Improved Shewhart-type runs-rules nonparametric sign charts

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Abstract

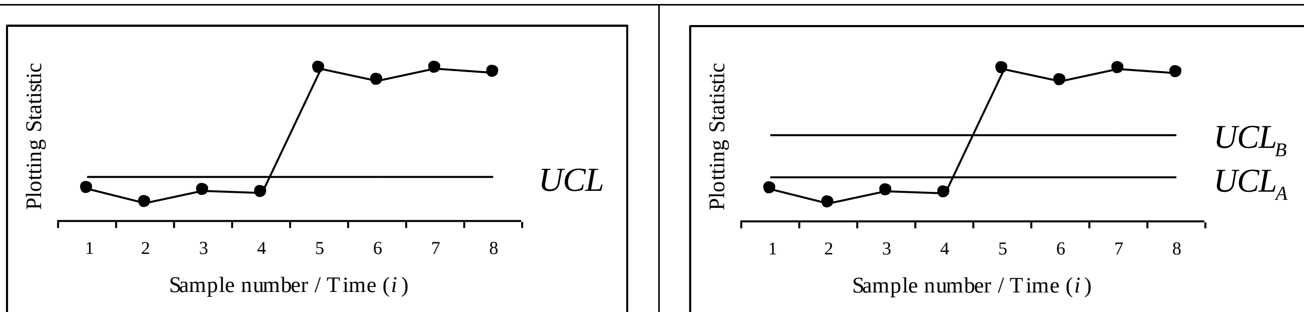
Runs-rules are typically incorporated in control charts to increase their sensitivity to detect small process shifts. However, a drawback of this approach is that runs-rules charts are unable to detect large shifts quickly. In this paper *improved* runs-rules are introduced to the nonparametric sign chart, to address this limitation. *Improved* runs-rules are incorporated to maintain sensitivity to small process shifts, while having the added ability to detect large shifts in the process more efficiently. Performance comparisons between sign charts with runs-rules and sign charts with *improved* runs-rules illustrate that the *improved* runs-rules are superior in performance for large shifts in the process, while maintaining the same sensitivity in the detection of small shifts.

Keywords: nonparametric; sign chart; Shewhart; control chart; runs-rules; *improved* runs-rules; average run-length (*ARL*); control limit; SPC.

1. Introduction

Amin et al. (1995) proposed the nonparametric Shewhart-type control charts based on the well-known sign test statistic (see e.g. Gibbons and Chakraborti (2010)) that can be used to monitor any known or specified percentile of a process; this chart is known as the sign chart. The sign chart is based on the classic *1-of-1* signalling rule i.e. the chart signals when the first plotting statistic plots on or outside the control limit(s) and uses only the most recent plotting statistic to determine if the process is in-control (IC) or out-of-control (OOC). Hence, the *1-of-1* sign chart is known to be relatively insensitive to small process shifts (see e.g. Klein (2000)). Human et al. (2010) addressed this shortcoming of the *1-of-1* sign chart by introducing several runs-type rules such as the *2-of-2* and *2-of-3* runs-rules and showed that their "runs-rules enhanced" sign charts outperform the *1-of-1* sign chart.

Since runs-rules enhanced charts need more than one plotting statistic to be examined (e.g. the *2-of-2* runs-rule needs two plotting statistics), they are not as good at quickly detecting large shifts. As an illustration, consider the upper one-sided *2-of-2* runs-rules enhanced sign chart illustrated in panel (a) of Figure 1. Note that even though there seems to be a substantial assignable cause present in the process from the fifth time point, the chart only signals at the sixth time point, upon the plotting of the sixth plotting statistic above the upper control limit (*UCL*). To remedy this, a second upper control limit can be introduced i.e. UCL_B, which is placed above the *UCL*, and the new chart is designed to signal as soon as a single plotting statistic plots on or above the UCL_B or if any two consecutive plotting statistics plot on or above the *UCL*. Hence, the new chart uses two upper control limits i.e. UCL_B and UCL_A, where the latter is taken to be the same as the *UCL*, which is the upper control limit associated with the *2-of-2* runs-rules sign chart. So, given the extra control limit, the new chart is expected to be as sensitive as the *2-of-2* runs-rules sign chart for detecting small process shifts but more sensitive to larger shifts. This new chart is called the upper one-side *improved 2-of-2* runs-rules sign chart and is shown in panel (b) of Figure 1; this new chart signals on the fifth plotting statistic.



(a) Illustration of an upper one-sided *2-of-2* runs-rules sign chart.

(b) Illustration of an upper one-sided *improved 2-of-2* runs-rules sign chart.

Figure 1: The enhanced and the *improved* runs-rules sign chart.

Khoo and Ariffin (2006) introduced *improved* runs-rules to the Shewhart-type $\overline{X}$ chart and showed that the resulting chart outperforms the $\overline{X}$ chart with runs-rules considered by Klein (2000). We show that this is also the case for the sign chart. It may be noted that Amin et al. (1995) studied the effect of adding warning limits to the sign chart following the work of Page (1962), Weindling et al. (1970) and Champ and Woodall (1987) for the $\overline{X}$ control charts. The motivation for introducing warning limits to the sign chart was to improve the efficiency in detecting small shifts in the process. Coincidentally, one of the sign charts of Amin et al. (1995) with warning limits is the same as the *improved 2-of-2* sign charts as considered here. However, the motivation for introducing our *improved* runs-rules chart is different here in that we want the chart to be able to detect larger shifts quicker. Additional contributions in this paper include the derivation of the entire run-length distribution using a Markov chain approach, its moments such as the average run-length (*ARL*) and the variance of run-length (*VRL*) as well as formulae for the false alarm rate (*FAR*). Moreover, we undertake an extensive performance analysis to compare the runs-rules and the *improved* runs-rules sign charts, which is not done in Amin et al. (1995). The results from our study lends support towards the *improved* runs-rules sign charts as they are seen to be superior in detecting larger process shifts, while maintaining the same sensitivity as the runs-rules in detecting small shifts.

The outline of the rest of the paper is as follows: In Section 2 the plotting statistic of the sign chart is given. In Section 3 notation is defined and a graphical illustration of the *improved* runs-rules chart is discussed. In Section 4 a summary of the signalling events are given. In Section 5 the formulae that are used to calculate the run-length distribution and some characteristics of the run-length distribution are discussed. In Section 6 the design of the *improved* runs-rules sign chart is provided. In Section 7 the performance of the *improved* runs-rules sign charts is discussed. We conclude the article with a summary in Section 8.

2. The *1-of-1* sign chart

Let $X_{i1}, X_{i2}, ..., X_{in}$ denote a random sample of size $n > 1$ at time $i = 1,2,3,...$. Assume that the samples are independent and that each observation follows a continuous distribution with a cumulative distribution function (cdf) denoted by $F_X(x)$ and the unique $100\pi^{th}$ $(0 < \pi < 1)$ percentile is denoted by $\theta = F_X^{-1}(\pi)$ where $0 < \pi < 1$. The plotting statistic is the classical sign statistic which is defined as:

$$T_i = \sum_{j=1}^{n} I(X_{ij} > \theta_0) \quad i = 1,2,3,... \tag{1}$$

$$\text{where} \quad I(X_{ij} > \theta_0) = \begin{cases} 1 & \text{if} \quad X_{ij} > \theta_0 \\ 0 & \text{if} \quad X_{ij} < \theta_0 \end{cases}$$

The plotting statistic in (1) is calculated for each sample and is plotted on the chart to determine if the process is IC or OOC. Note the following three important points:

i. The plotting statistic T_i denotes the number of observations larger than θ_0 in the i^{th} sample of size $n > 1$, where θ_0 denotes the known or the specified or the target value of the percentile of interest i.e. being monitored.

ii. T_i follows a Binomial distribution with parameters n and p, where $p = P(X_{ij} > \theta_0)$ is the probability that an observation is larger than θ_0.

iii. If the percentile of interest is equal to its specified value i.e. $\theta = \theta_0$, the process is said to be IC and in this case p is denoted by $p_0 = P(X_{ij} > \theta_0 \mid \text{IC})$.

iv. For illustration purposes we focus on the scenario where θ denotes the median and θ_0 denotes the median's known or specified value so that $p_0 = P(X_{ij} > \theta_0 \mid \text{IC}) = 0.5$; more is said about this later.

3. Runs-rules and improved runs-rules

In this section a graphical illustration of the *improved* runs-rules charts is discussed and the necessary notation is introduced which is later used to derive the properties of the charts via their run-length distributions. To this end, Figure 2 shows a two-sided *improved* runs-rules chart with two control limits on both sides of the center line (CL). From Figure 2 we observe the following:

i. The *inner* lower (upper) control limit of the *improved* runs-rules charts is denoted by LCL_A (UCL_A) and is taken to be same as the lower (upper) control limit of the runs-rules enhanced charts which is denoted by LCL (UCL) – compare, for example, the control charts in panels (a) and (b) of Figure 1. To avoid any confusion these control limits are denoted by LCL_A / LCL and UCL_A / UCL in Figure 2.

ii. The four control limits $LCL_B, LCL_A, UCL_A, UCL_B$ and the CL divide the vertical axis (i.e. the control region) into nine "zones".

These nine zones play a key role in the derivation of the run-length distributions of the *improved* runs-rules charts. To this end, the discrete random variable Z_i, $i = 1,2,3,...$ is defined; this variable can be any integer value between and including 1 to 9 and its value is determined by the zone in which the plotting statistic (T_i) falls. For example, if T_i falls in zone 5 at time or sample number 4 we have that $Z_4 = 5$. For completeness, the values that Z_i can attain and the corresponding probabilities are shown in Table 1. Note that, in Table 1 $Bin(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$ denotes the probability mass (pmf) function of the Binomial distribution with parameters n and $p = P(X_{ij} > \theta_0)$.

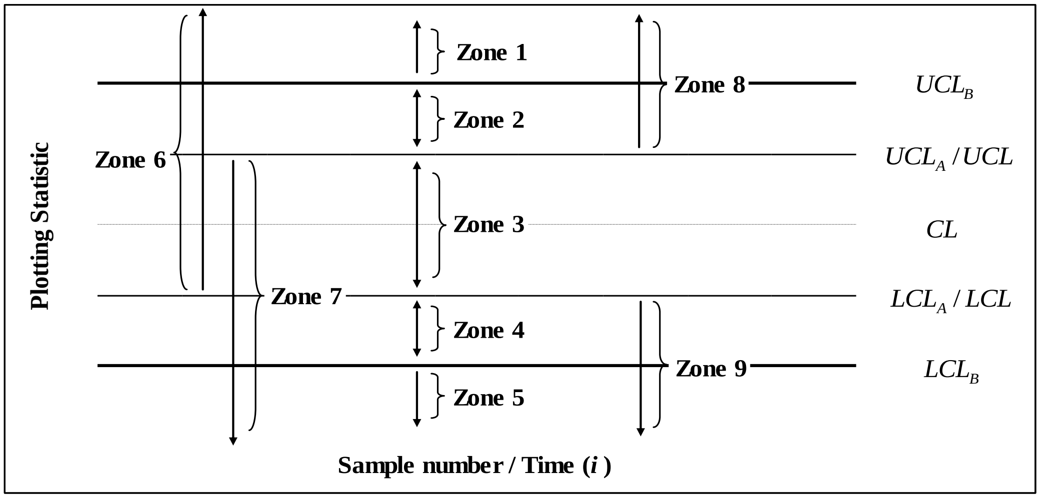


Figure 2: The different "zones" on the runs-rules and *improved* runs-rules control charts.

Table 1: Notation

Definition of Z_i	Corresponding Probability
$Z_i = 1$ if $UCL_B \le T_i$	$p_1 = P(Z_i = 1) = \sum_{UCL_B}^{n} Bin(x; n, p)$
$Z_i = 2$ if $UCL_A \le T_i < UCL_B$	$p_2 = P(Z_i = 2) = \sum_{UCL_A}^{UCL_B - 1} Bin(x; n, p)$
$Z_i = 3$ if $LCL_A / LCL < T_i < UCL_A / UCL$	$p_3 = P(Z_i = 3) = \sum_{LCL_A + 1}^{UCL_A - 1} Bin(x; n, p)$
$Z_i = 4$ if $LCL_B < T_i \le LCL_A$	$p_4 = P(Z_i = 4) = \sum_{LCL_B + 1}^{LCL_A} Bin(x; n, p)$
$Z_i = 5$ if $T_i \le LCL_B$	$p_5 = P(Z_i = 5) = \sum_{0}^{LCL_B} Bin(x; n, p)$
$Z_i = 6$ if $T_i > LCL_A / LCL$	$p_6 = P(Z_i = 6) = \sum_{LCL_A + 1}^{n} Bin(x; n, p)$
$Z_i = 7$ if $T_i < UCL_A / UCL$	$p_7 = P(Z_i = 7) = \sum_{0}^{UCL_A - 1} Bin(x; n, p)$
$Z_i = 8$ if $T_i \ge UCL$	$p_8 = P(Z_i = 8) = \sum_{UCL}^{n} Bin(x; n, p)$
$Z_i = 9$ if $T_i \le LCL$	$p_9 = P(Z_i = 9) = \sum_{0}^{LCL} Bin(x; n, p)$

4. Summary of the signalling events

To derive the properties of the control charts we need to properly define the signalling events of each chart. Hence, in this section the signalling events of the *1-of-1*, *2-of-2* and *improved 2-of-2* control charts are defined – these events are shown in Table 2.

To explain the notation that is used, consider for example the event $E^{1}_{1-of-1(U)}$; this is the signalling event of the *1-of-1* upper one-sided control chart. Note the following:

i. The subscripts indicate which chart is considered, i.e. the *1-of-1*, *2-of-2* etc. and whether it is an upper, lower or a two-sided chart (denoted by *U*, *L* or *T* in the brackets, respectively). In this case the upper one-sided control chart is considered.

ii. The superscript represents the signalling event number; this is necessary because a control chart can have multiple signalling events. In this case the chart has only a single signalling event, so the superscript is 1.

As further examples, consider the events $E^{2}_{2-of-2(T)}$ and $E^{3}_{I2-of-2(L)}$ - the former denotes the second signalling event of the two-sided *2-of-2* chart whereas the latter denotes the third signalling event of the lower one-sided *improved 2-of-2* chart (note the *I* that is in the beginning of the subscript which indicates that the *improved* chart's signalling event is considered).

Table 2: Signalling events of the *1-of-1*, *2-of-2* and *improved 2-of-2* control charts.

The *1-of-1* control charts	**The *2-of-2* control charts**	**The *improved 2-of-2* control charts**
Upper one-sided *1-of-1*	**Upper one-sided *2-of-2***	**Upper one-sided *improved 2-of-2***
$E^{1}_{1-of-1(U)} = \{Z_i = 8\}$	$E^{1}_{2-of-2(U)} = \{Z_{i-1} = 8, Z_i = 8\}$	$E^{1}_{I2-of-2(U)} = \{Z_i = 1\}$ or $E^{2}_{I2-of-2(U)} = \{Z_{i-1} = 2, Z_i = 2\}$
Lower one-sided *1-of-1*	**Lower one-sided *2-of-2***	**Lower one-sided *improved 2-of-2***
$E^{1}_{1-of-1(L)} = \{Z_i = 9\}$	$E^{1}_{2-of-2(L)} = \{Z_{i-1} = 9, Z_i = 9\}$	$E^{1}_{I2-of-2(L)} = \{Z_i = 5\}$ or $E^{2}_{I2-of-2(L)} = \{Z_{i-1} = 4, Z_i = 4\}$
Two-sided *1-of-1*	**Two-sided *2-of-2***	**Two-sided *improved 2-of-2***
$E^{1}_{1-of-1(T)} = \{Z_i = 8\}$ or $E^{2}_{1-of-1(T)} = \{Z_i = 9\}$	$E^{1}_{2-of-2(T)} = \{Z_{i-1} = 8, Z_i = 8\}$ or $E^{2}_{2-of-2(T)} = \{Z_{i-1} = 9, Z_i = 9\}$	$E^{1}_{I2-of-2(T)} = \{Z_i = 1\}$ or $E^{2}_{I2-of-2(T)} = \{Z_{i-1} = 2, Z_i = 2\}$ or $E^{3}_{I2-of-2(T)} = \{Z_i = 5\}$ or $E^{4}_{I2-of-2(T)} = \{Z_{i-1} = 4, Z_i = 4\}$

5. Run-length distribution of the sign charts

The run-length distribution and its various associated characteristics (such as the mean (*ARL*), the variance (*VRL*) etc.) reveal important information regarding the performance of a control chart (see e.g. Human and Graham, 2007). We use a Markov chain approach (see e.g. Fu and Lou, 2003) to derive the necessary results for the proposed sign charts. The essential transition probability matrices of all the control charts are given and the formulas that are used to calculate the elements (transition probabilities) of the essential transition probability matrices are explained.

Transition probabilities

Recall that the sign statistic T_i follows a Binomial distribution with parameters n and p where $p = P(X_{ij} > \theta)$. A probability (element) inside an essential transition probability matrix is associated with the probability of the plotting statistic T_i plotting inside a "zone" on the control chart. The probabilities p_i, $i = 1,2,...,9$ (as defined in Table 1) are required to set up the essential transition probability matrices and can be calculated using the expressions given in Table 1.

Essential transition probability matrices

The essential transition probability matrices for all the charts are provided in Table 3. Using these matrices, the run-length distribution as well as some associated characteristics can be calculated as follows.

Table 3: Essential Transition Probability Matrices.

***1-of-1* charts**	***2-of-2* charts**	***Improved 2-of-2* charts**
Upper *1-of-1* chart	**Upper *2-of-2* chart**	**Upper *improved 2-of-2* chart**
$\mathbf{Q}_{1-of-1(U)} = \begin{bmatrix} 0 & p_7 \\ 0 & p_7 \end{bmatrix}$	$\mathbf{Q}_{2-of-2(U)} = \begin{bmatrix} 0 & p_7 & p_8 \\ 0 & p_7 & p_8 \\ 0 & p_7 & 0 \end{bmatrix}$	$\mathbf{Q}_{I\,2-of-2(U)} = \begin{bmatrix} 0 & p_7 & p_2 \\ 0 & p_7 & p_2 \\ 0 & p_7 & 0 \end{bmatrix}$
Lower *1-of-1* chart	**Lower *2-of-2* chart**	**Lower *improved 2-of-2* chart**
$\mathbf{Q}_{1-of-1(L)} = \begin{bmatrix} 0 & p_9 \\ 0 & p_9 \end{bmatrix}$	$\mathbf{Q}_{2-of-2(L)} = \begin{bmatrix} 0 & p_6 & p_9 \\ 0 & p_6 & p_9 \\ 0 & p_6 & 0 \end{bmatrix}$	$\mathbf{Q}_{I\,2-of-2(L)} = \begin{bmatrix} 0 & p_6 & p_4 \\ 0 & p_6 & p_4 \\ 0 & p_6 & 0 \end{bmatrix}$
Two-sided *1-of-1* chart	**Two-sided *2-of-2* chart**	**Two-sided *improved 2-of-2* chart**

$\mathbf{Q}_{1-of-1(T)} = \begin{bmatrix} 0 & p_3 \\ 0 & p_3 \end{bmatrix}$	$\mathbf{Q}_{2-of-2(T)} = \begin{bmatrix} 0 & p_3 & p_8 & p_9 \\ 0 & p_3 & p_8 & p_9 \\ 0 & p_3 & 0 & p_9 \\ 0 & p_3 & p_8 & 0 \end{bmatrix}$	$\mathbf{Q}_{I\,2-of-2(T)} = \begin{bmatrix} 0 & p_3 & p_2 & p_4 \\ 0 & p_3 & p_2 & p_4 \\ 0 & p_3 & 0 & p_4 \\ 0 & p_3 & p_2 & 0 \end{bmatrix}$

Run-length distribution and various run-length characteristics

The run-length distribution formulae are as follows:

$$P(N = j) = \boldsymbol{\xi}\mathbf{Q}^{j-1}(\mathbf{I}-\mathbf{Q})\mathbf{1}, \text{ for } j = 1,2,3,\ldots \text{ with } \mathbf{Q}^0 = \mathbf{I} \quad \text{(pmf of the run-length distribution)} \quad (2)$$

$$P(N \le j) = 1 - \boldsymbol{\xi}\mathbf{Q}^{j}\mathbf{1}, \text{ for } j = 1,2,3,\ldots \quad \text{(cdf of the run-length distribution)} \quad (3)$$

$$E(N) = \boldsymbol{\xi}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1} \quad \text{(expected value of the run-length distribution = } ARL\text{)} \quad (4)$$

$$Var(N) = \boldsymbol{\xi}(\mathbf{I}+\mathbf{Q})(\mathbf{I}-\mathbf{Q})^{-2}\mathbf{1} - \left(\boldsymbol{\xi}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1}\right)^2 \quad \text{(variance of the run-length distribution = } VRL\text{)} \quad (5)$$

where $\mathbf{Q} = \mathbf{Q}_{h\times h}$ is the essential transition probability matrix for a given chart, $\boldsymbol{\xi} = \boldsymbol{\xi}_{1\times h} = (1,0,0,\ldots,0)$, $\mathbf{1} = \mathbf{1}_{h\times 1} = (1,1,1,\ldots,1)^T$ and $\mathbf{I} = \mathbf{I}_{h\times h}$ is the identity matrix. The variable N denotes the run-length random variable and h is an integer value representing the number of transient states i.e. the number of columns or rows in the matrix $\mathbf{Q}$.

Thus the run-length distribution and various associated characteristics of a chart under consideration can be calculated using equations (2), (3), (4) and (5) and by entering the probabilities $p_1, p_2, \ldots, p_9$ into the corresponding essential transition probability matrix (shown in Table 3). For the IC case (when the percentile of interest is the median), the $p_1, p_2, \ldots, p_9$ are calculated assuming $\theta = \theta_0$ and involves a $Bin(x; n, 0.5)$ distribution whereas for the OOC case run-length distribution and characteristics, the probabilities $p_1, p_2, \ldots, p_9$ are calculated using the $Bin(x; n, p)$ distribution where $p = P(X_{ij} > \theta)$ and $\theta \ne \theta_0$. For additional and more general elaborations on (2), (3), (4) and (5) refer to Fu and Lou (2003) and Fu, Spiring and Xie (2002).

False alarm rates

The *FAR* is the probability that a chart signals when the process is IC. The formula for the *FAR* of each chart is presented in Table 4. Note that in this case the probabilities $(p's)$ in Table 4 are to be calculated using the distribution of T_i given that the process is IC. Thus for example, p_1 is calculated from the expression given in Table 1 with $p = p_0$, where $p_0 = P(X_{ij} > \theta_0 \mid \text{IC})$. When θ_0 is the specified median of the IC distribution, $p_0 = 0.5$ and then $p_1(\text{IC}) = \sum_{UCL_B}^{n} Bin(x; n, 0.5)$.

Table 4: The False Alarm Rates of the *1-of-1, 2-of-2* and *improved 2-of-2* control charts.

For time	*FAR* of the *1-of-1* charts	*FAR* of the *2-of-2* charts	*FAR* of the *improved 2-of-2* charts
	Upper *1-of-1* chart	**Upper *2-of-2* chart**	**Upper *improved 2-of-2* chart**
1	p_8	0	p_1
2,3,4,…	p_8	p_8^2	$p_1 + p_2^2$
	Lower *1-of-1* chart	**Lower *2-of-2* chart**	**Lower *improved 2-of-2* chart**
1	p_9	0	p_5
2,3,4,…	p_9	p_9^2	$p_5 + p_4^2$
	Two-sided *1-of-1* chart	**Two-sided *2-of-2* chart**	**Two-sided *improved 2-of-2* chart**
1	$p_8 + p_9$	0	$p_1 + p_5$
2,3,4,…	$p_8 + p_9$	$p_8^2 + p_9^2$	$p_1 + p_5 + p_2^2 + p_4^2$

Note that in the IC case $\theta = \theta_0$ and hence the probability that the chart signals depends only on three things: (i) the sample size *n*, (ii) some or all of the control limits UCL_B, UCL_A / UCL, LCL_A / LCL and LCL_B, and (iii) the $100\pi^{th}$ percentile denoted by θ_0. Therefore the sign chart is distribution-free since the probability of a signal (and hence the IC run-length distribution) does not depend on the underlying process distribution as long as the process distribution is continuous.

In order to apply the charts in practice, the control limits are needed. This is discussed next.

6. Design of the proposed charts

Designing the chart, i.e., finding the control limits, is important for practical applications. In practice one is typically interested in a chart that has good properties, for instance a large IC *ARL* (ARL_0) (a small *FAR*) and a small OOC *ARL*. The focus in the remaining part of the paper is on the median chart since the median is by far the most popular percentile as a measure of the location and is a robust estimator of location. Hence, $\pi = 0.5$ so that $\theta = F_X^{-1}(0.5)$ and $p = p_0 = P(X_{ij} > \theta_0 \mid \text{IC}) = 0.5$. Charts based on other percentiles can be developed using a similar approach. The charting constants, i.e., the control limits are chosen such that the ARL_0 assumes some values that are informative to the quality practitioner. Note that, because the distribution plotting statistic is symmetric when the median is monitored i.e. $\left(T_i \sim Bin(n,0.5)\right)$, it follows naturally to select the control limits (charting constants) so that they are symmetric i.e. $LCL_B = a$, $LCL_A / LCL = b$, $UCL_A / UCL = c = n - b$ and $UCL_B = d = n - a$, where n is the sample size. Furthermore, note that with this choice of control limits the IC run-length distribution of the upper and lower *improved 2-of-2* charts are identical and therefore the IC performance of the lower and the upper *improved* one-sided sign charts, for monitoring the median, are also identical i.e. they are equal in distribution.

These IC characteristics of the *improved* runs-rules sign charts are calculated by evaluating exact expressions using Proc IML in SAS®9.2 and are shown in Table 5 for certain values of n, a and b. This table should greatly assist the practitioner in the design and implementation of the chart

Table 5: The in-control *ARL* and *FAR* of the upper one-sided, lower one-sided and two-sided *improved 2-of-2* sign charts for the median (*n*=5(1)10,15(5)25).

Sample Size	LCL_B	LCL_A	UCL_A	UCL_B	*I2-of-2* (U & L)			*I2-of-2* (Two-Sided)		
n	*a=n-d*	*b=n-c*	*c=n-b*	*d=n-a*	ARL_0	*FAR1*	*FAR234*	ARL_0	*FAR1*	*FAR234*
5	0	1	4	5	19.10	0.03125	0.05566	9.55	0.06250	0.11133
	0	2	3	5	5.53	0.03125	0.25098			
	1	2	3	4	3.82	0.18750	0.28516			
6	0	1	5	6	42.26	0.01563	0.02441	21.13	0.03125	0.04883
	0	2	4	6	10.34	0.01563	0.12329			
	1	2	4	5	6.50	0.10938	0.16431			
7	0	1	6	7	93.91	0.00781	0.01080	46.96	0.01563	0.02161
	0	2	5	7	21.24	0.00781	0.05566			
	1	2	5	6	11.68	0.06250	0.08942			
8	0	1	7	8	206.05	0.00391	0.00488	103.02	0.00781	0.00977
	0	2	6	8	47.07	0.00391	0.02368	23.54	0.00781	0.04736
	1	2	6	7	21.77	0.03516	0.04712			
9	0	1	8	9	443.11	0.00195	0.00226	221.55	0.00391	0.00452
	0	2	7	9	110.45	0.00195	0.00968	55.23	0.00391	0.01936
	1	2	7	8	41.41	0.01953	0.02448			
10	0	1	9	10	933.70	0.00098	0.00107	466.85	0.00195	0.00214
	0	2	8	10	269.22	0.00098	0.00386	134.61	0.00195	0.00772
	0	3	7	10	38.58	0.00098	0.03018			
	1	2	8	9	79.41	0.01074	0.01267	39.71	0.02148	0.02535
15	0	3	12	15	3001.87	0.00003	0.00034	1500.93	0.00006	0.00068
	0	4	11	15	299.43	0.00003	0.00354	149.71	0.00006	0.00707
	0	5	10	15	50.50	0.00003	0.02279	25.25	0.00006	0.04557
	1	3	12	14	1289.60	0.00049	0.00078	644.80	0.00098	0.00156
	1	4	11	14	266.81	0.00049	0.00394	133.41	0.00098	0.00788
	1	5	10	14	49.63	0.00049	0.02311	24.82	0.00098	0.04621
	2	3	12	13	257.55	0.00369	0.00389	128.77	0.00739	0.00777

	2	4	11	13	151.17	0.00369	0.00678	75.58	0.00739	0.01356
	2	5	10	13	44.29	0.00369	0.02536	22.15	0.00739	0.05071
	3	4	11	12	51.96	0.01758	0.01931	25.98	0.03516	0.03863
	4	5	10	11	14.94	0.05923	0.06763			
	0	5	15	20	2378.10	0.00000	0.00043	1189.05	0.00000	0.00086
	0	6	14	20	318.05	0.00000	0.00333	159.02	0.00000	0.00665
	0	7	13	20	65.35	0.00000	0.01732	32.67	0.00000	0.03463
	1	5	15	19	2278.88	0.00002	0.00045	1139.44	0.00004	0.00089
	1	6	14	19	316.33	0.00002	0.00334	158.17	0.00004	0.00668
	1	7	13	19	65.28	0.00002	0.01733	32.64	0.00004	0.03466
	2	5	15	18	1631.92	0.00020	0.00062	815.96	0.00040	0.00124
20	2	6	14	18	300.91	0.00020	0.00350	150.45	0.00040	0.00701
	2	7	13	18	64.69	0.00020	0.01746	32.34	0.00040	0.03493
	3	4	16	17	763.55	0.00129	0.00131	381.78	0.00258	0.00262
	3	6	14	17	232.75	0.00129	0.00447	116.37	0.00258	0.00893
	3	7	13	17	61.32	0.00129	0.01827	30.66	0.00258	0.03653
	4	5	15	16	163.28	0.00591	0.00613	81.64	0.01182	0.01226
	4	6	14	16	118.27	0.00591	0.00859	59.13	0.01182	0.01717
	4	7	13	16	50.15	0.00591	0.02170	25.07	0.01182	0.04341
	5	6	14	15	45.43	0.02069	0.02206	22.71	0.04139	0.04412
	0	7	18	25	2180.98	0.00000	0.00047	1090.49	0.00000	0.00094
	0	8	17	25	363.07	0.00000	0.00290	181.54	0.00000	0.00581
	0	9	16	25	84.64	0.00000	0.01317	42.32	0.00000	0.02634
	0	10	15	25	26.93	0.00000	0.04502			
	1	7	18	24	2177.59	0.00000	0.00047	1088.80	0.00000	0.00094
	1	8	17	24	362.98	0.00000	0.00290	181.49	0.00000	0.00581
	1	9	16	24	84.64	0.00000	0.01317	42.32	0.00000	0.02634
	1	10	15	24	26.93	0.00000	0.04502			
	2	7	18	23	2137.72	0.00001	0.00048	1068.86	0.00002	0.00096
	2	8	17	23	361.93	0.00001	0.00291	180.96	0.00002	0.00582
	2	9	16	23	84.59	0.00001	0.01318	42.29	0.00002	0.02636
	2	10	15	23	26.92	0.00001	0.04503			
	3	6	19	22				3837.93	0.00016	0.00026
25	3	7	18	22	1874.53	0.00008	0.00054	937.27	0.00016	0.00109
	3	8	17	22	354.02	0.00008	0.00297	177.01	0.00016	0.00594
	3	9	16	22	84.19	0.00008	0.01323	42.10	0.00016	0.02646
	3	10	15	22	26.89	0.00008	0.04506			
	4	5	20	21				1092.27	0.00091	0.00092
	4	6	19	21				995.98	0.00091	0.00100
	4	7	18	21	1117.50	0.00046	0.00090	558.75	0.00091	0.00181
	4	8	17	21	316.02	0.00046	0.00331	158.01	0.00091	0.00662
	4	9	16	21	82.10	0.00046	0.01352	41.05	0.00091	0.02704
	4	10	15	21	26.70	0.00046	0.04528			
	5	6	19	20	483.94	0.00204	0.00207	241.97	0.00408	0.00413
	5	7	18	20	413.98	0.00204	0.00242	206.99	0.00408	0.00485
	5	8	17	20	217.71	0.00204	0.00473	108.85	0.00408	0.00945
	5	9	16	20	74.31	0.00204	0.01475	37.15	0.00408	0.02949
	5	10	15	20	25.95	0.00204	0.04620			

	6	7	18	19	133.00	0.00732	0.00752	66.50	0.01463	0.01504
	6	8	17	19	106.52	0.00732	0.00948	53.26	0.01463	0.01897
	6	9	16	19	56.37	0.00732	0.01886	28.18	0.01463	0.03772
	7	8	17	18	44.15	0.02164	0.02268	22.08	0.04329	0.04536

Note that, because the sign statistic is discrete, it can only assume a finite number of values and consequently, the sign chart can only assume a finite number of *ARL* and *FAR* combinations for a certain value of *n* (i.e. the sample size). However, it should be noted that the possible *ARL* and *FAR* combinations increase as *n* increases. The sample size needs to be at least 9 and 10 for the one-sided and two-sided *improved 2-of-2* sign chart, respectively, so that enough of practically usable *ARL* and *FAR* combinations are obtained.

Example

An example where the median is monitored is presented to illustrate that the application of the sign charts. In this example the median of the inside diameter measurements of forged automobile engine piston rings are monitored. The observations that was used for this example is given on p.223 in Table 5.3 of Montgomery (2005), the observations was supplemented with additional observations given on p.250 in exercise 5.10 of Montgomery (2005). Note that the data is modified by grouping two consecutive samples of size five together to obtain nineteen samples (i.e. a total of 190 observations) of size $n = 10$ each.

In order to apply the *improved 2-of-2* runs-rules sign charts the charting constants are required. Table 6 is provided to aid in choosing appropriate charting constants.

Table 6: The in-control characteristics (*ARL* and *FAR*) of the two-sided *improved 2-of-2* sign chart for the median (*n*=10).

***I2-of-2* (Two-Sided)**						
LCL_B	LCL_A	UCL_A	UCL_B	ARL_0	***FAR1***	***FAR234***
a=n-d	***b=n-c***	***c=n-b***	***d=n-a***			
0	1	9	10	466.85	0.00195	0.00214
0	2	8	10	134.61	0.00195	0.00772
1	2	8	9	39.71	0.02148	0.02535

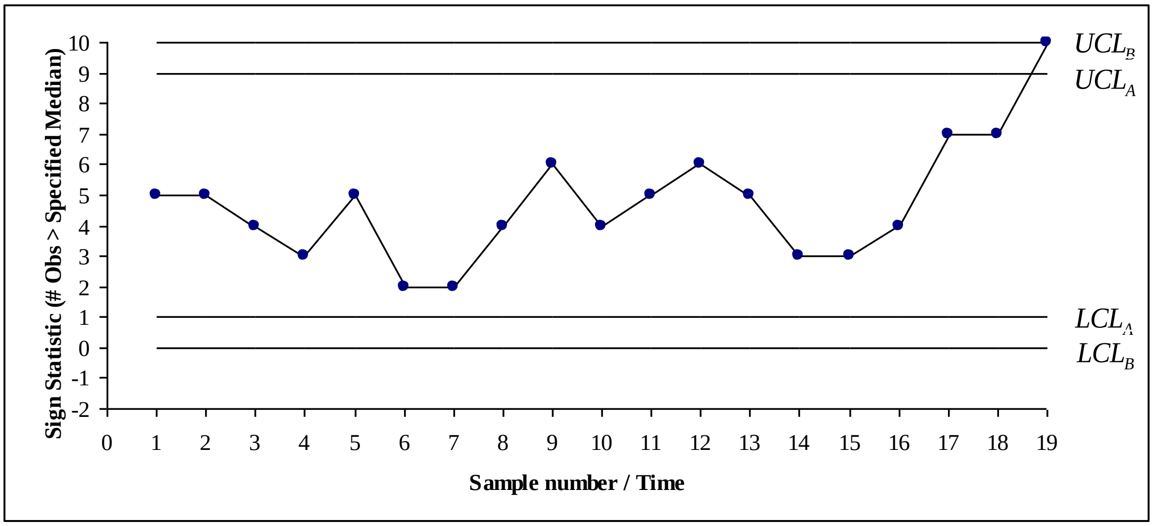


Figure 3: The two-sided *improved 2-of-2* sign chart for monitoring the median for Montgomery (2005) piston-ring data.

A plot of the sign statistics is presented in Figure 3. It is seen that the two-sided *improved 2-of-2* sign chart signals at the 19th sample. It may be noted that the conclusions are the same for the *1-of-1* and *improved 2-of-2* sign charts. Figure 3 is also a visual representation of the two-sided *1-of-1* sign and the two-sided *2-of-2* sign charts. However these charts have different signalling rules and consequently different run-length distributions and run-length characteristics but can be calculated using the methodology discussed in Section 5.

Note that the two-sided *1-of-1* sign chart signals at the 19th sample since this is the first time that a single plotting statistic plots on or above the *UCL*=9. However the two-sided *2-of-2* sign chart does not signal at the sample 19th sample since the chart signals following the event that two consecutive plotting statistics plot on or above the *UCL*=9.

7. Performance of the *improved* runs-rules sign charts

Performance analysis between two competing charts are performed when their ARL_0 and/or *FAR* are equal or, at least be approximately so. The chart that detects a shift in the process in the least amount of observation (smallest OOC *ARL*) is declared to be the best chart.

The *improved* runs-rules charts are compared to the runs-rules charts to illustrate the advantages of the *improved* runs-rules charts over the runs-rules charts. These performance comparisons were done by evaluating exact expressions using Proc IML in SAS®9.2.

Recall that our claims are that *improved* runs-rules sign charts are superior in performance to runs-rules sign charts for large shifts, while maintaining the same sensitivity in the detection of small shifts. Performance analysis is done to confirm these claims.

The performance comparisons between the *improved* runs-rules sign charts and runs-rules sign charts are done by considering the Normal distribution (which is symmetric), Students-*t* distribution (which symmetric with heavier tails than the Normal distribution) and the Exponential distribution (which is positively skewed) as the underlying process distributions, shown in Figure 4.

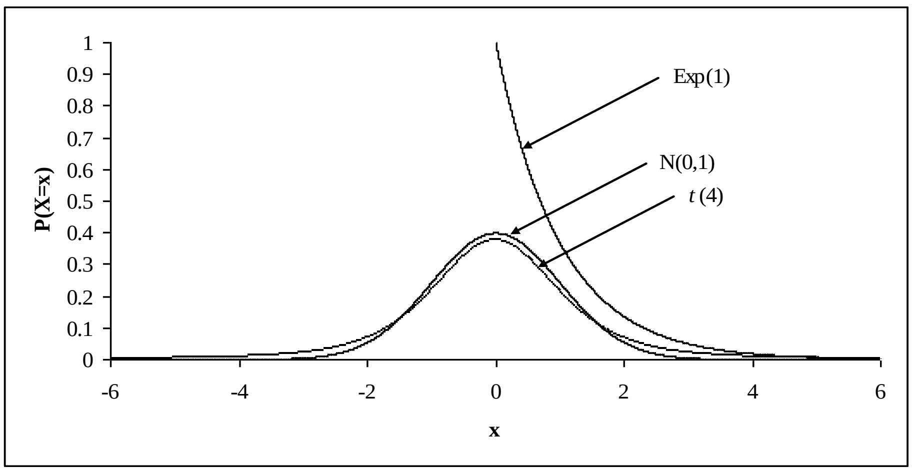


Figure 4: Different underlying process distributions used to perform performance comparisons.

Discussion on the performance analysis results

Considering Tables 7 to 11 the following observations can be made regarding the performance comparison between the runs-rules and *improved* runs-rules charts:

- From Tables 7, 8 and 9 we can see that the upper and lower one-sided *improved 2-of-2* sign charts are as sensitive to small process shifts compared to the upper and lower one-sided *2-of-2* sign charts, while having superior performance in the detection of large process shifts. From this it can be concluded that the one-sided *improved 2-of-2* sign charts are an improvement.
- From Tables 10 and 11 we can see that the two-sided *improved 2-of-2* sign chart is as sensitive to small process shifts compared to the two-sided *2-of-2* sign chart while having superior performance in the detection of large process shifts.

- From investigating the performance analysis one can conclude that the *improved* runs-rules sign charts are as sensitive to small shifts as the runs-rules sign charts, while having superior performance in the detection of large process shifts compared to runs-rules sign charts.

Remark 1

The OOC characteristics for the upper and lower one-sided *N*(0,1) and *t*(4) distributions are presented in the same Tables since the distributions are symmetric (i.e. the performance is identical). The shifts for the lower one-sided charts (downward shifts) are presented in brackets in the column labelled "Shift" (on the left hand side of the tables).

Table 7: The OOC characteristics of the upper one-sided and lower one-sided *improved 2-of-2* sign charts for the median (n=20) where LCL_B =1, LCL_A =6, UCL_A =14 and UCL_B =19.

***N*(0,1) Distribution**														
σ ***Units***	***2-of-2* (U&L)**							***I2-of-2* (U&L)**						
Shift	***ARL***	***SDRL***	5^{th}	Q_1	***MDRL***	Q_3	95^{th}	***ARL***	***SDRL***	5^{th}	Q_1	***MDRL***	Q_3	95^{th}
-0.2 (0.2)	8474.71	8473.22	436	2439	5875	11748	25385	8414.54	8413.06	433	2422	5833	11664	25205
-0.1 (0.1)	1449.97	1448.49	76	418	1005	2010	4341	1441.17	1439.70	75	416	999	1997	4314
0	318.13	316.68	18	93	221	440	950	316.33	314.89	18	92	220	438	945
0.1 (-0.1)	89.23	87.81	6	27	62	123	264	88.71	87.31	6	27	62	122	263
0.2 (-0.2)	31.71	30.34	3	10	22	43	92	31.51	30.15	3	10	22	43	92
0.3 (-0.3)	14.03	12.70	2	5	10	19	39	13.93	12.61	2	5	10	19	39
0.4 (-0.4)	7.53	6.22	2	3	6	10	20	7.46	6.17	2	3	6	10	20
0.5 (-0.5)	4.76	3.45	2	2	4	6	12	4.71	3.41	2	2	4	6	12
0.6 (-0.6)	3.44	2.09	2	2	3	4	8	3.38	2.06	2	2	2	4	8
0.7 (-0.7)	2.76	1.34	2	2	2	3	6	2.69	1.33	2	2	2	3	5
0.8 (-0.8)	2.39	0.89	2	2	2	2	4	2.30	0.90	1	2	2	2	4
0.9 (-0.9)	2.19	0.60	2	2	2	2	4	2.07	0.65	1	2	2	2	4
1 (-1)	2.09	0.40	2	2	2	2	3	1.92	0.52	1	2	2	2	2
1.1 (-1.1)	2.04	0.26	2	2	2	2	2	1.80	0.48	1	2	2	2	2
1.2 (-1.2)	2.02	0.16	2	2	2	2	2	1.70	0.49	1	1	2	2	2
1.3 (-1.3)	2.01	0.10	2	2	2	2	2	1.59	0.50	1	1	2	2	2
1.4 (-1.4)	2.00	0.06	2	2	2	2	2	1.49	0.50	1	1	1	2	2
1.5 (-1.5)	2.00	0.03	2	2	2	2	2	1.39	0.49	1	1	1	2	2
1.6 (-1.6)	2.00	0.02	2	2	2	2	2	1.30	0.46	1	1	1	2	2
1.7 (-1.7)	2.00	0.01	2	2	2	2	2	1.22	0.42	1	1	1	1	2
1.8 (-1.8)	2.00	0.00	2	2	2	2	2	1.16	0.37	1	1	1	1	2

Table 8: The OOC characteristics of the upper one-sided and lower one-sided *improved 2-of-2* sign charts for the median (*n*=20) where LCL_B =1, LCL_A =6, UCL_A =14 and UCL_B =19.

***T*(4) Distribution**														
σ Units	***2-of-2* (U&L)**							***I2-of-2* (U&L)**						
Shift	***ARL***	***SDRL***	***5th***	***Q_1***	***MDRL***	***Q_3***	***95th***	***ARL***	***SDRL***	***5th***	***Q_1***	***MDRL***	***Q_3***	***95th***
-0.2 (0.2)	30268.2	30266.7	1554	8709	20981	41960	90673	30017.7	30016.2	1541	8637	20807	41613	89922
-0.1 (0.1)	2516.02	2514.54	130	725	1744	3487	7534	2500.11	2498.64	130	720	1733	3465	7487
0	318.13	316.68	18	93	221	440	950	316.33	314.89	18	92	220	438	945
0.1 (-0.1)	61.97	60.58	4	19	43	85	183	61.61	60.22	4	19	43	85	182
0.2 (-0.2)	18.29	16.95	2	6	13	25	52	18.16	16.83	2	6	13	25	52
0.3 (-0.3)	7.76	6.45	2	3	6	10	21	7.69	6.39	2	3	6	10	20
0.4 (-0.4)	4.39	3.07	2	2	3	6	11	4.34	3.03	2	2	3	5	10
0.5 (-0.5)	3.07	1.70	2	2	2	4	6	3.01	1.67	2	2	2	4	6
0.6 (-0.6)	2.49	1.02	2	2	2	3	4	2.41	1.02	2	2	2	2	4
0.7 (-0.7)	2.22	0.65	2	2	2	2	4	2.11	0.69	1	2	2	2	4
0.8 (-0.8)	2.10	0.41	2	2	2	2	3	1.93	0.53	1	2	2	2	3
0.9 (-0.9)	2.04	0.26	2	2	2	2	2	1.81	0.48	1	2	2	2	2
1 (-1)	2.02	0.16	2	2	2	2	2	1.70	0.49	1	1	2	2	2
1.1 (-1.1)	2.01	0.10	2	2	2	2	2	1.60	0.50	1	1	2	2	2
1.2 (-1.2)	2.00	0.06	2	2	2	2	2	1.50	0.50	1	1	1	2	2
1.3 (-1.3)	2.00	0.04	2	2	2	2	2	1.41	0.49	1	1	1	2	2
1.4 (-1.4)	2.00	0.02	2	2	2	2	2	1.34	0.47	1	1	1	2	2
1.5 (-1.5)	2.00	0.01	2	2	2	2	2	1.27	0.44	1	1	1	2	2
1.6 (-1.6)	2.00	0.01	2	2	2	2	2	1.21	0.41	1	1	1	1	2
1.7 (-1.7)	2.00	0.00	2	2	2	2	2	1.17	0.37	1	1	1	1	2
1.8 (-1.8)	2.00	0.00	2	2	2	2	2	1.13	0.34	1	1	1	1	2
1.9 (-1.9)	2.00	0.00	2	2	2	2	2	1.10	0.30	1	1	1	1	2
2 (-2)	2.00	0.00	2	2	2	2	2	1.08	0.27	1	1	1	1	2
2.1 (-2.1)	2.00	0.00	2	2	2	2	2	1.06	0.24	1	1	1	1	2
2.2 (-2.2)	2.00	0.00	2	2	2	2	2	1.05	0.22	1	1	1	1	1

Table 9: The OOC characteristics of the upper one-sided *improved 2-of-2* sign chart for the median (*n*=20) where LCL_B =1, LCL_A =6, UCL_A =14 and UCL_B =19.

***Exp*(1) Distribution**														
σ Units	***2-of-2* (U)**							***I2-of-2* (U)**						
Shift	***ARL***	***SDRL***	***5th***	***Q_1***	***MDRL***	***Q_3***	***95th***	***ARL***	***SDRL***	***5th***	***Q_1***	***MDRL***	***Q_3***	***95th***
-0.2	14869.17	14867.68	764	4279	10307	20612	44541	14756.61	14755.12	758	4246	10229	20456	44204
-0.1	2008.01	2006.53	104	579	1392	2783	6013	1995.54	1994.07	104	575	1384	2766	5975
0	318.13	316.68	18	93	221	440	950	316.33	314.89	18	92	220	438	945
0.1	62.36	60.96	5	19	44	86	184	61.99	60.60	4	19	43	85	183
0.2	16.10	14.77	2	6	12	22	46	15.99	14.67	2	6	11	22	45
0.3	5.78	4.47	2	2	4	7	15	5.73	4.43	2	2	4	7	15
0.4	2.97	1.58	2	2	2	4	6	2.91	1.56	2	2	2	4	6
0.5	2.15	0.53	2	2	2	2	3	2.02	0.60	1	2	2	2	3
0.6	2.00	0.08	2	2	2	2	2	1.54	0.50	1	1	2	2	2
0.7	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1

Table 10: The OOC characteristics of the two-sided *improved 2-of-2* sign chart for the median (n=20) where LCL_B =1, LCL_A =6, UCL_A =14 and UCL_B =19.

***N*(0,1) Distribution**														
σ ***Units***	***2-of-2* (T)**							***I2-of-2* (T)**						
Shift	***ARL***	***SDRL***	5^{th}	Q_1	***MDRL***	Q_3	95^{th}	***ARL***	***SDRL***	5^{th}	Q_1	***MDRL***	Q_3	95^{th}
2.2	2.00	0.00	2	2	2	2	2	1.03	0.17	1	1	1	1	1
2.1	2.00	0.00	2	2	2	2	2	1.05	0.22	1	1	1	1	1
2	2.00	0.00	2	2	2	2	2	1.08	0.26	1	1	1	1	2
1.9	2.00	0.00	2	2	2	2	2	1.11	0.31	1	1	1	1	2
1.8	2.00	0.00	2	2	2	2	2	1.16	0.37	1	1	1	1	2
1.7	2.00	0.01	2	2	2	2	2	1.22	0.42	1	1	1	1	2
1.6	2.00	0.02	2	2	2	2	2	1.30	0.46	1	1	1	2	2
1.5	2.00	0.03	2	2	2	2	2	1.39	0.49	1	1	1	2	2
1.4	2.00	0.06	2	2	2	2	2	1.49	0.50	1	1	1	2	2
1.3	2.01	0.10	2	2	2	2	2	1.59	0.50	1	1	2	2	2
1.2	2.02	0.16	2	2	2	2	2	1.70	0.49	1	1	2	2	2
1.1	2.04	0.26	2	2	2	2	2	1.80	0.48	1	2	2	2	2
1	2.09	0.40	2	2	2	2	3	1.92	0.52	1	2	2	2	2
0.9	2.19	0.60	2	2	2	2	4	2.07	0.65	1	2	2	2	4
0.8	2.39	0.89	2	2	2	2	4	2.30	0.90	1	2	2	2	4
0.7	2.76	1.34	2	2	2	3	6	2.69	1.33	2	2	2	3	5
0.6	3.44	2.09	2	2	3	4	8	3.38	2.06	2	2	2	4	8
0.5	4.76	3.45	2	2	4	6	12	4.71	3.41	2	2	4	6	12
0.4	7.53	6.22	2	3	6	10	20	7.46	6.17	2	3	6	10	20
0.3	14.02	12.69	2	5	10	19	39	13.92	12.60	2	5	10	19	39
0.2	31.59	30.22	3	10	22	43	92	31.39	30.03	3	10	22	43	91
0.1	84.05	82.64	6	25	59	116	249	83.57	82.16	6	25	58	115	248
0	159.07	157.61	10	47	111	220	474	158.17	156.72	9	47	110	219	471
-0.1	84.05	82.64	6	25	59	116	249	83.57	82.16	6	25	58	115	248
-0.2	31.59	30.22	3	10	22	43	92	31.39	30.03	3	10	22	43	91
-0.3	14.02	12.69	2	5	10	19	39	13.92	12.60	2	5	10	19	39
-0.4	7.53	6.22	2	3	6	10	20	7.46	6.17	2	3	6	10	20
-0.5	4.76	3.45	2	2	4	6	12	4.71	3.41	2	2	4	6	12
-0.6	3.44	2.09	2	2	3	4	8	3.38	2.06	2	2	2	4	8
-0.7	2.76	1.34	2	2	2	3	6	2.69	1.33	2	2	2	3	5
-0.8	2.39	0.89	2	2	2	2	4	2.30	0.90	1	2	2	2	4
-0.9	2.19	0.60	2	2	2	2	4	2.07	0.65	1	2	2	2	4
-1	2.09	0.40	2	2	2	2	3	1.92	0.52	1	2	2	2	2
-1.1	2.04	0.26	2	2	2	2	2	1.80	0.48	1	2	2	2	2
-1.2	2.02	0.16	2	2	2	2	2	1.70	0.49	1	1	2	2	2
-1.3	2.01	0.10	2	2	2	2	2	1.59	0.50	1	1	2	2	2
-1.4	2.00	0.06	2	2	2	2	2	1.49	0.50	1	1	1	2	2
-1.5	2.00	0.03	2	2	2	2	2	1.39	0.49	1	1	1	2	2
-1.6	2.00	0.02	2	2	2	2	2	1.30	0.46	1	1	1	2	2
-1.7	2.00	0.01	2	2	2	2	2	1.22	0.42	1	1	1	1	2
-1.8	2.00	0.00	2	2	2	2	2	1.16	0.37	1	1	1	1	2
-1.9	2.00	0.00	2	2	2	2	2	1.11	0.31	1	1	1	1	2
-2	2.00	0.00	2	2	2	2	2	1.08	0.26	1	1	1	1	2

-2.1	2.00	0.00	2	2	2	2	2	1.05	0.22	1	1	1	1	1
-2.2	2.00	0.00	2	2	2	2	2	1.03	0.17	1	1	1	1	1

Table 11: The OOC characteristics of the two-sided *improved 2-of-2* sign chart for the median (n=20) where LCL_B =1, LCL_A =6, UCL_A =14 and UCL_B =19.

***Exp*(1) Distribution**														
σ Units	***2-of-2* (T)**							***I2-of-2* (T)**						
Shift	***ARL***	***SDRL***	***5^{th}***	***Q_1***	***MDRL***	***Q_3***	***95^{th}***	***ARL***	***SDRL***	***5^{th}***	***Q_1***	***MDRL***	***Q_3***	***95^{th}***
2.2	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
2.1	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
2	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.9	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.8	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.7	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.6	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.5	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.4	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.3	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.2	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1.1	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
1	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
0.9	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
0.8	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
0.7	2.00	0.00	2	2	2	2	2	1.00	0.00	1	1	1	1	1
0.6	2.00	0.08	2	2	2	2	2	1.54	0.50	1	1	2	2	2
0.5	2.15	0.53	2	2	2	2	3	2.02	0.60	1	2	2	2	3
0.4	2.97	1.58	2	2	2	4	6	2.91	1.56	2	2	2	4	6
0.3	5.78	4.47	2	2	4	7	15	5.72	4.43	2	2	4	7	15
0.2	16.10	14.76	2	6	12	22	46	15.99	14.66	2	6	11	22	45
0.1	60.83	59.43	4	18	43	84	179	60.48	59.08	4	18	42	83	178
0	159.07	157.61	10	47	111	220	474	158.17	156.72	9	47	110	219	471
-0.1	69.11	67.70	5	21	48	95	204	68.71	67.31	5	21	48	95	203
-0.2	24.42	23.06	2	8	17	33	70	24.26	22.91	2	8	17	33	70
-0.3	11.36	10.04	2	4	8	15	31	11.28	9.96	2	4	8	15	31
-0.4	6.60	5.29	2	3	5	9	17	6.54	5.24	2	3	5	9	17
-0.5	4.50	3.18	2	2	4	6	11	4.44	3.14	2	2	3	6	11
-0.6	3.44	2.09	2	2	3	4	8	3.39	2.06	2	2	2	4	8
-0.7	2.86	1.46	2	2	2	3	6	2.80	1.44	2	2	2	3	6
-0.8	2.52	1.06	2	2	2	3	5	2.45	1.06	2	2	2	3	4
-0.9	2.32	0.79	2	2	2	2	4	2.22	0.81	1	2	2	2	4
-1	2.19	0.60	2	2	2	2	4	2.07	0.65	1	2	2	2	4
-1.1	2.12	0.45	2	2	2	2	3	1.96	0.55	1	2	2	2	3
-1.2	2.07	0.35	2	2	2	2	2	1.88	0.50	1	2	2	2	2
-1.3	2.04	0.26	2	2	2	2	2	1.81	0.48	1	2	2	2	2
-1.4	2.02	0.20	2	2	2	2	2	1.74	0.48	1	1	2	2	2
-1.5	2.01	0.15	2	2	2	2	2	1.68	0.49	1	1	2	2	2
-1.6	2.01	0.11	2	2	2	2	2	1.62	0.50	1	1	2	2	2

-1.7	2.00	0.08	2	2	2	2	2	1.56	0.50	1	1	2	2	2
-1.8	2.00	0.06	2	2	2	2	2	1.50	0.50	1	1	2	2	2
-1.9	2.00	0.05	2	2	2	2	2	1.45	0.50	1	1	1	2	2
-2	2.00	0.03	2	2	2	2	2	1.40	0.49	1	1	1	2	2
-2.1	2.00	0.02	2	2	2	2	2	1.35	0.48	1	1	1	2	2
-2.2	2.00	0.02	2	2	2	2	2	1.31	0.46	1	1	1	2	2

Remark 2

Note that in order to perform performance analysis a sufficiently large sample size is required so that there is an outer set of control limits that has an absolute minimal influence on the ARL_0 of the *improved* runs-rules charts. This is done so that the ARL_0 for both runs-rules and *improved* runs-rules are almost exactly the same by choosing the inner set of control limits of the *improved* runs-rules charts and the control limits of the runs-rules charts the same. Consequently the sample size is chosen to be 20 in the performance analysis.

8. Summary and Conclusions

Improved runs-rules are introduced to the sign chart. The run-length distribution of the *improved* runs-rules sign charts are derived using a Markov chain approach. Performance analysis is carried out to illustrate that the *improved* runs-rules sign charts are superior in performance to runs-rules sign charts for large shifts in the process, while maintaining the same sensitivity in the detection of small shifts.

The performance analysis confirms that the *improved* runs-rules sign charts are superior in performance to runs-rules sign charts for large shifts, while maintaining the same sensitivity in the detection of small shifts. We conclude with a summary of the strengths of the *improved* runs-rules sign charts: 1) Does not require a specified underlying process distribution (nonparametric), 2) Does not require the variance of the process to be established, 3) Can monitor any desirable percentile of the underlying process distribution, 4) Does not require the actual measurements, but only the count of observations within each sample that are larger (or smaller than) the specified value of the percentile of interest, 5) Is as sensitive to small process shifts as existing runs-rules based sign charts but superior in the detection of large process shifts.

On balance it may be said that the *improved* runs-rules sign charts are slightly more complex than the runs-rules charts. However, there are rewards in terms of better performance.

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