MAX-EWMA CHART FOR AUTOCORRELATED PROCESSES (MEWMAP CHART)

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ABSTRACT

This paper proposes an exponentially weighted moving average (EWMA) control chart that is capable of detecting changes in both process mean and standard deviation for autocorrelated data (referred to as the Maximum Exponentially Weighted Moving Average Chart for Autocorrelated Process, or MEWMAP chart). This chart is based on fitting a time series model to the data, and then calculating the residuals. The observations are represented as a first-order autoregressive process plus a random error term. The Average Run Lengths (ARLs) for fixed decision intervals and reference values (h, k) are calculated. The proposed chart is compared with the Max-CUSUM chart for autocorrelated data proposed by Thaga (2003). Comparisons are based on the out-of-control ARLs. The MEWMAP chart detects moderate to large shifts in the mean and/or standard deviation at both low and high levels of autocorrelations more quickly than the Max-CUSUM chart for autocorrelated processes.

OPSOMMING

Die navorsing stel voor dat 'n eksponensiaal geweegde bewegende gemiddelde kontrolekaart gebruik word om verandering van prosesgemiddelde en – standaardafwyking van outogekorreleerde data te bepaal. Die kontrolekaart word gedryf deur passing van 'n tydreeks as datamodel met bepaling van residuwaardes. Met hierdie gegewens as vertrekpunt word gemiddelde looplengtes vir vaste besluitintervalle en verwysingwaardes (h, k) bereken. Die kontrolekaart bepaal matige en groot verskuiwings van waardes vir hoë en lae outokorrelasiewaardes heel snel.

1. INTRODUCTION

Statistical process control (SPC) charts such as the Shewhart control chart (Shewhart [14]), the cumulative sum control chart (Page [10]), and the exponentially weighted moving average control chart (Roberts [12]), are used to monitor product quality and to detect special events that may be indicators of out-of-control situations. These charts are designed on the assumption that a process being monitored will produce measurements that are independent and identically distributed over time, when only the inherent sources of variability are present in the system. However, in some applications the dynamics of the process will produce correlations in observations that are closely spaced in time. If the sampling interval used for process monitoring in these applications is short enough for the process dynamics to produce significant correlation, then this correlation can have a very serious effect on the properties of standard control charts (VanBrackle & Reynolds [16], Lu & Reynolds [7] and [8], Runger & Willemain [13], Atienza, Tang & Ang [2]).

Positive autocorrelation in observations can result in severe negative bias in traditional estimators of the standard deviation. This bias produces control limits that are much tighter than desired. This can result in a much higher average false alarm rate than expected. Furthermore, when observations are positively autocorrelated, when there is a shift in the process mean, only a fraction of the shift will be transferred to the residual mean, and the chart will not quickly detect this shift. It is therefore very important to take autocorrelation among observations into consideration when designing a process-monitoring scheme – in particular, control charts – in order to maximize full benefit from the control charts.

Recently, new control charts have been proposed for dealing with autocorrelated data. Two approaches have been advocated for dealing with this phenomenon. The first approach uses standard control charts on original observations, but adjusts the control limits and methods of estimating parameters to account for the autocorrelation in the observations (VanBackle & Reynolds [16], Lu & Reynolds [7]). This approach is particularly applicable when the level of autocorrelation is not high.

A second approach for dealing with autocorrelation fits a time series model to the process observations. The procedure forecasts observations from previous values and then computes the forecast errors or residuals. These residuals are then plotted on standard control charts, because – when the fitted time series model is the same as the true process model and the parameters are estimated without error – the residuals are independent and identically distributed normal random variables when the process is in control (Alwan & Roberts [1], Montgomery & Mastrangelo [9], Wadell, Moskowitz, & Plante [17], Lu & Reynolds [7], and Runger, Willemain & Prabhu [3]). Yashchin [19] recommends directly charting raw data when the level of autocorrelation is low; at a high level of autocorrelation he recommends some transformation procedures that create residuals. If a shift in the mean and/or standard deviation of the process occurs, this will cause a shift in the mean and/or standard deviation of the residuals. Control charts based on residuals seem to work best when the level of correlation is high. When the level of correlation is low, forecasting is more difficult and residual charts are not very effective at detecting process changes.

The studies mentioned above used several methods, such as simulation, asymptotic approximation, and direct calculation, to evaluate the properties of the control charts. A conclusion that can be drawn from these studies is that correlation between observations has a significant effect on the properties of the control charts. In particular, when the level of autocorrelation is high, control charts run for a long time before detecting shifts in the process parameters from in-control values.

The objective of this paper is to investigate control charts for simultaneously monitoring the process mean and variation using a single chart in the presence of autocorrelation. We propose an exponentially weighted moving average (EWMA) control chart for autocorrelated data that can simultaneously monitor shifts in the mean and standard deviation using a single plotting variable. This investigation is done for the case of processes that can be modeled as a first order autoregressive AR(1) process plus an additional random error, which can correspond to sampling, or measurement error. This model allows relatively accurate numerical techniques to be used to evaluate the properties of the control charts. Lu & Reynolds [7] proposed a simultaneous EWMA control chart for autocorrelated processes, which runs two control charts concurrently. Chen, Cheng & Xie [5] developed a single EWMA chart on the assumption that the process produces measurements that are independent over time when the process is in control.

Our proposed chart monitors the process by monitoring the residual means and standard deviations. The results show that by taking the autocorrelation structure of the process into consideration, the EWMA chart can effectively detect small shifts in the process mean and/or spread.

2. THE AR(1) PROCESS WITH AN ADDITIONAL RANDOM ERROR

This model has been used previously in a number of contexts, and has the advantage that it will account for the correlation between observations that are close together in time, for variability in the process mean over time, and for additional variability due to sampling or measurement error.

To model observations from an autocorrelated process, we use a model that has been discussed previously in quality control by authors such as Lu & Reynolds [7 and 8] and VanBackle & Reynolds [16]. For this model, X_t can be represented as

$$X_t = \mu_t + \varepsilon_t. \qquad t = 1, 2, \ldots \qquad (1)$$

Where X_t represents an observation taken from the process at sampling time; t, μ_t is the random process mean at sampling time; t and ε_t's are independent normal random errors with mean 0 and variance σ_ε^2. This model accounts for a correlation between samples that are close together in time, for variability in the process mean over time, and for additional variability due to sampling or measurement error. It is assumed that μ_t can be described as an AR(1) process, defined as

$$\mu_t = (1-\phi)\xi + \phi\mu_{t-1} + \alpha_t . \qquad t = 1, 2, \ldots \tag{2}$$

Where ξ is the overall process mean, α_t' s are independent normal random variables with mean 0 and variance σ_α^2, and ϕ is the autoregressive parameter satisfying $|\phi|<1$ for a process to be stationary. We assume that the starting value μ_0 follows a normal distribution with mean ξ and variance $\sigma_\mu^2 = \sigma_\alpha^2/(1-\phi^2)$. The distribution of X_t is therefore constant with mean ξ and variance given by $\text{Var}(X_t) = \text{Var}(\mu_t) + \text{Var}(\varepsilon_t)$. This variance is given as

$$\sigma_X^2 = \sigma_\mu^2 + \sigma_\varepsilon^2 = \frac{\sigma_\alpha^2}{1-\phi^2} + \sigma_\varepsilon^2 . \tag{3}$$

In this case σ_μ^2 represents long-term variability, and σ_ε^2 represents a combination of short-term variability and the variability associated with measurement error. When assessing processes following models (1) and (2), it is often convenient to consider the proportion of total process variability that is due to variation in μ_t and the proportion due to error variability. The proportion of the process variability due to variation in μ_t is defined as

$$\psi = \frac{\sigma_\mu^2}{\sigma_X^2} = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\varepsilon^2} .$$

The proportion of the variance due to ε_t is then 1 - ψ. The covariance between two observations that are *i* units apart is $\phi^i \sigma_\mu^2$, and the correlation between two adjacent observations is $\rho = \phi\psi$.

The AR(1) process with an additional random error is equivalent to a first order autoregressive moving average, ARMA(1,1) process (Box, Jenkins & Reinsel [3]), which can be written as

$$(1-\phi B)X_t = (1\text{-}\phi)\xi + (1-\theta B)\gamma_t . \tag{4}$$

Where γ_t' s are independent normal random variables with mean 0 and variance σ_γ^2, θ is the moving average parameter, ϕ is the autoregressive parameter defined in equation (2), and B is a backshift operator such that $BX_t = X_{t-1}$. If $\phi>0$, Koons & Foutz [6] estimate θ and σ_γ^2 as

$$\theta = \frac{\sigma_\alpha^2 + (1-\phi^2)\sigma_\varepsilon^2}{2\phi\sigma_\varepsilon^2} - \frac{1}{2}\sqrt{\left(\frac{\sigma_\alpha^2 + (1+\phi^2)\sigma_\varepsilon^2}{\phi\sigma_\varepsilon^2}\right)^2 - 4} \tag{5}$$

and

$$\sigma_\gamma^2 = \frac{\phi\sigma_\varepsilon^2}{\theta} \tag{6}$$

The standard time series estimation techniques can be used to estimate the parameters in the ARMA(1,1) model.

In some production processes, a large volume of items is produced in a single lot. In this situation, more than one observation is sampled each time. Let X_{ti} be the i^{th} observation at sampling time *t*. We assume that X_{ti} can be represented as

$$X_{ti} = \mu_t + \varepsilon'_{ti}. \tag{7}$$

Where the ε'_{ti}s are independent and identically distributed normal random variables with mean 0 and variance $\sigma_{\varepsilon'}^2$, and μ_t follows model (2). Lu & Reynolds [7] indicated that the sample means from this process will follow models (1) and (2) with $\sigma_\varepsilon^2 = \sigma_{\varepsilon'}^2 / n$.

In this article, we monitor the process mean and standard deviation by monitoring the residuals from a forecast. To do this, we first determine the distribution of the residuals when the process is in control. When the process is in control, the residual at observation *t* from the minimum mean square error forecast made at observation *t-1* is

$$e_t = X_t - \xi_0 - \phi(X_{t-1} - \xi_0) + \theta e_{t-1}. \tag{8}$$

Where ϕ and θ are parameters in the ARMA(1,1) model given in equation (4), and ξ_0 is the in-control process mean – that is, the residual at time *t* is the difference between X_t and the prediction of X_t based on the previous data.

If the fitted time series model is the same as the true process model and the parameters are estimated without error, then the residuals are independent and identically distributed normal random variables when the process is in control. We can then monitor the process by using standard control charts for independent observations using these residuals. If there is a step change in the process mean from the in-control value ξ_0 to ξ_1 between time $t = \tau - 1$ and $t = \tau$, the expectations of the residuals for various times are (Lu & Reynolds [7]):

$E(e_t) = 0,$ $\quad t < \tau$

$E(e_t) = \xi_1 - \xi_0,$ $\quad t = \tau$

and

$$E(e_t) = \frac{1-\phi+\phi^l(\phi-\theta)}{1-\theta}(\xi_1-\xi_0) \qquad t=\tau+l,\ l=1,2,\ldots \tag{9}$$

The asymptotic mean of these residuals is given as

$$\frac{1-\phi}{1-\theta}(\xi_1-\xi_0). \tag{10}$$

These residuals are independent and normally distributed with variance σ_γ^2. The expectation of the residuals after the shift occurs is a decreasing function of time. Also, as ϕ increases, a small fraction of shift in the process mean will be transferred to the mean of the residuals. As a result, the ability of the chart for residuals to detect the mean shift is reduced. On the other hand, the residuals chart is theoretically very appealing because it takes the serial correlation into account, and reduces the problem to the well-known case of a shift in the process mean for independent observations.

A change in the process variance can be attributed to a change in the autoregressive parameter ϕ, the individual observation random shock variance σ_ε^2, and/or change in variability of the random shocks associated with the mean σ_α^2. If between observations t-1 and t, σ_α^2 increases from its nominal value $\sigma_{\alpha 0}^2$ to $\sigma_{\alpha 1}^2$ and σ_ε^2 increases from its nominal value $\sigma_{\varepsilon 0}^2$ to $\sigma_{\varepsilon 1}^2$, with ϕ remaining unchanged, the model in equation (1) becomes

$$X_{t+l} = \mu_{t+l} + \varepsilon_{t+l} + \varepsilon_{t+l}^*, \qquad l = 0,1,\ \ldots \tag{11}$$

Where $\varepsilon_t^*, \varepsilon_{t+l}^*, \ldots$ is a sequence of independent normal random variables with a mean 0 and variance $\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2$, independent of the ε_{t-l}'s. The model in equation (2) for μ_t becomes

$$\mu_{t+l} = (1-\phi)\xi + \phi\mu_{t+l-1} + \alpha_{t+l} + \alpha_{t+l}^*, \qquad l = 0,1,\ \ldots \tag{12}$$

where $\alpha_t^*, \alpha_{t+l}^*, \ldots$ is a sequence of independent normal random variables with mean 0 and variance $\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2$, independent of the α_{t-l}'s. We can write μ_{t+1}, X_{t+1}, $\hat{X}_{t+1}$ and ε_{t+1} in terms of their corresponding in-control quantities, say $\mu_{t+1}^0, X_{t+1}^0, \hat{X}_{t+1}^0$ and ε_{t+1}^0 respectively. Therefore, using the model in equation (1) and equations (8), (9) and (10), it can be shown by induction that

$$\mu_{t+l} = \mu_{t+l}^0 + \sum_{i=0}^{l} \phi^{l-i} \alpha_{t+i}^*, \qquad l = 0,1, \ldots$$

$$X_{t+l} = X_{t+l}^0 + \xi_{t+l}^* + \sum_{i=0}^{l} \phi^{l-i} \alpha_{t+l}^*, \qquad l = 0,1, \ldots$$

$$\hat{X}_{t+l} = \hat{X}_{t+l}^0 + (\phi - \theta_0) \sum_{i=0}^{l} \theta_0^{l-i} \varepsilon_{t+i}^* + \sum_{i=0}^{l-1} (\phi^{l-i} - \theta_0^{l-i}) \alpha_{t+i}^*, \qquad l = 0,1, \ldots$$

$$e_t = e_t^0 + \varepsilon_t^* + \alpha_t^* \tag{13}$$

and

$$e_{t+l} = e_{t+l}^0 + \varepsilon_{t+l}^* - (\phi - \theta_0) \sum_{i=0}^{l-1} \theta_0^{l-i-1} \varepsilon_{t+i}^* + \sum_{i=0}^{l} \theta_0^{l-i} \alpha_{t+i}^* \qquad l = 0,1, \ldots \tag{14}$$

where θ_0 is the in-control value of θ. This shows that e_{t+l} is a function of ε_{t+i}^* and α_{t+i}^* for $i \le l$. Therefore the effect of a shift in the variance is to induce correlation in the residuals (Lu & Reynolds [7]).

Assuming that $\sigma_{\gamma 0}^2$ is the in-control value of σ_γ^2, then $Var(e_{t+l}^0) = \sigma_{\gamma 0}^2$. Therefore the variance of the residuals after the shift is

Var(e_t) = $\sigma_{\gamma 0}^2 + (\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2) + (\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2)$

And

Var(e_{t+l}) =

$$\sigma_{\gamma 0}^2 + \left[1 + (\phi - \theta_0)^2 \sum_{i=0}^{l-i} \theta_0^{2(l-i-1)}\right] \times (\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2) + \sum_{i=0}^{l} \theta_0^{2(l-i)} (\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2),$$

l = 1,2, …

The asymptotic variance of these residuals after the shift will increase to the limit

$$\text{Var}(e_t) = \sigma_{\gamma 0}^2 + \frac{\phi^2 - 2\phi\theta_0 + 1}{1 - \theta_0^2} (\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2) + \frac{\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2}{1 - \theta_0^2}. \tag{15}$$

If $\theta_0 = 1$, the second term in equation (15) will vanish, and therefore we will use the variance of the residual as Var(e_t) = $\sigma_{\gamma 0}^2 + (\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2) + (\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2)$

The residuals after these shifts are correlated normal random variables with an asymptotic mean in equation (10) and asymptotic variance given in equation (15). From equation (15), we can see that changes in σ_α^2 and σ_ε^2 have different impacts on the variability of the residuals. Given the parameters in the ARMA(1,1) model, for $\phi > 0$, σ_α^2 and σ_ε^2 can be obtained from (Reynolds, Arnold & Baik [11]):

$$\sigma_\alpha^2 = \frac{\sigma_\gamma(\phi - \theta)(1 - \phi\theta)}{\phi} \tag{16}$$

and

$$\sigma_\varepsilon^2 = \frac{\theta\sigma_\gamma^2}{\phi} \tag{17}$$

We can then fit the AR(1) plus random error model in equations (1) and (2), which is the model considered in this paper. We consider the case of positive autocorrelation, which is more prevalent than negative autocorrelation in control chart applications.

3. THE NEW CONTROL CHART

We propose a new exponentially weighted moving average control chart for residuals in this section. Let $X_i = X_{i1}, X_{i2}, \ldots, X_{in}$, $i = 1, 2, \ldots$, denote a sequence of samples of size n taken on a quality characteristic X. It is assumed that, for each i, $X_{i1}, X_{i2}, \ldots, X_{in}$ follows a normal distribution and is autocorrelated and can be expressed as in equation (7). We monitor the process by first fitting the time series model into the process observations, and then computing the residuals. Let ξ_0 and $\sigma_{\gamma 0}$ be the nominal process mean and standard deviation of the residuals for this fitted model. Assume that the process residual parameters ξ and σ_γ can be expressed as $\xi = \xi_0 + a\sigma_{\gamma 0}$ and $\sigma_\gamma = b\sigma_{\gamma 0}$, where $a = 0$ and $b = 1$ when the process is in-control; otherwise, the process has changed due to some assignable causes. Then a represents the shift in the process mean and b represents the shift in the process standard deviation.

Let $\bar{\xi}_i = (\xi_{i1} + \xi_{i2} + \ldots + \xi_{in_i})/n$ and $\text{MSE}_i = \sum_{j=1}^{n}(\xi_{ij} - \bar{\xi}_i)^2/n$ be the mean and variance for the i^{th} sample residuals respectively. These statistics are independently distributed, as are the sample residual values when the process is in-control. These two statistics follow different distributions. The EWMA charts for the mean and standard deviation are based on $\bar{\xi}_i$ and MSE_i respectively.

To develop an EWMA chart for the process mean and process standard deviation using residuals, we carry out the following transformations:

$$Z_i = \sqrt{n}\,\frac{(\bar{\xi}_i - \xi_0)}{\sigma_{\gamma 0}} \tag{18}$$

$$Y_i = \Phi^{-1}\left\{H\left[\frac{(n)MSE_i}{\sigma_{\gamma 0}^2}; n\right]\right\}. \tag{19}$$

Where $\Phi(z) = P(Z \le z)$, for Z ~ N(0, 1), the standard normal distribution. $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$, the cumulative distribution function of N(0, 1) and H(w; p) = P(W$\le$w|p) for W ~ χ_p^2, the chi-square distribution with *p* degrees of freedom.

Z_i and Y_i are independent and when *a* = 0 and *b* = 1, they follow the standard normal distribution. The EWMA statistics based on Z_i and Y_i are defined as:

$$U_i = (1-\lambda)U_{i-1} + \lambda Z_i, \tag{20}$$

$$V_i = (1-\lambda)V_{i-1} + \lambda Y_i \tag{21}$$

respectively, where U_0 and V_0 are starting values of the chart and $0 < \lambda < 1$. The parameter λ is called the smoothing parameter of the EWMA chart. Because Z_i and Y_i follow the same distribution, a new statistic for a new single control chart is defined as

$$M_i = \max[|U_i|, |V_i|] \tag{22}$$

If the process is operating out of control, the M_i's will be plotted outside the control limits, otherwise the M_i values are within the limits. Since M_i>0, we plot only the upper limit for this chart, and consider the process to be out of control if an M_i value is plotted above the upper control limit.

In statistical process control (SPC), we often use the average run length (ARL) of the chart to assess the performance of the scheme. This is the expected number of samples (or observations, if we take a single observation each time) required by the chart to signal an out-of-control situation. For a change in variability, we consider the effects of changes in σ_ε and σ_α separately to calculate the ARL. This is because the two parameters have different impacts on the level of variability of the process as shown in equation (11). The shifts in these parameters are considered for different values of ϕ.

As shown previously, when the process is in control, the residuals are independent and identically distributed normal random variables with mean $\xi_0 = 0$ and standard deviation $\sigma_{\gamma 0}$. If there is a change in the mean and/or standard deviation, the residuals are correlated normal random variables with mean given in equation (10) and variance given in equation (15). In this article we consider a situation of an increase in the mean, standard deviation, or both mean and standard deviation of the process.

4. DESIGN OF A MAX-EWMA CHART FOR AUTOCORRELATED PROCESS (MEWMAP CHART)

When the process is in control and $U_0 = V_0 = 0$, U_i and V_i can be written as:

$$U_i = \lambda \sum_{j=1}^{i-1} (1-\lambda)^j Z_{i-j} \tag{23}$$

$$V_i = \lambda \sum_{j=1}^{i-1} (1-\lambda)^j Y_{i-j} \tag{24}$$

It can be shown that

$$U_i \sim N(0, \sigma_{U_i}^2)$$
$$V_i \sim N(0, \sigma_{V_i}^2)$$

and

$$\sigma_{U_i}^2 = \sigma_{V_i}^2 = \frac{\lambda(1-(1-\lambda)^{2i})}{2-\lambda} \tag{25}$$

Where again λ is called the smoothing parameter of the EWMA chart.

Chen, Cheng & Xie [5] showed that the in-control cumulative distribution function of M_i is given as:

$$F(y; \sigma_{U_i}^2) = P(M_i \le y)$$

$$= P(|U_i| \le y, |V_i| \le y)$$

$$= P(|U_i| \le y) P(|V_i| \le y)$$

$$= \left[2\Phi\left(\frac{y}{\sigma_{U_i}} \right) \right]^2, \qquad y \ge 0 \tag{26}$$

The corresponding probability density function of M_i is the derivative of $F(y; \sigma_{U_i})$, and is given by

$$f(y; \sigma_{U_i}) = \frac{4}{\sigma_{U_i}^2} \phi\left(\frac{y}{\sigma_{U_i}} \right) \left[2\Phi\left(\frac{y}{\sigma_{U_i}} \right) - 1 \right] \tag{27}$$

Using cubic-spline numerical integration, Xie [18] computed the mean and variance of M_i and obtained

$$E(M_i) = \int_0^\infty y f(y;\sigma_{U_i})dy$$

$$= 1.128379\,\sigma_{U_i} \quad (28)$$

and

$$Var(M_i) = \int_0^\infty y^2 f(y;\sigma_{U_i})dy$$

$$= 0.363381\,\sigma_{U_i}^2 \quad (29)$$

respectively.

The upper control limit (UCL) is obtained as follows:

$$UCL = E(M_i) + L\sqrt{Var(M_i)}$$

$$= \sigma_{U_i}(1.128379 + L\sqrt{0.363381})$$

$$= \sqrt{\frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}}(1.128379 + 0.602810L) \quad (30)$$

where L is the width of the control limits. As i gets large, $[1-(1-\lambda)]$ in equation (30) approaches unity and the UCL approaches the steady-state value, given as:

$$UCL = \sqrt{\frac{\lambda}{2-\lambda}}(1.128379 + 0.602810L) \quad (31)$$

We use the statistic M_i to construct a new control chart. Because M_i is the maximum of the two statistics, we call this new chart the Maximum Exponentially Weighted Moving Average for Autocorrelated Process chart (MEWMAP chart).

To calculate the ARL of the new chart, we use the modified Markov chain procedure (VanBrackle & Reynolds [16]). For the AR(1) plus random error model investigated in this article for shifts in mean and/or standard deviation, we use the asymptotic mean and variance given in equation (10) and equation (15) respectively. For a given in-control ARL and a shift of the mean and/or standard deviation intended to be detected by the chart, we find the combination of the chart design parameters (λ, L) that gives the desired in-control ARL and also minimizes the out-of-control ARL. This guideline takes into consideration the autocorrelation structure between the variables.

$ARL_0 = 250$ *A*									
B	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	L	2.9163	2.7981	2.9881	2.9986	2.9989	2.9998	3.2539	3.2539
	λ	0.2801	0.1487	0.3787	0.8487	0.9954	0.9985	1.0000	1.0000
	ARL σ_α	250.20	38.29	16.02	7.51	3.71	2.01	1.58	1.38
	ARL σ_ε	250.20	38.29	16.02	7.51	3.71	2.01	1.58	1.38
1.25	L	2.8550	2.9784	2.9854	2.9993	2.9998	3.2539	3.2539	3.2539
	λ	0.3981	0.2014	0.4814	0.7735	0.8735	1.0000	1.0000	1.0000
	ARL σ_α	22.41	15.32	7.22	5.42	2.92	2.10	1.55	1.33
	ARL σ_ε	21.53	15.04	7.10	5.30	2.88	2.02	1.53	1.30
1.50	L	2.9276	2.9531	2.9854	2.9897	2.9914	2.9996	2.9996	2.9999
	λ	0.4536	0.5454	0.4814	0.9217	0.9616	0.9853	0.9853	0.9904
	ARL σ_α	9.15	6.10	5.58	2.99	2.34	1.71	1.54	1.29
	ARL σ_ε	8.60	6.05	5.45	2.92	2.30	1.68	1.54	1.29
2.00	L	2.9276	2.9670	2.9789	2.9897	2.9925	2.9996	2.9996	2.9996
	λ	0.4536	0.5731	0.6519	0.9217	0.9643	0.9853	0.9853	0.9853
	ARL σ_α	4.75	2.90	2.78	2.16	1.87	1.52	1.45	1.27
	ARL σ_ε	3.67	2.83	2.70	2.11	1.83	1.50	1.44	1.26
2.50	L	2.9276	2.9670	2.9789	2.9897	2.9925	2.9996	2.9996	2.9996
	λ	0.4536	0.5731	0.6519	0.9217	0.9643	0.9853	0.9853	0.9853
	ARL σ_α	3.48	2.40	2.26	1.81	1.77	1.40	1.38	1.24
	ARL σ_ε	3.31	2.34	2.19	1.77	1.73	1.38	1.30	1.22
3.00	L	2.9276	2.9670	2.9789	2.9897	2.9925	2.9996	2.9996	2.9996
	λ	0.4536	0.5731	0.6519	0.9217	0.9643	0.9853	0.9853	0.9853
	ARL σ_α	2.75	2.09	1.96	1.61	1.58	1.36	1.28	1.24
	ARL σ_ε	2.63	2.03	1.90	1.58	1.46	1.31	1.26	1.17
4.00	L	2.9276	2.9670	2.9789	2.9897	2.9925	2.9996	2.9996	2.9996
	λ	0.4536	0.5731	0.6519	0.9217	0.9643	0.9853	0.9853	0.9853
	ARL σ_α	2.08	1.77	1.64	1.41	1.33	1.23	1.22	1.21
	ARL σ_ε	2.00	1.69	1.60	1.39	1.33	1.21	1.20	1.20

Table 1: (λ, L) combinations and the corresponding ARL for the MEWMAP chart, with $\phi = 0.25$ and $\psi = 0.8$.

Table 1 gives the optimal combinations of λ and *L* for an in-control ARL fixed at 250 and the autoregressive parameter $\phi = 0.25$ with 80% of process variability due to variation in μ_t, and the correlation between adjacent observations $\rho = 0.2$. Without loss of generality, we take $\sigma_{\gamma 0} = 1$. We calculate the out-of-control ARL for the effect of changes in the standard deviation that is due to changes in σ_ε and

σ_α respectively. The smallest value of an out-of-control ARL is calculated with respect to a pair of specified shifts in both mean and standard deviation, using the optimal in-control ARL EWMA chart parameters. We assume that the process starts in an in-control state, and thus the initial value of the EWMA statistic is set at zero. For example, if one wants to have an in-control ARL of 250 and to guard against a $3\sigma_{\gamma 0}$ increase in mean and $2\sigma_{\gamma 0}$ increase in the process standard deviation with an increase in $\sigma_{\gamma 0}$ with $\phi = 0.25,$ i.e., *a* = 3 and *b* = 2, the optimal in-control chart parameter values are λ = 0.9853 and *L* = 2.9996. These shifts can be detected on the second sample inspection, i.e., the ARL is approximately two, as seen in Table 1.

Table 2 gives the optimal combinations of λ and *L* for an in-control ARL fixed at 250 and the autoregressive parameter $\phi = 0.75,$ with 80% of process variability due to variation in μ_t, and the correlation between adjacent observations equal to 0.6. We use the same procedure to calculate the ARL for this table as for Table 1.

Comparing these tables, it can be seen that at a low level of autocorrelation, the charts detect small shifts in the parameters more quickly than at a high level of autocorrelation. The scheme is slightly more sensitive to shifts in the standard deviation due to shifts in σ_ε than it is to shifts in the process standard deviation resulting from shifts in σ_α. This is due to the fact that an increase in σ_α increases the level of correlation between adjacent observations. This is because the variance of μ_t increases, and thus the proportion of total process variability due to variation in the autocorrelated mean increases.

An increase in σ_ε decreases the level of correlation between observations. This is particularly evident at higher levels of autocorrelations. This improves the performance of the MEWMAP chart. As would be expected, the chart detects small shifts with small values of λ, and large shifts are quickly detected when using large values of λ. When $\lambda = 1,$ the EWMA chart is equivalent to the Shewhart chart, where all the weight is given to the current observation and L = 3.2359 which gives an UCL = 3.0899, which is the same as the UCL for the Shewhart-type Max-chart proposed by Chen & Cheng [3].

5. COMPARISON WITH OTHER CHARTS

In this section we compare the MEWMAP chart with a single CUSUM chart for autocorrelate processes (Thaga [15]). These charts' parameters are adjusted so that their in-control ARLs are equal to 250. These charts are compared in Tables 3 and 4. We compare these charts for the autocorrelated processes when the values of ϕ – the correlation between μ_t and μ_{t-1} – are 0.25 and 0.75, with the proportion of variation in the process attributed to variation in μ_t fixed at 0.8.

$ARL_0 = 250$ *A*									
b	Parameter	0.00	0.25	0.50	1.00	1.50	2.00	2.50	3.00
1.00	L	2.9127	2.9241	2.9657	2.9883	2.9984	3.2539	3.2539	3.2539
	λ	0.1024	0.1265	0.2206	0.6206	0.9972	1.0000	1.0000	1.0000
	ARL σ_α	250.38	31.03	17.50	8.18	4.07	3.92	3.55	2.25
	ARL σ_ε	250.38	31.03	17.50	8.18	4.07	3.92	3.55	2.25
1.25	L	2.9121	2.9435	2.9764	2.9943	2.9981	2.9993	3.2539	3.2539
	λ	0.1443	0.1465	0.2456	0.6673	0.8854	0.9997	1.0000	1.0000
	ARL σ_α	49.60	22.79	11.04	8.87	3.76	2.69	2.05	1.60
	ARL σ_ε	38.87	20.60	9.80	7.13	3.36	2.62	1.97	1.40
1.50	L	2.9432	2.9432	2.9552	2.9764	2.9981	2.9993	2.9998	3.2539
	λ	0.1462	0.1462	0.2918	0.5692	0.8854	0.9997	0.9999	1.0000
	ARL σ_α	32.08	14.46	7.39	4.59	3.13	2.90	2.02	1.53
	ARL σ_ε	22.67	11.92	6.39	3.88	2.64	2.43	1.90	1.41
2.00	L	2.9432	2.9432	2.9552	2.9764	2.9981	2.9993	2.9998	3.2539
	λ	0.1462	0.1462	0.2918	0.5692	0.8854	0.9997	0.9999	1.0000
	ARL σ_α	18.99	10.92	5.94	3.56	2.44	2.25	1.93	1.54
	ARL σ_ε	12.77	8.64	4.82	2.77	1.98	1.84	1.81	1.43
2.50	L	2.9432	2.9432	2.9552	2.9764	2.9981	2.9993	2.9998	3.2539
	λ	0.1462	0.1462	0.2918	0.5692	0.8854	0.9997	0.9999	1.0000
	ARL σ_α	13.85	9.12	5.08	2.95	2.08	1.93	1.89	1.57
	ARL σ_ε	12.07	8.22	3.97	2.54	1.82	1.75	1.66	1.38
3.00	L	2.9432	2.9432	2.9552	2.9764	2.9981	2.9993	2.9998	2.9998
	λ	0.1462	0.1462	0.2918	0.5692	0.8854	0.9997	0.9999	0.9999
	ARL σ_α	11.11	8.02	4.45	2.54	1.86	1.74	1.60	1.56
	ARL σ_ε	6.99	5.97	3.18	1.92	1.59	1.54	1.47	1.33
4.00	L	2.9432	2.9432	2.9552	2.9764	2.9981	2.9993	2.9998	2.9998
	λ	0.1462	0.1462	0.2918	0.5692	0.8854	0.9999	0.9999	0.9999
	ARL σ_α	8.01	6.56	3.52	2.07	1.60	1.51	1.51	1.47
	ARL σ_ε	4.73	4.41	2.42	1.60	1.34	1.30	1.30	1.30

Table 2: (λ, L) combinations and the corresponding ARL for the MEWMAP chart, with $\phi = 0.75$ and $\psi = 0.8$.

The conclusion that can be made is that the Max-CUSUM chart performs better than the Max-EWMA chart for small shifts in the process mean and/or standard deviation at both low and high levels of autocorrelations, while the Max-EWMA outperforms the Max-CUSUM for moderate to large shifts in the process mean and/or standard deviation. Both schemes are more sensitive when change in process variability is due to change in σ_ε than they are to changes in process variability due to changes in σ_α.

		A							
		0		0.5		1		2	
b	Parameter	MEWP	MCAP	MEWP	MCAP	MEWP	MCAP	MEWP	MCAP
1.00	σ_ε^2 σ_α^2	250.20 250.20	250.76 250.76	16.02 16.02	10.16 10.16	7.51 7.51	4.99 4.99	2.01 2.01	1.97 1.97
1.25	σ_ε^2 σ_α^2	21.53 22.41	13.41 13.70	7.10 7.22	9.25 9.29	5.30 5.42	4.75 4.76	2.01 2.01	2.0 2.1
1.50	σ_ε^2 σ_α^2	8.60 9.15	10.76 11.12	5.45 5.58	8.76 8.84	2.92 2.99	4.57 4.60	1.68 1.71	2.12 2.12
2.00	σ_ε^2 σ_α^2	3.67 4.75	7.79 8.11	2.70 2.78	7.76 7.90	2.11 2.16	4.21 4.26	1.50 1.52	2.14 2.14
3.00	σ_ε^2 σ_α^2	2.63 2.75	5.27 5.48	1.90 1.96	6.14 6.32	1.58 1.61	3.62 3.68	1.31 1.36	2.14 2.14

MEWP: *Max-EWMA Chart for Autocorrelated Processes*.
MCAP: *Max-CUSUM Charts for Autocorrelated Processes*.

Table 3: Comparison of the MEWMAP chart with the MCAP chart with $\phi = 0.25$ and $\psi = 0.8$.

		a							
		0		0.5		1		2	
b	Parameter	MEWP	MCAP	MEWP	MCAP	MEWP	MCAP	MEWP	MCAP
1.00	σ_ε^2 σ_α^2	250.38 250.38	250.14 250.14	17.50 17.50	14.65 14.65	8.18 8.18	6.57 6.57	3.92 3.92	2.78 2.78
1.25	σ_ε^2 σ_α^2	38.87 49.60	25.36 37.45	9.80 11.04	12.84 13.60	7.13 7.87	6.08 6.30	2.62 2.69	2.74 2.76
1.50	σ_ε^2 σ_α^2	22.67 32.08	16.36 28.02	6.38 7.39	11.58 13.05	3.88 4.59	5.68 6.15	2.43 2.90	2.69 2.75
2.00	σ_ε^2 σ_α^2	12.77 18.99	9.62 17.74	4.82 5.94	9.45 11.82	2.77 3.56	4.95 5.77	1.84 2.25	2.60 2.71
3.00	σ_ε^2 σ_α^2	6.99 11.11	5.63 9.80	3.18 4.45	6.75 9.51	1.92 2.54	3.97 4.98	1.54 1.74	2.45 2.60

MEWP: *Max-EWMA Chart for Autocorrelated Processes*.
MCAP: *Max-CUSUM Charts for Autocorrelated Processes*.

Table 4: Comparison of the MEWMAP chart with the MCAP chart with $\phi = 0.75$ and $\psi = 0.8$.

6. CHARTING PROCEDURES

Since the residuals are independent normal random variables when the process is in control, the charting procedure for the Max-EWMA chart for Autocorrelated Process is similar to that of the Max-EWMA for independent observations (Chen, Cheng & Xie [5]). The successive EWMA values M_i are plotted against the sample numbers. If a point plots below the UCL, the process is said to be in control and the point is plotted as a dot. An out-of-control signal is given if any point plots above the UCL and is plotted as one of the characters defined below.

The charting procedure for the MEWMAP chart is as follows:

1. Fit the time series model into the data.

2. Find the (λ, L) combination for a given in-control ARL, ξ_0 and $\sigma_{\gamma 0}$.

3. If ξ_0 is not known, use the sample grand average $\bar{\bar{\xi}}$ to estimate it, where $\bar{\bar{\xi}}$ = $(\bar{\xi}_1 + \bar{\xi}_2 + \ldots + \bar{\xi}_m)/m$. If $\sigma_{\gamma 0}$ is unknown, use $\bar{R}/d_2$, where $\bar{R}$ = $(R_1 + R_2 + \ldots + R_m)/m$ is the average of the sample ranges. We can also use $\bar{S}/c_4$ to estimate $\sigma_{\gamma 0}$, where $\bar{S}$ = $(S_1 + S_2 + \ldots + S_m)/m$ is the average of the sample standard errors, $S_i = \sqrt{MSE_i}$, and d_2 and c_4 are statistically determined constants.

4. For each sample, compute Z_i and Y_i.

5. Compute U_i, V_i, M_i and the UCL.

6. Denote the sample points with a dot and plot them against the sample number if $M_i \leq$ UCL.

7. If any of the M_is is greater than the UCL, the following plotting characters should be used to show the direction as well as the statistic that is plotting above the interval.

   (i) If $|U_i| > UCL$ and $U_i > 0$, plot C+. This shows an increase in the process mean.

   (ii) If $|U_i| > UCL$ and $U_i < 0$, plot C-. This shows a decrease in the process mean.

   (iii) If $|V_i| > UCL$ and $V_i > 0$, plot S+. This shows an increase in the process standard deviation.

   (iv) If $|V_i| > UCL$ and $V_i < 0$, plot S-. This shows a decrease in the process standard deviation.

   (v) If $|U_i| > UCL$ and $|V_i| > UCL$, plot B++ if $U_i > 0$ and $V_i > 0$. This indicates an increase in both the mean and the standard deviation of the process.

(vi) If $|U_i| > UCL$ and $|V_i| > UCL$, plot B+- if $U_i > 0$ and $V_i < 0$. This indicates an increase in the mean and a decrease in the standard deviation of the process.

(vii) If $|U_i| > UCL$ and $|V_i| > UCL$, plot B-+ if $U_i < 0$ and $V_i > 0$. This indicates a decrease in the mean and an increase in the standard deviation of the process.

(viii) If $|U_i| > UCL$ and $|V_i| > UCL$, plot B--if $U_i < 0$ and $V_i < 0$. This shows a decrease in both the mean and the standard deviation of the process.

8. Once an out of control signal is given by the chart plotting outside the control limit, the process should be stopped, an investigation of the cause(s) of shift for each out-of-control point in the chart should be carried out, and remedial measure(s) needed to bring the process back into an in-control state should be implemented.

7. APPLICATION IN INDUSTRY

Observations from processes in the chemical and pharmaceutical industries are frequently autocorrelated. The standard Shewhart chart, the cumulative sum chart, and the exponentially weighted moving average chart are effective in detecting shifts in the process when measurements are independent and identically distributed over time. The proposed chart is an improved version of the traditional EWMA chart. It has an added advantage of being able to detect shifts in the process when the measurements are not independent over time.

The new chart can be used to monitor observations from processes in the chemical and pharmaceutical industries where observation comes in batches. In this case a special cause might produce an increase in within-batch variability, in between-batch variability, or in both.

8. AN EXAMPLE

To provide a visual picture of how the MEWMAP chart responds to various kinds of process changes, a set of simulated data is used. Specific process changes are introduced into the data, and the chart is plotted to monitor these changes in the parameters. The data set was generated using the first order autoregressive models in equations (1) and (2).

For a fixed sequence of α_t' s and ε_t' s, a shift in σ_α can be simulated by multiplying α_t in equation (2) by a constant. A change in σ_ε can be simulated by multiplying ε_t in equation (1) by a constant, and a change in the mean is simulated by adding a constant to the generated observations. This approach is discussed in the literature (Lu & Reynolds [7]). This procedure allows different types of process changes to be investigated on the same basic sequence of α_t' s and ε_t' s. In this example, we assume the autoregressive parameter ϕ remains constant.

We simulated 100 observations with the following parameters: ξ = 0, ϕ = 0.75, σ_α = 0.59 and σ_ε =0.5. This gives σ_X = 1.02 and ψ = 0.76. This implies that 76% of variability in the process is due to variation in μ_t and that the correlation between the adjacent observations is ρ = $\phi\,\psi$ = 0.57. Using equations (5) and (6), the corresponding parameters in the ARMA(1,1) model in equation (4) are θ = 0.27 and σ_γ = 0.83.

The MEWMAP chart for these simulated observations is drawn in Figure 1. All points fall within the acceptance region; thus the simulated process is in control. The chart's parameters (λ, L) are for an in-control ARL of 250 runs. The chart is designed to detect a 1σ increase in the process mean and 2σ increase in the process standard deviation.

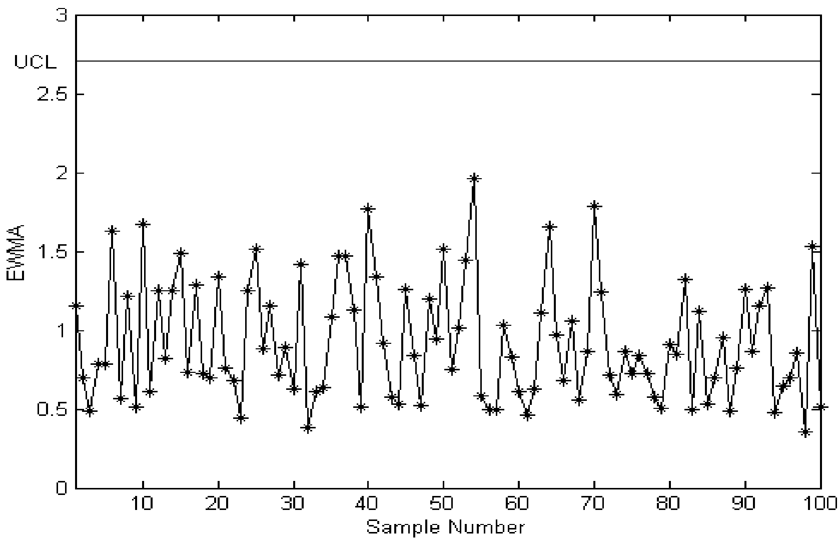

Figure 1: The MEWMAP chart for the in-control simulated values

Figure 2 shows the performance of this chart for a simultaneous shift in the process mean and standard deviation, with a shift in the standard deviation due to an increase in σ_α. The process mean is assumed to shift from 0 to 1, and σ_α increases from 0.59 to 0.97. This increase in σ_α results in an increase in the process standard deviation σ_X from 1.02 to 1.56. This corresponds to a 52% increase in the process standard deviation. This also leads to an increase in ψ from 0.76 to 0.90, and an increase in the correlation between adjacent observations from 0.57 to 0.675.

Therefore 90% of variation in the process is due to variation in μ_t. The increase in σ_α for the last 40 observations was accomplished by multiplying the last 40 values of α_t by the factor 0.97/0.59 = 1.644. Adding 1 to the last 40 observations of the new process observation X_t accomplishes the increase in the process mean for the last 40 observations. These shifts are signaled for the first time on the 67th observation.

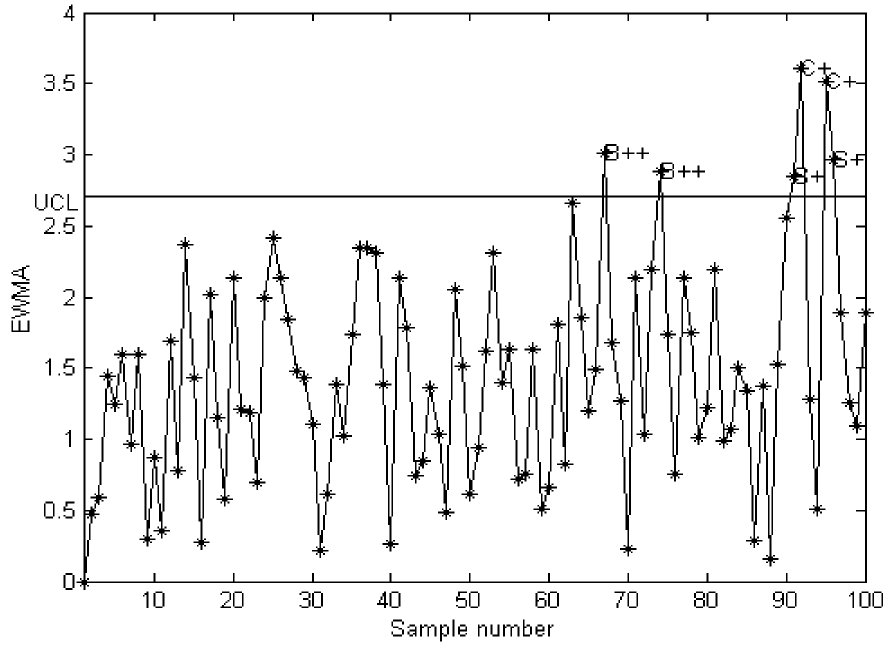

Figure 2: The MEWMAP chart for shift in the process mean and variability due to shift in σ_α.

An increase in both the mean and σ_ε is shown in Figure 3. We consider an increase in mean from 0 to 1 and an increase in σ_ε from 0.5 to 1. Assume that due to some special causes, these shifts occur immediately after the 60th observation and remain in effect for the rest of the process. This increase in σ_ε will result in an increase in the process standard deviation from 1.02 to 1.34, and this represents a 30% increase in the process standard deviation. Unlike the increase in σ_α, the increase in σ_ε results in a decrease in the correlation between adjacent observations from 0.57 to 0.33, and the value of the proportion of total process variability that is due to the variability in μ_t also decreases from 56% to 44%.

The increase in σ_ε for the last 40 observations was accomplished by multiplying the last 40 observations of the simulated ε_t values by 1/0.5 = 2.00, while the increase in the mean is accomplished by adding 1 to the last 40 observations of the new process value X_t. The chart signals a shift for the first time on the 62nd observation for an increase in both parameters.

Comparing the results of the simulated shifts, we realize that this chart quickly detects a simultaneous shift in the process mean and standard deviation with a shift in the process standard deviation due to shift in σ_ε. This is due to the fact that an increase in σ_ε results in a decrease in the correlation between adjacent observations, while an increase in σ_α results in an increase in this correlation structure.

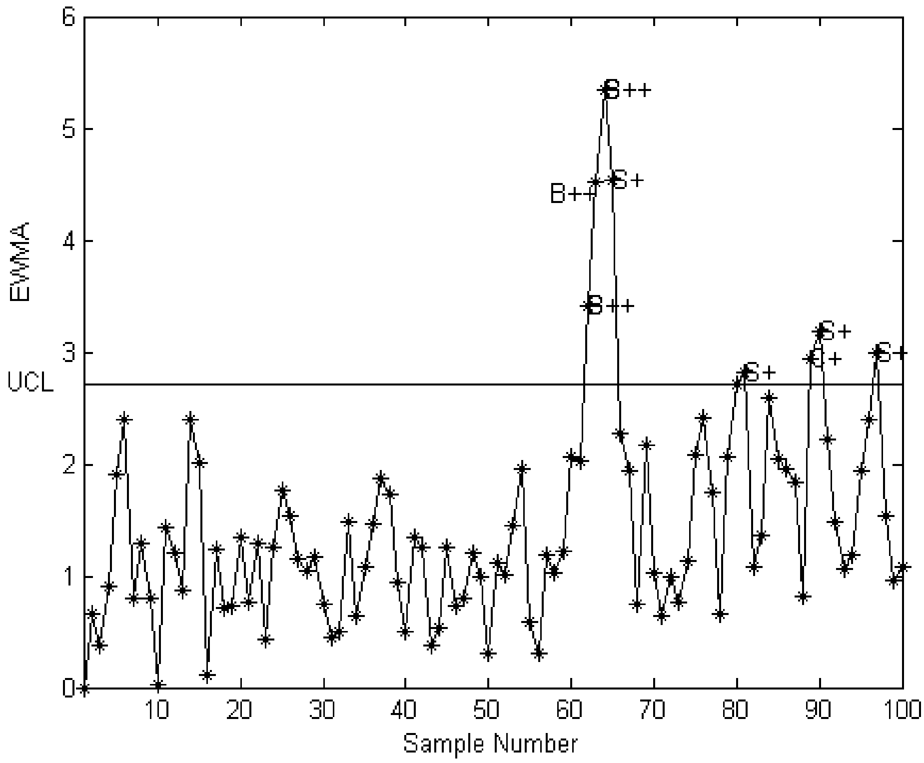

Figure 3: The MEWMAP chart for shift in the process mean and variability due to shift in σ_{ε}.

9. CONCLUSIONS

Although it is very difficult to draw general conclusions based on one set of data corresponding to one set of process parameters, the ARL results given in this article, together with charts plotted in Figures 2 and 3, allow some conclusions to be drawn.

The results reported here have shown that correlation among observations from a process can have significant effect on the performance of a EWMA control chart. Computer simulation of individual data from a first order autoregressive plus a random error model was used to show a pictorial display of the MEWMAP chart. The monitoring problem in this model is very complicated, as it requires more parameters than is the case when the observations are independent. We have shown how a change in one of the two components of residual variances and the process mean impacts on the overall process performance.

However, in many applications, a change in the process may be because of a combination of changes in these parameters. Therefore it becomes very difficult to diagnose the variance component that has caused the process variance to change. It might be necessary to estimate the residual variance at the point of the shift to see which component has shifted.

The MEWMAP chart that simultaneously monitors the process mean and standard deviation performs better than the Max-CUSUM chart for autocorrelated processes for moderate to large shifts in the process parameters. The MEWMAP chart is simple to construct: it uses the standard EWMA parameters because residuals are independent when the process is in-control. We therefore recommend this chart for autocorrelated data. The standard time series procedure discussed in Box, Jenkins & Reinsel [3] can be used to fit the model and calculate the residuals.

10. REFERENCES

[1] **Alwan, L.C. & Roberts, H.V**. (1988). Time series modeling for statistical process control. *Journal of Business and Economic Statistics*, **6**, 87-95.

[2] **Atienza, O.O.; Tang, L.C. & Ang, B.W**. (2002). A CUSUM scheme for autocorrelated observations. *Journal of Quality Technology*, **34**, 188-199.

[3] **Box, G.E.P., Jenkins G.M. & Reinsel, G.C.** (1994, Chap. 3). *Time series analysis, forecasting and control*, 3rd edition. Prentice-Hall, Englewood Cliffs, NJ.

[4] **Chen, G. & Cheng, S.W** (1998). Max chart: Combining X-Bar chart and S chart. *Statistica Sinica* **8**, 263-271.

[5] **Chen, G., Cheng, S.W. & Xie, H.** (2001). Monitoring process mean and variability with one EWMA chart. *Journal of Quality Technology*, **33**, 223-233.

[6] **Koons, B.K. & Foutz, R.V.** (1990). Estimating moving average parameters in the presence of measurement error. *Communications in Statistics – Theory and Methods*, **19**, 3179-3187.

[7] **Lu, C.W. & Reynolds, Jr, M.R.** (1999). Control charts for monitoring the mean and variance of autocorrelated processes. *Journal of Quality Technology*, **31**, 259-274.

[8] **Lu, C.W. & Reynolds, Jr, M.R**. (2001). CUSUM charts for monitoring an autocorrelated process. *Journal of Quality Technology*, **33**, 316-334.

[9] **Montgomery, D.C. & Mastrangelo, C.M**. (1991). Some statistical process control methods for autocorrelated data. *Journal of Quality Technology*, **23**, 179-193.

[10] **Page, E.S**. (1954). Continuous inspection schemes. *Biometrika*, **41**, 100-115.

[11] **Reynolds, Jr, M.R., Arnold, J.C. & Baik, J.W.** (1996). Variable sampling interval $\overline{X}$ charts in the presence of correlation. *Journal of Quality Technology*, **28**, 12-30.

[12] **Roberts, S.W.** (1959). Control charts tests based on geometric moving averages. *Technometrics*, **1**, 239-250.

[13] **Runger, G.C., Willemain, T.R. & Prabhu, S**. (1995). Average run lengths for CUSUM control charts applied to residuals. *Communications in Statistics – Theory and Methods*, **24**, 273-282.

[14] **Shewhart, W.A.** (1931*). Economic control of quality of manufactured product*. Van Nostrand Inc., New York.

[15] **Thaga, K.** (2003). Contributions to statistical quality control tools. PhD. thesis, University of Manitoba, Winnipeg, Canada.

[16] **VanBrackle,** III, L.N. & Reynolds, Jr, M.R. (1997). EWMA and CUSUM control charts in the presence of correlation. *Communications in Statistics – Simulation*, **26**, 979-1008.

[17] **Wardell, D.G., Moskowitz, H. & Plante, R.D.** (1994). Run-length distributions of special-cause control charts for correlated processes. *Technometrics*, **36**, 3-17.
[18] **Xie, H.** (1999). Contributions to qualimetry. PhD. Thesis, University of Manitoba, Winnipeg, Canada
[19] **Yashchin, E.** (1993). Performance of CUSUM control schemes for serially correlated observations. *Technometrics*, **35**, 37-52.