

Homogenization of partial differential equations: from multiple scale expansions to Tartar's H-Measures

by

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Declaration

I, the undersigned declare that the dissertation, which I hereby submit for the degree *Magister Scientiae in Mathematics* at the University of Pretoria, is my own work and has not previously been submitted by me or any other person for a degree at this or any other tertiary institution.

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Abstract

Homogenization theory has emerged over the last decades as a fundamental tool in the study of mathematical problems arising in processes taking place in highly heterogeneous media, such as composite materials, flow through porous medium, living tissues, just to cite a few. The main feature of these problems is the presence of multiple scales, notably microscopic and macroscopic scales.

A prominent and simplified theory of homogenization is period homogenization based on assumptions of periodic structure in the problems investigated. Since its inception, several challenges had to be overcome in the evolution of the theory. My dissertation was aimed at covering these challenges and the corresponding deep methods that were invented subsequently.

First, we study elliptic partial differential equations with periodic coefficients using the multiscale expansion and Tartar's method of oscillating test functions. Then we discuss nonlinear homogenization using the div-curl lemma, compensated compactness, Young measures and H-measures. We shall endeavour to motivate the emergence of these methods along their historical flow.

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To my late grandmother Monica Emereuwa and my youngest brother Onyekachi Emereuwa.

Notations

For the reader's convenience, listed here are some symbols, sets and function spaces used throughout this dissertation.

- If X is a Banach space, X' denotes its dual and its bidual is denoted as X'' i.e. the dual of the dual of X .
- $\{x_n\}$: A sequence of functions x_n .
- $x_n \rightharpoonup x$: $\{x_n\}$ converges weakly to x .
- $x_n \rightharpoonup^* x$: $\{x_n\}$ converges weakly* to x .
- $x_n \rightarrow x$: $\{x_n\}$ converges strongly to x .
- \bar{z} : The complex conjugate of a complex number z .
- $x \cdot y$: The dot product of two vectors x and y in \mathbb{R}^n .
- $\langle \cdot, \cdot \rangle_{X', X}$: The duality pairing between X' and X .
- δ_{ij} : Kronecker delta, equal to 1 if $i = j$ and equal to 0 if $i \neq j$ for $i, j = 1, \dots, n$.
- $\mathcal{F}(u)$: The Fourier transform of the function u , given by

$$\mathcal{F}(u) = \int_{\mathbb{R}^n} e^{-2i\pi(x \cdot \xi)} u(x) dx.$$

- a.e.: almost everywhere.

- Ω : An open bounded subset of \mathbb{R}^n .
- $|\Omega|$: The Lebesgue measure of Ω .
- $\partial\Omega$: The boundary of Ω .
- $C(\Omega)$: The space of continuous functions $u : \Omega \rightarrow \mathbb{R}$.
- $C_0(\Omega)$: The space of continuous functions $u : \Omega \rightarrow \mathbb{R}$ with compact support contained in Ω .
- $C^\infty(\Omega)$: The space of all infinitely differentiable functions $u : \Omega \rightarrow \mathbb{R}$.
- $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$: The space of all infinitely differentiable functions with compact support contained in Ω .
- $C_0(\mathbb{R}^n)$: The space of continuous functions converging to zero at infinity.
- Y, Y_1, Y_2, \dots, Y_n : Unit cells in \mathbb{R}^n .
- $C_{per}^\infty(Y)$: The restriction to Y of functions in $C^\infty(\mathbb{R}^n)$ that are Y -periodic.
- $L^p(\Omega; \mathbb{R}^n)$: The space of measurable vector functions $u : \Omega \rightarrow \mathbb{R}^n$ whose components belong to $L^p(\Omega)$.
- $L^\infty(\Omega; X)$: The space of functions $u : \Omega \rightarrow X$ such that

$$\|u\|_{L^\infty(\Omega; X)} = \operatorname{ess\,sup}_{x \in \Omega} \|u(x)\|_X < \infty.$$

- $\mathcal{M}(\Omega)$: the space of radon measures. The dual of $C(\Omega)$ up to isomorphism.
- The operator \mathcal{A} is defined as

$$\mathcal{A} = -\operatorname{div}(A(x)\nabla) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

where A is an $n \times n$ matrix.

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Chapter 1

Introduction

When materials with different physical or chemical properties are combined, they produce a material which may combine the essential properties of their individual constituents, the resulting material is called a composite material. Common examples of composite materials are concrete and plastic. An industrial example is Carbon-Fiber-Reinforced Polymer (CFRP). It is an expensive, light and extremely strong composite material made up of carbon fiber (for the strength) and polymer resin (to hold the carbon fibres together). It is mainly used wherever there is a need for high strength-to-weight ratio and rigidity e.g. automotive and aerospace engineering, sport goods and some modern bicycles and motorcycles. Some literature on CFRP include [TS03] and [Has03].

Nowadays in industries, composite materials are widely used because they have better properties according to the performance one looks for. In recent years, researchers study the use of composite materials in place of individual materials, for example, the use of $Fe_2O_3/CNTs$ as anode materials for Lithium-ion batteries in place of graphite [Sun13]. Due to the increasing use of composite materials, there is a huge need for mathematically understanding these materials. The heterogeneities in a composite material are smaller in size compared to the global size

of the composite material. So a composite material is said to be made up of two scales, a macroscopic scale representing the global behaviour of the material and the microscopic scale representing the heterogeneities.

In fields like physics, chemistry, material science and engineering disciplines, scientists study the physical properties/behaviour (e.g. temperature, heat conductivity or elasticity) of heterogeneous media, but the presence of the microstructure creates some difficulties. In other words, supposing the physical property of a heterogeneous material is modeled using partial differential equations, where the heterogeneities are represented by rapid oscillations in the coefficients of the equation. The oscillations present in the coefficients may cause severe difficulties while trying to solve the equations. A heterogeneous medium possessing a fine microstructure takes on the appearance of a homogeneous medium at first glance, which makes one think that at a macroscopic scale (the global form of the composite material), it will act like a homogeneous medium. A homogeneous medium which can be modeled using a partial differential equation without oscillations but will capture the properties in the microstructure. The process in which this equation is determined is known as homogenization. The theory of homogenization enables one to determine the macroscopic behaviour of a composite material from its microscopic behaviour. Hence, it enables one to understand how the properties on the microscopic level influence the macroscopic behaviour of a composite material.

The physical properties of heterogeneous media are usually modeled using partial differential equations. So the theory of homogenization can be seen as a collection of methods for approximating a heterogeneous problem by a homogeneous one. These methods employ different mathematical tools. For example, the mathematical tool 'weak convergence' can be used to model the relation between the macroscopic and the microscopic scales. The methods described in this dissertation involve the notion of weak convergence.

For an illustrative purpose. Suppose two conductors with different conductivities are crushed into fine powder and mixed together in a certain proportion. Either out of curiosity or for research purposes, one would want to know the conductivity of this mixture. Let us assume that the conductivity is modeled using a partial differential equation, and the change in conductivity in the mixture is captured using oscillations on the coefficients of the equation. The aim would be to derive an equation without oscillations that describes the conductivity of this mixture, taking into consideration the arrangement and the properties of the microscopic constituents, which in this case are the two conductors.

1.1 Research Objectives

Materials having different length scales can be reduced to a body covering a smooth domain on \mathbb{R}^n and modeled using differential equations. It is however difficult to determine the characteristics of this heterogeneous body since the coefficients on the microscopic level are rapidly oscillating functions.

In the major parts of this dissertation, we study the homogenization of an elliptic problem with periodic coefficient

$$\begin{aligned} -\operatorname{div} \left(a\left(\frac{x}{\epsilon}\right) \nabla u^\epsilon(x) \right) &= f(x) \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1.1}$$

where ϵ is a small parameter used to represent the heterogeneities.

In other words, we obtain a limit equation known as the effective or homogenized equation which describes the global behaviour of the heterogeneous body. Our approach is to study the corresponding sequence of the weak formulation of equation (1.1.1) using some of the methods developed over the years for this purpose.

In addition to studying the homogenization of an elliptic problem with periodic

coefficient, a chapter on nonlinear homogenization is included. This additional chapter is based on a few approaches developed by L. Tartar and F. Murat in that area. We discuss each approach with an application, then we highlight its shortcomings and why there is a need for improvement or a new approach to accommodate unanswered problems.

In summary, this dissertation serves as a survey of homogenization. It brings together two approaches to periodic homogenization and an overview on four approaches to nonlinear homogenization so that a reader with a limited knowledge of the concept of homogenization may understand the basic ideas

1.2 Dissertation Outline

In Chapter 2, the basic concepts of periodic homogenization are presented. We begin the chapter in Section 2.1 by describing homogenization with a simple example. While in Section 2.2, some methods of homogenization are briefly discussed.

Chapter 3 is devoted to definitions, theorems and important results used throughout the dissertation. The necessary functions spaces are defined.

In Chapter 4, the method of asymptotic expansion is introduced. We illustrate how the homogenized equation and solution is computed for an elliptic partial differential equation. To determine the accuracy of the result, we also compute the error estimate..

In Chapter 5, we consider the convergence theorem which will be proven using Tartar's method of oscillating test functions. This method involves passing to the limit in (1.1.1) by making appropriate choice of test functions, thereby eliminating products of weakly converging sequences that would cause problems. Section 5.3,

deals with the corrector result and as an example, the corrector matrix for a one dimensional case is computed.

Nonlinear homogenization approaches based on the research done by Luc Tartar and F. Murat are discussed in Chapter 6. In Section 6.1, we discuss the theory of compensated compactness and Young measures and give an example of its application to Maxwell equations and also state its shortcomings. H-measures are defined in Section 6.2.

Chapter 2

Homogenization

2.1 Introduction and Brief History

Broadly speaking, homogenization can be viewed as a mathematical theory in the field of partial differential equations used to study differential operators with rapidly oscillating coefficients, equations with rough random coefficients, equations in perforated domains, boundary value problems with rapidly varying boundary conditions and many other objects of practical and theoretical interests.

Different differential equations arise in the theory of homogenization. They could be linear partial differential equations with rapidly oscillating coefficients which may be periodic or non-periodic, see e.g. [Cio99]. The differential equations may be considered in domains with a complex microstructure such as perforated domains, see for instance in [Mar06] and [Zhi94]. They could be stochastic partial differential equations, see e.g. [Ich05], [San12]. One could also have nonlinear partial differential equations with oscillating, variable or constant coefficients. (We could even have differential equations with a combination of different conditions e.g. a differential equation in a perforated domain with periodic microstructure

see for instance in [Zhi94], [Ole92], [Sha82] etc.) The aim of homogenization is to be able to represent a complex, rapidly-varying heterogeneous medium with a simpler, slowly-varying homogeneous medium.

From as early as the 19th century, homogenization problems i.e. (finding homogeneous equation for heterogeneous medium) have been studied. They were initially associated with methods of nonlinear mechanics and ordinary differential equations developed by Poincaré. They came up in works of Maxwell [Max81] and Rayleigh [Ray92]. While Maxwell studied the effective conductivity of heterogeneous media, Rayleigh studied the effective conductivity of heterogeneous media with periodic inclusions. A good number of physicists have considered the problem. See for instance Hill [Hil64], Ilíushina [Ilí72], just to cite a few.

From the sixties, mathematicians began to intensively develop the theory of homogenization. For ordinary differential equations, contributions by Bogolyubov and Mitropolskii can be found in [Bog61]. In the aspect of partial differential equations, the work of Marchenko and Khruslov [Mar64], [Mar06] can be considered as pioneering. They introduced an approach that could handle boundary value problems in non-periodically structured domains with fine-grained boundaries with fine-grained domains. This direction in homogenization is more challenging. In the 1970s, several methods of homogenization were introduced and homogenization became an area of research in mathematics. Many of the results can be found in [BLP78], [Pan05], [KZO79], [Koz79] and [Bab75]. The theory of homogenization has numerous applications which include deriving the effective properties of composite materials, (see [Pob96] or [Mil02]) and the macroscopic modeling of microscopic systems. The interests of mathematicians in homogenization has led to the emergence of new ideas and concepts relevant to mathematics as well.

Some of these methods of homogenization include; The method of asymptotic expansion, G-convergence, H-convergence, Γ -convergence, Two-scale convergence.

The method of asymptotic expansion was introduced by engineers and mechanical scientists, (see for instance [Bog61]). It was later formalized to handle problems with periodic rapidly oscillating coefficients in [BLP78], (see also [Lar75], [Lar76], [Lar74] and [San80]). It is now a widely used method in physics and mechanics. The G-convergence was introduced by Spagnolo [Spa67]. It is an operator-like convergence that deals with the convergence of solutions to symmetric problems with periodic or non-periodic coefficients. The H-convergence of L. Tartar and F. Murat [Tar77], [Mur77], [MT97] is an extension of the G-convergence to non-symmetric problems. The Γ -convergence was introduced by De Giorgi [Gio84], (see also [Spa73] and [Bra02]), for the study of homogenization of functionals. It can be considered as the analog of G-convergence. Tartar's method of oscillating test functions was introduced by L. Tartar [Tar77]. It is a mathematically rigorous method that better handles problems containing the product of two weakly converging sequences, as passing to the limit in such product is a notorious problem.

For vector-valued case (systems of nonlinear PDEs). L. Tartar and F. Murat introduced the div-curl lemma as a tool for passing to the limit of the product of weakly converging vector fields. This lemma states that under certain conditions on the derivatives of the weakly converging fields, their product converges to the product of their limits in the sense of distributions, see e.g. [Tar79], [Tar09], [Zhi94], [Cio99]. This lemma has been applied to different problems in mathematics and physics, see for instance [Chr05] and [Zhi94]. The div-curl lemma was further extended to compensated compactness method by L. Tartar and F. Murat [Mur81], [Tar79], [Tar09]. Literature on this method and its applications include [Zhi94]. Unfortunately, the compensated compactness method can only handle problems with constant coefficients. So In the late 1980s, L. Tartar developed yet another approach known as H-measures. This was also introduced independently by P. Gérard under the name microlocal defect measures [Gé91].

In 1989, G. Nguetseng [Ngu89b] introduced the two-scale convergence for studying

boundary value problems with periodic rapidly oscillating coefficients. It was further developed by Allaire [All92]. In 1994, Mikelić, Bourgeat and Wright [Mik94] introduced stochastic two-scale convergence. Further development on the two-scale convergence can be found in [Neu96], [Ngu02] and [Hei11]. In recent years (precisely 2002), Cioranescu, Griso and Damlamian [Cio02], [Cio08] introduced the periodic unfolding method for the homogenization of periodic composites. And in 2003, the two-scale convergence was extended to tackle problems beyond the periodic setting by Nguetseng [Ngu03], [Ngu04] under the name Σ -convergence. Then in 2009, Wellander [Wel09] introduced the two-scale Fourier transform which is like a combination of the two-scale convergence, the periodic unfolding method and the Floquet-Bloch expansion approach to homogenization.

There is a wide range of excellent monographs and journal articles books and publications written in the area of homogenization that give insights to these methods, see e.g. [Pan97], [Bra98], [Zhi94], [San80], [All93], [All12], [Lu02], [Mur77], [Ngu89b], [Mas93], [Hom97], [Tar90]. Widely used text on the theory of periodic homogenization of partial differential equations are textbooks by Cioranescu and Donato [Cio99] and the so-called bible of homogenization by Bensoussan, Lions and Papanicolaou [BLP78]. More comments on some of these methods will be found in Section 2.2. A description of the asymptotic expansion and Tartar's method of oscillating test functions will be found in Chapters 3 and 4 respectively. And some of the methods by Tartar and Murat on nonlinear homogenization will be briefly described in Chapter 6.

2.1.1 The Concept of Homogenization

For a heterogeneous medium, we always assume that the length scale of the oscillation is far smaller than the size of the domain in which a physical phenomenon is investigated. This physical phenomenon can be described using differential equations with rapidly oscillating coefficients or differential equations with some other complicated structure e.g. being in a perforated domain. The structure of the differential equations makes solving its corresponding boundary value problem very difficult.

Supposing that the microscopic scale is much less than the scale of the heterogeneous material, then one could give a macroscopic description of the investigated phenomenon. In cases such as this, the material usually has some stable characteristics (e.g. heat conduction, elasticity etc) which may be significantly different from its characteristics on the microscopic scale. These stable characteristics are known as the effective or homogenized characteristics as they are mostly obtained using methods of homogenization or the relevant mean field methods, see for instance [OK13].

To mathematically describe a composite material, we assume that its characteristics on the microscopic scale depends on a small parameter ϵ which is the length scale of the microstructure (it is important to note that a composite material can possess several microscopic scales). The homogenized model of the phenomenon under investigation is obtained by an asymptotic analysis as $\epsilon \rightarrow 0$ of the problem. As it is, the limit of the solution to the original problem satisfies a new differential equation whose coefficients are expected to have better regularity in a simpler domain. In the following subsection, we illustrate periodic homogenization to a certain extent, since it will constitute most parts of this dissertation.

Periodic homogenization

Let us consider a two-phase composite material covering Ω , where one material is periodically distributed within the other. See Figure 2.1

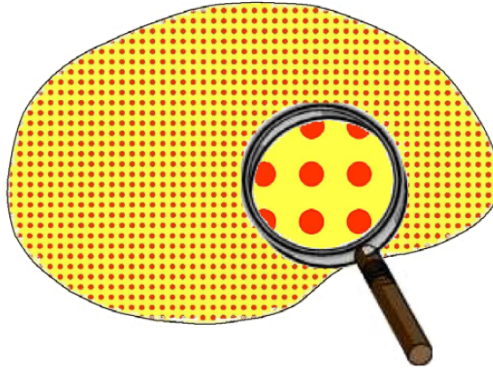


Figure 2.1: A microstructured material

Looking at this material, we see that the periodic inclusions appear microscopic compared to the size of Ω . To model this material, we assume that Ω can be divided into periodic cells with side length ϵ , a small parameter.

Let us denote by Y a periodicity cell of a fixed size and suggest that the microstructure is a disjoint union of translated ϵ -homotheties, ϵY of the periodicity cell. Then the whole space Ω can be covered with a disjoint union of translated sets of Y . Since the constitutive properties of the microstructured constituent repeat periodically, they can be described by a function $f : Y \mapsto \mathbb{R}^n$ which is extended by periodicity to the whole of Ω .

Most times, the unit cube $[0, 1)^n$ is used as the periodicity cell for simplicity, as in Figure 2.2.

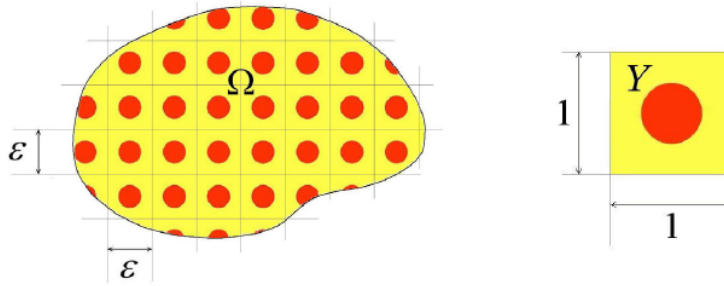


Figure 2.2: Illustration of the concept of periodicity

There are materials with periodic micro structures whose periodicity cells are not represented by unit cubes. For example, a material that possesses a honeycomb-like micro structure may be represented by a periodicity cell shaped like a hexagon i.e. its periodicity cell Y , is a regular hexagon. An example of such material is a large wire rope made from wires of similar diameter and arranged like a honeycomb. The microstructure in the cross-section of this material can be described using hexagonal periodic cells, where each wire is seen as the incircle of a regular hexagon as in Figure 2.3.

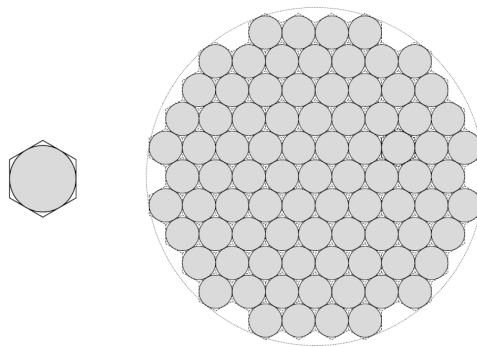


Figure 2.3: Hexagonal periodic cells in a cross section of a wire rope

There are also materials whose periodic micro structure cannot be described by the translation of periodic cells. For instance, the functionally graded annular disk can be described by the rotation of a periodic cell. See Figure 2.4.

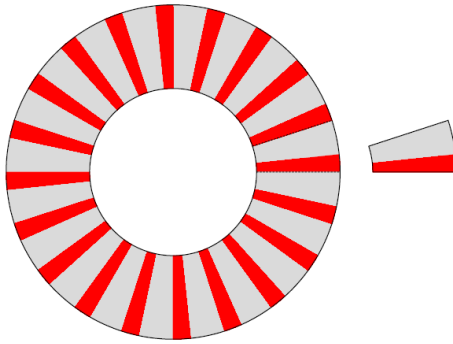


Figure 2.4: Functionally graded annular disk

Situations like this may be explained by changing to an appropriate coordinate system where the rotation leads to translations of an angle variable. For the functionally graded annular disk mentioned above, polar coordinates can be used.

There are also composites with heterogeneities occurring on several microscopic scales. Examples include, a fiber composite with thinner fibers inside each fiber and a composite with inclusions of different periodicities and sizes, see Figure 2.5. Homogenization problems of this sort are known as iterated homogenization problems. They were studied by Bensoussan [BLP78] and are being investigated by many other mathematicians, see for instance [Bra98] and [Cas12].

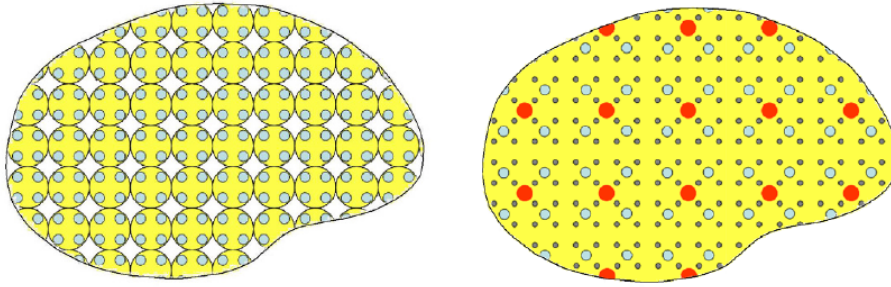


Figure 2.5: Material with more than one microscale

2.1.2 Illustrative example

Let us consider a composite material occupying $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$. Suppose this body has a periodic microstructure with periodicity cell $Y = [0, 1]^n$. Let $a(y)$ be a matrix-valued function that shows how the thermal conductivity changes in the periodicity cell. Substituting $\frac{x}{\epsilon}$ for y gives a function $a(\frac{x}{\epsilon})$ that oscillates periodically with period ϵ at any point x on Ω , thus describing the oscillations of the thermal conductivity in the material, where $\epsilon \in \{\epsilon_k\}_{k=0}^{\infty}$, a sequence of positive real numbers such that $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Supposing the material is placed in a medium with zero temperature and a heat source is introduced, given by a function f , the equilibrium temperature u^ϵ is the solution to the problem

$$\begin{aligned}
 -\operatorname{div}\left(a\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon(x)\right) &= f(x) \quad \text{in } \Omega \\
 u^\epsilon &= 0 \quad \text{on } \partial\Omega.
 \end{aligned}
 \tag{2.1.1}$$

For small values of ϵ , it will be difficult to solve equation (2.1.1) numerically, which makes it hard to find u^ϵ . Furthermore, the smaller ϵ gets, the smaller the heterogeneities become, and the material begins to take on the appearance of a homogeneous material, (See Figure 2.6). This makes one suspect that on the macroscopic scale, it would behave like a homogeneous material. Now we ask; as ϵ tends to zero, can we determine a limit equation similar to (2.1.1), representing

the heterogeneous material and satisfied by the limit u ? i.e. a problem of the form

$$\begin{aligned} -\operatorname{div}(\tilde{a}\nabla u) &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.1.2}$$

This problem is known as the homogenized or effective problem and \tilde{a} is the effective or homogenized thermal conductivity matrix.

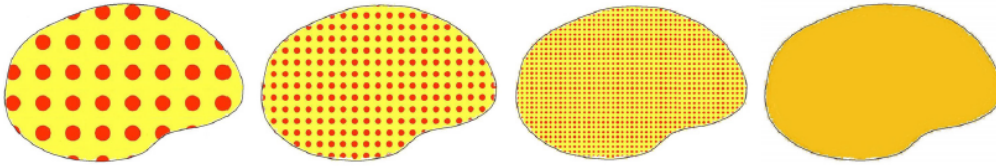


Figure 2.6: Homogenization

Some of the subsequent questions that arise are:

Q(1) Is the solution u a good approximation of u^ϵ for small enough ϵ ?

Q(2) Can \tilde{a} be estimated from $a(\frac{x}{\epsilon})$?

To investigate these questions, let us consider a one dimensional example.

Let $\Omega = (0, 1)$, $f(x) = x^2$, and

$$a(y) = \frac{1}{2 + \sin(2\pi y)}.$$

Substituting into (2.1.1) gives

$$-\frac{d}{dx} \left(\frac{1}{2 + \sin(2\pi(\frac{x}{\epsilon}))} \frac{d}{dx} u^\epsilon \right) = x^2.$$

Integrating both sides, we obtain

$$-\int_0^x \frac{d}{dt} \left(\frac{1}{2 + \sin(2\pi(\frac{t}{\epsilon}))} \frac{d}{dt} u^\epsilon \right) dt = \int_0^x t^2 dt.$$

Let $u^\epsilon(0) = 0$. then we get

$$-\frac{1}{2 + \sin(2\pi(\frac{x}{\epsilon}))} \frac{d}{dx} u^\epsilon(x) = \frac{x^3}{3},$$

this implies that

$$-\frac{d}{dx} u^\epsilon(x) = \frac{x^3}{3} (2 + \sin(2\pi(\frac{x}{\epsilon}))).$$

Integrating again gives

$$-\int_0^x \frac{d}{dt} u^\epsilon(t) dt = \frac{1}{3} \int_0^x t^3 (2 + \sin(2\pi(\frac{t}{\epsilon}))) dt,$$

so

$$-(u^\epsilon(x) - u^\epsilon(0)) = \frac{1}{3} \left(\frac{x^4}{2} + \int_0^x t^3 \sin(2\pi(\frac{t}{\epsilon})) dt \right).$$

Integrating by parts a few times leads to

$$\begin{aligned}
 u^\epsilon(x) = & \frac{\epsilon x^3}{6\pi} \cos(2\pi(\frac{x}{\epsilon})) - \frac{\epsilon^2 x^2}{4\pi^2} \sin(2\pi(\frac{x}{\epsilon})) - \frac{\epsilon^3 x}{4\pi^3} \cos(2\pi(\frac{x}{\epsilon})) \\
 & + \frac{\epsilon^4}{8\pi^4} \sin(2\pi(\frac{x}{\epsilon})) - \frac{x^4}{6}.
 \end{aligned}$$

The figures below show a graph of the solution u and the exact solution u^ϵ for different values of ϵ

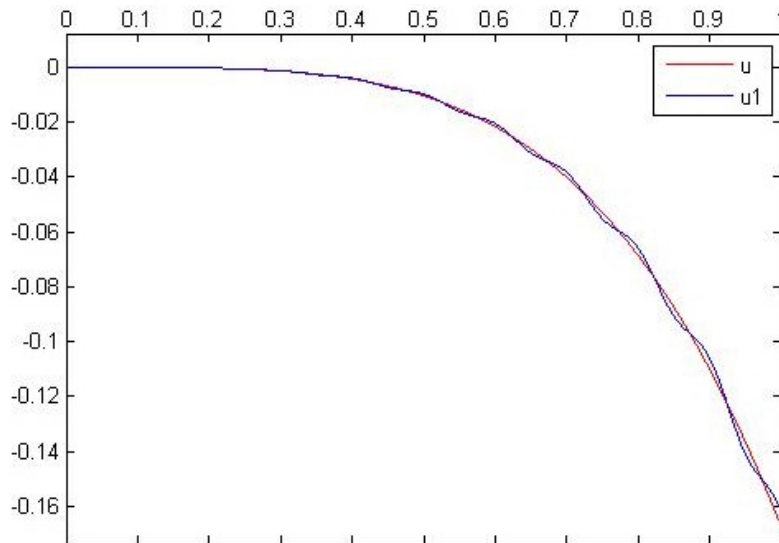


Figure 2.7: u and u_1 , where $u_1 = u^\epsilon$ for $\epsilon = 0.1$.

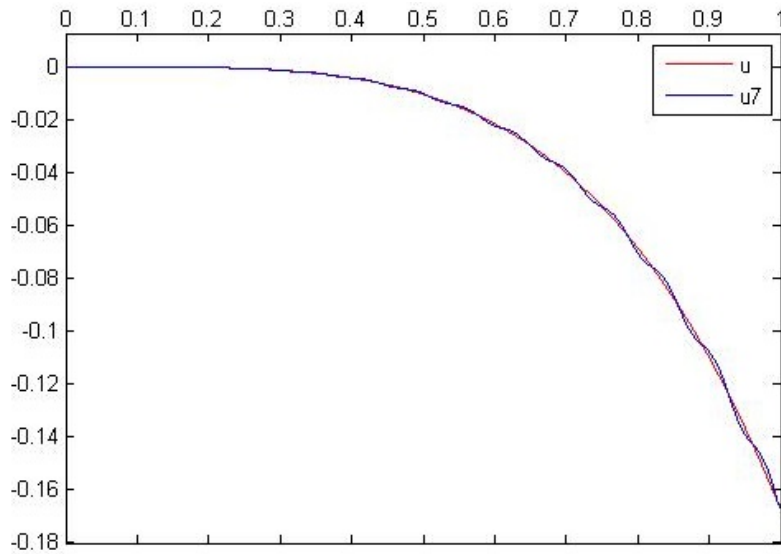


Figure 2.8: u and u_7 , where $u_7 = u^\epsilon$ for $\epsilon = 0.07$.

We see from Figures 2.7 - 2.9 that as ϵ becomes smaller, the curves for the approximation u and the exact solution u^ϵ fit better, i.e., the difference between the solutions u and u^ϵ becomes smaller. Even though this happens, the solution u does not capture the oscillations in u^ϵ , so there is still a difference between ∇u and ∇u^ϵ . To deal with this problem, we see in Chapter 5, Section 5.3, how the approximation u is adjusted with the introduction of a corrector matrix. See also [All92], [Cio99].

In periodic homogenization, various methods have been introduced to compute the limit equations and the limit matrix \tilde{a} . In essence, the theory of homogenization gives positive answers to Q(1) and Q(2).

In summary, the theory of homogenization is aimed at predicting the global properties of a material by taking into account the properties of the heterogeneous constituents, i.e., approximating a heterogeneous problem by a homogeneous one.

Remark. Not all linear partial differential equations used to model these effects have periodic coefficients. There are materials with non-periodic microstructures whose corresponding linear partial differential equations have non-periodic coeffi-

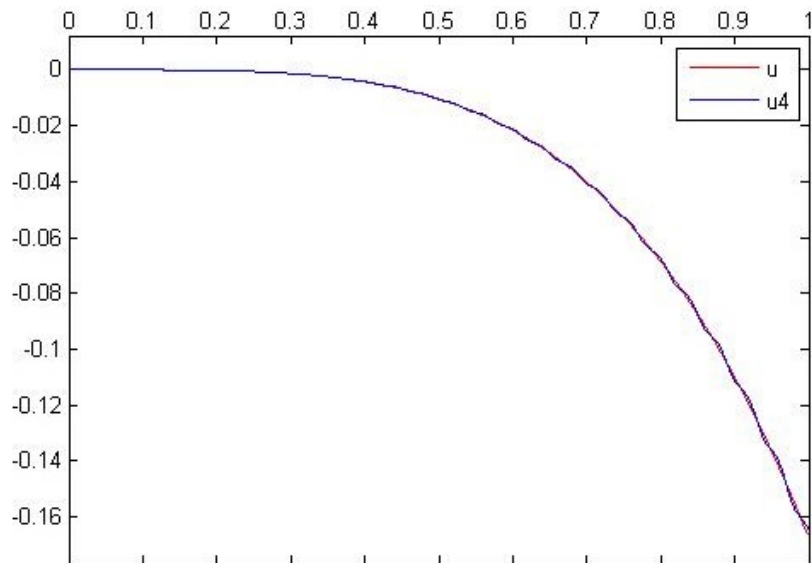


Figure 2.9: u and u_4 , where $u_4 = u^\epsilon$ for $\epsilon = 0.04$.

cients and there are also cases where nonlinear partial differential equations arise as well.

2.2 Homogenization techniques

A brief historic development of the methods of homogenization has been given in Section 2.1. This section contents a brief illustration of some these methods. We shall expand to a certain extent (in subsequent chapters) on two of these methods.

Method of Asymptotic Expansion: This is a classical method widely used in mechanics and physics. It was originally introduced for mechanical problems by engineers till mathematicians began to use it in the study of problems with periodic coefficients. The method of asymptotic expansion is based on the assumption that the solution u^ϵ to problem (2.1.1) is of the form

$$u^\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \dots$$

where $u_i = u_i(x, \frac{x}{\epsilon})$ are Y -periodic in the second variable. When this expansion is substituted into problem (2.1.1), the terms with equal powers of ϵ are equated and a series of problems is obtained. Solving these problems leads to the homogenized problem and the homogenized solution. This is mainly used for partial differential equations with periodic coefficients. See e.g. [San80], [Cio99], [Bak11], [BLP78], [All13] [Pan05], [And09]. This method can also be applied to equations with periodic oscillations on more than one microscopic scale.

G-Convergence: This is a method introduced by S. Spagnolo [Spa67] in the late 1960s for second-order elliptic and parabolic operators. It is an operator-like convergence defined as follows.

Suppose $\{a^\epsilon\} \in M(\alpha, \beta, \Omega)$ (see Definition 3.64) is a sequence of symmetric matrices. Then a^ϵ G-converges to $a^0 \in M(\alpha, \beta, \Omega)$ if and only if for all $f \in H^{-1}(\Omega)$, the solution u^ϵ of (2.1.1) is such that $u^\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, where u_0 is the unique solution of

$$\begin{aligned} -\operatorname{div}(a^0 \nabla u_0) &= f & \text{in } \Omega \\ u_0 &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.2.1}$$

The G-limit a^0 has appropriate properties that makes (2.2.1) uniquely solvable. G-convergence handles problems with symmetric matrices only and periodicity is not a necessary condition. For more on G-convergence, see [Spa76], [Def93], [Pan97], [KZO79], [Cio99] and [Spa73].

H-Convergence: This is an extension of G-convergence to problems involving non-symmetric matrices. In the 1970s, F. Murat [MT97] and L. Tartar [Tar77] introduced an additional condition i.e.,

$$a^\epsilon \nabla u^\epsilon \rightharpoonup a^0 \nabla u^0 \text{ weakly in } (L^2(\Omega))^n.$$

With this additional condition, G-convergence was generalized to problems involving non-symmetric matrices under the name H-convergence. The major difference between this approach and G-convergence is that, H-convergence is based

on the convergence of $a^\epsilon \nabla u^\epsilon$ and the solution u^ϵ , while G-convergence is based on the solution u^ϵ only. Like G-convergence, periodicity assumption is not required for H-convergence. For more on this method, see [Mur77], [MT97], [Pan97] and [Ole92]. Moreover, the H-convergence has been extended by Donato, Damlamian and Braine [Dam98] under the name H^0 -convergence to handle problems in perforated domains. see also [Dam04] and [Don99].

Tartar's method of oscillating test functions: Using equation (2.1.1), choosing $v \in H_0^1(\Omega)$, we have

$$\int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla v \, dx = \int_{\Omega} f \nabla v \, dx, \quad v \in H_0^1(\Omega).$$

While passing to the limit in the equation above, one may have to find the limit of the product of two weakly converging sequences which is a problem, as this product does not generally converge to the product of their limits. This method was introduced by Luc Tartar [Tar77]. It enables one to find the limit by using test functions obtained by periodizing the solutions to a cell problem. When the operator is non-symmetric, the adjoint operator of the cell problem is used, see for instance [Cio99]. Unlike the method of asymptotic expansions which leads to the homogenized problem and the homogenized solution, this method constructs the test function using only the knowledge of the cell problem and the homogenized problem is obtained independently. Constructing the test functions are not as easy as it seems, we refer to Chapter 5 for more details. For more on this method, see [MT97], [Tar09] and [Van81]

Two-scale convergence: This method of convergence was introduced by G. Nguetseng [Ngu89b], [Ngu89a] and was further developed by Allaire [All92]. It can be seen as a combination of the test functions method and the method of asymptotic expansion. It involves the convergence of integrals of the form

$$\int_{\Omega} v^\epsilon(x) \varphi\left(x, \frac{x}{\epsilon}\right) \, dx \rightarrow \int_{\Omega} \int_Y v^0(x, y) \varphi(x, y) \, dy \, dx \quad \forall \varphi \in L^2(\Omega \times Y),$$

up to a subsequence where $\{v^\epsilon\}$ is bounded in $L^2(\Omega)$ and $\varphi = \varphi(x, \frac{x}{\epsilon})$ is smooth and periodic in the second variable. The main feature of this convergence is that the limit v^0 contains both the macroscopic and the microscopic variable. Two-scale convergence was initially restricted to periodic homogenization until Bourgeat et al [Mik94] extended the method to stochastic two-scale convergence, see e.g. [Hei11], [Soó14]. In 2003,Nguetseng extended his two-scale convergence method to handle problems in an almost periodic setting. This extension was named Σ -convergence. For more on two-scale convergence and its applications, see [All92], [Fré11], [Ama98], [Cas02], [Cas00], [Neu96].

The Periodic Unfolding Method: This method was developed in 2002 by Cioranescu, Grisco and Damlamian [Cio02],[Cio08]. It is used for periodic homogenization problems including problems with several microscales and problems in perforated domains [Cio12], [DT13]. It could be seen as an extension of the two-scale convergence and has a broader range of applications, for more see [Fra10], [Fra12].

Compensated compactness: This was introduced by L. Tartar [Tar79] and F. Murat [Mur78] in the 1970s. First, they proved that under certain conditions on the derivatives of weakly converging sequences, the product of two of such sequences converge to the product of their limits in the sense of distributions. This result is known as the ***Div-curl lemma***, a prototype of the result is given below. Suppose Ω is a bounded subset of \mathbb{R}^n and $\{u_h\}$, $\{v_h\}$ are two vector-valued sequences in $(L^2(\Omega))^n$ such that

$$\begin{aligned} u_h &\rightharpoonup u \text{ weakly in } (L^2(\Omega))^n, \\ v_h &\rightharpoonup v \text{ weakly in } (L^2(\Omega))^n. \end{aligned}$$

If

$$\begin{aligned} \operatorname{div} u_h &\text{ is bounded in } L^2(\Omega), \\ \operatorname{curl} v_h &\text{ is bounded in } L^2(\Omega)^{n \times n}, \end{aligned}$$

then

$$u_h \cdot v_h \rightharpoonup u \cdot v \text{ in the sense of distributions.}$$

It is applicable to non-periodic problems and nonlinear homogenization problems. In the study of elliptic problems in divergence forms, this lemma comes in handy. However, one can not apply it to any quadratic product because it requires some specific conditions on the derivatives of the weakly converging quantities. We shall see a generalization of this result in Chapter 6. See [Tar79], [Mur79], [Eva90] for more details on Compensated Compactness.

Young measures: This tool was developed by L.C. Young [You37] in the 1930s, for treating problems of calculus of variation. Later, Tartar [Tar79] developed it as a tool for the analysis of nonlinear partial differential equations. This tool can be used to compute the weak limit of any nonlinear function of weakly converging fields i.e. if $u^\epsilon \rightharpoonup u$, weakly in $L^p(\Omega)$, $1 < p < \infty$, then there exists a subsequence (still denoted by ϵ), and a family of probability measures v_x on \mathbb{R} such that for all $f \in C^0(\Omega)$,

$$f(u^\epsilon(x)) \rightharpoonup \int_{\mathbb{R}} f(\lambda) v_x(\lambda) d\lambda \text{ weakly in } L^\infty(\Omega).$$

Young Measures however, do not capture the differential structure of the equation satisfied by the field u^ϵ , except for certain classes of conservation laws. See for instance [Tar79], [Bal89] and [DiP83].

H-measures: These devices were introduced independently by L. Tartar in [Tar90] and P. Gérard in [Gé91]. They can be seen as a middle ground between Young measures and compensated compactness. Unlike Young measures, H-measures inherit the differential structure of the problem studied and this results in the localization and transport properties of the H-measure, while unlike compensated compactness, no compensation is needed to be able to pass to the limit for quadratic products, see also [Fra06], [Mik02] and [Tar95]. .

The compensated compactness, Young measures and H-measures are further explained in Chapter 6.

Chapter 3

Preliminaries

This chapter contains a variety of definitions, theorems and some basic properties and results used in the forthcoming chapters. The notion of periodicity is introduced. The Lax-Milgram theorem and how it is applied is also shown. The proofs of the theorems and proposition can be found in [Hai11], [Cio99], [Eva98] and [Ole92].

3.1 Banach Spaces

Definition 3.1. Let E be a vector space. A mapping

$$\|\cdot\| : x \in E \mapsto \|x\| \in \mathbb{R}_+,$$

is called a *norm* if and only if

1. $\|x\| = 0 \iff x = 0$,
2. $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{R}$, $x \in E$,
3. $\|x + y\| \leq \|x\| + \|y\|$, for any $x, y \in E$.

Then E is called a normed space with its norm denoted by $\|\cdot\|_E$. Furthermore, E is called a Banach space if and only if it is complete with respect to the strong convergence, i.e.,

$$x_n \rightarrow x \text{ in } E \iff \|x_n - x\|_E \rightarrow 0.$$

Definition 3.2. Let H be a vector space. A mapping

$$(\cdot, \cdot)_H : (x, y) \in H \times H \longmapsto (x, y)_H \in \mathbb{R}$$

is called a *scalar product* if and only if

1. $(x, x)_H > 0 \iff x \neq 0$,
2. $(x, y)_H = (y, x)_H$, for any $x, y \in H$,
3. $(\lambda x + \mu y, z)_H = \lambda(x, z)_H + \mu(y, z)_H$, for any $\lambda, \mu \in \mathbb{R}$, $x, y, z \in H$.

Moreover, if H is a Banach space with respect to the norm associated with this scalar product, i.e.,

$$\|x\|_H = (x, x)_H^{\frac{1}{2}},$$

then H is called a Hilbert space.

Definition 3.3. Let V be a vector space over a field F . A quadratic form is a function $Q : V \rightarrow F$ such that

- $Q(kv) = k^2Q(v) \quad \forall v \in V, k \in F$
- $b_Q(u, v) \doteq \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$ is a symmetric bilinear form.

Definition 3.4. Any $n \times n$ symmetric matrix A determines a quadratic form q_A in n variables by the formula

$$q_A(x_1, \dots, x_n) = \sum_{i,j} A_{ij}x_ix_j.$$

Definition 3.5. Suppose X and Y are Banach spaces. Let $T : X \mapsto Y$ be a linear map such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$\forall x, y \in X$ and $\forall \alpha, \beta \in \mathbb{R}$. T is said to be *bounded* if and only if

$$\sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} < +\infty.$$

Definition 3.6. Let X be a Banach space. The set of all linear functionals on X is called the *dual space* of X and denoted by X' . The dual of the dual space X' is called the *bidual* of X and is denoted by X'' .

If $x' \in X'$, then the image $x'(x)$ of $x \in X$ is denoted by $\langle x', x \rangle_{X', X}$.

Definition 3.7. Let X be a Banach Space and suppose $J : X \mapsto X''$ is a canonical injection from X into X'' . If in addition, J is bijective then the space X is said to be *reflexive*. If X is reflexive, then X can be identified with X'' .

Definition 3.8. (Weak Convergence) Let X be a Banach Space. A sequence $\{x_n\}$ in X is said to *converge weakly* to x if and only if

$$x'(x_n) \rightarrow x'(x) \quad \forall x' \in X',$$

and this is written as

$$x_n \rightharpoonup x \text{ weakly in } X.$$

Proposition 3.9. *Let X be a Banach Space. Every weakly convergent sequence $\{x_n\}$ in X is bounded in X , i.e., there exists a constant C , independent of n , such that*

$$\|x_n\|_X \leq C, \quad \forall n \in \mathbb{N}$$

and the norm on X is lower semi-continuous with respect to weak convergence, that is

$$\|x\|_X \leq \liminf_{n \rightarrow \infty} \|x_n\|_X.$$

Proposition 3.10. *Suppose X is a Banach Space. If a sequence $\{x_n\}$ in X is such that*

$$x_n \rightarrow x \text{ strongly in } X,$$

then

$$(i) \ x_n \rightharpoonup x \text{ weakly in } X$$

$$(ii) \ \|x_n\|_X \rightarrow \|x\|_X.$$

Theorem 3.11 (Eberlein-Šmulian). *Suppose X is reflexive. Then every bounded sequence $\{x_n\}$ in X , has a weakly convergent subsequence, and if every weakly convergent subsequence has the same limit then the whole sequence converges weakly to that limit.*

Theorem 3.11 is referred to as the weak compactness theorem.

Proposition 3.12. *Let $\{x_n\} \subset X$ and $\{x'_n\} \subset X'$ be such that*

$$x_n \rightharpoonup x \text{ weakly in } X,$$

$$x'_n \rightarrow x' \text{ strongly in } X'.$$

Then

$$\lim_{n \rightarrow \infty} x'_n(x_n) = x'(x).$$

The result in the above proposition enables us to find the limit of a product of a weak and a strong convergent sequence.

Definition 3.13. (Weak* Convergence) Let X be a Banach space. A sequence $\{x'_n\}$ in X' converges weakly* to x' , if and only if

$$x'_n(x) \rightarrow x'(x) \quad \forall x \in X,$$

and this is written as

$$x'_n \rightharpoonup x' \text{ weakly* in } X'.$$

Proposition 3.14. *Every weakly* convergent sequence $\{x'_n\}$ in X' is bounded and the norm is lower semi-continuous with respect to the weak* convergence, i.e.,*

$$\|x'\|_{X'} \leq \liminf_{n \rightarrow \infty} \|x'_n\|_{X'}.$$

Theorem 3.15. *Let X be a separable Banach space (i.e. there exists a countable basis in X) with its dual given by X' . If $\{x'_n\}$ is a bounded sequence in X' , then,*

1. *There is a subsequence $\{x'_{n_k}\}$ of $\{x'_n\}$ and $x' \in X'$ such that*

$$x'_{n_k} \rightharpoonup x' \text{ weakly* in } X' \text{ as } k \rightarrow \infty.$$

2. *If each weakly* converging subsequence of $\{x'_n\}$ has the same limit x' , then the entire sequence $\{x'_n\}$ is weakly* convergent to x' , i.e.,*

$$x'_n \rightharpoonup x' \text{ weakly* in } X'.$$

Theorem 3.15 is referred to as weak* compactness theorem.

3.2 Function Spaces

3.2.1 L^p Spaces

Definition 3.16. Let $p \in \mathbb{R}$ with $1 \leq p \leq +\infty$. $L^p(\Omega)$ is defined as the class of measurable functions f on Ω such that for $1 \leq p < +\infty$,

$$\int_{\Omega} |f(x)|^p dx < +\infty,$$

with the norm,

$$\|f\|_{L^p(\Omega)} = \left[\int_{\Omega} |f(x)|^p dx \right]^{\frac{1}{p}}.$$

For $p = \infty$,

$$L^\infty(\Omega) = \{f | f : \Omega \mapsto \mathbb{R}, f \text{ is measurable and } \exists C \in \mathbb{R} \text{ with } |f(x)| < C \text{ a.e. } x \in \Omega\}$$

with the norm

$$\|f\|_{L^\infty(\Omega)} = \inf\{C, |f(x)| \leq C \text{ a.e. } x \in \Omega\}.$$

Proposition 3.17 (Hölder's inequality). *Suppose $f \in L^p$ and $g \in L^q$ where $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\Omega)$ and*

$$\int_{\Omega} |f(x)g(x)| dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

If $p = 2$, the inequality is known as the Cauchy-Schwarz's inequality.

Remark. q is referred to as the Hölder's conjugate of p .

Definition 3.18 (Plancherel's Identity). If $f \in L^2(\Omega)$, then

$$\int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |f(x)|^2 dx,$$

where $\mathcal{F}(f)$ is the Fourier transform of f .

Suppose $1 \leq p \leq q \leq +\infty$. Then $L^q(\Omega) \subset L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq c \|f\|_{L^q(\Omega)}$$

where c is a constant depending on $|\Omega|, p$ and q .

Definition 3.19. Let $1 \leq p \leq +\infty$. A sequence $\{f_n\}$ in $L^p(\Omega)$ converges weakly to f , written

$$f_n \rightharpoonup f \text{ weakly in } L^p(\Omega),$$

if

$$\int_{\Omega} f_n \phi dx \longrightarrow \int_{\Omega} f \phi dx, \quad \forall \phi \in L^q(\Omega)$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 3.20. Suppose $1 < p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\Omega)$. Then $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ iff

$$(i) \quad \|f_n\|_{L^p(\Omega)} \leq C$$

$$(ii) \quad \int_I f_n dx \rightarrow \int_I f dx \quad \text{for any interval } I \subset \Omega$$

Theorem 3.21. Suppose $1 \leq p < +\infty$ with q as its conjugate and let $f \in (L^p(\Omega))'$. Then there exists a unique $g \in L^q(\Omega)$ such that

$$\langle f, \phi \rangle_{(L^p(\Omega))', L^p(\Omega)} = \int_{\Omega} g(x)\phi(x) dx, \quad \forall \phi \in L^p(\Omega).$$

Furthermore,

$$\|g\|_{L^q(\Omega)} = \|f\|_{(L^p(\Omega))'}.$$

Remark. The above theorem allows $L^q(\Omega)$ to be identified with $(L^p(\Omega))'$ for $1 \leq p < +\infty$, so $(L^1(\Omega))' = L^\infty(\Omega)$ but $(L^\infty(\Omega))' \neq L^1(\Omega)$. Instead, $L^1(\Omega) \subset (L^\infty(\Omega))'$. The space $(L^\infty(\Omega))'$ is known as the space of Radon measures on Ω and is also denoted by $\mathcal{M}(\Omega)$.

Theorem 3.22. For $1 \leq p \leq +\infty$, $L^p(\Omega)$ is a Banach space. Moreover, $L^p(\Omega)$ is separable for $1 \leq p < +\infty$ and reflexive for $1 < p < +\infty$.

Remark. The Banach space $L^1(\Omega)$ is not reflexive except where Ω is made up of a finite number of atoms and $L^1(\Omega)$ is finite-dimensional.

Radon measures

The space $L^1(\Omega)$ has no compactness property. So, to be able to still work with it, $L^1(\Omega)$ is embedded into a larger space, the space of Radon measures. The space of Radon measures $\mathcal{M}(\Omega)$ is equal $(C(\Omega))'$, the dual of the space of continuous functions on a bounded open set Ω in \mathbb{R}^n .

Let us define a mapping $T : L^1(\Omega) \mapsto \mathcal{M}(\Omega)$, by

$$\langle Tf, u \rangle_{\mathcal{M}(\Omega), C(\Omega)} = \int_{\Omega} fu dx \quad \forall f \in L^1(\Omega), u \in C(\Omega),$$

with

$$\|Tf\|_{\mathcal{M}(\Omega)} = \sup_{u \in C(\Omega), \|u\| \leq 1} \int_{\Omega} fu \, dx = \|f\|_1.$$

One may identify $L^1(\Omega)$ with a subspace of $\mathcal{M}(\Omega)$. Now since $C(\Omega)$ is a separable space, there is a weak compactness property, i.e., if $\{f_n\}$ is a bounded sequence in $L^1(\Omega)$ then there is a subsequence $\{f_{n_k}\}$ and $\mu \in \mathcal{M}(\Omega)$ such that $f_{n_k} \rightharpoonup \mu$ weakly* in $\mathcal{M}(\Omega)$, i.e., $\int_{\Omega} f_{n_k} u \rightarrow \langle \mu, u \rangle$, $\forall u \in C(\Omega)$.

3.2.2 Sobolev Spaces

Definition 3.23. Let Ω be a bounded open set in \mathbb{R}^n . For a function $\phi : \Omega \mapsto \mathbb{R}$, the closed subset F of Ω , such that $\phi = 0$ a.e. on $\Omega \setminus F$ is known as the *support* of ϕ and is denoted by $\text{supp } \phi$ i.e

$$\text{supp } \phi = \overline{\{x \in \Omega : \phi(x) \neq 0\}}$$

Definition 3.24. Assume that $\{\varphi_n\}$ is a sequence in $\mathcal{D}(\Omega)$. We say that φ_n converges to φ in $\mathcal{D}(\Omega)$ if

- (i) there exists a compact set $K \subset \Omega$ such that $\forall n \in \mathbb{N}, \text{supp } \varphi_n \subset K$,
- (ii) for every multi-index α , $\{\partial^\alpha \varphi_n\}$ converges uniformly to $\partial^\alpha \varphi$ on K .

Theorem 3.25. For $1 \leq p < +\infty$, $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.

Theorem 3.26. Suppose $f \in L^1_{loc}(\Omega)$ is such that

$$\int_{\Omega} f(x)\phi(x) \, dx = 0, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Then $f = 0$, a.e. on Ω .

Definition 3.27. A mapping $T : \mathcal{D}(\Omega) \mapsto \mathbb{R}$ is called a *distribution* on Ω if

- (i) T is a linear form on $\mathcal{D}(\Omega)$, i.e. $T(\alpha\varphi_1 + \beta\varphi_2) = \alpha T(\varphi_1) + \beta T(\varphi_2)$, $\alpha, \beta \in \mathbb{R}$.

(ii) T is continuous on sequences, i.e., $\varphi_n \rightarrow \varphi$ in $\mathcal{D}(\Omega) \implies T(\varphi_n) \rightarrow T(\varphi)$.

The set of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$ and $T(\varphi) = \langle T, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$.

Definition 3.28. A sequence $\{T_n\}$ in $\mathcal{D}'(\Omega)$ is said to *converge* to T in $\mathcal{D}'(\Omega)$ in the sense of distributions if and only if

$$\langle T_n, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \longrightarrow \langle T, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

and we write

$$T_n \longrightarrow T \text{ in } \mathcal{D}'(\Omega).$$

Definition 3.29. For any $i = 1, \dots, n$, the derivative $\frac{\partial T}{\partial x_i}$ of $T \in \mathcal{D}'(\Omega)$ is defined by

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = - \left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Definition 3.30. Let $1 \leq p \leq +\infty$. The *Sobolev space* $W^{1,p}(\Omega)$ is defined as

$$\left\{ u \mid u \in L^p(\Omega), \frac{\partial u}{\partial x_i} \in L^p(\Omega), \quad i = 1, \dots, n \right\},$$

with derivatives taken in the sense of distributions i.e.

$$\forall v \in \mathcal{D}(\Omega), \quad \left\langle \frac{\partial u}{\partial x_i}, v \right\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = - \left\langle u, \frac{\partial v}{\partial x_i} \right\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$$

If $p = 2$, $W^{1,2}(\Omega)$ is written as $H^1(\Omega)$ i.e.

$$H^1(\Omega) = \left\{ u \mid u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), \quad i = 1, \dots, n \right\}.$$

Proposition 3.31. 1.) *The norm on the Sobolev space $W^{1,p}(\Omega)$ is given by*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}.$$

$W^{1,p}(\Omega)$ is a Banach space with the above norm.

2.) The space $H^1(\Omega)$ is equipped with the following scalar product,

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}, \quad \forall u, v \in H^1(\Omega), \quad (3.2.1)$$

and its norm is given by

$$\|u\|_{H^1(\Omega)} = \sqrt{(u, u)_{H^1(\Omega)}}.$$

The space $H^1(\Omega)$ is a Hilbert space.

Definition 3.32. The boundary of Ω , is said to be *Lipschitz continuous* if and only if there exist two real numbers, $c_1 > 0$, $c_2 > 0$ and M number of local co-ordinates (y'_r, y_{rn}) and local maps φ_r , $r = 1, \dots, M$ defined on the set

$$\Delta_r = \{y'_r \in \mathbb{R}^{n-1}, |y'_r| \leq c_1\},$$

such that for

$$\begin{aligned} \partial\Omega &= \bigcup_{r=1}^M \Gamma_r, \quad r = 1, \dots, M, \\ \Gamma_r &= \{(y'_r, y_{rn}); y_{rn} = \varphi_r(y'_r), y'_r \in \Delta_r\}, \end{aligned}$$

and

$$\begin{aligned} U_r^+ &= \{(y'_r, y_{rn}); \varphi_r(y'_r) < y_{rn} < \varphi_r(y'_r) + c_2, y'_r \in \Delta_r\} \subset \Omega, \\ U_r^- &= \{(y'_r, y_{rn}); \varphi_r(y'_r) - c_2 < y_{rn} < \varphi_r(y'_r), y'_r \in \Delta_r\} \subset \mathbb{R}^N \setminus \bar{\Omega}. \end{aligned}$$

φ_r is Lipschitz continuous if and only if there is a constant $K_r > 0$ such that

$$|\varphi_r(x'_r) - \varphi_r(y'_r)| \leq K_r |x'_r - y'_r|, \quad \forall x'_r, y'_r \in \Delta_r.$$

Theorem 3.33. Let $1 \leq p < \infty$ be a real number. Then $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. Furthermore, if $\partial\Omega$ is Lipschitz continuous, then $\mathcal{D}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Theorem 3.34 (Sobolev embedding theorem). Assume that $\partial\Omega$ is Lipschitz continuous. Then

1. if $1 \leq p < n$, $W^{1,p}(\Omega) \subset L^q(\Omega)$ with

- (a) compact injection for $q \in [1, s)$ where $\frac{1}{s} = \frac{1}{p} - \frac{1}{n}$
- (b) continuous injection for $q = s$,
2. if $p = n$, $W^{1,p}(\Omega) \subset L^q(\Omega)$ with compact injection if $q \in [1, +\infty)$,
3. if $p > n$, $W^{1,p}(\Omega) \subset C^0(\bar{\Omega})$ with compact injection.

Theorem 3.35. *The linear continuous operator $\gamma : H^1(\mathbb{R}^{n-1} \times \mathbb{R}_+) \mapsto L^2(\mathbb{R}^{n-1})$ such that*

$$\forall u \in H^1(\mathbb{R}^{n-1} \times \mathbb{R}_+) \cap C^0(\mathbb{R}^{n-1} \times \mathbb{R}_+), \quad \gamma(u) = u|_{\mathbb{R}^{n-1}}$$

is called the trace of u .

Theorem 3.36. *Suppose $\partial\Omega$ is Lipschitz continuous. Then the linear map*

$$\gamma : H^1(\Omega) \mapsto L^2(\partial\Omega)$$

such that

$$\forall u \in H^1 \cap C^0(\Omega), \quad \gamma(u) = u|_{\partial\Omega}.$$

is called the trace of u on $\partial\Omega$.

Definition 3.37. For $1 \leq p \leq \infty$, $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ with respect to the $W^{1,p}(\Omega)$ norm. For $p = 2$, $W_0^{1,2}(\Omega)$ is written as $H_0^1(\Omega)$.

The space $H_0^1(\Omega)$ is a subset of $H^1(\Omega)$. Hence $H_0^1(\Omega)$ is reflexive and a Hilbert space with the H^1 -scalar product (3.2.1). The Sobolev embedding theorem applies to this space as well.

Proposition 3.38 (Poincaré inequality). *Let Ω be a bounded open set. Then there is a constant depending on the diameter of Ω denoted C_Ω such that*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)}. \quad \forall u \in H_0^1(\Omega).$$

Thus, $\|\nabla u\|_{L^2(\Omega)}$ is the $H_0^1(\Omega)$ norm equivalent to the H^1 -norm.

If $\partial\Omega$ is Lipschitz continuous, the set $H^{\frac{1}{2}}(\Omega)$ is defined as the range of the operator γ , i.e, $H^{\frac{1}{2}}(\partial\Omega) = \gamma(H^1(\Omega))$.

Theorem 3.39. *Assume $\partial\Omega$ is Lipschitz continuous. Then $H^{\frac{1}{2}}(\partial\Omega)$ is a Banach space with the norm defined by*

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} = \int_{\partial\Omega} |u(x)|^2 ds_x + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+1}} ds_x ds_y.$$

Proposition 3.40. *Suppose $\partial\Omega$ is Lipschitz continuous. Then there is a constant $C_\gamma(\Omega)$ such that*

$$\|\gamma(u)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_\gamma(\Omega) \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Theorem 3.41 (Green formula). *Suppose $\partial\Omega$ is Lipschitz continuous. Then*

$$\int_{\Omega} \frac{\partial v}{\partial x_i}(x) u(x) dx = - \int_{\Omega} v(x) \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} \gamma(u) \gamma(v) n_i ds, \quad \forall u, v \in H^1(\Omega).$$

where $n = (n_1, \dots, n_n)$ is the outward unit normal vector on $\partial\Omega$ and $i = 1, \dots, n$.

Definition 3.42. Suppose Ω is connected. The space $\mathcal{W}(\Omega) = H^1(\Omega)/\mathbb{R}$ is defined as the space of equivalent classes, where

$$\forall u, v \in H^1(\Omega), \quad u \equiv v \iff u - v \text{ is a constant.}$$

Proposition 3.43. *Suppose Ω is connected. The quotient space $\mathcal{W}(\Omega)$ is a Banach space with the norm*

$$\|\dot{u}\|_{\mathcal{W}(\Omega)} = \|\nabla u\|_{L^2(\Omega)}, \quad u \in \dot{u}, \quad \dot{u} \in \mathcal{W}(\Omega),$$

and is a Hilbert space with the following inner product;

$$(u, v)_{\mathcal{W}(\Omega)} = \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}, \quad \forall u, v \in \mathcal{W}(\Omega).$$

Definition 3.44. The dual space of $H_0^1(\Omega)$ written as $H^{-1}(\Omega)$ is a Banach space equipped with the norm

$$\|F\|_{H^{-1}(\Omega)} = \sup_{v \neq 0} \frac{|\langle F, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}|}{\|v\|_{H_0^1(\Omega)}}, \quad \text{for } F \in H^{-1}(\Omega), v \in H_0^1(\Omega).$$

Proposition 3.45. *Suppose $F \in H^{-1}(\Omega)$. Then there exist $n + 1$ functions in $L^2(\Omega)$ such that*

$$F = f_0 + \sum_{i=1}^N \frac{\partial f_i}{\partial x_i}, \quad (3.2.2)$$

in the sense of distributions. Furthermore,

$$\|F\|_{H^{-1}(\Omega)}^2 = \inf \sum_{i=0}^n \|f_i\|_{L^2(\Omega)}^2.$$

where $(f_0, f_1, \dots, f_n) \in (L^2(\Omega))^{n+1}$.

Conversely, if $(f_0, f_1, \dots, f_n) \in (L^2(\Omega))^{n+1}$ is a vector, then any $F \in H^{-1}(\Omega)$ such that

$$\|F\|_{H^{-1}(\Omega)}^2 \leq \sum_{i=0}^n \|f_i\|_{L^2(\Omega)}^2.$$

is defined by (3.2.2).

3.2.3 Sobolev Spaces of Periodic Functions

This section contains a brief review of the class of periodic functions of the form

$$a_\epsilon = a\left(\frac{x}{\epsilon}\right),$$

where a is a periodic function and $\epsilon > 0$ takes its values in a sequence that tends to zero. Throughout this thesis, Y will denote a parallelepiped in \mathbb{R}^n defined by

$$Y = (0, l_1) \times \dots \times (0, l_n), \quad (3.2.3)$$

where l_1, \dots, l_n are given positive numbers, Y will be referred to as the periodicity cell.

Rapidly oscillating periodic functions

Definition 3.46. Let Y be defined by the relation (3.2.3) and f , a function defined a.e. on \mathbb{R}^n . The function f is called Y -periodic if

$$f(x + kl_i e_i) = f(x) \text{ a.e on } \mathbb{R}^n, \quad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, \dots, n\}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{R}^n . If $n = 1$, then f is said to be l_1 -periodic.

Definition 3.47. Let Ω be a bounded open set of \mathbb{R}^n and f a function in $L^1(\Omega)$. The *mean value of f over Ω* is the real number $\mathcal{M}_\Omega(f)$ given by

$$\mathcal{M}_\Omega(f) = \frac{1}{|\Omega|} \int_\Omega f(y) dy.$$

Proposition 3.48 (Poincaré-Wirtinger inequality). *If Ω is connected, then there is a constant depending on the diameter of Ω denoted by C_Ω , such that*

$$\|u - \mathcal{M}_\Omega(u)\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega).$$

where $\mathcal{M}_\Omega(u)$ denotes the mean value of u on Ω .

Lemma 3.49. *Let f be a Y -periodic function in $L^1(Y)$ and y_0 a fixed point in \mathbb{R}^n . Denote by Y_0 the translated set of Y , defined by*

$$Y_0 = y_0 + Y.$$

Set

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right) \text{ a.e on } \mathbb{R}^n.$$

Then

1. $\int_{Y_0} f(y) dy = \int_Y f(y) dy,$
2. $\int_{\epsilon Y_0} f_\epsilon(x) dx = \int_{\epsilon Y} f_\epsilon(x) dx = \epsilon^n \int_Y f(y) dy.$

Theorem 3.50. *Let $1 \leq p \leq +\infty$, and f be a Y -periodic function in $L^p(Y)$. Set*

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right) \text{ a.e on } \mathbb{R}^n.$$

If $p < +\infty$, then as $\epsilon \rightarrow 0$,

$$f_\epsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|} \int_Y f(y) dy \text{ weakly in } L^p(\omega),$$

for any bounded open subset ω of \mathbb{R}^n .

If $p = +\infty$, then as $\epsilon \rightarrow 0$,

$$f_\epsilon \xrightarrow{*} \mathcal{M}_Y(f) \text{ weakly* in } L^\infty(\mathbb{R}^n).$$

Example 3.51. Let $f(y)$ be a periodic function with period 2 defined on $(0, 2)$ such that

$$f(y) = \begin{cases} \rho & y \in (0, \frac{4}{5}) \\ \sigma & \text{otherwise.} \end{cases}$$

Let $f_\epsilon(x) = f(\frac{x}{\epsilon})$ with $x \in (a, b)$, where $a, b \in \mathbb{R}, a \neq b, \epsilon \in \{1/2^n\}$ and $n \in \mathbb{N}$.

Using Proposition 3.20, we compute the weak limit of $f_\epsilon(x)$.

$f_\epsilon(x)$ is bounded independent of ϵ , so (i) is satisfied. To check (ii), let $I = (a_1, b_1)$ be an interval in (a, b) , then we compute

$$\int_I f_\epsilon(x) dx = \int_{a_1}^{b_1} f_\epsilon(x) dx.$$

For any $\epsilon > 0$, there exist k and θ such that

$$b_1 = a_1 + 2k\epsilon + \theta_\epsilon, \quad k \in \mathbb{N}, \quad 0 \leq \theta < 2. \quad (3.2.4)$$

So

$$\begin{aligned} \int_{a_1}^{b_1} f_\epsilon(x) dx &= \int_{a_1}^{b_1} f\left(\frac{x}{\epsilon}\right) dx = \epsilon \int_{\frac{a_1}{\epsilon}}^{\frac{b_1}{\epsilon}} f(y) dy \\ &= \epsilon \int_{\frac{a_1}{\epsilon}}^{\frac{a_1}{\epsilon} + 2k} f(y) dy + \epsilon \int_{\frac{a_1}{\epsilon} + 2k}^{\frac{a_1}{\epsilon} + 2k + \theta} f(y) dy. \end{aligned}$$

Using Lemma 3.49, we have

$$\epsilon \int_{\frac{a_1}{\epsilon}}^{\frac{a_1}{\epsilon} + 2k} f(y) dy = \epsilon \sum_{h=1}^k \int_{\frac{a_1}{\epsilon} + 2(h-1)}^{\frac{a_1}{\epsilon} + 2h} f(y) dy = \epsilon k \int_0^2 f(y) dy,$$

from (3.2.4),

$$= \frac{b_1 - a_1 - \theta_\epsilon}{2} \int_0^2 f(y) dy.$$

On the other hand,

$$\left| \epsilon \int_{\frac{a_1}{\epsilon} + 2k}^{\frac{a_1}{\epsilon} + 2k + \theta} f(y) dy \right| \leq \epsilon \int_{\frac{a_1}{\epsilon} + 2k}^{\frac{a_1}{\epsilon} + 2k + 2} |f(y)| dy = \epsilon \int_0^2 |f(y)| dy.$$

Therefore, as $\epsilon \rightarrow 0$,

$$\begin{aligned} \int_{\frac{a_1}{\epsilon}}^{\frac{b_1}{\epsilon}} f(y) dy &\longrightarrow \frac{b_1 - a_1}{2} \int_0^2 f(y) dy \\ &= \frac{b_1 - a_1}{2} \left(\frac{4}{5}\rho + \frac{6}{5}\sigma \right). \end{aligned}$$

Using Proposition 3.20, this implies that

$$f_\epsilon \rightharpoonup \frac{1}{2} \left(\frac{4}{5}\rho + \frac{6}{5}\sigma \right) \text{ weakly in } L^2(a, b).$$

Notice that Theorem 3.50 makes computing the weak limit of a periodic function easier and more straightforward.

Using the same example and taking $Y = (0, 2)$, we have

$$\begin{aligned} f_\epsilon(x) \rightharpoonup \mathcal{M}_Y(f) &= \frac{1}{2} \int_0^2 f(y) dy \\ &= \frac{1}{2} \left(\frac{4}{5}\rho + \frac{6}{5}\sigma \right). \end{aligned}$$

So $f_\epsilon(x) \rightharpoonup \frac{1}{2} \left(\frac{4}{5}\rho + \frac{6}{5}\sigma \right)$ weakly in $L^2(a, b)$.

Suppose f and g are Y -periodic functions in $L^2(Y)$. Let

$$\begin{aligned} u_\epsilon(x) &= f\left(\frac{x}{\epsilon}\right) \text{ a.e. on } \mathbb{R}^n, \\ v_\epsilon(x) &= g\left(\frac{x}{\epsilon}\right) \text{ a.e. on } \mathbb{R}^n. \end{aligned}$$

Then by Theorem 3.50,

$$u_\epsilon(x) \rightharpoonup \mathcal{M}_Y(u) \text{ weakly in } L^2(Y),$$

and

$$v_\epsilon(x) \rightharpoonup \mathcal{M}_Y(v) \text{ weakly in } L^2(Y).$$

Again by Theorem 3.50,

$$v_\epsilon u_\epsilon = fg\left(\frac{\cdot}{\epsilon}\right) \rightharpoonup \mathcal{M}_Y(fg) \text{ weakly in } L^1(\omega),$$

where ω is an open bounded subset in \mathbb{R}^n .

$$\langle v_\epsilon, u_\epsilon \rangle_{L^2(Y), L^2(Y)} = \int_\omega v_\epsilon u_\epsilon dx \rightarrow |\omega| \mathcal{M}_Y(fg),$$

while

$$\langle \mathcal{M}_Y(u), \mathcal{M}_Y(v) \rangle_{L^2(Y), L^2(Y)} = \int_{\omega} \mathcal{M}_Y(f) \mathcal{M}_Y(g) dx = |\omega| \mathcal{M}(f) \mathcal{M}(g).$$

Using an example, we check if $\mathcal{M}_Y(fg) = \mathcal{M}_Y(f) \mathcal{M}_Y(g)$. From Example 3.51, let

$$\mathcal{M}_Y(g) = \mathcal{M}_Y(f) = \frac{1}{2} \left(\frac{4}{5} \rho + \frac{6}{5} \sigma \right).$$

Then

$$\begin{aligned} \mathcal{M}_Y(f) \mathcal{M}_Y(g) &= \left[\frac{1}{2} \left(\frac{4}{5} \rho + \frac{6}{5} \sigma \right) \right]^2 \\ &= \left(\frac{2}{5} \rho + \frac{3}{5} \sigma \right)^2 \\ &= \frac{1}{25} (4\rho^2 + 12\rho\sigma + 9\sigma^2). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{M}_Y(f^2) &= \frac{1}{2} \int_0^2 (f(y))^2 dy \\ &= \frac{1}{2} \left(\frac{4}{5} \rho^2 + \frac{6}{5} \sigma^2 \right) \\ &= \frac{1}{5} (2\rho^2 + 3\sigma^2). \end{aligned}$$

So from the above example, we see that in general, $\mathcal{M}_Y(fg) \neq \mathcal{M}_Y(f) \mathcal{M}_Y(g)$.

Remark. Suppose A is a Banach space with A' as its dual. If $\{a_\epsilon\} \subset A$ is a sequence converging weakly to $a \in A$ and $\{a'_\epsilon\} \subset A'$ is a sequence converging weakly to $a' \in A'$, then in general,

$$\langle a'_\epsilon, a_\epsilon \rangle_{A', A} \not\rightarrow \langle a', a \rangle_{A', A}.$$

The space $L^2(\Omega; L^2(Y))$ which can be written as $L^2(\Omega \times Y)$ is a Hilbert space with the inner product,

$$(u, v)_{L^2(\Omega \times Y)} = \int_{\Omega} \int_Y u(x, y) v(x, y) dy dx.$$

and a norm given by

$$\|u\|_{L^2(\Omega \times Y)} = \int_{\Omega} \int_Y |u(x, y)|^2 dy dx.$$

Let f be a function defined a.e on Y , its extension $f^\#$ by periodicity to \mathbb{R}^n is defined by

$$f^\#(x + kl_i e_i) = f(x) \text{ a.e on } Y, \quad k \in \mathbb{Z}, \quad \forall i \in \{1, \dots, n\},$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{R}^n .

The space $L^2(\Omega; C_{per}(Y))$ is a separable space dense in $L^2(\Omega; Y)$ with norm given by

$$\|u\|_{L^2(\Omega; C_{per}(Y))}^2 = \int_{\Omega} (\sup_{y \in Y} |u(x, y)|)^2 dx.$$

Theorem 3.52. [Pav08] Let $u_0 \in L^2(\Omega; C_{per}(Y))$, and define $u^\epsilon(x)$ by $u(x, \frac{x}{\epsilon})$ with $\epsilon > 0$. Then

1. $u^\epsilon \in L^2(\Omega)$ and $\|u^\epsilon\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega; C_{per}(Y))}$.
2. $u^\epsilon(x) \rightharpoonup \int_Y u_0(x, y) dy$ weakly in $L^2(\Omega)$ as $\epsilon \rightarrow 0$.
3. $\|u^\epsilon\|_{L^2(\Omega)} \rightarrow \|u_0\|_{L^2(\Omega \times Y)}$ as $\epsilon \rightarrow 0$.

Definition 3.53. The completion of $C_{per}^\infty(Y)$ with respect to the H^1 -norm is denoted by $H_{per}^1(Y)$.

Proposition 3.54. If $u \in H_{per}^1(Y)$, then the trace of u on the opposite faces of Y are equal.

Definition 3.55. The space

$$\mathcal{W}_{per}(Y) = H_{per}^1(Y)/\mathbb{R}$$

is defined as the space of classes of equivalence with respect to the relation

$$u \equiv v \iff u - v \text{ is a constant, } \forall u, v \in H_{per}^1(Y).$$

Proposition 3.56. *The norm on $\mathcal{W}_{per}(Y)$ is defined by*

$$\|\dot{u}\|_{\mathcal{W}_{per}(Y)} = \|\nabla u\|_{L^2(Y)}, \quad \forall u \in \dot{u}, \dot{u} \in \mathcal{W}_{per}(Y).$$

Furthermore, the dual space $(\mathcal{W}_{per}(Y))'$ can be identified with the set

$$\{F \in (H_{per}^1(Y))' \mid F(c) = 0, \quad \forall c \in \mathbb{R}\}.$$

where

$$\langle F, \dot{u} \rangle_{(\mathcal{W}_{per}(Y))', \mathcal{W}_{per}(Y)} = \langle F, u \rangle_{(H_{per}^1(Y))', H_{per}^1(Y)}, \quad \forall u \in \dot{u}, \forall \dot{u} \in \mathcal{W}_{per}(Y).$$

3.3 The Lax-Milgram Theory

The Lax-Milgram theorem is an essential tool which plays a crucial role in the theory of weak solutions of linear elliptic partial differential equations in divergence form. This section is devoted to the abstract formulation and few basic applications of the Lax-Milgram theorem. We use it in the proof of the existence and uniqueness of a solution to a boundary value problem, in the weak sense.

Definition 3.57. let H be a Hilbert space. The map B from $H \times H$ to \mathbb{R} is called a *bilinear form* on H iff, for any fixed $u \in H$, the following maps:

$$B(u, \cdot) : v \in H \longmapsto B(u, v) \in \mathbb{R},$$

$$B(\cdot, u) : v \in H \longmapsto B(v, u) \in \mathbb{R},$$

are linear.

Definition 3.58. A map B from $H \times H$ to \mathbb{R} is *bounded* on H if and only if there exists $C > 0$ such that

$$|B(u, v)| \leq C \|u\|_H \|v\|_H.$$

Proposition 3.59. *Let $B : H \times H \longmapsto \mathbb{R}$ be a bilinear form. Then B is bounded if and only if B is continuous on $H \times H$.*

Definition 3.60. A bilinear form B on H is called *symmetric* if and only if

$$B(u, v) = B(v, u), \quad \forall u, v \in H.$$

It is called *positive* if only if

$$B(u, u) \geq 0, \quad \forall u \in H.$$

It is called *H-elliptic or coercive* with a constant α , if and only if there exists $\alpha > 0$ such that

$$B(u, u) \geq \alpha \|u\|_H^2, \quad \forall u \in H.$$

Proposition 3.61. Let V be a vector space, any symmetric bilinear form b_Q defines a quadratic form i.e.

$$Q(u) = b_Q(u, u), \quad \forall u \in V.$$

Let H be a Hilbert space, B a bilinear form on H and $F \in H'$. Consider the problem

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ B(u, v) = \langle F, v \rangle_{H', H}, \quad \forall v \in H. \end{cases} \quad (3.3.1)$$

Problem (3.3.1) is known as a variational problem and $v \in H$ is a test function.

Definition 3.62. The problem (3.3.1) is said to be well-posed if it has a unique solution and there exists a constant $C > 0$ such that

$$\forall F \in H', \quad \|u\|_H \leq C \|F\|_{H'}.$$

Theorem 3.63 (Lax-Milgram Theorem). Let

$$B : H \times H \mapsto \mathbb{R}$$

be a bilinear form such that B is H -elliptic with constant α (see Definition 3.60) and $F \in H'$. Then there exists a unique solution $u \in H$ for the variational problem (3.3.1). Furthermore,

$$\|u\|_H \leq \frac{1}{\alpha} \|F\|_{H'}.$$

Definition 3.64. Let $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$. Let $M(\alpha, \beta, \Omega)$ be the set of $N \times N$ matrices $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(\Omega))^{N \times N}$ such that

$$\begin{cases} i.) & (A(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ ii.) & |A(x)\lambda| \leq \beta|\lambda|. \end{cases} \quad (3.3.2)$$

for any $\lambda \in \mathbb{R}^N$ and almost everywhere on Ω , where Ω is a bounded open set in \mathbb{R}^N .

In this dissertation, partial differential equations involving elliptic operator in divergence form are treated i.e.

$$\mathcal{A} = -\operatorname{div}(A(x)\nabla) = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right). \quad (3.3.3)$$

3.3.1 Dirichlet Problem

Homogeneous Dirichlet Problem

Solving a homogeneous Dirichlet problem means finding a function u defined on Ω that solves the following problem:

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.3.4)$$

where A is a positive matrix and $f \in H^{-1}(\Omega)$.

Assume that u is sufficiently smooth. Then we may multiply each side of the first part of the equation by a test function, $v \in D(\Omega)$. Integrating over Ω , using the Green formula and the condition that $u = 0$ on $\partial\Omega$, we get

$$\int_{\Omega} A\nabla u \nabla v \, dx = \int_{\Omega} f v \, dx.$$

The definition of A , the Cauchy-Schwarz inequality and the Poincaré inequality give

$$\int_{\Omega} A\nabla u \nabla v \, dx \leq \beta \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} = \beta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

So the solution u to problem (3.3.4) belongs to $H_0^1(\Omega)$.

Hence, the corresponding variational problem (or the weak formulation of problem (3.3.4)) is

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u, v) = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \end{cases} \quad (3.3.5)$$

where

$$a(u, v) = \int_{\Omega} A \nabla u \nabla v \, dx, \quad \forall u, v \in H_0^1(\Omega)$$

Theorem 3.65. *Suppose A is a matrix in $M(\alpha, \beta, \Omega)$ and $f \in H^{-1}(\Omega)$. Then there exists a unique solution $u \in H_0^1(\Omega)$ to problem (3.3.5). Furthermore,*

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)}. \quad (3.3.6)$$

Proof. First, check ellipticity and continuity of $a(\cdot, \cdot)$ in order to apply the Lax-Milgram theorem.

From the definition of A ,

$$a(u, u) = \int_{\Omega} A \nabla u \nabla u \, dx \geq \alpha \|\nabla u\|_{L^2(\Omega)}^2 = \alpha \|u\|_{H_0^1(\Omega)}^2.$$

Next, from the definition of A and the Cauchy-Schwarz inequality, one gets,

$$|a(u, v)| = \left| \int_{\Omega} A \nabla u \nabla v \, dx \right| \leq \beta \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} = \beta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

Applying Lax-Milgram theorem, the problem (3.3.5) has a unique solution $u \in H_0^1(\Omega)$.

Lastly, we prove estimate (3.3.6). From the ellipticity condition,

$$\begin{aligned} \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} &\leq a(u, v) = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\leq \|f\|_{H^{-1}(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

This implies that

$$\alpha \|u\|_{H_0^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}.$$

Hence

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)}.$$

□

Non-homogeneous Dirichlet Problem

Suppose $\partial\Omega$ is Lipschitz continuous. Let $f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Consider the following problem

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= f & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega. \end{aligned}$$

Using the notion of trace, an equivalent equation is

$$\begin{aligned} \operatorname{div}(A\nabla u) &= f & \text{in } \Omega, \\ \gamma(u) &= g & \text{on } \partial\Omega. \end{aligned}$$

Theorem 3.66. *Let A be a matrix in $M(\alpha, \beta, \Omega)$ and $\partial\Omega$ be Lipschitz continuous. Suppose $f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Then there exists a unique solution $u \in H^1(\Omega)$ to problem (3.3.1) and*

$$\|u\|_{H^1(\Omega)} \leq C_1 \|f\|_{H^{-1}(\Omega)} + C_2 \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}. \quad (3.3.7)$$

where $C_1 > 0$ and $C_2 > 0$ depends on Ω, α and β .

3.3.2 Periodic boundary conditions

As defined in 3.2.3, let Y be a parallelepiped in \mathbb{R}^n defined by

$$Y = (0, l_1) \times \dots \times (0, l_n),$$

where l_1, \dots, l_n are give positive numbers.

Suppose $f \in H^{-1}(\Omega)$ and $A \in M(\alpha, \beta, Y)$. Consider the problem

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= f & \text{in } Y, \\ u & \text{ } Y\text{-periodic.} \end{aligned} \quad (3.3.8)$$

The weak formulation of Problem (3.3.8) is

$$\begin{cases} \text{Find } \dot{u} \in \mathcal{W}_{per}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}, \dot{v}) = \langle f, \dot{v} \rangle_{(\mathcal{W}_{per}(Y))', \mathcal{W}_{per}(Y)} \quad \forall \dot{v} \in \mathcal{W}_{per}(Y), \end{cases} \quad (3.3.9)$$

where

$$a_Y(\dot{u}, \dot{v}) = \int_Y A \nabla u \nabla v \, dy \quad \forall u \in \dot{u}, \forall v \in \dot{v}. \forall \dot{u}, \dot{v} \in \mathcal{W}_{per}(Y).$$

Using the definition of A and the Lax-Milgram theorem, one can prove that there exists a solution $\dot{u} \in \mathcal{W}_{per}(Y)$ for problem (3.3.9). But the solution $\dot{u} \in \mathcal{W}_{per}(Y)$ implies that problem (3.3.8) has a solution in $H_{per}^1(Y)$ defined up to an additive constant. To make the solution unique, we remove this constant by choosing a representative element for the class of equivalence \dot{u} . In that light, we may propose that the solution has a zero mean value and problem (3.3.8) becomes;

$$\begin{aligned} -\operatorname{div}(A \nabla u) &= f \quad \text{in } Y, \\ u &\text{ } Y\text{-periodic,} \\ \mathcal{M}_Y(u) &= 0, \end{aligned} \tag{3.3.10}$$

with f still in $\mathcal{W}_{per}(Y)$ and the weak formulation is

$$\begin{cases} \text{Find } u \in W_{per}(Y) \text{ such that} \\ a_Y(u, v) = \int_Y A \nabla u \nabla v \, dy = \langle f, v \rangle_{(W_{per}(Y))', W_{per}(Y)} \\ \forall v \in W_{per}(Y), \end{cases} \tag{3.3.11}$$

where

$$W_{per}(Y) = \{u \in H_{per}^1(Y) : \mathcal{M}_Y(u) = 0\}.$$

Using the Poincaré inequality, $W_{per}(Y)$ is a Banach space with the norm given by

$$\|u\|_{W_{per}(Y)} = \|\nabla u\|_{L^2(Y)}, \quad \forall u \in W_{per}(Y),$$

and the pairing $\langle f, v \rangle_{(W_{per}(Y))', W_{per}(Y)}$ is well defined using proposition 3.56.

Theorem 3.67. *Suppose A is a matrix in $M(\alpha, \beta, Y)$ with Y -periodic coefficients and $f \in (W_{per}(Y))'$. Then there exists a unique solution for problem (3.3.11). Furthermore,*

$$\|u\|_{W_{per}(Y)} \leq \frac{1}{\alpha} \|f\|_{(W_{per}(Y))'} \tag{3.3.12}$$

Proof. First, we verify the conditions of the Lax-Milgram theorem i.e ellipticity and continuity of $a_Y(\cdot, \cdot)$ then we apply the theorem.

From the definition of A ,

$$a(u, u) = \int_Y A \nabla u \nabla u \, dy \geq \alpha \|\nabla u\|_{L^2(Y)}^2 = \alpha \|u\|_{W_{per}(Y)}^2.$$

This shows ellipticity.

Now for continuity; from the definition of A and the Cauchy-Schwarz inequality, one gets;

$$|a(u, v)| = \left| \int_Y A \nabla u \nabla v \, dy \right| \leq \beta \|\nabla u\|_{L^2(Y)} \|\nabla v\|_{L^2(Y)} = \beta \|u\|_{W_{per}(Y)} \|v\|_{W_{per}(Y)}.$$

Thus $a(\cdot, \cdot)$ is continuous.

So by the Lax-Milgram theorem, there exists a unique solution $u \in W_{per}(Y)$ for (3.3.10).

Lastly, for estimate (3.3.12), the ellipticity condition and the general Cauchy-Schwarz inequality gives

$$\begin{aligned} \alpha \|u\|_{W_{per}(Y)} \|v\|_{W_{per}(Y)} &\leq a(u, v) = \langle f, v \rangle_{(W_{per}(Y))', W_{per}(Y)} \\ &\leq \|f\|_{(W_{per}(Y))'} \|v\|_{W_{per}(Y)}. \end{aligned}$$

This implies that

$$\alpha \|u\|_{W_{per}(Y)} \leq \|f\|_{(W_{per}(Y))'}.$$

Hence

$$\|u\|_{W_{per}(Y)} \leq \frac{1}{\alpha} \|f\|_{(W_{per}(Y))'}.$$

□

Chapter 4

Method of Asymptotic Expansions

4.1 Introduction

The method of asymptotic expansion also known as the multiple-scale method is a method of homogenization widely used in physics, mechanics and mathematics. It was initially introduced by mechanical scientists and engineers for mechanical problems; see for instance [Bog61]. Later, it was used in the study of problems with periodic structures in [BLP78]. The basic idea behind this method is to postulate that the solution u^ϵ to the classical homogenization problem (2.1.1) is of the form

$$u^\epsilon = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \dots$$

where the terms in the expansion depends on both x and $y = \frac{x}{\epsilon}$, with x representing the macroscopic scale while $\frac{x}{\epsilon}$ represents the microscopic scale. Since the coefficients of the problem of which u^ϵ is the presumed solution, is Y -periodic, then one assumes that the terms in the expansion are also Y -periodic; i.e. each u_i is Y -periodic in the second variable $y = \frac{x}{\epsilon}$. Using this method, we obtain both the homogenized problem and the homogenized solution. However, the results are

heuristically obtained and the calculations involved in this method are long and cumbersome which makes it prone to errors. This makes it necessary to justify the results, like we do in Section 4.3; see for instance [Cio99], [Ole92].

Over the years, more mathematically rigorous methods have been introduced to obtain the limit problem. One of the first is Tartar's method of oscillating test functions [Tar77], this we shall see in Chapter 5. But the two-scale convergence by Nguetseng [Ngu89b] is so far one of the most powerful approaches and has been developed to go beyond periodic homogenization.

The object of the present section is to elaborate on the method of asymptotic expansions.

4.2 Derivation of the Homogenized problem and solution

Let us consider a linear second-order partial differential equation with the Dirichlet boundary condition: Let u^ϵ be the solution of the problem

$$\begin{aligned} \mathcal{A}_\epsilon u^\epsilon &= f \quad \text{in } \Omega, \\ u^\epsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.2.1}$$

where $f = f(x)$ is a smooth function in Ω independent of ϵ .

$$\mathcal{A}_\epsilon = -\operatorname{div}(A^\epsilon \nabla) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}^\epsilon(x) \frac{\partial}{\partial x_j} \right),$$

with $a_{ij}^\epsilon(x) = a_{ij}\left(\frac{x}{\epsilon}\right)$ a.e in \mathbb{R}^n , $\forall i, j = 1, \dots, n$.

The functions a_{ij} are assumed to be Y -Periodic, $\forall i, j = 1, \dots, n$. Y is known as the periodicity cell defined by $(0, l_i)^n$, where l_i ($i = 1, \dots, n$) are positive numbers, and $A = (a_{ij})_{1 \leq i, j \leq n} \in M(\alpha, \beta, Y)$ is such that for $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < \beta$, and

$$\begin{cases} i) & (A(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ ii) & |A(x)\lambda| \leq \beta|\lambda|, \end{cases}$$

for any $\lambda \in \mathbb{R}^n$ and a.e. in Y .

Suppose $u^\epsilon(x) = u(x, \frac{x}{\epsilon})$. By the chain rule, with $y = \frac{x}{\epsilon}$,

$$\frac{\partial u^\epsilon}{\partial x_i}(x) = \frac{\partial u}{\partial x_i}\left(x, \frac{x}{\epsilon}\right) = \frac{\partial u}{\partial x_i}\left(x, \frac{x}{\epsilon}\right) + \frac{1}{\epsilon} \frac{\partial u}{\partial y_i}\left(x, \frac{x}{\epsilon}\right).$$

So

$$\begin{aligned} \mathcal{A}_\epsilon u^\epsilon &= -\operatorname{div}(A^\epsilon \nabla u^\epsilon) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}^\epsilon(x) \frac{\partial u^\epsilon}{\partial x_j}(x) \right) \\ &= -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial u}{\partial x_j}\left(x, \frac{x}{\epsilon}\right) \right) \\ &= -\sum_{i,j=1}^n \left[\frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\epsilon}\right) \cdot \frac{\partial u}{\partial x_j}\left(x, \frac{x}{\epsilon}\right) + a_{ij}\left(\frac{x}{\epsilon}\right) \cdot \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_j}\left(x, \frac{x}{\epsilon}\right) \right]. \\ \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\epsilon}\right) &= \sum_{k=1}^n \frac{\partial}{\partial y_k} a_{ij}\left(\frac{x_k}{\epsilon}\right) \cdot \frac{\partial y_k}{\partial x_i} = \sum_{k=1}^n \frac{\partial}{\partial y_k} a_{ij}\left(\frac{x}{\epsilon}\right) \cdot \frac{\delta_{ik}}{\epsilon} = \frac{1}{\epsilon} \frac{\partial}{\partial y_i} a_{ij}(y). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{A}_\epsilon u^\epsilon &= -\sum_{i,j=1}^n \left[\frac{1}{\epsilon} \frac{\partial}{\partial y_i} a_{ij}(y) \left(\frac{\partial u^\epsilon}{\partial x_j}(x) + \frac{1}{\epsilon} \frac{\partial u^\epsilon}{\partial y_j}(x) \right) + a_{ij}(y) \frac{\partial}{\partial x_i} \left(\frac{\partial u^\epsilon}{\partial x_j}(x) + \frac{1}{\epsilon} \frac{\partial u^\epsilon}{\partial y_j}(x) \right) \right] \\ &= -\sum_{i,j=1}^n \left[\frac{1}{\epsilon} \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial u^\epsilon}{\partial x_j}(x) + \frac{1}{\epsilon^2} \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial u^\epsilon}{\partial y_j}(x) + \frac{\partial}{\partial x_i} a_{ij}(y) \frac{\partial u^\epsilon}{\partial x_j}(x) \right. \\ &\quad \left. + \frac{1}{\epsilon} \frac{\partial}{\partial x_i} a_{ij}(y) \frac{\partial u^\epsilon}{\partial y_j}(x) \right] \\ &= \left[\left(\frac{1}{\epsilon^2} \mathcal{A}_0 + \frac{1}{\epsilon} \mathcal{A}_1 + \mathcal{A}_2 \right) u \right] \left(x, \frac{x}{\epsilon} \right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_0 &= -\sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial}{\partial y_j} \right), \\ \mathcal{A}_1 &= -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial}{\partial y_j} \right) - \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial}{\partial x_j} \right), \\ \mathcal{A}_2 &= -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial}{\partial x_j} \right). \end{aligned} \tag{4.2.2}$$

Plugging the asymptotic expansion

$$u^\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \dots, \tag{4.2.3}$$

in Problem (4.2.1), one has

$$\mathcal{A}_\epsilon u^\epsilon = \left(\frac{1}{\epsilon^2} \mathcal{A}_0 + \frac{1}{\epsilon} \mathcal{A}_1 + \mathcal{A}_2 \right) \left(u_0 + \epsilon u_1 + \epsilon^2 u_2 \right) \left(x, \frac{x}{\epsilon} \right) = f \quad \text{in } \Omega.$$

Thus

$$\frac{1}{\epsilon^2} \mathcal{A}_0 u_0 + \frac{1}{\epsilon} \mathcal{A}_0 u_1 + \mathcal{A}_0 u_2 + \frac{1}{\epsilon} \mathcal{A}_1 u_0 + \mathcal{A}_1 u_1 + \epsilon \mathcal{A}_1 u_2 + \mathcal{A}_2 u_0 + \epsilon \mathcal{A}_2 u_1 + \epsilon^2 \mathcal{A}_2 u_2 = f.$$

Sorting and equating by the powers of the microscale parameter ϵ give

$$\begin{aligned} \mathcal{A}_0 u_0 &= 0, \\ \mathcal{A}_0 u_1 + \mathcal{A}_1 u_0 &= 0 \implies \mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0, \\ \mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0 &= f \implies \mathcal{A}_0 u_2 = f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0, \\ \mathcal{A}_1 u_2 + \mathcal{A}_2 u_1 &= 0 \implies \mathcal{A}_1 u_2 = -\mathcal{A}_2 u_1, \\ \mathcal{A}_2 u_2 &= 0. \end{aligned} \tag{4.2.4}$$

with the $u_i(x, y)$ Y -periodic in the second variable.

We notice that in (4.2.4), the three equations are given in terms of u_i ($i = 0, 1, 2$). This suggests that the problems can be solved in succession. The problems are similar to problem (3.3.8), so we either write its weak formulation in the form of (3.3.9), where its solution using Lax-Milgram theorem is a class of equivalence, or in the form (3.3.11), where its solution is a function with a zero mean value.

We start with the first problem,

$$\begin{aligned} \mathcal{A}_0 u_0 &= 0 \quad \text{in } Y \\ u_0 & \quad Y\text{-Periodic in } y. \end{aligned}$$

Its weak formulation is

$$\begin{cases} \text{Find } \dot{u}_0 \in \mathcal{W}_{per}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}_0, \dot{v}) = 0 \quad \forall \dot{v} \in \mathcal{W}_{per}(Y). \end{cases}$$

where

$$\dot{a}_Y(\dot{u}, \dot{v}) = \int_Y A \nabla u \nabla v \, dy, \quad \forall u \in \dot{u}, \forall v \in \dot{v}, \forall \dot{u}, \dot{v} \in \mathcal{W}_{per}(Y).$$

$\mathcal{W}_{per}(Y) = H^1_{per}(Y)/\mathbb{R}$ is the space of classes of equivalence. So \dot{u}_0 is a class of equivalence. Since

$$\dot{a}_Y(\dot{u}_0, \dot{v}) = \int_Y A \nabla u_0 \nabla v \, dy = 0, \quad \forall u_0 \in \dot{u}_0.$$

Then by the definition of A and the Lax-Milgram theorem, the unique solution is

$$\dot{u}_0 = \dot{0} \text{ in } \mathcal{W}_{per}(Y).$$

But by definition, $u_0 = u_0(x, y)$, this implies that the solutions are constant in y , so

$$u_0(x, y) = u_0(x) \quad \forall u_0 \in \dot{u}_0.$$

In the asymptotic expansion, we see u_0 as an oscillating function depending on the second variable $\frac{x}{\epsilon}$ but this computation shows that u_0 only depends on x . For this reason, we expect u_0 to be the homogenized solution (a function without oscillations). If we are able to find a problem which has u_0 as its solution, then we have found the homogenized problem.

Next,

$$\begin{aligned} \mathcal{A}_0 u_1 &= -\mathcal{A}_1 u_0 \quad \text{in } Y, \\ u_1 &\quad Y\text{-Periodic,} \end{aligned}$$

then from equation (4.2.2),

$$\begin{aligned} \mathcal{A}_1 u_0 &= - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial u_0}{\partial y_j}(x) \right) - \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial u_0}{\partial x_j}(x) \right) \\ &= - \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial y_i}(y) \frac{\partial u_0}{\partial x_j}(x). \end{aligned}$$

So

$$\mathcal{A}_0 u_1 = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial y_i}(y) \frac{\partial u_0}{\partial x_j}(x), \quad \text{with } u_1 \quad Y\text{-Periodic.} \quad (4.2.5)$$

and the weak formulation is

$$\begin{cases} \text{Find } \dot{u}_1 \in \mathcal{W}_{per}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}_1, \dot{\varphi}) = \langle F, \dot{\varphi} \rangle_{(\mathcal{W}_{per}(Y))', \mathcal{W}_{per}(Y)} \quad \forall \dot{\varphi} \in \mathcal{W}_{per}(Y), \end{cases} \quad (4.2.6)$$

where

$$\dot{a}_Y(\dot{u}_1, \dot{\varphi}) = \int_Y A \nabla u_1 \nabla \varphi \, dy, \quad \forall u_1 \in \dot{u}_1, \forall \varphi \in \dot{\varphi}, \quad \forall \dot{\varphi}, \forall \dot{u}_1 \in \mathcal{W}_{per}(Y),$$

and

$$\langle F, \dot{\varphi} \rangle_{(\mathcal{W}_{per}(Y))', \mathcal{W}_{per}(Y)} = \sum_{i,j=1}^n \frac{\partial u_0}{\partial x_j} \int_Y a_{ij}(y) \frac{\partial \varphi}{\partial y_i} \, dy \quad \forall \varphi \in \dot{\varphi}, \quad \forall \dot{\varphi} \in \mathcal{W}_{per}(Y).$$

By the definition of A and the Lax-Milgram theorem, the problem (4.2.6) has a unique solution $\dot{u}_1 \in \mathcal{W}_{per}(Y)$.

From equation (4.2.2), \mathcal{A}_0 involves only the y variable, so let us separate the variables. We choose a function $\dot{\chi}_j \in \mathcal{W}_{per}(Y)$ and propose that u_1 is of the form

$$-\sum_{j=1}^n \dot{\chi}_j(y) \frac{\partial u_0}{\partial x_j}(x) \quad \text{in } \mathcal{W}_{per}(Y).$$

So

$$\mathcal{A}_0 u_1 = \mathcal{A}_0 \left(-\sum_{j=1}^n \dot{\chi}_j(y) \frac{\partial u_0}{\partial x_j}(x) \right) = -\sum_{j=1}^n \mathcal{A}_0 \dot{\chi}_j(y) \frac{\partial u_0}{\partial x_j}(x).$$

By equation (4.2.5),

$$\mathcal{A}_0 u_1 = -\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial y_i}(y) \frac{\partial u_0}{\partial x_j}(x) = -\sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial a_{ij}}{\partial y_i}(y) \right) \frac{\partial u_0}{\partial x_j}(x).$$

Thus

$$\mathcal{A}_0 \dot{\chi}_j(y) = -\sum_{i=1}^n \frac{\partial a_{ij}}{\partial y_i}(y).$$

So $\dot{\chi}_j$ satisfies

$$\begin{cases} \mathcal{A}_0 \dot{\chi}_j = -\sum_{i=1}^n \frac{\partial a_{ij}}{\partial y_i} & \text{in } Y \\ \dot{\chi}_j & Y\text{-Periodic.} \end{cases}$$

Since $\dot{\chi}_j$ is an equivalence class, we can choose a representative $\chi_j \in \dot{\chi}_j$ so that we can have the weak formulation of the form (3.3.11). Then by the Lax-Milgram theorem, there exists a unique solution $\chi_j \in \mathcal{W}_{per}(Y)$ to the problem

$$\begin{cases} \text{Find } \chi_j \in \mathcal{W}_{per}(Y) \text{ such that} \\ a_Y(\chi_j, \varphi) = \sum_{i=1}^n \int_Y \frac{\partial a_{ij}}{\partial y_i} \varphi \, dy = \sum_{i=1}^n \int_Y a_{ij} \frac{\partial \varphi}{\partial y_i}, \, dy \\ \varphi \in \mathcal{W}_{per}(Y), \end{cases} \quad (4.2.7)$$

where

$$W_{per}(Y) = \{\varphi \in H_{per}^1(Y); \mathcal{M}_Y(\varphi) = 0\}.$$

Now if χ_j is a solution for (4.2.7), then $\chi_j + c_j$ is also a solution where c_j is a function of x alone so

$$u_1(x, y) = - \sum_{j=1}^n \chi_j(y) \frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x),$$

where \tilde{u}_1 denotes an arbitrary function of x alone, i.e.,

$$\tilde{u}_1(x) \in \dot{0} \quad \text{in } \mathcal{W}_{per}(Y).$$

We now deal with the problem

$$\begin{cases} \mathcal{A}_0 u_2 = f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 & \text{in } Y, \\ u_2 \quad Y\text{-periodic} & \text{in } y. \end{cases} \quad (4.2.8)$$

From equation (4.2.2), it follows that

$$f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 = f + \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial u_1}{\partial x_j} \right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(y) \left(\frac{\partial u_1}{\partial y_j} + \frac{\partial u_0}{\partial x_j} \right) \right),$$

thus the weak formulation of (4.2.8) is

$$\begin{cases} \text{Find } \hat{u}_2 \in \mathcal{W}_{per}(Y) \text{ such that} \\ \hat{a}_Y(\hat{u}_2, \hat{v}) = \langle M, \hat{v} \rangle_{(\mathcal{W}_{per}(Y))', \mathcal{W}_{per}(Y)} \quad \forall \hat{v} \in \mathcal{W}_{per}(Y), \end{cases}$$

where

$$\hat{a}_Y(\hat{u}, \hat{v}) = \int_Y A \nabla u \nabla v \, dy \quad \forall u \in \hat{u}, \forall v \in \hat{v}, \forall \hat{u}, \hat{v} \in \mathcal{W}_{per}(Y),$$

and

This problem will be well-posed (See definition 3.62 and proposition 3.56) if $M \in (\mathcal{W}_{per}(Y))'$, i.e. if

$$\langle M, 1 \rangle_{(H_{per}^1(Y))', H_{per}^1(Y)} = 0,$$

so

$$\int_Y f \, dy = \sum_{i,j=1}^n \int_Y \frac{\partial}{\partial x_i} \left(a_{ij}(y) \left(\frac{\partial u_1}{\partial y_i} + \frac{\partial u_0}{\partial x_j} \right) \right) dy.$$

For

$$u_1(x, y) = - \sum_{k=1}^n \chi_k(y) \frac{\partial u_0}{\partial x_k}(x),$$

$$\frac{\partial u_1}{\partial y_j} = - \frac{\partial}{\partial y_j} \left(\sum_{k=1}^n \chi_k(y) \frac{\partial u_0}{\partial x_k}(x) \right) = - \sum_{k=1}^n \frac{\partial \chi_k}{\partial y_j} \frac{\partial u_0}{\partial x_k}.$$

so

$$\begin{aligned} |Y|f &= - \sum_{i,j,k=1}^n \int_Y \frac{\partial}{\partial x_i} \left(a_{ij}(y) \left(- \frac{\partial \chi_k}{\partial y_j} \frac{\partial u_0}{\partial x_k} + \frac{\partial u_0}{\partial x_j} \right) \right) dy \\ &= - \sum_{i,j,k=1}^n \int_Y \frac{\partial}{\partial x_i} \left(- a_{ij}(y) \frac{\partial \chi_k}{\partial y_j} \frac{\partial u_0}{\partial x_k} + a_{ik}(y) \frac{\partial u_0}{\partial x_k} \right) dy \\ &= - \sum_{i,j,k=1}^n \int_Y - a_{ij}(y) \frac{\partial \chi_k}{\partial y_j} \frac{\partial^2 u_0}{\partial x_i \partial x_k} + a_{ik}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_k} dy \\ &= - \sum_{i,j,k=1}^n \int_Y \left(a_{ik}(y) - a_{ij}(y) \frac{\partial \chi_k}{\partial y_j} \right) \frac{\partial^2 u_0}{\partial x_i \partial x_k} dy. \end{aligned}$$

Thus

$$- \sum_{i,k=1}^n \left[\sum_{j=1}^n \int_Y \left(a_{ik} - a_{ij} \frac{\partial \chi_k}{\partial y_j} \right) dy \right] \frac{\partial^2 u_0}{\partial x_i \partial x_k} = |Y|f. \quad (4.2.9)$$

Let us introduce a_{ik}^0 and assume that

$$\sum_{j=i}^n \int_Y \left(a_{ik} - a_{ij} \frac{\partial \chi_k}{\partial y_j} \right) dy = \int_Y (a - a \nabla_y \chi_k) dy = \int_Y a_{ik}^0 dy = |Y|a_{ik}^0,$$

where we set $a - a \nabla_y \chi_k = a_{ik}^0$. From problem (4.2.7), χ_j is the solution to the following problem

$$\int_Y a \nabla \chi_k \nabla v \, dy = \int_Y a \nabla v \, dy = \int_Y a e_k \nabla v \, dy \quad \forall v \in W_{per}(Y). \quad (4.2.10)$$

This implies that

$$\int_Y a (\nabla \chi_k - e_k) \nabla v \, dy = 0. \quad (4.2.11)$$

Then equation (4.2.9) becomes

$$- \sum_{i,k=1}^n \frac{\partial}{\partial x_i} \left(a_{ik}^0 \frac{\partial u_0}{\partial x_k} \right) = f \quad (4.2.12)$$

where a_{ik}^0 is a constant matrix. Next we check that it is elliptic. For this we use the lemma and the proposition below

Lemma 4.1. [Pav08],[Cio99] *Let the matrix $A^0 = (a_{ik}^0)$ be defined as*

$$a_{ik}^0 = \sum_{j=1}^n a_{ij} - a_{ij} \frac{\partial \chi_k}{\partial y_j}.$$

Then

$$a_{ik}^0 = a_1(\chi_i - y_i, \chi_k - y_k) = \frac{1}{|Y|} a_Y(\chi_i - y_i, \chi_i - y_k).$$

($a_1(\phi, \varphi)$ is the bilinear form corresponding to \mathcal{A}_1)

Proof.

$$\begin{aligned} \frac{1}{|Y|} a_Y(\chi_i - y_i, \chi_k - y_k) &= \frac{1}{|Y|} \int_Y a \nabla(\chi_i - y_i) \nabla(\chi_k - Y_k) dy \\ &= \frac{1}{|Y|} \int_Y a(\nabla \chi_i - e_i)(\nabla \chi_k - e_k) dy \\ &= \frac{1}{|Y|} \int_Y a \nabla \chi_i \cdot \nabla \chi_k - a \nabla \chi_i \cdot e_k - a e_i \cdot \nabla \chi_k + a e_i \cdot e_k dy \\ &= \frac{1}{|Y|} \int_Y a \nabla \chi_i \cdot \nabla \chi_k - a e_i \cdot \nabla \chi_k - a \nabla \chi_i \cdot e_k + a e_i \cdot e_k dy \\ &= \frac{1}{|Y|} \int_Y a(\nabla \chi_i - e_i) \nabla \chi_k + a e_i \cdot e_k - a \nabla \chi_i \cdot e_k dy \end{aligned}$$

using equation (4.2.11), we deduce that

$$\begin{aligned} \frac{1}{|Y|} \int_Y a_Y(\chi_i - y_i, \chi_k - y_j) dy &= \int_Y \frac{1}{|Y|} a e_i \cdot e_k - a \nabla \chi_i \cdot e_k dy \\ &= \frac{1}{|Y|} \int_Y (a e_i - a \nabla \chi_i) e_k dy \\ &= a_{ik}^0. \end{aligned}$$

□

Proposition 4.2. [Cio99],[Pav08] *The matrix A^0 defined in the above lemma is positive definite and there exists $\alpha > 0$ such that $\langle A\xi, \xi \rangle \geq \alpha |\xi|^2$, $\forall \xi \in \mathbb{R}^n$.*

Proof. let $\xi \in \mathbb{R}^n$, $a_{ik}^0 = a_1(\chi_i - y_i, \chi_k - y_k)$. Then

$$\langle a_{ik}^0 \xi, \xi \rangle = a_1(w, w),$$

where $w = \xi(\chi_i - y_i)$. So by the ellipticity of a ,

$$a_1(w, w) \geq \frac{\alpha}{|Y|} \int_Y |\nabla w|^2 dy \geq 0$$

Suppose

$$\langle a_{ik}^0 \xi, \xi \rangle = 0, \quad \forall \xi \in \mathbb{R}^n, \quad \xi \neq 0.$$

Then $|\nabla w| = 0 \iff w = \xi(\chi_i - y_i) = c$, a constant. So

$$\xi \cdot y = \xi \chi - c.$$

But the right hand side is periodic which implies that the left hand side should be periodic as well. This is only possible if $\xi = 0$ which is a contradiction. Hence, the matrix is positive definite. \square

We have verified that a_{ik}^0 is elliptic, so by the Lax-Milgram theorem, there exists a unique solution $u_0 \in H_0^1(\Omega)$ to problem (4.2.12). The solution u_0 is the homogenized solution and problem (4.2.12) is the homogenized problem.

Observe that $u_i(x, y)$ can be determined successively. To obtain u_2 , we use the equations of u_1 and u_0 . So solving for u_2 as we did for u_1 , we get

$$u_2 = \sum_{k,l=1}^n \zeta^{k,l}(y) \frac{\partial^2 u_0}{\partial x_k \partial x_l} + \tilde{u}_2(x),$$

where $\tilde{u}_2(x) \in u_2$ is independent of y .

Consequently the expansion (4.2.3) takes the form

$$u^\epsilon(x) = u_0(x) - \epsilon \sum_{k=1}^n \chi_k \left(\frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_k}(x) + \epsilon^2 \sum_{k,l=1}^n \zeta^{k,l} \left(\frac{x}{\epsilon} \right) \frac{\partial^2 u_0}{\partial x_k \partial x_l}(x) + \dots$$

4.3 Error estimate

We have obtained the homogenized equation and the homogenized solution. But for the accuracy of the asymptotic expansion, it is important to estimate the

difference between the solution u^ϵ and the asymptotic expansion in appropriate norm. This gives the error estimate. The smaller the norm, the more accurate the expansion is. We prove the following estimate.

$$\left\| u^\epsilon(x) - \left(u_0(x) - \epsilon \sum_{k=1}^n \chi_k\left(\frac{x}{\epsilon}\right) + \epsilon^2 \sum_{k,l=1}^n \zeta^{kl}\left(\frac{x}{\epsilon}\right) \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}, \quad (4.3.1)$$

where C is a constant independent of ϵ .

Let

$$Z_\epsilon(x) = u^\epsilon(x) - (u_0 + \epsilon u_1 + \epsilon^2 u_2)\left(x, \frac{x}{\epsilon}\right),$$

with

$$\begin{aligned} u_0(x, y) &= u_0(x), \\ u_1(x, y) &= - \sum_{l=1}^n \chi_l(y) \frac{\partial u_0}{\partial x_l}, \\ u_2(x, y) &= \sum_{k,l=1}^n \zeta^{kl}(y) \frac{\partial^2 u_0}{\partial x_k \partial x_l}, \end{aligned}$$

$$\text{where } y = \frac{x}{\epsilon}.$$

Take

$$\mathcal{A}_\epsilon = -\text{div}(A^\epsilon \nabla) = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}^\epsilon \frac{\partial}{\partial x_j}).$$

Then

$$\begin{aligned} \mathcal{A}_\epsilon Z_\epsilon(x) &= \mathcal{A}_\epsilon u^\epsilon - \mathcal{A}_\epsilon (u_0 + \epsilon u_1 + \epsilon^2 u_2)\left(x, \frac{x}{\epsilon}\right) \\ &= [\mathcal{A}_\epsilon u^\epsilon - (\epsilon^{-2} \mathcal{A}_0 + \epsilon^{-1} \mathcal{A}_1 + \mathcal{A}_2)(u_0 + \epsilon u_1 + \epsilon^2 u_2)]\left(x, \frac{x}{\epsilon}\right) \\ &= \mathcal{A}_\epsilon u^\epsilon - \epsilon^{-2} \mathcal{A}_0 u_0 - \epsilon^{-1} (\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0) - (\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0) \\ &\quad - \epsilon (\mathcal{A}_1 u_2 + \mathcal{A}_2 u_1) - \epsilon^2 \mathcal{A}_2 u_2. \end{aligned}$$

Using relation (4.2.4), we get

$$\mathcal{A}_\epsilon Z_\epsilon(x) = \left(-\epsilon (\mathcal{A}_2 u_1 + \mathcal{A}_1 u_2) - \epsilon^2 \mathcal{A}_2 u_2 \right) \left(x, \frac{x}{\epsilon} \right).$$

From (4.2.2), we have

$$\begin{aligned}
 \mathcal{A}_1 u_2 &= - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial u_2}{\partial y_j} \right) - \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial u_2}{\partial x_j} \right) \\
 &= - \sum_{i,j}^n \frac{\partial}{\partial x_i} \left[a_{ij}(y) \frac{\partial}{\partial y_j} \left(\sum_{k,l=1}^n \zeta^{kl}(y) \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \right] \\
 &\quad - \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left[a_{ij}(y) \frac{\partial}{\partial x_j} \left(\sum_{k,l=1}^n \zeta^{kl}(y) \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \right] \\
 &= - \sum_{i,j,k,l=1}^n a_{ij}(y) \frac{\partial \zeta^{kl}}{\partial y_j}(y) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l}(x) \\
 &\quad - \sum_{i,j,k,l=1}^N \frac{\partial}{\partial y_i} \left(a_{ij}(y) \zeta^{k,l}(y) \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l}(x).
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_2 u_1 &= - \sum_{i,k=1}^n \left(a_{ik}(y) \frac{\partial u_1}{\partial x_k} \right) = \sum_{i,k=1}^n \left(a_{ik}(y) \frac{\partial}{\partial x_k} \left(\sum_{l=1}^n \chi_l(y) \frac{\partial u_0}{\partial x_l} \right) \right) \\
 &= \sum_{i,k=1}^n \left(a_{ik}(y) \sum_{l=1}^n \chi_l(y) \frac{\partial^2 u_0}{\partial x_k \partial x_l}(x) \right) \\
 &= \sum_{i,k,l=1}^n a_{ik}(y) \chi_l \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l}(x).
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_2 u_2 &= - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial u_2}{\partial x_j} \right) \\
 &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial}{\partial x_j} \left(\sum_{k,l=1}^n \zeta^{kl}(y) \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \right) \\
 &= - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(y) \left(\sum_{k,l=1}^n \zeta^{kl}(y) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l}(x) \right) \right) \\
 &= - \sum_{i,j,k,l=1}^n a_{ij}(y) \zeta^{kl}(y) \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l}(x).
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_1 u_2 &= - \sum_{i,j,k,l=1}^n a_{ij} \left(\frac{x}{\epsilon} \right) \frac{\partial \zeta^{kl}}{\partial y_j} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l}(x) \\
 &\quad - \sum_{i,j,k,l=1}^n \frac{\partial}{\partial y_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l}(x).
 \end{aligned}$$

Since $\frac{\partial}{\partial x_i} \left(\frac{x}{\epsilon} \right) = \frac{1}{\epsilon} \frac{\partial}{\partial y_i} \left(\frac{x}{\epsilon} \right)$, $\frac{\partial}{\partial y_i} = \epsilon \frac{\partial}{\partial x_i} \left(\frac{x}{\epsilon} \right)$. Then

$$\begin{aligned}
 \sum_{i,j,k,l=1}^n \frac{\partial}{\partial y_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) &= \epsilon \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) \\
 &= \epsilon \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) \right) \\
 &\quad - \epsilon \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l} (x) \\
 &= \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) \right) \\
 &\quad + \epsilon \mathcal{A}_2 u_2.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \mathcal{A}_\epsilon Z_\epsilon &= -\epsilon \sum_{i,j,k,l=1}^n a_{ik} \left(\frac{x}{\epsilon} \right) \chi_l \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l} (x) + \epsilon \sum_{i,j,k,l=1}^n a_{ij} \left(\frac{x}{\epsilon} \right) \frac{\partial \zeta^{kl}}{\partial y_j} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l} (x) \\
 &\quad + \epsilon^2 \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) \right) + \epsilon^2 \mathcal{A}_2 u_2 \left(\frac{x}{\epsilon} \right) - \epsilon^2 \mathcal{A}_2 u_2 \left(\frac{x}{\epsilon} \right) \\
 &= -\epsilon \sum_{i,j,k,l=1}^n a_{ik} \left(\frac{x}{\epsilon} \right) \chi_l \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l} (x) + \epsilon \sum_{i,j,k,l=1}^n a_{ij} \left(\frac{x}{\epsilon} \right) \frac{\partial \zeta^{kl}}{\partial y_j} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l} (x) \\
 &\quad + \epsilon^2 \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) \right).
 \end{aligned}$$

Given that u^ϵ and u_0 vanish on $\partial\Omega$, then $Z_\epsilon = (\epsilon u_1 - \epsilon^2 u_2)(x, \frac{x}{\epsilon})$ on $\partial\Omega$ and Z_ϵ satisfies

$$\begin{cases} \mathcal{A}_\epsilon Z_\epsilon = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}^\epsilon \frac{\partial Z_\epsilon}{\partial x_j} \right) = \epsilon F^\epsilon & \text{in } \Omega, \\ Z_\epsilon = \epsilon G^\epsilon & \text{on } \partial\Omega. \end{cases}$$

where

$$\begin{aligned}
 F^\epsilon &= \sum_{i,j,k,l=1}^n \left[-a_{ik} \left(\frac{x}{\epsilon} \right) \chi_l \left(\frac{x}{\epsilon} \right) + a_{ij} \left(\frac{x}{\epsilon} \right) \frac{\partial \zeta^{kl}}{\partial y_j} \left(\frac{x}{\epsilon} \right) \right] \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l} (x) \\
 &\quad + \epsilon \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_i} \left[a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_l} (x) \right], \\
 G^\epsilon &= \sum_{k=1}^n \chi_k \left(\frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_k} (x) - \epsilon \sum_{k,l=1}^n \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial^2 u_0}{\partial x_k \partial x_l} (x).
 \end{aligned}$$

The above problem is a non-homogeneous Dirichlet problem.

To be able to use Theorem 3.66, we check that $F^\epsilon \in H^{-1}(\Omega)$ (see proposition 3.45), i.e., that there exists $n + 1$ functions in $L^2(\Omega)$ such that

$$F^\epsilon = F_0^\epsilon + \epsilon \sum_{i=1}^n \frac{\partial F_i^\epsilon}{\partial x_i}$$

and $G^\epsilon \in H^{\frac{1}{2}}(\partial\Omega)$. Then estimate (3.3.7) can be used and we have

$$\|Z_\epsilon\|_{H^1(\Omega)} \leq C_1 \epsilon \|F^\epsilon\|_{H^{-1}(\Omega)} + C_2 \epsilon \|G^\epsilon\|_{H^{\frac{1}{2}}(\Omega)}, \quad (4.3.2)$$

where

$$F_0^\epsilon = \sum_{i,j,k,l=1}^n \left[-a_{ik} \left(\frac{x}{\epsilon} \right) \chi_l \left(\frac{x}{\epsilon} \right) + a_{ij} \left(\frac{x}{\epsilon} \right) \frac{\partial \zeta^{kl}}{\partial y_j} \left(\frac{x}{\epsilon} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_l} (x) \right].$$

Let us estimate the components of F^ϵ and G^ϵ . Since $A \in L^\infty(Y)$, $f \in H^{-1}(\Omega)$ and u_0 is the solution of an elliptic equation with constant coefficient, the regularity theory of second order elliptic equations ensures that the derivatives of u_0 belong to $L^\infty(\Omega)$; see [Gil01]. Furthermore, $\chi_l, \zeta^{kl} \in \mathcal{W}_{per}(Y) \subset H^1_{per}(Y) \subset H^1(Y)$ by definition. Then

$$\|F_0^\epsilon\|_{L^2(\Omega)} \leq \|\partial^3 u_0\|_{L^\infty(\Omega)} \left\| \sum_{i,j,k,l=1}^n \left[-a_{ik} \left(\frac{\cdot}{\epsilon} \right) \chi_l \left(\frac{\cdot}{\epsilon} \right) + a_{ij} \left(\frac{\cdot}{\epsilon} \right) \frac{\partial \zeta^{kl}}{\partial y_j} \left(\frac{\cdot}{\epsilon} \right) \right] \right\|_{L^2(\Omega)} \leq C,$$

with C independent of ϵ .

Also

$$F_i^\epsilon = \sum_{j,k,l=1}^n a_{ij} \left(\frac{x}{\epsilon} \right) \zeta^{kl} \left(\frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_j \partial x_k \partial x_l} (x),$$

with

$$\|F_i^\epsilon\|_{L^2(\Omega)} \leq \|\partial^3 u_0\|_{L^\infty(\Omega)} \left\| \sum_{j,k,l=1}^n a_{ij} \left(\frac{\cdot}{\epsilon} \right) \zeta^{kl} \left(\frac{\cdot}{\epsilon} \right) \right\|_{L^2(\Omega)} \leq C,$$

where C is independent of ϵ .

Combining these facts we deduce that F^ϵ belongs to $H^{-1}(\Omega)$ and

$$\|F^\epsilon\|_{H^{-1}(\Omega)}^2 = \inf \sum_{i=0}^n \|F_i^\epsilon\|_{L^2(\Omega)}^2 \leq c_1 \quad (4.3.3)$$

with the constant c_1 independent of ϵ .

For G^ϵ , let show that

$$\|G^\epsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq c_2\epsilon^{-\frac{1}{2}}, \quad (4.3.4)$$

following an argument from [Ole92](Chapter 2, proof of Theorem 1.2).

Let us define a function κ_ϵ as follows, let κ_ϵ be such that $0 \leq \kappa_\epsilon \leq 1$,

$$\begin{aligned} \kappa_\epsilon(x) &= 1 \text{ if } \rho(x, \partial\Omega) \leq \epsilon, \\ \kappa_\epsilon(x) &= 0 \text{ if } \rho(x, \partial\Omega) \geq 2\epsilon, \\ \|\kappa_\epsilon\|_{L^\infty(\Omega)} &\leq c_2\epsilon^{-1}. \end{aligned}$$

Set

$$\Psi_\epsilon = \kappa_\epsilon G^\epsilon.$$

Then by definition of κ_ϵ ,

$$\text{supp } \Psi_\epsilon = \{x : \rho(x, \partial\Omega) \leq 2\epsilon\}, \text{ which we will denote by } U_\epsilon.$$

Now, we show that $\Psi_\epsilon \in H^1(\Omega)$ and $\|\Psi_\epsilon\|_{H^1(U_\epsilon)} \leq c_3\epsilon^{-\frac{1}{2}}$ with c_3 independent of ϵ .

Using the H^1 -norm, we have

$$\|\Psi_\epsilon\|_{H^1(U_\epsilon)} = \|\Psi_\epsilon\|_{L^2(U_\epsilon)} + \|\nabla\Psi_\epsilon\|_{L^2(U_\epsilon)}.$$

From the definition of κ_ϵ and the regularity on u_0 ,

$$\begin{aligned} \|\Psi_\epsilon\|_{L^2(U_\epsilon)} &= \|\kappa_\epsilon G^\epsilon\|_{L^2(U_\epsilon)} \leq c_4, \\ \|\nabla\Psi_\epsilon\|_{L^2(U_\epsilon)} &= \|\nabla(\kappa_\epsilon G^\epsilon)\|_{L^2(U_\epsilon)} \leq \epsilon^{-1}c_5\|u_0\|_{H^1(U_\epsilon)} + c_6, \end{aligned} \quad (4.3.5)$$

with c_5, c_6 independent of ϵ .

At this stage we use the following lemma.

Lemma 4.3. [Ole92] *Suppose $\partial\Omega$ is Lipschitz continuous and $B_\delta = \{x \in \Omega, \rho(x, \partial\Omega) < \delta\}$, $\delta > 0$. Then there is a $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and every $v \in H^1(\Omega)$ one has*

$$\|v\|_{L^2(B_\delta)} \leq c\delta^{\frac{1}{2}}\|v\|_{H^1(\Omega)},$$

where c is a constant independent of δ and v .

We then get

$$\|u_0\|_{H^1(U_\epsilon)} \leq c_7 \epsilon^{\frac{1}{2}} \|u_0\|_{H^1(\Omega)}.$$

Therefore by (4.3.5) we have;

$$\begin{aligned} \|\Psi_\epsilon\|_{H^1(U_\epsilon)} &\leq c_8 + \epsilon^{-1} c_5 \|u_0\|_{H^1(U_\epsilon)} \\ &\leq c_8 + \epsilon^{-1} c_5 (c_7 \epsilon^{\frac{1}{2}} \|u_0\|_{H^1(\Omega)}) \\ &\leq c_9 \epsilon^{-\frac{1}{2}} \end{aligned}$$

On $\partial\Omega$, $\Psi_\epsilon = G^\epsilon$, so one has

$$\|G^\epsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|\Psi_\epsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_\gamma(\Omega) \|\Psi_\epsilon\|_{H^1(\Omega)} = C_\gamma(\Omega) \|\Psi_\epsilon\|_{H^1(U_\epsilon)} \leq c_9 \epsilon^{-\frac{1}{2}}.$$

This proves (4.3.4).

Using the estimate (4.3.2), we conclude that

$$\|Z_\epsilon\|_{H^1(\Omega)} \leq C_1 \epsilon \|F^\epsilon\|_{H^1(\Omega)} + C_2 \epsilon \|G^\epsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \epsilon C_1 c_1 + \epsilon^{\frac{1}{2}} C_2 c_2 \leq c \epsilon^{\frac{1}{2}},$$

which is the claimed estimate.

The method of asymptotic expansion is very powerful but formal. It can be used without prior knowledge about specific properties of the solution to the micro structured problem. Thus, it is used only to 'guess' the form of the homogenized equation. Although we can get the desired result, the derivation does not contain a strict proof. Some methods have been developed to rigorously provide a proof of the convergence. A more general and powerful approach is Tartar's method of oscillating test functions developed by Luc Tartar [Tar77]. This we shall discuss in the next chapter for a linear periodic homogenization problem. Another drawback is the rigorous justification of the asymptotic expansion of a solution to a problem, this is usually very difficult. Despite the shortcomings, the method of asymptotic expansions remains commonly used in mathematics literature e.g. [Roh10] and engineering literature e.g. [Mar11]. Furthermore, the multiscale expansion method can also handle problems involving more than two scales.

Chapter 5

Tartar's method of Oscillating Test functions

When looking for solutions to asymptotic problems, one may encounter the product of two converging sequences. If this product consists of a strongly convergent sequence and a weakly convergent sequence, then Proposition (3.12) can be used to find the limit. But the case where this product consists of two weakly convergent sequences, passing to the limit is rather difficult. Tartar's method of oscillating test functions enables one to find the limit by canceling out any products of two weakly converging sequences.

We consider the same problem as in Chapter 4, i.e.,

$$\begin{cases} -\operatorname{div}(A^\epsilon \nabla u^\epsilon) &= f & \text{in } \Omega, \\ u^\epsilon &= 0 & \text{on } \partial\Omega. \end{cases} \quad (5.0.1)$$

Under the assumption that the condition on the data still hold here, by Theorem 3.65, there exists a unique weak solution u^ϵ belonging to $H_0^1(\Omega)$ for a fixed ϵ , with $f \in H^{-1}(\Omega)$ such that

$$\int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \quad (5.0.2)$$

and the estimate

$$\|u^\epsilon\|_{H_0^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}, \quad (5.0.3)$$

holds.

Using Theorem 3.11, there is a weakly converging subsequence of $\{u^\epsilon\}$ which we still denote by $\{u^\epsilon\}$ and an element $u^0 \in H_0^1(\Omega)$ such that

$$u^\epsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega). \quad (5.0.4)$$

By Sobolev embedding theorem,

$$u^\epsilon \rightarrow u^0 \quad \text{strongly in } L^2(\Omega). \quad (5.0.5)$$

Let us introduce the vector function

$$\xi^\epsilon = (\xi_1^\epsilon, \dots, \xi_n^\epsilon) = \left(\sum_{j=1}^n a_{1j}^\epsilon \frac{\partial u^\epsilon}{\partial x_j}, \dots, \sum_{j=1}^n a_{nj}^\epsilon \frac{\partial u^\epsilon}{\partial x_j} \right) = A^\epsilon \nabla u^\epsilon.$$

Then (5.0.2) implies that

$$\int_{\Omega} \xi^\epsilon \nabla v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (5.0.6)$$

Since $A \in M(\alpha, \beta, \Omega)$, we have

$$\begin{aligned} \int_{\Omega} \xi^\epsilon \nabla v \, dx &= \int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla v \, dx \leq \beta \|\nabla u^\epsilon\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &= \beta \|u^\epsilon\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

But

$$\|u^\epsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)},$$

so

$$\|\xi^\epsilon\|_{L^2(\Omega)} \leq \frac{\beta}{\alpha} \|f\|_{H^{-1}(\Omega)}.$$

Thus (ξ^ϵ) is a uniformly bounded sequence in $(L^2(\Omega))^n$.

Again by Theorem 3.11, there exists a subsequence of $\{\xi^\epsilon\}$ which we still denote by $\{\xi^\epsilon\}$ and $\xi^0 \in L^2(\Omega)$ such that

$$\xi^\epsilon \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^n. \quad (5.0.7)$$

Hence passing to the limit in (5.0.6) gives

$$\int_{\Omega} \xi^0 \nabla v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad v \in H_0^1(\Omega). \quad (5.0.8)$$

And this is a weak formulation of the equation

$$-\operatorname{div} \xi^0 = f \quad \text{in } \Omega. \quad (5.0.9)$$

The main goal of this chapter is to identify ξ^0 .

5.1 The convergence theorem

The results in the previous chapter are based on the multiple scale expansions which is heuristic in essence. Here we consider the following convergence theorem which we prove using Tartar's method of oscillating test functions. We show the proof for when the operator \mathcal{A} is symmetric.

Theorem 5.1. *Let u^ϵ be the weak solution of problem (5.0.1), with $f \in L^2(\Omega)$ and $A^\epsilon \in M(\alpha, \beta, \Omega)$ is Y -periodic.*

Then

- $u^\epsilon \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$,
- $A^\epsilon \nabla u^\epsilon \rightharpoonup A^0 \nabla u^0$ weakly in $(L^2(\Omega))^n$.

Furthermore, $u^0 \in H_0^1(\Omega)$ is the weak solution to the homogenized problem:

$$\begin{aligned} -\operatorname{div}(A^0 \nabla u^0) &= f \quad \text{in } \Omega, \\ u^0 &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (5.1.1)$$

and

$$A^0 = (a_{ij}^0)_{1 \leq i, j \leq n} = \frac{1}{|Y|} \int_Y a_{ij}(y) \, dy - \frac{1}{|Y|} \sum_{k=1}^n \int_Y a_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \, dy, \quad (5.1.2)$$

where χ_j is the weak solution to the cell problem:

$$\begin{aligned} -\operatorname{div}(A(y)\nabla\chi_j) &= -\operatorname{div}(A(y)e_j) \quad \text{in } Y, \\ \chi_j & \text{ } Y\text{-periodic.} \end{aligned} \tag{5.1.3}$$

Remark. The identification of ξ^0 in (5.0.9) is provided by equation (5.1.1); $\xi^0 = A^0\nabla u^0$.

5.2 Proof by Tartar's method of oscillating test functions

From the computation above,

$$A^\epsilon\nabla u^\epsilon = \xi^\epsilon \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^n.$$

The aim here is to show that ξ^0 in (5.0.7) is equal to $A^0\nabla u^0$. Hence the claim of Theorem 5.1 will be proved. From (5.0.4), we have that $u^\epsilon \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$. This implies that ∇u^ϵ is bounded in $L^2(\Omega)^n$, which further implies that up to a subsequence, $\nabla u^\epsilon \rightharpoonup \nabla u^0$ weakly in $(L^2(\Omega))^n$. If A^ϵ converges strongly to A^0 , then we can pass to the limit using Proposition 3.12. But dealing with composite materials, one cannot have a strong convergence of the matrix A^ϵ .

From the membership of A^ϵ to $M(\alpha, \beta, \Omega)$, one has weakly* convergence of A^ϵ to A^0 in $L^\infty(\Omega)^{n \times n}$, which implies weak convergence in $L^2(\Omega)^{n \times n}$ to A^0 .

That leaves us to finding the limit of the product of two weakly convergent sequences $A^\epsilon\nabla u^\epsilon$. As mentioned earlier, this is not straightforward and generally, the product of two weakly convergences does not converge to the product of their limit, hence we employ the method of oscillating test functions introduced by Luc Tartar [Tar77].

Proof. This method involves periodizing the solution of a cell problem. So we

consider the cell problem in the convergence theorem, i.e.,

$$\begin{aligned} -\operatorname{div} (A(y)\nabla\chi_j) &= -\operatorname{div} (A(y)e_j) \quad \text{in } Y, \\ \chi_j & \text{ } Y\text{-Periodic.} \end{aligned} \tag{5.2.1}$$

Remark. Strictly speaking, we get this cell problem from the asymptotic expansions method.

Its corresponding weak formulation reads

$$\begin{aligned} \text{Find } \chi_j \in W_{per}(Y) \text{ such that} \\ \int_Y A(y)\nabla_y\chi_j\nabla v \, dy = \int_Y A(y) e_j\nabla v \, dy, \quad \forall v \in W_{per}(Y). \end{aligned}$$

We shall need the following results.

Lemma 5.2. [Cio99] *Let u be a function in $H^1_{per}(Y)$. Then its extension by periodicity belongs to $H^1(\omega)$ for any bounded open subset ω of \mathbb{R}^n .*

Lemma 5.3. [Cio99] *Let $A \in M(\alpha, \beta, Y)$ and $h \in (L^2(Y))^n$. Suppose $u \in W_{per}(Y)$ is the solution to the following problem:*

$$\int_Y A\nabla u\nabla v \, dy = \int_Y h\nabla v \, dy, \quad \forall v \in W_{per}(Y).$$

Then the extension of u denoted by u^\sharp is the unique solution to the following problem:

$$\begin{aligned} -\operatorname{div} (A\nabla u^\sharp) &= -\operatorname{div} h \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \\ u^\sharp & \text{ } Y\text{-periodic,} \\ \mathcal{M}_Y(u^\sharp) &= 0. \end{aligned}$$

If one extends the solution χ_j of problem (5.2.1) by periodicity to \mathbb{R}^n and still denote it by χ_j , then by Lemma 5.3, this extension is the unique solution to the following problem:

$$\begin{aligned} -\operatorname{div} (A\nabla\chi_j) &= -\operatorname{div} (A(y) e_j) \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \\ \chi_j & \text{ } Y\text{-periodic,} \\ \mathcal{M}_Y(\chi_j) &= 0, \end{aligned}$$

where

$$\int_{\mathbb{R}^n} A\left(\frac{x}{\epsilon}\right) \nabla_y \chi_j\left(\frac{x}{\epsilon}\right) \nabla v(x) dx = \int_{\mathbb{R}^n} A\left(\frac{x}{\epsilon}\right) e_j \nabla v(x) dx, \quad \forall v \in \mathcal{D}(\mathbb{R}^n).$$

This implies that

$$\int_{\mathbb{R}^n} A\left(\frac{x}{\epsilon}\right) \left(e_j - \nabla_y \chi_j\left(\frac{x}{\epsilon}\right) \right) \nabla v(x) dx = 0. \quad (5.2.2)$$

Let

$$w_j(x) = x_j - \chi_j\left(\frac{x}{\epsilon}\right), \quad (5.2.3)$$

and set

$$w_j^\epsilon(x) = \epsilon w_j\left(\frac{x}{\epsilon}\right) = x_j - \epsilon \chi_j\left(\frac{x}{\epsilon}\right), \quad \text{for } j = 1, \dots, n. \quad (5.2.4)$$

Then from equations (5.2.2) and (5.2.4),

$$\int_{\mathbb{R}^n} A^\epsilon(x) \nabla w_j^\epsilon(x) \nabla v(x) dx = 0, \quad \forall v \in \mathcal{D}(\mathbb{R}^n),$$

where $A\left(\frac{x}{\epsilon}\right) = A^\epsilon(x)$.

By definition of $H_0^1(\Omega)$, i.e., $H_0^1(\Omega) = \overline{D(\Omega)}$, with respect to the H^1 -norm, one has

$$\int_{\Omega} A^\epsilon(x) \nabla w_j^\epsilon(x) \nabla v(x) dx = 0, \quad \forall v \in H_0^1(\Omega). \quad (5.2.5)$$

Since $\chi_j\left(\frac{x}{\epsilon}\right)$ is Y -periodic, then by Theorem 3.50,

$$w_j^\epsilon(x) \rightharpoonup \mathcal{M}_Y(w_j) \quad \text{weakly in } L^2(\Omega),$$

and by (5.2.3),

$$\mathcal{M}_Y(w_j) = \mathcal{M}_Y(x_j - \chi_j) = x_j - \mathcal{M}_Y(\chi_j),$$

but $\mathcal{M}_Y(\chi_j) = 0$, so

$$w_j^\epsilon \rightharpoonup x_j \quad \text{weakly in } L^2(\Omega),$$

and

$$(\nabla_x w_j^\epsilon)(x) = (\nabla_x w_j)\left(\frac{x}{\epsilon}\right) = \epsilon \left(\frac{1}{\epsilon} \nabla_y w_j \right)\left(\frac{x}{\epsilon}\right) = (\nabla_y w_j)\left(\frac{x}{\epsilon}\right).$$

Moreover, $\nabla_y w_j$ is Y -periodic so by Theorem 3.50,

$$\nabla_x w_j^\epsilon \rightharpoonup \mathcal{M}_Y(\nabla_y w_j) \quad \text{weakly in } L^2(\Omega),$$

where

$$\mathcal{M}_Y(\nabla_y w_j) = \mathcal{M}(e_j - \nabla_y \chi_j) = e_j - \mathcal{M}_Y(\nabla_y \chi_j). \quad (5.2.6)$$

By Green's formula,

$$\mathcal{M}_Y(\nabla_y \chi_j) = \frac{1}{|Y|} \left(\int_Y \nabla_y \chi_j(y) dy \right) = \frac{1}{|Y|} \left(\int_Y \chi_j(y) \nabla_y 1 dy + \int_{\partial Y} \chi_j(y) \cdot n ds_y \right) = 0.$$

Thus (5.2.6) implies that

$$\nabla_x w_j^\epsilon \rightharpoonup e_j \quad \text{weakly in } (L^2(\Omega))^n. \quad (5.2.7)$$

Consequently,

$$w_j^\epsilon \rightharpoonup x_j \quad \text{weakly in } (H^1(\Omega))^n, \quad (5.2.8)$$

and by Sobolev embedding theorem,

$$w_j^\epsilon \rightarrow x_j \quad \text{strongly in } (L^2(\Omega))^n. \quad (5.2.9)$$

For $\varphi \in D(\mathbb{R}^n)$, let us choose $v = \varphi w_j^\epsilon$ in equation (5.0.2) and $v = \varphi u^\epsilon$ in equation (5.2.5) to get

$$\begin{aligned} \int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla(\varphi w_j^\epsilon) dx &= \int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla \varphi w_j^\epsilon dx + \int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla w_j^\epsilon \varphi dx \\ &= \langle f, \varphi w_j^\epsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned} \quad (5.2.10)$$

and

$$\begin{aligned} \int_{\Omega} A^\epsilon \nabla w_j^\epsilon \nabla(\varphi u^\epsilon) dx &= \int_{\Omega} A^\epsilon \nabla w_j^\epsilon \nabla \varphi u^\epsilon dx + \int_{\Omega} A^\epsilon \nabla w_j^\epsilon \nabla u^\epsilon \varphi dx \\ &= 0. \end{aligned} \quad (5.2.11)$$

By the symmetry of A ,

$$\int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla w_j^\epsilon \varphi dx = \int_{\Omega} A^\epsilon \nabla w_j^\epsilon \nabla u^\epsilon \varphi dx.$$

So subtracting equation (5.2.11) from equation (5.2.10) gives

$$\int_{\Omega} A^\epsilon \nabla u^\epsilon \nabla \varphi w_j^\epsilon dx - \int_{\Omega} A^\epsilon \nabla w_j^\epsilon \nabla \varphi u^\epsilon dx = \langle f, \varphi w_j^\epsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (5.2.12)$$

Now we pass to the limit as $\epsilon \rightarrow 0$.

For the first term in equation (5.2.12), equations (5.0.7) and (5.2.9) give

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} A^{\epsilon} \nabla u^{\epsilon} \nabla \varphi w_j^{\epsilon} dx = \int_{\Omega} \xi^0 \nabla \varphi x_j dx. \quad (5.2.13)$$

For the second term,

$$\begin{aligned} (A^{\epsilon}(x) \nabla w_j^{\epsilon}(x))_k &= \sum_{i=1}^n a_{ik} \left(\frac{x}{\epsilon}\right) \frac{\partial w_j^{\epsilon}}{\partial x_i}(x) \\ &= \sum_{i=1}^n a_{ik} \left(\frac{x}{\epsilon}\right) \frac{\partial}{\partial x_i} (x_j - \epsilon \chi_j \left(\frac{x}{\epsilon}\right)) \\ &= \sum_{i=1}^n a_{ik} \left(\frac{x}{\epsilon}\right) \left(\delta_{ij} - \frac{\partial}{\partial y_i} \chi_j(y)\right), \quad y = \frac{x}{\epsilon} \\ &= a_{jk} - \sum_{i=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_i}. \end{aligned}$$

Thus we have the following convergence in $(L^2(\Omega))^n$;

$$\begin{aligned} (A^{\epsilon}(x) \nabla w_j^{\epsilon}(x))_k &\rightharpoonup \mathcal{M}_Y(a_{jk}) - \mathcal{M}_Y \left(\sum_{i=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_i} \right) \\ &= \frac{1}{|Y|} \int_Y a_{jk}(y) dy - \frac{1}{|Y|} \sum_{i=1}^n \int_Y a_{ik}(y) \frac{\partial \chi_j}{\partial y_i}(y) dy \\ &= A_{jk}^0. \end{aligned}$$

Now using equation (5.0.5),

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} A^{\epsilon} \nabla w_j^{\epsilon} \nabla \varphi u^{\epsilon} dx = \int_{\Omega} A^0 \nabla \varphi u^0 dx. \quad (5.2.14)$$

Lastly we deal with the right hand side of (5.2.12)

From equation (5.2.8), we have,

$$\lim_{\epsilon \rightarrow 0} \langle f, \varphi w_j^{\epsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle f, \varphi(x_j) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Having obtained the limits of all the terms in equation (5.2.12), we finally get

$$\int_{\Omega} \xi^0 \nabla \varphi x_j dx - \int_{\Omega} A^0 \nabla \varphi u^0 dx = \langle f, \varphi(x_j) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

This can be rewritten as

$$\begin{aligned} \int_{\Omega} \xi^0 \nabla(\varphi x_j) dx - \int_{\Omega} \xi^0 e_j \varphi dx - \int_{\Omega} A^0 \nabla \varphi u^0 dx \\ = \langle f, \varphi(x_j) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega). \end{aligned} \quad (5.2.15)$$

But equation (5.0.8), i.e.,

$$\int_{\Omega} \xi^0 \nabla v dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

implies that

$$\int_{\Omega} \xi^0 \nabla(\varphi x_j) dx = \langle f, \varphi x_j \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \varphi \in \mathcal{D}(\Omega).$$

Hence, it follows from (5.2.15) that

$$\int_{\Omega} \xi^0 e_j \varphi dx = - \int_{\Omega} A^0 \nabla \varphi u^0 dx.$$

But

$$- \int_{\Omega} A^0 \nabla \varphi u^0 dx = \int_{\Omega} A^0 \nabla u^0 \varphi dx.$$

So

$$\int_{\Omega} (\xi^0 e_j - A^0 \nabla u^0) \varphi dx = 0.$$

Hence we conclude that

$$\xi^0 = A^0 \nabla u^0.$$

□

Remark. The symmetric structure of the operator \mathcal{A} was crucial in ensuring the cancellation of troubling terms. That restriction may be done away with. Indeed tartar's method works for non-symmetric operators. The adjoint operator then plays a crucial role in the derivation of the homogenized problem.

5.3 Correctors

As can be seen in Theorem 5.1, the convergence of u^ϵ to u^0 is weak in $H_0^1(\Omega)$. This means that

$$\nabla u^\epsilon - \nabla u^0 \rightharpoonup 0 \text{ weakly in } (L^2(\Omega))^n$$

In general, the above convergence cannot be improved. However, the term ∇u^0 can be adjusted with the introduction of a corrector matrix to get a strong convergence. In this section, we address that issue.

Let us denote by C^ϵ , the corrector matrix whose entries are defined by

$$\begin{aligned} C_{ij}^\epsilon(x) &= C_{ij}\left(\frac{x}{\epsilon}\right) \text{ a.e. on } \Omega \\ C_{ij}(y) &= \frac{\partial w_j}{\partial y_i}(y) = \delta_{ij} - \frac{\partial \chi_j}{\partial y_i}(y) \text{ a.e. on } Y. \end{aligned} \quad (5.3.1)$$

where w_j and χ_j are given by (5.1.3) and (5.2.3). We have the following

Proposition 5.4. *Let C^ϵ be defined by 5.3.1 and A^0 be given as in Theorem 5.1.*

Then

$$\begin{cases} C^\epsilon \rightharpoonup I \text{ weakly in } (L^2(\Omega))^n \\ A^\epsilon C^\epsilon \rightharpoonup A^0 \text{ weakly in } (L^2(\Omega))^n. \end{cases}$$

where I is the unit $n \times n$ matrix.

Proof. See [Cio99]. □

Based on the above proposition and the convergence:

$$\nabla u^\epsilon - \nabla u^0 \rightharpoonup 0 \text{ weakly in } (L^2(\Omega))^n,$$

we have that

$$\nabla u^\epsilon - C^\epsilon \nabla u^0 \rightharpoonup 0 \text{ weakly in } (L^1(\Omega))^n. \quad (5.3.2)$$

Indeed, since $C^\epsilon \nabla u^0 \in L^1(\Omega)$, one has

$$\int_{\Omega} C^\epsilon \nabla u^0 \varphi \, dx \longrightarrow \int_{\Omega} \nabla u^0 \varphi \, dx, \quad \forall \varphi \in L^\infty(\Omega).$$

As stated earlier, the introduction of the corrector matrix is to obtain a strong convergence in (5.3.2).

Theorem 5.5. *Let u^ϵ, u^0 and A^0 be given by Theorem 5.1. Then*

$$\nabla u^\epsilon - C^\epsilon \nabla u^0 \longrightarrow 0 \text{ strongly in } (L^1(\Omega))^n.$$

To prove this theorem, we need the following proposition.

Proposition 5.6. *Let u^ϵ, u^0 and A^0 be given by Theorem 5.1. Then there is a constant C , independent of ϵ , such that*

$$\limsup_{\epsilon \rightarrow 0} \|\nabla u^\epsilon - C^\epsilon \phi\|_{L^2(\Omega)} \leq C \|\nabla u^0 - \phi\|_{L^2(\Omega)} \quad \forall \phi \in (D(\Omega))^n$$

Proof of Theorem 5.5. Recall that $D(\Omega)$ is dense in $L^2(\Omega)$, so there exists $\phi_\delta \in (D(\Omega))^n$ and an arbitrary $\delta > 0$ such that

$$\|\nabla u^0 - \phi_\delta\|_{L^2(\Omega)} \leq \delta. \quad (5.3.3)$$

By triangular inequality, Proposition 5.6 and (5.3.3), we have

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \|\nabla u^\epsilon - C^\epsilon \nabla u^0\|_{L^1(\Omega)} \\ & \leq \limsup_{\epsilon \rightarrow 0} [\|\nabla u^\epsilon - C^\epsilon \phi_\delta\|_{L^1(\Omega)} + \|C^\epsilon \phi_\delta - C^\epsilon \nabla u^0\|_{L^1(\Omega)}] \\ & \leq \limsup_{\epsilon \rightarrow 0} c_1 \|\nabla u^\epsilon - C^\epsilon \phi_\delta\|_{L^2(\Omega)} + c_2 \|\nabla u^0 - \phi_\delta\|_{L^2(\Omega)} \\ & \leq c \|\nabla u^0 - \phi_\delta\|_{L^2(\Omega)} \\ & \leq c_3 \delta. \end{aligned}$$

□

Example 5.7. Let us give the corrector matrix for the one-dimensional case studied in [Spa67].

Let (d_1, d_2) be an interval in \mathbb{R}^n . Consider the following problem;

$$\begin{aligned} -\frac{d}{dx} \left(a^\epsilon \frac{du^\epsilon}{dx} \right) &= f \quad \text{in } (d_1, d_2), \\ u^\epsilon(d_1) &= u^\epsilon(d_2) = 0. \end{aligned} \quad (5.3.4)$$

Suppose a is a positive function in $L^\infty(\Omega)$ such that

$$\begin{aligned} a & \text{ is } l_1\text{-periodic,} \\ 0 < \alpha & \leq a(x) \leq \beta < +\infty, \end{aligned} \quad (5.3.5)$$

where α, β are constants and

$$a^\epsilon(x) = a\left(\frac{x}{\epsilon}\right). \quad (5.3.6)$$

Then the following result holds.

Theorem 5.8. [Cio99] Suppose $f \in L^2(d_1, d_2)$ and a^ϵ is defined by equations (5.3.5) and (5.3.6). Let $u^\epsilon \in H_0^1(d_1, d_2)$ be the solution of problem (5.3.4). Then

$$u^\epsilon \rightharpoonup u^0 \text{ weakly in } H_0^1(d_1, d_2),$$

where u^0 is the unique solution of the following problem

$$\begin{aligned} -\frac{d}{dx} \left(\frac{1}{\mathcal{M}_{(0,l_1)}\left(\frac{1}{a}\right)} \frac{du^0}{dx} \right) &= f, \\ u^0(d_1) &= u^0(d_2) = 0. \end{aligned}$$

Proposition 5.9. [Cio99] With the same assumptions as Theorem 5.8,

$$\frac{1}{\mathcal{M}_{(0,l_1)}\left(\frac{1}{a}\right)} = \mathcal{M}_{(0,l_1)} \left(a - a \frac{d\chi}{dy} \right),$$

where χ is the weak solution of the following problem

$$\begin{cases} -\frac{d}{dy} \left(a(y) \frac{d\chi}{dy} \right) = -\frac{d}{dy} (a(y)) & \text{in } (0, l_1), \\ \chi & l_1\text{-periodic}, \\ \mathcal{M}_{(0,l_1)}(\chi) &= 0, \end{cases}$$

and is given by

$$\chi(y) = -\frac{1}{\mathcal{M}_{(0,l_1)}\left(\frac{1}{a}\right)} \int_0^y \frac{1}{a(t)} dt + y + c$$

where c is the constant for which $\mathcal{M}_{(0,l_1)}(\chi) = 0$.

Using (5.3.1), the corrector for this one-dimensional case is given by

$$C(y) = \frac{dw}{dy} = \frac{d}{dy} (y - \chi(y)),$$

where using proposition 5.9 and Theorem 5.8 give

$$C(y) = \frac{1}{\mathcal{M}_{(0,l_1)}\left(\frac{1}{a}\right)} \frac{1}{a(y)} = \frac{a_0(y)}{a(y)}.$$

Remark. The corrector $C^\epsilon \nabla u^0$ appears also in the asymptotic expansion of u^ϵ .

Indeed using the expansion in Chapter 4, we see that

$$\nabla u^\epsilon(x) = \nabla u_0(x) - \sum_{k=1}^N \nabla_y \chi_k \left(\frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_k}(x) - \epsilon \sum_{k=1}^N \chi_k \left(\frac{x}{\epsilon} \right) \nabla \left(\frac{\partial u_0}{\partial x_k} \right)(x) + \dots$$

From the definition of C^ϵ ,

$$\nabla u^\epsilon(x) = C^\epsilon(x) \nabla u_0(x) - \epsilon \sum_{k=1}^N \chi_k \left(\frac{x}{\epsilon} \right) \nabla \left(\frac{\partial u_0}{\partial x_k} \right) (x) + \dots$$

which makes $C^\epsilon(x) \nabla u_0(x)$, the first term in the asymptotic expansion of ∇u^ϵ .

Even though we considered an elliptic problem with periodic coefficient, Tartar's method can be applied to parabolic homogenization problems. Tartar's method of oscillating test functions may be used for nonlinear monotone problems. But the process of constructing test functions to cancel out problems is not applicable. This created the need for another tool. To this end, the theory of compensated compactness was introduced by L. Tartar [Tar79] and F. Murat [Mur78]. This we shall discuss in the next chapter.

Chapter 6

Homogenization of Nonlinear Partial Differential Equations

We discussed in previous chapters the method of asymptotic expansions and Tartar's method of oscillating test functions for linear homogenization problems; the asymptotic expansions method is a heuristic method and Tartar's method was one of the first mathematically rigorous methods that placed the asymptotic expansion method on firm theoretical grounds. Due to the nature of the problems, constructing test functions to cancel out troubling terms in nonlinear homogenization problems may lead to insurmountable challenges. In view of the prevalence of nonlinear partial differential equations in the modeling of most natural processes, it became imperative to develop new tools in Homogenization.

In the 1970s, L. Tartar and F. Murat introduced the theory of compensated Compactness to handle nonlinear homogenization problems involving the product of two weakly converging fields. Here, certain conditions on the derivatives of the weakly converging fields compensate for the lack of strong convergence. Due to the restrictions on the compensated compactness method, another mathematical tool was introduced in the 1980s by L. Tartar under the name H-measures. But

this tool was also introduced independently by P. Gérard under the name Microlocal defect measures.

Following the works of L. Tartar and F. Murat on nonlinear homogenization, we discuss in Section 6.1, the compensated compactness results and in Section 6.2, the H-measures.

6.1 The Compensated Compactness Theory

In this Section, we briefly discuss the development of the compensated compactness theory.

6.1.1 Div-Curl Lemma

As shown in Section 3.2.3, the product of two weakly convergent sequences does not generally converge to the product of their limits. For elliptic equations involving scalar-valued solutions, we saw in Chapter 5 how Tartar's method of oscillating test functions can be used to pass to the limit of such a product. For vector-valued functions, the lemma below known as the div-curl lemma introduced by F. Murat [Mur79], [Mur78] and L. Tartar [Tar79] enables one to pass to the limit of the product of two weakly convergent sequences of vector fields provided that the sequences satisfy certain conditions.

Definition 6.1. Given a vector $w \in (L^2(\Omega))^n$ such that $w = (w_1, \dots, w_n) = (w_i : i = 1, \dots, n)$. The matrix $(\text{curl } w)_{ij}$ is defined by

$$(\text{curl } w)_{ij} = \frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i} \quad \text{for } i, j = 1, \dots, n.$$

Theorem 6.2 (Div-Curl Lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\{u^k\}$ and $\{w^k\}$ be vector-valued bounded sequences on Ω such that $\{u^k\} \rightharpoonup u$ weakly in $(L^2(\Omega))^n$ and $\{w^k\} \rightharpoonup w$ weakly in $(L^2(\Omega))^n$. If*

$$H(1) \quad \text{div } u^k \text{ lies in a compact subset of } H^{-1}(\Omega),$$

$$H(2) \quad (\text{curl } w^k)_{ij} \text{ lies in a compact subset of } H^{-1}(\Omega)^{n \times n},$$

then

$$u^k \cdot w^k \rightharpoonup u \cdot w \quad \text{in the sense of distributions}$$

i.e

$$\int_{\Omega} (u^k \cdot w^k) \varphi \, dx \mapsto \int_{\Omega} (u \cdot w) \varphi \, dx, \quad \forall \varphi \in D(\Omega),$$

where

$$u \cdot w = \sum_{i=1}^n u_i w_i, \quad \text{for } u = (u_1, \dots, u_n), \quad w = (w_1, \dots, w_n)$$

Proof. (We show the proof following Evans [Eva90])

Let us consider the vector field $v^k \in (H^2(\Omega))^n$ weakly solving

$$\begin{aligned} -\Delta v^k &= w^k & \text{in } \Omega, \\ v^k &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{6.1.1}$$

$\{w^k\}$ is bounded in $(L^2(\Omega))^n$ implies that $\{v^k\}$ is bounded in $(H^2(\Omega))^n$. This follows from Theorem 3.65, by differentiating with respect to x_i in (6.1.1) then $\frac{\partial w^k}{\partial x_i} \in (H^{-1}(\Omega))^n$ and subsequently $\frac{\partial v^k}{\partial x_i} \in (H^1(\Omega))^n$ thus, $v^k \in (H^2(\Omega))^n$.

Set

$$\begin{aligned} z^k &= -\text{div } v^k, \\ y^k &= w^k - \nabla z^k. \end{aligned}$$

Then

z^k is bounded in $H^1(\Omega)$. Furthermore, if $1 \leq i \leq n$,

$$\begin{aligned} y_i^k &= w_i^k - \nabla v^k \\ &= w_i^k - z_{ix_i}^k \\ &= -v_{lx_j x_j}^k + v_{jx_i x_j}^k \\ &= (v_{jx_i}^k - z_{lx_j}^k)_{x_j}. \end{aligned} \tag{6.1.2}$$

From H(2) and (6.1.1), $(\text{curl } v^k)$ lies in a compact subset of $(H^{-1}(\Omega))^{n \times n}$. Thus from (6.1.2), it follows that y^k is bounded in $(L^2(\Omega))^n$.

Using Theorem 3.11, there is a subsequence of $\{z^k\}$ which we will still denote by $\{z^k\}$ and $z \in H^1(\Omega)$ such that

$$z^k \rightharpoonup z \text{ weakly in } H^1(\Omega). \quad (6.1.3)$$

Also, there is a subsequence of y^k which we still denote by y^k and $y \in (L^2(\Omega))^n$ such that

$$y^k \rightarrow y \text{ strongly in } (L^2(\Omega))^n. \quad (6.1.4)$$

Here, we take $z = \text{div } v$ and $y = w - \nabla z$, where $u \in (H^2(\Omega))^n$ is the solution of

$$\begin{aligned} -\Delta v &= w & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Now take $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} u^k \cdot w^k \varphi \, dx &= \int_{\Omega} u^k \cdot (y^k + \nabla z^k) \varphi \, dx \\ &= \int_{\Omega} u^k \cdot y^k \varphi \, dx + \int_{\Omega} u^k \cdot \nabla z^k \varphi \, dx. \end{aligned}$$

From relation (6.1.4),

$$\int_{\Omega} u^k \cdot y^k \varphi \, dx \longrightarrow \int_{\Omega} u \cdot y \varphi \, dx,$$

and from relation (6.1.3) and H(1), we have

$$\begin{aligned} \int_{\Omega} u^k \cdot \nabla z^k \varphi \, dx &= \int_{\Omega} u^k \cdot \nabla(z^k \varphi) \, dx - \int_{\Omega} u^k \cdot z^k \nabla \varphi \, dx \\ &= -\langle \text{div } u^k, z^k \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\Omega} u^k \cdot z^k \nabla \varphi \, dx \\ &\longrightarrow -\langle \text{div } u, z \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\Omega} u \cdot z \nabla \varphi \, dx \\ &= \int_{\Omega} u \cdot \nabla(z \varphi) \, dx - \int_{\Omega} u \cdot z \nabla \varphi \, dx \\ &= \int_{\Omega} u \cdot \nabla z \varphi \, dx. \end{aligned}$$

This implies that

$$\int_{\Omega} u^k \cdot (y^k + \nabla z^k) \varphi \, dx \longrightarrow \int_{\Omega} u \cdot (y + \nabla z) \varphi \, dx = \int_{\Omega} (u \cdot w) \varphi \, dx.$$

And this proves the Lemma. □

As an illustration of the application of the div-curl lemma, we state a prototype of the div-curl lemma and apply it to an example based on Maxwell's equations.

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\{u^k\}$ and $\{v^k\}$ be vector-valued sequences in Ω such that $\{u^k\} \rightharpoonup u$ weakly in $(L^2(\Omega))^n$ and $\{v^k\} \rightharpoonup w$ weakly in $(L^2(\Omega))^n$. If $\operatorname{div} u^k$ is bounded in $L^2(\Omega)$ and $\operatorname{curl} v^k$ is bounded in $(L^2(\Omega))^n$, then*

$$u^k \cdot v^k \rightharpoonup u \cdot v \quad \text{in the sense of distributions.}$$

Example 6.4. Let $\Omega \subset \mathbb{R}^n$ be occupied by a non-homogeneous body, with electric field E and electrostatic potential u such that

$$E = -\nabla u. \tag{6.1.5}$$

Suppose

$$u(x) = u_0(x) + \epsilon u_1\left(\frac{x}{\epsilon}\right),$$

where u_0 and u_1 are smooth functions such that u_1 is periodic with period Y and ϵ is a small parameter representing the microscale.

Let $\frac{x}{\epsilon} = y$. Then

$$E = -\nabla u = -\nabla u_0(x) - \nabla_y u_1\left(\frac{x}{\epsilon}\right).$$

Now if $u_1 \in L^2(\Omega)$, then by applying Theorem 3.50, one has

$$u_1\left(\frac{x}{\epsilon}\right) \rightharpoonup \mathcal{M}_\Omega(u_1) \text{ (a constant) weakly in } L^2(\Omega) \text{ as } \epsilon \rightarrow 0.$$

This implies that

$$\nabla_y u_1\left(\frac{x}{\epsilon}\right) \rightharpoonup 0 \text{ weakly in } L^2(\Omega).$$

So u can be weakly approximated by $u_0 \in L^2(\Omega)$.

On the other hand, if $E \in L^2(\Omega)$, then from relation (6.1.5), one has

$$-\nabla u \in L^2(\Omega),$$

which implies that

$$u \in H^1(\Omega).$$

The constitutive equations in electrodynamics are given by the Maxwell's equations

$$E = -\nabla u, \quad D = aE, \quad \operatorname{div} D = \rho, \quad \operatorname{curl} E = -\frac{\partial B}{\partial t},$$

where a is a constant (permittivity of free state), D is the electric induction field, ρ is the charge density, and B is the magnetic field.

Assume that measurements can be taken in terms of convergences, i.e. for ρ, E, D and u ,

$$\rho_\epsilon \rightharpoonup \rho_0 \text{ weakly in } L^2(\Omega),$$

$$E_\epsilon \rightharpoonup E_0 \text{ weakly in } L^2(\Omega),$$

$$D_\epsilon \rightharpoonup D_0 \text{ weakly in } L^2(\Omega),$$

$$u_\epsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega).$$

We have the electrostatic density e given by $E \cdot D$. Now the question we try to answer is;

Does $e_\epsilon = E_\epsilon \cdot D_\epsilon$ converge to $e_0 = E_0 \cdot D_0$ in any sense?

First let us assume that the magnetic field of the non homogeneous body is constant and the medium is such that the perturbation of the Maxwell's equations give

$$\begin{cases} \operatorname{curl} E_\epsilon = F_\epsilon, \\ \operatorname{div} D_\epsilon = \rho_\epsilon, \end{cases}$$

where F_ϵ is bounded in $L^2(\Omega)$.

We can see that the conditions for the div-curl lemma is satisfied i.e the curl $E_\epsilon = F_\epsilon$ is bounded in $L^2(\Omega)$ and $\operatorname{div} D_\epsilon = \rho_\epsilon$ is bounded in $L^2(\Omega)$.

Using the lemma, we have

$$E_\epsilon \cdot D_\epsilon \rightharpoonup E_0 \cdot D_0 \text{ in the sense of distributions,}$$

which gives the question a positive answer.

There are different variants of the div-curl lemma that can be applied to various problems (see for instance [Bal10], [Daf05] and [Chr05]). In the study of elliptic equations with divergent forms, the div-curl lemma has proven successful,

since the requirements/conditions on the derivatives of the weakly converging fields "compensate" for the lack of strong convergence. However, the div-curl lemma cannot be used on any quadratic product, because of its specific requirements on the derivatives of the weakly converging sequences.

Suppose one encounters the composition of a real valued function F defined on \mathbb{R}^n , and a sequence $\{u_n\}$ which converges weakly to u . One would want to know if there is a relation between the limit of $F(u_n)$ and u . In general, $F(u_n)$ does not converge to $F(u)$ except if F is an affine function. A powerful tool to overcome this challenge is provided by the theory of Young measures described in the next subsection.

6.1.2 Parametrized Measures (Young Measures)

Young measures were developed by L.C Young [You37]. They were initially used for treating problems of calculus of variations, until L. Tartar [Tar79] developed it as a tool for the analysis of nonlinear partial differential equations. Young measures can be used to compute the weak limit of any function of weakly converging fields. Additional information on Young measures can be found in [Bal89], [Ped00], [Gia98], just to cite a few.

Definition 6.5. Let K be a bounded open set in \mathbb{R}^n and let $u : \Omega \mapsto \mathbb{R}^n$ be a measurable function such that $u(x) \in K$ a.e. We define a measure μ on $\Omega \times \mathbb{R}^n$ by

$$\langle \mu, \phi(x, \lambda) \rangle = \int_{\Omega} \phi(x, u(x)) dx.$$

for all continuous function ϕ with compact support contained in $\Omega \times \mathbb{R}^n$. μ is known as the Radon measure or the generalized measure associated to u .

Proposition 6.6. *The Radon measure μ has the following properties.*

- (i) $\mu \geq 0$ i.e. $\langle \mu, \phi \rangle \geq 0$ if $\phi \geq 0$.
- (ii) $\text{supp } \mu \subset \overline{\text{graph } u}$ i.e. if $\phi = 0$ on $\text{graph } u$, then $\langle \mu, \phi \rangle = 0$.
- (iii) The projection on Ω of $\mu = \text{proj}_\Omega \mu = dx$,
 i.e. if $\phi(x, \lambda) = \psi(x)$ then $\langle \mu, \phi \rangle = \int_\Omega \psi(x) dx$.

Theorem 6.7. [Tar79],[Tar95] *Let K be a bounded set in \mathbb{R}^m and Ω , a bounded open set in \mathbb{R}^n . Let $u_j : \Omega \mapsto \mathbb{R}^m$ be a sequence such that $u_j(x) \in K$ a.e.. Then there exists a subsequence $\{u_{j_k}\}$ and a family of probability measures $\{\nu_x\}_{x \in \Omega}$ (i.e., $\nu_x \geq 0$, $\nu_x(\mathbb{R}^n) = 1$) with $\text{supp } \nu_x \subset \overline{K}$, such that for F , a continuous function on \mathbb{R}^n ,*

$$F(u_{j_k}) \xrightarrow{*} \bar{f} \text{ weakly* in } L^\infty(\Omega), \text{ as } k \rightarrow \infty,$$

where

$$\bar{f}(x) = \langle \nu_x, F(\lambda) \rangle = \int_{\mathbb{R}^m} \nu_x(\lambda) F(\lambda) d\lambda \text{ a.e. .}$$

The family $\{\nu_x\}_{x \in \Omega}$ is called the Young measure associated to the subsequence $\{u_{j_k}\}$.

Proof. Let us associate to u_j the measure μ_j in the following way

$$\langle \mu_j, \phi(x, \lambda) \rangle = \int_\Omega \phi(x, u_j(x)) dx, \quad \forall \phi \in C(\Omega \times \mathbb{R}^m).$$

Since Ω is bounded, one may extract a subsequence μ_{j_k} and a nonnegative measure μ such that

$$\mu_{j_k} \xrightarrow{*} \mu \text{ weakly*}, \text{ i.e. } \langle \mu_{j_k}, \phi \rangle \longrightarrow \langle \mu, \phi \rangle, \quad \forall \phi \in C(\Omega \times \mathbb{R}^m).$$

then we check the properties of μ ,

- (i) For all $\phi \geq 0$, $\langle \mu, \phi \rangle = \lim_{j \rightarrow \infty} \langle \mu_j, \phi \rangle \geq 0$. So $\mu = \int_\Omega \phi(x, u_j(x)) dx \geq 0$.
- (ii) If $\phi = 0$ on $\Omega \times \overline{K}$, then $\langle \mu, \phi \rangle = 0$ which implies that $\text{supp } \mu \subset \Omega \times \overline{K}$.
- (iii) Suppose $\phi(x, \lambda) = \psi(x)$. Then $\langle \mu, \phi \rangle = \lim_{j \rightarrow \infty} \langle \mu_j, \phi \rangle$
 $= \lim_{j \rightarrow \infty} \int_\Omega \psi(x) dx = \int_\Omega \psi(x) dx$. Hence, $\text{Proj}_\Omega \mu = dx$

These properties imply that μ is absolutely continuous with respect to the Lebesgue measure dx , so by the Radon-Nikodym theorem, there exists a family of probability measures $\{\nu_x\}$ such that

$$\mu = \int_{\Omega} \nu_x dx, \quad \text{i.e. } \langle \mu, \phi(x, \lambda) \rangle = \int_{\Omega} \langle \nu_x, \phi(x, \lambda) \rangle dx.$$

Now suppose $F(u_{j_k}) \rightharpoonup \bar{f}$. Then for all $\psi \in C(\Omega)$, one has

$$\langle \mu_{j_k}, \psi(x)F(\lambda) \rangle \longrightarrow \langle \mu, \psi(x)F(\lambda) \rangle = \int_{\Omega} \langle \nu_x, F(\lambda) \rangle \psi(x) dx$$

But

$$\langle \mu_{j_k}, \psi(x)F(\lambda) \rangle = \int_{\Omega} \psi(x)F(u_{j_k}(x)) dx = \int_{\Omega} \psi(x)\bar{f}(x) dx$$

Therefore,

$$\bar{f}(x) = \langle \nu_x, F(\lambda) \rangle = \int_{\mathbb{R}^m} \nu_x(\lambda)F(\lambda) d\lambda.$$

This proves the theorem. □

Example 6.8. Let $u_k(x)$ be a sequence defined by

$$u_k(x) = \sin(kx), \quad x \in [0, 1]$$

According to Theorem 6.7, the associated Young measure ν_x is such that, for any function $F \in C([-1, 1])$,

$$F(u_k) \xrightarrow{*} \int_{-1}^1 F(y) d\nu_x(y) \quad \text{in } L^\infty([0, 1]), \quad (6.1.6)$$

Since $\sin(x)$ is a periodic function with period 2π , $F(\sin(x))$ is also periodic with period 2π . Thus by Theorem 3.50, we have

$$F(u_k) \xrightarrow{*} \bar{F}, \quad \text{in } L^\infty([0, 1]), \quad (6.1.7)$$

where

$$\bar{F} = \frac{1}{2\pi} \int_0^{2\pi} F(\sin(z)) dz. \quad (6.1.8)$$

We need to express \bar{F} as the right hand side of (6.1.6). The idea is to use an appropriate substitution of the type $y = \sin(z)$. However, $\sin(z)$ is not invertible

on $[0, 2\pi]$, but on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Fortunately, by splitting the integral in (6.1.8) suitably, we can succeed. We have

$$\int_0^{2\pi} F(\sin(z)) dz = \int_0^{\frac{\pi}{2}} F(\sin(z)) dz + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} F(\sin(z)) dz + \int_{\frac{3\pi}{2}}^{2\pi} F(\sin(z)) dz. \quad (6.1.9)$$

Using the substitution $y = \sin(z)$ for $z \in [0, \frac{\pi}{2}]$, we get $z = \arcsin(y)$. Thus since

$$dz = \frac{dy}{\sqrt{1-y^2}} \quad \text{and} \quad y \in [0, 1],$$

we have that

$$\int_0^{\frac{\pi}{2}} F(\sin(z)) dz = \int_0^1 F(y) \frac{dy}{\sqrt{1-y^2}}. \quad (6.1.10)$$

For the second integral in the right hand side of (6.1.9), we first use the translation $z \rightarrow z - \pi$. Then

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} F(\sin(z)) dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\sin(z + \pi)) dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(-\sin(z)) dz \quad (6.1.11)$$

Next using the substitution $y = -\sin(z)$, we have

$$\begin{aligned} z &= -\arcsin(y), & dz &= -\frac{dy}{\sqrt{1-y^2}}, \\ -\sin(-\frac{\pi}{2}) &= 1, & -\sin(\frac{\pi}{2}) &= -1. \end{aligned}$$

Thus

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(-\sin(z)) dz = \int_1^{-1} F(y) \frac{-dy}{\sqrt{1-y^2}} = \int_{-1}^1 F(y) \frac{dy}{\sqrt{1-y^2}}.$$

Hence it follows from (6.1.11) that

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} F(\sin(z)) dz = \int_{-1}^1 F(y) \frac{dy}{\sqrt{1-y^2}}. \quad (6.1.12)$$

For the last term in the right hand side of (6.1.9), we again use a translation $z \rightarrow z - 2\pi$ which gives

$$\int_{\frac{3\pi}{2}}^{2\pi} F(\sin(z)) dz = \int_{-\frac{\pi}{2}}^0 F(\sin(z + 2\pi)) dz = \int_{-\frac{\pi}{2}}^0 F(\sin(z)) dz.$$

Setting $y = \sin(z)$ and noting that $\sin(-\frac{\pi}{2}) = -1$, $\sin(0) = 0$, we get

$$\int_{-\frac{\pi}{2}}^0 F(\sin(z)) dz = \int_{-1}^0 F(y) \frac{dy}{\sqrt{1-y^2}}.$$

Thus

$$\int_{\frac{3\pi}{2}}^{2\pi} F(\sin(z)) dz = \int_{-1}^0 F(y) \frac{dy}{\sqrt{1-y^2}} \quad (6.1.13)$$

Combining (6.1.8), (6.1.9), (6.1.10), (6.1.12) and (6.1.13), we get

$$\begin{aligned} \bar{F} &= \frac{1}{2\pi} \left[\int_{-1}^1 \frac{F(y) dy}{\sqrt{1-y^2}} + \int_0^1 \frac{F(y) dy}{\sqrt{1-y^2}} + \int_{-1}^0 \frac{F(y) dy}{\sqrt{1-y^2}} \right] \\ &= \int_{-1}^1 F(y) \frac{dy}{\pi\sqrt{1-y^2}}. \end{aligned}$$

This shows that the measure ν_x in (6.1.6) is equal to the measure ν given by

$$\nu(dy) = \frac{dy}{\pi\sqrt{1-y^2}},$$

That is

$$\nu(I) = \frac{1}{\pi} \int_I \frac{dy}{\sqrt{1-y^2}}$$

for any measurable set $I \subset [0, 1]$.

Parametrized measures or Young measures provide the weak* limit of sequences after extracting a subsequence, but only for relations that may be non-linear but pointwise. Unfortunately, they are not suitable for relations with differential structures as they are unable to capture the differential structure of the equation, except for certain classes of conservation laws; see for instance [Tar79], [Eva90] and [DiP83].

6.1.3 Compensated Compactness

The inability of Young Measures to handle convergence problems involving differential structures motivated the need for the development of another mathematical tool. This led to the theory of compensated compactness introduced by L. Tartar and F. Murat as an extension of the div-curl lemma. But it is only suitable for problems involving linear balance equations which are partial differential equations with constant coefficients.

Definition 6.9. The set Υ is defined by

$$\Upsilon = \{(\lambda, \xi) : \lambda \in \mathbb{R}^m, \xi \in \mathbb{R}^n \text{ such that } \sum_{j=1}^m \sum_{k=1}^n A_{ijk} \lambda_j \xi_k = 0 \text{ for } i = 1, \dots, q\}.$$

Definition 6.10. The set Λ is defined by

$$\Lambda = \{(\lambda \in \mathbb{R}^m \text{ such that there exists } \xi \in \mathbb{R}^n \setminus 0, (\lambda, \xi) \in \Upsilon\}. \quad (6.1.14)$$

Theorem 6.11 (Quadratic Theorem). *Let Ω be a bounded open set in \mathbb{R}^n and let Q be a real quadratic form on \mathbb{R}^m such that*

$$Q(\lambda) \geq 0 \quad \forall \lambda \in \Lambda. \quad (6.1.15)$$

Let $\{u^\epsilon\}$ be a sequence such that

A(1) $u^\epsilon \rightharpoonup u$ weakly in $L^2(\Omega)$,

A(2) $\sum_{j=1}^m \sum_{k=1}^n A_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k}$ is compact in the strong topology of $H_{loc}^{-1}(\Omega)$ for $i = 1, \dots, q$.

If $Q(u^\epsilon) \rightharpoonup \mu$ in the sense of distributions (μ may be a measure), then

$$\mu \geq Q(u) \quad \text{in the sense of measures .}$$

Proof. See [Tar79]. □

Corollary. *Suppose Q is a real quadratic form on \mathbb{R}^m such that*

$$Q(\lambda) = 0 \quad \forall \lambda \in \Lambda.$$

If u^ϵ is a sequence such that $u_i^\epsilon \rightharpoonup u_i$ weakly in $L^2(\Omega)$ and

$$\sum_{j=1}^m \sum_{k=1}^n A_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} \in H_{loc}^{-1}(\Omega) \quad \text{for } i = 1, \dots, q.$$

then

$$Q(u^\epsilon) \rightharpoonup Q(u) \quad \text{weakly in the sense of measures.}$$

Example 6.12. Let $u^n = (E^n, D^n)$ and $u^\infty = (E^\infty, D^\infty)$ be such that $u^n \rightharpoonup u^\infty$ in $(L^2(\Omega))^{2n}$ and

$$\frac{\partial E_i^n}{\partial x_j} - \frac{\partial E_j^n}{\partial x_i}, \quad \sum_j \frac{\partial D_j^n}{\partial x_j} \quad \text{belong to } (H^{-1}(\Omega))^{2n},$$

where E is the electric field and D is the electric induction of a given non-homogeneous body covering $\Omega \subset \mathbb{R}^n$.

The set Λ is given by

$$\Lambda = \{(E, D) : \xi \neq 0, \xi_j E_i - \xi_i E_j = 0 \forall i, j \quad \text{and} \quad \sum_j \xi_j D_j = 0\}.$$

Then we have that

$$Q(u) = E \cdot D$$

is quadratic and

$$Q(u) = 0.$$

So using Corollary 6.1.3, we deduce that

$$Q(u^n) = E^n \cdot D^n \xrightarrow{*} Q(u^\infty) = E^\infty \cdot D^\infty \quad \text{weakly* in } \mathcal{M}(\Omega).$$

The fact that the theory of compensated compactness is restricted to partial differential equations with constant coefficients, made room for another mathematical tool that can handle partial differential equations with variable coefficients, the H-Measures.

6.2 H-Measures

While Compensated compactness is only used in the case of partial differential equations with constant variables. Young measures computes the weak limit of any nonlinear function of weakly converging sequences. But it does not capture the differential structure of the equation satisfied by the sequence. This drawback

lead to the creation of another powerful tool, the H-measures.

H-Measures were introduced independently by Luc Tartar [Tar90], as an extension of Compensated compactness, and by P. Gérard [Gé91], under the name "Microlocal defect measures" to compute the weak limit of quadratic products of oscillating fields. H-measures can be looked at as a middle ground between Young measures which computes limits but fails to capture differential structure, and compensated compactness which only handles differential equations with constant coefficients. H-measures however, computes the limits of partial differential equations with variable coefficients.

6.2.1 Introduction

H-Measures only apply to sequences of functions that converge weakly to zero. Let $u_\epsilon : \Omega \rightarrow \mathbb{R}^p$ be a sequence of vector-valued functions defined on Ω , an open subset of \mathbb{R}^n , such that $u_\epsilon \rightharpoonup 0$ weakly in $(L^2(\Omega))^p$. The main idea behind H-measure is to extract subsequences $(u_{i_\epsilon})_{i=1}^p$. So that for all $i, j = 1, \dots, p$, we can define a family of complex Radon measures μ^{ij} on $\Omega \times S^{n-1}$ by

$$\langle \mu^{ij}, \phi_1 \bar{\phi}_2 \otimes \psi \rangle = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \mathcal{F}(\phi_1 u_{i_\epsilon})(\xi) \overline{\mathcal{F}(\phi_2 u_{j_\epsilon})(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi,$$

where \mathcal{F} is the Fourier transform, $\psi(\xi) \in C(S^{n-1})$ is a test function used to localize in the direction of ξ and $\phi_1, \phi_2 \in C_0(\Omega)$ are test functions used to localize the oscillations in the space variable x . The sequence u_ϵ is extended by zero outside Ω so that any concentration effects on the boundary $\partial\Omega$ will not be missed. But this does not affect the convergence of the sequence as the sequence u_ϵ still converges weakly to zero in $(L^2(\mathbb{R}^n))^p$ and ϕ_1, ϕ_2 now belong to $C_0(\mathbb{R}^n)$.

A class of Pseudo-differential operators is needed to define the H-measures, but because the classical pseudo-differential operators are not sufficient, a new class of pseudo-differential operators is needed. They are known as pseudo-differential

operators of order zero. The next subsection contains some pseudo-differential calculus needed in the study, followed by some definitions and basic properties of H-measures.

6.2.2 Pseudo-Differential Operators

Let $a \in C(S^{n-1})$, the space of continuous functions on S^{n-1} the unit sphere and $b \in C_0(\mathbb{R}^n)$, the space of continuous functions converging to zero at infinity. Let P_a and M_b be bounded linear continuous operators in $L^2(\mathbb{R}^n)$ associated to a and b in the following way:

$$\begin{aligned} \mathcal{F}(P_a u)(\xi) &= a\left(\frac{\xi}{|\xi|}\right) \mathcal{F}u(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^n, \\ (M_b u)(x) &= b(x)u(x) \quad \text{a.e. } x \in \mathbb{R}^n \end{aligned} \tag{6.2.1}$$

Lemma 6.13. *If P_a and M_b are given as above, then $C = P_a M_b - M_b P_a$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Definition 6.14. A continuous function P on $\mathbb{R}^n \times S^{n-1}$ written as

$$P(x, \xi) = \sum_{n=1}^{\infty} b_n(x) a_n(\xi),$$

is known as an *admissible symbol*, and

$$\sum_{n=1}^{\infty} \|a_n\|_{L^\infty(S^{n-1})} \|b_n\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

where $a_n \in C(S^{n-1})$ and $b_n \in C_0(\mathbb{R}^n)$.

Definition 6.15. The operator $Q = \sum_{n=1}^{\infty} A_n B_n$ is defined by

$$\mathcal{F}(Qu)(\xi) = \sum_{n=1}^{\infty} a_n\left(\frac{\xi}{|\xi|}\right) \mathcal{F}(b_n u)(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^n \quad \text{if } u \in L^2(\mathbb{R}^n),$$

and

$$\begin{aligned} \mathcal{F}(Qu)(\xi) &= \sum_{n=1}^{\infty} a_n\left(\frac{\xi}{|\xi|}\right) \int_{\mathbb{R}^n} b_n(x) u(x) e^{-2i\pi x \xi} dx, \\ &\quad \forall \xi \in \mathbb{R}^n \quad \text{if } u \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n). \end{aligned}$$

Using the symbol P ,

$$\mathcal{F}(Qu)(\xi) = \int_{\mathbb{R}^n} P\left(x, \frac{\xi}{|\xi|}\right) u(x) e^{-2i\pi x\xi} dx,$$

$$\forall \xi \in \mathbb{R}^n \quad \text{if } u \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).$$

Q is called the *standard operator with symbol P* .

6.2.3 Existences and Properties of H-Measures

Let Ω be an open set in \mathbb{R}^n and U_ϵ be a vector-valued sequence of functions converging weakly to 0 in $L^2(\Omega; \mathbb{R}^n)$. Let us extend the sequence U_ϵ by zero outside Ω so that the sequence still converges weakly to zero.

Theorem 6.16 (Existence of H-Measures). *There exists a subsequence of $\{U_\epsilon\}$ (which we still denote by $\{U_\epsilon\}$) and a family of complex-valued Radon measures μ^{ij} on $\mathbb{R}^n \times S^{n-1}$ such that for all $\phi_1, \phi_2 \in C_0(\mathbb{R}^n)$ and $\psi \in C(S^{n-1})$, one has*

$$\langle \mu^{ij}, \phi_1 \bar{\phi}_2 \otimes \psi \rangle = \lim_{\epsilon \rightarrow \infty} \int_{\mathbb{R}^N} \mathcal{F}(\phi_1 U_{i\epsilon})(\xi) \overline{\mathcal{F}(\phi_2 U_{j\epsilon})(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi, \quad i, j = 1, \dots, n.$$

$\mu = (\mu^{ij})$ is a matrix valued measure called the *H-measure associated with the subsequence*.

Proof. Show that the limit of the right hand side depends on the product $\phi_1 \bar{\phi}_2$ and defines a Radon measure. From (6.2.1),

$$\phi_1 U_{i\epsilon} = M_{\phi_1} U_{i\epsilon}, \quad \phi_2 U_{j\epsilon} = M_{\phi_2} U_{j\epsilon}, \quad \mathcal{F}(\phi_1 U_{i\epsilon})(\xi) \psi\left(\frac{\xi}{|\xi|}\right) = \mathcal{F}(P_\psi M_{\phi_1} U_{i\epsilon}).$$

So

$$\int_{\mathbb{R}^n} \mathcal{F}(\phi_1 U_{i\epsilon})(\xi) \overline{\mathcal{F}(\phi_2 U_{j\epsilon})(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi = \int_{\mathbb{R}^n} \mathcal{F}(P_\psi M_{\phi_1} U_{i\epsilon})(\xi) \overline{\mathcal{F}(M_{\phi_2} U_{j\epsilon})(\xi)} d\xi.$$

By Plancherel's identity,

$$\int_{\mathbb{R}^n} \mathcal{F}(P_\psi M_{\phi_1} U_{i\epsilon})(\xi) \overline{\mathcal{F}(M_{\phi_2} U_{j\epsilon})(\xi)} d\xi = \int_{\mathbb{R}^n} (P_\psi M_{\phi_1} U_{i\epsilon})(x) \overline{(M_{\phi_2} U_{j\epsilon})(x)} dx.$$

If one commutes P_ψ and M_{ϕ_1} using Lemma 6.13, one gets,

$P_\psi M_{\phi_1} - M_{\phi_1} P_\psi = C$, a compact operator, which transforms the weakly convergent sequence $U_{i\epsilon}$ into a strongly convergent sequence while keeping the limit zero.

Therefore, what we are looking for is

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} M_{\phi_1} M_{\bar{\phi}_2} P_\psi U_{i\epsilon} U_{j\epsilon} dx$$

But

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} M_{\phi_1} M_{\bar{\phi}_2} P_\psi U_{i\epsilon} U_{j\epsilon} dx &\leq B \|\phi_1 \bar{\phi}_2\|_{C_0(\mathbb{R}^n)} \|U_{i\epsilon} U_{j\epsilon}\|_{L^2(\mathbb{R}^n)} \|\psi\|_{C(S^{n-1})} \\ &\leq C, \end{aligned}$$

where $B > 0$, $C > 0$ are constants. By a diagonal argument, there exist a subsequence for a countable dense set of functions ϕ_1, ϕ_2 and ψ that converges.

The limit is linear and depends on $\phi_1 \bar{\phi}_2$.

This defines a bilinear continuous form on $C_0(\mathbb{R}^n) \times C(S^{n-1})$ written

$$\langle \mu^{ij}, \phi_1 \bar{\phi}_2 \otimes \psi \rangle.$$

Next we show that μ^{ij} is a measure using the Lemma below.

Lemma 6.17. *Let X and Y be two locally compact manifold and let B be a continuous bilinear form on $C(X) \times C(Y)$. If $f > 0$ and $g > 0$ implies that $B(f, g) > 0$, then there exists a Radon measure m on $X \times Y$ such that*

$$B(f, g) = \langle m, f \otimes g \rangle, \quad \forall f \in C(X), g \in C(Y).$$

Proof. See [Tar90] □

Using the extracted sequence, let

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \mathcal{F}(\phi_1 U_{i\epsilon})(\xi) \overline{\mathcal{F}(\phi_2 U_{j\epsilon})(\xi)} \psi \left(\frac{\xi}{|\xi|} \right) d\xi = B^{ij}(\phi_1 \bar{\phi}_2, \psi),$$

where B^{ij} is a continuous bilinear form on $C_0(\mathbb{R}^n) \times C(S^{n-1})$.

If for every choice of complex numbers λ_j , the bilinear form B is such that $B =$

$\sum_{jk} \lambda_j \bar{\lambda}_k B^{jk}$, we have

$$\sum_{jk} \lambda_j \bar{\lambda}_k B^{jk}(|\phi|^2, \psi) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \left| \sum_j \lambda_j \mathcal{F}(\phi U_{j\epsilon}) \right|^2 \psi \left(\frac{\xi}{|\xi|} \right) d\xi.$$

This implies that for $\phi \geq 0$ and $\psi \geq 0$, $B(\phi, \psi) \geq 0$. So by Lemma 6.17, there is a Radon measure m on $\mathbb{R}^n \times S^{n-1}$ such that

$$B(\phi_1 \bar{\phi}_2, \psi) = \langle m, \phi_1 \bar{\phi}_2 \otimes \psi \rangle,$$

where B defines the Radon measure m which in turn defines the measures μ^{jk} that we can identify by writing $m = \sum_{jk} \lambda_j \bar{\lambda}_k \mu^{jk}$.

□

Theorem 6.18 (Localization of H-measures). *Suppose U_ϵ is a sequence converging weakly to zero in $L^2(\mathbb{R}^n; \mathbb{R}^p)$ and define an H-measure μ . If U_ϵ is such that*

$$\sum_{j=1}^p \sum_{k=1}^n \frac{\partial}{\partial x_k} (A_{jk}(x) U_{j\epsilon}) \rightarrow 0 \quad \text{strongly in } H_{loc}^{-1}(\Omega),$$

then

$$\sum_{j=1}^p \sum_{k=1}^n A_{jk}(x) \xi_k \mu^{jm} = 0 \quad \text{in } \Omega \times S^{n-1} \quad \forall m.$$

where A_{jk} are continuous in Ω .

Example 6.19. Let z be a point in \mathbb{R}^n . Consider the sequence $\{u_\epsilon\}$ obtained by a translation of the function $f \in L^2(\mathbb{R}^n)$ followed by a scaling;

$$u_\epsilon(x) = \epsilon^{-\frac{n}{2}} f\left(\frac{x-z}{\epsilon}\right), \quad \epsilon > 0. \quad (6.2.2)$$

These functions characterize a concentration effect at the point z . We shall construct the H-measure associated to $\{u_\epsilon\}$. We start by showing that u_ϵ converges to zero weakly in $L^2(\mathbb{R}^n)$. We have,

$$\int_{\mathbb{R}^n} u_\epsilon^2(x) dx = \epsilon^{-n} \int_{\mathbb{R}^n} f^2\left(\frac{x-z}{\epsilon}\right) dx.$$

Using the change of variables

$$y = \frac{x-z}{\epsilon}, \quad x = \epsilon y + z, \quad dx = \epsilon^n dy, \quad (6.2.3)$$

we get

$$\|u_\epsilon\|_{L^2(\mathbb{R}^n)}^2 = \epsilon^{-n} \epsilon^n \int_{\mathbb{R}^\times} f^2(y) dy = \int_{\mathbb{R}^\times} f^2(y) dy.$$

Thus $\{u_\epsilon\}$ is uniformly bounded in $L^2(\mathbb{R}^n)$. Let us extract a subsequence which we denote again by $\{u_\epsilon\}$, we have that there exists $u \in L^2(\mathbb{R}^n)$ such that

$$u_\epsilon \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^n),$$

we also have

$$u_\epsilon \rightharpoonup u \quad \text{in } L^1(\mathbb{R}^n). \quad (6.2.4)$$

But thanks to (6.2.3),

$$\|u_\epsilon\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \epsilon^{-\frac{n}{2}} \left| f\left(\frac{x-z}{\epsilon}\right) \right| dx = \epsilon^{\frac{n}{2}} \int_{\mathbb{R}^n} |f(y)| dy.$$

Thus

$$\lim_{\epsilon \rightarrow 0} \|u_\epsilon\|_{L^1(\mathbb{R}^n)} = 0.$$

This implies that $U_\epsilon \rightarrow 0$ strongly in $L^1(\mathbb{R}^n)$.

But then $u_\epsilon \rightharpoonup 0$ weakly in $L^1(\mathbb{R}^n)$. By the uniqueness of weak limits, we deduce from (6.2.4) that $u = 0$. Hence

$$u_\epsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}^n).$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$. We show that

$$\phi u_\epsilon - \phi(z) u_\epsilon \longrightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}^n). \quad (6.2.5)$$

By (6.2.3), we have

$$\int_{\mathbb{R}^n} |\phi(x) u_\epsilon(x)|^2 dx = \int_{\mathbb{R}^n} \epsilon^{-n} \phi^2(x) \left| f\left(\frac{x-z}{\epsilon}\right) \right|^2 dx = \epsilon^{-n} \epsilon^n \int_{\mathbb{R}^n} \phi^2(\epsilon y + z) |f(y)|^2 dy.$$

Since $\phi \in C_0^\infty$, by continuity, we deduce that

$$\int_{\mathbb{R}^n} |\phi(x) u_\epsilon(x)|^2 dx \longrightarrow \int_{\mathbb{R}^n} |\phi(x)|^2 |f(y)|^2 dy.$$

By (6.2.3) again we have

$$\int_{\mathbb{R}^n} \phi(x) |f(y)|^2 dy = \phi^2(z) \int_{\mathbb{R}^n} \epsilon^\epsilon \left| f\left(\frac{x-z}{\epsilon}\right) \right|^2 dx = \int_{\mathbb{R}^n} |\phi(z) u_\epsilon(x)|^2 dx,$$

thus

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} (|\phi(x)u_\epsilon(x)|^2 - |\phi(z)u_\epsilon(x)|^2) dx = 0.$$

hence (6.2.5) follows.

By Parseval's identity, (6.2.5) implies that

$$\mathcal{F}(\phi u_\epsilon)(\xi) - \mathcal{F}(\phi(z)u_\epsilon)(\xi) \rightarrow 0 \text{ in } L^2(\mathbb{R}^n).$$

Thus for $\psi \in C(S^{n-1})$, we get

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} (|\mathcal{F}(\phi u_\epsilon)(\xi)|^2 - |\mathcal{F}(\phi(z)u_\epsilon)(\xi)|^2) \psi\left(\frac{\xi}{|\xi|}\right) d\xi \\ & \leq C \left(\int_{\mathbb{R}^n} (|\mathcal{F}(\phi u_\epsilon)(\xi)|^2 + |\mathcal{F}(\phi(z)u_\epsilon)(\xi)|^2) d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\mathcal{F}(\phi u_\epsilon)(\xi) - \mathcal{F}(\phi(z)u_\epsilon)(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ & \leq C \|\mathcal{F}(\phi u_\epsilon) - \mathcal{F}(\phi(z)u_\epsilon)\|_{L^2(\mathbb{R}^n)}; \end{aligned}$$

this converges to zero.

Thus

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} |\mathcal{F}(\phi u_\epsilon)(\xi)|^2 \psi\left(\frac{\xi}{|\xi|}\right) d\xi = \lim_{\epsilon \rightarrow 0} |\phi(z)|^2 \int_{\mathbb{R}^n} |(\mathcal{F}u_\epsilon)(\xi)|^2 \psi\left(\frac{\xi}{|\xi|}\right) d\xi. \quad (6.2.6)$$

Let us use the change of variables (6.2.3) to rewrite $(\mathcal{F}u_\epsilon)(\xi)$. We have

$$\begin{aligned} (\mathcal{F}u_\epsilon)(\xi) &= \epsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} f\left(\frac{x-z}{\epsilon}\right) dx \\ &= \epsilon^{-\frac{n}{2}} \epsilon^n \int_{\mathbb{R}^n} e^{-i(\epsilon y+z)\xi} f(y) dy \\ &= \epsilon^{\frac{n}{2}} e^{-iz\xi} \int_{\mathbb{R}^n} e^{-iy\xi} f(y) dy \\ &= \epsilon^{\frac{n}{2}} e^{-iz\xi} (\mathcal{F}f)(\epsilon\xi). \end{aligned}$$

Thus $|\mathcal{F}u_\epsilon(z)|^2 = \epsilon^n |\mathcal{F}f(\epsilon\xi)|^2$, since $|e^{-iz\xi}| = 1$. The right hand side of (6.2.6) becomes

$$\lim_{\epsilon \rightarrow 0} |\phi(z)|^2 \int_{\mathbb{R}^n} \epsilon^n |\mathcal{F}f(\epsilon\xi)|^2 \psi\left(\frac{\xi}{|\xi|}\right) d\xi.$$

Let us denote it by J .

Using the change of variables $\xi' = \epsilon\xi$, $d\xi = \epsilon^{-n} d\xi'$. J can be rewritten as

$$\begin{aligned} J &= \lim_{\epsilon \rightarrow 0} |\phi(z)|^2 \int_{\mathbb{R}^n} \epsilon^n \epsilon^{-n} |\mathcal{F}f(\xi')|^2 \psi\left(\frac{\epsilon\xi'}{|\epsilon\xi'|}\right) d\xi' \\ &= |\phi(z)|^2 \int_{\mathbb{R}^n} |\mathcal{F}f(\xi)|^2 \psi\left(\frac{\xi}{|\xi|}\right) d\xi. \end{aligned}$$

Next we use spherical coordinates with $\xi = t\omega$, $\omega \in S^{n-1}$, $\omega = \frac{\xi}{|\xi|}$, $d\xi = t^{n-1} dt d\omega$.

Then

$$\begin{aligned} J &= |\phi(z)|^2 \int_{S^{n-1}} \int_0^\infty t^{n-1} |\mathcal{F}f(t\omega)|^2 \psi(\omega) dt d\omega \\ &= |\phi(z)|^2 \int_{S^{n-1}} \nu(\omega) \psi(\omega) d\omega \end{aligned} \quad (6.2.7)$$

where $\nu(\omega) = \int_0^\infty t^{n-1} |(\mathcal{F}f)(t\omega)|^2 dt$. By (6.2.6), we have shown that

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} |\mathcal{F}\phi u_\epsilon(\xi)|^2 \psi\left(\frac{\xi}{|\xi|}\right) d\xi = \langle \mu, |\psi|^2 \otimes \psi \rangle,$$

where $\mu = \delta_z \otimes \nu$. This is the H-measure associated with u_ϵ . Using the definition of the Dirac-delta function δ_z we check indeed, that

$$\begin{aligned} \langle \delta_z \otimes \nu, |\phi|^2 \otimes \psi \rangle &= \int_{\mathbb{R}^n} \int_{S^{n-1}} |\phi(x)|^2 \left(\int_0^\infty t^{n-1} |\mathcal{F}f(t\omega)|^2 dt \right) \psi(\omega) d\omega_{\delta_z} dx \\ &= |\phi(z)|^2 \int_{S^{n-1}} \nu(\omega) \psi(\omega) d\omega \end{aligned}$$

This corresponds to (6.2.7).

Example 6.20. let u_ϵ be a scalar sequence defined by

$$u_\epsilon(x) = v\left(x, \frac{x}{\epsilon}\right), \quad x \in \mathbb{R}^n,$$

with $v(x, y)$ periodic in the y variable and $\epsilon > 0$ takes values in a sequence that converges to zero.

Let us assume that v is continuous in x with values in $L^2(Y)$ and

$$\int_Y v dy = 0,$$

so $u_\epsilon(x) \rightharpoonup 0$ weakly in $L^2_{loc}(\mathbb{R}^n)$.

We consider the Fourier transform expansion of v in the y variable

$$v(x, y) = \sum_{m \in \mathbb{Z}^n} v_m e^{2i\pi(m \cdot y)},$$

where $v_0 = 0$ by hypothesis so that $u_\epsilon \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^n)$. Under these assumptions, it can be shown that the H-measure μ associated with u_ϵ is defined by

$$\langle \mu, \phi \rangle = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \int_{\mathbb{R}^n} |v_m(x)|^2 \phi\left(x, \frac{m}{|m|}\right) dx,$$

for all continuous function ϕ on $\mathbb{R}^n \times S^{n-1}$ with compact support in x . μ then takes the form

$$\mu = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} |v_m|^2 \otimes \delta_{\frac{m}{|m|}}.$$

Conclusion and Further research

In this dissertation, we endeavour to cover the theory of periodic homogenization from its genesis characterized by the multiple scale expansions method to the cutting edge and deep theory of Tartar's H-measures. Although the multiscale expansion is widely used, it only handles problems with periodic coefficients. The method is heuristic in nature and involves lots of calculations making it error-prone.

Tartar's method of oscillating test functions, a mathematically rigorous method was introduced later and it placed the multiscale expansions on firm theoretical grounds. But its success was mainly limited to scalar problems (equations). For the study of homogenization problems relevant to systems of partial differential equations, a new tool was needed. This led to the invention of the div-curl lemma by L. Tartar and F. Murat. The div-curl lemma is however not applicable to convergences of composition of weakly converging sequences and general nonlinear functions so Tartar adapted Young measures to that case.

With the div curl lemma, one is able to find the weak limit of the product of two weakly converging sequences of vector-valued functions, but with appropriate conditions on the sequences. The compensated compactness method was later introduced as an extension of the div-curl lemma but unfortunately, it is restricted to problems with constant coefficients. Then the H-measures was introduced to accommodate problems with variable coefficients. So far H-measures has been used

for hyperbolic equations and in the study of oscillations. There is also a variant of H-measures for parabolic equations.

As future research, we intend to develop H-measures for randomly perturbed partial differential equations with rapidly oscillating coefficients. The recently developed notion of multi-scale H-measures by L. Tartar [Tar15] opens new avenues for potential applications. The relation between Wigner measures and H-measures remains a controversial issue. We do seriously also consider potential research on Wigner measures in a stochastic framework. For background on Wigner measures we refer to [Lio93] and to [Zha08]. Homogenization of stochastic partial differential equations is still at its infancy. Crucial pioneering work in that direction have been undertaken by M. Sango and his coworkers; see for instance [San15], [San12], [San14], [San02].

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