned}$$

Similar reasoning can be used to estimate I_2. Namely,

$$\begin{aligned} I_2 &\leqslant \mathbb{E}\left[\|\Xi\|_{L^2(\mathbb{D})} \left\| \int_0^T [G(t, u) - G^\varepsilon(t, u)] dW(t) \right\|_{L^2(\mathbb{D})} \right] \\ &\leqslant C\mathbb{E}\left[\int_0^T \|G(t, u) - G^\varepsilon(t, u)\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))} dt \right]^{\frac{1}{2}}. \end{aligned}$$

Letting ε tend to zero in the above inequalities and making use of convergence (3.105), (3.101) yields first I_1 converges to zero as $\nu \to \infty$ and by (3.118) I_2 converge to zero as required. Additionally we must show that I_3 converges to zero as ε goes to zero and $\nu \to \infty$. In order to do so, we use (3.118) to get

$$\lim_{\varepsilon \to 0} \left[\mathbb{E} \lim_{\nu \to \infty} \left(\Xi, \int_0^T G^\varepsilon(t, u_{m_\nu}) \circ \mathcal{J}^{-1} dW_{m_\nu}(t) \right) - \mathbb{E}\left(\Xi, \int_0^T G^\varepsilon(t, u) dW(t) \right) \right] = 0.$$

Hence, $I_3 = 0$ as $\varepsilon \to 0$ and $\nu \to \infty$. Hence letting $\varepsilon \to 0$ and $\nu \to \infty$ in (3.119), we deduce that that the left hand side of (3.119) converges to zero. From this we achieve (3.117) which proves 3.106. In its turn, 3.106 enables us to prove (3.103). Thus, (3.102) is thereby proved.

Next, after integrating by parts in the first term of (3.37), we get

$$\begin{aligned} &- \int_0^T (u_{m_\nu}, w_j) \frac{d\varphi}{dt} dt + \int_0^T \langle A(t, u_{m_\nu}), w_j \rangle \varphi dt \\ = &\int_0^T \langle f(t, u_{m_\nu}), w_j \rangle \varphi dt + \sum_{k=1}^{m_\nu} \int_0^T (G(t, u_{m_\nu}) e_k, w_j) \varphi d\beta^k_{m_\nu}(t) \\ &+ (u_{m_\nu}(0), w_j) \varphi(0) - (u_{m_\nu}(T), w_j) \varphi(T), \end{aligned}$$

for any $\varphi \in C^1([0, T])$. This holds, if we replace w_j by any of their linear combinations for all $j = 1, 2, \ldots$. Since H_m is dense in $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$, passing to the limit in the resulting relation as $\nu \to \infty$, and making use of all the above convergence results, we obtain $\forall v \in \mathring{W}^{1,p(\cdot)}(\mathbb{D})$

$$\begin{aligned} &- \int_0^T (u, v) \frac{d\varphi}{dt} dt + \int_0^T \langle \chi, v \rangle \varphi dt \\ = &\int_0^T \langle f(t, u), v \rangle \varphi dt + \sum_{k=1}^{\infty} \int_0^T (G(t, u) w_k, v) \varphi d\beta_k(t) + (u_0, v)\varphi(0) - (\daleth, v)\varphi(T). \end{aligned}$$

It follows from this that

$$du + \chi(t)dt = f(t,u)dt + G(t,u)dW(t), \tag{3.120}$$

in the sense of distribution in $L^{r/(r-1)}\left(0,T; W^{-1,r/(r-1)}(\mathbb{D})\right)$, therefore in $\mathring{V}'(Q)$. A natural integral representation of (3.120) can be expressed as:

$$u(t) + \int_0^t \chi(s)ds = u_0 + \int_0^t f(s,u)ds + \int_0^t G(s,u)dW(s).$$

(3.120) should be understood as an abbreviation of the above integral form.
Using the weak formulation of (3.120) by testing it over $(0,T)\times D$ by the product $v\,(x)\,\varphi\,(t)$ with $v \in \mathring{W}^{1,p(\cdot)}$ and $\varphi \in C^1\,([0,T])$ and replacing w_j by v in (3.37) duly multiplied by $\varphi\,(t)$ and integrated over $(0,T)\times\mathbb{D}$, we get that

$$\begin{aligned}
&\int_0^T (d(u_{m_\nu} - u), v)\,\varphi(t)dt + \int_0^T \langle A(t,u_{m_\nu}) - \chi, v\rangle \varphi dt \\
= \; &\int_0^T \langle f(t,u_{m_\nu}) - f(t,u), v\rangle \varphi dt + \sum_{k=1}^{m_\nu} \int_0^T (G(t,u_{m_\nu})e_k, v)\varphi d\beta^k_{m_\nu}(t) \\
&-\sum_{k=1}^{\infty} \int_0^T (G(t,u)w_k, v)\varphi d\beta_k(t).
\end{aligned}$$

Hence, using integration by parts we get

$$\begin{aligned}
&-\int_0^T (u_{m_\nu}(t) - u(t), v)\,\frac{d\varphi(t)}{dt}dt + \int_0^T \langle A(t,u_{m_\nu}) - \chi, v\rangle \varphi dt \\
= \; &\int_0^T \langle f(t,u_{m_\nu}) - f(t,u), v\rangle \varphi dt + \sum_{k=1}^{m_\nu} \int_0^T (G(t,u_{m_\nu})e_k, v)\varphi d\beta^k_{m_\nu}(t) - \\
&\sum_{k=1}^{\infty} \int_0^T (G(t,u)w_k, v)\varphi d\beta_k(t) + (u_{m_\nu}(0) - u_0, v)\varphi(0) - (u_{m_\nu}(T) - u(T), v)\varphi(T).
\end{aligned} \tag{3.121}$$

Noting that $(u(t), v) \in C([0,T])$ for a. e. $\omega \in \Omega$, and passing to the limit in this relation as $\nu \longrightarrow \infty$ using the convergence (3.94), (3.98)-(3.99), we obtain that

$$(u(0) - u_0, v)\varphi(0) - (\gimel - u(T), v)\varphi(T) = 0, \;\; \mathbb{P} - a.s.$$

Since $u(0) = u_0$ and v is arbitrary, we can choose φ so that $\varphi(T) = 1$ and $\varphi(0) = 0$ to obtain

$$u(T) = \gimel.$$

3.4.4.1 Monotonicity Method

Here, we would like to show that in (3.94), $\chi(t,\omega) = A(t,u(\omega))$, which requires arguments of monotone operators that are well developed in [123], [155] in order to solve stochastic evolutionary equations and [178] for their deterministic counterparts.

Recall that, (3.42) remains valid if we replace the full sequence u_m by the newly obtained subsequence u_{m_ν}. That is,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u_{m_\nu}(\tau)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle A(\tau,u_{m_\nu}),u_{m_\nu}\rangle d\tau \\
&= \mathbb{E}\int_0^t \langle f(\tau,u_{m_\nu}),u_{m_\nu}\rangle d\tau + \frac{1}{2}\mathbb{E}\sum_{j=1}^{m_\nu}\sum_{k=1}^{m_\nu}\int_0^t (G(\tau,u_{m_\nu})w_k,w_j)^2\, d\tau \\
&+ \mathbb{E}\sum_{k=1}^{m_\nu}\int_0^t (G(\tau,u_{m_\nu})e_k,u_{m_\nu})\, d\beta^k_{m_\nu} + \frac{1}{2}\|u_{m_\nu}(0)\|^2_{L^2(\mathbb{D})}.
\end{aligned} \tag{3.122}$$

For an arbitrary function v in $\mathring{V}(Q)$, it follows from (3.122) that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u_{m_\nu}(t)-v(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle A(\tau,u_{m_\nu})-A(\tau,v),u_{m_\nu}(\tau)-v(\tau)\rangle d\tau \\
&= \mathbb{E}\int_0^t \langle f(\tau,u_{m_\nu}),u_{m_\nu}\rangle d\tau + \frac{1}{2}\mathbb{E}\sum_{j=1}^{m_\nu}\sum_{k=1}^{m_\nu}\int_0^t (G(\tau,u_{m_\nu})w_k,w_j)^2\, d\tau + \frac{1}{2}\|u_{m_\nu}(0)\|^2_{L^2(\mathbb{D})} \\
&+ \mathbb{E}\sum_{k=1}^{m_\nu}\int_0^t (G(\tau,u_{m_\nu})e_k,u_{m_\nu})\, d\beta^k_{m_\nu} - \mathbb{E}\int_0^t \langle A(\tau,u_{m_\nu}),v(\tau)\rangle d\tau - \mathbb{E}\,(u_{m_\nu}(t),v(t)) \\
&- \mathbb{E}\int_0^t \langle A(\tau,v),u_{m_\nu}(\tau)\rangle d\tau + \mathbb{E}\int_0^t \langle A(\tau,v),v(\tau)\rangle d\tau + \frac{1}{2}\mathbb{E}\|v(t)\|^2_{L^2(\mathbb{D})}.
\end{aligned} \tag{3.123}$$

To this end, for an arbitrary function v in $\mathring{V}(Q_T)$, we set

$$X_\nu = \mathbb{E}\int_0^t \langle A(\tau,u_{m_\nu})-A(\tau,v),u_{m_\nu}(\tau)-v(\tau)\rangle d\tau + \frac{1}{2}\mathbb{E}\|u_{m_\nu}(t)-v(t)\|^2_{L^2(\mathbb{D})}. \tag{3.124}$$

Similarly as the above reasoning, writing (3.120) in integral form as an equality between random variables with values in $W^{-1,q(\cdot)}(\mathbb{D})$, and applying Itö's formula to the corresponding relation yields

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle \chi(\tau),u(\tau)\rangle d\tau \\
&= \mathbb{E}\int_0^t \langle f(\tau,u),u\rangle d\tau + \frac{1}{2}\mathbb{E}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_0^t (G(\tau,u)w_k,w_j)^2\, d\tau \\
&+ \mathbb{E}\sum_{k=1}^{\infty}\int_0^t (G(\tau,u)w_k,u)\, d\beta_k + \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{D})}.
\end{aligned} \tag{3.125}$$

Let $v \in \overset{\circ}{V}(Q)$ be an arbitrary function, similarly as above, we use (3.125) to get

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u(t)-v(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, v), u(\tau) - v(\tau)\rangle d\tau \\
&= \mathbb{E}\int_0^t \langle f(\tau, u), u\rangle d\tau + \frac{1}{2}\mathbb{E}\int_0^t \|G(\tau, u)\|^2_{\mathcal{L}_2(\mathbb{K}, L^2(\mathbb{D}))} d\tau + \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{D})} \\
&+ \mathbb{E}\sum_{k=1}^{\infty}\int_0^t (G(\tau, u)w_k, u)\, d\beta_k - \mathbb{E}\int_0^t \langle \chi(\tau), v(\tau)\rangle d\tau - \mathbb{E}\,(u(t), v(t)) \\
&- \mathbb{E}\int_0^t \langle A(\tau, v), u(\tau)\rangle d\tau + \mathbb{E}\int_0^t \langle A(\tau, v), v(\tau)\rangle d\tau + \frac{1}{2}\mathbb{E}\|v(t)\|^2_{L^2(\mathbb{D})}.
\end{aligned} \tag{3.126}$$

To this end, let $v \in \overset{\circ}{V}(Q)$ and set

$$X_\infty = \frac{1}{2}\mathbb{E}\|u(t)-v(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, v), u(\tau) - v(\tau)\rangle d\tau.$$

It is easy to check using the definition of the operator A that

$$\begin{aligned}
&2\langle A(t, u_{m_\nu}) - A(t, v), u_{m_\nu} - v) \\
&\geqslant \sum_{i=1}^n \int_0^t\int_{\mathbb{D}} \left[\left|\frac{\partial u_{m_\nu}}{\partial x_i}\right|^{p(\cdot)-2} - \left|\frac{\partial v}{\partial x_i}\right|^{p(\cdot)-2}\right]\left[\left(\frac{\partial u_{m_\nu}}{\partial x_i}\right)^2 - \left(\frac{\partial v}{\partial x_i}\right)^2\right] dx d\tau \geqslant 0;
\end{aligned}$$

which implies that the operator $A(t, .)$ is monotone. This monotonicity enables us to deduce that $X_\nu \geqslant 0$, since $\mathbb{E}\|u_{m_\nu}(t) - v(t)\|^2_{L^2(\mathbb{D})} \geqslant 0$.
Passing to the limit as $\nu \longrightarrow \infty$ in (3.122) and using (3.124), gives

$$\begin{aligned}
\limsup_\nu X_\nu &\geqslant \limsup_\nu \langle A(t, u_{m_\nu}) - A(t, v), u_{m_\nu} - v) \\
&\geqslant \frac{1}{2}\limsup_\nu \sum_{i=1}^n \int_0^t\int_{\mathbb{D}} \left[\left|\frac{\partial u_{m_\nu}}{\partial x_i}\right|^{p(x)-2} - \left|\frac{\partial v}{\partial x_i}\right|^{p(\cdot)-2}\right]\left[\left(\frac{\partial u_{m_\nu}}{\partial x_i}\right)^2 - \left(\frac{\partial v}{\partial x_i}\right)^2\right] dx d\tau \\
&= \frac{1}{2}\sum_{i=1}^n \int_0^t\int_{\mathbb{D}} \left[\left|\frac{\partial u}{\partial x_i}\right|^{p(\cdot)-2} - \left|\frac{\partial v}{\partial x_i}\right|^{p(\cdot)-2}\right]\left[\left(\frac{\partial u}{\partial x_i}\right)^2 - \left(\frac{\partial v}{\partial x_i}\right)^2\right] dx d\tau \geqslant 0;
\end{aligned} \tag{3.127}$$

since $\frac{\partial u_{m_\nu}}{\partial x_i}$ converges weakly to $\frac{\partial u}{\partial x_i}$ in $L^{p(\cdot)}((0,T)\times\mathbb{D})$ for each $i = 1, \ldots, n$ with $n \in \mathbb{N}$.
From this, we deduce on one hand that

$$\limsup_\nu X_\nu \geqslant 0. \tag{3.128}$$

Owing to the condition (3.21) on G, the estimates of Lemma 18, the almost everywhere convergence of u_{m_ν} to u on $[0,T]\times\mathbb{D}\times\Omega$, we see that $(G(t, u_{m_\nu}))$ is uniformly integrable in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))))$ and $G(t,u_{m_\nu})$ converges to $G(t,u)$ almost everywhere on $[0,T]\times\mathbb{D}\times\Omega$. Therefore, relying on Vitali's theorem, we get that

$$G(t,u_{m_\nu}) \longrightarrow G(t,u) \text{ strongly in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D})))\right). \tag{3.129}$$

On the other hand, combining (3.129) and (3.128) with (3.126), we get

$$\begin{aligned} 0 \leqslant \limsup_{\nu} X_\nu \leqslant{}& \mathbb{E}\int_0^t \langle f(\tau,u),u\rangle d\tau + \frac{1}{2}\mathbb{E}\int_0^t \|G(\tau,u)\|^2_{\mathcal{L}_2(\mathbb{K},L^2(\mathbb{D}))}d\tau + \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{D})}\\ &+\mathbb{E}\sum_{k=1}^{\infty}\int_0^t (G(\tau,u)w_k,u)\,d\beta_k - \mathbb{E}\int_0^t\langle\chi(\tau),v(\tau)\rangle d\tau - \mathbb{E}\,(u(t),v(t))\\ &-\mathbb{E}\int_0^t\langle A(\tau,v),u(\tau)\rangle d\tau + \mathbb{E}\int_0^t\langle A(\tau,v),v(\tau)\rangle d\tau + \frac{1}{2}\mathbb{E}\|v(t)\|^2_{L^2(\mathbb{D})}\\ ={}& X_\infty. \end{aligned} \tag{3.130}$$

Hence

$$0 \leqslant X_\infty = \frac{1}{2}\mathbb{E}\|u(t)-v(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t\langle \chi(\tau)-A(\tau,v),u(\tau)-v(\tau)\rangle d\tau. \tag{3.131}$$

The fact that the expression of χ in (3.131) is still unknown to us, forces us to use (3.130). For this purpose, let us consider the functions $u,v,w\in \mathring{V}(Q)$ such that $v=u-\alpha w$, where $\alpha>0$ is a constant. First, we get that

$$\|v\|_{\mathring{V}(Q)} = \|u-\alpha w\|_{\mathring{V}(Q)} \leqslant \|u\|_{\mathring{V}(Q)} + \|w\|_{\mathring{V}(Q)},$$

provided that $|\alpha|\leqslant 1$. By Remark 8, since $u-\alpha w\in\mathring{V}(Q)$, we also have

$$\|A(t,u-\alpha w)\|_{\left(\mathring{V}(Q)\right)'} \leqslant C.$$

Here, our main focus is to show that the operator A defined from $\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ to $(\mathring{W}^{1,p(\cdot)}(\mathbb{D}))'$ is semicontinuous; that is, for any $u(t),v(t),w(t)\in\mathring{W}^{1,p(\cdot)}(\mathbb{D})$ the mapping $\mathcal{T}:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $\alpha\mapsto\mathcal{T}(\alpha)=\langle A(t,u-\alpha v),w\rangle$ is continuous. In fact, with this specific purpose in mind, we concentrate on the second term on the right hand side of (3.131); hence we have

$$\begin{aligned} \lim_{\alpha\to 0}\mathbb{E}\int_0^t\langle A(\tau,v),v(\tau)-u(\tau)\rangle d\tau &= \lim_{\alpha\longrightarrow 0}\mathbb{E}\int_0^t\langle A(\tau,u(\tau)-\alpha w(\tau)),-\alpha w(\tau)\rangle d\tau\\ &= -\lim_{\alpha\longrightarrow 0}\alpha\mathbb{E}\int_0^t\sum_{i=1}^n\int_{\mathbb{D}}\left|\frac{\partial}{\partial x_i}(u-\alpha w)\right|^{p(\cdot)-2}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}dxd\tau\\ &= 0. \end{aligned}$$

This ensures that the operator A is semicontinuous. With $v \in \overset{\circ}{V}(Q)$ as expressed above, it follows from (3.131) that

$$0 \leqslant X_\infty = \frac{1}{2}\mathbb{E}\|-\alpha w(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, u - \alpha w), \alpha w\rangle d\tau.$$

Dividing both sides of this inequality by α results in

$$0 \leqslant X_\infty = \frac{\alpha}{2}\mathbb{E}\|w(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, u - \alpha w), w\rangle d\tau. \tag{3.132}$$

Using the property of semicontinuity of A and then passing to the limit as $\alpha \to 0$ in (3.132) we obtain

$$\begin{aligned} 0 \leqslant \lim_{\alpha\to 0} X_\infty &= \lim_{\alpha \to 0}\left\{\frac{\alpha}{2}\mathbb{E}\|w(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, u - \alpha w), w\rangle d\tau\right\} \\ &= \mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, u), w\rangle d\tau \end{aligned}$$

Hence, we deduce that

$$\mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, u), w\rangle d\tau \geqslant 0;$$

as $\nu \to 0$, with w being an arbitrary function belonging to $\overset{\circ}{V}(Q)$.
Then by the Lebesgue's theorem, we get

$$\mathbb{E}\int_0^t \langle \chi(\tau) - A(\tau, u), w\rangle d\tau = 0; \text{ for } \tau \in (0, t) \text{ a.e..}$$

We conclude from this that $\chi(t) = A(t, u)$ and hence the integral identity (3.36) is valid. That is by passing to the limit in (3.82) and substituting $\chi(t) = A(t, u)$ in (3.110) one gets that

$$du + A(t, u)dt = f(t, u)dt + G(t, u)dW \text{ in } L^2(0, T; L^2(\mathbb{D})).$$

Thus u is a solution of the problem (3.1)-(3.3) in the sense of Definition 30.

Chapter 4

Weak and strong probabilistic solutions for a class of strongly nonlinear stochastic parabolic problems

4.1 Introductory background

An important class of SPDEs which has so far not been studied by experts in the field is the stochastic counterpart of nonlinear parabolic equations which originated in the work of Brezis-Browder [34, 33] under the name of strongly nonlinear equations. The main feature of these equations is characterized by the presence of nonlinear terms which are unbounded perturbations of zero-th order, making it impossible to treat the resulting problem by means such as those used in works cited in the previous paragraphs. Brezis and Browder introduced a suitable regularization through appropriate truncations and thanks to compactness arguments, they derived the needed existence result. Further advances in the study of these equations are due to Landes and Mustonen; see [129], [130], [131] and [133].

The goal of the present paper is to initiate the study of strongly nonlinear parabolic equations in the sense of Brezis-Browder, in the stochastic framework.

Namely, we consider on $(0,T)\times\mathbb{D}$ with $\mathbb{D}\subset\mathbb{R}^m$, $m\geqslant 1$ being an open bounded subset with sufficiently regular boundary $\partial\mathbb{D}$ the higher-order stochastic quasilinear parabolic problem

$$(P)\begin{cases} du+[A_t(u)+g(t,x,u)]\,dt = f(t)dt+G(t,u)dW(t) & \text{in } Q_T\\ u(x,0)=0 & \text{in } \mathbb{D}\\ \dfrac{\partial^j u}{\partial N^j}=0 & \text{on } (0,T)\times\partial\mathbb{D}, \end{cases}$$

where, Q_T is the cylinder $(0,T)\times\mathbb{D}$, $\mathbb{D}\subset\mathbb{R}^m$ an open bounded set with sufficiently regular boundary $\partial\mathbb{D}$, $u = u(t,x)$ is the unknown function, the linear term f, the nonlinear term G and the perturbation g are given, W is a m-dimensional Wiener process; $\dfrac{\partial^j u}{\partial N^j} = 0$ is the Dirichlet boundary condition where $\dfrac{\partial^j u}{\partial N^j}$ is the j^{th} normal derivative of u with $0 \leqslant j \leqslant m-1$ and A_t is an elliptic operator of order $2m$ in the generalized divergence form. That is,

$$A_t(u) = \sum_{|\beta| \leqslant m} (-1)^{|\beta|} D^{\beta} A_{\beta}(t, x, u, Du, \ldots, D^m u),$$

for each $t \in [0,T]$ with the coefficient functions A_β of the leading operator A_t satisfying the Carathéodory conditions, that is each $A_\beta(t,x,\eta,\zeta)$ is measurable in (t,x) and continuous in η and ζ. Here ξ is an element of the vector space $\mathbb{R}^m$ of m-jets on $\mathbb{R}^m$ which assumes the representation $\xi = \{\xi_\beta : |\beta| \leqslant m\}$. To each ξ, there corresponds a couple (η,ζ), with $\eta = \{\eta_\beta : |\beta| \leqslant m-1\}$ and $\zeta = \{\zeta_\beta : |\beta| = m\}$.

Our main results are the construction of a probabilistic weak solution under rather general conditions on the nonlinear intensity $G\,(t,u)$ of the noise followed by the existence of a probabilistic strong solution for the problem (P) under monotonicity condition on A_t and Lipschitz condition on $G\,(t,u)$; here, the perturbation $g(t,x,u)$ is unbounded and of zero-th order. Thus a direct approach through Galerkin approximation is hopeless. In fact even Itô's formula is prohibited in that case. We therefore introduce a regularization through truncations which reduces the problem (P) to a sequence of problems (called (P_k) in the sequel) which fits the class of SPDEs studied by Krylov and Rozovskii [123]. Unfortunately, the estimates from [123] will not be enough for our purpose, since they depend on k, when applied to the sequence (u_k) of solutions of (P_k). Therefore we establish appropriate uniform a priori estimates of (u_k). Once this is achieved, we are then able to appeal to some analytic and probabilistic compactness results to extract converging subsequences from (u_k). Large part of the work is devoted to the passage to the limit in (P_k) which turns out to be very delicate. A combination of stopping time technics, with measure theoretical arguments and a result on pseudo monotone operators due to Browder [39] enable us to show that a sequence of solutions of (P_k) converges in suitable topologies to the requested weak solution for our original problem. Under some additional assumptions we establish the pathwise uniqueness of weak probabilistic solutions and appeal to an infinite-dimensional version of the famous Yamada-Watanabe's result [211] due to Röckner [175] and Ondrejat [152] to derive the existence of a unique strong probabilistic solution. To the best of our knowledge, the results obtained in the present paper are novel. Beside the novelty of the results, several difficulties which are due to the stochastic nature of the problem and therefore absent in [33, 34], had to be overcome.

This chapter is organized as follows. In section 2, we introduce needed function spaces, some probabilistic compactness results, we state our first main result on the existence of probabilistic weak solutions. In view of the importance of the pioneering results in [123], we summarized them in the same section; we also introduce the regularization of (P). Section 3 is devoted to the proof of the existence of probabilistic weak solutions of problem (P). We subdivide it into subsections in which we derive some a priori estimates, compactness results on spaces of probability measures, tightness properties and application of Prokhorov and skorokhod results. In the same section, we deal with the delicate passage to the limit using among other ideas, the pseudo-monotonicity method. In the last section 4, we prove the existence of strong probabilistic solutions to our problem via the pathwise uniqueness of probabilistic weak solutions and Yamada-Watanabe's famous theorem.

4.2 The weak probabilistic solution

4.2.1 Preliminaries

We start this subsection by introducing needed function spaces.

We denote by $\|f\|_X$ the norm of a function $f(x)$ in a space X. Let V be a reflexive Banach space and H a separable Hilbert space that can be identified with its dual H^* by mean of the Riesz representation Theorem. We assume that V is continuously and densely embedded into H. We denote by V^* the dual space of V and write the symbol $\langle \cdot, \cdot \rangle$ for the duality pairing between V and its dual space V^*. Let $(\cdot, \cdot)$ be the inner product in the Hilbert space H. Then we have

$$V \subset H \equiv H^* \subset V^*.$$

Here each space is dense in the following one. In addition to the above Gelfand triple, we also have

$$\langle u \, , \, v \rangle = (u \, , \, v)_H \, , \text{ for any } u \in H, v \in V.$$

We shall introduce some function spaces.

Throughout $W_0^{m,p}(\mathbb{D})$ and $W^{-m,p'}(\mathbb{D})$ will denote the usual Sobolev space and its dual.

4.2.2 Assumptions and formulation of main result

We obtain the the above Gelfand triplet by the following specification: We set $V = W_0^{m,p}(\mathbb{D})$, $H = L^2(\mathbb{D})$ and $V' = W^{-m,p'}(\mathbb{D})$ with p', the conjugate of p and $p \geqslant 1$. By Rellich-Kondrachov embedding theorm, V is compactly embedded in H which in its turn continuously embedded in V'.

We consider the operator family $A_t : W_0^{m,p}(\mathbb{D}) \longrightarrow W^{-m,p'}(\mathbb{D})$, defined by

$$\langle A_t(u), v\rangle = \sum_{|\beta|\leqslant m} \int_{\mathbb{D}} A_\beta \left(t, x, u, Du, \ldots, D^m u\right) D^\beta v dx,$$

for any $u, v \in W_0^{m,p}(\mathbb{D})$, and for any $t \in [0, T]$.

We now formulate the structure conditions on A_β. Let $Q = [0, T] \times \mathbb{D}$ For $2 \leqslant p < \infty$ we have

(i) There exists a constant $c_0 > 0$ and a measurable function $h_0 \in L^{p'}(Q)$ such that

$$|A_\beta(t, x, \xi)| \leqslant c_0\{|\xi_\beta|^{p-1} + h_0(t, x)\},$$

for all $|\beta| \leqslant m$, $\forall (t, x, \xi) \in Q \times \mathbb{R}^m$ with $\xi_\beta = \{D^\beta u, |\beta| \leqslant m - 1\}$

(ii) For all $(t, x) \in Q$, all lower-order jets $\eta \in R^{n_1}$, and higher-order jets $\zeta \neq \zeta^{\#}$ in $\mathbb{R}^{n_2}$ with $n_1 + n_2 = m$,

$$\sum_{|\beta|=m} \left[A_\beta(t, x, (\eta, \zeta)) - A_\beta(t, x, (\eta, \zeta^{\#}))\right] \left(\zeta_\beta - \zeta_\beta^{\#}\right) > 0,$$

where $\zeta = (\eta, \zeta)$ with $\eta = \{D^\beta u : |\beta| = m$ and ζ is of the form ξ_β as in (i), that is $\zeta = \{D^\beta u, |\beta| \leqslant m - 1\}$.

(iii) There exists $c_1 > 0$ and a measurable function $h_1 \in L^1(Q)$ such that for all $(t, x) \in Q$ and all $\xi \in \mathbb{R}^m$, we have

$$\sum_{|\beta|\leqslant m} A_\beta(t, x, \xi)\xi_\beta \geq c_1|\xi|^p - h_1(t, x).$$

We require on the strongly nonlinear perturbation term $g(t, x, u)$ the following conditions.

(iv) (1) The function $g(t, x, u)$ is measurable in (t, x), and continuous in u.

$$rg(t, x, r) \geq 0;$$

(2) There exists a continuous nondecreasing function $h : \mathbb{R} \longrightarrow \mathbb{R}$ with $h(0) = 0$ such that for all $(t, x) \in Q$, $r \in \mathbb{R}$, and a fixed constant c_2,

$$|g(t, x, r)| \leqslant |h(r)| \leqslant c_2\{|g(\cdot, x, r)| + |r|^{p-1} + 1\} \tag{4.1}$$

(v) The nonlinear intensity of the noise G satisfies the conditions:

$$\begin{aligned} &G(\cdot, u) : (0, T) \longrightarrow H^m, \quad \text{measurable}, \\ &\text{a.e. } t,\ G(t, \cdot) \quad H \longrightarrow H^m, \quad \text{continuous}, \\ &\|G(t, u)\|_{H^m} \leqslant C(1 + \|u(t)\|_H), \end{aligned} \tag{4.2}$$

where C is a positive constant.

(vi) We assume that f is measurable in (t, x) and belongs to the evolution space $L^{p'}(0, T; L^{p'}(\mathbb{D}))$. there exists a positive constant C such that

$$\int_0^T \|f(t)\|_{L^{p'}(D)}^{p'} dt \leq C, \quad \mathbb{P} - \text{a.s.} \tag{4.3}$$

Next, we define the concept of probabilistic weak solution for the problem (P).

Definition 31. A probabilistic weak solution of the problem (P) is a system

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P}, W, u),$$

where

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $\mathcal{F}_t$ is a filtration on it,

(2) W is a m-dimensional $\mathcal{F}_t$- standard Wiener process,

(3) $(\omega, t) \to u(\omega, t)$ is progressively measurable,

(4)

$$u \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; H)) \cap L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; V)),\ \forall p \in [2, \infty)$$

(5) for *all* $t \in [0, T]$, $u(t)$ satisfies the integral identity

$$\begin{aligned} &(u(t), v) + \int_0^t \langle A_s(u), v\rangle ds + \int_0^t \int_{\mathbb{D}} g(s, u) v(s, x)\, dx ds \\ &= \int_0^t \langle f(s), v\rangle ds + \left(\int_0^t G(s, u(s))\, dW(s), v\right), \forall v \in V,\ \mathbb{P} - a.s. \end{aligned} \tag{4.4}$$

Our first main result is

Theorem 33. *Assume that the conditions (i)-(vi) are satisfied. Then problem (P) has a weak solution in the sense of the above definition.*

By the definition 31, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener process W are unknown alongside the process u.

4.2.3 Regularization of problem (P) and Krylov-Rozovskii's result

As explained earlier, the standard methods of monotonicity and compactness used in order to solve nonlinear parabolic equations in the fundamental works of Minty [145], Browder [38] , Vishik [207], Lions [138] are not directly applicable to the deterministic version of problem (P); Brezis and Browder [34] had to rely on a regularization argument and other tools in their study of that problem. In the case at hand here, we shall also rely on a regularization which will reduce our problem to sequence of problems which fall in the class studied by Krylov and Rozovskii [123] (called (P_k) in the sequel). Our required weak solution will be constructed as a limit (in appropriate sense) of solutions of the regularized problem.

In order to introduce the regularization we introduce an intermediary probability space $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$ with a prescribed Wiener process $\bar{W}(t)$.

We define the truncated perturbation functions g_k of g by setting

$$g_k(u) =: T_k\left(g\left(u\right)\right) = \begin{cases} g(u) & \text{if} \quad |g(u)| < k, \\ k & \text{if} \quad g(u) \geq k \\ -k & \text{if} \quad g(u) \leqslant -k. \end{cases}$$

A useful fact is that $T_k(\cdot)$ is Lipschitz with respect to $(\cdot)$. Our regularized problem reads as follows:

$$(P_k) \begin{cases} du + \left[A_t(u) + g_k(t,x,u)\right]dt & = & f(t)dt + G(t,u)d\bar{W}(t) & \text{in} \quad Q_T \\ u(x,0) & = & 0, & \text{in} \quad \mathbb{D} \\ \dfrac{\partial^j u}{\partial N^j} & = & 0, & \text{on} \quad \partial\mathbb{D} \times (0,T). \end{cases}$$

By the properties of truncations, we have that $g_k \in L^\infty(Q_T)$, $\bar{\mathbb{P}}$-a.s. Moreover, from the definition of g_k and the assumption $g(t,x,r)r \geq 0$, one can obviously check that $g_k(t,x,r)r \geq 0$. Now, problem (P_k) is more regular than problem (P) in the sense that the L^∞-norm of the nonlinear term g_k is under control; this is in sharp contrast with the unboundedness of g.

We now summarize the results from the celebrated work of Krylov and Rozovskii [123] which are of interest to us.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(\mathcal{F}_t)_{t\in[0,T]}$ a filtration $(T > 0)$, $\mathbb{D}$ a bounded domain in $\mathbb{R}^n$ with sufficiently regular boundary $\partial\mathbb{D}$. Let $z(t)$ be a square-integrable martingale with values in $H = L^2(\mathbb{D})$ such that z is continuous in t, $W(t)$ a Wiener process with values in some separable Hilbert space E. Consider in $[0,T] \times \mathbb{D}$ the following problem

$$\begin{aligned} du = -(-1)^{|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|} D^{\alpha_1} \cdot \ldots \cdot D^{\alpha_m} A_{\alpha_1\ldots\alpha_m}\left(D^{\beta_1} \cdot \ldots \cdot D^{\beta_m} u\right) + \\ + B(D^{\beta_1} \cdot \ldots \cdot D^{\beta_m} u)dW(t,\omega) + dz(t,x,\omega) \end{aligned} \tag{4.5}$$

$$u(0,x,\omega) = u_0(x,\omega), \quad x \in \mathbb{D}, \tag{4.6}$$

$$D^{\beta_0} \cdot \ldots \cdot D^{\beta_{m-1}} u_{|_S} = 0, \quad \text{for all} \quad \beta_0, \ldots, \beta_{m-1}; \tag{4.7}$$

such that $|\beta_0| + \cdots |\beta_{m-1}| \leqslant m-1$.

It is well-known [123] that the operator A_t given by

$$A_t(u) = -(-1)^{|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|} D^{\alpha_1} \cdot \ldots \cdot D^{\alpha_m} A_{\alpha_1\ldots\alpha_m} \left(D^{\beta_1} \cdot \ldots D^{\beta_m} u\right)$$

and the function B satisfy the following conditions:
A_1) For all $v, v_1, v_2 \in V = W_0^{m,p}(\mathbb{D})$, $(t,\omega) \in [0,T] \times \Omega$, the function $\lambda \mapsto vA_t(v_1 + \lambda v_2, t, x)$ is continuous on $\mathbb{R}$.
A_2)

$$2\langle v_1 - v_2, A_t(v_1) - A_t(v_2)\rangle + \|B(v_1) - B(v_2)\|_Q^2 \leqslant K\|v_1 - v_2\|_H^p.$$

A_3) For a non-negative function $\Psi \in L^1(\mathbb{D})$, we have the coercivity condition

$$2\langle v, A_t(v)\rangle + \|B(v)\|_Q^2 + K\|v\|_V^p \leqslant \Psi + K\|v\|_H^2,$$

where $\|\cdot\|_Q$ stands for the norm in $\mathcal{L}_Q(E,H)$, the space of all linear operators $\Phi : Q^{1/2}(E) \to H$.

In [123], an existence result of strong solution of problem (4.5)-(4.7) was obtained under the additional assumptions:
A_4)

$$|A_{\alpha_1\ldots\alpha_m}(\xi,t,x,\omega)| \leqslant \Psi^{\frac{1}{q}}(t,x,\omega) + K \sum_{\beta_1,\ldots,\beta_m} |\xi^{\beta_1,\ldots,\beta_m}|^{p-1}.$$

A_5)

$$|B(\xi,t,x,\omega)|_E^2 \leqslant \Psi(t,x,\omega) + K \sum_{\beta_1,\ldots,\beta_m} |\xi^{\beta_1,\ldots,\beta_m}|^{p} + K|\xi^{0,\ldots,0}|^2.$$

(A_6) The mapping $(x,\omega) \to u_0(x,\omega)$ is $\mathcal{B}(\mathbb{D}) \times \mathcal{F}_0$;

$$||u_0(\omega)||_{L^2(\mathbb{D})} < \infty, \text{ for a.e. } \omega. \ \|\Psi(t,\omega)\|_{L^1(\mathbb{D})} < \infty, \text{for all } t, \text{ a.e. } \omega.$$

Definition 32. A strong solution of problem(4.5)-(4.7) is a process $u(\omega,t)$ with values in $L^2(\mathbb{D})$ defined on $\Omega \times [0,T]$, strongly continuous in H with respect to t (up to identification with a continuous modification), progressively measurable and satisfying

1) $u \in W_0^{m,p}(\mathbb{D})$ for a.e. $(\omega,t) \in \Omega \times [0,T]$,

$$\mathbb{E}\int_0^T \left(\|u(t)\|_{W_0^{m,p}(\mathbb{D})}^p + \|u(t)\|_{L_2(\mathbb{D})}^2\right) dt < \infty,$$

and

2)

$$u(t,\omega) = u_0 + \int_0^t A(u(s), s, \omega)ds + \int_0^t B(u(s,\omega), s, \omega)dW(s,\omega) + z(t,\omega)$$

in $W^{-m,p'}(\mathbb{D})$, $\forall t \in [0,T]$, and $\mathbb{P}$-a.s.

Theorem 34. *Under the above assumptions* (A_1)*-*(A_6)*, a strong solution of problem (4.5)-(4.7) exists in the sense of Definition 32.*

The initial step of the proof of this theorem is a suitable regularization of problem (P) on intermediary probability space $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$ with a prescribed Wiener process $\bar{W}(t)$ and we denote by $\bar{\mathcal{F}}_t$ the filtration generated by the Wiener process $\bar{W}$. The strongly nonlinear perturbing term g does not induce any mapping from $W_0^{m,p}(\mathbb{D})$ to it is dual, $W^{-m,p'}(\mathbb{D})$ since no a priori particular growth restriction is imposed. For this purpose, we truncate this perturbation:

Letting $g_k(t,u) := \mathcal{T}_k(g(t,u))$ be the truncation of the function g at levels $k \in \mathbb{N}$, then for all $u, v \in W_0^{m,p}(\mathbb{D})$, define the mapping $v \mapsto \mathcal{R}_k(t,u,v)$ by setting

$$\mathcal{R}_k(t,u,v) = \int_0^t \int_{\mathbb{D}} g_k(s,u)v(s,x)dxds.$$

Next, we set $\mathcal{A}(t,v) = A_t(v) + g_k(t,v)$ and as usual, we simply use $\mathcal{A}(t,v)$ to mean the map $\bar{\omega} \mapsto \mathcal{A}(t,v,\bar{\omega})$.

Since g_k is the truncation of g, by (iv), it is straightforward

$$r g_k(\cdot,x,r) \geq 0 \text{ and } |g_k(\cdot,x,r)| \leqslant |g(\cdot,x,r)| \leqslant |h(r)| \text{ on } [0,T] \times \bar{\Omega}. \tag{4.8}$$

For any $R > 0$, we have

$$|g_k(t,u)| = |g_k(t,u)| I_{(t,x):u \leqslant R} + |g_k(t,u)| I_{(t,x):u>R}. \tag{4.9}$$

This implies that

$$|g_k(t,u)| \leqslant \sup_{|r \leqslant |R} g_k(t,u) \leqslant h(R) \ \forall R \in [0,\infty). \tag{4.10}$$

By virtue of assumption (4.1) and Hölder's inequality, we derive the following estimate:

$$\begin{aligned}
|\mathbb{R}_k(t,u,v)| &\leqslant \int_0^t \int_{\mathbb{D}} \left[k I_{\{(s,x):\ meas(s,x) \leqslant k\}} + |g(u)|\right] |v(s,x)| dxds \\
&\leqslant k \int_0^t \int_{\mathbb{D}} |v(s,x)| dxds + \int_0^t \int_{\mathbb{D}} |h(u)||v(s,x)| dxds \\
&\leqslant k(meas(Q_t))^{1/p'} \left[\int_0^t \int_{\mathbb{D}} |v(s,x)|^p dxds\right]^{1/p} \\
&\quad + h(R)k(meas(Q_t))^{1/p'} \left[\int_0^t \int_{\mathbb{D}} |v(s,x)|^p dxds\right]^{1/p}.
\end{aligned} \tag{4.11}$$

Therefore

$$|\mathbb{R}_k(t,u,v)| \leqslant C(k)||v(s,x)||_{L^p(\mathbb{D})}.$$

Thus for any $t \in (0,T)$, the mapping $v \mapsto \mathcal{R}_k(t,u,v)$ defines an element of the dual space $W^{-m,p'}(\mathbb{D})$. For the sake of simplicity, let us denote it by $\mathcal{M}_k$. As a result of this, it is easy to show following similar arguments as in [33, Proposition 2, page 593], [39] and [208, page 127] that the operator $\mathcal{A}+\mathcal{M}-g_k$ is pseudomonotone. We refer to Browder [39] for the notion of pseudomonotonicity. We can easily prove using the parabolicity condition (ii) that the operator A_t is monotone i.e.,

$$\langle A_t(u)-A_t(v), u-v\rangle \geqslant 0 \text{ for any } u,v \in W_0^{m,p}(\mathbb{D}). \tag{4.12}$$

It thus far follows from the monotonicity of A_t that $\mathcal{A}$ is pseudomonotone as well.

On the other hand, we have

$$|g_k(t,x,u(t,x))| \leqslant \sup_{|r|\leqslant R}|g_k(t,x,u)| + R^{-1}ug_k(t,x,u(t,x)), \tag{4.13}$$

where $I_{\{(t,x):|u(t,x)|\leqslant R\}}$ is the indicator function. It follows from condition (iv), that

$$|g_k(t,x,u)| \leqslant |g(t,x,u)| \leqslant |h(u)| \leqslant C\{h(R)+|h(-R)|\} \tag{4.14}$$

on the set $\{(t,x): u(t,x)\leqslant R\}$.

Our regularized problem reads as follows:

Theorem 34 is a particular case of a more general result obtained in [123] for a stochastic evolution equation involving abstract operators in the framework of Sobolev spaces. Assume that (4.12) holds, then it is easy to check that problem (P_k) is a particular case of problem (4.5)-(4.7), with $\mathcal{A}(t,u,\bar{\omega}) = A_t + g_k$, $B = G$ and $z = 0$. In our case we only need the following assumptions:

(H_1) $(\mathcal{A},G)$-Coercitivity condition: there exist $c_3 > 0$, $c_4 \in \mathbb{R}$ and there exists an $(\bar{\mathcal{F}}_t)$-adapted process $h_1 \in L^1(Q\times\bar{\Omega})$ such that for all $(t,x)\in Q$, $v \in W_0^{m,p}(\mathbb{D})$

$$2\int_{\mathbb{D}}\mathcal{A}(t,v)v dx + ||G(t,v)||^2_{(L^2(\mathbb{D}))^m} - h_1(t) + c_3||v(t)||^p_{W_0^{m,p}(\mathbb{D})} \leqslant c_4||v(t)||^2_{L^2(\mathbb{D})} \text{ on } \bar{\Omega}.$$

(H_2) Hemicontinuity: for all $t\in[0,T]$, $\bar{\omega}\in\bar{\Omega}$ and $u,v,w \in W_0^{m,p}(\mathbb{D})$ the map $\lambda \mapsto \langle\mathcal{A}(t,u+\lambda v), w\rangle$ is continuous on $\mathbb{R}$.

(H_3) We assume that f is an $(\bar{\mathcal{F}}_t)$-adapted process and there exists $c_5 > 0$ such that

$$\int_0^T ||f(t,\bar{\omega})||^{p'}_{L^{p'}(\mathbb{D})} dt \leqslant c_5 \ \bar{\mathbb{P}} - \text{a.s.}$$

here $(L^2(\mathbb{D}))^m$ denotes the m copies of the space $L^2(\mathbb{D})$, $W^{-m,p'}(\mathbb{D})$ is the dual of $W_0^{m,p}(\mathbb{D})$ and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W_0^{m,p}(\mathbb{D})$ and $W^{-m,p'}(\mathbb{D})$ and $L^{p'}(\mathbb{D})$ denotes the dual of $L^p(\mathbb{D})$ and we need to make notation clear, instead of using the map $\omega \mapsto G(t,x,v,\bar{\omega})$ we write $G(t,v)$. Analogously, this applies also to the functions $g(t,x,v,\bar{\omega})$, $g_k(t,x,v,\bar{\omega})$, $f(t,x,\bar{\omega})$, $\mathcal{A}(t,x,v,\bar{\omega})$ and so on.

Let us show that the operator $\mathcal{A}$ satisfies a boundedness condition. In order to do so, we are interested only in the term

$$\langle A_t(u), u \rangle + \int_{\mathbb{D}} u g_k(t,u) dx$$

for now. For this purpose, we multiply the first equation in (P_k) by u then integrate over $[0,t]$ modulo a stopping time arguments. Then, we remark from (iii), (4.1) (4.8), (4.9), (4.14) and the compact embedding $W_0^{m,p}(\mathbb{D}) \subset\subset L^p(\mathbb{D})$ that

$$\begin{aligned}
&\int_0^t \langle \mathcal{A}(s,(u),u\rangle ds \\
&= \int_0^t \langle A_s(u), u \rangle ds + \int_0^t \int_{\mathbb{D}} u(s) g_k(s,u) dx ds \\
&\leqslant c_0 \int_0^t ||u(s)||^{p-1}_{W_0^{m,p}(\mathbb{D})} ds + c_0 \int_0^t \int_{\mathbb{D}} h_0(s,x) dx ds + \int_0^t \int_{\mathbb{D}} |u(s)||g(s,u)| ds \\
&\leqslant c_0 \int_0^t ||u(s)||^{p-1}_{W_0^{m,p}(\mathbb{D})} ds + c_0 ||h_0||_{L^1(Q_t)} + K(R) \int_0^t \int_{\mathbb{D}} \{|g(s,u(s))| + |u(s,x)|^{p-1} + 1\} ds \\
&\leqslant c_0 \int_0^t ||u(s)||^{p-1}_{W_0^{m,p}(\mathbb{D})} ds + c_0 ||h_0||_{L^1(Q_t)} + K(R) \left[h(R) meas(Q_t) + \int_0^t ||u(s)||^{p-1}_{L^p(\mathbb{D})} ds + meas(Q_t) \right] \\
&\leqslant c_6 \int_0^t ||u(s)||^{p-1}_{W_0^{m,p}(\mathbb{D})} ds + c_0 ||h_0||_{L^1(Q_t)}.
\end{aligned}$$

Thus, there exists a constant $c_7 > 0$ and an $(\bar{\mathcal{F}}_t)$-adapted function $h_3 \in L^{p'}(Q)$ such that for any $v \in W_0^{m,p}(\mathbb{D})$

$$||\mathcal{A}(t,v)||_{W^{-m,p'}(\mathbb{D})} \leqslant c_7 ||v(t)||^{p-1}_{W_0^{m,p}(\mathbb{D})} + h_3(t) \text{ on } \bar{\Omega}. \tag{4.15}$$

We remark from (H_1) and (4.15) that for all $v \in W_0^{m,p}(\mathbb{D})$ and $t \in [0,T]$

$$||G(t,v)||^2_{(L^2(\mathbb{D}))^m} \leqslant c_4 ||v(t)||^2_{L^2(\mathbb{D})} + h_1(t) + 2h_3(t)||v(t)||_{W_0^{m,p}(\mathbb{D})} + (2c_7 - c_3)||v(t)||^p_{W_0^{m,p}(\mathbb{D})} \text{ on } \bar{\Omega} \tag{4.16}$$

In order to be inline with the assumptions in [123], we set

$$\Psi = f + h_1 + h_3.$$

It is easily seen that Ψ is $\bar{\mathcal{F}}_t$-adapted. Using the embedding $L^{p'}(\mathbb{D}) \subset L^1(\mathbb{D})$ is continuous, it is straightforward that there exists a constant $c_9 > 0$ such that

$$||\Psi(t, \bar{\omega}||_{L^1(\mathbb{D})} \leqslant c_9 \text{ on } \bar{\Omega}.$$

A strong solution of problem (P_k) is defined as follows.

Definition 33. By a strong solution of problem (P_k), we mean a process $u : \bar{\Omega}\times[0,T] \longrightarrow H$, strongly continuous in H with respect to t (up to identification with a continuous modification), progressively measurable and satisfying

(1) $u \in V$ and for almost every (ω, t) we have

$$\bar{\mathbb{E}} \int_0^T \left(\|u(t)\|_V^p + \|u(t)\|_{L^2(\mathbb{D})}^2 \right) dt < \infty.$$

(2) for every $t \in [0,T]$, for all $v \in W_0^{m,p}(\mathbb{D})$, $u(t)$ satisfies the weak formulation of the problem (P_k)

$$\begin{aligned}(u(t), v) + \int_0^t \langle A_s(u(s)), v\rangle ds \int_{Q_t} g_k(s, u(s))v ds = \int_0^t \langle f(s, u(s)), v\rangle ds + \\ + \int_{Q_t} G(s, u(s))v dx d\bar{W}(s).\end{aligned}$$

It is easy to see that under our assumptions (i)-(vi), problem (P_k) satisfies the above conditions of Krylov and Rozovskii. Thus Theorem 34 implies

Theorem 35. *Under the hypotheses* (i)-(vi)*, (H_1)-(H_3) and if in addition (4.12) holds for each fixed $k \in \mathbb{N}$, the truncated problem (P_k) has a strong solution u_k in the sense of Definition 33.*

This result will be an important building block for the proof of the existence of weak probabilistic solutions for problem (P).

4.3 Proof of Theorem 33

4.3.1 Uniform A Priori Estimates

We start with the proof of key uniform a priori estimates. Throughout u_k is assumed to satisfy problem (P_k). Therefore the conditions made on (P_k) will be assumed to hold. Also all generic constants independently will be denoted by C and they may change from line to line. We have

Lemma 18. *There exists a constant C (independent of k) such that the following estimates hold for u_k*

$$\bar{\mathbb{E}} \sup_{0\leqslant t\leqslant T} \|u_k(t)\|^2_{L^2(\mathbb{D})} \leqslant C, \tag{4.17}$$

$$\bar{\mathbb{E}} \int_0^T \|u_k(t)\|^p_{W_0^{m,p}(\mathbb{D})} dt \leqslant C. \tag{4.18}$$

Proof. For a fixed $k \in \mathbb{N}$, we define the $\bar{\mathcal{F}}_t$-stopping times

$$\tau_k^N = \begin{cases} \inf\{t > 0 : \|u_k(t)\|_{L^2(\mathbb{D})} \geq N\} & \text{if} \quad \{\omega \in \Omega : \|u_k(t)\|_{L^2(\mathbb{D})} \geq N\} \neq \emptyset \\ T & \text{otherwise.} \end{cases}$$

By applying Itô's formula to the function $\|u_k(t)\|^2_{L^2(\mathbb{D})}$, we have

$$\begin{aligned} \|u_k(t)\|^2_{L^2(\mathbb{D})} &+ 2\sum_{|\beta|\leqslant m} \int_{Q_t} A_\beta\left(s, x, u_k(s), Du_k(s), \ldots, D^m u_k(s)\right) D^\beta u_k(s) dx ds + \\ &+ 2\int_0^t \int_{\mathbb{D}} g_k(s, u_k(s)) u_k(s) dx ds \\ &= 2\int_0^t \langle f(s), u_k(s)\rangle ds + \int_0^t ||G(s, u_k(s))||^2_{H^m} ds + 2\int_0^t \left(G(s, u_k(s)), u_k(s)\right) d\bar{W}(s). \end{aligned} \tag{4.19}$$

Hence, using assumption (iii) on A_t, we get

$$\begin{aligned} &\|u_k(t)\|^2_{L^2(\mathbb{D})} + 2c_1 \int_0^t \|u_k(s)\|^p_V ds - 2c_1 \int_{Q_t} h_1(s, x) dx ds + 2\int_{Q_t} g_k(u_k) u_k(s) dx ds \\ &\leqslant 2C \int_0^t \|f(s)\|_{V'} \|u_k(s)\|_V ds + \int_0^t \|G(s, u_k(s))\|^2_{H^m} ds + 2\int_0^t \left(G(s, u_k(s)), u_k(s)\right) d\bar{W}(s). \end{aligned} \tag{4.20}$$

Taking the supremum over $[0, t \wedge \tau_k^N]$, it follows from (4.19), (4.20), Young's inequality and assumption (v) that

$$\sup_{0\leqslant s\leqslant t\wedge\tau_k^N}\|u_k(s)\|^2_{L^2(\mathbb{D})}+2c_1\int_0^{t\wedge\tau_k^N}\|u_k(s)\|^p_V ds+2\int_0^{t\wedge\tau_k^N}\int_{\mathbb{D}} g_k(u_k)u_k(s)dxds \tag{4.21}$$
$$\leqslant 2C_\varepsilon C\int_0^{t\wedge\tau_k^N}\|f(s)\|^{p'}_{V'}ds+\varepsilon^p\int_0^{t\wedge\tau_k^N}\|u_k(s)\|^p_V ds+C\int_0^{t\wedge\tau_k^N}\left[1+\|u_k(s)\|^2_{L^2(\mathbb{D})}\right]ds$$
$$+2c_1\|h_1\|_{L^1(Q_{t\wedge\tau_k^N})}+2\sup_{s\in[0,t\wedge\tau_k^N]}\int_{Q_s}G(r,u_k(x,r))u_k(x,r)dxd\bar{W}(r),$$

where $a\wedge b=\min\{a,b\}$, and ε is an arbitrary positive number.
For the stochastic integral appearing on the right side of (4.21) we proceed first by taking mathematical expectation in both sides of (4.21) then, in view of assumption (v), and applying the Burkholder-Davis-Gundy, Cauchy-Schwarz and Young's inequalities, we have

$$2\bar{\mathbb{E}}\sup_{s\in[0,t\wedge\tau_k^N]}\left|\int_0^s\left(G(r,u_k(r)),u_k(r)\right)d\bar{W}(r)\right|$$
$$\leqslant 2\bar{\mathbb{E}}\left[\int_0^s\left(G(r,u_k(r)),u_k(r)\right)^2dr\right]^{\frac{1}{2}}$$
$$\leqslant 2\bar{\mathbb{E}}\left[\int_0^{t\wedge\tau_k^N}\|G(s,u_k(s))\|^2_{(L^2(\mathbb{D}))^m}\|u_k(s)\|^2_{L^2(\mathbb{D})}ds\right]^{\frac{1}{2}}$$
$$\leqslant \bar{\mathbb{E}}2\sup_{t\wedge\tau_k^N\in[0,t\wedge\tau_k^N]}\|u_k(s)\|_{L^2(\mathbb{D})}\left[\int_0^{t\wedge\tau_k^N}\|G(s,u_k(s))\|^2_{(L^2(\mathbb{D}))^m}ds\right]^{\frac{1}{2}}$$
$$\leqslant \eta\bar{\mathbb{E}}\sup_{s\in[0,t\wedge\tau_k^N]}\|u_k(s)\|^2_{L^2(\mathbb{D})}+C_\eta\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^N}\|G(s,u_k(s))\|^2_{(L^2(\mathbb{D}))^m}ds$$
$$\leqslant \eta\bar{\mathbb{E}}\sup_{s\in[0,t\wedge\tau_k^N]}\|u_k(s)\|^2_{L^2(\mathbb{D})}+C_\eta\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^N}\left[1+\|u_k(s)\|^2_{L^2(\mathbb{D})}\right]ds$$
$$=\eta\bar{\mathbb{E}}\sup_{s\in[0,t\wedge\tau_k^N]}\|u_k(s)\|^2_{L^2(\mathbb{D})}+C_\eta\left(t\wedge\tau_k^N\right)+C_\eta\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^N}\|u_k(s)\|^2_{L^2(\mathbb{D})}ds.$$

Combining all these estimates, we obtain

$$\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\|u_k(s)\|^2_{L^2(\mathbb{D})}+2c_3\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds$$
$$\leqslant 2\varepsilon^p\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds+2C_\varepsilon C\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}\|f(s)\|^{p'}_{W^{-m,p'}(\mathbb{D})}ds+C_4\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}\|u_k(s)\|^2_{L^2(\mathbb{D})}ds$$
$$+2c_8\bar{\mathbb{E}}\|h_1(s)\|_{L^1(Q_{t\wedge\tau_N})}+\eta C\bar{\mathbb{E}}\sup_{s\in t\wedge\tau_N}\|u_k(s)\|^2_{L^2(\mathbb{D})}+2\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}||h_3(s)||_{L^{p'}(\mathbb{D})}||v||_{W_0^{m,p}(\mathbb{D})}ds. \tag{4.22}$$

Noting that by Young's inequality

$$\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}||h_3(s)||_{L^{p'}(\mathbb{D})}||v||_{W_0^{m,p}(\mathbb{D})}ds \leqslant C_\varepsilon\bar{\mathbb{E}}\int_{Q_{t\wedge\tau_N}}||h_3(s)||^{p'}_{L^{p'}(\mathbb{D})}ds+\varepsilon^p\bar{\mathbb{E}}\int_0^{t\wedge\tau_N}||v(s)||^p_{W_0^{m,p}(\mathbb{D})}ds. \tag{4.23}$$

Combining assumption (H_4) with the estimates (4.22), (4.23) and in view of (vi), (4.2) and the fact that $g_k(s,x,u_k(s))u_k(s)\geq 0$ we deduce from the corresponding relation by appropriate choices of ε and η that

$$\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t\wedge\tau_k^N}\|u_k(s)\|^2_{L^2(\mathbb{D})}+2c_1\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^N}\|u_k(s)\|^p_V ds\leqslant C(T)+C\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^N}\|u_k(s)\|^2_{L^2(\mathbb{D})}ds.$$

Then Gronwall's inequality implies that for all $s\in[0,t\wedge\tau_k^N]$ and each $k\in\mathbb{N}$

$$\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t\wedge\tau_k^N}\|u_k(s)\|^2_{L^2(\mathbb{D})}+\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^N}\|u_k(s)\|^p_V ds\leqslant C. \tag{4.24}$$

Hence, passing to the limit in (4.24), as $N\to\infty$ and using the fact that $t\wedge\tau_k^N\to t,\ \mathbb{P}-a.s.$ we get (4.17) and (4.18). Thus, the lemma is proved. $\square$

The next result provides us with some higher-integrability estimates.

Lemma 19. *u_k satisfies the estimates*

$$\bar{\mathbb{E}}\sup_{t\in[0,T]}\|u_k(t)\|^q_{L^2(\mathbb{D})}\leqslant C, \tag{4.25}$$

$$\bar{\mathbb{E}}\int_0^t\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds\leqslant C; \tag{4.26}$$

for any $q\geq 2$; C is a constant independent of k.

Proof. We apply Itô's formula to the function $\left(\|u_k(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}}$. In view of (4.19), we get

$$\begin{aligned}
&d\left(\|u_k(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}}= \qquad (4.27)\\
&=q\left(\|u_k(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}-1}\left[-\langle A_t(u_k),u_k\rangle-\int_{\mathbb{D}}u_k(t)g_k(t,u_k(t))dx\right]dt+\\
&+q\left(\|u_k(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}-1}\left[\langle f(t),u_k(t)\rangle dt+\int_{\mathbb{D}}u_k(t)G(t,u_k(t))dxd\bar{W}(t)\right]+\\
&+q\frac{q-2}{2}\left(\|u_k(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}-2}\left(G(t,u_k(t)),u_k(t)\right)^2dt\\
&+\frac{q}{2}\left(\|u_k(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}-1}\|G(t,u_k(t))\|^2_{(L^2(\mathbb{D}))^m}dt.
\end{aligned}$$

Moreover, integrating (4.27) over $(0,t)$ yields

$$\begin{aligned}
&\|u_k(t)\|^q_{L^2(\mathbb{D})} + q\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\left[\langle A_s(u_k),u_k\rangle + \int_{\mathbb{D}} u_k(s)g_k(s,u_k(s))dx\right]ds \\
&\leqslant q\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\left[\langle f(s),u_k(s)\rangle ds + \int_{\mathbb{D}} u_k(s)G(s,u_k(s))dxd\bar{W}(s)\right] \\
&+ q\frac{q-2}{2}\int_0^t \|u_k(s)\|^{q-4}_{L^2(\mathbb{D})}\|G(s,u_k(s))\|^2_{(L^2(\mathbb{D}))^m}\|u_k(s)\|^2_{L^2(\mathbb{D})}ds \\
&+ \frac{q}{2}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|G(s,u_k(s))\|^2_{(L^2(\mathbb{D}))^m}ds. \qquad (4.28)
\end{aligned}$$

Using the coercivity condition of A_t, taking the supremum over $s\in[0,t]$ and mathematical expectation on both sides of (4.28) leads to

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^q_{L^2(\mathbb{D})} + q\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\left[c_1\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})} + \int_{\mathbb{D}} u_k g_k(s,u_k)dx\right]ds \qquad (4.29)\\
&\leqslant q\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\left[\langle f,u_k\rangle + \frac{1}{2}\|G(u_k)\|^2_{(L^2(\mathbb{D}))^m}\right]ds + q\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\int_{\mathbb{D}} h_1(s,x)dxds \\
&+ q\frac{(q-2)}{4}\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|G(u_k)\|^2_{(L^2(\mathbb{D}))^m}ds + q\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\int_0^s \|u_k(r)\|^{q-2}_{L^2(\mathbb{D})}\left(G(u_k),u_k\right)d\bar{W}.
\end{aligned}$$

We tackle the terms of (4.28) one by one. By Young's inequality with an arbitrary $\varepsilon>0$

$$\begin{aligned}
&\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\langle f(s),u_k(s)\rangle ds \qquad (4.30)\\
&\leqslant \bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|f(s)\|_{W^{-m,p'}(\mathbb{D})}\|u_k(s)\|_{W_0^{m,p}(\mathbb{D})}ds \\
&\leqslant \bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\left[C(\varepsilon)\|f(s)\|^{p'}_{W^{-m,p'}(\mathbb{D})} + \varepsilon\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}\right]ds \\
&\leqslant \bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}C(\varepsilon)\|f(s)\|^{p'}_{W^{-m,p'}(\mathbb{D})}ds + \varepsilon\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds \\
&\leqslant CC(\varepsilon)\bar{\mathbb{E}}\left[\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\int_0^t \|f(s)\|^{p'}_{W^{-m,p'}(\mathbb{D})}ds\right] + \varepsilon\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds.
\end{aligned}$$

For the first term on the right hand side of (4.30), we first use the assumption (vi) and then apply Young's inequality to get

$$\begin{aligned}
\bar{\mathbb{E}}\left[\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\int_0^t \|f(s)\|^{p'}_{W^{-m,p'}(\mathbb{D})}ds\right] &\leqslant C\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})} \\
&\leqslant C_\varepsilon + \varepsilon\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^q_{L^2(\mathbb{D})},
\end{aligned}$$

with any $\varepsilon > 0$. As in the above case, by (4.2) and Young's inequality, we have

$$\begin{aligned}\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|G(u_k)\|^2_{(L^2(\mathbb{D}))^m}ds &\leqslant \bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\left[1+\|u_k(s)\|_{L^2(\mathbb{D})}\right]^2 ds\\ &\leqslant 2\bar{\mathbb{E}}\int_0^t \left[\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}+\|u_k(s)\|^q_{L^2(\mathbb{D})}\right]ds\\ &\leqslant CT + C\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^q_{L^2(\mathbb{D})}ds. \end{aligned} \tag{4.31}$$

For the stochastic term we use Burkholder-Davis-Gundy, Young's and Hölder's inequalities. We have, for any $\varepsilon > 0$,

$$\begin{aligned}&\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s \|u_k(r)\|^{q-2}_{L^2(\mathbb{D})}\int_{\mathbb{D}}G(r,u_k)u_k dx d\bar{W}(r)\right| \qquad (4.32)\\ &\leqslant C\bar{\mathbb{E}}\left[\int_0^t \|u_k(s)\|^{2q-2}_{L^2(\mathbb{D})}\|G(s,u_k(s))\|^2_{(L^2(\mathbb{D}))^m}ds\right]^{\frac{1}{2}}\\ &\leq C\bar{\mathbb{E}}\sup_{0\leq s\leq t}\|u_k(s)\|^{q-1}_{L^2(\mathbb{D})}\left[\int_0^t\left(1+\|u_k(s)\|^2_{L^2(\mathbb{D})}\right)ds\right]^{\frac{1}{2}}\\ &\leqslant \varepsilon\bar{\mathbb{E}}\sup_{0\leq s\leq t}\|u_k(s)\|^q_{L^2(\mathbb{D})}+C_\varepsilon\bar{\mathbb{E}}\left[\int_0^t\left(1+\|u_k(s)\|^2_{L^2(\mathbb{D})}\right)ds\right]^{\frac{q}{2}}\\ &\leqslant \varepsilon\bar{\mathbb{E}}\sup_{0\leq s\leq t}\|u_k(s)\|^q_{L^2(\mathbb{D})}+C_\varepsilon T^{\frac{q-2}{2}}\bar{\mathbb{E}}\left[\int_0^t\left(1+\|u_k(s)\|^2_{L^2(\mathbb{D})}\right)^{\frac{q}{2}}ds\right]\\ &\leqslant \varepsilon\bar{\mathbb{E}}\sup_{0\leq s\leq t}\|u_k(s)\|^q_{L^2(\mathbb{D})}+C_\varepsilon T^{\frac{q-2}{q}}\left[T+\bar{\mathbb{E}}\int_0^t\|u_k(s)\|^q_{L^2(\mathbb{D})}ds\right].\end{aligned}$$

In view of condition (iii) and application of Young's inequality, we obtain with any $\varepsilon > 0$

$$\bar{\mathbb{E}}\int_0^t \|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\int_{\mathbb{D}}h_1(x,s)dxds \leqslant \varepsilon\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^q_{L^2(\mathbb{D})}+C_\varepsilon. \tag{4.33}$$

Collecting the results of estimates (4.30)-(4.33), it then follows from (4.29) that for sufficiently small $\varepsilon > 0$

$$\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^q_{L^2(\mathbb{D})}+\bar{\mathbb{E}}\int_0^t\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds \leqslant C\left(T\right)+C\bar{\mathbb{E}}\int_0^t\|u_k(s)\|^q_{L^2(\mathbb{D})}ds.$$

Thus using Gronwall's inequality we deduce that

$$\bar{\mathbb{E}}\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^q_{L^2(\mathbb{D})} \leqslant C, \tag{4.34}$$

$$\bar{\mathbb{E}}\int_0^t\|u_k(s)\|^{q-2}_{L^2(\mathbb{D})}\|u_k(s)\|^p_{W_0^{m,p}(\mathbb{D})}ds \leqslant C. \tag{4.35}$$

The proof of the lemma is thereby complete. □

We now prove one of the most important ingredient which essentially central in proving the tightness property of the Galerkin approximating sequence.

Lemma 20. *For all $k \geq 1$, we have*

$$\bar{\mathbb{E}} \sup_{|\theta|\leqslant\delta} \int_0^T \|u_k(t+\theta) - u_k(t)\|^{p'}_{W^{-m,p'}(\mathbb{D})} dt \leqslant C\delta^{1/(p-1)},$$

for any sufficiently small $\delta > 0$; C is a positive constant independent of k.

Proof. Suppose that $\theta > 0$ so that $t + \theta \in [0, T]$ for $t \in [0, T]$ and $u_k(t + \theta)$ is well defined. u_k is extended by zero outside $(0, T)$. A similar reasoning can be done whenever $\theta < 0$.

We have

$$u_k(t+\theta) - u_k(t) = -\int_t^{t+\theta} [A_s(u_k) + g_k(u_k)]\, ds + \int_t^{t+\theta} f(s) ds + \int_t^{t+\theta} G(u_k) d\bar{W}(s),$$

thanks to problem (P_k). We set $V = W_0^{m,p}(\mathbb{D})$ and $V^* = W^{-m,p'}(\mathbb{D})$. By definition

$$\|u_k(t+\theta) - u_k(t)\|_{W^{-m,p'}(\mathbb{D})} = \sup_{\varphi \in W_0^{m,p}(\mathbb{D}) : \|\varphi\|_{W_0^{m,p}(\mathbb{D})} = 1} \int_{\mathbb{D}} [u_k(t+\theta) - u_k(t)]\, \varphi(x)\, dx.$$

It then follows from Fubini's theorem (or Jensen's inequality) that

$$\begin{aligned} \|u_k(t+\theta) - u_k(t)\|_{V'} \leqslant & \int_t^{t+\theta} \|A_s(u_k)\|_{V'}\, ds + \int_t^{t+\theta} \|g_k(s, u_k)\|_{V'}\, ds + \\ & + \int_t^{t+\theta} \|f(s)\|_{V'}\, ds + \left\| \int_t^{t+\theta} G(s, u_k) d\bar{W}(s) \right\|_{V'}. \end{aligned} \tag{4.36}$$

Firstly, by assumption (i) and Hölder's inequality

$$\begin{aligned} \int_t^{t+\theta} \|A_s(u_k)\|_{V'}\, ds &\leqslant \int_t^{t+\theta} c_0 \left\{ \|u_k(s)\|^{p-1}_{V'} + \int_{\mathbb{D}} h_0(s,x) dx \right\} ds \\ &\leqslant c_0 \int_t^{t+\theta} \|u_k(s)\|^{p-1}_V ds + c_0 \int_t^{t+\theta} \int_{\mathbb{D}} h_0(s,x) dx ds \\ &\leqslant c_0 \theta^{\frac{1}{p}} \left[\int_t^{t+\theta} \|u_k(s)\|^p_V ds \right]^{\frac{1}{p'}} + c_0 \theta^{\frac{1}{p}} \left[\int_t^{t+\theta} \int_{\mathbb{D}} h_0^{p'}(s,x) dx ds \right]^{\frac{1}{p'}}. \end{aligned} \tag{4.37}$$

Similarly, we have

$$\int_t^{t+\theta} \|f(s)\|_{V'} ds \leqslant \theta^{\frac{1}{p}} \left[\int_t^{t+\theta} \|f(s)\|^{p'}_{V'} ds \right]^{\frac{1}{p'}}. \tag{4.38}$$

By the Sobolev embedding, we have

$$\begin{aligned}
&\left\|\int_t^{t+\theta} g_k(s,u_k)ds\right\|_{V'}\\
&\leqslant \int_t^{t+\theta} \|g_k(s,u_k)\|_{V'}\,ds \leqslant \theta^{\frac{1}{p}}\left[\int_t^{t+\theta}\|g_k(s,u_k)\|_{V'}^{p'}ds\right]^{\frac{1}{p'}}\\
&\leqslant \theta^{\frac{1}{p}}\left[\int_t^{t+\theta}\|g_k(s,u_k)\|_{L^{p'}(\mathbb{D})}^{p'}ds\right]^{\frac{1}{p'}} \leqslant \theta^{\frac{1}{p}}\left[\int_t^{t+\theta}\|g_k(s,u_k)\|_{L^{\infty}(\mathbb{D})}^{p'}ds\right]^{\frac{1}{p'}}. \qquad (4.39)
\end{aligned}$$

Combining (4.36)-(4.39) and taking the supremum over $\delta \in [0,1)$ and the mathematical expectation, we get

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{0\leq\theta\leqslant\delta}\|u_k(t+\theta)-u_k(t)\|_{V'}^{p'} \qquad (4.40)\\
&\leqslant c_0\delta^{\frac{p'}{p}}\bar{\mathbb{E}}\int_t^{t+\delta}\|u_k(s)\|_V^p ds + c_0\delta^{\frac{p'}{p}}\bar{\mathbb{E}}\int_t^{t+\delta}\|h_0(s)\|_{L^{p'}(\mathbb{D})}^{p'}ds\\
&+\delta^{\frac{p'}{p}}\bar{\mathbb{E}}\int_t^{t+\delta}\|g_k(s,u_k)\|_{L^\infty(\mathbb{D})}^{p'}ds + C\delta^{\frac{p'}{p}}\bar{\mathbb{E}}\int_t^{t+\delta}\|f(s)\|_{V'}^{p'}ds + \bar{\mathbb{E}}\sup_{0\leq\theta\leqslant\delta}\left\|\int_t^{t+\theta}G(s,u_k)d\bar{W}(s)\right\|_{V'}^{p'}.
\end{aligned}$$

For the last term on the right hand side of (4.40), we use the definition of the norm in $W^{-m,p'}(\mathbb{D})$, Fubini's theorem and Burkholder-Davis-Gundy's and Cauchy-Schwarz's inequalities. We have

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{0\leq\theta\leqslant\delta}\left\|\int_t^{t+\theta}G(s,u_k)d\bar{W}(s)\right\|_{W^{-m,p'}(\mathbb{D})}^{p'}\\
&\leq \bar{\mathbb{E}}\sup_{0\leq\theta\leqslant\delta}\left(\int_t^{t+\theta}\left(\int_{\mathbb{D}}G(s,u_k)\varphi dx\right)d\bar{W}(s)\right)^{p'}\\
&\leq \sup_{\varphi\in W_0^{m,p}(\mathbb{D}):||\varphi||_{W_0^{m,p}(\mathbb{D})}=1}\bar{\mathbb{E}}\left(\int_t^{t+\delta}\left(\int_{\mathbb{D}}G(s,u_k)\varphi dx\right)^2 ds\right)^{p'/2}\\
&\leq \sup_{\varphi\in W_0^{m,p}(\mathbb{D}):||\varphi||_{W_0^{m,p}(\mathbb{D})}=1}\bar{\mathbb{E}}\left(\int_t^{t+\delta}||\varphi||_{L^2(\mathbb{D})}^2\,||G(s,u_k)||_{(L^2(\mathbb{D}))^m}^2\,ds\right)^{p'/2}\\
&\leqslant C\bar{\mathbb{E}}\left(\int_t^{t+\delta}||G(s,u_k)||_{(L^2(\mathbb{D}))^m}^2\,ds\right)^{p'/2}\\
&\leqslant C\bar{\mathbb{E}}\left[\delta+\int_t^{t+\delta}\|u_k(s)\|_{L^2(\mathbb{D})}^2\,ds\right]^{\frac{p'}{2}}\\
&\leqslant C\left[\delta^{\frac{p'}{2}}+\delta^{\frac{p'}{2}}\bar{\mathbb{E}}\left(\sup_{t\leq s\leq t+\delta}\|u_k(s)\|_{L^2(\mathbb{D})}^{p'}\right)\right]\leq C\delta^{\frac{p'}{2}}, \qquad (4.41)
\end{aligned}$$

where we have used the estimate (4.25). Now, integrating (4.40) over $[0, T-\delta]$ and using (4.41), we get

$$\begin{aligned}
&\int_0^T \bar{\mathbb{E}} \sup_{\theta\leqslant\delta} \|u_k(t+\theta)-u_k(t)\|_{V'}^{p'} dt \\
\leq\ & C\delta^{\frac{1}{p-1}} \int_0^T \bar{\mathbb{E}} \int_t^{t+\delta} \|u_k(s)\|_V^p ds dt + C\delta^{\frac{1}{p-1}} \int_0^T \bar{\mathbb{E}} \int_t^{t+\delta} \|h_0(s)\|_{L^{p'}(\mathbb{D})}^{p'} ds dt \\
& + C\delta^{\frac{1}{p-1}} \int_0^T \bar{\mathbb{E}} \int_t^{t+\delta} \|g_k(s,u_k)\|_{L^\infty(\mathbb{D})}^{p'} ds dt + C\delta^{\frac{1}{p-1}} \int_0^T \bar{\mathbb{E}} \int_t^{t+\delta} \|f(s)\|_{V'}^{p'} ds dt + C\delta^{\frac{p'}{2}}.
\end{aligned}$$

At this stage we choose δ such that $\delta < (1/k)^{p'}$, so that we have a uniform control with respect to k on the term involving $\|g_k(s,u_k)\|_{L^\infty(\mathbb{D})}$.

In view of Lemma 19, the conditions on the data and the fact that $\dfrac{1}{p-1} \leqslant \dfrac{p'}{2}$ for $p \geq 2$, we get that

$$\int_0^T \bar{\mathbb{E}} \sup_{\theta\leqslant\delta} \|u_k(t+\theta)-u_k(t)\|_{W^{-m,p'}(\mathbb{D})}^{p'} dt \leq C\delta^{\frac{1}{p-1}}.$$

This completes the proof of the lemma. □

4.3.2 Compactness Results

We start this subsection by introducing some auxiliary spaces which will be needed for the compactness of probability measures generated by the pair $(\bar{W}, u_k)$.

Following [22], for any sequences μ_n, ν_n such that μ_n, $\nu_n \geq 0$ and μ_n, $\nu_n \longrightarrow 0$ as $n \longrightarrow \infty$, we define the set U_{μ_n,ν_n} of functions

$$\varphi \in L^p\left(0,T; W_0^{m,p}(\mathbb{D})\right) \cap L^\infty\left(0,T; L^2(\mathbb{D})\right)$$

such that

$$\sup_n \frac{1}{\nu_n} \sup_{|\theta|\leqslant\mu_n} \left(\int_0^T \|\varphi(t+\theta)-\varphi(t)\|_{W^{-m,p'}(\mathbb{D})}^{p'} dt \right)^{\frac{1}{p'}} < \infty,$$

where $p \in [1,\infty)$ and p' its Hölder's conjugate. We endow U_{μ_n,μ_n} with the norm

$$\begin{aligned}
\|\varphi\|_{U_{\mu_n,\nu_n}} =\ & \sup_{0\leqslant t\leqslant T} \|\varphi(t)\|_{L^2(\mathbb{D})} + \left(\int_0^T \|\varphi(t)\|_{W_0^{m,p}(\mathbb{D})}^p dt \right)^{\frac{1}{p}} + \\
& + \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta|\leqslant\mu_n} \int_0^T \|\varphi(t+\theta)-\varphi(t)\|_{W^{-m,p'}(\mathbb{D})}^{p'} dt \right)^{\frac{1}{p'}}.
\end{aligned}$$

U_{μ_n,ν_n} is then a Banach space.
We have the following compactness result from [21] which is interesting in its own right.

Lemma 21. *The set U_{μ_n,ν_n} just defined above is a compact subset of $L^2\left(0,T;L^2(\mathbb{D})\right)$.*

Let $1\leqslant p<\infty$. The space U_{p,μ_n,ν_n} consists of random variables φ on $\left(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}}\right)$ such that

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{0\leqslant t\leqslant T}\|\varphi(t)\|^2_{L^2(\mathbb{D})}<\infty,\\
&\bar{\mathbb{E}}\int_0^T\|\varphi(t)\|^p_{W_0^{m,p}(\mathbb{D})}dt<\infty,\\
&\bar{\mathbb{E}}\sup_n\frac{1}{\nu_n}\left(\sup_{|\theta|\leqslant\mu_n}\int_0^T\|\varphi(t+\theta)-\varphi(t)\|^{p'}_{W^{-m,p'}(\mathbb{D})}dt\right)^{\frac{1}{p'}}<\infty.
\end{aligned}$$

U_{p,μ_n,ν_n} is a Banach space under the norm

$$\begin{aligned}
\|\varphi\|_{U_{p,\mu_n,\nu_n}}=&\left(\bar{\mathbb{E}}\sup_{0\leqslant t\leqslant T}\|\varphi(t)\|^2_{L^2(\mathbb{D})}\right)^{\frac{1}{2}}+\left(\bar{\mathbb{E}}\int_0^T\|\varphi(t)\|^p_{W_0^{m,p}(\mathbb{D})}dt\right)^{\frac{1}{p}}+\\
&+\bar{\mathbb{E}}\sup_n\frac{1}{\nu_n}\left(\sup_{|\theta|\leqslant\mu_n}\int_0^T\|\varphi(t+\theta)-\varphi(t)\|^{p'}_{W^{-m,p'}(\mathbb{D})}dt\right)^{\frac{1}{p'}}.
\end{aligned}$$

Here φ is extended by zero outside the interval $[0,T]$.

The a priori estimates established in the previous Lemmas allow us to assert that for any $p\geq 2$, and for μ_n, ν_n such that the series $\sum_{n=1}^{\infty}\dfrac{(\mu_n)^{p'/p}}{\nu_n}$ converges, the sequence $\{u_k\ :\ k\in\mathbb{N}\}$ remain in a bounded subset of U_{p,μ_n,ν_n}. We extend φ by zero outside $[0,T]$.

Next, we shall prove the tightness property thanks to the finite difference estimate in the dual space $W^{-m,p'}(\mathbb{D})$ proved in Lemma 20. For more information about similar version of this proof, we refer for instance to [7, 22, 21, 67, 68, 168, 186, 182, 185].

4.3.3 Tightness result

Now, let $\mathcal{S}=C\left([0,T];\mathbb{R}^m\right)\times L^2(0,T;L^2(\mathbb{D}))$ and $\mathcal{B}(\mathcal{S})$ the σ-algebra of the Borel sets of $\mathcal{S}$. For each k, we construct the probability measure Λ_k on $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ as follows.
Consider the mapping

$$\varphi:\ \bar{\omega}\mapsto\left(\bar{W}(.,\bar{\omega}),u_k(.,\bar{\omega})\right)$$

defined on $\left(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}}\right)$ and taking values in $(\mathcal{S},\mathcal{B}(\mathcal{S}))$. Then

$$\Lambda_k(\mathbb{A})=\bar{\mathbb{P}}(\varphi^{-1}(\mathbb{A}))\text{ for all }\mathbb{A}\in\mathcal{B}(\mathcal{S}).$$

We now formulate the following key tightness result for our work.

Lemma 22. *The family of probability measures* $\{\Lambda_k\}_{k=1}^{\infty}$ *is tight on* $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. *That is, for any* $\varepsilon > 0$, *there exist some compact subsets* $\Sigma_\varepsilon \subset C([0,T];\mathbb{R}^m)$ *and* $Z_\varepsilon \subset L^2(0,T;L^2(\mathbb{D}))$ *such that*

$$\Lambda_k(\Sigma_\varepsilon \times Z_\varepsilon) \geq 1-\varepsilon, \ \forall k \in \mathbb{N}.$$

Proof. We shall find for a given $\varepsilon > 0$ a compact subset $\mathcal{K}_\varepsilon \subset \mathcal{S}$ such that

$$\Lambda_k\left((\bar{W}, u_k) \notin \mathcal{K}_\varepsilon\right) < \varepsilon. \tag{4.42}$$

It is enough to show that there exists two subsets $\Sigma_\varepsilon \subset C(0,T;\mathbb{R}^m)$ and $Z_\varepsilon \subset L^2(0,T;L^2(\mathbb{D}))$ such that

$$\bar{\mathbb{P}}\left(\bar{\omega} : \bar{W}(\cdot,\bar{\omega}) \in \Sigma_\varepsilon; u_k(\cdot,\bar{\omega}) \in Z_\varepsilon\right) \geqslant 1-\varepsilon. \tag{4.43}$$

It is sufficient to prove that

$$\bar{\mathbb{P}}(\bar{\omega} : \bar{W} \notin \Sigma_\varepsilon) \leqslant \frac{\varepsilon}{2}; \tag{4.44}$$

$$\bar{\mathbb{P}}(\bar{\omega} : u_k \notin Z_\varepsilon) \leqslant \frac{\varepsilon}{2}. \tag{4.45}$$

In order to prove (4.44), we define $\Theta_\delta^k(\bar{W})$, the modulus of continuity of $\bar{W}$ by

$$\Theta_\delta^k(\bar{W}) := \sup\left\{|\bar{W}(t) - \bar{W}(s)| : |t-s| \leqslant \delta; s,t \in [0,T]\right\}, \tag{4.46}$$

and the set

$$\Sigma_\varepsilon = \left\{\bar{W} : \sup_{t\in[0,T]} |\bar{W}(t)| \leqslant M_\varepsilon, \Theta_{\delta_N}^k(\bar{W}) \leqslant \frac{L_\varepsilon}{N} \text{ with } \delta < \delta_N, \forall N\right\}$$

where $\delta_N = \dfrac{T}{N^6}$ is the length of $\left\{\dfrac{j}{N^6}\right\}$, the subdivision of the interval $[0,T]$ and $L_\varepsilon, M_\varepsilon$ are positive constants to be chosen later. Indeed, the set Σ_ε is compact thanks to Arzela-Ascoli's result.

We have

$$\begin{aligned}
\bar{\mathbb{P}}(\bar{\omega} : \bar{W} \notin \Sigma_\varepsilon) &= \bar{\mathbb{P}}\left(\bar{\omega} : \left\{\sup_{0\leqslant t\leqslant T} |W(t,\bar{\omega})| > M_\varepsilon\right\} \bigcup \left\{\Theta_{\delta_N}^k(\bar{W}(\bar{\omega})) > \frac{L_\varepsilon}{N}\right\}\right) \\
&= \bar{\mathbb{P}}\left(\bar{\omega} : \left\{\sup_{0\leqslant t\leqslant T} |W(t,\bar{\omega})| > M_\varepsilon\right\}\right) + \bar{\mathbb{P}}\left(\bigcup_{N\in\mathbb{N}} \left\{\Theta_{\delta_N}^k(\bar{W}(\bar{\omega})) > \frac{L_\varepsilon}{N}\right\}\right) \\
&\leqslant \frac{C}{M_\varepsilon} \bar{\mathbb{E}} \sup_{0\leqslant t\leqslant T} |\bar{W}(t)| + \sum_{n\in\mathbb{N}} \bar{\mathbb{P}}\left(\bar{\omega} : \left\{\bar{W}(\cdot) : \Theta_{\delta_N}^k(\bar{W}) > \frac{L_\varepsilon}{N}\right\}\right).
\end{aligned}$$

Let $I=[s,t]\subset[0,T]$. We take the subdivision of of $[0,T]$ with length $\frac{T}{N^6}$. Let $\mathcal{I}_j$ be the j^{th} subsets of $[0,T]$. Using this subdivision, we obtain

$$\begin{aligned}
&\bar{\mathbb{P}}\left(\bar{\omega}:\bar{W}(t,\bar{\omega})\notin\Sigma_\varepsilon\right)\\
&\leqslant \frac{C}{M_\varepsilon}\bar{\mathbb{E}}\sup_{0\leqslant t\leqslant T}|\bar{W}(t)|+\sum_{N\in\mathbb{N}}\bar{\mathbb{P}}\left(\bigcup_{j=1}^{[TN^6]}\left(\bar{W}(t,\bar{\omega}):\sup_{s,t\in\mathcal{I}_j}|\bar{W}(t)-\bar{W}(s)|>L_\varepsilon N^{-1}\right)\right)\\
&\leqslant \frac{C}{M_\varepsilon}\bar{\mathbb{E}}\sup_{0\leqslant t\leqslant T}|\bar{W}(t)|+\sum_{N\in\mathbb{N}}\left(\sum_{j=1}^{[TN^6]}\bar{\mathbb{P}}\left(\bar{W}(t,\bar{\omega}):\sup_{s,t\in\mathcal{I}_j}|\bar{W}(t)-\bar{W}(s)|>L_\varepsilon N^{-1}\right)\right).
\end{aligned}$$

We recall the following classical result

$$\bar{\mathbb{E}}\left(|\bar{W}(t)-\bar{W}(s)|^{2n}\right)\leqslant(2n-1)!(t-s)^n,\ \forall n\in\mathbb{N},s,t\in\mathcal{I}_j. \tag{4.47}$$

Using this result and Markov's inequality, we have

$$\begin{aligned}
&\bar{\mathbb{P}}\left(\bar{\omega}:\bar{W}(t,\bar{\omega})\notin\Sigma_\varepsilon\right)\\
&\leqslant \frac{C}{M_\varepsilon}\sqrt{T}+\sum_{N\in\mathbb{N}}\sum_{j=1}^{[TN^6]}\left(\frac{N}{L_\varepsilon}\right)^4\bar{\mathbb{E}}\sup_{s,t\in\mathcal{I}_j}|\bar{W}(t)-\bar{W}(s)|^4\\
&\leqslant \frac{C}{M_\varepsilon}\sqrt{T}+9T^2L_\varepsilon^{-4}\sum_{N\in\mathbb{N}}\sum_{j=1}^{[TN^6]}N^{-12}N^4\\
&\leqslant \frac{C}{M_\varepsilon}\sqrt{T}+T^2CL_\varepsilon^{-4}\sum_{N\in\mathbb{N}}N^{-2}.
\end{aligned}$$

We choose

$$M_\varepsilon>\frac{4C\sqrt{T}}{\varepsilon}\text{ and }L_\varepsilon^4>\frac{4C_T}{\varepsilon}\sum_{N\in\mathbb{N}}\frac{1}{N^2},$$

to deduce the prove of (4.44).
For (4.45), we choose $Z_\varepsilon\subset U_{\mu_N,\nu_N}$ to be the set of the form: Z_ε is the set of elements $z\in U_{\mu_N,\nu_N}$ satisfying

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}||z(t)||^2_{L^2(\mathbb{D})}\leqslant P_\varepsilon,\int_0^T||z(t)||^p_{W_0^{m,p}(\mathbb{D})}dt\leqslant q_\varepsilon,\\
&\sup_{|\theta|\leqslant\mu_N}\int_0^{T-\mu_N}||z(t+\theta)-z(t)||^{p'}_{W^{-m,p'}(\mathbb{D})}dt\leqslant r_\varepsilon\nu_N.
\end{aligned}$$

where $P_\varepsilon, q_\varepsilon, r_\varepsilon$ are positive constant to be chosen later on and μ_N and ν_N can be chosen so that $\mu_n, \nu_N \to 0$ as $N \to \infty$ and satisfying

$$\sum_{N\in\mathbb{N}} \frac{\mu_N^{\frac{1}{p-1}}}{\nu_N} < \infty.$$

For instance, we take

$$\nu_N = \frac{1}{N}, \mu_N = \frac{1}{N^\kappa} \text{ for } \kappa > 2(p-1).$$

Thus Z_ε is a compact subset of $L^2(0,T;L^2(\mathbb{D}))$ by Lemma 21.

$$\begin{aligned}
&\bar{\mathbb{P}}\left(\bar\omega : u_k(t,\bar\omega) \notin Z_\varepsilon\right) \\
&\leqslant \bar{\mathbb{P}}\left(\sup_{0\leqslant t\leqslant T} ||u_k(t,\bar\omega)||^2_{L^2(\mathbb{D})} > P_\varepsilon\right) + \bar{\mathbb{P}}\left(\int_0^T ||u_k(t,\bar\omega)||^p_{W_0^{m,p}(\mathbb{D})} dt > q_\varepsilon\right) + \\
&+ \bar{\mathbb{P}}\left(\bigcup_{N\in\mathbb{N}} \sup_{|\theta|\leqslant\mu_N} \int_0^{T-\mu_N} ||u_k(t+\theta,\bar\omega) - u_k(t,\bar\omega)||^{p'}_{W^{-m,p'}(\mathbb{D})} > r_\varepsilon \nu_N\right)
\end{aligned}$$

Once more using Markov's inequality, we have

$$\begin{aligned}
&\bar{\mathbb{P}}\left(\bar\omega : u_k(t,\bar\omega) \notin Z_\varepsilon\right) \\
&\leqslant \frac{1}{P_\varepsilon}\bar{\mathbb{E}} \sup_{0\leqslant t\leqslant T} ||u_k(t,\bar\omega)||^2_{L^2(\mathbb{D})} + \frac{1}{q_\varepsilon}\bar{\mathbb{E}} \int_0^T ||u_k(t,\bar\omega)||^p_{W_0^{m,p}(\mathbb{D})} dt + \\
&+ \sum_{N=1}^{\infty} \frac{1}{r_\varepsilon \nu_N} \bar{\mathbb{E}} \sup_{|\theta|\leqslant\mu_N} \int_0^T ||u_k(t+\theta) - u_k(t)||^{p'}_{W^{-m,p'}(\mathbb{D})} dt
\end{aligned}$$

since

$$\int_0^{T-\mu_N} ||u_k(t+\theta) - u_k(t)||^{p'}_{W^{-m,p'}(\mathbb{D})} dt \leqslant \int_0^T ||u_k(t+\theta) - u_k(t)||^{p'}_{W^{-m,p'}(\mathbb{D})} dt.$$

By virtue of Lemma 18, 19, 20, we deduce that

$$\bar{\mathbb{P}}\left(\bar\omega : u_k(t,\bar\omega) \notin Z_\varepsilon\right) \leqslant C\left(\frac{1}{P_\varepsilon} + \frac{1}{q_\varepsilon} + \frac{1}{r_\varepsilon}\right).$$

We choose $P_\varepsilon, q_\varepsilon, r_\varepsilon$ so that

$$P_\varepsilon, q_\varepsilon, r_\varepsilon > \frac{6C}{\varepsilon}$$

yields the proof of (4.45); hence (4.43) which thus proves (4.42) by taking $\mathcal{K}_\varepsilon = \Sigma_\varepsilon \times Z_\varepsilon$. From this, the proof of Lemma 22 is concluded. $\square$

4.3.4 Application of Prokhorov and Skorokhod theorems

The tightness of the family of probability measures Λ_k proved in the previous subsection and Prokhorov's theorem imply that $\{\Lambda_k\}_{k=1}^{\infty}$ is relatively compact in the set of probability measures in $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, equipped with the weak convergence topology. Therefore, we can extract a subsequence $\{\Lambda_{k_i}\}_{i=1}^{\infty}$which weakly converges to a probability measure Λ. That is,

$$\Lambda_{k_i} \to \Lambda \text{ weakly}$$

i.e., for any continuous and bounded function $\varphi : \mathcal{S} \to \mathbb{R}$ we have

$$\int_{\mathcal{S}} \varphi(\bar{W}(t), z(t)) d\Lambda_{k_i} \to \int_{\mathcal{S}} \varphi(\bar{W}(t), z(t)) d\Lambda.$$

Hence by Skorokhod's theorem, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and pairs of random variables (W_{k_i}, u_{k_i}) and (W, u) on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in S such that

$$\text{the probability law of } (W_{k_i}, u_{k_i}) \text{ is } \Lambda_{k_i}, \tag{4.48}$$

$$W_{k_i}(., \omega) \longrightarrow W(., \omega) \text{ in } C([0,T]; \mathbb{R}^m), \text{ as } i \longrightarrow \infty, \ \mathbb{P} - \text{a.s.}, \tag{4.49}$$

$$u_{k_i}(., \omega) \longrightarrow u(., \omega) \text{ in } L^2\left(0, T; L^2(\mathbb{D})\right), \text{ as } i \longrightarrow \infty, \ \mathbb{P} - \text{a.s.}, \tag{4.50}$$

$$\text{the probability law of } (W, u) \text{ is } \Lambda. \tag{4.51}$$

Next, we choose for the filtration $(\mathcal{F}_t)$ by setting

$$\mathcal{F}_t = \sigma\{(W(s), u(s)) : \ 0 \le s \le t\}.$$

It turns out, according to similar reasoning used in [7, 22, 21, 67, 68, 168, 185, 182, 185]; one can prove that W is a m-dimensional $\mathcal{F}_t$-standard Wiener.
Following these references, one can also prove that the pair (W_{k_i}, u_{k_i}) satisfies the the following equation (the so-called weak formulation)

$$\begin{aligned} &(u_{k_i}(t), v) + \int_0^t \langle A_s(u_{k_i}(s)), v\rangle ds + \int_0^t (g_{k_i}(s, u_{k_i}), v) ds \\ = &\int_0^t \langle f(s), v\rangle ds + \int_0^t (G(s, u_{k_i}), v) dW_{k_i}(s), \ \ \forall v \in W_0^{m,p}(\mathbb{D}) \ \mathbb{P} - \text{a.s.}, \end{aligned} \tag{4.52}$$

Unlike the corresponding case in Chapter 3, we cannot directly use Lemma 7 (Chapter 2) since we are not in an infinite dimensional situation. Therefore, the approach in the just cited papers become necessary.

4.3.5 Existence of weak probabilistic solution

We split this subsection into steps.

Step1. This step is devoted to some weak convergence results. Owing to (4.52), Lemmas 18, 19 and 20 can be applied to u_{k_i}. That is, for any $p, q \in [2, \infty)$, we have

$$\mathbb{E} \int_0^T \|u_{k_i}(t)\|^p_{W_0^{m,p}(\mathbb{D})} dt \leqslant C, \tag{4.53}$$

$$\mathbb{E} \sup_{t\in[0,T]} \|u_{k_i}(t)\|^q_{L^2(\mathbb{D})} \leqslant C, \tag{4.54}$$

$$\mathbb{E} \sup_{|\theta|\leqslant\delta} \int_0^T \|u_{k_i}(t+\theta) - u_{k_i}(t)\|^{p'}_{W^{-m,p'}(\mathbb{D})} dt \leqslant C\delta^{1/(p-1)}. \tag{4.55}$$

Thus, there exists a new subsequence of u_{k_i} which we still denote by the same symbol u_{k_i} such that

$$u_{k_i} \rightharpoonup u \text{ weakly in } L^p\left(\Omega, \mathcal{F}, \mathbb{P}; L^p\left(0, T; W_0^{m,p}(\mathbb{D})\right)\right), \tag{4.56}$$

$$u_{k_i} \rightharpoonup u \text{ weakly in } L^q\left(\Omega, \mathcal{F}, \mathbb{P}; L^r\left(0, T; L^2(\mathbb{D})\right)\right) \ \forall r \in [2, \infty), \tag{4.57}$$

$$u_{k_i}(\omega) \rightharpoonup u(\omega) \text{ weakly star in } L^\infty\left(0, T; L^2(\mathbb{D})\right).$$

Furthermore u satisfies

$$\mathbb{E} \int_0^T \|u(t)\|^p_{W_0^{m,p}(\mathbb{D})} dt \leqslant C,$$

$$\mathbb{E} \left(\int_0^T \|u(t)\|^r_{L^2(\mathbb{D})} dt \right)^{q/r} \leqslant C \ \forall r \in [2, \infty),$$

$$||u(\omega)||_{L^\infty(0,T;L^2(\mathbb{D}))} < \infty, \ \mathbb{P} - a.s.$$

From Lemma 18 and the conditions on the operator A_t, we have

$$\mathbb{E} \int_0^T \|A_t(u_{k_i}(t))\|^{p'}_{W^{-m,p'}(\mathbb{D})} dt \leqslant C.$$

Hence there exists a random function $\varpi \in L^{p'}\left(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}\left(0, T; W^{-m,p'}(\mathbb{D})\right)\right)$ such that up to extraction of a subsequence

$$A_t(u_{k_i}(t)) \rightharpoonup \varpi(t) \text{ weakly in } L^{p'}\left(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}\left(0, T; W^{-m,p'}(\mathbb{D})\right)\right). \tag{4.58}$$

Following standard arguments as in Chapter 3, one shows that

$$u_{k_i}(T) \longrightarrow u(T) \text{ weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})). \tag{4.59}$$

Thanks to the higher integrability (4.54) and Vitali's theorem, we obtain

$$u_{k_i} \longrightarrow u \text{ strongly in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; L^2(\mathbb{D}))\right) \tag{4.60}$$
$$\text{and almost everywhere in } \Omega \times Q_T.$$

Thus, there exists a new subsequence still denoted u_{k_i} for simplicity of notation such that for almost every (t,ω) we have

$$u_{k_i} \longrightarrow u \text{ strongly in } L^2(\mathbb{D}) \text{ (with respect to the measure } d\mathbb{P}\times dt). \tag{4.61}$$

Step 2. In the next lines we prove that the sequence $\{g_{k_i}(u_{k_i})\}$ is uniformly integrable in $L^1(\Omega,\mathcal{F},\mathbb{P};L^1(Q_t))$ for any $t\in[0,T]$.

By applying Ito's formula to $\|u_{k_i}(t)\|^2_{L^2(\mathbb{D})}$ and using the equation (4.52), we get

$$2\int_{Q_t} g_{k_i}(u_{k_i})u_{k_i}dxds = -2\sum_{|\beta|\leqslant m}\int_{Q_t} A_\beta\left(u_{k_i},Du_{k_i},\dots,D^m u_{k_i}\right)D^\beta u_{k_i}dxds+$$
$$2\int_0^t\langle f(s),u_{k_i}\rangle ds+\int_0^t\|G(s,u_{k_i})\|^2_{(L^2(\mathbb{D}))^m}+2\int_{Q_t}G(s,u_{k_i})u_{k_i}dxdW_{k_i}(s)-\|u_{k_i}(t)\|^2_{L^2(\mathbb{D})}. \tag{4.62}$$

Owing to Lemma 18 and the conditions on A_t, it follows from this relation and Burkholder-Gundy-Davis's inequality that

$$\mathbb{E}\int_{Q_t} g_{k_i}(u_{k_i})u_{k_i}\,dx\,ds\leq C \text{ for all } i\in\mathbb{N}. \tag{4.63}$$

Hence $\{g_{k_i}(u_{k_i})u_{k_i}\}$ is bounded in $L^1(\Omega,\mathcal{F},\mathbb{P};L^1(Q_t))$.

A crucial role will be played by condition (iv). Indeed, for any $R>0$, we have

$$|g_{k_i}(t,x,u_{k_i}(t,x))| = |g_{k_i}(t,x,u_{k_i}(t,x))|I_{\{(t,x):u_{k_i}<R\}}+|g_{k_i}(t,x,u_{k_i}(t,x))|I_{\{(t,x):u_{k_i}\geqslant R\}}.$$

This in conjunction with the sign condition $u_{k_i}g_{k_i}(u_{k_i})\geqslant 0$ implies that

$$|g_{k_i}(t,x,u_{k_i}(t,x))|\leqslant \sup_{|s|\leqslant R}|g_{k_i}(t,x,s)|+R^{-1}u_{k_i}g_{k_i}(t,x,u_{k_i}(t,x)); \tag{4.64}$$

where $I_{\{(t,x):|u(t,x)|<R\}}$ is the indicator function. For all $i\in\mathbb{N}$, we know from condition (iv), that

$$|g_{k_i}(r)|\leqslant|g(r)|\leqslant|h(r)|\leqslant C\{h(R)+|h(-R)|\} \tag{4.65}$$

provided that $r\leqslant R$. Hence for any subset Σ of Q_t

$$\mathbb{E}\int_\Sigma|g_{k_i}(u_{k_i})|\,dxds\leqslant K(R)\text{ meas}(\Sigma)+R^{-1}C, \tag{4.66}$$

thanks to (iv), (4.64) and (4.65). Thus $\{g_{k_i}(u_{k_k})\}$ is bounded in $L^1(\Omega,\mathcal{F},\mathbb{P};L^1(\Sigma))$. Hence $\forall\varepsilon>0$, $\exists\delta>0$ and a set $\Sigma_\varepsilon\subset Q_T$ with meas$(\Sigma_\varepsilon)<\infty$ such that for all $i\in\mathbb{N}$

$$\mathbb{E}\int_\Sigma|g_{k_i}(u_{k_i})|\,dxds\leqslant\varepsilon;$$

and

$$\mathbb{E}\int_{\Sigma_\varepsilon^c} |g_{k_i}(u_{k_i})|\, dxds \leqslant \varepsilon.$$

Hence $\{g_{k_i}(u_{k_i})\}$ is equi-integrable in $L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_t))$. This notion of equi-integrability is of fundamental significance. Thus, by virtue of De La Vallee Poussin principles, the sequence $\{g_{k_i}(t, x, u_{k_i})\}_{i\in\mathbb{N}}$ is uniformly integrable in $L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_t))$. Now we rely on a deep subject in functional analysis known as the Dunford-Pettis theorem; which enables us to deduce that $\{g_{k_i}(t, x, u_{k_i})\}_{i\in\mathbb{N}}$ is relatively sequentially compact for the weak topology $\sigma(L^1, L^\infty)$. The following is a consequence of (4.60) and (4.61):

$$u_{k_i}(\omega) \longrightarrow u(\omega)\, \text{a.e. in}\, \Omega \times Q_t. \tag{4.67}$$

Moreover, in view of the Lipschitzity of $T_{k_i}(g_{k_i})$, we have

$$|g_{k_i}(u_{k_i}) - g_{k_i}(u)| \leq C\, |g(u_{k_i}) - g(u)|\,, \tag{4.68}$$

pointwise; C is independent of i. Since $g(u)$ is continuous with respect to u, it follows from (4.67) that

$$g(u_{k_i}) \longrightarrow g(u) \text{ a.e. in } \Omega \times Q_t.$$

This together with (4.68) imply that

$$g_{k_i}(u_{k_i}) - g_{k_i}(u) \to 0, \text{ a.e. in } \Omega \times Q_t.$$

But

$$g_{k_i}(u) \longrightarrow g(u)\, \text{a.e. in}\, \Omega \times Q_t. \tag{4.69}$$

Hence, we get

$$g_{k_i}(u_{k_i}) \longrightarrow g(u)\, \text{a.e. in } \Omega \times Q_t. \tag{4.70}$$

By the Vitali convergence theorem, however the equi-uniform integrability of the sequence $\{g_{k_i}(u_{k_i})\}$ in conjunction with the a.e. convergence (4.70) yields the following strong convergence:

$$g_{k_i}(u_{k_i}) \longrightarrow g(u) \text{ in } L^1\left(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_t)\right).$$

Thus, it follows that

$$g_{k_i}(u_{k_i}) \longrightarrow g(u) \text{ strongly in } L^1\left(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_T)\right). \tag{4.71}$$

Hence $g(u) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(Q_T))$.

Step 3. We prove in this section the convergence of the stochastic integral

$$\int_0^t G(s, u_{k_i}) dW_{k_i}(s).$$

We intend to use integration by parts. But since the integrand is not smooth with respect to t, we introduce a suitable regularization in order to overcome that obstacle. For that purpose, let ϱ be a standard mollifier, we define

$$G^{\varepsilon}(t,u)=\frac{1}{\varepsilon}\int_0^T\varrho\left(\frac{s-t}{\varepsilon}\right)G(s,u)ds;$$

G^{ε} is smooth in t and continuous in u and we have the uniform estimate:

$$\mathbb{E}\int_0^T\|G^{\varepsilon}(t,u)\|^2_{(L^2(\mathbb{D}))^m}dt\leqslant\mathbb{E}\int_0^T\|G(t,u)\|^2_{(L^2(\mathbb{D}))^m}dt \tag{4.72}$$

and

$$G^{\varepsilon}(.,u)\longrightarrow G(.,u)\text{ in }L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2\left(0,T;\left(L^2(\mathbb{D})\right)^m\right)\right), \tag{4.73}$$

as $\varepsilon\to 0$.

Integrating by parts, we get

$$\int_0^t G^{\varepsilon}(s,u_{k_i})dW_{k_i}(s)=G^{\varepsilon}(t,u_{k_i})W_{k_i}(t)-\int_0^t G^{\varepsilon\prime}(s,u_{k_i})W_{k_i}(s)ds, \tag{4.74}$$

By Fubini's theorem, Burkholder-Davis-Gundy's inequality and (4.72), we have

$$\mathbb{E}\left\|\int_0^t G^{\varepsilon}(s,u_{k_i})dW_{k_i}(s)\right\|^2_{L^2(\mathbb{D})}\leqslant\mathbb{E}\int_0^t\|G^{\varepsilon}(s,u_{k_i})\|^2_{(L^2(\mathbb{D}))^m}ds\leqslant C. \tag{4.75}$$

Similarly,

$$\int_0^t G^{\varepsilon}(s,u)dW(s)=G^{\varepsilon}(t,u)W(t)-\int_0^t G^{\varepsilon\prime}(s,u)W(s)ds. \tag{4.76}$$

Owing to (4.61), we have that

$$G^{\varepsilon}(t,u_{k_i})\longrightarrow G^{\varepsilon}(t,u),\ \text{almost everywhere in }\Omega\times(0,T). \tag{4.77}$$

It then follows from the definition of G^{ε} ($G^{\varepsilon\prime}(t,u)$ is still continuous in u), (4.74) and (4.49) that

$$\lim_{i\longrightarrow\infty}\int_0^t G^{\varepsilon}(s,u_{k_i})dW_{k_i}(s)=G^{\varepsilon}(t,u)W(t)-\int_0^t G^{\varepsilon\prime}(s,u)W(s)ds, \tag{4.78}$$

pointwise in x for almost all ω. Hence, by (4.76) and (4.78), we get

$$\lim_{i\longrightarrow\infty}\int_0^t G^{\varepsilon}(s,u_{k_i})dW_{k_i}(s)\to\int_0^t G^{\varepsilon}(s,u)dW(s), \tag{4.79}$$

pointwise in x for almost all ω.

By (4.75), the sequence of stochastic integrals $\left(\int_0^t G^{\varepsilon}(s,u_{k_i})dW_{k_i}(s)\right)_{i\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D}))$ for any $t\in[0,T]$, then it is uniformly integrable in the space $L^r(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$ for any $1 \leq r < 2$. Combining this with (4.79), we are able to use Vitali's theorem in order to obtain that

$$\int_0^t G^\varepsilon(s, u_{k_i}) dW_{k_i}(s) \to \int_0^t G^\varepsilon(s, u) dW(s) \text{ strongly in } L^r\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})\right). \tag{4.80}$$

On the other hand, we also have that

$$\int_0^t G^\varepsilon(s, u_{k_i}) dW_{k_i}(s) \rightharpoonup \psi(t) \text{ weakly in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})\right).$$

Therefore

$$\int_0^t G^\varepsilon(s, u_{k_i}) dW_{k_i}(s) \rightharpoonup \psi(t) \text{ weakly in } L^r\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})\right).$$

Since the convergence (4.80) holds also weakly in $L^r(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$, hence we have that

$$\psi(t) = \int_0^t G^\varepsilon(s, u) dW(s),$$

by uniqueness of weak limits. Thus

$$\int_0^t G^\varepsilon(s, u_{k_i}) dW_{k_i}(s) \rightharpoonup \int_0^t G^\varepsilon(s, u) dW(s) \text{ weakly in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D})\right).$$

This can be expressed as: for fixed ε we let i tends to ∞ to have, for any $\kappa \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$

$$\mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u_{k_i}) dW_{k_i}(s)\right) \longrightarrow \mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u) dW(s)\right). \tag{4.81}$$

We obviously have that the sequence $\left(\int_0^t G(s, u_{k_i}) dW_{k_i}(s)\right)_{i \in \mathbb{N}}$ is uniformly bounded in $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$. Thus, there exists $\eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$ such that for any $\kappa \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$

$$\mathbb{E}\left(\kappa, \int_0^t G(s, u_{k_i}) dW_{k_i}(s)\right) \longrightarrow \mathbb{E}(\kappa, \eta) \text{ as } i \longrightarrow \infty.$$

Lastly we need to prove that $\int_0^t G(s, u) dW(s) = \eta$. For that purpose we write (4.81) as follows:

$$\mathbb{E}\left(\kappa, \int_0^t G(s, u_{k_i}) dW_{k_i}(s) - \int_0^t G(s, u) dW(s)\right) = I_1 + I_2 + I_3,$$

where κ is an arbitrary element of $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(\mathbb{D}))$ and

$$I_1 = \mathbb{E}\left(\kappa, \int_0^t [G(s, u_{k_i}) - G^\varepsilon(s, u_{k_i})] dW_{k_i}(s)\right), \; I_2 = \mathbb{E}\left(\kappa, \int_0^t [G^\varepsilon(s, u) - G(s, u)] dW(s)\right),$$
$$I_3 = \mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u_{k_i}) dW_{k_i}(s) - \int_0^t G^\varepsilon(s, u) dW(s)\right).$$

By Burkholder-Davis-Gundy's inequality

$$\begin{aligned} I_1 &\leqslant \mathbb{E}\|\kappa\|_{L^2(\mathbb{D})}\left\|\int_0^t G(s,u_{k_i})-G^\varepsilon(s,u_{k_i})dW_{k_i}(s)\right\|_{L^2(\mathbb{D})} \\ &\leqslant C\mathbb{E}\left[\int_0^t \|G(s,u_{k_i})-G^\varepsilon(s,u_{k_i})\|^2_{(L^2(\mathbb{D}))^m}\,ds\right]^{\frac{1}{2}}; \end{aligned}$$

and

$$\begin{aligned} I_2 &\leqslant \mathbb{E}\|\kappa\|_{L^2(\mathbb{D})}\left\|\int_0^t G(s,u)-G^\varepsilon(s,u)dW(s)\right\|_{L^2(\mathbb{D})} \\ &\leqslant C\mathbb{E}\left[\int_0^t \|G(s,u)-G^\varepsilon(s,u)\|^2_{(L^2(\mathbb{D}))^m}\,ds\right]^{\frac{1}{2}}. \end{aligned}$$

Passing to the limit as $\varepsilon \longrightarrow 0$ in the above inequalities and using (4.73), we get that $\lim_{\varepsilon\longrightarrow 0}(I_1+I_2)=0$. By (4.81), we have

$$I_3=\mathbb{E}\left(\kappa,\int_0^t G^\varepsilon(s,u_{k_i})dW_{k_i}(s)-\int_0^t G^\varepsilon(s,u)dW(s)\right)\to 0,$$

Thus, it follows from (4.3.5) that

$$\int_0^t G(s,u_{k_i})dW_{k_i}(s)\rightharpoonup\int_0^t G(s,u)dW(s) \text{ weakly in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right). \tag{4.82}$$

Step 4. We start by introducing the notion of pseudo monotone operator which is needed in order to replace the ordinary monotonicity condition. Let $\mathbb{X}$ be a Banach space and $\mathbb{X}'$ its dual. Let us denote by $\langle\cdot\,,\,\cdot\rangle_{\mathbb{X}\times\mathbb{X}'}$ the duality pairing between $\mathbb{X}$ and $\mathbb{X}'$.

Definition 34. Let $\mathcal{T}$ be a mapping from a reflexive Banach space $\mathbb{X}$ to its dual space $\mathbb{X}'$, which is continuous from finite dimensional subspaces of $\mathbb{X}$ to $\mathbb{X}'$ endowed with the weak topology. $\mathcal{T}$ said to be pseudo-monotone, if for any sequence $\{u_j\}_{j\in\mathbb{N}}\subset\mathbb{X}$ with $u_j\rightharpoonup u$ in $\mathbb{X}$ and

$$\overline{\lim}\langle\mathcal{T}(u_j),u_j-u\rangle_{\mathbb{X}\times\mathbb{X}'}\leqslant 0,$$

then

$$\mathcal{T}(u_j)\longrightarrow\mathcal{T}(u) \text{ weakly in } \mathbb{X}'$$

while

$$\langle\mathcal{T}(u_j),u_j-u\rangle_{\mathbb{X}\times\mathbb{X}'}\longrightarrow 0 \text{ i.e. } \langle\mathcal{T}(u_j),u_j\rangle_{\mathbb{X}\times\mathbb{X}'}\longrightarrow\langle\mathcal{T}(u),u\rangle_{\mathbb{X}\times\mathbb{X}'}.$$

The theory of pseudo monotone operators is well established since the 1960's; details could be found for instance in [32] and [39].

We intend to use it in order to prove that $\varpi(t) = A_t(u)$, following [39], [34] and [33]. A crucial step toward that goal is to prove that

$$\limsup_{k_i \longrightarrow \infty} \mathbb{E} \int_0^t \langle A_s(u_{k_i}), u_{k_i}(s) - u(s) \rangle ds \leqslant 0.$$

We recall the equation (4.52)

$$du_{k_i} + A_t(u_{k_i})dt + g_{k_i}(t, u_{k_i})dt = f(t)dt + G(t, u_{k_i})dw_{k_i}.$$

Let

$$v \in L^p\left(\Omega, \mathcal{F}, \mathbb{P}, L^p(0,T; W_0^{m,p}(\mathbb{D})) \cap L^\infty(Q_T) \cap C^1(0,T, L^2(\mathbb{D}))\right)$$

be any function such that $v(0) = 0$. Since $dv = \dfrac{\partial v}{\partial t} dt$, then

$$d(u_{k_i} - v) = -\left[A_t(u_{k_i}) + g_{k_i}(t, u_{k_i}) - f(t) + \frac{\partial v}{\partial t}\right] dt + G(t, u_{k_i}) dW_{k_i};$$

and by Ito's formula applied to the function $\|u_{k_i}(t) - v(t)\|^2_{L^2(\mathbb{D})}$, we have

$$\begin{aligned} d\|u_{k_i}(t) - v(t)\|^2_{L^2(\mathbb{D})} &= 2\int_{\mathbb{D}} (u_{k_i}(t) - v(t)) \left[-A_t(u_{k_i})dt - g_{k_i}(t, u_{k_i})dt + f(t)dt + G(t, u_{k_i})dW_{k_i}\right] dx \\ &\quad - 2\int_{\mathbb{D}} (u_{k_i}(t) - v(t)) \frac{\partial v}{\partial t} dt + \|G(t, u_{k_i})\|^2_{(L^2(\mathbb{D}))^m} dt, \end{aligned}$$

for all $(t, \omega) \in [0,T] \times \Omega$.

Integrating this relation over $(0,t)$ for $t \in [0,T]$ yields

$$\mathbb{E}\|u_{k_i}(t) - v(t)\|^2_{L^2(\mathbb{D})} + 2\mathbb{E}\int_0^t \langle A_s(u_{k_i}), u_{k_i}(s) - v(s)\rangle ds + 2\mathbb{E}\int_{Q_t} g_{k_i}(u_{k_i})[u_{k_i}(s) - v(s)] dx ds \tag{4.83}$$

$$\begin{aligned} &= 2\mathbb{E}\int_0^t \langle f(s), u_{k_i}(s) - v(s)\rangle ds + 2\mathbb{E}\int_{Q_t} G(s, u_{k_i}) \left[u_{k_i}(s) - v(s)\right] dW_{k_i}(s) dx + \\ &+ \mathbb{E}\int_0^t \|G(s, u_{k_i})\|^2_{(L^2(\mathbb{D}))^m} ds - 2\mathbb{E}\int_0^t \left(\frac{\partial v}{\partial t}, u_{k_i}(s) - v(s)\right) ds. \end{aligned}$$

Passing to the limit in (4.52) using (4.49), (4.58), (4.56), (4.57), (4.61), (4.70), (4.67) and (4.71) we have

$$du(t) + \varpi(t)dt + g(s,u)dt = f(t)dt + G(t,u)dW_t \text{ in the sense of distributions on } Q_T.$$

Therefore with v as above, we have

$$d(u(t) - v(t)) = -\left[\varpi(t) + g(t,u) - f(t) + \frac{\partial v}{\partial t}\right] dt + G(t,u) dW.$$

By Ito's formula applied to the function $\|u(t)-v(t)\|^2_{L^2(\mathbb{D})}$, we have that

$$\mathbb{E}\|u(t)-v(t)\|^2_{L^2(\mathbb{D})} + 2\mathbb{E}\int_0^t \langle \varpi(s), u(s)-v(s)\rangle ds + 2\mathbb{E}\int_{Q_t} g(s,u)[u(s)-v(s)]dxds \tag{4.84}$$
$$= 2\mathbb{E}\int_0^t \langle f(s), u(s)-v(s)\rangle ds + 2\mathbb{E}\int_0^t (G(s,u), u(s)-v(s))\, dW$$
$$-\, 2\mathbb{E}\int_0^t \left(\frac{\partial v}{\partial t}, u(s)-v(s)\right) ds + \mathbb{E}\int_0^t \|G(s,u)\|^2_{(L^2(\mathbb{D}))^m} ds;$$

it should be noted that the integral

$$\int_{Q_t} g(s,u)[u(s)-v(s)]dxds$$

is meaningful thanks to the fact that $u(s)-v(s) \in L^\infty(Q_T)$ and $g(s,u) \in L^1(Q_T)$, $\mathbb{P}$-a.s. We know that

$$\int_0^t \langle A_s(u_{k_i}), u_{k_i}(s)-v(s)\rangle ds = \int_0^t \langle A_s(u_{k_i}), u_{k_i}(s)-u(s)\rangle ds + \int_0^t \langle A_s(u_{k_i}), u(s)-v(s)\rangle ds.$$

This combined with (4.83) and taking mathematical expectation implies that

$$\mathbb{E}\int_0^t \langle A_s(u_{k_i}), u_{k_i}(s)-u(s)\rangle ds + \mathbb{E}\int_{Q_t} \{u_{k_i} g_{k_i}(u_{k_i}) - ug(u)\}\, dxds \tag{4.85}$$
$$= -\frac{1}{2}\mathbb{E}\|u_{k_i}(t)-v(t)\|^2_{L^2(\mathbb{D})} + \mathbb{E}\int_{Q_t} \{g_{k_i}(u_{k_i})v - g(u)\,u)\}\, dxds +$$
$$+\,\mathbb{E}\int_0^t \langle f(s), u_{k_i}(s)-v(s)\rangle ds - \mathbb{E}\int_0^t \left(\frac{\partial v}{\partial t}, u_{k_i}(s)-v(s)\right) ds +$$
$$+\,\mathbb{E}\int_{Q_t} [u_{k_i}(s)-v(s)]G(s,u_{k_i})dW_{k_i}dx + \frac{1}{2}\mathbb{E}\int_0^t \|G(s,u_{k_i})\|^2_{(L^2(\mathbb{D}))^m} ds$$
$$-\,\mathbb{E}\int_0^t \langle A_s(u_{k_i}), u(s)-v(s)\rangle ds.$$

Owing to the condition (v) on G, the estimates (4.54), the almost everywhere convergence of u_{k_i} to u on $\Omega\times[0,T]\times\mathbb{D}$, we see that $(G(s,u_{k_i}))$ is uniformly integrable in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(Q_T))$ and $G(s,u_{k_i})$ converges to $G(s,u)$ almost everywhere on $\Omega\times[0,T]\times\mathbb{D}$. Therefore Vitali's theorem implies that

$$G(s,u_{k_i}) \to G(s,u) \text{ strongly in } L^2\left(\Omega,\mathcal{F},\mathbb{P};L^2(\mathbb{D})\right). \tag{4.86}$$

In view of (4.49)-(4.61), (4.58), (4.82) and (4.86), we deduce from (4.85) that

$$\begin{aligned}
&\limsup_{i\longrightarrow\infty}\left\{\mathbb{E}\int_0^t\langle A_s(u_{k_i}),u_{k_i}(s)-u(s)\rangle ds+\mathbb{E}\int_{Q_t}\{u_{k_i}g_{k_i}(u_{k_i})-ug(u)\}\,dxds\right\}\\
&\leqslant-\frac{1}{2}\mathbb{E}\|u(t)-v(t)\|^2_{L^2(\mathbb{D})}-\mathbb{E}\int_{Q_t}[u(s)-v(s)]g(s,u)\,dxds+\\
&\quad+\mathbb{E}\int_0^t\langle f(s),u(s)-v(s)\rangle ds-\mathbb{E}\int_0^t\left(\frac{\partial v}{\partial t},u(s)-v(s)\right)ds+\\
&\quad+\mathbb{E}\int_{Q_t}[u(s)-v(s)]G(s,u)dW_sdx+\frac{1}{2}\mathbb{E}\int_0^t\|G(s,u)\|^2_{(L^2(\mathbb{D}))^m}ds\\
&\quad-\mathbb{E}\int_0^t\langle\varpi(s),u(s)-v(s)\rangle ds.
\end{aligned}\tag{4.87}$$

Now thanks to (4.84), the right hand side in the above relation vanishes. Hence we get that

$$\limsup_{i\longrightarrow\infty}\left\{\mathbb{E}\int_0^t\langle A_s(u_{k_i}),u_{k_i}(s)-u(s)\rangle ds+\mathbb{E}\int_{Q_t}\{u_{k_i}g_{k_i}(u_{k_i})-ug(u)\}\,dxds\right\}\leqslant 0.\tag{4.88}$$

By (4.60) and (4.71), we have that $u_{k_i}g_{k_i}(u_{k_i})$ converges to $ug(u)$ almost everywhere. Thus it follows by Fatou's lemma that

$$\mathbb{E}\int_{Q_t}ug(u)dxds\le\liminf_i\mathbb{E}\int_{Q_t}u_{k_i}g_{k_i}(u_{k_i})dxds.$$

This and (4.63) imply that $ug(u)\in L^1(\Omega,\mathcal{F},\mathbb{P};L^1(Q_T))$ and

$$0\le\liminf_{i\longrightarrow\infty}\mathbb{E}\int_{Q_t}\{u_{k_i}g_{k_i}(u_{k_i})-ug(u)\}dxds\le\limsup_{i\longrightarrow\infty}\mathbb{E}\int_{Q_t}\{u_{k_i}g_{k_i}(u_{k_i})-ug(u)\}dxds.$$

Therefore, it follows from (4.88) that

$$\limsup_{i\longrightarrow\infty}\mathbb{E}\int_0^t\langle A_s(u_{k_i}),u_{k_i}(s)-u(s)\rangle ds\le 0.\tag{4.89}$$

According to Theorem 1 from [39] under the conditions (i), (ii) and (iii), the operator A_t is pseudo-monotone thus the relation (4.89) implies that $\varpi(t)=A_t(u)$. This completes the proof that $(\Omega,\mathcal{F},\mathbb{P},W,u)$ is a weak probabilistic solution of problem (P) in the definition 31. Furthermore we have the following energy estimate

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u(t)\|^2_{L^2(\mathbb{D})}+\mathbb{E}\int_0^t\langle A_s(u),u(s)\rangle ds+\mathbb{E}\int_{Q_t}u(s)g(s,u)dxds\\
&\qquad=\mathbb{E}\int_0^t\langle f(s),u\rangle ds+\mathbb{E}\int_{Q_t}u(s)G(s,u)dxdW(s)+\frac{1}{2}\mathbb{E}\int_{Q_t}\|G(s,u)\|^2_{(L^2(\mathbb{D}))^m}ds,
\end{aligned}$$

for all $t\in[0,T]$. This implies that $u\in L^2(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T,L^2(D)))$ and the limiting process u belong to the space $L^q(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;L^2(\mathbb{D})))$ for any $q\geqslant 2$.

4.4 Strong solution

In this section we establish the pathwise uniqueness of solutions of the problem (P) and using arguments from Yamada-Watanabe's classical result to derive the existence of a strong probabilistic solution for our problem. This will be possible thanks to the existence of a weak probabilistic solution (proved in the previous section) and the pathwise uniqueness to be proved shortly. For relevant information in the finite dimensional case, we refer to [110] and [211]. The papers by Röckner, Schmuland and Zhang [175] and Ondrejat [152] deals with the Banach space version in great generality. We shall make use of the result in [175].

Before we state the result on pathwise uniqueness we introduce the following

Definition 35. We say that pathwise uniqueness holds for problem (P), if whenever the systems $(\Omega, \mathcal{F}, (\mathcal{F}_t,)\mathbb{P}, W, u_1)$ and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, W, u_2)$ are two weak solutions of the said problem on the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with the same initial condition, then

$$u_2(t) = u_1(t), \ t \in [0, T], \ \mathbb{P} - a.s..$$

That is, there exists a set

$$N \in \sigma \left\{ \bigcup_{t \in [0,T]} \mathcal{F}_t \right\}$$

with $\mathbb{P}(N) = 0$ such that the set $\{\omega : \ u_2(\omega, t) = u_1(\omega, t)\} \subset N$ for all $t \in [0, T]$.

We shall need an additional condition on the perturbation term $g(t, r)$ in (P) that we borrow from [34]. Namely, we require that

(ii)" The operator family A_t is monotone, i.e.,

$$\langle A_t(u) - A_t(v), u - v \rangle \geqslant 0 \text{ for all } u, v \in W_0^{m,p}(\mathbb{D}).$$

(iv)' The function $g(t, x, r)$ is non-decreasing in r, progressively measurable and in addition, the sign condition on g is preserved. That is,

$$rg(t, x, r) \geq 0 \text{ on } \Omega$$

.

Next, we introduce a function Γ by setting

$$\Gamma(t, x, r) = \int_0^r g(t, x, s) ds \text{ on } \Omega.$$

Γ is continuous and convex in r and non-negative for all (t, r). By construction we have that $\Gamma(t, x, 0) = 0$. Using the fundamental theorem of Calculus, we assert that the function Γ is differentiable with $\Gamma'(t,x,r) = g(t,x,r)$. From now on, we use the notation $g(t,r)$ instead of $g(t,x,r)$ and similarly in order to keep the notation simple, we sometimes omit the x variable in general in most of the function of r.

We also need an extra condition on G; the Lipschitz condition. Namely, there exists a constant $L > 0$ such that

$$\|G(t,u_1) - G(t,u_2)\|_{(L^2(\mathbb{D}))^m} \leq L\|u_1(t) - u_2(t)\|_{L^2(\mathbb{D})}. \tag{4.90}$$

Our pathwise uniqueness is stated in the following

Theorem 36. *Under assumptions Theorem 33, hypotheses (H_1)-(H_3), (ii)", (4.90) and (iv)'. then the pathwise uniqueness in the sense of definition 35 holds.*

The function Γ defined above plays a central role in proving the theorem.

Proof. For notational purpose, we simply write (W,u) for short instead of using the system $(\Omega,\mathcal{F},\mathbb{P};W,u)$. Let (W,u_1) and (W,u_2) be two weak solutions with

$$u_1, u_2 \in L^p\left(\Omega,\mathcal{F},\mathbb{P}; L^p\left(0,T; W_0^{m,p}(\mathbb{D})\right)\right) \cap L^q\left(\Omega,\mathcal{F},\mathbb{P}; L^\infty\left(0,T; L^2(\mathbb{D})\right)\right),$$

for any $q \in [2,\infty)$, $p \geq 2$. For any

$$v \in L^p\left(\Omega,\mathcal{F},\mathbb{P}; L^p\left(0,T; W_0^{m,p}(\mathbb{D})\right) \cap L^\infty\left(Q_T\right)\right),$$

it holds that

$$d(u-v) = -\left[A_t(u) + g(t,u) - f(t)\right]dt - dv + G(t,u(t))dW_t. \tag{4.91}$$

We substitute u by u_1 and u_2 in (4.91) and multiply the corresponding relations by $u_1 - v$ and $u_2 - v$ to get

$$\begin{aligned}
&(u_1-v)d(u_1-v) + (u_1-v)[A_t(u_1) - f(t)]dt = \\
&= (v-u_1)g(u_1)dt + (u_1-v)G(t,u_1(t))dW - (u_1-v)dv, \qquad (4.92)\\
&(u_2-v)d(u_2-v) + (u_2-v)[A_t(u_2) - f(t)]dt = \\
&= (v-u_2)g(u_2)dt + (u_2-v)G(t,u_2(t))dW - (u_2-v)dv. \qquad (4.93)
\end{aligned}$$

Arguing similarly as in [33], we express v as the mean of u_1 and u_2 i.e., $v = \frac{1}{2}(u_1+u_2)$. We then have $u_1 - v = \frac{1}{2}(u_1-u_2)$ and $u_2 - v = -\frac{1}{2}(u_1-u_2)$. Substituting the expressions of $u_1 - v$ and $u_2 - v$ back into (4.92) and (4.93), respectively, we get

$$\begin{aligned}
&(u_1-u_2)d(u_1-u_2) + 2(u_1-u_2)[A_t(u_1) - f(t)]dt \\
&= 4(v-u_1)g(u_1)dt + 2(u_1-u_2)G(t,u_1)dW - 4(u_1-v)dv, \\
&(u_1-u_2)d(u_1-u_2) + 2(u_1-u_2)[-A_t(u_2) + f(t)]dt \\
&= 4(v-u_2)g(u_2)dt - 2(u_1-u_2)G(t,u_2)dW - 4(u_2-v)dv.
\end{aligned}$$

Adding these two equalities and applying Ito's formula to the function $\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}$ we get

$$d\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}+2\langle(u_1-u_2),[A_t(u_1)-A_t(u_2)]\rangle dt$$
$$=4\int_{\mathbb{D}}(v-u_1)g(u_1)dxdt+4\int_{\mathbb{D}}(v-u_2)g(u_2)dxdt+2\int_{\mathbb{D}}(u_1-u_2)[G(t,u_1)-G(t,u_2)]dWdx+$$
$$+\|G(t,u_1)-G(t,u_2)\|^2_{(L^2(\mathbb{D}))^m}dt.$$

Integrating this relation over $[0,t]$, taking the supremum over the interval $[0,T]$ and using (H_1) leads to

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}+2\mathbb{E}\int_0^t\langle(u_1-u_2),A_s(u_1)-A_s(u_2)\rangle ds\\&\le 4\mathbb{E}\int_{Q_t}(v-u_1)g(u_1)dxds+4\mathbb{E}\int_{Q_t}(v-u_2)g(u_2)dxds+\\&+2\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{D}}(u_1-u_2)[G(s,u_1)-G(s,u_2)]dW_sdx\right|+C\mathbb{E}\int_0^t\|u_1(s)-u_2(s)\|^2_{L^2(\mathbb{D})}ds.\end{aligned}\tag{4.94}$$

Since by definition, $\Gamma'(t,u)=g(t,u)$, we use $(iv)'$ to obtain the following properties which are straightforward consequences of the convexity of the function Γ:

$$g(t,u_1)(v-u_1)\leqslant\Gamma(t,v)-\Gamma(t,u_1),\tag{4.95}$$
$$g(t,u_2)(v-u_2)\leqslant\Gamma(t,v)-\Gamma(t,u_2),\tag{4.96}$$
$$\Gamma\left(t,\frac{1}{2}(u_1+u_2)\right)\le\frac{1}{2}\left\{\Gamma(t,u_1)+\Gamma(t,u_2)\right\}.\tag{4.97}$$

It follows from (4.90), (4.94), Burkholder-Davis-Gundy, Young's inequalities and (4.95)-(4.97) that

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}+2\mathbb{E}\int_0^t\langle(u_1-u_2),A_s(u_1)-A_s(u_2)\rangle ds\\&\le 8\mathbb{E}\int_{Q_t}\left[\Gamma(s,x,v)-\left\{\frac{1}{2}\Gamma(s,x,u_1)+\frac{1}{2}\Gamma(s,x,u_2)\right\}\right]dxds+2\epsilon\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}+\\&+C_\epsilon\mathbb{E}\int_0^T\|G(t,u_1)-G(t,u_2)\|^2_{(L^2(\mathbb{D}))^m}dt+C\mathbb{E}\int_0^T\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}dt.\end{aligned}\tag{4.98}$$

Using the Lipschitz condition on G, (4.90) we assert that there exists a positive constant L

such that

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}+2\mathbb{E}\int_0^t\langle(u_1-u_2),A_s(u_1)-A_s(u_2)\rangle ds\\
&\le 8\mathbb{E}\int_{Q_t}\left[\Gamma(s,v)-\left\{\frac{1}{2}\Gamma(s,u_1)+\frac{1}{2}\Gamma(s,u_2)\right\}\right]dxds+2\epsilon\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}+\\
&+C_\epsilon L\mathbb{E}\int_0^T\|u_1(t)-u_2(t)\|^2_{(L^2(\mathbb{D}))^m}dt+C\mathbb{E}\int_0^T\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}dt. \qquad (4.99)
\end{aligned}
$$

From (4.97), we obviously have

$$
\mathbb{E}\int_{Q_t}\left[\Gamma(s,v)-\left\{\frac{1}{2}\Gamma(s,u_1)+\frac{1}{2}\Gamma(s,u_2)\right\}\right]dxds\le 0.
$$

The assumption (ii) on the strict monotonicity of A_t gives

$$
\mathbb{E}\int_0^t\langle(u_1-u_2),A_s(u_1)-A_s(u_2)\rangle ds>0.
$$

Combining the last two inequalities with an appropriate choice of ϵ yields

$$
\mathbb{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}\le C\mathbb{E}\int_0^T\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{D})}dt.
$$

Thanks to Gronwall's Lemma, we conclude that for any $t\in[0,T]$, we have $u_1(t)=u_2(t)$, $\mathbb{P}$-a.s..
This leads to the desired conclusion. $\square$

Our main result of this section is stated in the following

Theorem 37. *Let the assumptions in Theorem 33 and Theorem 36. Then problem (P) admits a unique strong solution u, in the sense that the system* $(\Omega,\mathcal{F},\mathbb{P},W,u)$ *is a weak solution of (P) and u is adapted to the filtration generated by the Wiener process W.*

The proof of this result follows from the celebrated theorem of Yamada-Watanabe originally proved in [211] (see also [110]), in the finite dimensional case. According to Yamada-Watanabe's theorem, weak probabilistic solution and pathwise uniqueness give rise to the existence of unique probabilistic strong solution. The result has since been established in the infinite-dimensional setting by many authors; we refer to Röckner, Schmuland and Zhang [175] for further details in this direction. A Banach space version of the Yamada-Watanabe Theorem in the infinite dimensional framework of mild solutions we refer for instance to Ondreját [152] for more information. It is the version obtained in [175] that enables us to conclude the proof of the theorem.

Conclusion and Future Perspectives

In this thesis, we studied firstly the existence of weak probabilistic solution to various classes of stochastic quasilinear parabolic problems driven by infinite-dimensional Wiener processes of cylindrical type. Secondly, we proved the existence and uniqueness of weak and strong probabilistic solutions to strongly nonlinear stochastic problems of Brézis-Browder type, characterized by the presence of a nonlinear unbounded perturbation of zero-th order.

We established in the first part, the existence of a probabilistic weak solution (also known as martingale solution) for a certain class of stochastic quasilinear parabolic equations (3.1) in a framework of probabilistic evolution spaces involving the spaces $\mathring{W}^{1,p(x)}(\mathbb{D})$. The main feature of those equations is characterized by the presence of a nonlinear elliptic part admitting nonstandard growth. Our SPDE is subjected to infinite dimensional Wiener processes of cylindrical type and the nonlinear forcing terms do not satisfy Lipschitzity conditions.

Our framework is based on a construction of an approximating sequence to the weak probabilistic solution of the problem under consideration using Galerkin method. In the proof, we combined the Galerkin method with some deep analytic (Aubin-Simon's type) and probabilistic compactness results (Prokhorov-Skorohod). Using results from Skorohod, we firstly proved that the Galerkin problems admit solutions. Secondly, we derived appropriate uniform a priori estimates for the approximating solutions $(u_m)_{m\in\mathbb{N}}$. The just mentioned compactness results were obtained thanks to the derivation of these crucial uniform a priori bounds on (u_m). We experienced many difficulties when proving the tightness of the probability measures generated by the Galerkin approximate solutions. This is achieved thanks to results from [166, 192, 157, 191]. Thirdly we pass to the limit in the Galerkin equations by extracting a subsequence of the original sequence that converges weakly. Finally, we adapt arguments of monotone operators to show that the limiting process u is a solution of the problem (3.1)-(3.3).

For the investigations of SPDE's governed by finite dimensional Wiener processes, we refer to the works [7] and [19]. Bauzet [19] obtained existence and uniqueness for SPDE's subjected to one dimensional Wiener processes in the functional setting of Lebesgue-Sobolev's type spaces with variable exponents $p(t, x)$. The dissertation [7] dealt with the case of d-dimensional multiplicative white noise. The paper [8] established both the weak and strong probabilistic solutions.

In the second part of the thesis, we investigated an important class of SPDEs which has so far not been studied by experts in the field is the stochastic counterpart of nonlinear parabolic equations which originated in the work of Brézis-Browder [34, 33] under the name of strongly nonlinear equations. The main feature of those equations is characterized by the presence of nonlinear terms which are unbounded perturbation of zeroth order (having no growth restrictions), making it impossible to study the resulting problem by directly using means such as Galerkin's approximation, the monotonicity method, for instance. We adapted to the stochastic case, a regularization process through truncations which reduced problem (P) to a sequence of more regular problems, (P_k). Thanks to results from [123], we proved that (P_k) admit solutions u_k. We established needed uniform a priori estimates of $(u_k)_{k\geqslant 1}$ solutions to (P_k) which enabled us to appeal to some profound analytic and probabilistic compactness results. Thanks to a subsequent passage to the limit involving a result on pseudomonotone operators due to Browder [39], we showed that a sequence of solutions of (P_k) converges in suitable topologies to the requested probabilistic weak solution for our original problem. Under some additional assumptions of Lipschitizity on $\mathcal{A}$ and the intensity of the noise, we establish the pathwise uniqueness of weak probabilistic solutions and appeal to an infinite-dimensional version of the famous Yamada-Watanabe's result [211] due to Röckner, Schmuland and Zhang [175] and Ondreját [152] to derive the existence of a unique strong probabilistic solution.
To the best of our knowledge, there have not been any attempts so far for the study of the present problems. Thus, this is the first thesis dealing with the actual situations.

Future work

1) Consider the problems dealt with in the thesis in the framework of jump noises.

2) A thorough investigation of stochastic electrorheological fluids by extending the important works of Růžička [177], Rajagopal [164] and Lars Diening [72]. Both continuous and jump noises will be explored.

3) The issues of regularity in the context of these SPDEs might also be investigated.

4) The numerical analysis of these models by applying a splitting-up method for stochastic PDEs, see for instance [26, 22, 27], [28], [69], [100] and [183].

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