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Common fixed points in partially ordered modular function spaces

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Abstract
The purpose of this paper is to study the existence and uniqueness of common fixed point results in partially ordered modular function spaces.
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1 Introduction

Study of modular spaces was initiated by Nakano [1] in connection with the theory of order spaces which was further generalized by Musielak and Orlicz [2]. The study of fixed points of mappings on complete metric spaces equipped with a partial ordering $\preceq$ was first investigated in 2004 by Ran and Reurings [3], and then by Nieto and Rodriguez-Lopez [4]. They applied their results to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions (see also [5]). The study of this theory in the context of modular function spaces was initiated by Khamsi *et al.* [6] (see also [7] and [8]). Kuaket and Kumam [9] and Mongkolkeha and Kumam [10–12], considered and proved some fixed point and common fixed point results for generalized contraction mappings in modular spaces. Also, Kumam [13] obtained some fixed point theorems for non-expansive mappings in arbitrary modular spaces. Recently, Kutabi and Latif [14] studied fixed points of multivalued maps in modular function spaces.

The study of common fixed points of mappings satisfying certain contractive conditions in the setup of partially ordered metric spaces can be employed to establish the existence of solutions of many types of operator equations, such as differential and integral equations. There are a few examples given in the following papers: [15–20] and references mentioned therein. The objective of this paper is to initiate the study of common fixed point results in partially ordered modular function spaces. As an application of our results, we study the property Q for mappings involved herein.

2 Preliminaries

Some basic facts and notations about modular spaces are recalled from [21].

Definition 2.1 Let X be a real (or complex) vector space. A functional $\rho : X \to [0,\infty]$ is called modular if, for any x, y in X, the following hold:

(m_1) $\rho(x) = 0$ if and only if $x = 0$.
(m_2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$.
(m_3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ provided that $\alpha + \beta = 1$, and $\alpha, \beta \ge 0$.

If (m_3) is replaced by $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$, and $\alpha, \beta \geq 0$, then ρ is called convex modular.

The vector space X_ρ given by

$$X_\rho = \{x \in X; \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

is called a modular space. Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance.

Modular space X_ρ can be equipped with an F-norm defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \leq \alpha\right\}.$$

If ρ is convex modular, then

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}$$

defines a norm on the modular space X_ρ and is called the Luxemburg norm.

Define the ρ-ball, centered at $x \in X_\rho$ with radius r, as

$$B_\rho(x, r) = \{h \in X\rho; \rho(x - h) \leq r\}.$$

Definition 2.2 A function modular is said to satisfy Δ_2-type condition, if there exists $K > 0$ such that for any $x \in X_\rho$, we have $\rho(2x) \leq K\rho(x)$.

Definition 2.3 ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 2.4 Let X_ρ be a modular space. The sequence $\{x_n\} \subset X_\rho$ is called:

(t_1) ρ-convergent to $x \in X_\rho$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(t_2) ρ-Cauchy, if $\rho(x_n - x_m) \to 0$ as n and $m \to \infty$.

Note that ρ-convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality. In fact, one can show that this will happen if and only if ρ satisfies the Δ_2-type condition.

It is well known that [6, 22] under the Δ_2-condition the norm convergence and modular convergence are equivalent. The same is true when we deal with the Δ_2-type condition. Throughout this paper, we assume that modular function ρ is convex and satisfies the Δ_2-type condition. We also state the following definition and results given in [7].

Definition 2.5 The growth function w_ρ of a function modular ρ is defined as

$$w_\rho(t) = \sup\left\{\frac{\rho(tx)}{\rho(x)}, x \in X_\rho \setminus \{0\}\right\} \quad \text{for all } 0 \leq t < \infty.$$

Observe that $w_\rho(t) \leq 1$ for all $t \in [0, 1]$.

Lemma 2.6 *The growth function ω has the following properties*:

(g_1) $\omega(t) < \infty$, *for each* $t \in [0, \infty)$.
(g_2) $\omega : [0, \infty) \to [0, \infty)$ *is a convex, strictly increasing function. So, it is continuous.*
(g_3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta)$; *for all* $\alpha, \beta \in [0, \infty)$.
(g_4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta)$; *for all* $\alpha\beta \in [0, \infty)$, *where* ω^{-1} *is the inverse function of* ω.

The following lemma shows that the growth function can be used to give an upper bound for $\|x\|_\rho$ for each $x \in X_\rho$.

Lemma 2.7 *Let ρ be a convex modular function satisfying the Δ_2-type condition. Then*

$$\|x\|_\rho \leq \frac{1}{\omega^{-1}(\frac{1}{\rho(x)})},$$

whenever $x \in X_\rho$.

Let S and T be two self-maps on a modular function space X_ρ. A point $x \in X_\rho$ is called (1) a fixed point of S if $S(x) = x$; (2) a *coincidence point of a pair* (S, T) if $Sx = Tx$; (3) a *common fixed point of a pair* (S, T) if $x = Sx = Tx$. If $w = Sx = Tx$ for some x in X_ρ, then w is called a point of coincidence of S and T.

The pair (S, T) is said to be compatible if $\rho(STx_n - TSx_n) \to 0$ as $n \to \infty$, whenever $\{x_n\}$ is a sequence in X such that $\{Sx_n\}$ and $\{Tx_n\}$ are ρ-convergent to $t \in X_\rho$.

A pair (S, T) is said to be ρ-weakly compatible if S and T commute at their coincidence points.

We denote the set of fixed points of S by $\mathrm{Fix}(S)$.

Definition 2.8 Let $(X_\rho, \preccurlyeq)$ be a partially modular ordered space. A pair (T_1, T_2) of self-maps of X_ρ is said to be ρ-weakly increasing if $T_1(f) \leq T_2T_1(f)$ and $T_2(f) \leq T_1T_2(f)$ for all $f \in X_\rho$.

Definition 2.9 Let $(X_\rho, \preccurlyeq)$ be a partially modular ordered space and T_1, T_2 be two self-maps on X_ρ. An order pair (T_1, T_2) is said to be partially ρ-weakly increasing if $T_1(f) \leq T_2T_1(f)$ for all $f \in X_\rho$.

The pair (T_1, T_2) is ρ-weakly increasing if and only if the ordered pairs (T_1, T_2) and (T_2, T_1) are partially ρ-weakly increasing.

Definition 2.10 Let $(X_\rho, \preccurlyeq)$ be a partially modular ordered space. A mapping T_1 is said to be ρ-weak annihilator of T_2 if $T_1T_2(f) \leq f$ for all $f \in x_\rho$.

Definition 2.11 Let $(X_\rho, \preccurlyeq)$ be a partially ordered modular space. A mapping T_1 is said to be ρ-dominating if $f \leq T_1f$ for all $f \in X_\rho$.

Definition 2.12 Let $(X_\rho, \preccurlyeq)$ be a partially modular ordered space and T_1, T_2, T_3 be three self-maps on X_ρ, such that $T_1X_\rho \subseteq T_3X_\rho$ and $T_2X_\rho \subseteq T_3X_\rho$. We say that T_1 and T_2 are ρ-weakly increasing with respect to T_3 if and only if for all $f \in X_\rho$, we have $T_1f \leq T_2g$ for all $g \in T_3^{-1}(T_1f)$, and $T_2f \leq T_1g$, for all $g \in T_3^{-1}(T_2f)$, where $T_3^{-1}(f) = \{h \in X_\rho \mid T_3h = f\}$ for all $f \in X_\rho$.

Definition 2.13 Let $(X_\rho, \preccurlyeq)$ be a partially modular ordered space and T_1, T_2 ,T_3 be three self-maps on X_ρ such that $T_1 X_\rho \subseteq T_3 X_\rho$. We say that T_1 and T_2 are partially ρ-weakly increasing with respect to T_3 if for all $f \in X_\rho$, we have $T_1 f \preceq T_2 g$, for all $g \in T_3^{-1}(T_1 f)$.

Definition 2.14 Let X be a vector space. Then $(X, \preceq, \rho)$ is called an ordered modular function space iff: (i) ρ is convex modular function on X and (ii) $\preceq$ is a partial order on X.

Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

3 Common fixed point results

We begin with a common fixed point theorem for two pairs of partially weakly increasing functions on an ordered modular function spaces. It may regarded as the main result of this article.

Theorem 3.1 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space and S, I, T, and J self-maps on X_ρ such that $S(X_\rho) \subseteq J(X_\rho)$ and $I(X_\rho) \subseteq T(X_\rho)$. Suppose that (J, S) and (I, T) are ρ-weakly increasing, and the dominating maps S and T are weak annihilators of J and I, respectively. If for every two comparable elements $f, g \in X_\rho$*

$$\rho(Sf - Tg) \leq \alpha\rho(If - Jg) \tag{3.1}$$

is satisfied, then S, I, T, and J have a common fixed point provided that for a non-decreasing sequence $\{f_n\}$ with $f_n \preceq g_n$ for all n and $g_n \to g$ implies that $f_n \preceq g$ and either

(a) *$\{S, I\}$ are ρ-compatible, S or I is ρ-continuous and $\{T, J\}$ are ρ-weakly compatible*;

(b) *$\{T, J\}$ are ρ-compatible, T or J is ρ-continuous and $\{S, I\}$ are ρ-weakly compatible.*

Moreover, the set of common fixed points of S, I, T, and J is well ordered if and only if S, I, T, and J have one and only one common fixed point.

Proof (a) Let $f_0 \in X_\rho$. Construct sequences f_n and g_n in X_ρ, such that $g_{2n-1} = Sf_{2n-2} = Jf_{2n-1}$ and $g_{2n} = Tf_{2n-1} = If_{2n}$. This can be done because $S(X_\rho) \subseteq J(X_\rho)$ and $I(X_\rho) \subseteq T(X_\rho)$. Since S is a ρ-dominating map and the pair (J, S) is partially ρ-weakly increasing so $f_{2n-2} \preceq Sf_{2n-2} = Jf_{2n-1} \preceq S(Jf_{2n-1})$. Also, S is a ρ-weak annihilator of J so $f_{2n-2} \preceq Sf_{2n-2} = Jf_{2n-1} \preceq S(Jf_{2n-1}) \preceq f_{2n-1}$. This implies that $f_{2n-2} \preceq f_{2n-1}$. Since T is a dominating map, $f_{2n-1} \preceq Tf_{2n-1} = If_{2n}$. As (I, T) is a pair of partially weakly increasing mappings, $If_{2n} \preceq T(If_{2n})$ and $f_{2n-1} \preceq Tf_{2n-1} = If_{2n} \preceq T(If_{2n})$. Also T is a weak annihilator of I, so we have $f_{2n-1} \preceq Tf_{2n-1} = If_{2n} \preceq T(If_{2n}) \preceq f_{2n}$. This implies that $f_{2n-1} \preceq f_{2n}$. Hence for all $n \geq 1$ we have $f_n \preceq f_{n+1}$. Suppose that $\rho(g_{2n} - g_{2n+1}) > 0$ for every n. If not, then $\rho(g_{2n} - g_{2n+1}) = 0$ implies that $g_{2n} - g_{2n+1} = 0$, that is, $g_{2n} = g_{2n+1}$ for some n. Now from inequality (3.1) we have

$$\rho(g_{2n+1} - g_{2n+2}) = \rho(Sf_{2n} - Tf_{2n+1}) \leq \alpha\rho(If_{2n} - Jf_{2n+1}) = \alpha\rho(g_{2n} - g_{2n+1})$$

and therefore $\rho(g_{2n+1} - g_{2n+2}) = 0$. So $g_{2n+1} = g_{2n+2}$ and so on. Thus $\{g_n\}$ becomes a constant sequence and g_{2n} is a required common fixed point of given mappings. Assume that $\rho(g_{2n+1} - g_{2n+2}) \neq 0$ for each n. From (3.1), we obtain

$$\begin{aligned}\rho(g_{2n+1} - g_{2n+2}) &= \rho(Sf_{2n} - Tf_{2n+1}) \\ &\leq \alpha\rho(If_{2n} - Jf_{2n+1}) = \alpha\rho(g_{2n} - g_{2n+1})\end{aligned}$$

$$\begin{aligned}
&= \alpha\rho(Sf_{2n-1} - Tf_{2n}) \\
&\leq \alpha^2\rho(If_{2n-1} - Jf_{2n}) = \alpha^2\rho(g_{2n-1} - g_{2n}) \\
&= \alpha^2\rho(Sf_{2n-2} - Tf_{2n-1}) \leq \alpha^3\rho(If_{2n-2} - Jf_{2n-1}) \\
&= \alpha^3(g_{2n-2} - g_{2n-1}) = \alpha^3\rho(Sf_{2n-3} - Tf_{2n-2}) \\
&\leq \alpha^4\rho(If_{2n-3} - Jf_{2n-2}) = \alpha^4\rho(g_{2n-3} - g_{2n-2}).
\end{aligned}$$

Inductively, we have $\rho(g_{2n+1} - g_{2n+2}) \leq \alpha^n\rho(g_{n+1} - g_{n+2})$, which implies that

$$\frac{1}{\alpha^n\rho(g_{n+1} - g_{n+2})} \leq \frac{1}{\rho(g_{2n+1} - g_{2n+2})}.$$

Using Lemma 2.7, we have

$$\|g_{2n+1} - g_{2n+2}\|_\rho \leq \frac{1}{\omega^{-1}(\frac{1}{g_{2n+1}-g_{2n+2}})}$$

and

$$\omega^{-1}\left(\frac{1}{\alpha^n\rho(g_{n+1} - g_{n+2})}\right) \leq \omega^{-1}\left(\frac{1}{\rho(g_{2n+1} - g_{2n+2})}\right).$$

Employing the properties of the growth function, we obtain

$$\omega^{-1}\left(\frac{1}{\alpha}\right)^n\omega^{-1}\left(\frac{1}{\rho(g_{n+1} - g_{n+2})}\right) \leq \omega^{-1}\left(\frac{1}{\rho(g_{2n+1} - g_{2n+2})}\right),$$

which implies that

$$\begin{aligned}
\|g_{2n+1} - g_{2n+2}\|_\rho &\leq \frac{1}{\omega^{-1}(\frac{1}{\alpha})^n\omega^{-1}(\frac{1}{\rho(g_{n+1}-g_{n+2})})} \\
&= \frac{1}{[\omega^{-1}(\frac{1}{\alpha})]^n\omega^{-1}(\frac{1}{\rho(g_{n+1}-g_{n+2})})}.
\end{aligned}$$

As $\omega(1) = 1$, and $\alpha < 1$ so $1 < \omega^{-1}(\frac{1}{\alpha})$, and $\frac{1}{\omega^{-1}(\frac{1}{\alpha})} < 1$. This shows that $\{g_{2n}\}$ is a Cauchy sequence in $(X_\rho, \|\cdot\|_\rho)$. There exists $h \in X_\rho$ such that $\|g_{2n} - h\|_\rho \to 0$. That is, the sequence $\{g_{2n}\}$ is norm convergent to $h \in X_\rho$. Since the Δ_2-condition implies equivalence of norm and modular convergence, $\{g_{2n}\}$ is modular convergent to $h \in X_\rho$. Therefore $\{g_{2n}\} \xrightarrow{\rho} h$. Thus, we have $h = \lim_{n\to\infty} g_{2n+1} = \lim_{n\to\infty} Jf_{2n+1} = \lim_{n\to\infty} Sf_{2n}$ and $h = \lim_{n\to\infty} g_{2n+2} = \lim_{n\to\infty} Tf_{2n+1} = \lim_{n\to\infty} If_{2n+2}$. Assume that I is continuous. Since $\{S, I\}$ are ρ-compatible, we have $\lim_{n\to\infty} SIf_{2n+2} = \lim_{n\to\infty} ISf_{2n+2} = Ih$. As T is a ρ-dominating map, $f_{2n+1} \leq Tf_{2n+1} = If_{2n+2}$, that is, $f_{2n+1} \leq If_{2n+2}$. Therefore we have

$$\rho(SIf_{2n+2} - If_{2n+2}) = \rho(SIf_{2n+2} - Tf_{2n+1}) \leq \alpha\rho(IIf_{2n+2} - Jf_{2n+1}),$$

which on taking the limit as $n \to \infty$ gives $\rho(Ih - h) \leq \alpha\rho(Ih - h)$. That is, $(1-\alpha)\rho(Ih - h) \leq 0$. As $\alpha < 1$, so $\rho(Ih - h) \leq 0$ implies that $Ih = h$. Since T is ρ-dominating, $f_{2n+1} \leq Tf_{2n+1}$. Also,

$Tf_{2n+1} = g_{2n+2} \to h$ as $n \to \infty$ implies that $f_{2n+1} \preceq h$. So, we have

$$\begin{aligned}\rho(Sh - g_{2n+2}) &= \rho(Sh - If_{2n+2}) = \rho(Sh - Tf_{2n+1}) \\ &\le \alpha\rho(Ih - Jf_{2n+1}) = \alpha\rho(Ih - g_{2n+1}).\end{aligned}$$

On taking the limit as $n \to \infty$, we obtain

$$\rho(Sh - h) \le \alpha\rho(Ih - h) = 0.$$

Hence $\rho(Sh - h) = 0$ and $Sh = h$. As $S(X_\rho) \subseteq J(X_\rho)$, there exists a point $k \in X_\rho$ such that $Sh = Jk$. Suppose that $Tk \neq Jk$. Since S is ρ-dominating, (J, S) is partially ρ-weakly increasing, and S is a ρ-weak annihilator of J, so we have $h \preceq Sh = Jk \preceq SJk \preceq k$, that is, $h \preceq k$. Thus, we have

$$\begin{aligned}\rho(h - Tk) &= \rho(Jk - Tk) = \rho(Sh - Tk) \\ &\le \alpha\rho(Ih - Jk) = \alpha\rho(h - Jk) = 0\end{aligned}$$

giving $h = Tk$. Since $\{T, J\}$ are ρ-weakly compatible, $Th = TSh = TJk = JTk = Jh$. Thus h is a coincidence point of T and J. As S is a ρ-dominating map, $f_{2n} \preceq Sf_{2n}$. Now $Sf_{2n} \to h$ as $n \to \infty$ implies that $f_{2n} \preceq h$. Now from (3.1), we have

$$\rho(Sf_{2n} - Th) \le \alpha\rho(If_{2n} - Jh),$$

which, on taking the limit as $n \to \infty$, gives

$$\rho(h - Th) \le \alpha\rho(h - Jh) = \alpha\rho(h - Th),$$
$$\rho(h - Th) \le \alpha\rho(h - Th)$$

or $(1 - \alpha)\rho(h - Th) \le 0$, as $\alpha < 1$, so $h = Th$. Thus, $Sh = Ih = Th = Jh = h$. That is, h is a common fixed point of S, T, I, and J.

(b) Similarly the result follows when (b) holds.

Now suppose that the set of common fixed points of S, I, T, and J is well ordered. We claim that the common fixed point of S, I, T, and J is unique. Assume to the contrary that these maps have two common fixed points u and v, that is,

$$Su = Iu = Tu = Ju = u \quad \text{and} \quad Sv = Iv = Tv = Jv = v.$$

From inequality (3.1), we have

$$\rho(u - v) = \rho(Su - Tv) \le \alpha\rho(Iu - Jv) = \alpha\rho(u - v).$$

Thus

$$(1 - \alpha)\rho(u - v) \le 0 \text{ implies that } \rho(u - v) = 0, \text{which further implies that } u = v.$$

Hence uniqueness is proved. The converse is straightforward. □

Corollary 3.2 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space and S, I, and J self-maps on X such that $S(X) \subseteq J(X)$, and $I(X) \subseteq S(X)$. Suppose that (J, S) and (I, S) are partially ρ-weakly increasing and the dominating map S is a weak annihilator of J and I. If for every two comparable elements $f, g \in X$*

$$\rho(Sf - Sg) \leq \alpha\rho(If - Jg)$$

is satisfied, then S, I, and J have a common fixed point provided that for a non-decreasing sequence $\{f_n\}$ with $f_n \leq g_n$ for all n and $g_n \to g$ implies that $f_n \leq g$ and either

(a) *$\{S, I\}$ are ρ-compatible, S or I is ρ-continuous and $\{S, J\}$ are ρ-weakly compatible;*

(b) *$\{S, J\}$ are ρ-compatible, S or J is ρ-continuous and $\{S, I\}$ are ρ-weakly compatible.*

Moreover, the set of the common fixed points of S, I, and J is well ordered if and only if S, I, and J have one and only one common fixed point.

Corollary 3.3 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space and S, T, and J self-maps on X such that $S(L_\rho) \subseteq J(L_\rho)$ and $J(L_\rho) \subseteq T(L_\rho)$. Suppose that (J, S) and (J, T) are partially ρ-weakly increasing, and the dominating maps S and T are weak annihilators of J. If for every two comparable elements $f, g \in L_\rho$*

$$\rho(Sf - Tg) \leq \alpha\rho(Jf - Jg)$$

is satisfied, then S, T, and J have a common fixed point provided that for a non-decreasing sequence $\{f_n\}$ with $f_n \leq g_n$ for all n and $g_n \to g$ implies that $f_n \leq g$ and either

(a) *$\{S, J\}$ are ρ-compatible, S or J is ρ-continuous and $\{T, J\}$ are ρ-weakly compatible;*

(b) *$\{T, J\}$ are ρ-compatible, T or J is ρ-continuous and $\{S, J\}$ are ρ-weakly compatible.*

Moreover, the set of common fixed points of S, T, and J is well ordered if and only if S, T, and J have one and only one common fixed point.

Corollary 3.4 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space, S and J self-maps on X such that $S(L_\rho) \subseteq J(L_\rho)$ and $J(L_\rho) \subseteq S(L_\rho)$. Suppose that (J, S) is a partially ρ-weakly increasing and the dominating map S is a weak annihilator of J. If for every two comparable elements $f, g \in L_\rho$*

$$\rho(Sf - Sg) \leq \alpha\rho(Jf - Jg)$$

is satisfied, then S and J have a common fixed point provided that for a non-decreasing sequence $\{f_n\}$ with $f_n \leq g_n$ for all n and $g_n \to g$ it is implied that $f_n \leq g$ and either $\{S, J\}$ are ρ-compatible, S or J is ρ-continuous and $\{S, J\}$ are ρ-weakly compatible. Moreover, the set of common fixed points of S and J is well ordered if and only if S and J have one and only one common fixed point.

Theorem 3.5 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space and S, T, I, and J continuous self-maps on X_ρ. Suppose that (S, I) and (T, J) are ρ-compatible, (S, T) and (T, S) are ρ-partially weakly increasing with respect to J and I, respectively, and*

$$\rho(Sf - Tg) \leq \alpha\rho(If - Jg)$$

holds for every $f, g \in X_\rho$ for which If and Jg are comparable. Then the pairs (S, I) and (T, J) have a coincidence point $h \in X$. Moreover if Ih and Jh are comparable, then $h \in X$ is a coincidence point of S, T, I, and J.

Proof Let f_0 be an arbitrary point in X. Construct the sequences $\{f_n\}$ and $\{g_n\}$ in X such that $g_{2n} = Sf_{2n} = Jf_{2n+1}$ and $g_{2n+1} = Tf_{2n+1} = If_{2n+2}$. As (S, T) is ρ-partially weakly increasing with respect to J, from $f_{2n+1} \in J^{-1}(Sf_{2n})$ we have

$$Jf_{2n+1} = Sf_{2n} \preceq Tf_{2n+1} = If_{2n+2}.$$

Since (T, S) is ρ-partially weakly increasing with respect to I, from $f_{2n+2} \in I^{-1}(Tf_{2n+1})$, we have

$$If_{2n+2} = Tf_{2n+1} \preceq Sf_{2n+2} = Jf_{2n+3}.$$

Hence $Jf_1 \preceq If_2 \preceq Jf_3 \preceq \cdots \preceq Jf_{2n+1} \preceq If_{2n+2} \preceq Jf_{2n+3} \preceq \cdots$, that is, $g_0 \preceq g_1 \preceq g_2 \preceq \cdots \preceq g_{2n} \preceq g_{2n+1} \preceq g_{2n+2} \cdots$. Following similar arguments to those given in Theorem 3.1, we obtain $\lim_{n\to\infty} \rho(g_n - g_{n+1}) = 0$ and $\{g_n\}$ is a Cauchy sequence. Now we show the existence of the coincidence point for the pairs (S, I) and (T, J). To prove this, we proceed as follows: Since X is complete, there exists $h \in X_\rho$ such that $\lim_{n\to\infty} g_n = h$. That is, $\lim_{n\to\infty} \rho(If_{2n} - h) = \lim_{n\to\infty} \rho(Sf_{2n} - h) = \lim_{n\to\infty} \rho(If_{2n+2} - h) = \lim_{n\to\infty} \rho(Tf_{2n+1} - h) = \lim_{n\to\infty} \rho(Jf_{2n+1} - h) = 0$. By compatibility of (S, I) and (T, J), we have

$$\lim_{n\to\infty} \rho\bigl(I(Sf_{2n}) - S(If_{2n})\bigr) = \lim_{n\to\infty} \rho\bigl(J(Tf_{2n+1}) - T(Jf_{2n+1})\bigr) = 0.$$

By continuity of S, T, I, and J, we have $\lim_{n\to\infty} \rho(S(If_{2n}) - Sh) = \lim_{n\to\infty} \rho(T(Jf_{2n+1}) - Th) = 0$. Note that

$$\begin{aligned}\rho(Ih - Sh) &\leq \omega(3)\rho(Ih - ISf_{2n}) + \omega(3)\rho(ISf_{2n} - SIf_{2n}) + \omega(3)\rho(SIf_{2n} - Sh)\\ &= \omega(3)\bigl[\rho(Ih - ISf_{2n}) + \rho(ISf_{2n} - SIf_{2n}) + \rho(SIf_{2n} - Sh)\bigr],\end{aligned}$$

which on taking the limit as $n \to \infty$ implies that $\rho(Ih - Sh) = 0$, that is, $Ih - Sh = 0$ and $Ih = Sh$. Similarly,

$$\rho(Jh - Th) \leq \omega(3)\bigl[\rho(Jh - JTf_{2n}) + \rho(JTf_{2n} - TJf_{2n}) + \rho(TJf_{2n} - Th)\bigr],$$

which on taking the limit as $n \to \infty$ implies that $\rho(Jh - Th) = 0$, that is, $Jh - Th = 0$ and $Jh = Th$. Next we show that $Jh = Ih$. Assume to the contrary $Jh \neq Ih$, that is, $\rho(Ih - Jh) > 0$. By the given assumption, we have

$$\rho(Ih - Jh) = \rho(Sh - Th) \leq \alpha\rho(Ih - Jh);$$

a contradiction. Hence $Jh = Ih$, therefore, $Sh = Th = Ih = Jh$. Hence h is a coincidence point of S, I, T, and J. □

Corollary 3.6 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space and S and T, continuous self-maps on X_ρ, (S, T) and (T, S) are ρ-partially weakly increasing with respect* *to identity mapping on X, and*

$$\rho(Sf - Tg) \leq \alpha\rho(f - g)$$

holds for every $f, g \in X_\rho$ for which f and g are comparable. Then the pair (S, T) has a common fixed point.

Corollary 3.7 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space and S and J continuous self-maps on X_ρ. Suppose that (S,J) is ρ-compatible, S is ρ-partially weakly increasing with respect to J, and*

$$\rho(Sf - Sg) \leq \alpha\rho(Jf - Jg)$$

holds for every $f, g \in X_\rho$ for which Jf and Jg are comparable. Then the pair (S,J) has a coincidence point $h \in X$.

Example 3.8 Assume that $X_\rho = \ell^1$, where $\rho(x) = \|x\|$ for $x \in \ell^1$. For $x, y \in \ell^1$, define $x \preceq y$ if and only if $x \geq y$.

Let $S,J : \ell^1 \to \ell^1$ be defined as

$$S(x) = \left(\frac{1}{6}x_1, \frac{1}{6}x_2, 0, 0, 0, \dots\right),$$
$$J(x) = \left(\frac{1}{3}x_1, \frac{1}{3}x_2, 0, 0, 0, \dots\right).$$

Note that

$$\begin{aligned}\rho(Sx - Sy) = \|Sx - Ty\| &= \left\|\left(\frac{1}{6}x_1 - \frac{1}{6}y_1, \frac{1}{6}x_2 - \frac{1}{6}y_2, 0, 0, \dots\right)\right\| \\ &= \frac{1}{6}\big[|x_1 - y_1| + |x_2 - y_2|\big] \leq \frac{2}{9}\big[|x_1 - y_1| + |x_2 - y_2|\big] \\ &= \frac{2}{3}\left[\frac{1}{3}|x_1 - y_1| + \frac{1}{3}|x_2 - y_2|\right] = \frac{2}{3}\rho(Jx - Jy).\end{aligned}$$

That is,

$$\rho(Sx - Ty) \leq \frac{2}{3}\rho(Jx - Jy).$$

Note that ρ-compatibility of (S,J) follows immediately. Also S is ρ-weakly partially increasing with respect to J. Indeed, $S(X_\rho) \subseteq J(X_\rho)$ and $Sf \preceq Sg$ for all $g \in J^{-1}(Sf)$. Therefore all conditions of Corollary 3.7 are satisfied. However, the pair (S,J) has $(0, 0, 0, \dots, 0, \dots)$ as a coincidence and a common fixed point.

4 Periodic point results

If S is a map which has a fixed point f, then f is also a fixed point of S^n for every natural number n. However, the converse is false. If a map satisfies $F(S) = F(S^n)$ for each $n \in N$, where $F(S)$ denotes a set of all fixed point of S, then it is said to have property P [23]. We shall say that S and T have property Q if $F(S) \cap F(T) = F(S^n) \cap F(T^n)$. The set $O(T,\infty) = \{f, Tf, T^2f, \ldots\}$ is called the *orbit of* T. The set $O(T,S,\infty) = \{f, Tf, Sf, T^2f, S^2f, \ldots\}$ is called the *orbit of* T *and* S.

As an application of our results in Section 2, we provide the following periodic point theorems.

Theorem 4.1 *Let S be a non-decreasing self-map of a complete ordered modular function space $(X, \preceq, \rho)$, satisfying*

$$\rho(Sf - S^2f) \le \alpha\rho(f - Sf)$$

for all $f \in X$, or (ii) *with strict inequality,* $\alpha = 1$ *and for all $f \in X$, $f \neq Sf$. If, $F(S) \neq \phi$, then S has property P provided that $f \preceq Sf$ for any $f \in F(S^n)$.*

Proof We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $f \in F(S^n)$. Then $f \preceq Sf$, so a non-decreasing characteristic of the mapping S implies that $O(f,\infty)$ is a well-ordered subset of X. Suppose that S satisfies (i). Then

$$\begin{aligned}
\rho(f - Sf) &= \rho\bigl(S\bigl(S^{n-1}f\bigr) - S^2\bigl(S^{n-1}f\bigr)\bigr) \\
&= \rho\bigl(S^2\bigl(S^{n-1}f\bigr) - Sf\bigr) \\
&\le \alpha\rho\bigl(S^{n-1}f - S^nf\bigr) \\
&\le \alpha^2\rho\bigl(S^{n-2}f - S^{n-1}f\bigr) \\
&\le \cdots \le \alpha^n\rho(f - Sf).
\end{aligned}$$

Now the right-hand side of the above inequality approaches zero as $n \to \infty$. Hence $\rho(f - Sf) = 0$, and $f = Sf$. Suppose that S satisfies (ii). If $Sf = f$, then there is nothing to prove. Suppose, if possible, that $Sf \neq f$. Then a repetition of the argument for case (i) leads to

$$\rho(f - Sf) < \rho(f - Sf);$$

a contradiction. Therefore, in all cases, $f = Sf$. □

Theorem 4.2 *Let $(X, \preceq, \rho)$ be a complete ordered modular function space. Let the mappings S and T be as in Corollary* 3.6. *Then S and T have property Q provided that $O(T,S,\infty)$ for every $f \in F(S^n) \cap F(T^n)$.*

Proof From Corollary 3.6, S and T have a common fixed point in X. Let $f \in F(S^n) \cap F(T^n)$. Now,

$$\begin{aligned}
\rho(f - Sf) &= \rho\bigl(T\bigl(T^{n-1}f\bigr) - S\bigl(S^nf\bigr)\bigr) \\
&\le \alpha\rho\bigl(T^{n-1}f - S^nf\bigr) = \alpha\rho\bigl(T^{n-1}f - f\bigr),
\end{aligned}$$

and we have

$$\rho(f - Sf) \le \alpha\rho\bigl(T^{n-1}f - f\bigr) \le \cdots \le \alpha^n\rho(f - Sf).$$

Now the right-hand side of the above inequality approaches zero as $n \to \infty$. Hence $\rho(f - Sf) = 0$, and $f = Sf$. Now,

$$\rho(f - Tf) = \rho(Sf - Tf) \leq \alpha\rho(f - f)$$

give $\rho(f - Tf) = 0$, and $f = Tf$. □

Remark Recently, Paknazar *et al.* [24] gave the existence of the solutions of the integral equations in modular function spaces. Hajji and Hanebaly [25] also applied their fixed point result to obtain the solution of perturbed integral equations in modular function spaces (see also [26]). Our results can also be employed to solve such integral equations in the framework of complete ordered modular function spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

1. Nakano, H: Modulared Semi-Ordered Spaces. Maruzen, Tokyo (1950)
2. Musielak, J, Orlicz, W: On modular spaces. Stud. Math. **18**, 591-597 (1959)
3. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
4. Nieto, JJ, Rodriguez-Lopez, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
5. Nieto, JJ, Rodriguez-Lopez, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. **23**(12), 2205-2212 (2007)
6. Khamsi, MA, Kozolowski, WK, Reich, S: Fixed point theory in modular function spaces. Nonlinear Anal. **14**, 935-953 (1990)
7. Benavides, TD, Khamsi, MA, Samadi, S: Asymptotically regular mappings in modular function spaces. Sci. Math. Jpn. **53**, 295-304 (2001)
8. Khamsi, MA: A convexity property in modular function spaces. Math. Jpn. **44**, 269-279 (1996)
9. Kuaket, K, Kumam, P: Fixed points of asymptotic pointwise contractions in modular spaces. Appl. Math. Lett. **24**, 1795-1798 (2011)
10. Mongkolkeha, C, Kumam, P: Fixed point and common fixed point theorems for generalized weak contraction mappings of integral type in modular spaces. Int. J. Math. Math. Sci. **2011**, Article ID 705943 (2011)
11. Mongkolkeha, C, Kumam, P: Common fixed points for generalized weak contraction mappings in modular spaces. Sci. Math. Jpn. **e-2012**, 117-127 (2012)
12. Mongkolkeha, C, Kumam, P: Some fixed point results for generalized weak contraction mappings in modular spaces. Int. J. Anal. **2013**, Article ID 247378 (2013). doi:10.1155/2013/247378
13. Kumam, P: Fixed point theorem for non-expansive mappings in modular spaces. Arch. Math. **40**, 345-353 (2004)
14. Kutbi, MA, Latif, A: Fixed points of multivalued mappings in modular function spaces. Fixed Point Theory Appl. **2009**, Article ID 786357 (2009)
15. Abbas, M, Khamsi, MA, Khan, AR: Common fixed point and invariant approximation in hyperbolic ordered metric spaces. Fixed Point Theory Appl. **2011**, 25 (2011)
16. Abbas, M, Khan, AR, Nemeth, SZ: Complementarity problems via common fixed points in vector lattices. Fixed Point Theory Appl. **2012**, 60 (2012)
17. Abbas, M, Nazir, T, Radenovic, S: Common fixed points of four maps in partially ordered metric spaces. Appl. Math. Lett. **24**, 1520-1526 (2011)

18. Altun, I, Damjanovic, B, Djoric, D: Fixed point and common fixed point theorems on ordered cone metric spaces. Appl. Math. Lett. **23**, 310-316 (2010)
19. Esmaily, J, Vaezpour, SM, Rhoades, BE: Coincidence point theorem for generalized weakly contractions in ordered metric spaces. Appl. Math. Comput. **219**, 1536-1548 (2012)
20. Harandi, AA, Emami, H: A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. Nonlinear Anal. **72**, 2238-2242 (2010)
21. Kozlowski, WM: Modular Function Spaces. Dekker, New York (1988)
22. Khamsi, MA: Fixed point theory in modular function spaces. In: Recent Advances on Metric Fixed Point Theory, pp. 31-58. Universidad de Sevilla, Sevilla (1996)
23. Jeong, GS, Rhoades, BE: Maps for which $F(T) = F(T^n)$. Fixed Point Theory Appl. **6**, 87-131 (2005)
24. Paknazar, M, Eshaghi, M, Cho, YJ, Vaezpour, SM: A Pata-type fixed point theorem in modular spaces with application. Fixed Point Theory Appl. **2013**, 239 (2013)
25. Hajji, A, Hanebaly, E: Perturbed integral equations in modular function spaces. Electron. J. Qual. Theory Differ. Equ. **2003**, 20 (2003)
26. Taleb, AA, Hanebaly, E: A fixed point theorem and its application to integral equations in modular function spaces. Proc. Am. Math. Soc. **127**, 2335-2342 (1999)

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